Chapter 1

Introduction

1.1 Introduction

The field of *algebraic logic* assumed its modern systematic form, known as *abstract algebraic logic*, with the appearance of the pioneering "Memoirs" monograph of Blok and Pigozzi [36]. In this celebrated monograph one can find clearly discernible the seeds and the foundations of almost all subsequent developments in the field and, consequently, also, the foundations on which most parts of the work and of the developments detailed in the present monograph are based.

Related to the term "abstract algebraic logic", another of the pioneers of the field, Josep Maria Font, in a more recent textbook, titled "Abstract Algebraic Logic An Introductory Textbook" [89], advocates that the name should continue to be simply *algebraic logic* and that, as is the case with most other fields of Mathematics, Logic and Science, the abstraction, to which the term "abstract" refers, is part of the natural evolution of the same field, and should not be construed as constituting a special subfield justifying a special naming or rebranding.

In a similar sense, one may share the same belief for *categorical abstract algebraic logic*, which is also another natural evolution of algebraic logic and, therefore, according to this point of view, should also be referred to, simply, as *algebraic logic*. It may, in fact, be preferable to refer to the underlying formalizations of the logical systems treated in each particular context than to rebrand the entire field. So instead of referring to "abstract algebraic logic", we may say "algebraic logic as applied to sentential logics" (or "to deductive systems") and, similarly, "algebraic logic as applied to logics formalized as institutions or π-institutions", instead of using "categorical abstract algebraic logic" for the latter. For now, however, the traditional names have stuck and have been used widely, with well-discernible meanings, and we use them freely, as is also done in [89].

In "traditional" algebraic logic, which may be viewed to have started with the work of Tarski [5], the underlying formalism consists of *sentential logics* or *deductive systems*. These are pairs $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is an algebraic language (a set of operation symbols with specified finite arities) and $\vdash_{\mathcal{S}}$ is a *consequence relation* on the absolutely free algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ generated by a countable set V of variables. That is, $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$, satisfies the following, for all $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

Inflation: $\Gamma \vdash_{\mathcal{S}} \varphi$, for all $\varphi \in \Gamma$;

Monotonicity: $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash_{\mathcal{S}} \varphi$;

Idempotency: $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Delta \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$, imply, $\Delta \vdash_{\mathcal{S}} \varphi$;

Structurality: $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma(\varphi)$, for all endomorphisms $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Equivalently, $\mathcal{S}$ may be expressed in terms of a *structural closure operator* $C_{\mathcal{S}}$, i.e., a function $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$, satisfying, for all $\Gamma \cup \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$:

Inflation: $\Gamma \subseteq C_{\mathcal{S}}(\Gamma)$;

Monotonicity: $C_{\mathcal{S}}(\Gamma) \subseteq C_{\mathcal{S}}(\Delta)$, for all $\Gamma \subseteq \Delta$;

Idempotency: $C_{\mathcal{S}}(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(\Gamma)$;

Structurality: $\sigma(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(\sigma(\Gamma))$, for all endomorphisms $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

The equivalence is established by setting, on the one hand, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$C_{\mathcal{S}}(\Gamma) = \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \Gamma \vdash_{\mathcal{S}} \varphi\},$$

and, on the other, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \phi \in C_{\mathcal{S}}(\Gamma).$$

The reliance on sentential logics as the underlying formalism of the theory persists when passing to abstract algebraic logic. The reader is referred to the aforementioned [36, 89], as well as to the standard reference [65] by Janusz Czelakowski, another pioneer in the field, all clearly showcasing the primary role of this framework in all related developments and investigations.

By contrast, in this monograph the underlying logical formalism consists of π-institutions [34]. This formalism encompasses systems with varying signatures and quantifiers in a more direct way than allowed by the formalism of sentential logics (see Appendix C of [36], as well as the work on cylindric [16, 28] and polyadic algebras [9] and related work at the institutional level [109, 110, 111, 112] based and/or closely related to these). The structure of a π-institution forms a modification of the structure of an institution [26, 42], which was introduced in computer science to formalize logical systems for specification and programming, based on semantics. Diaconescu's monograph [80] offers a comprehensive advanced study of institutions and presents a multitude of model theoretic results that can be abstracted from first-order, and other specific logical systems, to the institutional level. On the other hand, in π-institutions, the framework is stripped of the semantic, or model theoretic, aspects and the focus is on the syntax, thus recovering the essential features of the sentential logic framework, without, however, shedding the versatility afforded, and the advantage gained, by incorporating in the object language multiple signatures and signature-changing morphisms. In fact, this inclusion is what gives the area its distinctive and unique character inside (abstract) algebraic logic. This is apparent in all aspects of our studies.

To make clearer the exact relationship between sentential logics and π-institutions, and showcase the fact that the former constitute very narrow special cases of the latter, let us recall the definition of a π-institution. A *π-institution*, as originally defined in [34], is a triple $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where

- $\mathbf{Sign}$ is an arbitrary category, whose objects are called *signatures* and its morphisms *signature morphisms*;
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor giving, for each signature $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{SEN}(\Sigma)$ of Σ-*sentences*;
- For every $\Sigma \in |\mathbf{Sign}|$, $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a closure operator, such that the collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ satisfies the property of *structurality*, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,
$$\mathrm{SEN}(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)).$$

In the modified (enriched) form that is used in the present monograph, and which was (essentially) introduced in [115], there is an additional component N, which represents a *category of natural transformations* on the sentence functor SEN. Roughly speaking, this category corresponds to clones of algebraic operations on $\{\mathrm{SEN}(\Sigma) : \Sigma \in |\mathbf{Sign}|\}$, under the assumption that all operations are defined uniformly and naturally over all $\mathrm{SEN}(\Sigma)$, for $\Sigma \in |\mathbf{Sign}|$. This accords in style with the algebraic theories of Lawvere [10], which are closely related to the Eilenberg-Moore [12] and the Kleisli [13] constructions. For more details on these, one may consult the classic texts by Mac Lane [17], Pareigis [15], Borceux [46] and Barr and Wells [58]. Thus, we are studying logical systems formalized as quadruples $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, N, C\rangle$, which are further recast as pairs
$$\mathcal{I} = \langle \mathbf{F}, C\rangle,$$
where

- $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ expresses the algebraic structure, corresponding to the absolutely free algebra in the case of deductive systems, and
- $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is a family of closure operators, satisfying structurality, which is referred to as a *closure system*, and corresponds to the closure $C_\mathcal{S}$ in the case of sentential logics.

Suppose now that $\mathcal{S} = \langle \mathcal{L}, C_\mathcal{S}\rangle$ is a sentential logic. The standard rendering of it as a π-institution
$$\mathcal{I}^\mathcal{S} = \langle \mathbf{F}^\mathcal{L}, C^\mathcal{S}\rangle,$$
with $\mathbf{F}^\mathcal{L} = \langle \mathbf{Sign}^\mathcal{L}, \mathrm{SEN}^\mathcal{L}, N^\mathcal{L}\rangle$, is given by defining the four components as follows:

- $\mathbf{Sign}^\mathcal{L}$ is a trivial category, with object, say, V;

- $\mathrm{SEN}^{\mathcal{L}} : \mathbf{Sign}^{\mathcal{L}} \to \mathbf{Set}$ is given by $\mathrm{SEN}^{\mathcal{L}}(V) = \mathrm{Fm}_{\mathcal{L}}(V)$;
- $N^{\mathcal{L}}$ is the clone of all $\mathcal{L}$-operations on $\mathrm{Fm}_{\mathcal{L}}(V)$;
- $C^{\mathcal{S}}_V = C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$.

It is worth noting that $\mathbf{F}^{\mathcal{L}}$ only depends on $\mathcal{L}$ and V, as was to be expected (since it was deemed to correspond to the algebraic structure), and the deductive apparatus is reflected entirely in the definition of $C^{\mathcal{S}}$. Moreover, the formalism on the logical side does not incorporate substitutions in the object language, even though, since $C^{\mathcal{S}}_V = C_{\mathcal{S}}$ and the latter is structural, we have, for every endomorphism $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\sigma(C^{\mathcal{S}}_V(\Phi)) \subseteq C^{\mathcal{S}}_V(\sigma(\Phi)),$$

for all $\Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$. On the algebraic side, on the other hand, e.g., when congruences are to be determined, the inclusion of the clone $N^{\mathcal{L}}$, reflecting the algebraic $\mathcal{L}$-structure, forces congruences at the institutional level to exactly correspond to the familiar $\mathcal{L}$-congruences on the formula algebra in the universal algebraic sense.

The reasons why one might want to develop a theory of algebraization for logical systems formalized as institutions or π-institutions parallel the motivations provided by Blok and Pigozzi [36] for developing a theory of algebraizability for sentential logics.

One of the main motivations is providing a classification of logical systems based on the strength of the ties of their deductive apparatuses with those corresponding to algebraic deductive systems, i.e., deductive systems whose closure systems are induced by algebraic structures. Preferably, when the definitions applicable in the context of logical systems formalized as π-institutions specialize in the way outlined above to π-institutions associated with deductive systems, one would be able to recover the well-known algebraic (or Leibniz) hierarchy of abstract algebraic logic [65, 89]. The finitary and finitely algebraizable sentential logics of [36] form a special class in this hierarchy. In [89], this property is termed *Blok-Pigozzi algebraizability* (see Definition 3.39 of [89]).

Another desideratum is that the definitions should be as general as possible so that, given virtually any π-institution, one would be able, at least in principle, to classify it in one or more of the classes of the hierarchy, based on the strength of its algebraic properties.

Further, an additional reassurance would be provided if the definitions supplied turned out to be robust in the sense that one would be able to obtain, at least for several, if not for most, of them, different characterizations depending on the various viewpoints taken. This was clearly and successfully undertaken in [36] for the class of algebraizable deductive systems. In fact, Blok and Pigozzi obtained several different characterizations whose variety and strength played a major role in convincing other researchers that their definitions were chosen wisely and, as a result, in establishing firmly the new trends in the field and, thus, contributing, in large part, to virtually all subsequent developments. It is hoped that pursuits along the same lines here will prove, at least moderately, successful with respect to similar criteria. In particular, it is hoped that the characterizations of many of the classes presented in this monograph in a variety of ways will prove to many of the readers and to, present and future, researchers in the field satisfactory and motivating, as was the case with the work of Blok and Pigozzi [36].

One last motivation, equally important, however, in significance, comes by taking an adversarial point of view. As Blok and Pigozzi realized when studying sentential logics, and is certainly true also for logics formalized as π-institutions, since they encompass sentential logics, is the fact that many logical systems of historical and/or practical significance failed to be amenable to classical methods of algebraization, such as, e.g., the Lindenbaum-Tarski process. Naturally one is inclined to ask whether those systems can be algebraized in some alternative way, using different techniques, or whether the failure in their algebraization is due to intrinsic reasons. That is, one would like to investigate whether those systems have some innate characteristics, e.g., pertaining to their structural properties, that many, if not all, of them share and that decide their algebraizability status. This is reminiscent of the extensive and intensive research in computational complexity theory in separating various complexity classes [102, 100, 103, 101, 104, 99], where common features and rigorous criteria are sought for classification of problems in hierarchies of complexity classes. As is the case there, such an analysis and rigorous classification presupposes the existence of a formal definition of algebraizability (and of other related properties) so as to delineate formal boundaries and establish criteria that could potentially be used to falsify claims of algebraizability for some logical systems. Such criteria would point to shortcomings and defects of some logical systems as related to qualitative requirements that a logic should satisfy in order to qualify for membership in a corresponding class. It is believed that the definitions adopted here are helpful in establishing such criteria and in setting up boundaries. The examples that are scattered throughout seem to support this assertion, but, of course, the jury is out as far as gathering further evidence in support of, or in criticism and opposition to, this claim.

The notion of algebraizability adopted in this monograph is inspired by the one established for deductive systems by Blok and Pigozzi in [36]. Apart from the technical complications inherent in passing from the sentential to the institutional framework, one substantial difference is that we distinguish between a treatment based on the Leibniz operator, referred to as **semantic**, as contrasted to the one based on interpretations from logic to algebra and vice-versa, which is termed **syntactic**, since it is based on natural transformations corresponding to term operations on the free algebra of terms. In the sentential logic framework, such a distinction is only apparent, since, as it turns out, the two approaches are equivalent and, hence, interchangeable. On the other hand, in π-institutions, the added flexibility afforded in the relation between morphisms (which are treated in the object language in the category of signatures) and clone operations (also part of the framework, but added a posteriori to enhance the algebraic character of the intended studies) means that the syntactic concepts dominate (i.e., are, in general, stronger) than their corresponding semantic counterparts.

The role that theories play in sentential logics is subsumed here by theory families, which consist of deductively closed sets of sentences, one for each signature. They form a complete lattice $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$, when ordered by signature-wise inclusion $\leq$. To each theory family is associated a congruence system, a collection of equivalence relations on formulas, one for each signature, that satisfy both the congruence property (or substitution property) and invariance under signature morphisms. These also form a complete lattice under signature-wise inclusions, which is denoted by $\mathbf{ConSys}(\mathcal{I}) = \langle \mathrm{ConSys}(\mathcal{I}), \leq \rangle$. The congruence system selected is the largest one compatible with the given theory family and is termed, by analogy with the sentential logic framework, the *Leibniz congruence system* associated with the theory family.

Starting from **semantics**, we say that a π-institution is *algebraizable* if it satisfies two conditions that impose very intimate ties between the lattice of theory families of the π-institution and that of the congruence systems determined by a class of algebraic systems. The first condition is that the Leibniz operator is monotone on theory families. The second is that it is order-reflecting.

On the **syntactic** side, a π-institution is *algebraizable* if, on the one hand, the Leibniz congruence systems are definable via a collection of natural transformations in two arguments and, on the other, if the theory families are definable via a collection of natural transformations in a single argument. In general, parametric arguments are allowed and, by restricting those, we obtain potentially narrower classes.

One of the main theorems established by Blok and Pigozzi in [36] is the characterization of algebraizable sentential logics via the existence of an isomorphism between the theory lattice of the deductive system and the equational theory lattice associated to a class of algebras, which also commutes with substitutions. A characterization along similar lines is established here for logical systems formalized as π-institutions (see, e.g., Section 4.3 or Section 12.4, even though other related forms appear in other places in the monograph, as will be discussed in the overview). In the literature several forms of this theorem and a host of generalizations of increasing power (or generality) have been discussed. A sample list includes [74, 36, 41, 108, 76, 82, 94]. The majority of these deal with deductive equivalence of logical systems, and related lattice-theoretic algebraic structures. They encompass the characterization of algebraizability mentioned above and deal with the case in which mutual interpretations between logical structures induce isomorphisms between lattices of theories and vice versa, under some constraints and special hypotheses, depending on the context under consideration.

Another major characterization theorem provided in [36] for the notion of algebraizability asserts that, roughly speaking, in the context of sentential logics, the aforementioned analogs of the semantic and of the syntactic notions are equivalent. That is, the algebraization attained via the definability of theories and congruences via sets of equations and formulas, respectively, coincides with that ensured by the Leibniz operator being monotone and order reflecting on the lattice of the theories of the logic. This characterization, when abstracted to logics formalized as π-institutions, continues to hold under special provisos, namely, under the hypotheses that the π-institution under consideration has a rich enough supply of natural transformations or, more formally, as will be studied in detail in the monograph, that it has a Leibniz binary reflexive core and an adequate Suszko core.

In [36] as well as in many other works in the field, a considerable amount of emphasis has been placed on, and a substantial amount of effort expended in, studying specific logical systems of historical and/or practical interest from the point of view of algebraizability. This was only natural, given, on the one hand, the desire to showcase the applicability of the theory on logics of particular interest in traditional studies, and, on the other, the urge to investigate the power of falsifiability that the theory provides for those concrete logical systems that had resisted previous attempts at algebraization.

Our point of view, however, is slightly different and, as a consequence, we do not deal with or present such examples. Firstly, the majority of logical systems of historical and/or practical interest have already been dealt with in existing literature. Secondly, since our treatment abstracts and subsumes that of sentential logics and, considerably generalizes it, as was shown above, our goal is not to look at the more concrete, already encompassed by the study of the algebraization of deductive systems, but, rather, to look into the more abstract and discern what can be carried over to that level and how its validity and its applicability compares when applied to new systems and new examples which do not fit exactly, or do not conform at all to, the sentential logic framework. However, these aims and the mode of treatment they motivate should in no way be construed as underestimating the significance, or underplaying the beauty and elegance, of the studies concerned with the concrete and the more specific. After all, it is on those studies that the abstract is based, to those studies' insights, ideas and methodology that an enormous scientific debt is due, and from those studies' successes, and widespread recognition and appreciation, that a relative confidence is drawn regarding the potential usefulness and applicability of the more general framework presented, and elaborated on, in this monograph.

1.2 Fin de siècle: The Golden Age

We give an account of some of the major developments in abstract algebraic logic that occurred mostly, but not exclusively, around the last two decades of the 20th century. This period may be thought of as constituting the golden age of algebraic logic, in the sense that, during this time, there is clearly discernible a passage from an ad-hoc, case-by-case algebraic treatment of logical systems to a well-organized field, with a powerful arsenal of universally applicable concepts, methods and techniques, culminating to the classification of logics in an algebraic hierarchy, known as the Leibniz hierarchy. Needless to say, the foundations for this success were laid much earlier. Likewise, the development continued, and many important results around, and complementing, the main theory were obtained later, into the new millennium, and the area continues to be active. In order to avoid, in our short exposition, reinventing the wheel, we base this account on preexisting sources. We draw the material primarily from the, perhaps best-known, survey of the field by Font, Jansana and Pigozzi [70] and, when needed and/or convenient, the two existing specialized books on the subject by Czelakowski [65] and by Font [89].

Algebraic logic has its origins in the work of George Boole [1, 2], who formalized the "laws of thought" in an algebraic way. The intuition governing this process was made mathematically precise by Tarski [5, 6, 8]. Tarski used the key idea of Lindenbaum of identifying formulas of a logical language with the terms of the absolutely free algebra formed using the logical connectives as operation symbols [3] to give a precise connection between classical propositional calculus and Boolean algebras. This formed the paradigmatic example from which significant inspiration was drawn and on which subsequent developments were based. Furthermore, it served as a kind of testbed for comparing, trying, modifying and calibrating new ideas, methods and techniques. The way Boolean algebras arose as the algebraic counterparts of classical propositional calculus has become known as the Lindenbaum-Tarski method. It has subsequently been used to "algebraize" a variety of propositional systems.

A conceptual shift occurred around 1950 when Rasiowa and Sikorski [7, 21] (see, also, the historical surveys [60, 75]), among others, realized that the Lindenbaum-Tarski method could be applied not only to isolated logics but, rather, to classes of logical systems with an implication connective satisfying certain properties. In passing from a "per logic" or "a la carte" treatment to one addressing classes specified by some abstract properties, one discerns clearly for the first time the seeds of what, later, became known as "abstract algebraic logic". Papers that may be thought of as protoabstract, in the sense that they advance further the main ideas of Rasiowa and Sikorski towards the modern truly abstract era, were the one by Prucnal and Wronski [20] introducing equivalential logics, the ones by Czelakowski introducing protoalgebraic logics [27, 30] and further studying equivalential logics [24, 25] and the one by Blok and Pigozzi [29] studying protoalgebraic logics.

The seismic shift, one might say, in firmly founding and establishing the modern era came in the 1980s with the work of Blok and Pigozzi, which led eventually to the publication of their famous, seminal "Memoirs" monograph [36]. In a way analogous to the preceding three passages, from classical logic and Boolean algebra to the Lindenbaum-Tarski method, from the Lindenbaum-Tarski method to dealing with implicative logics and from implicative logics to abstract properties of deduction, Blok and Pigozzi were able to distil the essential spirit of the association between logic and algebra and, thus, extract and formalize the concept of an algebraizable logic in modern abstract terms and provide landmark characterizations. On the way, they established a very general process of algebraization, applicable to arbitrary logical systems, which has been, since, further refined and used to create the Leibniz hierarchy, often considered the pinnacle - certainly a milestone and a gem - of algebraic logic in general.

Before returning to provide a more detailed account, we take a small break to recount those features of the theory that distinguish the abstract approach from the more traditional treatments and give it its special character. First, as alluded to previously, instead of applying the Lindenbaum-Tarski process in an ad-hoc way, on a case-by-case basis, or, as in Rasiowa's work, to a class of logics sharing a specific connective satisfying certain properties, it applies the abstract process to arbitrary sentential logics and, according to the outcome, classifies them into classes reflecting the closeness of the ties between them and the corresponding algebraic counterparts. In establishing this association and performing the resulting classification, it opens, in parallel, two distinct but closely interrelated directions. On the one hand, it motivates the study of classes of algebras arising as algebraic counterparts of either single or groups of logical systems. On the other, it allows investigating the exact correspondence between metalogical properties of the logical systems at hand and algebraic properties of the classes of their algebraic counterparts.

By now a plethora of works falling distinctly in each of these three directions exist and many will appear as references in the more detailed account that will follow. But to give some indication and pointers, we mention a few of the earliest ones that may be viewed as ground breaking. Concerning the process of algebraization itself and the classification, one should mention Blok and Pigozzi's [29, 36], Czelakowski's [24, 25], Herrmann's [44, 54, 55] and Font and Jansana's [53]. Concerning the study of classes of algebraic counterparts arising from the abstract algebraization process, one should mention [39, 40] dealing with the conjunction-disjunction fragment of classical propositional calculus, as well as Jansana's study of selfextensional logics in [72, 77], with clear precedents in Font and Jansana's [53]. Finally, paradigmatic examples of the study of metalogical and corresponding algebraic properties constitute several works addressing forms of the deduction-detachment theorem, e.g., Czelakowski's [27, 30] and Blok and Pigozzi's [33, 38, 64], the work of Blok and Hoogland on the Beth poperty [73], as well as the work of Czelakowski and Pigozzi concerning interpolation and amalgamation properties [59].

1.3 Outline of Contents by Chapter

We give an outline of the contents of the monograph focusing on the main points of each chapter and describing them by section, using some formal notation, but without providing formal definitions, which will be presented in the main body of the text. This section is very closely related to other sections. First, in Section 1.4, we give a very concise summary, only mentioning the main overarching topics discussed in each chapter. Second, at the beginning of each chapter, a similar overview is provided focusing only on the specific chapter, with the exception that, in those introductions, being closer to the formal treatment, an even more informal narrative is adopted and a concerted effort is made to keep notation at a minimum.

1.3.1 Chapter 2

Chapter 2 presents the basic elements of the theory of algebraic systems, of π-institutions and of the interaction between logical and algebraic structures. These constitute the foundations and form the backbone of our theory throughout the monograph.

Section 2.1 gives an informal introduction to the chapter, akin to the introduction presented here, only containing a little less of formal notation and being more on the narrative, informal, side.

Section 2.2 is the first main section of the chapter. Here, we start by introducing the notion of a *sentence functor* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, which is simply a set-valued functor on an arbitrary category of signatures. It formalizes the carriers on which both algebras and logical systems are based, akin to the underlying universe of a universal algebra. Then we consider *sentence families* of sentence functors, which are families $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of subsets of sentences, one for each signature. These formalize distinguished sets of sentences when one considers logical structures, much like the distinguished sets in logical matrices. A *sentence system* is a sentence family T which is invariant under the action of signature morphisms. Two canonical ways of obtaining from a given sentence family T a sentence system consist of taking the largest sentence system $\overleftarrow{T}$ included in the family T and taking the smallest sentence system $\overrightarrow{T}$ that includes the sentence family T. Sentence functors are related via *sentence morphisms*, which are pairs $\langle F, \alpha \rangle$, F being a functor between the categories of signatures and α a natural transformation mapping sentences to sentences, taking into account the effect of

F. *Special morphisms* are those with surjective and full signature components and *surjective* ones are special ones whose sentence components are also surjective.

We then turn to *relation families* $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ over sentence functors. Those assume the place of binary relations. Of the highest interest and importance are *equivalence families* and *equivalence systems*, i.e., equivalence families invariant under the action of signature morphisms. They induce partitions on the components of sentence functors. Equivalence families and systems have important interactions and connections with both sentence families and with morphisms. The notion that relates an equivalence family with a sentence family is that of *compatibility*. An equivalence family R is *compatible* with a sentence family T if each component of the sentence family is a union of blocks of the equivalence family on the same component. The connection between equivalence systems and morphisms goes through the notion of kernels. Namely, the *kernel* $\mathrm{Ker}(\langle F,\alpha\rangle)$ of a morphism $\langle F,\alpha\rangle$ between two sentence functors forms an equivalence system on the domain.

If a set is equipped with operations, we get an algebraic structure. On this algebraic structure, one may reason in an algebraic way about any of the operations that are in its clone, i.e., that can be generated by applying the fundamental operations and the projections and composing them in arbitrary ways. In an analogous fashion, if a sentence functor $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ is equipped with a category of natural transformations N, which corresponds to the clone of algebraic operations on an algebra, one obtains an *algebraic system* $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$. As algebras play a fundamental role in both logical and algebraic aspects of the traditional theory, so do algebraic systems in the theory developed in the monograph. The role of free algebra is played in this context by that of a *base algebraic system* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$. Moreover, the notion of morphism extends from the context of sentence functors to the context of algebraic systems. The additional stipulation is that they also preserve the algebraic structure that turns the sentence functor into an algebraic system, i.e., that they satisfy the well-known *replacement* or *congruence condition*.

In traditional treatments, in specific contexts, all algebras are considered to be over the same algebraic signature, which is fully captured by the absolutely free algebra over that signature. In the present context, this similarity is captured by fixing a base algebraic system $\mathbf{F}$, as above, and considering only $\mathbf{F}$-*algebraic systems*, which are algebraic systems that, roughly speaking, have similar clones of operations with $\mathbf{F}$ and whose sentences are all images of sentences of $\mathbf{F}$ under a fixed algebraic system morphism $\langle F,\alpha\rangle$. Formally, these are expressed as pairs $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$, where $\langle F,\alpha\rangle:\mathbf{F}\to\mathbf{A}$ is a surjective algebraic system morphism. The notion of morphism extends further to morphisms between $\mathbf{F}$-algebraic systems.

In Section 2.3, the limelight falls on *congruence systems*, which play in this context the same role that congruences play in the context of universal algebras. The least congruence system on an algebraic system $\mathbf{A}$ is the identity congruence system $\Delta^{\mathbf{A}}$ and the largest one is the full relation system, written $\nabla^{\mathbf{A}}$. These form the min and max elements, respectively, of the complete lattice of congruence systems $\mathbf{ConSys(A)}$ on $\mathbf{A}$. The kernel $\mathrm{Ker}(\langle F,\alpha\rangle)$ of a morphism $\langle F,\alpha\rangle : \mathbf{A}\to\mathbf{B}$ between two algebraic systems forms a congruence system on $\mathbf{A}$. Moreover, congruence systems allow the definition of *quotient algebraic systems*. And, for every algebraic system $\mathbf{A}$ and every one of its quotient systems $\mathbf{A}^\theta := \mathbf{A}/\theta$, there is a canonical morphism $\langle I,\pi^\theta\rangle : \mathbf{A}\to\mathbf{A}^\theta$ onto the quotient algebraic system, whose kernel is exactly the congruence system θ that gave rise to the quotient. All these properties reflect well known properties from the context of congruences and quotients of universal algebras.

Congruence systems inherit from equivalence families the relation of compatibility with given sentence families. The critical property to be established is that for a given sentence family T on an algebraic system $\mathbf{A}$, there exists a largest congruence system on $\mathbf{A}$ that is compatible with T. This is called the *Leibniz congruence system of T on* $\mathbf{A}$, is denoted by $\Omega^{\mathbf{A}}(T)$ and plays the role that Leibniz congruences play in the context of traditional abstract algebraic logic. As such, its role in characterizing many of the classes in the algebraic hierarchy studied in the monograph is ubiquitous and, as a consequence, the whole hierarchy is known as the *Leibniz hierarchy*. After introducing the *Leibniz operator* on an algebraic system, we establish two important results concerning it. The first, inspired by a result from the traditional treatment, provides a characterization of the Leibniz operator in terms of the category of natural transformations (i.e., clone operations) of the algebraic system and the sentence family. Roughly speaking it asserts that a pair of sentences are Leibniz related if and only if they are indistinguishable modulo T with respect to the available algebraic apparatus. The second addresses specifically the categorical framework and asserts that the Leibniz congruence system of a sentence family T is dominated by the Leibniz congruence system of the largest sentence system $\overleftarrow{T}$ contained in the sentence family, i.e., that $\Omega^{\mathbf{A}}(T)\leq\Omega^{\mathbf{A}}(\overleftarrow{T})$. The value of this observation in establishing refinements of the traditional hierarchy, as reflected in the present context, is critical and hard to overestimate. Also of importance is the fact that the surjective morphisms between algebraic systems, which form the focus of our work, respect Leibniz congruence systems, in the sense that, if $\langle F,\alpha\rangle : \mathbf{A}\to\mathbf{B}$ is a surjective morphism and T is a sentence family on $\mathbf{B}$, then $\Omega^{\mathbf{A}}(\alpha^{-1}(T)) = \alpha^{-1}(\Omega^{\mathbf{B}}(T))$. Finally, it is worth noting that, in general, the intersection of the Leibniz congruence systems of a collection of sentence families is contained in the Leibniz congruence of the intersection of those sentence families. Significantly, though, the reverse inclusion holds universally on sentence families if and only if the Leibniz operator is monotone on sentence families, a property that does not always hold. In fact, the latter property is used in a critical way, when restricted to special kinds of sentence families, to determine some of the most important classes of logical systems located close to the bottom of the algebraic hierarchy. In addition, it is of great historical significance in many of the most important classical developments in the field.

In Section 2.4, we focus on *congruence systems relative to given classes of algebraic systems*. Given a class K of algebraic systems, all over the same base algebraic system, that is, possessing, in some sense, the same algebraic signature, a congruence system θ on an algebraic system $\mathbf{A}$, not necessarily belonging to K, is called a *congruence system relative to* K or a K-*congruence system* if the quotient $\mathbf{A}^\theta$ belongs to the class K. Naturally, if $\mathbf{A} \in \mathsf{K}$ and the class K happens to be closed under morphic images, then congruence systems relative to K coincide with arbitrary congruence systems. The section introduces another important notion in this context. That of an algebraic system $\mathbf{A}$ being a *subdirect intersection* of a collection of algebraic systems. This means that there exists surjective morphisms $\langle H^i, \gamma^i \rangle : \mathbf{A} \to \mathbf{A}^i$ from the algebraic system to each of the algebraic systems in the given collection and, moreover, the intersection of the kernels of those morphisms is the identity congruence on $\mathbf{A}$. Closure of a class of algebraic systems under subdirect intersections ensures that the collection of congruence systems relative to the class is closed under intersections. Additionally, if the class K contains a trivial algebraic system, then the nabla congruence system happens to be a relative congruence system. Therefore, possession of a trivial algebraic system together with closure under subdirect intersections ensures that the collection of all congruence systems relative to the class forms a complete lattice under signature-wise inclusion.

Suppose that the class K contains a trivial algebraic system and is closed under subdirect intersections so that it makes sense to associate with a given relation family X on its base algebraic system the least congruence system $\Theta^{\mathsf{K}}(X)$ relative to K containing X. An alternative, equally natural, way to associate a congruence system with X is to consider the closure $D^{\mathsf{K}}(X)$ under equational consequence relative to the algebraic systems in the class K. It is proven in this section that the two closures give rise to the same congruence system on the base algebraic system $\mathbf{F}$.

In Section 2.5, we study *varieties of algebraic systems*. There are two possibilities in adopting a choice for the entities that would play the role of equations in this context. The first is to view pairs of sentences as equations. The second is to adopt pairs of natural transformations in the clone as equations. The ones of the latter type are called *natural equations* to differentiate them from those of the former kind which are simply referred to as *equations*. We define formally the notion of *satisfiability* of a given equation and of a given natural equation in an algebraic system and that of validity of a natural equation in an algebraic system. Depending on whether we use equations or natural equations to determine a class of algebraic systems through satisfiability, we obtain two different kinds of varieties. Varieties determined by families of equations are called *semantic varieties*. Those determined by collections of natural equations are called *syntactic varieties*. It turns out that, in general, every syntactic variety is also a semantic variety. The opposite implication does not hold in general. The section concludes by presenting a sufficient condition on the structure of a base algebraic system that ensures that the classes of semantic and syntactic varieties over it coincide.

Much of the work in the first sections of Chapter 1 focuses on the algebraic framework that underlies both the logical and the algebraic aspects of the theory in the monograph. In Section 2.6, we turn to the study of *π-institutions*, the underlying structure of the logical aspects of our theory. The entire monograph assumes that all logical systems are formalized as π-institutions and its main goal is to study the process of their algebraization and to detail the various classes in the hierarchy that is formed by examining their algebraic character. It is needless, thus, to point out the importance of Section 2.6, as it presents the foundational aspects of the logical side of the theory.

We start, here, by defining the notion of *π-institution*. It is a pair $\mathcal{I} = \langle \mathbf{F}, C\rangle$ consisting of a base algebraic system $\mathbf{F}$ and a closure system C on the sentence functor of $\mathbf{F}$. It generalizes the Tarskian concept of a deductive system in that it allows multiple signatures and accommodates morphisms between signatures. To take into account the logical structure imposed on top of the underlying algebraic structure in this context, sentence families and systems are subsumed by *theory families* and *theory systems*. These are sentence families (systems, respectively) each of whose components is closed under logical deduction. The least among these is called the *theorem system* of $\mathcal{I}$. It turns out that, due to the property of structurality, which is key in the study of π-institutions, given a theory family T, $\overleftarrow{T}$ is also closed under deduction, whence it forms that largest theory system included in T. On the other hand, $\overrightarrow{T}$ fails to be closed under deduction in general. That is the reason why the smallest theory system including T is not simply $\overrightarrow{T}$ but, rather, $C(\overrightarrow{T})$.

An important derived concept is that of the π-institution that has as its theory families those theory families of $\mathcal{I} = \langle \mathbf{F}, C\rangle$ which include a given theory system T of $\mathcal{I}$. This is denoted by $\mathcal{I}^T = \langle \mathbf{F}, C^T\rangle$. The construction results in a π-institution whose theorem system is identical with the theory system T of $\mathcal{I}$.

As is the case in most mathematical contexts, objects are accompanied by morphisms between them that preserve the structure of interest in each particular context. *Morphisms between π-institutions* are algebraic morphisms between the underlying algebraic systems that, in addition, preserve the logical structure in the sense that the forward image of the logical closure of a set of sentences is included in the closure of the image of the same set of sentences. Among the most useful characterizations is that a given algebraic morphism is logical if and only if preimages of theory families of the target institution under the morphism constitute theory families of the domain π-institution.

In Section 2.7, we turn to those structures that are intermediate between logic and algebra and facilitate the interplay and the establishment of meaningful ties between the two domains. These are *matrix families*, which correspond to the ordinary logical matrices in the traditional treatment. Roughly speaking a *matrix family* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ consists of an algebraic system $\mathcal{A}$ together with a sentence family T of the algebraic system. If the sentence family is a system, i.e., invariant under signature morphisms, then the matrix family is called a *matrix system*. Their role is twofold. On the one hand, a given collection of matrix families M, over a base algebraic system $\mathbf{F}$, may be used to define a closure system C^{M}, and hence a π-institution structure $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$, on $\mathbf{F}$. On the other, given a π-institution structure $\mathcal{I}$ on $\mathbf{F}$, we may define the class $\mathrm{MatFam}(\mathcal{I})$ of all matrix families whose sentence families are closed under the deductive apparatus of the π-institution. Such sentence families are termed *$\mathcal{I}$-filter families* and, if they happen to be systems, then they are called *$\mathcal{I}$-filter systems*. The collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ of all filter families over the same underlying algebraic system $\mathcal{A}$, ordered by component-wise inclusion, forms a complete lattice and the collection of all filter systems on that same algebraic system forms a complete sublattice of the complete lattice of all filter families.

Among the main results presented in this section are the ones relating morphisms between algebraic systems with preservation of filter families. More precisely, the inverse image of a filter family is a filter family. The situation is more complicated when it comes to direct images. First of all, it only makes sense to define the direct image of a filter family in case the signature functor is an isomorphism. Second, it turns out that, in that case, for the image to also be a filter family on the target algebraic system, we must require additional restrictions. A sufficient condition is that the kernel system of the algebraic morphism be compatible with the filter family in the domain.

This result has particular consequences for the most important type of morphisms considered in the monograph, the canonical quotient morphisms associated with congruence systems on an algebraic system. It asserts that, given a filter family T on the quotient $\mathcal{A}^\theta$, the inverse image $\pi^{\theta^{-1}}(T)$ under the quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$ is a filter family on $\mathcal{A}$ and that, moreover, if the congruence system θ is compatible with a filter family T on $\mathcal{A}$, then the quotient T/θ is a filter family on $\mathcal{A}^\theta$.

We consider, by particularizing even further, the Leibniz quotient morphisms, which are those morphisms defined using the Leibniz congruence system that is compatible with a given filter family on the domain. Since, by definition, the Leibniz congruence system $\Omega^{\mathcal{A}}(T)$ associated with a given sentence family T is compatible with that sentence family, it follows that a filter family T on $\mathcal{A}$ gives rise, by passing to the Leibniz quotient $\mathcal{A}/\Omega^{\mathcal{A}}(T)$, to a filter family in the quotient. The corresponding matrix family $\mathfrak{A}/\theta = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle$ is called a (*Leibniz*) *reduced matrix family*.

The section closes by defining two classes of matrix families and two classes of algebraic systems that play a key role when investigating the algebraic nature of a given π-institution $\mathcal{I}$. The first is the class $\mathrm{MatFam}^*(\mathcal{I})$ of all *Leibniz reduced matrix families* associated with the given π-institution. The second is the class $\mathrm{MatSys}^*(\mathcal{I})$ of all *Leibniz reduced matrix systems*. Finally, on the algebraic side, by considering all algebraic system reducts of the reduced matrix families, we get the class $\mathrm{AlgSys}^*(\mathcal{I})$ of all *family reduced algebraic systems* and, by considering all algebraic system reducts of the reduced matrix systems, we get the class $\mathrm{AlgSys}^\bullet(\mathcal{I})$ of all *system reduced algebraic systems*.

Section 2.8 studies the two related concepts of *axiomatic extensions* and of *filter extensions*. An *axiomatic extension* $\mathcal{I}'$ of a given π-institution $\mathcal{I}$ is a π-institution over the same base algebraic system whose closure system is obtained by that of $\mathcal{I}$ by adding more axioms. More precisely, the consequences $C'(X)$ of a family of sentences X under $\mathcal{I}'$ are the consequences under $\mathcal{I}$ of the same family of sentences, augmented by some fixed family of sentences T, i.e., $C'(X) = C(X \cup T)$. The sentences in T are viewed as axioms in $\mathcal{I}'$. A *filter extension* arises in a similar way. The difference is that one considers filter families over arbitrary algebraic systems and not just theory families over the base algebraic system.

One of the first results in this section provides a characterization of axiomatic extensions. It asserts that axiomatic extensions are characterized by preservation of all those theories that include the theorem system of the extension. An alternative, lifting the condition to arbitrary algebraic systems, asserts that being an axiomatic extension is tantamount to the preservation of filterhood over all algebraic systems, for all those filters that include the least filter over the extension.

The last part of the section deals with *filter generation* over a given matrix family modulo a given π-institution $\mathcal{I}$. It defines the concept and formalizes, in a rather technical proposition, how generation of filters and surjective matrix family morphisms interact.

Section 2.9 turns the focus back to those structures that, like matrix families, play a critical role as intermediate structures in connecting the logical with the algebraic aspects of the theory. *Generalized matrix families* correspond to the generalized matrices of classical algebraic logic and, like generalized matrices, play a critical role in identifying classes of algebraic systems that may be naturally associated with given π-institutions (or classes of π-institutions). The way this association is established sheds light on the strength of ties between the two and on the nature of their interaction, e.g., by revealing which properties may be expected to be shared by the two or transferred from one to the other.

A *generalized matrix family* $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ consists of an underlying algebraic system $\mathcal{A}$ and a collection of sentence families $\mathcal{T}$ of the algebraic system. Such structures may also be used in two ways. They may serve in defining a closure system on a base algebraic system and, therefore, a π-institution structure. On the other hand, given a π-institution $\mathcal{I}$, we may associate with it the collection $\mathrm{GMatFam}(\mathcal{I})$ of those generalized matrices all of whose sentence families are filter families of the π-institution. With any generalized matrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, one may associate its *Tarski congruence system* $\widetilde{\Omega}(\mathbb{A})$ or $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$, an abstraction of the Tarski congruence systems associated with generalized matrices in classical abstract algebraic logic. *Tarski congruence systems* constitute the largest congruence systems on the base algebraic system compatible with all sentence families of the generalized matrix family. Taking the quotient $\mathbb{A}/\widetilde{\Omega}(\mathbb{A})$ of the generalized matrix family by its Tarski congruence system gives a new generalized matrix family $\mathbb{A}^*$, which is called the *Tarski reduction* of $\mathbb{A}$. A *Tarski reduced matrix family* is one that is isomorphic to its reduction, i.e., one whose Tarski congruence system is the identity congruence system on the underlying algebraic system.

There is a close connection between Tarski congruence systems and Leibniz congruence systems. Each generalized matrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ may be viewed as a bundle of matrix families $\{\langle \mathcal{A}, T \rangle : T \in \mathcal{T}\}$, i.e., of those matrix families whose sentence families belong to the collection of sentence families of the generalized matrix family. In that case, the Tarksi congruence system of the generalized matrix family is the intersection (in the component-wise sense) of the Leibniz congruence systems of all matrix families in the corresponding bundle, i.e., $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

In a similar way to Tarski congruence systems, one may also consider *Suszko congruence systems* $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T)$ associated with ordinary matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, and these are also introduced in Section 2.9. Suszko congruence systems of matrix families are defined only in a relative way, by viewing the matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ as being part of a bundle expressed as a generalized matrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$. Then the *Suszko congruence system* of the matrix family is identical to the Tarski congruence system $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}^T)$ of the bundle $\langle \mathcal{A}, \mathcal{T}^T \rangle$ consisting of only those sentence families that include (in the component-wise ordering) the sentence family T of the matrix family. Of course, expressed in terms of Lebniz congruence systems, the Suszko congruence system is the intersection of the Leibniz congruence systems of all matrix families determined by the sentence families in the given bundle that include that of the matrix family under consideration, i.e., $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T) = \bigcap_{T \leq T' \in \mathcal{T}} \Omega^{\mathcal{A}}(T')$. As was the case with Tarski congruence systems, we may consider the *Suszko reduction* $\mathfrak{A}^{\mathsf{Su}}$ of a given matrix family $\mathfrak{A}$, obtained by dividing out by the Suszko congruence system $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T)$. And, likewise, we call a matrix family *Suszko reduced*, when its Suszko congruence system is the identity congruence system on the underlying algebraic system.

Part of the significance of the Tarski and of the Suszko operators in algebraic logic is that they form one of the main mechanisms of selecting the "natural" class of algebraic systems to be associated with a given π-institution. Briefly and sketchily, starting from a π-institution $\mathcal{I}$, we obtain the collection $\mathrm{GMatFam}(\mathcal{I})$ of all generalized matrix families $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ whose sentence families $T \in \mathcal{T}$ are filter families of the π-institution. We then compute the Tarski reductions $\mathbb{A}^*$ by dividing out by the corresponding Tarski congruences $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$. This process gives rise to the class $\mathrm{GMatFam}^*(\mathcal{I})$ of all Tarski reduced generalized matrix families and to the class $\mathrm{AlgSys}(\mathcal{I})$ of all their underlying algebraic systems. This class subsumes, in the π-institution framework, the class of algebras which has long been viewed, in the traditional framework, as the most appropriate one to be associated with a given logic and, hence, as constituting the "natural" choice for the algebraic counterpart of the sentential logic. As it turns out, using a similar path, but relying on the Suszko operator, rather than on the Tarski operator, gives rise to exactly the same class of algebraic systems. Tracing the analogous process, one starts from a given π-institution $\mathcal{I}$ and considers all matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, viewed as part of the bundle $\mathbb{A} = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ of all matrix families associated with the π-institution. Then, one considers the Suszko reductions $\mathfrak{A}^{\mathsf{Su}}$ by dividing out by the corresponding Suszko congruence systems $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. The class of Suszko reduced matrix families obtained in this way is denoted by $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$. It can then be shown that the class of all algebraic reducts of the matrix families in $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ coincides with the class $\mathrm{AlgSys}(\mathcal{I})$.

In Sections 2.7 and 2.9, using the classes of Leibniz reduced matrix families and of Tarski reduced generalized matrix families associated with a given π-institution $\mathcal{I}$, we are able to define the two classes $\mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}(\mathcal{I})$ of algebraic systems associated with the π-institution. In Section 2.10, we take up the study of two additional classes of algebraic systems that may be perceived as counterparts of a given π-institution and compare them with those already defined.

Both new classes are based on a single algebraic system, namely the algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$ resulting by considering the quotient of the base algebraic system $\mathcal{F}$ by the Tarski congruence of the collection of all theory families of $\mathcal{I}$. Using this quotient algebraic system, the two classes are formed as the two kinds of varieties that may be generated by it. The first type, called the *semantic variety*, denoted by $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathbb{V}^{\mathsf{Sem}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))$, is the class of all algebraic systems that satisfy all equations that are satisfied by $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, i.e., all equations included in $\widetilde{\Omega}(\mathcal{I})$. The second type, called the *syntactic variety*, denoted by $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) = \mathbb{V}^{\mathsf{Syn}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))$, is the class of all algebraic systems that satisfy all natural equations that are satisfied by $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$.

Some results relating the four classes are presented. There is a linear hierarchy of inclusions that is not very difficult to establish. The class $\mathrm{AlgSys}^*(\mathcal{I})$ is the smallest class, followed by $\mathrm{AlgSys}(\mathcal{I})$, which is, in turn, included in

$\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$, whereas $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I})$ is the largest of the four classes considered. It turns out that all four classes generate the same syntactic variety of algebraic systems, which is identical to $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I})$, since it constitutes already a syntactic variety by definition. The section concludes with an important result showing that the class $\mathrm{AlgSys}(\mathcal{I})$ - perhaps the most important class associated with $\mathcal{I}$ - is closed under subdirect intersections and contains a trivial algebraic system. The usefulness of this fact is that it enables consideration, on any given algebraic system, of the least congruence system relative to $\mathrm{AlgSys}(\mathcal{I})$ generated by a prespecified relation family.

In Section 2.11, we study *equivalence families* and *systems* that are induced by sentence families or collections of sentence families of an algebraic system. The most fundamental among these is the *Frege equivalence family* $\lambda^{\mathbf{A}}(T)$ associated with a sentence family T of an algebraic system $\mathbf{A}$. It identifies two sentences if they are both inside or both outside the sentence family. Its companion *Frege equivalence system* $\Lambda^{\mathbf{A}}(T)$ is the largest equivalence system included in $\lambda^{\mathbf{A}}(T)$. The two Frege equivalences are intimately connected with the Leibniz congruence system $\Omega^{\mathbf{A}}(T)$, the latter being the largest congruence system contained in either of $\lambda^{\mathbf{A}}(T)$ or $\Lambda^{\mathbf{A}}(T)$.

In a way analogous to the extensions of the Leibniz congruence system that give rise to the Tarski and Suszko congruence systems, the Frege relations give rise to two more equivalences with similar roles. Given a collection $\mathcal{T}$ of sentence families of $\mathbf{A}$, the *Carnap equivalence family* $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ identifies two sentences if they are equivalent modulo T (in the Frege sense) for all $T \in \mathcal{T}$, i.e., $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T}) = \bigcap_{T\in\mathcal{T}} \lambda^{\mathbf{A}}(T)$. The *Carnap equivalence system* $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$ is the largest equivalence system included in $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$. The relation connecting Leibniz congruence systems with the Frege equivalences persists here as well, but with the Suszko congruence system in place of the Leibniz one. That is, the Suszko congruence system $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is the largest congruence system contained in either of $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ or $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$.

Finally, reminiscent of the passage from the Tarski to the Suszko congruence system, given a collection of sentence families $\mathcal{T}$ and $T \in \mathcal{T}$, the *Lindenbaum equivalence family* $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$ is the relation family identifying two sentences if they are equivalent modulo every $T' \in \mathcal{T}$, such that $T \leq T'$. The *Lindenbaum equivalence system* $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(T)$ is the largest equivalence system contained in $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$, and the Suszko congruence system $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(T)$ turns out to be the largest congruence system included in either of $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$ or $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(T)$.

The Carnap operators, viewed as operators on collections of sentence families on the same algebraic system, are monotone. The same applies to the Lindenbaum operators, viewed as operators on sentence families relative to the same collection of sentence families. However, the Frege operators do not satisfy a monotonicity property.

In Section 2.12, we are discussing *algebraic subsystems* and *π-subinstitutions*. The starting point is the observation that an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ may contain a *universe*, i.e., a functor $\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}$, such that $\mathrm{SEN}' \leq \mathrm{SEN}$ and closed under the action of natural transformations in N. Then, it is clear that this universe may be used to define an *algebraic subsystem* $\mathbf{A}'$ of $\mathbf{A}$ and, as it turns out, there exists a *canonical injection morphism* $\langle I, j\rangle : \mathbf{A}' \to \mathbf{A}$. Apart from detecting the existence of universes, there is a natural way to generate a universe starting from a given sentence family T of $\mathbf{A}$. This consists of passing, first, to the least sentence system $\overrightarrow{T}$ containing T and, then, closing $\overrightarrow{T}$ under the clone operations in N. This two-step process gives rise to a universe $\nu^{\mathbf{A}}(\overrightarrow{T})$. In case the algebraic system $\mathbf{A}$ supports a π-institution $\mathcal{I} = \langle \mathbf{A}, C\rangle$, then one obtains, for each algebraic subsystem $\mathbf{A}'$ of $\mathbf{A}$, a π-*subsinstitution* $\mathcal{I}' = \langle \mathbf{A}', C'\rangle$ by restricting the action of C on elements of $\mathbf{A}'$. It can be shown that the theory families of $\mathcal{I}'$ are exactly the restrictions of those of $\mathcal{I}$ on the universe giving rise to $\mathbf{A}'$. The section ends with some results relating Leibniz congruence systems of theory families of $\mathcal{I}$ with those of the corresponding theory families of $\mathcal{I}'$. A similar result also holds for Leibniz congruence systems of corresponding filter families of the two π-institutions.

Sections 2.13-2.15 deal with aspects of the "syntactic" apparatus of an algebraic system, i.e., with properties of the natural transformations viewed as term functions. Section 2.13 introduces the framework and studies some connections with the definability of the Leibniz congruence systems. Section 2.14 explores various modes of definability and details their relative power. Section 2.15 studies the effect of parameters and shows that two different possible ways of obtain a parameterless collection of natural transformations out of a given parametric one are essentially equivalent. We provide, next, some more details by section.

Section 2.13 introduces the concepts of *distinguished arguments* and of *parametric arguments* of a collection E of natural transformations. This is a conceptual distinction which becomes important in practice when one differentiates the role they each play when the collection of natural transformations is used to transform sentences, i.e., to produce new sets of sentences from tuples of given ones. The new family of sentences produced from a tuple of sentences $\vec{\phi}$ (possibly with the aid of parameters) is denoted by $E_\Sigma[\vec{\phi}]$, where Σ is the signature of $\vec{\phi}$. Another mode of transformation uses a dual or inverse construction. Namely, given a sentence family T, we consider the set $\overleftarrow{E}(T)$ consisting of all tuples $\vec{\phi}$, such that $E_\Sigma[\vec{\phi}] \leq T$. These tuples all share the same length, which equals the number of distinguished arguments of the transformations in E. The construction has some important properties, e.g., $\overleftarrow{E}$, viewed as an operator on sentence families is monotone and, moreover, commutes with inverse surjective morphisms. But, perhaps, its most important property is that, if E has two distinguished arguments and T is such that $\overleftarrow{E}(T)$ is reflexive, then $\overleftarrow{E}(T)$ includes the Leibniz congruence system $\Omega^{\mathbf{A}}(T)$ of T. Consequently, if $\overleftarrow{E}(T)$ is itself a congruence system compatible with T, then it coincides with $\Omega^{\mathbf{A}}(T)$. Thus, in this case, we may say that, in a specific sense, the Leibniz operator of T is *definable* using the natural transformations in E. We view this as a syntactic definability condition, which plays an important role in establishing the algebraic classification of π-institutions "by syntactic means" in subsequent chapters.

In Section 2.14 we continue the study of natural transformations as means of transforming tuples of sentences to sentences. We look as four possible ways of relating, via a fixed collection E of natural transformations with k distinguished arguments, a k-tuple of sentences $\vec{\phi}$ to a sentence family T. The simplest, *E-local membership*, asserts that $E_\Sigma(\vec{\phi},\vec{\chi}) \subseteq T_\Sigma$, for all values $\vec{\chi}$ of the parametric arguments. The second, *E-global membership*, asserts that $E_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}),\vec{\chi}) \subseteq T_{\Sigma'}$ holds for all signatures Σ', all morphisms $f : \Sigma \to \Sigma'$ and all appropriate values of the parameters $\vec{\chi}$. The remaining two, *left E-local membership* and *left E-global membership* mimic the preceding ones except that they use membership in $\overleftarrow{T}$ instead of membership in T. Closer scrutiny of the four modes reveals that the two global memberships are equivalent, followed in strength by left local membership, which, in turn, implies local membership. When a membership property holds for all $\vec{\phi}$, then we attribute it to the collection E itself. In this sense, it turns out that global, local, left global and left local memberships of E in T all coincide.

In Section 2.15, starting from a given collection S of natural transformations, possibly including parametric arguments, we study ways of obtaining a collection that is parameter-free. Here, two of the most natural, for our purposes, ways of doing this turn out to be equivalent, and, hence, release us from the obligation to distinguish between which one is applied in any specific context. Let us assume that S is taken to have k distinguished arguments. Then one way of obtaining from S a parameter-free collection is to replace all parametric arguments with k-ary natural transformations. This results in a collection $\dot{S}$ of k-ary, i.e., parameter-free, natural transformations. The second method builds on the notion of an *anti-monotone property* of natural transformations. These are properties P that a natural transformation either does or does not satisfy and for which an anti-monotonicity property holds, namely, if for all tuples of sentences $\vec{\phi}$, the family of transforms of $\vec{\phi}$ under σ is included in the family of transforms of $\vec{\phi}$ under τ, then τ satisfying P implies that σ also satisfies P. If P also denotes the class of all natural transformations satisfying property P, then we let $\widehat{P}$ be the subclass of P consisting of the parameter-free members of P. The section concludes with the assertion that, for anti-monotone properties P, both constructions $\dot{P}$ and $\widehat{P}$ give the same class of parameter-free natural transformations associated with P.

In Section 2.16, we study *finitarity*. This property holds for a π-institution $\mathcal{I}$ if every sentence ϕ that is derivable from a set Φ of sentences can be derived from some finite subset Φ' of Φ. Finitarity holds for the overwhelming majority of the logics considered in the literature. So it has played a central role in algebraic logic, even though much of the more abstract body of the theory is formalized and developed in a way that encompasses arbitrary, that is, not necessarily finitary, logical systems. A characterization of finitarity using the property of *continuity* is provided. We say that a collection of theory families is *directed* if every finite subcollection is included in some theory family in the collection. A π-institution is *continuous* if the union of a directed collection of theory families is also a theory family. Finitarity and continuity, as it turns out, are equivalent properties.

In the second part of the section, given a finitary π-institution $\mathcal{I}$, we provide a construction of the filter family $C^{\mathcal{I},\mathcal{A}}(X)$ generated by a sentence family X of $\mathcal{A}$. Taking advantage of the finitarity of $\mathcal{I}$, the filter family may be obtained by an incremental process, each step of which adds in the filter family sentences of $\mathcal{A}$ which are derivable, in a certain sense, by finite subsets of sentences that have already been included in the filter family at previous stages of the construction. In this way, the family $\Xi^{\mathcal{I},\mathcal{A}}(X)$ is obtained as the union of the families obtained at all stages and it can be shown that $C^{\mathcal{I},\mathcal{A}}(X) = \Xi^{\mathcal{I},\mathcal{A}}(X)$.

In the last two sections, Sections 2.17 and 2.18, we study *equational consequences* and provide analogs of some well-known fundamental results of universal algebra for classes of algebraic systems.

In Section 2.17, we look at closure families on pairs of sentences, i.e., equations, over a base algebraic system $\mathbf{F}$ that are induced by classes of $\mathbf{F}$-algebraic systems. Given a class K of $\mathbf{F}$-algebraic systems, we say that an equation $\phi \approx \psi$ is a *consequence* of a set E of equations *relative to* K if every algebraic system in K satisfying E also satisfies $\phi \approx \psi$. The resulting consequence family is denoted by D^{K}. It is not necessarily a closure system since it may fail to be structural. It is shown, however, that its theory families are exactly the congruence systems on $\mathbf{F}$ relative to the class K.

The second part of the section deals with a process of generating the closure of a family of equations E relative to an equational axiomatic system Q in an incremental way. Roughly speaking, it formalizes the process of closing under reflexivity, symmetry and transitivity, as well as under replacement and the action of signature morphisms. The family of equations obtained under this step-wise process from axioms Q and hypotheses E is denoted by $\Xi^Q(E)$. In the final result of the section, it is shown that the operator Ξ^Q coincides with D^{K} when Q is taken to be the collection of all equations satisfied by all algebraic systems in K.

Section 2.18, the closing section of the chapter, is inspired by universal algebra. It provides characterizations, in the spirit of Birkhoff's variety and Mal'cev's quasivariety theorems, of classes of algebraic systems defined by equations, quasiequations and generalized quasiequations, also referred to as guasiequations. The section begins by formally defining *equations*, *quasiequations* and *guasiequations* in the context of π-institutions. The relation of *satisfaction* of a syntactic entity of either of the above types in an algebraic system is also formally defined. In the usual way, these satisfaction relations establish Galois connections. The closed sets on the syntactic side form *equational*, *quasiequational* and *guasiequational theories*, whereas, on the semantic side, one obtains *equational*, *quasiequational* and *guasiequational classes of algebraic systems*, respectively. These are, respectively, the classes closed under the semantic variety $\mathbb{V}^{\mathsf{Sem}}$, semantic quasivariety $\mathbb{Q}^{\mathsf{Sem}}$ and semantic guasivariety $\mathbb{G}^{\mathsf{Sem}}$ operators.

To formulate characterizations of these classes, we introduce and study four operators on classes of algebraic systems. Let K be a class of algebraic systems. First, we say that an algebraic system $\mathcal{A}$ is K-*certified* if, for each signature Σ, there exists an algebraic system $\mathcal{A}^{\Sigma}$ in the class K that satisfies exactly the same equations of signature Σ as $\mathcal{A}$. The class K is said to be *abstract* or *closed under* K-*certifications* if every K-certified algebraic system is in K. The operator $\mathbb{C}$ is a closure operator and, if $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, then $\mathcal{A}$ satisfies all guasiequations satisfied by K. Moreover, if K is guasiequational, then it is an abstract class. Next, we say that an algebraic system $\mathcal{A}$ is *directedly* K-*certified* if, for each signature Σ, there exists a collection of algebraic systems $\{\mathcal{A}^{\Sigma,i} : i \in I\}$ in the class K that satisfy two conditions: On the one hand, the collection of all finite sets of equations satisfied by some $\mathcal{A}^{\Sigma,i}$, $i \in I$, is directed and, on the other, the union of all those sets is exactly the set of equations of signature Σ satisfied by $\mathcal{A}$. The class K is said to be *directedly abstract* or *closed under directed* K-*certifications* if every directedly K-certified algebraic system is in K. The operator $\mathbb{C}^*$ is a closure operator. It is shown that, if $\mathcal{A}$ is directedly K-certified, then it satisfies all quasiequations satisfied by K and, furthermore, that directed abstraction is a necessary condition for a class of algebraic systems to be a quasiequational class.

The third operator on classes of algebraic systems is that of taking *subdirect intersections* $\overset{\triangleleft}{\mathbb{I}\!\mathbb{I}}$. *Subdirect intersections* are collections of morphisms $\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i$, $i \in I$, with the same domain, the intersection of whose kernels is the identity system on $\mathcal{A}$. In that case, we also say that $\mathcal{A}$ is a *subdirect intersection* of the $\mathcal{A}^i$'s. This also turns out to be a closure operator on classes of algebraic systems and, in fact, closure under $\overset{\triangleleft}{\mathbb{I}\!\mathbb{I}}$ is necessary for a class to be guasiequational. The last operator considered is that of taking *morphic images*, denoted by $\mathbb{H}$. It also forms a closure operator on classes of algebraic systems and closure under $\mathbb{H}$ is necessary for a class to be an equational class.

The four operators serve in formulating the Birkhoff-style characterizations referred to previously for equational, quasiequational and guasiequational classes. Guasiequational classes are characterized as those that are abstract and closed under subdirect intersections. Quasiequational classes are those that are directedly abstract and closed under subdirect intersections. Finally, equational classes are characterized as those that are closed under subdirect intersections and morphic images. The section concludes with some additional characterizations of these three classes involving the structure of the subcollection $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ of the complete lattice $\mathrm{ConSys}(\mathcal{F})$. All of those additional results are based on the main characterizations described above.

1.3.2 Chapter 3

In Chapter 3 we start in earnest the study of the Leibniz hierarchy of π-institutions. Chapters 3-9 deal with the *semantic Leibniz hierarchy*. Here the classes are defined using properties of the Leibniz operator on theory families/systems of a π-institution. Chapters 11-17, on the other hand, deal with the *syntactic Leibniz hierarchy* in which classes are defined using collections of natural transformations satisfying specific definability properties. We shall see that "corresponding" classes in the two hierarchies may not coincide, but, nevertheless, the two hierarchies are closely connected - in fact may be seen as forming parts of a single hierarchy - and they are both modeled on the Leibniz hierarchy of sentential logics.

In Section 3.2, we study three properties. The first two are fundamental because they introduce concepts and terminology that play a critical role throughout the monograph. The third is used to establish classes of π-institutions at the very bottom of the hierarchy which abstract all other classes considered later in this and in subsequent chapters.

The first property if *systemicity*. A π-institution $\mathcal{I}$ is called *systemic* if every theory family of $\mathcal{I}$ is actually a theory system, i.e., if $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Recalling from Chapter 2 that, given a theory family T of $\mathcal{I}$, $\overleftarrow{T}$ is the largest theory system included in T, $\mathcal{I}$ is systemic if and only if, for every theory family T, $\overleftarrow{T} = T$. Yet another characterization asserts that, for every Σ-sentence ϕ of $\mathcal{I}$, the least theory family $C(\phi)$ of $\mathcal{I}$ generated by ϕ contains all translates of ϕ under arbitrary signature morphisms. One of the reasons why systemicity plays such a critical role is that, for a systemic π-institution, it suffices to restrict attention to theory systems, i.e., one may take invariance under signature morphisms for granted.

The second property is *stability*. It may be thought of as the counterpart of systemicity when focus shifts from theory families to corresponding Leibniz congruence systems. A π-institution $\mathcal{I}$ is *stable* if, for all theory families T, $\Omega(\overleftarrow{T}) = \Omega(T)$. Of course, every systemic π-institution is stable, and this implication is proper. Both systemicity and stability *transfer*. This means that a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is systemic if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, every $\mathcal{I}$-filter family of $\mathcal{A}$ is a filter system. Similarly, $\mathcal{I}$ is stable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and every $\mathcal{I}$-filter family T of $\mathcal{A}$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(T)$. These two transfer results are only the first of a host of, so-called, *transfer theorems* that are proved in the sequel for the majority of properties used to define classes in the Leibniz hierarchy. Having established the pattern and exhibited the main idea, we only mention such results briefly from now on, postponing the details for the main account in the relevant sections of the text.

The third property we study in Section 3.2 is *loyalty*. Unlike systemicity and stability, loyalty comes, as is typical for many subsequently introduced properties, in multiple flavors. To establish the pattern that will be followed in the presentation throughout, we introduce, first, the four versions, termed *family*, *left*, *right* and *system*. They may or may not be all different. So we study their properties, show which ones, if any, coincide, establish general implications between those that are not equivalent, and show, via examples, that these implications are proper, i.e., that no further collapsing of the subhierarchy based on these properties is possible.

A π-institution $\mathcal{I}$ is *family loyal* if, for all theory families T, T' of $\mathcal{I}$, $T \nless T'$ or $\Omega(T) \ngtr \Omega(T')$, or, equivalently, if it is not the case that $T < T'$ and $\Omega(T) > \Omega(T')$. If Ω, viewed as an operator mapping theory families to congruence systems, is either order preserving or order reflecting, then it is necessarily family loyal. So this property abstracts both monotonicity and reflectivity of Ω. Since both monotonicity and reflectivity play important roles in specifying classes in the Leibniz hierarchy, this observation provides partial justification for considering loyalty as a common abstraction. Here, as in all subsequently defined properties, once the family version is introduced, the other three versions follow by applying similar modifications. To obtain the *left version* one replaces, on the theory family side, all theory families by their arrow versions. So $\mathcal{I}$ is *left loyal* if, for all theory families T, T', $\overleftarrow{T} \nless \overleftarrow{T'}$ or $\Omega(T) \ngtr \Omega(T')$. To obtain the *right version*, a similar replacement is applied on the congruence system side. Thus, $\mathcal{I}$ is *right loyal* if, for all theory families T, T', $T \nless T'$ or $\Omega(\overleftarrow{T}) \ngtr \Omega(\overleftarrow{T'})$. Finally, the *system version* is obtained by imposing the same condition as in the family version, but restricting its application to theory systems, instead of insisting that it hold for all theory families. Accordingly, $\mathcal{I}$ is *system loyal* if, for all theory *systems* T, T' of $\mathcal{I}$, $T \nless T'$ or $\Omega(T) \ngtr \Omega(T')$.

Family loyalty properly implies stability. Moreover, family loyalty implies left loyalty, which, in turn, implies system and right loyalty, the latter two being equivalent properties. System loyalty, together with systemicity, imply family loyalty. That is, as is the case with virtually all properties introduced in the monograph, imposing systemicity has the effect of collapsing the entire four-class subhierarchy into a single class. This observation can be applied to obtain a backbone - or a bird's eye view - of the Leibniz hierarchy without worrying about the refinements and subdivisions due to the different flavors of each property. Section 3.2 concludes by showing that all three distinct versions of loyalty transfer, i.e., that a given π-institution has a certain loyalty property if the corresponding defining condition holds for all pairs of filter families (or systems) on arbitrary $\mathbf{F}$-algebraic systems.

In Section 3.3, we study *monotonicity properties*. A π-institution $\mathcal{I}$ is *family monotone* if, for all theory families $T, T', T \leq T'$ implies $\Omega(T) \leq \Omega(T')$, i.e., if the Leibniz operator on theory families is order preserving. In accordance with the general framework outlined above for loyalty, $\mathcal{I}$ is *left monotone* if, for all T, T', $\overleftarrow{T} \leq \overleftarrow{T'}$ implies $\Omega(T) \leq \Omega(T')$, *right monotone* if, for all T, T', $T \leq T'$ implies $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$ and *system monotone* if the same condition defining family monotonicity is restricted to theory systems, i.e., if the Leibniz operator on theory systems is order preserving. It is shown that family monotonicity implies stability. Most importantly, family and left monotonicity coincide as do system and right monotonicity. Following terminology inherited from sentential logics, we term π-institutions that satisfy family monotonicity *protoalgebraic* and those that satisfy the system version *prealgebraic*. Protoalgebraicity is equivalent to prealgebraicity plus stability. In particular, every protoalgebraic π-institution is prealgebraic, and this inclusion is proper. Both monotonicity properties transfer. Finally, pursuing connections with classes introduced in Section 3.2, we show that protoalgebraicity implies family loyalty, whereas prealgebraicity is sufficient for system loyalty.

In Sections 3.4 and 3.5, we study versions of a property called *complete monotonicity*. This is a property dual to complete order reflectivity, a property that characterizes truth equationality in the sentential framework. Given a sentential logic $\mathcal{S}$, complete order reflectivity stipulates that, for every collection $\mathcal{T} \cup \{T'\}$ of theories of $\mathcal{S}$, if $\bigcap_{T \in \mathcal{T}} \Omega(T) \subseteq \Omega(T')$, then $\bigcap \mathcal{T} \subseteq T'$. Since, in both the lattice of theories and that of congruences, meet and intersection coincide, but, on both theories and congruences, join is not the same as union, one may obtain two "dual" versions of complete order reflectivity. The first, following a set-theoretic approach, says that, for all $\mathcal{T} \cup \{T'\}$, $T' \subseteq \bigcup \mathcal{T}$ implies $\Omega(T') \subseteq \bigcup_{T \in \mathcal{T}} \Omega(T)$. The second, taking a lattice-theoretic point of view, asserts that, for all $\mathcal{T} \cup \{T'\}$, $T' \leq \bigvee \mathcal{T}$ implies $\Omega(T') \leq \bigvee_{T \in \mathcal{T}} \Omega(T)$, where the join in the hypothesis is taken in the complete lattice of theories of $\mathcal{S}$ and the one in the conclusion in the complete lattice of congruences on the formula algebra. In Section 3.4 we study an analog of the former property and in Section 3.5 an analog of the latter in the context of logics formalized as π-institutions. A few more details follow in the next two paragraphs.

In Section 3.4, we look at *complete* $\bigcup$*-monotonicity*, which is abbreviated as $c^\cup$*-monotonicity* or, simply, *c-monotonicity*. A π-institution is *family* $c^\cup$*-monotone* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigcup \mathcal{T}$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. *Left* and *right* $c^\cup$*-monotonicities* are obtained by replacing in the hypothesis and in the conclusion, respectively, every theory family occurring by its arrow version. Finally, *system* $c^\cup$*-monotonicity* is defined by the same condition as the family version, but applied exclusively to collections of theory systems. Family $c^\cup$-monotonicity implies stability, as does left $c^\cup$-monotonicity. Moreover, the family version is equivalent to the conjunction of the left and right versions and either of the latter implies system $c^\cup$-monotonicity. All four $c^\cup$-monotonicity properties transfer. And, whereas the left version is sufficient for protoalgebraicity, the system version implies only prealgebraicity.

In Section 3.5, we continue the study of complete monotonicity but switch from complete $\bigcup$-monotonicity to *complete* $\bigvee$*-monotonicity*, which is abbreviated as $c^\vee$*-monotonicity*. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family* $c^\vee$*-monotone* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$, where $\bigvee^{\mathcal{I}}$ denotes the join in the complete lattice of theory families of $\mathcal{I}$ and $\bigvee^{\mathbf{F}}$ the join in the complete lattice of congruence systems on $\mathbf{F}$. Again, following the general pattern, $\mathcal{I}$ is *left* $c^\vee$*-monotone* if, for all $\mathcal{T} \cup \{T'\}$, $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T\in\mathcal{T}} \overleftarrow{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$ and is *right* $c^\vee$*-monotone* if, for all $\mathcal{T} \cup \{T'\}$, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. Finally, $\mathcal{I}$ is *system* $c^\vee$*-monotone* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory *systems* of $\mathcal{I}$, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$. Again, either family or left $c^\vee$-monotonicity implies stability. The family version is equivalent to the conjunction of the left and right versions and either of those two implies system $c^\vee$-monotonicity. Left $c^\vee$-monotonicity implies protoalgebraicity and system $c^\vee$-monotonicity implies prealgebraicity.

Contrary to what the similarities of results pertaining to $c^\vee$-monotonicity with those of Section 3.4 on $c^\cup$-monotonicity may suggest, there are also significant differences between the two complete monotonicity properties. One instance concerns transfer theorems. Unlike $c^\cup$-monotonicity, $c^\vee$-monotonicity properties do not transfer in general. This is due to the fact that, unlike unions, joins do not commute with inverse surjective morphisms between algebraic systems. A second difference, which affords, perhaps, partial justification for introducing and discussing both types of properties in some detail, is that corresponding classes of π-institutions are incomparable. E.g., there exists a family $c^\vee$-monotone π-institution which is not family $c^\cup$-monotone and vice-versa.

In Section 3.6, we study *injectivity*. A π-institution $\mathcal{I}$ is *family injective* if, for all theory families T, T', $\Omega(T) = \Omega(T')$ implies $T = T'$, i.e., if the Leibniz operator is injective on theory families. It is *left injective* if, for all T, T', $\Omega(T) = \Omega(T')$ implies $\overleftarrow{T} = \overleftarrow{T'}$ and *right injective* if, for all T, T', $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$ implies $T = T'$. Finally, it is *system injective* if the Leibniz operator is injective on theory systems. Right injectivity is the strongest of the four injectivity properties and it implies systemicity. It is followed by family injectivity, then left injectivity, which implies system injectivity. System injectivity together with systemicity is equivalent to right injectivity, whereas, together with stability, which is weaker than systemicity, it implies left injectivity. All four injectivity properties transfer.

In Section 3.7, we turn to *reflectivity properties*. A π-institution $\mathcal{I}$ is *family reflective* if, for all theory families T, T' of $\mathcal{I}$, $\Omega(T) \leq \Omega(T')$ implies

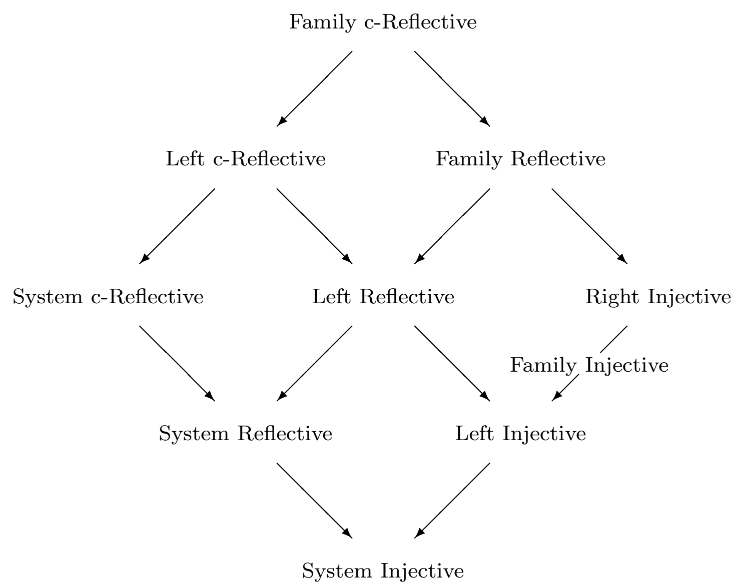
$T \le T'$, i.e., if the Leibniz operator on theory families is order reflecting. If, for all T, T', $\Omega(T) \le \Omega(T')$ implies $\overleftarrow{T} \le \overleftarrow{T'}$, then $\mathcal{I}$ is *left reflective*, whereas, if, for all T, T', $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$ implies $T \le T'$, $\mathcal{I}$ is *right reflective*. *System reflectivity* stipulates the order reflectivity of the Leibniz operator on theory systems. It turns out that family or right reflectivity imply systemicity. This allows showing that the two are actually equivalent properties. They imply left reflectivity, which, in turn, implies system reflectivity. System reflectivity, coupled with stability, implies left reflectivity, whereas, together with systemicity, it becomes equivalent to family reflectivity. All four versions transfer. Section 3.7 ends by relating reflectivity with the injectivity properties, introduced in Section 3.6, and with the loyalty properties, introduced in Section 3.2. More precisely, it is shown that family/right, left and system reflectivity imply, respectively, right, left and system injectivity and that family/right, left and system reflectivity imply, respectively, family, left and system loyalty.

Section 3.8, the last section of Chapter 3, introduces *complete reflectivity properties*, abbreviated to *c-reflectivity*. These form a generalization of the reflectivity properties of Section 3.7. Complete reflectivity originates in the work of Raftery, where it is used to characterize truth equationality in the context of sentential logics. A π-institution is *family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families of $\mathcal{I}$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$ implies $\bigcap \mathcal{T} \le T'$. It is *left c-reflective* if, for all $\mathcal{T} \cup \{T'\}$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$ implies $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \le \overleftarrow{T'}$ and *right c-reflective* if, for all $\mathcal{T} \cup \{T'\}$, $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$ implies $\bigcap_{T \in \mathcal{T}} \mathcal{T} \le T'$. *System c-reflectivity* is defined using the same condition as family c-reflectivity restricted to collections of theory systems. As was the case with reflectivity, either family or right c-reflectivity implies systemicity and this enables showing that the family and right versions are equivalent. They imply left c-reflectivity, which, in turn, implies the system version. System c-reflectivity and systemicity are jointly equivalent to family c-reflectivity, whereas system c-reflectivity, augmented with stability, implies left c-reflectivity. All complete reflectivity properties transfer and, as is apparent from the relevant definitions, each version of c-reflectivity implies the corresponding reflectivity version.

1.3.3 Chapter 4

In Chapter 4, we visit weak prealgebraizability and weak algebraizability properties of π-institutions. These create a subhierarchy of π-institutions whose members roughly correspond to the weakly algebraizable logics in the sentential logic framework. Weak prealgebraizability classes arise when coupling family monotonicity with either of injectivity, reflectivity or complete reflectivity properties. Analogously, weak algebraizability results by combining system monotonicity with injectivity, reflectivity or complete reflectivity.

Before describing the versions of weak prealgebraizability and algebraizability in more detail, we mention, firstly, that the term "weak" refers to the use of monotonicity, as opposed to the stronger notion of equivalentiality, in the definitions, and remind, secondly, the reader of the hierarchy, established in Chapter 3, of the various flavors of injectivity, reflectivity and c-reflectivity properties, which assumed the form depicted in the diagram.



In Section 4.2, we define the classes of *weakly prealgebraizable π-institutions*. Each class results by imposing prealgebraicity (system monotonicity) and one of the ten flavors of injectivity, reflectivity and complete reflectivity shown in the preceding hierarchy. Since prealgebraicity is shared by all classes, the deciding factor in the subhierarchy is the type of injectivity, reflectivity or c-reflectivity imposed. Thus, a priori, one obtains ten potentially distinct classes whose hierarchy reflects that shown in the preceding diagram. We name the corresponding property "weak X prealgebraizability", or "WX prealgebraizability" for short, where the string X stands for one of SI, LI, FI, RI for system, left, family, right injectivity, respectively, SR, LR, FR for system, left, family reflectivity, respectively, or SC, LC, FC for system, left, family c-reflectivity, respectively.

In our first result, we show that prealgebraicity is sufficient to identify all system versions, which forces the collapsing of the classes of WSI, WSR and WSC prealgebraizable π-institutions. We call the corresponding property *WS prealgebraizability*. In what sets a pattern for subsequent work in this chapter, it is shown that WS prealgebraizability transfers and, further, a characterization is obtained via properties of the Leibniz operator $\Omega^{\mathcal{A}}$, viewed as a mapping between ordered sets, for arbitrary $\mathbf{F}$-algebraic systems $\mathcal{A}$. More precisely, it is shown that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is WS prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$

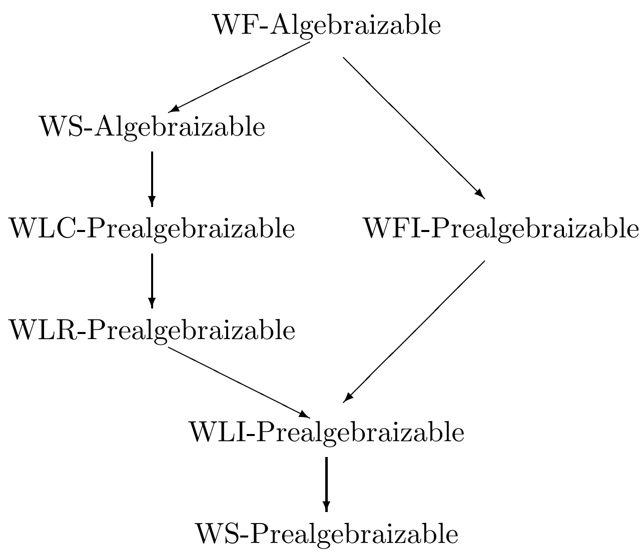
is an order embedding. Next, it is shown that, in view of prealgebraicity, family reflectivity implies family c-reflectivity and this leads to the identification of WFR prealgebraizability and WFC prealgebraizability. Moreover, under protoalgebraicity, family injectivity implies family reflectivity. This enables showing that both WFR and WRI prealgebraizability are characterized as the conjunction of WFI prealgebraizability and systemicity and, hence, are identical properties. Both WFR and WFI prealgebraizability transfer. Moreover, the WFI version is characterized by the property that, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is a bijection on filter families, restricting to an order embedding on filter systems, whereas the WFR version is characterized by the condition that, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is an order isomorphism.

At this point, the hierarchy has been reduced to six classes, since, as it turned out, all three system classes are identical and the three family plus the WRI prealgebraizability collapse down to two classes. The only classes not put under the microscope yet are those defined using the left versions of injectivity, reflectivity and c-reflectivity. We return to them after a short break that gives a glimpse of further possible reductions under special circumstances. Namely, it is proven that, under systemicity, the entire hierarchy collapses to a single class and that, under stability, it collapses down to two classes, as the only properties that can be distinguished are the family (but including also WRI prealgebraizability) from the remaining versions.

Returning to the left properties, Section 4.2 concludes by showing that all three transfer and by providing characterizations along the lines outlined previously, using $\Omega^{\mathcal{A}}$. More precisely, it is shown that $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is WLC (WLR, WLI, respectively) prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left completely order reflecting (left order reflecting, left injective, respectively) surjection, restricting to an order embedding on theory systems.

In Section 4.3, we study versions of weak algebraizability. These combine protoalgebraicity (family monotonicity) with the various versions of injectivity, reflectivity and complete reflectivity. Since protoalgebraicity dominates prealgebraicity, it is clear that one obtains at least as many identifications between the ten apparent weak algebraizability properties as those established between corresponding weak prealgebraizability properties in Section 4.2. However, the situation under closer scrutiny turns out to be much more radical. Since protoalgebraicity is strong enough to yield stability, the emerging landscape was anticipated by the previously mentioned collapse of the weak prealgebraizability hierarchy down to two classes in the presence of stability. Similarly, under protoalgebraicity and, hence, stability, all three weak family algebraizability properties together with WRI algebraizability collapse to a single property, termed *WF algebraizability*. Further, all remaining six left and system versions also collapse to a single property we call *WS algebraizability*. Both of these properties transfer. Also, for both one may obtain Leibniz operator type characterizations. More specifically, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is WS algebraizable iff it is stable and, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, whereas it is WF algebraizable iff, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

Observing that the characterization of WF algebraizability is identical with that obtained for WF prealgebraizability, we conclude that the top classes in the weak prealgebraizability and weak algebraizability subhierarchies actually coincide. Thus, by fusing these two subhierarchies, one obtains a total of seven potentially distinct classes, which form the combined hierarchy depicted in the diagram.



1.3.4 Chapter 5

In Chapter 5, we deal with classes of π-institutions that result from weakly prealgebraizable and weakly algebraizable π-institutions when the properties of prealgebraicity (system monotonicity) and protoalgebraicity (family monotonicity) are strengthened to preequivalentiality and equivalentiality, respectively. The strengthening, i.e., the passage from proto- (or pre-) algebraicity to (pre)equivalentiality, involves adding the condition of either family or system extensionality. Depending on which of these two properties is imposed, one obtains two parallel hierarchies, one on top of the other, both of which reflect the structure of the weak (pre)algebraizability hierarchy, described in Chapter 4.

In Section 5.2, we introduce and study *extensionality*. The definition requires the notion of subsystem of an algebraic system $\mathbf{F}$ generated by a given sentence family X, which is denoted by $\langle X \rangle$ and was introduced in Section 2.12. A π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is called *family extensional* if, for all sentence families X of $\mathbf{F}$ and all theory families T of $\mathcal{I}$, $\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle)$. It is called *system extensional* if the same condition holds, but T is quantified over all theory systems of $\mathcal{I}$, instead of ranging over arbitrary theory families. Since system extensionality specializes family extensionality, every family extensional π-institution is also system extensional. It is, moreover, the case that system extensionality, coupled with stability, implies family extensionality. Extensionality is very useful because, when satisfied, it causes certain properties that hold in a π-institution to be inherited by all its subinstitutions. For instance, under system extensionality, stability propagates from a π-institution $\mathcal{I}$ to all its subinstitutions $\mathcal{I}' \leq \mathcal{I}$. Additionally, system or family extensionality causes prealgebraicity or protoalgebraicity, respectively, to be inherited by all subinstitutions of a given π-institution. Both versions of extensionality transfer. The section closes by looking at *2-extensionality*, an apparently weaker condition than extensionality, which, however, turns out to be equivalent to it. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family 2-extensional* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and every theory family T of $\mathcal{I}$, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$ if and only if $\langle \phi, \psi\rangle \in \Omega_\Sigma^{\langle \phi, \psi\rangle}(T \cap \langle \phi, \psi\rangle)$. *System 2-extensionality* is defined by the same condition in which T is quantified over theory systems. A π-institution is family/system extensional if and only if it is family/system 2-extensional, respectively.

In Section 5.3, we study *Leibniz commutativity*. The notion relies on the concepts of *extension* and *logical extension*. Given an algebraic system $\mathbf{F}$ and a sentence family X of $\mathbf{F}$, an *extension* is an algebraic system morphism of the form $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$, where $\langle X\rangle$ is the algebraic subsystem of $\mathbf{F}$ generated by X and I is the identity functor on signatures. Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, an extension $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$ is called *logical*, denoted $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, if, for every signature Σ and all $\Phi \subseteq \langle X\rangle_\Sigma$, $\alpha_\Sigma(C_\Sigma^{\langle X\rangle}(\Phi)) \subseteq C_\Sigma(\alpha_\Sigma(\Phi))$, where $C^{\langle X\rangle}$ is the restriction of C on $\langle X\rangle$, discussed in detail in Section 2.12. A characterization of this notion asserts that $\langle I, \alpha\rangle$ is logical if and only if α^{-1} preserves theory families, i.e., if $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}^{\langle X\rangle})$, for every $T \in \mathrm{ThFam}(\mathcal{I})$.

Logical extensions form the background for introducing the property of *Leibniz commutativity*, or, simply, *commutativity*. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is called *family commuting* if the Leibniz operator on theory families commutes with logical extensions, i.e., if, for every sentence family X of $\mathbf{F}$, every logical extension $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ and all $T' \in \mathrm{ThFam}(\mathcal{I}^{\langle X\rangle})$, $\alpha(\Omega^{\langle X\rangle}(T')) \leq \Omega(C(\alpha(T')))$. Applying the same condition, where T' ranges over all theory systems of $\mathcal{I}^{\langle X\rangle}$, defines *system commutativity*. A closely related concept is that of *inverse Leibniz commutativity*, or, simply, *inverse commutativity*. A π-institution $\mathcal{I}$ is *family inverse commuting* if, for every sentence family X, every logical extension $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, and all $T \in \mathrm{ThFam}(\mathcal{I})$, $\alpha^{-1}(\Omega(T)) = \Omega^{\langle X\rangle}(\alpha^{-1}(T))$. The same condition, imposed on theory systems only, defines *system inverse commutativity*. The fact that injection morphisms $\langle I, j\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ of subinstitutions into their parent institutions are logical extensions allows us to show that family/system inverse commutativity implies family/system extensionality, respectively. It is clear that the family version implies the system version and, as it turns out, the system version augmented by stability implies the family version. What is important for our purposes, and the reason why both direct and inverse commutativity properties are studied, is that under pre/proto-algebraicity, respectively, system/family commutativity is equivalent to system/family inverse commutativity. Moreover, in a result strengthening the relationship mentioned above, it is proven that family/system inverse commutativity and family/system extensionality, respectively, are actually equivalent properties. This section concludes by showing that both versions of inverse commutativity transfer.

In Section 5.4, we introduce *equivalentiality*. This is the section we have been preparing for by studying extensionality and commutativity in Sections 5.2 and 5.3, respectively. *Equivalentiality* is the result of coupling monotonicity with extensionality. Since each of those two properties comes in two flavors, there are, a priori, four possible versions of equivalentiality. *Family equivalentiality* combines protoalgebraicity with family extensionality. *System equivalentiality* keeps protoalgebraicity but uses system extensionality. *Family* and *system preequivalentiality* are defined analogously, but here one uses prealgebraicity instead of protoalgebraicity. Since protoalgebraicity is strong enough to imply stability, it turns out that family and system equivalentiality coincide. This property is referred to simply as *equivalentiality*. Thus, we get three properties in this hierarchy, namely, in decreasing order of potency, equivalentiality, family preequivalentiality and system preequivalentiality. Moreover, equivalentiality is equivalent to system preequivalentiality plus stability. All three properties transfer. There also exist characterizations of equivalentiality and preequivalentiality by conditions imposed on the Leibniz operator on filter families/systems, respectively, on arbitrary **F**-algebraic systems. Finally, as is clear by the corresponding definitions, equivalentiality dominates protoalgebraicity and preequivalentiality dominates prealgebraicity.

In Section 5.5, by replacing prealgebraicity by preequivalentiality, we obtain from the weak prealgebraizability hierarchy of Section 4.2 two parallel *prealgebraizability hierarchies*. The term "prealgebraizability" in both refers to the fact that preequivalentiality, as opposed to equivalentiality, is applied. In one of the two hierarchies, "family prealgebraizability" refers to the application of family preequivalentiality, whereas in "prealgebraizability", it is understood that (system) preequivalentiality is applied. The five classes in the first hierarchy are termed *XF prealgebraizable* and in the second *X prealgebraizable*, where X is one of the following strings, suggesting the imposition of an additional property on the Leibniz operator.

- LC for left completely reflective;
- LR for left reflective;

- FI for family injective;
- LI for left injective; and
- S for system (system completely reflective, system reflective or system injective, which are all equivalent in view of prealgebraicity).

It is shown that systemicity causes the total collapse of the hierarchy into a single class, whereas stability collapses the two family injectivity classes, FI and FIF prealgebraizability, and, also, all eight remaining classes and, therefore, leads to a 2-class hierarchy. Moreover, it is proven that all ten properties transfer. The remainder of this section is devoted to providing characterizations of each of the ten classes using order theoretic properties of the Leibniz operator viewed as a mapping from lattices of filters systems/families to lattices of congruence systems over arbitrary **F**-algebraic systems. We focus only on a couple of pairs to give a flavor of the type of results obtained, and refer the reader to the main text for a full account. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is FIF prealgebraizable if and only if, for all **F**-algebraic systems $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a bijection commuting with inverse logical extensions, which restricts to an order embedding on filter systems. A similar characterization is obtained for FI prealgebraizability, but with a subtle important change: $\mathcal{I}$ is FI prealgebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a bijection, which restricts to an order embedding commuting with inverse logical morphisms on filter systems. Analogously, for the left reflectivity classes, we get, on the one hand, that $\mathcal{I}$ is LRF prealgebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection commuting with inverse logical extensions, which restricts to an order embedding on filter systems, and, on the other, noting again the same subtle change, $\mathcal{I}$ is LR prealgebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection, which restricts to an order embedding commuting with inverse logical extensions on filter systems. Characterizations of the remaining six classes follow a similar pattern.

In Section 5.6, we switch from prealgebraizability to *algebraizability*. Dropping "pre" signifies using equivalentiality instead of the weaker preequivalentiality property. Equivalentiality encompasses protoalgebraicity and, under protoalgebraicity, only two classes of the ten potentially different ones are actually distinct. Accordingly, we get *family algebraizability*, or, simply, *F algebraizability*, when family injectivity is added, and *system algebraizability*, or, simply, *algebraizability*, when system injectivity is added. Family algebraizability is equivalent to algebraizability plus systemicity. Both properties transfer. Finally, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is algebraizable if and only if it is stable and, for every **F**-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions, whereas $\mathcal{I}$ is family algebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions.

1.3.5 Chapter 6

The motivating force behind the considerations in this chapter is the observation that, since for a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, with $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)$, no π-institution without theorems can satisfy any of the injectivity, reflectivity of complete reflectivity properties introduced in Chapter 3. The question naturally arises whether, in that case, the existence of theory families with empty components is the only reason causing the lack of these properties or whether the π-institution in question would still not satisfy them even if theory families with empty components were in some way "discarded" or "bypassed". We choose two ways in which this circumvention may be accomplished, and study the various flavors of injectivity, reflectivity and complete reflectivity properties that result.

In Section 6.2, we introduce and study the relation of rough equivalence between theory families of a π-institution. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a theory family T of $\mathcal{I}$, we define the *rough companion* (*rough associate* or *rough representative*) $\widetilde{T}$ of T as the theory family resulting from T by replacing all empty Σ-components of T by the corresponding set $\mathrm{SEN}^\flat(\Sigma)$ of Σ-sentences. We say that two theory families T and T' are *roughly equivalent*, written $T \sim T'$, if $\widetilde{T} = \widetilde{T'}$. The rough equivalence class of T is denoted by $\widetilde{[T]}$ and $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ denotes the collection of all rough equivalence classes. When one considers the restriction of rough equivalence on theory systems, the corresponding rough equivalence class is denoted by $\widetilde{\lfloor T \rfloor}$ and the collection of all these classes by $\widetilde{\mathrm{ThSys}}(\mathcal{I})$. Reasoning with rough equivalence classes is one way of bypassing theory families with empty components. An alternative way is to ignore those theory families that have at least one empty component. This is accomplished by considering the collections $\mathrm{ThFam}^{\not\flat}(\mathcal{I})$ and $\mathrm{ThSys}^{\not\flat}(\mathcal{I})$ of all theory families and theory systems, respectively, none of whose components is empty.

The usefulness of rough equivalence in considering properties of the Leibniz operator stems from the fact that, for every theory family T, $\Omega(T) = \Omega(\widetilde{T})$. As a consequence, the Leibniz operator is constant on each rough equivalence class. It is fairly obvious that the rough companion $\widetilde{T}$ of a theory family T is the maximum element in the class $\widetilde{[T]}$. However, even if T happens to be a theory system, $\widetilde{T}$ may not be one. On the other hand, it can be shown that, even in that case, $\widetilde{\lfloor T \rfloor}$ has a maximum element, which, of course, does not coincide with $\widetilde{T}$. An unfortunate fact, when considering the operators $\overleftarrow{\ }$ and $\widetilde{\ }$ in the same context is that, even if two theory families T and T' are roughly equivalent, the same may not hold for $\overleftarrow{T}$ and $\overleftarrow{T'}$. On the positive side, if $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and T is an $\mathcal{I}$-filter family of $\mathcal{A}$, we do have $\alpha^{-1}(\widetilde{T}) = \widetilde{\alpha^{-1}(T)}$. This implies that the action of α^{-1} preserves rough equivalence, i.e., if T and T' are $\mathcal{I}$-filter families of $\mathcal{A}$, with $T \sim T'$, then $\alpha^{-1}(T) \sim \alpha^{-1}(T')$, the latter being roughly equivalent theory families of $\mathcal{I}$.

In Section 6.3, we look at some notions combining systemicity with rough equivalence. They form a hierarchy weakening systemicity in the absence of theorems. In the presence of theorems, however, all concepts considered coincide. We say that a π-institution $\mathcal{I}$ is *roughly systemic* if, for every theory family T, $\overleftarrow{T}$ is roughly equivalent to T, i.e., $\overleftarrow{T} \sim T$. We say $\mathcal{I}$ is *narrowly systemic* if, for every theory family T in $\mathrm{ThFam}^{\natural}(\mathcal{I})$ (i.e., with all components nonempty), $\overleftarrow{T} = T$. Finally, we say that $\mathcal{I}$ is *exclusively systemic* if, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, $\overleftarrow{T} = T$. Systemicity is the strongest of these four conditions, followed by rough and narrow systemicity, which are incomparable in strength, and each of these two implies exclusive systemicity. Moreover, as mentioned previously, exclusive systemicity in the presence of theorems implies systemicity and, therefore, in that case, the entire hierarchy collapses to a single class.

In Section 6.4, we formalize and study various versions of rough injectivity, resulting by combining injectivity of the Leibniz operator with rough equivalence. The easiest to grasp is rough family injectivity. A π-institution $\mathcal{I}$ is *roughly family injective* if, for all theory families T, T', $\Omega(T) = \Omega(T')$ implies $T \sim T'$. *Rough left injectivity* results by replacing in the conclusion of the implication defining rough family injectivity T and T' by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. *Rough right injectivity* arises by a similar replacement in the hypothesis. Finally, *rough system injectivity* imposes the same condition as the family version, but restricts its application to theory systems. Rough right injectivity implies rough systemicity, but the converse fails in general. The rough injectivity hierarchy turns out to be more complex than the injectivity hierarchy studied in Section 3.6. There, it was shown that right injectivity implies family injectivity, which implies left injectivity, which, in turn, implies system injectivity, giving rise to a linear injectivity hierarchy. On the other hand, in the rough case, it is shown that rough right injectivity implies rough family injectivity, which implies the system version, and, in addition, rough left injectivity also implies the system version. Moreover, rough right injectivity is equivalent to rough system injectivity plus rough systemicity. Rough system injectivity, supplemented with stability, implies rough left injectivity. Each of the four rough injectivity properties, together with the availability of theorems, is equivalent to the corresponding injectivity property. The section concludes by establishing that all four rough injectivity properties transfer and by providing characterizations of rough family and rough system injectivity via the Leibniz operator Ω, viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.5, we switch to a different version of injectivity properties, the overarching motivation still remaining that of bypassing theory families with empty components. *Narrow family injectivity* is defined by imposing the injectivity of the Leibniz operator on $\mathrm{ThFam}^{\nleq}(\mathcal{I})$, i.e., by stipulating that, for all $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ implies $T = T'$. *Narrow left injectivity* replaces T, T' in the conclusion by $\overleftarrow{T}, \overleftarrow{T'}$, respectively, whereas *narrow right injectivity* applies the same replacement in the hypothesis. Finally, *narrow system injectivity* enforces the same condition as that of narrow family injectivity, but restricts its scope on theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$. Narrow right injectivity implies exclusive systemicity, but does not imply any of the stronger versions of rough or narrow systemicity. With narrow injectivity, we recover the linearity of the injectivity hierarchy that was lost in passing to rough injectivity. That is, narrow right injectivity implies narrow family injectivity, which implies narrow left injectivity, which, in turn, implies the system version. Moreover, narrow system injectivity, supplemented by narrow systemicity, implies narrow right injectivity. It turns out that narrow family injectivity is equivalent to rough family injectivity. On the other hand, the two left injectivity properties, narrow left and rough left injectivity, ar incomparable, i.e., none implies the other. Some order is regained when looking at the right versions, where rough right injectivity implies narrow right injectivity. This order is maintained at the system level in which rough system injectivity also implies narrow system injectivity. As was the case with rough injectivity, each narrow injectivity property, supplemented with the existence of theorems, is equivalent to the corresponding injectivity property. Moreover, all four narrow injectivity properties transfer. Finally, the family and system versions have characterizations in terms of the injectivity of Ω, viewed as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Sections 6.4 and 6.5, we looked at the rough and narrow injectivity hierarchies. Following this paradigm, in Sections 6.6 and 6.7, we introduce and study the rough and narrow reflectivity properties and, then, in Sections 6.8 and 6.9, the rough and narrow complete reflectivity properties.

In Section 6.6, we turn to rough reflectivity. Once more, the family version is the easiest to describe. A π-institution is called *roughly family reflective* if, for all theory families T, T', $\Omega(T) \leq \Omega(T')$ implies $\widetilde{T} \leq \widetilde{T'}$. *Rough left reflectivity* results by replacing T, T' in the conclusion by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively. *Rough right reflectivity* applies the same change in the hypothesis. Finally, *rough system reflectivity* imposes the same implication as the family version, but only on theory systems. Rough right reflectivity implies rough systemicity. It also implies rough family reflectivity, which implies rough system reflectivity. Rough left reflectivity also implies the system version. Rough right reflectivity is actually equivalent to the system version plus rough systemicity. On the other hand, rough system reflectivity and stability imply rough left reflectivity. It is straightforward to see, based on the relevant defining conditions, that each of the four rough reflectivity versions implies the corresponding rough injectivity version. Furthermore, each rough reflectivity version, supplemented by the existence of theorems, is equivalent to the corresponding reflectivity property. The section concludes with a proof that all four rough reflectivity properties transfer and with characterizations of rough family and rough system reflectivity in terms of the Leibniz operator, viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.7, we look at narrow reflectivity properties. These constitute alternatives to rough reflectivity when dealing with reflectivity properties while attempting to bypass theory families with empty components. A π-institution is *narrowly family reflective* if, for all theory families T, T' in $\mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$. As before, *narrow left reflectivity* results by replacing T, T' in the conclusion by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, and *narrow right reflectivity* by performing the same replacement in the hypothesis instead. Finally, *narrow system reflectivity* stipulates that, for all $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$. Narrow family reflectivity implies exclusive systemicity. As was the case with narrow injectivity properties, narrow reflectivity properties also align into a linear hierarchy. The strongest is narrow right reflectivity, followed by narrow family reflectivity, then by the left version and, at the tail, by narrow system reflectivity. The weakest one, narrow system reflectivity, supplemented by narrow systemicity, implies narrow right reflectivity. The relationships between corresponding rough and narrow versions of reflectivity follow those established in Section 6.5 between corresponding rough and narrow injectivity properties. First, rough family and narrow family reflectivity are equivalent. On the opposite end, the left versions turn out to be incomparable. Somewhere in between, for both the right and system versions, it turns out that the rough property implies the narrow one. Not surprisingly, each narrow reflectivity property implies the corresponding narrow injectivity property. Moreover, a given narrow reflectivity property is equivalent to the corresponding reflectivity property in the presence of theorems. All four narrow reflectivity properties transfer. Finally, characterizations are provided of narrow family and narrow system reflectivity in terms of the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.8, we turn to complete reflectivity (c-reflectivity) properties starting with rough complete reflectivity. A π-institution $\mathcal{I}$ is *roughly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. The *left version* results by replacing each theory family by its arrow counterpart in the conclusion, whereas the *right one* by applying the same change in the hypothesis instead. Finally, the *system version* stipulates that the same condition as that defining the family version applies, but $\mathcal{T} \cup \{T'\}$ is allowed to range over collections of theory systems instead of arbitrary theory families. Paralleling the rough reflectivity hierarchy, rough right c-reflectivity implies rough family c-reflectivity, which implies rough system c-reflectivity, while the left version also implies the system version. In fact, rough right c-reflectivity is equivalent to rough system c-reflectivity plus rough systemicity, whereas rough system c-reflectivity, together with stability, imply rough left c-reflectivity. It is clear that each rough c-reflectivity property generalizes the corresponding rough reflectivity property. It is also not difficult to show that each rough c-reflectivity property, in the presence of theorems, coincides with the corresponding c-reflectivity property. All four rough c-reflectivity properties transfer and, as before, characterizations may be formulated of the family and system versions in terms of the Leibniz operator, perceived as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

Section 6.9 deals with narrow complete reflectivity. A π-institution $\mathcal{I}$ is *narrowly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap \mathcal{T} \leq T'$. Once more, the *left version* arises by replacing all theory families in the conclusion by their arrow counterparts and, similarly, the *right version* by performing the same change in the hypothesis. *Narrow system c-reflectivity* imposes the same condition as the family version, but restricted to collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\lightning}(\mathcal{I})$. As with narrow reflectivity, the narrow c-reflectivity hierarchy is linear. The right version is the strongest, followed by the family, then the left and, finally, the system version. In addition, narrow system c-reflectivity, together with narrow systemicity, implies the right version. Comparisons between the rough c-reflectivity and the narrow c-reflectivity classes also follow the pattern revealed for corresponding reflectivity properties. In accordance, rough family and narrow family c-reflectivity are equivalent, rough left and narrow left c-reflectivity are incomparable, whereas the rough right and rough system versions imply, respectively, the narrow right and narrow system versions. As with their rough counterparts in Section 6.8, all four narrow c-reflectivity properties coincide with the corresponding c-reflectivity properties in the presence of theorems. Furthermore, all four narrow c-reflectivity properties transfer. The family and system versions have characterizations via the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ and $\mathrm{ThSys}^{\lightning}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$, analogous to the ones obtained for both narrow injectivity and narrow reflectivity.

The last section of the chapter, Section 6.10, contains some characterizations of the property of a π-institution possessing theorems. This is closely connected to the overarching ideas governing the properties investigated in Sections 6.2-6.9, which aimed at rectifying the "pathologies" introduced by the absence of theorems. The availability of theorems is characterized by the injectivity of the Frege equivalence family operator, as well as by both the injectivity and the complete reflectivity of the Lindenbaum equivalence family operator, both applied to the collection of theory families of the π-

institution. These operators were introduced in Section 2.11. Possession of theorems transfers to the collections of all $\mathcal{I}$-filter families over arbitrary **F**-algebraic systems.

1.3.6 Chapter 7

In Chapter 7, we further pursue our endeavor of making properties in the lower bottom of the algebraic hierarchy suitable for the study of π-institutions that do not have theorems. Similarly to Chapter 6, we employ rough equivalence and narrowness to achieve this goal, but, unlike in Chapter 6, the focus here is on monotonicity and complete monotonicity properties, rather than on injectivity, reflectivity and complete reflectivity properties.

In Section 7.2, we define a *stability hierarchy*, which serves, in the sequel, to formalize properties of some of the classes in the monotonicity and complete monotonicity hierarchies. Recall that a π-institution $\mathcal{I}$ is *stable* if, for all theory families $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$. Weakening this notion, we call $\mathcal{I}$ *narrowly stable* if the same equation holds, provided $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, i.e., the scope is restricted to theory families all of whose components are nonempty. A further weakening insists that the same equation hold for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, i.e., it further restricts the scope of the quantification to theory families all of whose components are nonempty and whose arrow counterparts also have all components nonempty. Clearly, stability implies narrow stability, which, in turn, implies the last property, which is termed *exclusive stability*. It is shown that both implications are strict.

In Section 7.3, we study the *rough monotonicity hierarchy*. Recall that, given a π-institution $\mathcal{I}$ and a theory family T of $\mathcal{I}$, $\widetilde{T}$ denotes the *rough companion* of the theory family T, which is the theory family resulting from T by replacing all empty Σ-components of T by $\mathrm{SEN}^\flat(\Sigma)$. Two theory families T and T' are *roughly equivalent* if they have the same rough companion. This is equivalent to saying that if T and T' differ at some signature Σ, they one has an empty Σ-component, whereas the other has $\mathrm{SEN}^\flat(\Sigma)$ as its Σ-component. A π-institution $\mathcal{I}$ is *roughly family monotone* if, for all theory families $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$. *Rough left monotonicity* results by replacing T, T' in the hypothesis by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, and *rough right monotonicity* by applying the same replacement in the conclusion. *Rough system monotonicity* stipulates that the original implication hold, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$. It turns out that rough left monotonicity implies both rough family and rough right monotonicity and that each of the latter two implies the system version. Additionally, the strongest version, rough left monotonicity, is equivalent to the weakest, system, version, together with stability. Recall from Section 3.3 that family and left monotonicity are equivalent and this property was termed *protoalgebraicity*.

Recall also, from the same section, that system and right monotonicity are equivalent and this property was called *prealgebraicity*. Protoalgebraicity implies rough left monotonicity, whereas prealgebraicity implies rough right monotonicity. Tighter connections can be established under some fairly general hypotheses. For non almost inconsistent π-institutions, protoalgebraicity is equivalent to rough family or rough left monotonicity, coupled with the availability of theorems. Similarly, for π-institutions having a theory family $T \neq \mathrm{SEN}^\flat$, with $\overleftarrow{T} \neq \overline{\varnothing}$, prealgebraicity is equivalent to rough right or rough system monotonicity, supplemented with the availability of theorems. All four rough monotonicity properties transfer. E.g., $\mathcal{I}$ is roughly family monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{I}$-filter families $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Both rough family and rough system monotonicity can be characterized using properties of the Leibniz operator viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.4, we switch from rough monotonicity to *narrow monotonicity properties*. These constitute an alternative approach to bypassing theory families and theory systems with one or more empty components. We say that a π-institution $\mathcal{I}$ is *narrowly family monotone* if, for all theory families T, T', with all components nonempty, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. The *left version* results by replacing T, T' by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, in the hypothesis and the *right version* by performing the same replacement in the conclusion instead. *Narrow system monotonicity* stipulates that, for all $T, T' \in \mathrm{ThSys}^{\not\varnothing}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. Narrow left monotonicity implies narrow family monotonicity, which implies narrow system monotonicity, while the latter is also a consequence of narrow right monotonicity. Narrow left monotonicity is strong enough to yield exclusive stability, which, however, is the weakest of the three stability versions studied in Section 7.2. Under narrow systemicity, introduced in Section 6.3, the narrow monotonicity hierarchy collapses to a single class. Protoalgebraicity implies narrow left monotonicity and prealgebraicity implies the right version. In this case as well, tighter connections are possible under additional, fairly general, hypotheses, as was the case with rough monotonicity properties. Namely, under the hypothesis that $\mathcal{I}$ is not almost inconsistent, protoalgebraicity is equivalent to narrow left or narrow family monotonicity, coupled with the existence of theorems. And, provided that $\mathcal{I}$ possess a theory system $T \neq \overline{\varnothing}, \mathrm{SEN}^\flat$, prealgebraicity is equivalent to narrow right or narrow system monotonicity, together with the availability of theorems. Of central interest here is whether and how the rough monotonicity properties are related to the narrow monotonicity properties. In comparing the two hierarchies, we discover that the two family versions are equivalent, whereas each of the three remaining rough monotonicity properties implies the corresponding narrow monotonicity property. All four narrow monotonicity properties transfer. Finally, characterizations of the family and the system versions may be formulated in terms of the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.5, we return to roughness, but study complete monotonicity (c-monotonicity) instead of monotonicity properties. *Rough family c-monotonicity* stipulates that, for all collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$ implies $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. *Rough left c-monotonicity* and *rough right c-monotonicity* result by replacing in the hypothesis and in the conclusion, respectively, all theory families by their arrow versions. *Rough system c-monotonicity* imposes the same condition as does the family version, but restricts its applicability on collections $\mathcal{T} \cup \{T'\}$ consisting of theory systems. Here, it turns out that each of the left, family and right versions implies the system version. Moreover, rough left c-monotonicity is equivalent to the conjunction of rough system c-monotonicity and stability. It is also the case that, under stability, the rough family and rough right c-monotonicity properties coincide and that, under rough systemicity, the entire rough c-monotonicity hierarchy collapses to a single class. From the definitions, it is obvious that each of the four rough c-monotonicity properties implies the corresponding rough monotonicity version. It is also the case that each c-monotonicity property implies its rough c-monotonicity counterpart. Once more, for non almost inconsistent π-institutions, family (left c-monotonicity, respectively) is equivalent to the conjunction of rough family (rough left, respectively) c-monotonicity and the existence of theorems. Furthermore, if $\mathcal{I}$ possesses a theory family $T \neq \mathrm{SEN}^\flat$, such that $\overleftarrow{T} \neq \overline{\varnothing}$, then system (right, respectively) c-monotonicity is equivalent to rough system (right, respectively) c-monotonicity plus the existence of theorems. All four rough c-monotonicity properties transfer and one may, in this case also, recast the family and system versions in terms of properties of the Leibniz operator seen as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.6, we switch from rough versions of c-monotonicity to narrow versions of the same property. A π-institution $\mathcal{I}$ is called *narrowly family c-monotone* if, for all collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $T' \leq \bigcup_{T\in\mathcal{T}} T$ implies $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. In the *left version*, all theory families are replaced in the hypothesis by their arrow counterparts and, in the *right version*, the same change is applied in the conclusion. The *system version* stipulates that the implication above hold for all collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Each of the left, family and right versions implies the system version. Moreover, each of the four c-monotonicity versions implies the corresponding narrow c-monotonicity version. As was the case in relating rough and narrow monotonicity classes in Section 7.4, rough family c-monotonicity is equivalent to narrow family c-monotonicity, whereas each of the other three rough c-monotonicity properties implies the corresponding narrow c-monotonicity version. From the definitions, it is clear that a narrow c-monotonicity property implies its narrow monotonicity counterpart, the latter being a special-ization of the former. All four narrow c-monotonicity properties transfer. In closing, both the family and the system versions have characterizations in terms of properties of the Leibniz operator perceived as a mapping from $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, respectively, into $\mathrm{ConSys}^*(\mathcal{I})$.

1.3.7 Chapter 8

In Chapter 8, we undertake the study of *regularity*. Roughly speaking, it is the property stipulating that, whenever two sentences belong to a theory family of a given π-institution, they must be identified modulo the Leibniz congruence system relative to that theory family. When, in addition to regularity, availability of theorems is also postulated, the property of *assertionality* is obtained. Assertionality strengthens complete reflectivity and, as a result, it can be used to strengthen (weak) (pre)algebraizability properties. These strengthenings and their associated hierarchies are under the microscope in Sections 8.4-8.7. The classes of π-institutions obtained here are among the most powerful classes in the semantic hierarchy of π-institutions, i.e., satisfy the strongest properties and are included in most of the other classes in the hierarchy.

In Section 8.2, we introduce *regularity*. As was the case with other properties in preceding chapters, regularity comes in four different versions. Once more, we begin from the easiest to describe, the family version. A π-institution $\mathcal{I}$ is *family regular* if, for all theory families T, all signatures Σ and all Σ-sentences ϕ an ψ, if $\phi, \psi \in T_\Sigma$, then $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. *Left regularity* results by replacing T in the hypothesis by $\overleftarrow{T}$, *right regularity* by performing the same replacement in the conclusion instead, whereas *system regularity* stipulates that the implication hold for all theory systems T. Family regularity is the strongest of the four properties, followed by right regularity, which implies left regularity, which, in turn, implies the system version. Thus, regularity properties are stratified into a linear hierarchy. System regularity plus stability imply left regularity, and right regularity plus stability yield family regularity. It follows that, under stability, the four-class hierarchy is reduced to two classes. On the other hand, system regularity plus systemicity clearly yield family regularity, whence, systemicity leads to a collapse of the regularity hierarchy into a single class. The family, left and system versions have elegant characterizations in terms of the Suszko operator and one of its variants. E.g., a π-institution $\mathcal{I}$ is family regular if and only if, for every signature Σ and all Σ-sentences ϕ and ψ, $\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi))$, where $C(\phi, \psi)$ is the least theory family of $\mathcal{I}$ containing ϕ and ψ. All four regularity properties transfer. For instance, with regards to the right version, $\mathcal{I}$ is right regular if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ in $\mathcal{A}$ and all Σ-sentences ϕ and ψ, $\phi, \psi \in T_\Sigma$ implies $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T})$. The other three transfer results are formalized similarly.

Finally, the family and system versions may be characterized by the property that the filter family (system, respectively) in any reduced matrix family (system, respectively) is at most a singleton, in the sense that it consists of components with at most one element each.

In Section 8.3, we study *assertionality*. This is the property resulting from regularity by adding the requirement that theorems exist. Accordingly, four versions of assertionality are a priori obtained, depending on which of the four versions of regularity is postulated. They are termed *family*, *right*, *left* and *system assertionality* and, based on the hierarchy of regularity properties of Section 8.2, these also form a linear hierarchy, with the family version at the top, followed by the right, then the left and, finally, the system version at the bottom of the hierarchy. Assertionality is characterized by asserting, roughly speaking, that each theory family is fully determined by its Leibniz congruence system as the equivalence class of any theorem. Even though, a priori, there are four assertionality versions, there is a reduction holding without proviso. More precisely, it can be shown that right assertionality implies systemicity and this entails that right and family assertionality are equivalent. This property implies left assertionality, which, in turn, implies the system version. Moreover, the latter supplied with systemicity, implies family assertionality. By the definitions, it is clear that each assertionality version implies the corresponding regularity version. What is, however, more interesting, albeit not much more challenging to demonstrate, is that each assertionality property implies the corresponding complete reflectivity (c-reflectivity) property (see Section 3.8). So the assertionality properties may be viewed as further strengthening the hierarchy of reflectivity and c-reflectivity properties, studied in Sections 3.7 and 3.8. All three different assertionality properties transfer. Again, indicative of the flavor, a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is, e.g., left assertional if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is left assertional, meaning that, on the one hand, the least $\mathcal{I}$-filter family of $\mathcal{A}$ has all components nonempty and, on the other, that, for all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ and all Σ-sentences ϕ and ψ, such that $\phi, \psi \in T_\Sigma$, one has $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$. The section concludes with characterizations of the family and system versions, analogous to the ones provided in the conclusion of Section 8.2 for regularity. Namely, it is shown that $\mathcal{I}$ is family (system) assertional if and only if the filter family (system, respectively) of every reduced matrix family (system, respectively) is a singleton (i.e., consists of singleton components).

In Sections 8.4-8.7, we take advantage of the role of assertionality in strengthening of c-reflectivity to obtain strengthened versions of weak (pre)-algebraizability and (pre)algebraizability properties. The first two are obtained by combining assertionality properties with pre- or protoalgebraicity, whereas the latter are obtained by using (pre)equivalentiality instead.

In Section 8.4, we look at *regular weak prealgebraizability* properties. These result from adding to prealgebraicity (i.e., system monotonicity) a version of assertionality. Since there are three distinct versions of assertionality, one obtains three distinct corresponding versions of regular weak prealgebraizability. A π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *prealgebraizable* if it is prealgebraic and family assertional. It is *regularly weakly left* (RWL) *prealgebraizable* if it is prealgebraic and left assertional and it is *regularly weakly system* (RWS) *prealgebraizable* if it is prealgebraic and system assertional. Since the distinguishing feature between these three properties is the type of assertionality imposed, the assertionality hierarchy immediately yields that RWF prealgebraizability implies RWL prealgebraizability, which, in turn, implies RWS prealgebraizability. Equally clear from the definitions is the fact that RWF/L/S prealgebraizability implies, respectively, family/left/system assertionality. Additionally, the fact that each assertionality property implies its c-reflectivity counterpart entails that RWF/L/S prealgebraizability implies, respectively, WF/L/SC prealgebraizability (see Section 4.2). All three versions of regular weak prealgebraizability transfer. The section concludes with characterizations of the three versions based on the Leibniz operator viewed as a mapping between ordered sets of filter families/systems and congruence systems. To provide a flavor, we look at RWF prealgebraizability. The characterization states that $\mathcal{I}$ is RWF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton.

In Section 8.5, we study *regular weak algebraizability*. The properties here are obtained from the regular weak prealgebraizability properties of Section 8.4 by upgrading prealgebraicity to protoalgebraicity. Accordingly, a π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *algebraizable* if it is protoalgebraic and family assertional, it is *regularly weakly left* (RWL) *algebraizable* if it is protoalgebraic and left assertional and it is *regularly weakly system* (RWS) *algebraizable* if it is protoalgebraic and system assertional. Notice that, since these properties constitute enhancements of the properties of Section 8.4, the right version has been absorbed within the family version. Here, however, protoalgebraicity, which, unlike prealgebraicity, implies stability, forces, in addition, the identification of the left and the system versions. Thus, there are only two distinct regular weak algebraizability properties, regular weak family (equivalently, right) algebraizability being the strongest and regular weak system (equivalently, left) algebraizability the weakest of the two. In comparing this two-step hierarchy with that of regular weak prealgebraizability properties, we discover that the two family versions coincide, whereas regular weak system algebraizability implies regular weak left prealgebraizability. As a consequence, the combined regular weak (pre)algebraizability hierarchy consists of four classes that are linearly ordered. Moreover, essentially due to the fact that assertionality properties imply c-reflectivity properties, each of the two regular weak algebraizability classes are included in the corresponding weak algebraizability classes. Both regular weak algebraizability properties transfer. Finally, both have characterizations in terms of the Leibniz operator seen as a mapping between ordered sets. E.g., $\mathcal{I}$ is RWS algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphim, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton.

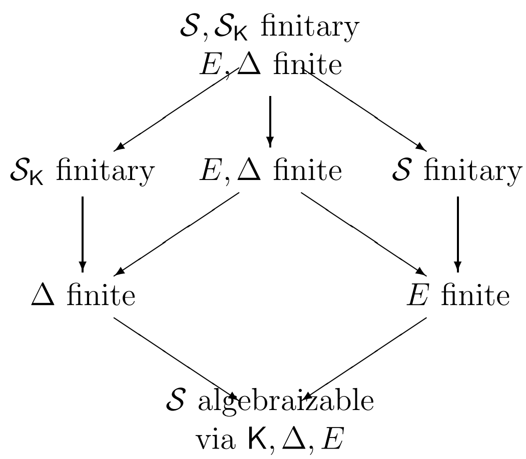
In Section 8.6, we introduce *regular prealgebraizability* properties. These are obtained by combining assertionality properties with preequivalentiality. Recalling that preequivalentiality is obtained by adding system extensionality to prealgebraicity, an alternative point of view is that regular prealgebraizability is obtained from regular weak prealgebraizability, studied in Section 8.4, by adding system extensionality. A π-institution $\mathcal{I}$ is *regularly family* (RF) *prealgebraizable* if it is preequivalential and family assertional, it is *regularly left* (RL) *prealgebraizable* if it is preequivalential and left assertional and it is *regularly system* (RS) *prealgebraizable* if it is preequivalential and system assertional. Based on the linear hierarchy of assertionality properties, we obtain a linear hierarchy of regular prealgebraizability properties, with RF prealgebraizability at the apex, followed by RL prealgebraizability, while RS prealgebraizability is at the bottom. Since preequivalentiality strengthens prealgebraicity, RF/L/S prealgebraizability implies, respectively, RWF/L/S prealgebraizability. Moreover, since each version of assertionality implies the corresponding c-reflectivity version, RF/L/S prealgebraizability implies, respectively, family/ left c-reflective/ system prealgebraizability (see Section 5.5). All three versions transfer. Finally, characterization theorems may be formulated for each of the three properties in terms of the Leibniz operator viewed as a mapping between ordered sets. To provide, once more, a preview, we mention the form this characterization takes in the case of regular left prealgebraizability. A π-institution $\mathcal{I}$ is regularly left prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system, $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding, commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overleftarrow{T}/\Omega^{\mathcal{A}}(T)$ is a singleton.

In Section 8.7, the last section of Chapter 8, we turn to the study of *regular algebraizability* properties, which combine equivalentiality with assertionality. Equivalentiality forms a common strengthening of both protoalgebraicity and preequivalentiality. Even though one obtains, a priori, three versions of regular algebraizability, only two are distinct. We say that $\mathcal{I}$ is *regularly family* (RF) *algebraizable* if it is equivalential and family assertional, *regularly left* (RL) *algebraizable* if it is equivalential and left assertional, and *regularly system* (RS) *algebraizable* if it is equivalential and system assertional. Regular left and regular system algebraizability coincide and, as a result, the regular algebraizability hierarchy consists of the class of RF algebraizable π-institutions and its proper subclass of RS algebraizable π-institutions. In comparing regular algebraizability with regular prealgebraizability properties, we discover that the two family versions are equivalent and that regular system algebraizability implies regular left prealgebraizability. Further, in comparing regular algebraizability with regular weak algebraizability properties, we obtain, based on equivalentiality's dominant position over protoalgebraicity, that RF/S algebraizability implies, respectively, RWF/S algebraizability. In the ultimate comparison between subhierarchies, based on the fact that assertionality implies c-reflectivity, we obtain that RF/S algebraizability implies, respectively, F/S algebraizability. The section closes with the same type of theorems as previous sections. Namely, it is shown that both versions of regular algebraizability transfer from a π-institution to the filter families/systems over arbitrary $\mathbf{F}$-algebraic systems and characterizations of both versions are obtained in terms of the Leibniz operator perceived as a mapping between ordered sets. The family version, e.g., asserts that $\mathcal{I}$ is regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton.

1.3.8 Chapter 9

In Chapter 9, we undertake the study of finitarity properties of weakly family algebraizable π-institutions. Here we draw inspiration by the analysis of corresponding properties of algebraizable sentential logics.

According to the theory of algebraization of sentential logics, a, not necessarily finitary, algebraizable sentential logic $\mathcal{S}$ is algebraized via an equivalence that relates its consequence relation with the equational consequence of a generalized quasivariety K. The relation of equivalence is established via a possibly infinite set of defining equations $E(x)$ in a single variable x, which serve to translate formulas into equations, and a possibly infinite set $\Delta(x,y)$ of equivalence formulas in two variables x and y, which serve to translate equations into formulas. Besides constituting interpretations between the two consequences, they should be mutually inverse in a specific sense. In examining the relationships between the various finitarity conditions that may hold, namely, $\mathcal{S}$ finitary, $\mathcal{S}_{\mathsf{K}}$ (the equational deductive system induced by K) finitary, $E(x)$ finite and $\Delta(x,y)$ finite, one may show that they are related by the implications depicted in the following diagram (see p. 137 in Section 3.4 of [89]).

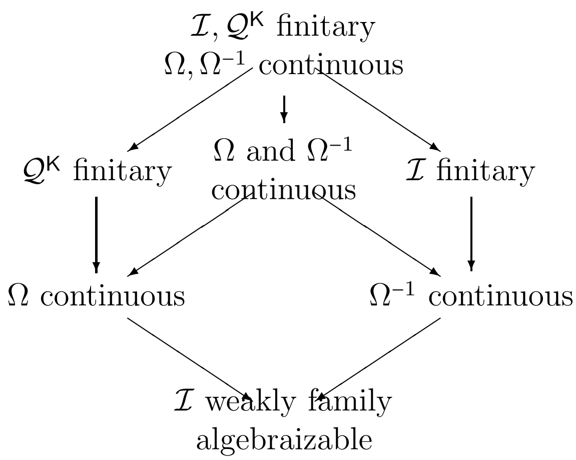


In the framework of sentential logics, roughly speaking, syntactic and semantic properties, i.e., those imposing the existence of transformations, such as $E(x)$ and $\Delta(x, y)$, satisfying certain properties, and those defined by order-theoretic properties of the Leibniz operator go hand-in-hand, in a tight correspondence. This is not the case in the framework of logics formalized as π-institutions. So in this chapter, the goal is to translate the sentential finitarity conditions to corresponding semantic properties and to establish an analogous hierarchy for weakly family algebraizable π-institutions. We also use examples from the sentential framework, recasting them as π-institutions, to obtain logical systems that serve to separate the classes of π-institutions specified by these finitarity properties.

In Section 9.2, the concept of *π-structure* is introduced, which abstracts that of a π-institution by removing the requirement of structurality. For π-structures, and, hence, also for π-institutions, the *finitary companion* is constructed, which is the π-structure over the same base algebraic system that has the largest finitary closure family included in the closure family of the given π-structure. *Locally finitely generated theory families* are defined and they are used to characterize those sentence families of a π-structure that are theory families of its finitary companion. These turn out to be exactly those sentence families that are unions of directed collections of locally finitely generated theory families of the given π-structure.

In Section 9.3, we investigate under which provisos, if any, the properties that define weak family algebraizability, i.e., protoalgebraicity and family reflectivity, are inherited by the finitay companion from the original π-structure and vice-versa. It is shown, first, that protoalgebraicity and family reflectivity are propagated from the finitary companion up to the parent π-structure unconditionally. On the other hand, the reverse inheritance requires additional conditions. To this end, the concept of *continuity* of the Leibniz and of the inverse Leibniz operator are introduced. The latter, of course, makes sense only if the π-institution under consideration is such that its Leibniz operator is an isomorphism, e.g., when it is weakly family algebraizable, which is precisely the case we focus on. If the Leibniz operator is continuous, it is easy to see that the π-institution is protoalgebraic. So continuity of the Leibniz operator actually strengthens protoalgebraicity. If, in addition to continuity, finiteness of the signature category is postulated, then the finitary companion is also protoalgebraic. Finally, it is shown that, if a π-institution, with a finite category of signatures, is weakly family algebraizable and both its Leibniz and inverse Leibniz operators are continuous, then its finitary companion is also weakly family algebraizable.

In Section 9.4, we undertake a detailed study of the interrelationships of the four finitarity properties pertaining to weakly family algebraizable π-institutions. These are the finitarity of the π-institution itself, the finitarity of its equational counterpart, the continuity of the Leibniz operator and the continuity of the inverse Leibniz operator, which is well-defined precisely because the π-institution is assumed to be weakly family algebraizable. These four properties are appropriate abstractions in the semantical institutional context of the properties of an algebraizable sentential logic being finitary, of its equivalent algebraic semantics being a quasivariety, of the set of equivalence formulas being finite and of the set of defining equations being finite, respectively. The close analogy is reflected in the fact that the results and hierarchy obtained here parallel the ones that hold for the corresponding properties in the sentential context. Our results come, as do their sentential counterparts, in dual pairs. In the first, it is shown that, for a weakly family algebraizable π-institution $\mathcal{I}$, the finitarity of $\mathcal{I}$ implies the continuity of its inverse Leibniz operator and, dually, the finitarity of the equational π-structure $\mathcal{Q}^{\mathsf{K}}$ induced by $\mathsf{K} := \mathrm{AlgSys}(\mathcal{I})$ implies the continuity of the Leibniz operator itself. Next, it is shown that, under weak family algebraizability, the finitarity of $\mathcal{I}$ and the continuity of the Leibniz operator imply that the equational counterpart is also finitary. Dually, the finitarity of the equational counterpart and the continuity of the inverse Leibniz operator imply that $\mathcal{I}$ itself is finitary. These implications lead to the following conditional equivalences, all applying to weakly family algebraizable π-institutions. For continuous Leibniz and inverse Leibniz operators, a π-institution is finitary if and only if its algebraic counterpart is finitary. For a finitary π-institution, its counterpart is finitary if and only if its Leibniz operator is continuous. Finally, if the algebraic counterpart of a π-institution is finitary, then the π-institution itself is finitary if and only if its inverse Leibniz operator is continuous. These outcomes lead to a finitarity hierarchy for weakly family algebraizable π-institutions paralleling the hierarchy depicted above for sentential logics.



What remains to be done is separate the classes of π-institutions constituting the finitarity hierarchy. For this task, given the analogies established with the sentential framework, we seek inspiration from the realm of sentential logics.

In Section 9.5, we revisit three sentential logics that serve in separating the classes that form the finitarity hierarchy in the sentential framework. The classes related by the vertical arrows are separated by Łukasiewicz's infinite valued logic. This is a non-finitary, semantically defined sentential logic. It is algebraizable with a non-finitary equivalent algebraic semantics. On the other hand, both sets of defining equations and equivalence formulas are finite. The classes connected by the southeast arrows are separated using a finitary logic introduced by Dellunde and defined via a Hilbert calculus. It is regularly algebraizable via a singleton set of defining equations but a necessarily infinite set of equivalence formulas. Finally, the classes related by the southwest arrows of the diagram are separated using a non-finitary logic semantically defined by Raftery. This logic has a finitary equivalent algebraic semantics (actually a variety) and is algebraized via a finite set of equivalence formulas but a necessarily infinite set of defining equations. Even though we could certainly rely on well-written accounts from the literature to simply refer to these logics, we chose to recount all details, based on those original references. The Introduction to Chapter 9 and the main body contain more information, as well as appropriate references.

In Section 9.6, the three sentential logics of Section 9.5 are recast as π-institutions, according to the general procedure outlined in Section 1.1. The resulting π-institutions serve, in turn, in separating the corresponding classes appearing in the finitarity hierarchy of weakly family algebraizable π-institutions. Further evidencing the analogies described between the two finitarity hierarchies, the π-institution encapsulating Łukasiewicz's logic separates the classes of π-institutions connected via vertical arrows, the one incorporating Dellunde's logic separates classes along the southeast arrows, while the one arising from Raftery's logic separates classes related by the southwest arrows in the institutional finitarity hierarchy.

1.3.9 Chapter 10

In Chapter 10, we turn attention to syntactic aspects of logics formalized as π-institutions. Even though the study of the syntactic algebraic hierarchy per se does not start in earnest until Chapter 11, in Chapter 10 we introduce the basic elements and lay the groundwork. In the process we provide some analysis of relevant properties and exhibit some smaller subhierarchies involving those properties. Much of the work done in the chapter has its roots in the work of Boole on the algebraization of the laws of thought [2]. Closer origins can be traced in the work of Lindenbaum [3] and of Tarski [5] (see, also, [**?**]) on the algebraization of propositional calculus and subsequent related work of Rasiowa and Sikorski [11] on the algebraization of implicative logics. A more recent work that influenced the contents presented here is the work of Cintula and Noguera on implicational semilinear logics [83, 88, 93].

In Section 10.2, we introduce *sets of natural transformations* $I^\flat$ which play the role of the sets of formulas in the sentential context. These transformations, depending on whether or not some of their arguments are considered primary or distinguished and some secondary, are either *parametric* or *parameter free*.

In Section 10.3, we introduce the property of a set $I^\flat$ of natural transformations on a base algebraic system $\mathbf{F}$, with two distinguished arguments, being *reflexive* in a π-institution $\mathcal{I}$ based on $\mathbf{F}$. Normally, each of the properties studied in the chapter has four versions. These four versions are the result of a duality between theory families versus theory systems and a duality between considering the property locally versus globally. However, reflexivity is defined by reference to theorem systems only. Moreover, its global version collapses to the local one. So only one version of reflexivity need be considered, as opposed to the other properties studied in the chapter. The first result of some importance, mainly because it serves as a harbinger of similar results to follow concerning other properties in later sections, is a *transfer theorem*. Namely, it is shown that a collection $I^\flat$ of natural transformations is reflexive in a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the corresponding collection I is reflexive in $\mathcal{A}$. The latter means that I is reflexive in the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$, where $C^{\mathcal{I},\mathcal{A}}$ is the closure system whose theory families are the $\mathcal{I}$-filter families on $\mathcal{A}$. We summarize this, and similar results later on, by saying that "*reflexivity transfers*".

In Section 10.4, we turn to the property of *symmetry*. Here we do have four versions of the property. *Local family symmetry* refers to the version defined using theory families and applying the natural transformations only locally. This means that applications of signature morphisms to change signatures, while applying the transformations, are not allowed. *Local system* *symmetry* is also defined locally, but using only theory systems instead of arbitrary theory families. The third version, *global family symmetry*, uses again arbitrary theory families, but, in addition, allows changes between signatures in the main arguments of the transformations via the application of signature morphisms. As, perhaps, anticipated, the fourth, and final, version is *global system symmetry*, still allowing changes of signatures but restricting attention to theory systems. The first result of the section establishes the hierarchy of symmetry properties, i.e., it explores how these symmetry properties compare in terms of their relative strength. It is shown that the local family symmetry implies the local system symmetry, which, in turn, implies both the global family and the global system versions, the latter two being equivalent in strength. Further, if the π-institution $\mathcal{I}$ is systemic, i.e., all theory families happen to be theory systems, then the local family and the local system symmetry properties coincide, whereas, if the set $I^\flat$ of natural transformations is parameter free, then the local system version coincides with the global versions. The section closes by establishing a transfer theorem for all three, generally distinct, versions of symmetry. Namely, it is shown that a set $I^\flat$ of natural transformations satisfies a certain symmetry property in $\mathcal{I} = \langle \mathbf{F}, C \rangle$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the corresponding set I has the same symmetry property in $\mathcal{A}$, in the same sense described previously for reflexivity.

In Section 10.5, we study versions of *transitivity*. Again one has, a priori, four versions based on whether arbitrary theory families or only theory systems are considered and on whether the local or the global mode is adopted. In this way we get *local family transitivity*, *local system transitivity*, *global family transitivity* and *global system transitivity*. As was the case with the symmetry hierarchy, local family transitivity is the strongest of the four versions, followed by the local system version, while the two global versions are the weakest and turn out to be mutually equivalent. Once more, under systemicity of the π-institution $\mathcal{I}$, the local family and local system versions coincide and, under the hypothesis that the set $I^\flat$ of natural transformations is parameter free, local system transitivity is identified with the global versions. The section ends by establishing that all transitivity versions transfer from a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathbf{F}$-algebraic systems, with the same meaning as before.

In Section 10.6, we take a look at collection $I^\flat$ of natural transformations possessing various forms of the *equivalence* property. As the name suggests, each form requires (the only version of) reflexivity, a version of symmetry and a version of transitivity. Since there are three distinct versions for symmetry and three for transitivity, we obtain nine versions of equivalence. Finally, since reflexivity, all versions of symmetry and all versions of transitivity transfer, we obtain as a consequence that all nine versions of equivalence also transfer.

Section 10.7 deals with the *antisymmetry* property. This property, as was the case with reflexivity, is defined using only the theorem system of the π-institution. As a result, the family/system duality is not relevant. As opposed, however, to reflexivity, the local/global aspect is still at play. So we obtain two versions of antisymmetry, *local antisymmetry* and *global antisymmetry*, of which the first implies the second. In addition, if the collection of natural transformations $I^\flat$ happens to be parameter free, the two versions coincide. On the other hand, antisymmetry is the first property studied in the chapter which does not transfer.

In Section 10.8, we study various versions of the *order* property. These require (the only version of) reflexivity, a version of antisymmetry and a version of transitivity. Recalling from Sections 10.7 and 10.5 that there exist two versions of antisymmetry and three of transitivity, we obtain six versions of order. Moreover, the antisymmetry and the transitivity hierarchies, established in those sections, give rise to a hierarchy of order properties. The six property hierarchy collapses to four properties under systemicity, whereas parameter freeness has a more drastic effect reducing the hierarchy to a two class one. Finally, the non-transference of antisymmetry causes the order properties to not transfer.

In Section 10.9, we study the *compatibility* property. The meaning is the the binary relation defined by a collection $I^\flat$ of natural transformations in a π-institution $\mathcal{I}$ is compatible with all natural transformations in $N^\flat$. One has again four flavors of the property, *local family*, *local system*, *global family* and *global system compatibility*. Here, also, local family compatibility turns out to be the strongest, followed by local system compatibility, which implies the global versions, which are mutually equivalent. Moreover, if $\mathcal{I}$ is systemic, the local versions collapse and, if $I^\flat$ is parameter free, local system compatibility and the global versions coincide. The section concludes with a result asserting that all three different compatibility properties transfer.

In Section 10.10, we combine versions of the equivalence properties of Section 10.6 with versions of the compatibility properties of Section 10.9 to obtain a subhierarchy of *congruence* properties. From the nine possible equivalence properties of Section 10.6, we restrict attention to the three "uniform" ones, i.e., those formed by applying the same type of symmetry and transitivity. Combining these with the three versions of compatibility, we get nine versions of congruence. If we had considered all nine versions of equivalence, we would have had, a priori, twenty-seven versions of congruence. As detailed previously, and in a similar way for similar reasons, systemicity collapses the subhierarchy down to four classes. Parameter freeness has an analogous effect. Finally, since all equivalence and compatibility properties transfer, so do all versions of congruence.

In Section 10.11, we study versions of *modus ponens* (MP). There are, a priori, four versions of the property. The first is *local family MP*. A set of natural transformations $I^\flat$ has this property in a π-institution $\mathcal{I}$ if, for every theory family T of $\mathcal{I}$, all signatures Σ and all Σ-sentences ϕ and ψ,

$$\phi \in T_\Sigma \text{ and } I^\flat_\Sigma(\phi,\psi,\vec{\chi}) \subseteq T_\Sigma, \text{ for all } \vec{\chi}, \text{ imply } \psi \in T_\Sigma.$$

Restricting only to theory systems, we get the second version, *local system MP*. The remaining two versions, *global family* and *global system MP*, adopt the global point of view. E.g., the global family version requires that

$$\phi \in T_\Sigma \text{ and } I^\flat_\Sigma[\phi,\psi] \leq T \text{ imply } \psi \in T_\Sigma,$$

where $I^\flat_\Sigma[\phi,\psi] = \{I^\flat_{\Sigma,\Sigma'}[\phi,\psi]\}_{\Sigma' \in |\mathbf{Sign}^\flat|}$, such that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$,

$$I^\flat_{\Sigma,\Sigma'}[\phi,\psi] = \{I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) : f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\}.$$

That is, one is allowed changes of signatures via signature morphisms and all resulting formulas must be in T for the second hypothesis to hold. As far as the MP hierarchy is concerned, the strongest property is the local family MP. It implies both the local system MP and the global family MP, which are independent in strength and each implies the local system MP, which is the weakest of the four versions. Under systemicity, the hierarchy shrinks to two versions, whereas, for parameter free natural transformations, we get a three step linear hierarchy, since the local system and the global system versions coincide. Finally, it is shown that all four versions of MP transfer from a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathbf{F}$-algebraic systems, in the same sense as previously discussed for other properties.

Section 10.12 deals with various versions of the *syntactic protoalgebraicity* property of a collection $I^\flat$ of natural transformations in a π-institution $\mathcal{I}$. These properties are a result of combining a uniform version of congruence, that is one of local family, local system or global congruence, with one of the four versions of modus ponens, studied in Section 10.11. Thus, we obtain a subhierarchy of twelve syntactic protoalgebraicity properties. Even though this subhierarchy is complicated, consisting, roughly speaking, of three rhombi, one on top of the other, systemicity reduces it to a single rhombus, whereas lack of parameters results in a six property hierarchy. We opt not to study in detail the entire twelve property hierarchy and, instead, single out the uniform properties, which constitute a four property subhierarchy forming a rhombus. This degenerates into a two chain in the case of systemicity and a three element linear hierarchy in the parameter free case. Again the section ends with a result asserting that all syntactic protoalgebraicity properties transfer.

In Section 10.13, we switch to the study of *invertibility* properties. There are four versions, based, once more, on the family/system and the local/global dualities. To get a flavor of these properties, let us look more closely at *local family invertibility*. We say that a collection $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations, with two distinguished arguments, has the local family invertibility in a π-institution $\mathcal{I}$ if there exists a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations, with a single distinguished argument, such that, for all theory families T, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in T_\Sigma \quad \text{iff} \quad \overleftrightarrow{I}^\flat_\Sigma(\tau^\flat(\phi,\vec{\chi}),\vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma),$$

where $\overleftrightarrow{I}^\flat$ is the "symmetrization" of $I^\flat$, i.e., the collection obtained by adding to $I^\flat$ those natural transformations resulting from it by interchanging the two distinguished arguments. The *local system version* is obtained by restricting attention to theory systems. The *global versions* allow changes of signatures via the application of signature morphisms, both in the main argument position of $\tau^\flat$ and in the two main argument positions of $\overleftrightarrow{I}^\flat$. The hierarchy of invertibility properties has two maximal elements, namely, the local family and the global family versions. The local family implies the local system version and each of the local system and the global family versions imply global system invertibility, the weakest of the four versions. For systemic π-institutions the hierarchy collapses to a two element chain. For parameter free collections of natural transformations, things seem to be more complicated. The four element hierarchy reduces to a three element V-shaped hierarchy, provided that there exists a parameter free witnessing collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $\mathcal{I}$. Finally, it is shown that all versions of invertibility transfer.

In Section 10.14, starting from the four uniform syntactic protoalgebraicity properties (congruence + modus ponens) of Section 10.12 and the four invertibility properties of Section 10.13, we define, via combinations, sixteen *syntactic algebraizability* properties. These sixteen properties form a rather involved hierarchy. Under systemicity, it devolves into a rhombus, whereas parameter freeness still leaves nine a priori distinct properties. Since this hierarchy is quite large, we only study in more detail the "uniform" versions, i.e., those obtained by applying the same versions of syntactic protoalgebraicity and invertibility properties. The subhierarchy obtained in this way consists of four properties. Maximal among those are the local family and the global family versions. The first implies the local system version and each of these three implies the global system version, which is the weakest in strength. These smaller four class subhierarchy yields a two element chain in case of a systemic π-institution and a three element V-shaped hierarchy in case of parameter free natural transformations. Closing the section is the result asserting that all syntactic algebraizability properties transfer.

In Section 10.15, we introduce and study versions of the *regularity* property. We define four versions, *local family*, *local system*, *global family* and *global system regularity*. To give an idea of the property, we say, e.g., that a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$, with two distinguished arguments, has the local family regularity in a π-institution $\mathcal{I}$ if, for every theory family T, every signature Σ and all Σ-sentences ϕ, ψ,

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

Of these properties, global family regularity is the strongest. It implies local family regularity, which, in turn, implies the system versions, which turn out to be mutually equivalent. The section ends with a proof of the assertion that all three distinct regularity properties transfer.

In Section 10.16, we look at *syntactic regularity* properties. These are properties obtained by combining a certain type of syntactic protoalgebraicity with one of the three different regularity properties. There are numerous different properties that one could study. Just to give a quick idea, there are three regularity versions. Moreover, syntactic protoalgebraicity (PA) results by combining a congruence property with a type of modus ponens. Further, a congruence property results by combining an equivalence property with a compatibility property. This brief analysis leads to the following summary of possibilities, presented in tabular form.

Property	Section	No. of Versions	No. of Uniform
Equivalence	10.6	9	
Compatibility	10.9	3	
Congruence	10.10	27	(Equivalence) 9
Modus Ponens	10.11	4	
Syntactic PA	10.12	108	(Congruence) 12
Regularity	10.15	3	
Syntactic Regularity	10.16	324	(Syntactic PA) 12

The table shows that one could study, in total, 324 a priori different versions of syntactic regularity. However, applying uniformity at the syntactic protoalgebraicity level, we get four uniform syntactic protoalgebraicity properties, which result in only 12 syntactic regularity properties. With the goal of keeping the detailed study within manageable limits, but also to focus on the seemingly most important properties, we restrict ourselves to the subhierarchy obtained by stepping even further. By applying the same type of uniform syntactic protoalgebraicity and regularity, we only obtain four different "uniform" syntactic regularity versions. These are *local family*, *local system*, *global family* and *global system syntactic regularity*. Of those, the local family and global family versions are maximal elements in the subhierarchy. The former dominates the local system version and either of local system or global family syntactic regularity implies the global system version, it being the weakest of the four versions. We also remark that systemicity reduces the subhierarchy to a two element chain, whereas parameter freeness yields a three element V-shaped hierarchy, since the two system versions collapse to a single property. In the final result of the section, it is shown that all versions of syntactic regularity transfer.

In Section 10.17, we study versions of a property we call *modus fortis*, but better known as the Rasiowa or the Wójcicki property. Again, a priori, we have four versions, *local family*, *local system*, *global family* and *global system modus fortis*. To provide an idea of the property, we say, e.g., that a collection $I^\flat$ of natural transformations, with two distinguished arguments, has the local family modus fortis in a π-institution $\mathcal{I}$ if, for all theory families T, every signature Σ and all Σ-sentences ϕ, ψ,

$$\psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

In this case, the global family modus fortis is the strongest version. It implies the local family version, which implies both the global and local system versions, which are mutually equivalent. Under systemicity, all modus fortis properties collapse to a single property. Moreover, all three different modus fortis properties transfer.

In Section 10.18, we look at versions of the *Rasiowa property*, formed by combining a version of syntactic protoalgebraicity with one of the three versions of modus fortis. As in Section 10.16 on syntactic regularity, to keep the number of versions within reasonable limits, we only consider the four "uniform" versions of syntactic protoalgebraicity. Together with the three available versions of modus fortis, they yield an, a priori, twelve member subhierarchy of Rasiowa properties. However, these apparently distinct properties are all shown to be identical. Thus, a posteriori (after a detailed analysis), there is only one Rasiowa property. The key factor in this drastic collapse is the observation that the only π-institutions in which a collection of natural transformations has the Rasiowa property are the inconsistent ones, i.e., those whose only theory family consists of the sets of all sentences over each signature. In the result closing the section, we show that the Rasiowa property also transfers.

In Section 10.19, we compare the three different regularity properties, studied in section 10.15, with the three different modus fortis properties of Section 10.17. It is shown that each of the three versions (global family, local family, system) of the modus fortis implies the corresponding version of regularity. In Section 10.20, we compare each of the four versions of the invertibility property, studied in Section 10.13, with the three versions of regularity. Comparisons, here, require some additional hypotheses. Namely, we assume the existence of natural theorems in the π-institution $\mathcal{I}$ and, also, some form of modus ponens (Section 10.11). We show that, if a collection of natural transformations has a certain type of modus ponens and the same type of regularity, then it also has the corresponding type of invertibility. Roughly speaking, the key observation, inherited from the study of the classical connectives in implicative logics, is that, given a natural theorem $\top^\flat$, the natural transformation $\langle \top^\flat, \iota \rangle : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ acts as an "inverse" to the collection $I^\flat$ satisfying modus ponens and regularity, so that $I^\flat$ satisfies invertibility. Chapter 10 closes with Section 10.21, which provides some details on the relationships of those properties involving syntactic protoalgebraicity. These are the Rasiowa property (syntactic protoalgebraicity + modus fortis), versions of syntactic regularity (syntactic protoalgebraicity + regularity) and versions of syntactic algebraizability (syntactic protoalgebraicity + invertibility).

1.3.10 Chapter 11

In Chapter 11, we begin the study of those classes of π-institutions that are defined syntactically. Syntactic definitions, in this sense, involve the existence of a collection of natural transformations satisfying certain properties among those introduced and studied in Chapter 10. More precisely, Chapter 11 looks at classes of the syntactic hierarchy that are analogs, and related in certain ways, to those classes near the bottom of the semantic hierarchy, introduced and studied in Chapter 3.

In Section 11.2, we study *syntactic prealgebraicity*. This is the syntactic counterpart of (semantic) prealgebraicity. It is a formalization in the π-institution framework of the property of protoalgebraicity for sentential logics [29] (see, also, [53, 65, 89]). The definition stipulates that for a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, with $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, to be syntactically prealgebraic it must possess a collection $I^\flat$ of natural transformations, with two distinguished arguments, satisfying reflexivity, global system transitivity, global system compatibility and global system modus ponens (see Chapter 10, Sections 10.3, 10.5, 10.9 and 10.11, respectively). These properties suffice to ensure that the binary relation system $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ on the sentence functor $\mathrm{SEN}^\flat$ induced by the symmetrization $\overleftrightarrow{I}^\flat$ of $I^\flat$ and by any theory system T of $\mathcal{I}$ is a congruence system on $\mathbf{F}$, compatible with T, and, in fact, is equal to $\Omega(T)$. The fact that the Leibniz congruence system associated with any theory system of $\mathcal{I}$ is defined, in this way, via the set $\overleftrightarrow{I}^\flat$ of natural transformations implies that Ω is monotone on theory systems. Hence, by definition (see Chapter 3, Section 3.3) $\mathcal{I}$ is prealgebraic. On the other hand, the well known equivalence of the two notions when applied to sentential logics is not generally true for arbitrary π-institutions.

To prove a characterization of syntactic prealgebraicity, we introduce a key syntactic concept. Given a π-institution $\mathcal{I}$, we define the *reflexive core* $R^{\mathcal{I}}$ of $\mathcal{I}$ as the collection of all natural transformations, with two distinguished arguments, that define a reflexive relation modulo the theorem system of $\mathcal{I}$ (and, thus, a fortiori, modulo every theory family of $\mathcal{I}$). It turns out that the relation $\overleftarrow{R^{\mathcal{I}}}(T)$ induced by the reflexive core of $\mathcal{I}$ and any theory family T of $\mathcal{I}$ includes the Leibniz congruence system $\Omega(T)$ of T. Moreover, $\overleftarrow{R^{\mathcal{I}}}(T)$ is symmetric and satisfies the compatibility property in $\mathbf{F}$. So among those properties that define syntactic prealgebraicity the one that is of uncertain status for $R^{\mathcal{I}}$ is modus ponens. We show that syntactic prealgebraicity is characterized by $R^{\mathcal{I}}$ having the global system modus ponens in $\mathcal{I}$. Following from this is an alternative characterization. Namely, $\mathcal{I}$ is syntactically prealgebraic if and only if the reflexive core $R^{\mathcal{I}}$ defines Leibniz congruence systems of theory systems of $\mathcal{I}$.

Having provided characterizations of syntactic prealgebraicity, we focus on pinpointing the exact property that separates (semantic) prealgebraicity from syntactic prealgebraicity. We mentioned that, even though the two properties are identical in the context of sentential logics, syntactic prealgebraicity strictly implies the semantic version for arbitrary π-institutions. So we attempt to find the characteristic difference, i.e., their separating property. Interestingly, this turns out to be a "hybrid" property, straddling syntax and semantics. We consider, again, the reflexive core $R^{\mathcal{I}}$. For a given signature Σ and arbitrary Σ-sentences ϕ, ψ, we may construct the theory system $C(R^{\mathcal{I}}_\Sigma[\phi, \psi])$. We say that $R^{\mathcal{I}}$ is *Leibniz* if, for all Σ and all ϕ, ψ,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

It can be shown that, if the reflexive core possesses the global system modus ponens, then it is Leibniz and that this implication is strict in general. However, if the π-institution is prealgebraic, then the reverse implication also holds. In fact, it is shown that syntactic prealgebraicity is characterized by (semantic) prealgebraicity and having a Leibniz reflexive core.

Section 11.2 closes with a proof of the assertion that syntactic prealgebraicity transfers from a π-institution to every $\mathbf{F}$-algebraic system. This means, as outlined previously, that if $\mathcal{I}$ is syntactically prealgebraic, with witnessing collection of natural transformations $I^\flat$, then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution model $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is syntactically prealgebraic, with witnessing collection of natural transformations I (corresponding to $I^\flat$ in $\mathcal{I}$). The property is a consequence of the fact, proven in Chapter 10, that all constituent properties defining syntactic prealgebraicity transfer.

In Section 11.3, we study *syntactic protoalgebraicity*, which is the syntactic analog of protoalgebraicity, studied in Chapter 3, and a stronger version of syntactic prealgebraicity, introduced in Section 11.2. A π-institution $\mathcal{I}$ is syntactically protoalgebraic if there exists a collection $I^\flat$ of natural transformations in $\mathcal{I}$, with two distinguished arguments, that satisfies reflexivity, global family transitivity, global family compatibility and global family modus ponens. Recall that for syntactic prealgebraicity, the corresponding system versions of these properties are required. Although, on the face of it, this introduces three points of difference in the two definitions, this is not really the case. In Sections 10.5 and 10.9 of Chapter 10, it is shown that global family and global system transitivity are identical properties and the same holds for global family and global system compatibility. Hence, the difference between syntactic protoalgebraicity and syntactic prealgebraicity is limited solely on the version of the modus ponens property required in each case. As was done in Section 11.2 for theory systems, it is shown that the witnessing collection $I^\flat$ of natural transformations in a syntactically protoalgebraic π-institution $\mathcal{I}$ defines Leibniz congruence systems of theory families, i.e., for every theory family T of $\mathcal{I}$, $\overleftarrow{\overleftrightarrow{I}^\flat}(T) = \Omega(T)$, where, once more, $\overleftrightarrow{I}^\flat$ denotes the "symmetrization" of $I^\flat$. This property enables one to show that, if $\mathcal{I}$ is syntactically protoalgebraic, then the Leibniz operator on theory families is monotone and, hence, $\mathcal{I}$ is (semantically) protoalgebraic.

In the second part of the section, the reflexive core $R^{\mathcal{I}}$ of the π-institution $\mathcal{I}$ is used to characterize syntactic protoalgebraicity. It is shown that $\mathcal{I}$ is syntactically protoalgebraic if and only if $R^{\mathcal{I}}$ has the global family modus ponens and, moreover, if that is the case, $R^{\mathcal{I}}$ constitutes a witnessing collection of natural transformations in $\mathcal{I}$ and, as a consequence, defines Leibniz congruence systems of theory families in $\mathcal{I}$. We also use the reflexive core to delineate the boundary between syntactic protoalgebraicity and semantic protoalgebraicity. To this end, we use, again, the property of the reflexive core being Leibniz. In general, if $R^{\mathcal{I}}$ has the global family modus ponens, then it is Leibniz. The reverse implication does not hold in general, but it does hold if the π-institution $\mathcal{I}$ is protoalgebraic. In fact, it is proven that $\mathcal{I}$ is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core. Thus, having a Leibniz reflexive core is the characteristic property separating the semantic and syntactic versions of protoalgebraicity. Similarly, and as was the case with the semantic notions, it turns out that the property that separates syntactic protoalgebraicity from syntactic prealgebraicity is stability, a semantic property that was introduced in Section 3.2 of Chapter 3. Section 11.3 concludes with the assertion that syntactic protoalgebraicity transfers.

Section 11.4 introduces four classes of matrix families that serve as semantics for a given π-institution $\mathcal{I}$. The first is the class $\mathrm{LMatFam}^*(\mathcal{I})$ of all reduced Lindenbaum $\mathcal{I}$-matrix families. The second is the class $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$ of all Suszko reduced Lindenbaum $\mathcal{I}$-matrix families. The third is the class $\mathrm{MatFam}^*(\mathcal{I})$ of all reduced $\mathcal{I}$-matrix families and the last one is the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ of all Suszko reduced $\mathcal{I}$-matrix families. We say that an arbitrary class M of matrix families is a *matrix semantics* for a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ if $C = C^{\mathsf{M}}$, that is, if, for all signatures Σ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, we have $\phi \in C_\Sigma(\Phi)$ if and only if, for all $\langle \mathcal{A}, T\rangle \in \mathsf{M}$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, all signatures Σ' and all signature morphisms $f : \Sigma \to \Sigma'$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

It is shown that each of the four classes of matrix families introduced above constitutes a matrix semantics for $\mathcal{I}$. The corresponding classes of algebraic system reducts of the matrix models in each of these four matrix family classes are denoted, respectively, by $\mathrm{LAlgSys}^*(\mathcal{I})$, $\mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I})$, $\mathrm{AlgSys}^*(\mathcal{I})$

and $\mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$. It is not difficult to see, based on the relevant definitions, that the four classes of matrix models satisfy the inclusions

$$\begin{array}{rcl}\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}) & \subseteq & \mathrm{LMatMat}^*(\mathcal{I}),\\ \mathrm{LMatFam}^*(\mathcal{I}) & \subseteq & \mathrm{MatFam}^*(\mathcal{I}),\\ \mathrm{MatFam}^*(\mathcal{I}) & \subseteq & \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}).\end{array}$$

These inclusions yield corresponding inclusions between the classes of algebraic system reducts, namely we have also $\mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}) \subseteq \mathrm{LAlgSys}^*(\mathcal{I})$, $\mathrm{LAlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$.

In Section 11.5, we study the notion of *algebraic semantics* for a π-institution $\mathcal{I}$ and the closely related notion of *equational definability of truth*. These were among the key concepts of the theory of algebraizable logics of Blok and Pigozzi [36]. They form dual notions to the definability of Leibniz congruence systems, which was studied in Sections 11.2 and 11.3. In the context of algebraizability, the two duals are studied together and they are required to be closely interrelated. On the other hand, in the general theory of algebraic logic, each is studied on its own and has its own role and significance in defining and studying the algebraic hierarchy of logics. In the context of sentential logics, equational definability of truth was pursued by Raftery [78], whereas the definability of the Leibniz congruence forms part of both protoalgebraic [29] and equivalential logics [24, 25] (see, also, [65]).

Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, a class K of $\mathbf{F}$-algebraic systems determines a closure family C^{K} on $\mathbf{F}$. If there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of pairs of natural transformations, with a single distinguished argument, that serves to interpret C into C^{K}, then K is said to be a $\tau^\flat$-algebraic semantics for $\mathcal{I}$. Interpretation in this sense means that, for every signature Σ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]).$$

On the other hand, given a class M of $\mathbf{F}$-matrix families, we say that truth is $\tau^\flat$-equationally definable in the class if, for every matrix family $\langle \mathcal{A}, T\rangle \in \mathrm{M}$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, all signatures Σ of $\mathcal{A}$ and all Σ-sentences ϕ,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Delta^{\mathcal{A}},$$

$\Delta^{\mathcal{A}}$ denoting, as usual, the identity relation system on $\mathcal{A}$. Having a $\tau^\flat$-algebraic semantics and having a matrix (family) semantics in which truth is $\tau^\flat$-equationally definable are shown to be equivalent. This result goes back to the work of Blok and Pigozzi (see, e.g., Theorem 2.4 of [36]). In Section 11.4, it was shown that the classes

$$\mathrm{LMatFam}^*(\mathcal{I}), \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}), \mathrm{MatFam}^*(\mathcal{I}), \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$$

form matrix semantics for a π-institution $\mathcal{I}$. Moreover, the classes

$$\mathrm{LAlgSys}^*(\mathcal{I}), \mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}), \mathrm{AlgSys}^*(\mathcal{I}), \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I}).$$

were defined as the corresponding classes of underlying algebraic systems. Thus, the aforementioned equivalence implies that, should any of the four matrix family classes have truth equationally definable, the corresponding class of algebraic systems would be an algebraic semantics for $\mathcal{I}$.

In Section 11.6, we return to the syntactic hierarchy. We define four a priori different notions of *truth equationality* for a π-institution $\mathcal{I}$. All four involve truth being equationally refinable in one of the four matrix family classes introduced in Section 11.4. For instance, we say that $\mathcal{I}$ is *Leibniz truth equational* if, for some collection $\tau^\flat$ of natural transformations, truth is $\tau^\flat$-definable in the class $\mathrm{LMatFam}^*(\mathcal{I})$ of Lindenbaum Leibniz reduced matrices of $\mathcal{I}$, in the sense that, for all $\langle\mathcal{F}/\Omega(T), T/\Omega(T)\rangle \in \mathrm{LMatFam}^*(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}.$$

If, instead of $\mathrm{LMatFam}^*(\mathcal{I})$, we take $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, then we obtain *Suszko truth equationality* and, similarly, by considering, respectively, $\mathrm{MatFam}^*(\mathcal{I})$ or $\mathrm{MatFam}^{\mathrm{Su}}(\mathcal{I})$, we obtain *universal Leibniz truth equationality* and *universal Suszko truth equationality*. Of these four truth equationality properties, it turns out that three are equivalent. In fact, Leibniz and universal Leibniz truth equationality and universal Suszko truth equationality turn out to be mutually equivalent. So the truth equationality subhierarchy, as it were, consists of two different classes of which Leibniz truth equationality implies Suszko truth equationality. Finally, based on what was shown in Section 11.5 on the relation between matrix and algebraic semantics, we may conclude that, if a π-institution $\mathcal{I}$ is Leibniz truth equational, all three classes of algebraic systems $\mathrm{LAlgSys}^*(\mathcal{I})$, $\mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$ form algebraic semantics for $\mathcal{I}$.

Section 11.7 is a bridge between Sections 11.6 and 11.8. In Section 11.6, we study versions of truth equationality, a syntactic property of a π-institution. In Section 11.8, we wish to relate truth equationality with complete reflectivity (c-reflectivity), a semantic property, various versions of which were introduced and studied in Chapter 3, Section 3.8. The bridge section establishes some relations between truth equationality and having an algebraic semantics, on the one hand, and gives some characterizations of family c-reflectivity, on the other. We describe briefly this second part.

Let $\mathcal{I} = \langle\mathbf{F}, C\rangle$ be a π-institution. We say that the Suszko operator in $\mathcal{I}$ is *universally family injective* if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter families T and T' on $\mathcal{A}$,

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \quad \text{implies} \quad T = T'.$$

Moreover, we say that the Suszko operator has *universal family minimality* if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and every $\mathcal{I}$-filter family T on $\mathcal{A}$, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the least filter family on $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. It is shown that universal family injectivity of the Suszko operator is equivalent to universal family minimality. In Section 3.8 of Chapter 3, it is shown that family c-reflectivity transfers, that is, it is equivalent to the property that, for every **F**-algebraic system $\mathcal{A}$ and all collections $\mathcal{T} \cup \{T'\}$ of $\mathcal{I}$-filter families on $\mathcal{A}$,

$$\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \text{implies} \quad \bigcap \mathcal{T} \leq T'.$$

This property may be termed *universal family c-reflectivity*. We show that it is also equivalent to the assertion that, for every **F**-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter families T, T' on $\mathcal{A}$,

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \text{implies} \quad T \leq T'.$$

The equivalence of the last two properties allows us to show that family c-reflectivity is equivalent to the universal injectivity of the Suszko operator in $\mathcal{I}$. In summary, the properties of universal family injectivity of the Suszko operator, of universal family minimality of the Suszko operator and of $\mathcal{I}$ being family c-reflective are mutually equivalent properties.

Section 11.8 relates truth equationality with family c-reflectivity. This relation is analogous, and, in a certain sense, dual, to the relation between syntactic protoalgebraicity and (semantic) protoalgebraicity, which was detailed in Section 11.3. A key result, here, is that, if a π-institution $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat$, then the witnessing equations reflect theory families, in the sense that, for every theory family T of $\mathcal{I}$, $\overleftarrow{\tau^\flat}(\Omega(T)) = T$, where

$$\overleftarrow{\tau^\flat_\Sigma}(\Omega(T)) = \{\phi \in \mathrm{SEN}^\flat(\Sigma) : \tau^\flat_\Sigma[\phi] \leq \Omega(T)\}.$$

This "reflectivity" property of $\tau^\flat$ allows showing that truth equationality implies family c-reflectivity. The result is the analog of the one asserting that syntactic protoalgebraicity implies protoalgebraicity in the context of Section 11.3. Pursuing the analogy further, we seek a characterization of truth equationality as well as a characteristic property separating truth equationality from family c-reflectivity. The analogous tasks in the context of protoalgebraicity were accomplished using the reflexive core $R^{\mathcal{I}}$ of $\mathcal{I}$. The notion that accomplishes our goals here and that corresponds to the reflexive core is the Suszko core $S^{\mathcal{I}}$ of the π-institution $\mathcal{I}$.

The *Suszko core* $S^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ consists of all natural transformations $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, that satisfy, for every theory family T of $\mathcal{I}$,

$$\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T),$$

i.e., that map a theory family into the Suszko congruence system of the theory family. This defining property turns out to be equivalent to the condition that, for all signatures Σ and all Σ-sentences ϕ,

$$\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)),$$

where $C(\phi)$ is the theory family generated by ϕ. Based on the definition, one is able to show that, roughly speaking, $S^{\mathcal{I}}$ satisfies a kind of congruence property at the level of natural transformations. For instance, it satisfies transitivity, in the sense that, if $\langle\delta^\flat,\epsilon^\flat\rangle$ and $\langle\epsilon^\flat,\zeta^\flat\rangle$ are pair in $S^{\mathcal{I}}$, then $\langle\delta^\flat,\zeta^\flat\rangle$ in also a pair in $S^{\mathcal{I}}$. Additionally, the Suszko core satisfies the property that, for every theory family T of $\mathcal{I}$, $T \leq \overleftarrow{S^{\mathcal{I}}}(\Omega(T))$. The reverse inclusion does not hold in general, but it is one of the key properties that we are looking for. We say that the Suszko core $S^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ is *soluble* if, for every theory family T of $\mathcal{I}$, $\overleftarrow{S^{\mathcal{I}}}(\Omega(T)) \leq T$. Solubility of the Suszko core plays an analogous role to that played by global family modus ponens of the reflexive core in the context of protoalgebraicity. It is, e.g., the property that characterizes truth equationality. More precisely, if $\mathcal{I}$ is truth equational then $S^{\mathcal{I}}$ is soluble and, conversely, if $S^{\mathcal{I}}$ is soluble, then $\mathcal{I}$ is truth equational, with witnessing equations $S^{\mathcal{I}}$.

The last task we set was separating truth equationality from family c-reflectivity. The key property here is the adequacy property of the Suszko core, which corresponds to the Leibniz property of the reflexive core in the context of protoalgebraicity. Briefly, it can be seen that, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

This inclusion is proper in general. We say that the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$ is *adequate* if the reverse inclusion also holds. Note that adequacy of $S^{\mathcal{I}}$ expresses its capability to define Suszko congruence systems in terms of specifically picked Leibniz congruence systems. Solubility of the Suszko core implies its adequacy. Even though this implication is strict in general, the converse holds under the proviso that $\mathcal{I}$ is family c-reflective. Furthermore, adequacy of the Suszko core is the property separating truth equationality from family c-reflectivity, in the sense that $\mathcal{I}$ is truth equational if and only if it is family c-reflective and has an adequate Suszko core. Section 11.8 closes with a transfer theorem for truth equationality.

In Section 11.9, we look at a version of truth equationality and c-reflectivity that do not force the π-institution to be systemic, as is the case with truth equationality and family c-reflectivity, studied in Section 11.8. Recall from Chapter 3, Section 3.8, that a π-institution $\mathcal{I}$ is *left c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'},$$

where $\overleftarrow{T}$ denotes the largest theory system included in the theory family T (see Chapter 2). This property is weaker than family c-reflectivity and stronger than system c-reflectivity and is not strong enough to imply systemicity, as is family c-reflectivity. On the other hand, we say that $\mathcal{I}$ is *left truth equational* if there exists a collection of natural transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with a single distinguished argument, such that, for every theory family T, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

$\tau^\flat$ are called the *witnessing equations*. Even though, unlike the case of truth equationality, witnessing equations need not reflect theory families, they do so "up to arrow", i.e., we have, for every theory family T, $\overleftarrow{\tau^\flat}(\Omega(T)) = \overleftarrow{T}$. This is, however, enough to establish that left truth equationality implies left c-reflectivity. This implication is strict in general. So we seek a characterization of left truth equationality and a condition separating left truth equationality from left c-reflectivity, as done in Section 11.9 for truth equationality and family c-reflectivity. Here, a modified version of the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$ is used. It is called the *left Suszko core* and it is defined by

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

The left Suszko core is also called the *system Suszko core*, since it can be equivalently defined by

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

It is important to observe that, on the one hand, $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$ and, hence, $\overleftarrow{L^{\mathcal{I}}}(\Omega(T)) \leq \overleftarrow{S^{\mathcal{I}}}(\Omega(T))$, and that, on the other, in systemic π-institutions, $S^{\mathcal{I}} = L^{\mathcal{I}}$. One can show that, for every family T, $\overleftarrow{T} \leq \overleftarrow{L^{\mathcal{I}}}(\Omega(T))$. In general, this is a proper inclusion. If the reverse inclusion also holds, then $L^{\mathcal{I}}$ is said to be *left soluble*. This amounts to saying that, for all theory systems T of $\mathcal{I}$, $\overleftarrow{L^{\mathcal{I}}}(\Omega(T)) = T$. Moreover, for systemic π-institutions, left solubility of the left Suszko core is identical to solubility of the Suszko core. In general, however, the latter is a stronger property than the former. It turns out that left solubility of the left Suszko core characterizes left truth equationality. Further, if $L^{\mathcal{I}}$ is left soluble, then $L^{\mathcal{I}}$ acts as a collection of witnessing equations for the left truth equationality of $\mathcal{I}$. We turn, finally, to the task of separating left c-reflectivity from left truth equationality. We show that the left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ satisfies, for all signatures Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})),$$

where $\overrightarrow{\phi}$ is the least sentence system of $\mathcal{I}$ containing ϕ, and that this is a proper inclusion in general. If the reverse inclusion holds, we say that the left Suszko core $L^{\mathcal{I}}$ is *left adequate*. Left solubility of $L^{\mathcal{I}}$ implies left adequacy and this is, in general, a strict implication. On the other hand, left adequacy implies left solubility, provided that $\mathcal{I}$ is left c-reflective. More decisively,

$L^{\mathcal{I}}$ is left soluble if and only if $\mathcal{I}$ is left c-reflective and $L^{\mathcal{I}}$ is left adequate. By the equivalence of left truth equationality with left solubility of $L^{\mathcal{I}}$, this yields that left adequacy of $L^{\mathcal{I}}$ is the separating property between left truth equationality and left c-reflectivity.

Section 11.10 continues (and concludes) the work of Sections 11.8 and 11.9, relating versions of truth equationality with those of c-reflectivity. More precisely, we turn to system truth equationality, the weakest version, which, like left truth equationality, can accommodate non-systemic π-institutions. We say that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is *system c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory systems of $\mathcal{I}$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

On the other hand, $\mathcal{I}$ is *system truth equational* if there exists a collection of natural transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with a single distinguished argument, called *witnessing equations*, such that, for every theory system T of $\mathcal{I}$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

This property implies that $\tau^\flat$ reflects theory systems, in the sense that, for every theory system T, $\overleftarrow{\tau^\flat}(\Omega(T)) = T$, which leads to the conclusion that system truth equationality implies system c-reflectivity. Since this implication is strict in general, taking after similar work from Sections 11.8 and 11.9, we provide both a characterization of system truth equationality and of those system c-reflective π-institutions that are system truth equational. The role played by the Suszko core in Section 11.8 and by the left Suszko core in Section 11.9 is played here by the system core of $\mathcal{I}$. For any theory system T of $\mathcal{I}$, define the *systemic Suszko congruence system* of T in $\mathcal{I}$ by

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\}.$$

Then define the *system core* of $\mathcal{I}$ by

$$Z^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

It turns out that $S^{\mathcal{I}} \subseteq L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$. Moreover, we have that, for all theory systems T of $\mathcal{I}$, $T \leq \overleftarrow{Z^{\mathcal{I}}}(\Omega(T))$, this being, in general, a proper inclusion. In case the reverse inclusion also holds, we say that the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is *soluble*. The solubility of the system core characterizes system truth equationality, that is, $\mathcal{I}$ is system truth equational if and only if its system core $Z^{\mathcal{I}}$ is soluble. Finally, to delineate the boundary between system truth equationality and system c-reflectivity, we resort to a property of the system core called adequacy, analogous to the adequacy of the Suszko core and of the left adequacy of the left Suszko core. In general, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

If the reverse inclusion also holds, we say that the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is *adequate*. Solubility of $Z^{\mathcal{I}}$ implies its adequacy. Further, if $\mathcal{I}$ is system c-reflective, then adequacy of $Z^{\mathcal{I}}$ implies its solubility. This leads to the promised relation between system truth equationality and system c-reflectivity. Namely, it is shown that $\mathcal{I}$ is system truth equational if and only if it is system c-reflective and has an adequate system core.

1.3.11 Chapter 12

In Chapter **??**, we develop the syntactic machinery of translations, interpretations and equivalences in order to characterize syntactic analogs of the various types of prealgebraizable and algebraizable π-institutions studied in Chapter 4. The roots of these investigations can be traced back to the famous "Memoir monograph" of Blok and Pigozzi [36]. Of course various advancements, refinements and generalizations have been presented in the literature since then, e.g., in [41, 53, 50, 74, 82] and in the categorical setting [106]. The monograph [65] and the textbook [89] give a more comprehensive list of more recent approaches pertaining to the sentential logic case.

Section 12.2 introduces *π-structures*, a variant of π-institutions that may lack structurality. By analogy with the terminology concerning theory families and theory systems, their closures are termed *closure families* rather than closure systems. Introduced, also, are *transformations* between algebraic systems and *interpretations* between π-structures. Let us consider, for concreteness, two algebraic systems $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$. A *translation* $\alpha : \mathbf{K} \to \mathbf{K}'$ from $\mathbf{K}$ to $\mathbf{K}'$ is a collection $\alpha = \{\alpha_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of mappings

$$\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

from Σ-sentences of $\mathbf{K}$ to sentence families of $\mathbf{K}'$. Now, given two π-structures $\mathcal{K} = \langle \mathbf{K}, D\rangle$ based on $\mathbf{K}$ and $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ based on $\mathbf{K}'$, a translation $\alpha : \mathbf{K} \to \mathbf{K}'$ is an *interpretation* $\alpha : \mathcal{K} \to \mathcal{K}'$ from $\mathcal{K}$ to $\mathcal{K}'$ if, for every signature Σ of $\mathcal{K}$ and all Σ-sentences $\Phi \cup \{\phi\}$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma[\phi] \leq D'(\alpha_\Sigma[\Phi]).$$

If such an α exists, then $\mathcal{K}$ is said to be *interpretable in* $\mathcal{K}'$. Two interpretations $\alpha : \mathcal{K} \to \mathcal{K}'$ and $\beta : \mathcal{K}' \to \mathcal{K}$, acting in opposite directions, are called *inverses* and $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ a *conjugate pair* if, for all $\Sigma \in |\mathbf{Sign}|$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\psi \in \mathrm{SEN}'(\Sigma')$,

$$D(\phi) = D(\beta[\alpha_\Sigma[\phi]]) \quad \text{and} \quad D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]]).$$

Two π-structures are called *equivalent* if there exist inverse interpretations in both directions. As in the classical case of sentential logics, not all four conditions required for equivalence are independent. The first of the interpretation conditions and the second invertibility condition imply the other two and vive versa. This was originally pointed out by Blok and Pigozzi in Corollary 2.9 of [36] in the context of interpretations between a sentential logic and an equational logic over the same algebraic signature.

Given a translation $\alpha : \mathbf{K} \to \mathbf{K}'$, one may define a mapping

$$\alpha^* : \mathrm{SenFam}(\mathbf{K}') \to \mathrm{SenFam}(\mathbf{K}),$$

called the *residual* of α. Such a mapping, when applied to interpretations $\alpha : \mathcal{K} \to \mathcal{K}'$ between π-structures $\mathcal{K}$ and $\mathcal{K}'$, yields a mapping $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$. Additionally, if $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$, then $\mathrm{ThFam}(\mathcal{K}') \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} \mathrm{ThFam}(\mathcal{K})$ is a pair of inverse mappings. These establish an order isomorphism between the complete lattice of theory families of $\mathcal{K}'$ and the complete lattice of theory families of $\mathcal{K}$. The process by which a conjugate pair of interpretations α and β gives rise to the order isomorphisms α^* and β^* can be reversed. This direction occupies the second part of Section 12.2.

Suppose $h : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ is an order isomorphism. We define translations $\overrightarrow{h} = \{\overrightarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} : \mathbf{K} \to \mathbf{K}'$ and $\overleftarrow{h} = \{\overleftarrow{h}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|} : \mathbf{K}' \to \mathbf{K}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$, all $\phi \in \mathrm{SEN}(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\psi \in \mathrm{SEN}'(\Sigma')$,

$$\overrightarrow{h}_\Sigma(\phi) = h^{-1}(D(\phi)) \quad \text{and} \quad \overleftarrow{h}_{\Sigma'}(\psi) = h(D'(\psi)).$$

It is shown that $\overrightarrow{h} : \mathcal{K} \to \mathcal{K}'$ and $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ are interpretations and, moreover, that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ forms a conjugate pair. Finally, it is shown that the residuals $\overrightarrow{h}^*$ and $\overleftarrow{h}^*$ coincide with h and h^{-1}, respectively.

In Section 12.3, we single out a special type of translation, which is closely connected to the syntactic apparatus of a given algebraic system, in particular of one that is the base algebraic system on which a π-institution is built. More precisely, instead of considering arbitrary maps from a sentence functor into the sentence families over another sentence functor, we specialize to the case in which the mappings occur between different powers of the same sentence functor, akin to the framework of k-deductive systems of Blok and Pigozzi [41]. In this special case, arbitrary translations may be replaced by *transformations*, i.e., sets of natural transformations, possibly with parameters. Those are said to be *natural*, i.e., are called *natural transformations*, if they are parameter free. Inherited by the special case of transformations from the general case of translations from Section 12.2 are the notions of interpretability and of equivalence between π-structures. Virtually all results involving arbitrary translations and interpretations are specialized here to the case of transformations and natural transformations and of interpretations induced by transformations. Accompanying those is the notion of a *transformational order isomorphism* between lattices of theory families which reflects the fact that those isomorphisms are induced by transformations rather than by arbitrary translations. It is shown that, if there exists a transformational order isomorphism between lattices of theory families, then the pair of transformations that induce the isomorphism do form a conjugate pair between the corresponding π-structures and, consequently, these two π-structures are equivalent. Similar results apply if arbitrary transformations are replaced by natural transformations.

In the sentential logic framework, the passage from arbitrary interpretations to ones between k-deductive systems serves in transitioning from general equivalences to equivalences between a deductive system and an equational deductive system, which defines algebraizability (albeit the development of this passage occurred in reverse chronological order). Similarly, here, the passage from arbitrary translations and interpretations to transformations and interpretations based on transformations serves an analogous purpose.

The second part of Section 12.3 introduces *equational π-structures*. Roughly speaking, those are the ones in which reflexivity, symmetry, transitivity, compatibility (or congruence) and invariance under signature morphisms hold. In a characterization theorem governing these structures, it is shown that a π-structure is equational if and only if its closure systems are congruence systems on the underlying algebraic system of the π-structure if and only if it is one induced on the underlying algebraic system $\mathbf{F}$ by a certain class of $\mathbf{F}$-algebraic systems. The notion of an equational π-structure facilitates connecting the framework of equivalence between π-structures with that of syntactic protoalgebraicity and of (family) truth equationality, studied in Sections 11.3 and 11.6, respectively, of Chapter 11. Namely, it is shown that, if a π-institution is equivalent to an equational π-structure over the same underlying algebraic system via a conjugate pair of transformations, then it is syntactically protoalgebraic and truth equational with witnessing transformations and equations those constituting the conjugate pair. Finally, in an analog of a well known result of Blk and Pigozzi (Theorem 2.15 of [36]), it is shown that equivalence of a π-institution with an equational π-structure via transformations can only occur in an essentially unique way in the sense that any two such π-structures must be identical and, furthermore, the transformations serving the purpose must be unique up to deductive closure.

In Chapter 4, we defined the class of weakly family algebraizable (WF algebraizable) π-institutions. Weak family algebraizability is characterized by protoalgebraicity (family monotonicity) and family injectivity of the Leibniz operator. In Section 12.4, the goal is to define a syntactic counterpart of this class. Critical here are the results obtained in Chapter 11 which connect syntactic protoalgebraicity and protoalgebraicity via the Leibniz property of the reflexive core $R^{\mathcal{I}}$ and, also, truth equationality and family c-reflectivity via the adequacy property of the Suszko core $S^{\mathcal{I}}$. Starting from these, we say that a π-institution $\mathcal{I}$ is *$R^{\mathcal{I}}S^{\mathcal{I}}$-fortified* if it has a Leibniz reflexive core and an adequate Suszko core and, further, declare that a π-institution is *syntactically weakly family algebraizable* (*syntactically WF algebraizable*) if it is protoalgebraic, family injective and $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified. This definition seems to be the "right" one for a syntactic counterpart of weakly family algebraizable π-institutions. According to this, e.g., syntactically WF algebraizable π-institutions are characterized as those which are, at the same time, syntactically protoalgebraic and family truth equational. Moreover, they are exactly those that are WF algebraizable and $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified. Another characterization obtained from this and one for WF algebraizability asserts that $\mathcal{I}$ is syntactically WF algebraizable if and only if it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. Yet another characterization uses the notion of equivalence with an equational π-structure introduced in Section 12.3. Recall from Chapter 2, Section 2.7, that $\mathrm{AlgSys}^*(\mathcal{I})$ is the class of all $\mathbf{F}$-algebraic systems that are algebraic system reducts of reduced $\mathcal{I}$-matrix families. We define the collection $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ of congruence systems on $\mathcal{A}$ whose quotients are algebraic systems in $\mathrm{AlgSys}^*(\mathcal{I})$. An alternative characterization of $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is that it consists of all those congruence systems on $\mathcal{A}$ that are Leibniz congruence systems of some $\mathcal{I}$-filter family of $\mathcal{A}$. In general, the collection of all $\mathcal{I}^*$-congruence systems on an $\mathbf{F}$-algebraic system $\mathcal{A}$ is not closed under signature-wise intersections and, therefore, does not form a closure family on $\mathcal{A}^2$. However, this turns out to be the case if the π-institution happens to be protoalgebraic. Specializing to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, we get that, provided $\mathcal{I}$ is protoalgebraic, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ is closed under arbitrary intersections. This enables us to define an equational π-structure $\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^2, D^{\mathcal{I}*} \rangle$, where $D^{\mathcal{I}*}$ is the closure operator associated with the closure family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$. The first indication that this definition serves well our purposes is the result asserting that, if a π-institution is syntactically WF algebraizable, then, with $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$, with two distinguished arguments, being the collection of witnessing transformations of the syntactic protoalgebraicity of $\mathcal{I}$ and with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ with a single distinguished argument, being the collection of witnessing equations of the truth equationality of $\mathcal{I}$, $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via the conjugate pair $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. Putting many preceding results together, we obtain a fundamental characterization of syntactic WF algebraizability, tracing its origins to the seminal "Memoirs monograph" [36] of Blok and Pigozzi. Namely, a π-institution $\mathcal{I}$ is syntactically WF algebraizable if and only if it is equivalent to its associated equational π-struture $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ of transformations. More generally, $\mathcal{I}$ is syntactically WF algebraizable if and only if it is equivalent to an equational π-structure via a conjugate pair of transformations. Finally, this generalized version implies that $\mathcal{I}$ is syntactically WF algebraizable if and only if there exists a transformational isomorphism between its lattice of theory families and the lattice of theory families of an equational π-structure.

Syntactic weak family algebraizability encompasses family reflectivity, which implies systemicity. Thus, if a π-institution is syntactically WF algebraizable, it must necessarily be systemic. This leads to the idea of considering weaker types of syntactic algebraizability admitting also non-systemic π-institutions. This endeavor is pursued in Sections 12.5 and 12.6. Section 12.5 considers *syntactic weak algebraizability*. The development parallels that of Section 12.4.

We say that a π-institution is *$R^{\mathcal{I}}L^{\mathcal{I}}$-fortified* if it has a Leibniz reflexive core $R^{\mathcal{I}}$ and a left adequate left Suszko core $L^{\mathcal{I}}$. $\mathcal{I}$ is *syntactically weakly algebraizable* (*syntactically W algebraizable*) if it is protoalgebraic, system injective and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. Syntactic weak algebraizability is characterized as the conjunction of syntactic protoalgebraicity and either system or left truth equationality. Moreover, $\mathcal{I}$ is syntactically W algebraizable if and only if it is weakly algebraizable and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. Yet another characterization using isomorphisms asserts that $\mathcal{I}$ is syntactically W algebraizable if and only if it is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$: $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

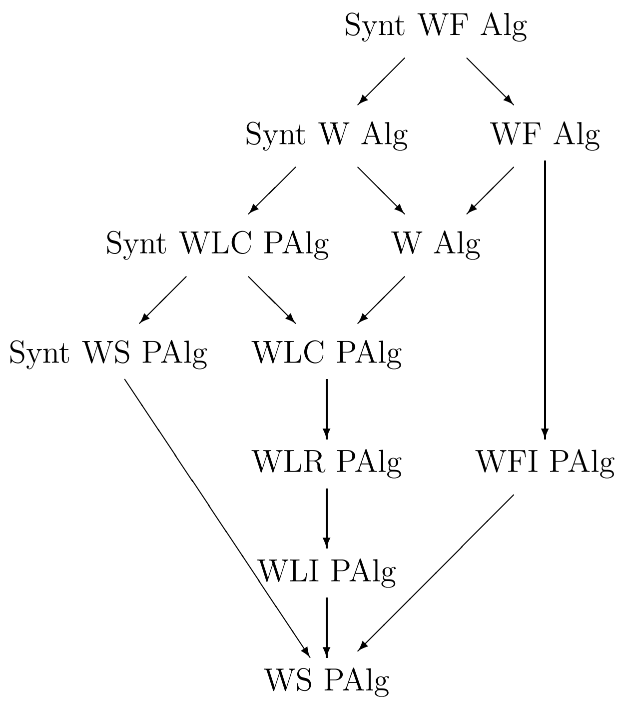
In the second part of Section 12.5, we give a characterization of syntactic W algebraizability using the equational counterpart $\mathcal{Q}^{\mathcal{I}*}$ of a π-institution $\mathcal{I}$. This mimics in style the fundamental characterization of syntactic weak family algebraizability, presented in Section 12.4. First, given a π-institution $\mathcal{I}$, we define a π-structure $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$, called the *systemic skeleton* of $\mathcal{I}$, by taking $K^{\mathcal{I}}$ to be the closure operator whose theory families are the theory systems of $\mathcal{I}$. This π-structure is not a π-institution in general, since $K^{\mathcal{I}}$ may fail to be structural. We show, nevertheless, that, if a π-institution $\mathcal{I}$ is syntactically W algebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its equational counterpart $\mathcal{Q}^{\mathcal{I}*}$ via the conjugate pair $(\tau^\flat, \overleftrightarrow{I^\flat})$ of transformations, where $I^\flat$ witnesses syntactic protoalgebraicity and $\tau^\flat$ witnesses left truth equationality. Conversely, if $\mathcal{I}$ is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to the equational counterpart $\mathcal{Q}^{\mathcal{I}*}$ of $\mathcal{I}$ via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically weakly algebraizable. More generally, we show that syntactic weak algebraizability is characterized by stability plus the existence of an equivalence between its systemic skeleton and an equational π-structure via a conjugate pair of transformations. This characterization yields also that $\mathcal{I}$ is syntactically W algebraizable if and only if it is stable and there exists a transformational order isomorphism between the lattice of its theory systems and that of the theory families of an equational π-institution.

Syntactic weak algebraizability is broad enough to accommodate non-systemic π-institutions, but it is still too narrow to allow non-stable π-

institutions to be considered. So in Section 12.6, we further weaken the required conditions so as to be able to also include non-stable π-institutions. We say that a π-institution $\mathcal{I}$ is *$R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified* if its reflexive core $R^{\mathcal{I}}$ is Leibniz and its system core $Z^{\mathcal{I}}$ is adequate. Then $\mathcal{I}$ is called *syntactically weakly system prealgebraizable* (*syntactically WS prealgebraizable*) if it is prealgebraic, system injective and $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified. According to this definition, $\mathcal{I}$ is syntactically WS prealgebraizable if and only if it is syntactically prealgebraic and system truth equational. Moreover, $\mathcal{I}$ is syntactically WS prealgebraizable if and only if it is weakly system prealgebraizable and $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified. In terms of isomorphisms between theory and congruence lattices, we get that $\mathcal{I}$ is syntactically WS prealgebraizable if and only if it is $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding.

To characterize syntactic WS prealgebraizability via an equivalence between π-structures, we use again the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ of $\mathcal{I}$, but we need to introduce an equational π-structure induced by $\mathcal{I}$ different from the equational π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with $\mathcal{I}$. To this end, instead of the collection $\mathrm{AlgSys}^*(\mathcal{I})$ of all algebraic system reducts of reduced $\mathcal{I}$-filter families, we use the collection $\mathrm{AlgSys}^\bullet(\mathcal{I})$ of all algebraic system reducts of reduced $\mathcal{I}$-filter systems. Then we define the class of $\mathcal{I}^\bullet$-congruence systems $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ on an $\mathbf{F}$-algebraic system $\mathcal{A}$ as the collection of all those congruence systems θ on $\mathcal{A}$, such that $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$. Alternatively, $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ may be characterized as the collection of all congruence systems on $\mathcal{A}$ which are images under $\Omega^{\mathcal{A}}$ of $\mathcal{I}$-filter systems on $\mathcal{A}$. Assuming prealgebraicity of a π-institution $\mathcal{I}$, we can show that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ is closed under arbitrary signature-wise intersections. In particular, this holds for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, whence one may consider the *systemic equational π-structure $\mathcal{Q}^{\mathcal{I}\bullet} = \langle \mathbf{F}^2, D^{\mathcal{I}\bullet}\rangle$ associated with* $\mathcal{I}$, where $D^{\mathcal{I}\bullet}$ is the closure operator corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$. With this machinery at hand, it is shown that, if $\mathcal{I}$ is syntactically WS prealgebraizable, then $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via the conjugate pair $(\tau^\flat, \overleftrightarrow{I^\flat})$ of transformations, where $I^\flat$ witnesses the syntactic prealgebraicity of $\mathcal{I}$ and $\tau^\flat$ witnesses the system truth equationality of $\mathcal{I}$. Conversely, if the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ of $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically WS prealgebraizable. More generally, $\mathcal{I}$ is syntactically WS prealgebraizable if and only if its systemic skeleton is equivalent to an equational π-structure via a conjugate pair of transformations. This characterization enables us to show that syntactic WS prealgebraizability is also characterized by the existence of a transformational order isomorphism between the lattice of theory systems of $\mathcal{I}$ and that of the theory families of an equational π-structure.

In Section 12.5, we studied syntactically weakly algebraizable π-institutions, one of the motivations being opening syntactic weak algebraizability to non-systemic π-institutions. Subsequently, in Section 12.6, we studied syntactically weakly system prealgebraizable π-institutions, weaker than syntactically weakly algebraizable ones, able to also encompass some non-stable π-institutions. Section 12.7 aims to bridge a gap between syntactic WS prealgebraizability and syntactic W algebraizability, which becomes apparent when one compares with the corresponding semantic hierarchy of Chapter 4. Recall that a π-institution $\mathcal{I}$ is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified if its reflexive core $R^{\mathcal{I}}$ is Leibniz and its left Suszko core $L^{\mathcal{I}}$ is left adequate. We call $\mathcal{I}$ *syntactically weakly left c-reflectively prealgebraizable* (*syntactically WLC prealgebraizable*) if it is prealgebraic, left c-reflective and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. Based on previously obtained characterizations of syntactic prealgebraicity and of left truth equationality, we get that syntactic WLC prealgebraizability is the conjunction of syntactic prealgebraicity and of left truth equationality. Moreover, $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is WLC prealgebraizable and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. In terms of morphisms between lattices of theory systems and of congruence systems, $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left completely order reflecting surjection, restricting to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Further, in terms of equivalences, syntactic WLC prealgebraizability is characterized by left truth equationality plus an equivalence of the systemic skeleton of the π-institution with an equational π-structure via a conjugate pair of transformations. In terms of isomorphisms between lattices of theory families, we have that a π-institution $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism between the lattice of its theory systems and that of an equational π-structure. In concluding the section and the chapter, a complete subhierarchy of the various (semantic) prealgebraizability and syntactic prealgebraizability classes is presented. This is reproduced here in lieu of a summary and to provide an overview of the location of the syntactic classes in the overall hierarchy.



1.3.12 Chapter 13

Chapter 13 deals with those classes in the syntactic hierarchy which may be defined using parameter free collections of natural transformations. They include both classes towards the bottom of the hierarchy, such as, e.g., syntactically preequivalential and syntactically equivalential π-institutions, and classes towards the top that include syntactically prealgebraizable and syntactically algebraizable π-institutions. The breadth of classes encompassed in this way is one of the reasons why this chapter is rather long. The role of parameters and parameterlessness was initially understood in the sentential logic framework with the works of Czelakowski on equivalential logics [24, 25], of Blok and Pigozzi on protoalgebraic logics [29] and later in the context of deduction theorems [27, 30, 33] (see, also, the monograph [65]).

In Section 13.2, we introduce the *binary reflexive core* $B^{\mathcal{I}}$ of a π-institution $\mathcal{I}$. This is a version of the reflexive core, used to characterize syntactic prealgebraicity and syntactic protoalgebraicity in Sections 11.2 and 11.3, respectively, but without parameters. Since, as it turns out, $B^{\mathcal{I}}$ coincides with the collection $\ddot{R}^{\mathcal{I}}$ obtained from $R^{\mathcal{I}}$ by substituting for the parameters arbitrary parameter free natural transformations, we drop the notation $B^{\mathcal{I}}$ and denote the binary reflexive core by $\ddot{R}^{\mathcal{I}}$ throughout.

In Section 13.3, we study *syntactic preequivalentiality*, a syntactic property corresponding to equivalentiality, studied in Chapter 5. We say that a π-institution $\mathcal{I}$ if *syntactically preequivalential* if there exists a parameter free collection $I^\flat$ of natural transformations in $N^\flat$, such that $I^\flat$ has reflexivity, global system transitivity, global system compatibility and the global system modus ponens in $\mathcal{I}$. In that case, $I^\flat$ is called a set of *witnessing transformations*. Parameter freeness implies that syntactic preequivalentiality is inherited by π-subinstitutions of a syntactically preequivalential π-institution. Moreover, as syntactic preequivalentiality is a form of parameterless syntactic prealgebraicity, the set $\overleftrightarrow{I^\flat}$ defines Leibniz congruence systems of theory systems, without parameters. Furthermore, syntactic preequivalentiality transfers from a π-institution to all generalized matrix families of the form $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$.

Syntactic preequivalentiality implies (semantic) preequivalentiality. Any set of witnessing transformations is included in the binary reflexive core $\ddot{R}^{\mathcal{I}}$ of $\mathcal{I}$, which, provided that $\mathcal{I}$ is syntactically preequivalential, has the system modus ponens in $\mathcal{I}$. It also possesses another important property called *extensionality*. This means that, for every theory system T, $\overleftarrow{\ddot{R}^{\mathcal{I}}}(T) = \overleftarrow{R^{\mathcal{I}}}(T)$. Since, as $\ddot{R}^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, the inclusion $\overleftarrow{R^{\mathcal{I}}}(T) \leq \overleftarrow{\ddot{R}^{\mathcal{I}}}(T)$ holds universally. Thus, extensionality asserts that $\overleftarrow{\ddot{R}^{\mathcal{I}}}(T) \leq \overleftarrow{R^{\mathcal{I}}}(T)$. It turns out that the binary reflexive core having the system modus ponens and extensionality characterize syntactic preequivalentiality. Another characterization asserts its equivalence with the definability of Leibniz congruence systems of theory systems by the binary reflexive core.

The last part of Section 13.3 delineates the boundary between syntactic equivalentiality and equivalentiality. For this a variant of the Leibniz property of the reflexive core is needed. More precisely, we say that $\ddot{R}^{\mathcal{I}}$ is *locally Leibniz* if, for every signature Σ and all Σ-sentences ϕ,ψ,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma^{\langle\phi,\psi\rangle}(C(\ddot{R}_\Sigma^{\mathcal{I}}[\phi,\psi]) \cap \langle\phi,\psi\rangle).$$

It is shown that $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional, then $\ddot{R}^{\mathcal{I}}$ is locally Leibniz. Conversely, if $\mathcal{I}$ is preequivalential, then, if $\ddot{R}^{\mathcal{I}}$ is locally Leibniz, then it has the system modus ponens and is extensional. These implications help us characterize syntactically preequivalential π-institutions as those that are preequivalential and have a locally Leibniz binary reflexive core.

In Section 13.4, we study *syntactic equivalentiality*. It results from syntactic preequivalentiality by replacing the system versions of the defining properties by the corresponding global family versions. Otherwise, the development mirrors that of Section 13.3. Syntactic equivalentiality implies both syntactic preequivalentiality and (semantic) equivalentiality. In the first characterization theorem, it is shown that a π-institution $\mathcal{I}$ is syntactically equivalential if and only if its binary reflexive core $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional. Another characterization asserts that $\mathcal{I}$ is syntactically equivalential if and only if its binary reflexive core defines Leibniz congruence systems of theory families in $\mathcal{I}$. Finally, it is shown that the condition that separates syntactic equivalentiality from equivalentiality is the local Leibniz property of the binary reflexive core of $\mathcal{I}$.

In Section 13.5, we study *strong (family) truth equationality*, a version of (family) truth equationality in which the witnessing equations are parameter free. To provide a characterization, analogous to the one given in Section 11.8 for truth equationality, we use the unary Suszko core $\dot{S}^{\mathcal{I}}$ of a π-institution $\mathcal{I}$, also a version of the Suszko core $S^{\mathcal{I}}$ without parameters. The unary Suszko core satisfies the property that, for every theory family T of $\mathcal{I}$, $T \le \overleftarrow{\dot{S}^{\mathcal{I}}}(\Omega(T))$. If the reverse inclusion also holds for every theory family, then we say that $\dot{S}^{\mathcal{I}}$ is *soluble*. In fact, the solubility property of the unary Suszko core of a π-institution characterizes strong truth equationality. Another characterization states that $\mathcal{I}$ is strongly truth equational if and only if $\dot{S}^{\mathcal{I}}$ defines theory families via their Leibniz congruence systems. Another property that relates the unary Suszko core with both the Leibniz and the Suszko congruence systems is that, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_{\Sigma}[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

If, for all signatures Σ and all Σ-sentences ϕ, the reverse inclusion also holds, then $\dot{S}^{\mathcal{I}}$ is said to be *adequate*. Solubility of $\dot{S}^{\mathcal{I}}$ always implies its adequacy. In general, this is a strict implication. However, provided that $\mathcal{I}$ is family completely reflective, then adequacy of $\dot{S}^{\mathcal{I}}$ implies its solubility. In fact, we get that $\mathcal{I}$ is strongly truth equational if and only if it is family completely reflective and $\dot{S}^{\mathcal{I}}$ is adequate. Thus, adequacy of $\dot{S}^{\mathcal{I}}$ is the property that separates strong truth equationality from family c-reflectivity.

In Section 13.6, we study *strong left truth equationality*, a property that has the same relation to left truth equationality as strong (family) truth equationality, studied in Section 13.5, has to (family) truth equationality. A π-institution is *strongly left truth equational* if it satisfies a condition similar to that defining left truth equationality, but with witnessing equations without parameters. Strong truth equationality is strong enough to imply systemicity and, as a consequence, it implies strong left truth equationality. Given a π-institution $\mathcal{I}$, we introduce the *unary left Suszko core* $\dot{L}^{\mathcal{I}}$ of $\mathcal{I}$, analogous to the left Suszko core $L^{\mathcal{I}}$, but without parameters. It is always the case that, for every theory family T, $\overleftarrow{T} \le \overleftarrow{\dot{L}^{\mathcal{I}}}(\Omega(T))$. *Left solubility* is the property that holds, by definition, if the reverse inclusion is also valid. We prove that $\mathcal{I}$ is strongly left truth equational if and only if the unary left Suszko core of $\mathcal{I}$ is left soluble. A second characterization asserts that $\mathcal{I}$ is strongly left truth equational if and only if the unary left Suszko core defines theory families in $\mathcal{I}$ "up to arrow", meaning that, for every theory family T of $\mathcal{I}$, $\overleftarrow{\dot{L}^{\mathcal{I}}}(\Omega(T)) = \overleftarrow{T}$. To relate strong left truth equationality with its weaker semantic analog, left complete reflectivity, we introduce an analog of the adequacy property for the unary left Suszko core. This simulates the left adequacy property, introduced in Section 11.9, in relation to the left Suszko core $L^{\mathcal{I}}$. In general, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

If the reverse inclusion is also valid, we say that the unary left Suszko core $\dot{L}^{\mathcal{I}}$ is *left adequate*. Again, in general, the solubility of the unary left Suszko core implies its left adequacy. The reverse implication holds provided that $\mathcal{I}$ is left completely reflective. In fact, as it turns out, $\mathcal{I}$ is strongly left truth equational if and only if it is left c-reflective and has a left adequate unary left Suszko core. That is, left adequacy of the left Suszko core is the property dividing strong left truth equationality from left complete reflectivity.

In Section 13.7, we study a third version of strong truth equationality, *strong system truth equationality*. As the name suggests, it is defined by the same condition as strong (family) truth equationality, but restricted to theory systems. Recall that in order to characterize system truth equationality in Section 11.10, we introduced the "systemic" Tarski congruence system of a theory system T in a π-institution $\mathcal{I}$,

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\}.$$

Based on this concept, the *unary system core* $\dot{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is defined in a similar way as the system core, but without allowing parameters. In general, it holds that, for every theory system T of $\mathcal{I}$, $T \leq \overleftarrow{\dot{Z}^{\mathcal{I}}}(\Omega(T))$. If the reverse inclusion is valid, we say that the unary system core is *soluble*. Solubility of the unary system core $\dot{Z}^{\mathcal{I}}$ of $\mathcal{I}$ intrinsically characterizes strong system truth equationality of $\mathcal{I}$. A second characterization asserts that $\mathcal{I}$ is strongly system truth equational if and only if $\dot{Z}^{\mathcal{I}}$ defines theory systems in terms of Leibniz congruence systems in $\mathcal{I}$.

Strong system truth equationality implies system complete reflectivity, which is its semantic counterpart. The second part of Section 13.7 seeks to pinpoint a property that characterizes strong system truth equationality inside system c-reflectivity. As was the case with strong (family) truth equationality and strong left truth equationality, this property turns out to be the adequacy of the unary system core. In general, for all signatures Σ and all Σ-sentences ϕ, $\dot{Z}^{\mathcal{I}}$ satisfies

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

If the reverse inclusion happens to be valid, then we say that $\dot{Z}^{\mathcal{I}}$ is *adequate*. Solubility of $\dot{Z}^{\mathcal{I}}$ implies its adequacy. A sufficient condition for the converse implication to hold is that $\mathcal{I}$ be system c-reflective. In fact, it is shown that a π-institution $\mathcal{I}$ is strongly system truth equational if and only if it is system c-reflective and has an adequate unary system core. Thus, in a certain sense, adequacy of the unary system core is the exact property characterizing the class of strongly system truth equational π-institutions as a subclass of the class of system c-reflective π-institutions.

In Chapter 5, we defined the (semantic) prealgebraizability hierarchy, which involved system monotonicity (prealgebraicity), a certain kind of extensionality and a certain type of injectivity, reflectivity or c-reflectivity. In Section 13.8, we delve into the study of syntactic analogs of classes in this hierarchy. The syntactic analogs entail definability, in some way, of Leibniz congruence systems and of truth. If the former is via parameter free collections of natural transformations, we obtain *syntactic prealgebraizability* classes. On the other hand, if the latter involves parameter free collections of witnessing equations, we talk about *syntactic antiprealgebraizability*.

We say that a π-institution $\mathcal{I}$ is *$\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified* if it has a locally Leibniz binary reflexive core $\ddot{R}^{\mathcal{I}}$ and a left adequate unary left Suszko core $\dot{L}^{\mathcal{I}}$. Then $\mathcal{I}$ is *syntactically strongly left prealgebraizable* if it is preequivalential, left c-reflective and $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified. Building on previous work on syntactic preequivalentiality and strong left truth equationality, it is shown that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is syntactically preequivalential and strongly left truth equational. As concerns its relation with its semantic counterpart, LC prealgebraizability, studied in Section 5.5, it is shown that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is LC prealgebraizable and $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified. A third characterization assumes the form of a transfer theorem. It asserts that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective. A fourth characterization uses mappings between lattices of theory families and of congruence systems. According to it, $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left completely order reflecting surjection which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ that commutes with inverse logical extensions. Yet another characterization involves equivalences between π-structures and equational π-structures associated, in some way, with the π-institution $\mathcal{I}$. We obtain that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of natural transformations. More generally, the same characterization holds, as long as some equivalence exists between $\mathcal{K}^{\mathcal{I}}$ and an equational π-structure via a conjugate pair of natural transformations. This characterization yields also that syntactic strong left prealgebraizability is characterized by strong left truth equationality and the existence of an order isomorphism $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$ induced by a conjugate pair of natural transformations, where $\mathcal{Q}$ is an equational π-structure.

In the second part of Section 13.8, two weaker properties of π-institutions than syntactic strong left prealgebraizability are introduced. One is obtained by dropping the requirement that the transformations witnessing syntactic prealgebraicity be parameter free and the other by doing the same on the left truth equationality side. These give the notions of *syntactically left prealgebraizable* and *syntactically left antiprealgebraizable* π-institutions, which constitute both superclasses of the syntactically strongly left prealgebraizable ones. After introducing those classes, several characterizations, similar to the ones described above for syntactically strongly left prealgebraizable π-institutions, are provided for each of these two classes.

In Section 13.9, we study versions of syntactic system prealgebraizability, lying below syntactic left prealgebraizability, studied in Section 13.8. As in Section 13.8, we introduce, first, the strongest version which requires both the collection of witnessing transformations and the collection of witnessing equations to be parameter free. A π-institution $\mathcal{I}$ is said to be *$\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified* if it has a locally Leibniz binary reflexive core $\ddot{R}^{\mathcal{I}}$ and an adequate unary system core $\dot{Z}^{\mathcal{I}}$. $\mathcal{I}$ is *syntactically strongly system prealgebraizable* if it is preequivalential (i.e., prealgebraic and system extensional), system c-reflective and $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. In terms of lower classes in the syntactic hierarchy, we get that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if it is syntactically preequivalential and strongly system truth equational. In relation to the corresponding semantic class, $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if it is system prealgebraizable and $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. In terms of transference of the constituent properties, $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is monotone on theory systems, system extensional and system c-reflective. As far as using mappings between lattices of theory families and of congruence systems, we get that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding which commutes with inverse logical extensions. In terms of equivalences with an equational π-structure, $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an equational π-structure via a conjugate pair of natural transformations. This yields, additionally, that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if there exists a transformational order isomorphism $h : \mathbf{ThSys}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$ induced by a conjugate pair of natural transformations, where $\mathcal{Q}$ is an equational π-structure.

As was the case in Section 13.8, here, also, below syntactic strong system prealgebraizability we have two properties that result from it by weakening the requirement of parameterlessness either on the side of syntactic preequivalentiality or on the side of strong system truth equationality. In the first instance, we obtain *syntactic system prealgebraizability* and, in the second, *syntactic system antiprealgebraizability*. For these two classes we obtain characterizations paralleling those given for syntactic strong system prealgebraizability in the preceding paragraph. More concretely, we get characterizations in term of constituent properties in the syntactic hierarchy, corresponding properties in the semantic hierarchy, a kind of transfer theorem of constituent properties, in terms of mappings between filter systems and congruence systems, in terms of transformational equivalences with an equational π-structure and, finally, in terms of mappings between theory systems and theory families of equational π-structures.

In Section 13.10, we switch from syntactic prealgebraizability properties, which involve prealgebraicity (system monotonicity), to syntactic algebraizability properties, involving protoalgebraicity (family monotonicity). There are a total of six classes, here, forming an echelon formation of two tiers, the ones at the top involving family c-reflectivity, whereas those at the bottom demanding only system c-reflectivity. In Section 13.10, the properties of the top tier are studied. The three remaining syntactic algebraizability classes are studied in Section 13.11.

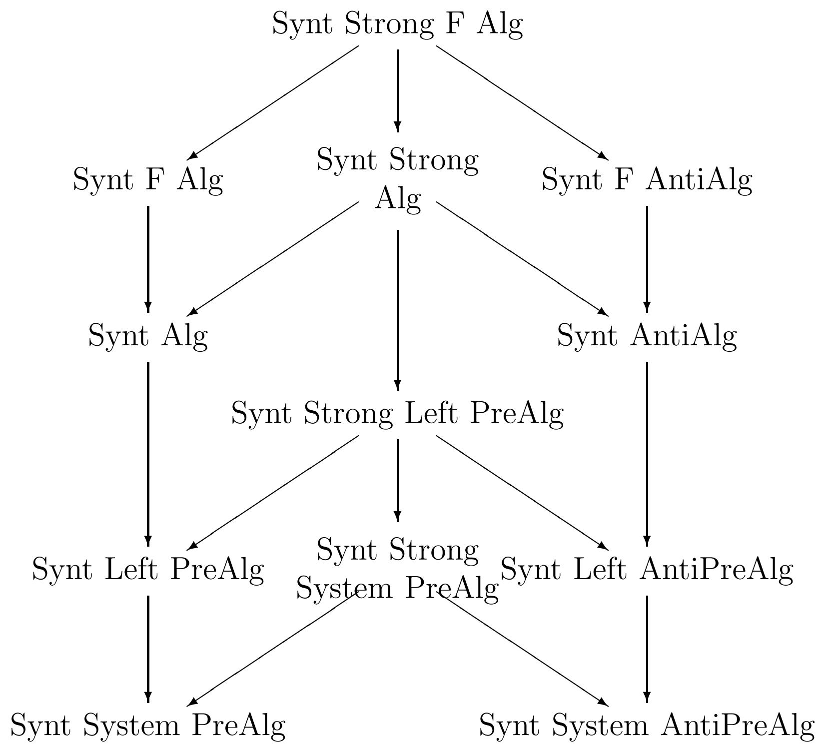
A π-institution $\mathcal{I}$ is said to be *$\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified* if it has a locally Leibniz binary reflexive core and an adequate unary Suszko core. It is *syntactically strongly family algebraizable* if it is equivalential (i.e., protoalgebraic and family extensional), family c-reflective and $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified. In terms of syntactic classes, we obtain that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if it is syntactically equivalential and strongly (family) truth equational. On the other hand, in terms of semantic classes, we get that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if it is strongly family algebraizable and $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified. With respect to transference of constituent properties, we get that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is monotone and injective on $\mathcal{I}$-filter families and family extensional. As far as mappings between posets of theory families and of congruence systems go, $\mathcal{I}$ is syntactically strongly family algebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions. Additionally, looking at equivalences with equational π-structures, we get that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if it is equivalent to an equational π-structure via a conjugate pair of natural transformations. This characterization yields, also, that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if there exists a transformational order isomorphism $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$, induced by a conjugate pair of natural transformations, where $\mathcal{Q}$ is an equational π-structure.

In the second part of Section 13.10, as was the case with Sections 13.8 and 13.9, we study the two classes obtained by relaxing the parameterlessness requirement on one of the two sides (equivalentiality or strong truth equationality) in the definition of syntactic strong family algebraizability. First, we say that $\mathcal{I}$ is *$\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified* if it has a locally Leibniz binary reflexive core and an adequate Suszko core and that it is *$R^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified* if it has a Leibniz reflexive core and an adequate unary Suszko core. Based on these, we define *syntactic family algebraizability* as the property of being equivalential, family c-reflective and $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and *syntactic family antialgebraizability* as the property of being protoalgebraic, family c-reflective and $R^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified. For both of these classes of π-institutions we obtain characterizations akin to those obtained for syntactic strong family algebraizability in the first part of the section.

In Section 13.11, we complete the work on syntactic algebraizability started in Section 13.10. We define, here, the three classes of syntactically algebraizable π-institutions that lie between those of the syntactic family algebraizability subhierarchy, introduced in Section 13.10, and those of the syntactic left prealgebraizability subhierarchy, introduced in Section 13.8. We start with the highest class of the syntactic algebraizability hierarchy. We say that a π-institution $\mathcal{I}$ is *$\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified* if its binary reflexive core is locally Leibniz and its unary system core is adequate. $\mathcal{I}$ is *syntactically strongly algebraizable* if it is equivalential, system c-reflective and $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. In terms of lower classes of the syntactic hierarchy, we get that $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is syntactically equivalential and strongly system truth equational. In terms of its semantic counterpart, we obtain that $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is algebraizable and $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. As far as transference of basic properties, we have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and system c-reflective. With respect to mappings between posets of theory families and of congruence systems, $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions. In terms of equivalences with equational π-structures, $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an equational π-structure via a conjugate pair of natural transformations. This implies that $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is stable and there exists a transformational order isomorphism $h : \mathbf{ThSys}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$ induced by a conjugate pair of natural transformations, where $\mathcal{Q}$ is an equational π-structure.

In the second part of Section 13.11, we relax the parameterlessness requirements so that two wider classes of syntactic algebraizability, lying below the one studied in the first part, are obtained. We say that a π-institution $\mathcal{I}$ is *$\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified* if it has a locally Leibniz binary reflexive core and an adequate system core, whereas it is *$R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified* if it has a Leibniz reflexive core and an adequate unary system core. Using these properties, we say that $\mathcal{I}$ is *syntactically algebraizable* if it is equivalential, system c-reflective and $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified. Moreover, $\mathcal{I}$ is *syntactically antialgebraizable* if it is protoalgebraic, system c-reflective and $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. For those two classes, lying below the class of syntactically strongly algebraizable π-institutions, we devise multiple characterizations along similar lines with those presented in the first part of the section for syntactic strong algebraizability.

We recapitulate pictorially the hierarchy of all twelve classes introduced in Chapter 13 in the accompanying diagram.



1.3.13 Chapter 14

In Chapter 14, the study of syntactic analogs of the semantic hierarchies studied in Chapter 6 are undertaken. These fall under the classes of the syntactic truth equationality subhierarchy detailed in Sections 11.6-11.10. Thus, they occupy positions close to the right bottom of the algebraic hierarchy of π-institutions. The study of both semantic and syntactic properties of those classes takes after a very similar study in the context of sentential logics initiated in Moraschini's Ph.D. Dissertation [90].

In Section 14.2, we study *rough* or *narrow truth equationality*, the syntactic analog of rough and of narroe complete reflectivity, studied in Sections 6.8-6.9. Rough truth equationality has the same relation to truth equationality as rough c-reflectivity has to c-reflectivity. That is, it is essentially truth equationality applied only to theory families all of whose components are nonempty.

In summarizing some of the key concepts, we recall that, for a theory family T of a π-institution $\mathcal{I}$, $\widetilde{T}$ denotes the theory family that results from T by replacing all empty components by the corresponding entire sets of sentences. Moreover, we denote by $\mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, the collection of all theory families of $\mathcal{I}$ all of whose components are nonempty. Notice that, if $\mathcal{I}$

has theorems, then we have, for every theory family T, $\widetilde{T} = T$ and, also, $\mathrm{ThFam}^{\nparallel}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Among others, an important observation is that, for every theory family T of $\mathcal{I}$, $\Omega(\widetilde{T}) = \Omega(T)$. These definitions and observations, and a quite detailed analysis of all relevant concepts, were the subject of the first sections of Chapter 6.

We say that a π-institution $\mathcal{I}$ is *roughly* or *narrowly* (*family*) *truth equational* if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all theory families T of $\mathcal{I}$, all signatures Σ and all Σ-sentences ϕ,

$$\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}.$$

This condition is equivalent to the assertion that for all theory families T of $\mathcal{I}$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Truth equationality is separated from rough truth equationality by the existence of theorems. More precisely, it is shown that a π-institution $\mathcal{I}$ is truth equational if and only if it is roughly truth equational and has theorems. Rough truth equationality implies rough family c-reflectivity. To intrinsically characterize rough truth equationality, we use a specially adapted version of the Suszko core defined by restricting attention to theory families without empty components. The *rough Suszko core* $S^{\mathcal{I}\nparallel}$ of $\mathcal{I}$ is defined by

$$S^{\mathcal{I}\nparallel} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\widetilde{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}))\}.$$

Equivalently, $S^{\mathcal{I}\nparallel} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}$. It always holds that, for all $T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, $T \leq \overleftarrow{S^{\mathcal{I}\nparallel}}(\Omega(T))$. If the reverse inclusion happens to be valid, we say that $S^{\mathcal{I}\nparallel}$ is *soluble*. Solubility is equivalent to the statement that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} = \overleftarrow{S^{\mathcal{I}\nparallel}}(\Omega(T))$. We show that $\mathcal{I}$ is roughly truth equational if and only if its rough Suszko core is soluble. Another characterization asserts that $\mathcal{I}$ is roughly truth equational if and only if $S^{\mathcal{I}\nparallel}$ roughly defines theory families, in the sense that, for all $T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, $T = \overleftarrow{S^{\mathcal{I}\nparallel}}(\Omega(T))$.

In a π-institution $\mathcal{I}$, we have, in general, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : S^{\mathcal{I}\nparallel}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

If the reverse inclusion holds, then we say that $S^{\mathcal{I}\nparallel}$ is *adequate*. Solubility of $S^{\mathcal{I}\nparallel}$ implies its adequacy. Provided that $\mathcal{I}$ is roughly family c-reflective, the reverse implication also holds, that is, adequacy of $S^{\mathcal{I}\nparallel}$ implies its solubility. These implications enable us to obtain a characterization of rough truth equationality to the effect that $\mathcal{I}$ is roughly truth equational if and only if it is roughly family c-reflective and has an adequate rough Suszko core. Section 14.2 closes with a result that, besides providing yet another characterization, may be viewed as a transfer theorem. It asserts that $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is roughly truth equational, with witnessing equations $\tau^{\mathcal{A}}$.

In Section 14.3, we study *rough left truth equationality*. It has the same relation to rough left c-reflectivity, its semantic counterpart, as rough truth equationality has to rough c-reflectivity. Additionally, it has the same relation to rough truth equationality as left truth equationality has to truth equationality. Roughly speaking, rough left truth equationality results from left truth equationality by bypassing theory families with empty components.

A π-institution $\mathcal{I}$ is *roughly left truth equational* if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for every theory family T, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in \overleftarrow{\widetilde{T}}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

According to this definition, $\mathcal{I}$ is left truth equational if and only if it is roughly left truth equational and has theorems. It turns out that, if $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$, then, for all theory families T, $\overleftarrow{\widetilde{T}} = \overleftarrow{\tau^\flat}(\Omega(T))$. Recalling that $\mathcal{I}$ is roughly left c-reflective if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}},$$

we can show that rough left truth equationality implies rough left c-reflectivity. To characterize intrinsically rough left truth equationality, we rely on the rough left Suszko core. The *rough left Suszko core* $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is the collection of all natural transformations $\sigma^\flat$ in $N^\flat$, such that, for every signature Σ and all Σ-sentences ϕ, $\sigma^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : \phi \in \overleftarrow{\widetilde{T}}_\Sigma\}$. The definition implies that, for every theory family T, $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{L}^{\mathcal{I}}}(\Omega(T))$. If the reverse inclusion is valid, we say that $\widetilde{L}^{\mathcal{I}}$ is *left soluble*. Left solubility of $\widetilde{L}^{\mathcal{I}}$ intrinsically characterizes rough left truth equationality. Another characterization involves rough definability of theory families up to arrow, in the sense that, for every theory family T, $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{L}^{\mathcal{I}}}(\Omega(T))$.

The rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ satisfies, in general, the property that, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \bigcap\{\Omega(T) : \phi \in \overleftarrow{\widetilde{T}}_\Sigma\}.$$

If the reverse inclusion is valid, then we say that $\widetilde{L}^{\mathcal{I}}$ is *left adequate*. Left solubility of $\widetilde{L}^{\mathcal{I}}$ implies its left adequacy. If $\mathcal{I}$ happen to be left c-reflective, then left adequacy implies left solubility as well. These results yield that $\mathcal{I}$ is roughly left truth equational if and only if it is roughly left c-reflective and has a left adequate rough left Suszko core. The section closes with a result asserting that rough left truth equationality transfers, in a sense similar to that detailed previously for rough truth equationality.

In Section 14.4, we look at an alternative way to deal with theory families having empty components. Whereas rough left truth equationality, studied in Section 14.3, picks the largest rough representative of a given theory family, if the family happens to have empty components, *narrow left truth equationality* ignores theory families with empty components by working with the collection $\mathrm{ThFam}^{\not\ell}(\mathcal{I})$. Accordingly, a π-institution $\mathcal{I}$ is *narrowly left truth equational* if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}^{\not\ell}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

By paying the price of introducing rough representatives on the left (and implicitly on the right), we may write this condition equivalently as the stipulation that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in \overleftarrow{\widetilde{T}}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

As was the case with rough left truth equationality, we get, here also, that a π-institution $\mathcal{I}$ is left truth equational if and only if it is narrowly left truth equational and has theorems. Narrow left truth equationality affords expressions for theory families in terms of Leibniz congruence systems by the rules $\overleftarrow{T} = \overleftarrow{\tau^\flat}(\Omega(T))$, for all $T \in \mathrm{ThFam}^{\not\ell}(\mathcal{I})$, and, again, avoiding the restriction, $\overleftarrow{\widetilde{T}} = \overleftarrow{\tau^\flat}(\Omega(T))$, for all $T \in \mathrm{ThFam}(\mathcal{I})$. Additionally, narrow left truth equationality implies narrow left c-reflectivity, its semantic counterpart. To intrinsically characterize narrow left truth equationality, another modification of the left Suszko core is introduced. The *narrow left Suszko core* $L^{\mathcal{I}\not\ell}$ of $\mathcal{I}$ consists of all $\sigma^\flat$ in $N^\flat$, such that, for all $T \in \mathrm{ThFam}^{\not\ell}(\mathcal{I})$, $\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. This definition ensures that, for all $T \in \mathrm{ThFam}^{\not\ell}(\mathcal{I})$, $\overleftarrow{T} \leq \overleftarrow{L^{\mathcal{I}\not\ell}}(\Omega(T))$. If the reverse inclusion is valid, we say that $L^{\mathcal{I}\not\ell}$ is *left soluble*. Let solubility of $L^{\mathcal{I}\not\ell}$ characterizes narrow left truth equationality. Another characterization asserts that $\mathcal{I}$ is narrowly left truth equational if and only if $L^{\mathcal{I}\not\ell}$ narrowly defines theory families of $\mathcal{I}$ up to arrow, in the sense that, for all $T \in \mathrm{ThFam}^{\not\ell}(\mathcal{I})$, $\overleftarrow{T} = \overleftarrow{L^{\mathcal{I}\not\ell}}(\Omega(T))$.

In general, the narrow left Suszko core satisfies, for all signatures Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : L^{\mathcal{I}\not\ell}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

We say that $L^{\mathcal{I}\natural}$ is *left adequate* if the reverse inclusion is also valid. Left solubility of $L^{\mathcal{I}\natural}$ implies its left adequacy. The converse implication holds, provided that $\mathcal{I}$ is narrowly left c-reflective. Using these results, we show that $\mathcal{I}$ is narrowly left truth equational if and only if it is narrowly left c-reflective and has a left adequate narrow left Suszko core. The section ends with a transfer theorem to the effect that $\mathcal{I}$ is narrowly left truth equational if and only if, for every **F**-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, all signatures Σ and all Σ-sentences ϕ, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T)$.

In Section 14.5, we study *rough system truth equationality*. This property is the syntactic analog of rough system c-reflectivity and lies below rough left truth equationality. It is defined similarly to system truth equationality, but avoids theory systems with empty components, by using roughly equivalent representatives. A π-institution $\mathcal{I}$ is *roughly system truth equational* if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all theory systems T, all signatures Σ, and all Σ-sentences ϕ,

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

A π-institution is system truth equational if and only if it is roughly system truth equational and has theorems. Rough system truth equationality yields, for every theory system T, an expression of $\widetilde{T}$ in terms of the Leibniz congruence system of T, via the witnessing equations, $\widetilde{T} = \overleftarrow{\tau^\flat}(\Omega(T))$. This gives that rough system truth equationality implies rough system c-reflectivity, its semantic counterpart. In this case, to derive an intrinsic characterization of rough system truth equationality, we introduce the *rough system core* $\widetilde{Z}^{\mathcal{I}}$ of a π-institution $\mathcal{I}$. It consists of all natural transformations $\sigma^\flat$ in $N^\flat$, such that, for every signature Σ and all Σ-sentences ϕ,

$$\sigma^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}.$$

This definition implies that, for every theory system T, $\widetilde{T} \leq \overleftarrow{\widetilde{Z}^{\mathcal{I}}}(\Omega(T))$. $\widetilde{Z}^{\mathcal{I}}$ is said to be *soluble* if the reverse inclusion is also valid. Here, also, we have that $\mathcal{I}$ is roughly system truth equational if and only if $\widetilde{Z}^{\mathcal{I}}$ is soluble. Another characterization asserts that $\mathcal{I}$ is roughly system truth equational if and only if $\widetilde{Z}^{\mathcal{I}}$ roughly defines theory systems, in the sense that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \overleftarrow{\widetilde{Z}^{\mathcal{I}}}(\Omega(T))$.

Similarly to the case of rough left truth equationality, the rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ satisfies, in general, for all signatures Σ and all Σ-sentences ϕ,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}\\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}.\end{aligned}$$

If the reverse inclusion is also valid, then we say that the rough system core is *adequate*. Solubility of $\widetilde{Z}^{\mathcal{I}}$ implies its adequacy. Moreover, the converse holds if $\mathcal{I}$ happens to be roughly system c-reflective. It follows that $\mathcal{I}$ is roughly system truth equational if and only if it is roughly system c-reflective and has an adequate rough system core. the section concludes with a transfer theorem for rough system truth equationality similar in spirit to the one described previously for narrow left truth equationality.

In Section 14.6, we switch to *narrow system truth equationality*. As was the case with narrow left truth equationality, narrow system truth equationality results from system truth equationality by bypassing theory systems with empty components. We say that a π-institution $\mathcal{I}$ is *narrowly system truth equational* if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for every theory system T, for which $\widetilde{T}$ is also a theory system, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Another, equivalent, formulation asserts that, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

A π-institution $\mathcal{I}$ is system truth equational if and only if it is narrowly truth system truth equational and has theorems. Recall that the analogous statement is also valid for rough system truth equationality in place of narrow system truth equationality. Narrow system truth equationality via witnessing equations $\tau^\flat$ is equivalent to having, for all $T \in \mathrm{ThSys}(\mathcal{I})$, with $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \overleftarrow{\tau^\flat}(\Omega(T))$ and, also, to having, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$ $T = \overleftarrow{\tau^\flat}(\Omega(T))$. These expressions allow us, with little effort, to show that narrow system truth equationality implies narrow system c-reflectivity, which constitutes its semantic analog. To intrinsically characterize narrow system truth equationality, we use another modification of the system core, called *narrow system core* and denoted by $Z^{\mathcal{I}\nleq}$, consisting of all those natural transformations $\sigma^\flat$ in $N^\flat$, such that, for every theory system T, with $\widetilde{T}$ also a theory system, $\sigma^\flat[\widetilde{T}] \leq \widehat{\Omega}^{\mathcal{I}}(T)$, where $\widehat{\Omega}^{\mathcal{I}}$ denotes the systemic Suszko operator associated with the π-institution $\mathcal{I}$. An equivalent definition asserts that $Z^{\mathcal{I}\nleq}$ consists of those natural transformations $\sigma^\flat$ in $N^\flat$, such that, for every $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T)$. It can be seen that, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $T \leq \overleftarrow{Z^{\mathcal{I}\nleq}}(\Omega(T))$. *Solubility* of $Z^{\mathcal{I}\nleq}$ is the property that holds when the reverse inclusion is also valid. Equivalently, $Z^{\mathcal{I}\nleq}$ is soluble if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, with $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \overleftarrow{Z^{\mathcal{I}\nleq}}(\Omega(T))$. Solubility of the narrow system core intrinsically characterizes narrow system truth equationality. Another characterization states that $\mathcal{I}$ is narrowly system truth equational if and only if $Z^{\mathcal{I}\nleq}$ narrowly defines theory systems , i.e., for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $T = \overleftarrow{Z^{\mathcal{I}\nleq}}(\Omega(T))$.

We mentioned that narrow system truth equationality implies narrow system c-reflectivity. To pinpoint the separating line between the two classes, we use another property of the narrow system core, akin to the property of adequacy of the system core. To define it, we first need, once more, a modified version of the systemic Suszko operator $\widehat{\Omega}^{\mathcal{I}}$, called the *narrow systemic Suszko operator*, given, for all $T \in \mathrm{ThSys}(\mathcal{I})$, by

$$\widehat{\Omega}^{\mathcal{I}\nmid}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})\}.$$

If $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, then $\widehat{\Omega}^{\mathcal{I}\nmid}(T) = \widehat{\Omega}^{\mathcal{I}}(T)$. In general, the narrow systemic Suszko operator satisfies, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } Z_{\Sigma}^{\mathcal{I}\nmid}[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})).$$

If the reverse inclusion happen to be valid, then we say that $Z^{\mathcal{I}\nmid}$ is *adequate*. Solubility of $Z^{\mathcal{I}\nmid}$ implies its adequacy and, moreover, if $\mathcal{I}$ is narrowly system c-reflective, the reverse implication also holds. These yield that a π-institution $\mathcal{I}$ is narrowly system truth equational if and only if it is narrowly system c-reflective and has an adequate narrow system core. This section ends as well with a transfer theorem for narrow system truth equationality.

In Section 14.7, the last section of the chapter, we study the availability and some properties of *natural theorems*. The reason they are studied here is that availability of natural theorems is a property stronger than having theorems and the way used to characterize availability of natural theorems inside the class of π-institutions with theorems is reminiscent of the way truth equationality is characterized in the class of c-reflective π-institutions. Recall that, by convention, is a π-institution $\mathcal{I}$ has theorems, then, for every signature Σ, $\mathrm{Thm}_{\Sigma}(\mathcal{I}) \coloneqq C_{\Sigma}(\varnothing) \neq \varnothing$. *Having natural theorems* is a stronger property defined by the stipulation that there exists a natural transformation $\vartheta^{\flat}$ in $N^{\flat}$, such that, for all signatures Σ and all tuples of Σ-sentences $\vec{\phi}$ of appropriate arity,

$$\vartheta_{\Sigma}^{\flat}(\vec{\phi}) \in \mathrm{Thm}_{\Sigma}(\mathcal{I}).$$

The collection of natural theorems of $\mathcal{I}$ is denoted by $\mathrm{NThm}(\mathcal{I})$. We show that, if $\mathcal{I}$ has natural theorems, then it has at least one that is at most unary. A characterization theorem asserts that $\vartheta^{\flat}$ is a natural theorem if and only if, for every theory family T, all signatures Σ and all Σ-sentneces $\phi, \vec{\chi}$,

$$\phi \in T_{\Sigma} \quad \text{iff} \quad \langle \phi, \vartheta_{\Sigma}^{\flat}(\vec{\chi})\rangle \in \lambda_{\Sigma}(T).$$

Further, it is shown that, if $\vartheta^{\flat}$ is a natural theorem, then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\vartheta^{\mathcal{A}}$ is a natural theorem of $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$. This allows another characterization of natural theorems. Namely, $\vartheta^{\flat}$ is a natural theorem if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in T_{\Sigma} \quad \text{iff} \quad \langle \phi, \vartheta_{\Sigma}^{\mathcal{A}}(\vec{\chi})\rangle \in \widetilde{\lambda}_{\Sigma}^{\mathcal{I},\mathcal{A}}(T), \text{ for all } \vec{\chi} \in \mathrm{SEN}(\Sigma).$$

Based on this, we may insert an intermediate step between the availability of natural theorems and the availability of theorems. More precisely, if $\mathcal{I}$ has natural theorems, there exists $\tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ and all Σ-sentences ϕ,

$$\phi \in T_\Sigma \quad \text{iff} \quad (\forall \vec{\chi} \in \mathrm{SEN}(\Sigma))(\tau^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T))$$

and, moreover, if the displayed condition holds, then $\mathcal{I}$ has theorems. To clarify further the relationship between the existence of theorems and that of natural theorems, we introduce an "analog" of the Suszko core, called the *Lindenbaum core*, because the Lindenbaum operator $\widetilde{\lambda}^{\mathcal{I}}$ is the analogous local equivalence operator to the Suszko congruence operator $\widetilde{\Omega}^{\mathcal{I}}$ (see the table at the end of Section 2.11). The *Lindenbaum core* $F^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ consists of all natural transformations $\sigma^\flat$ in $N^\flat$, such that, for all theory families T, all signatures Σ and all Σ-sentences $\vec{\chi}$,

$$\{\langle \phi, \sigma^\flat_\Sigma(\vec{\chi})\rangle : \phi \in T_\Sigma\} \subseteq \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T).$$

The Lindenbaum operator is denoted by $F^{\mathcal{I}}$ as the L is not available because of the left Suszko core $L^{\mathcal{I}}$ and the F is a reminder of the Frege operator, which is the basic equivalence operator on which the Lindenbaum operator is built. We show that, if $\mathcal{I}$ has natural theorems, then $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$. In general, for every signature Σ and all Σ-sentences ϕ,

$$\bigcap\{\lambda(T) : \{\langle \phi, F^{\mathcal{I}}_\Sigma(\vec{\chi})\rangle : \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\} \subseteq \lambda_\Sigma(T)\} \leq \widetilde{\lambda}^{\mathcal{I}}(C(\phi)).$$

If the reverse inclusion is valid, the we say that the Lindenbaum core $F^{\mathcal{I}}$ is *adequate*. This property is significant because it delineates the boundary between having theorems and having natural theorems. Indeed, it is shown that $\mathcal{I}$ has natural theorems if and only if it has theorems and its Lindenbaum core is adequate. The section (and the chapter) concludes with a theorem to the effect that availability of natural theorems transfers from a π-institution to all $\mathcal{I}$-gmatrix families of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$.

1.3.14 Chapter 15

Chapter 15 deals with the syntactic rough/narrow monotonicity hierarchy, which forms the syntactic analog of the rough/narrow monotonicity hierarchy, studied in Chapter 7. We look at the classes of syntactically roughly/narrowly family monotone, syntactically roughly/narrowly system monotone and syntactically narrowly right monotone π-institutions.

In Section 15.2, we study *syntactic narrow family monotonicity*, the syntactic analog of rough/narrow family monotonicity. To introduce this class, one needs, first, to either be able to apply the constituent properties of natural transformations modulo rough representatives of theory families or relativize them to the collection $\mathrm{ThFam}^{\not\,}(\mathcal{I})$ of theory families without empty components. Let us take, for instance, reflexivity. Consider a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ is *roughly family reflexive* if, for every theory family T, all signatures Σ and all Σ-sentences ϕ,

$$I^\flat_\Sigma[\phi,\phi] \le \widetilde{T}.$$

We say that $I^\flat$ is *narrowly family reflexive* if, for all $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$I^\flat_\Sigma[\phi,\phi] \le T.$$

It is shown that these two properties are identical. Similar results are proven for rough/narrow family symmetry, rough/narrow family transitivity, rough/narrow family compatibility and rough/narrow family modus ponens, which are all defined analogously. We say that a π-institution $\mathcal{I}$ is *syntactically roughly/narrowly family monotone* if there exists a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies narrow family reflexivity, narrow family transitivity, narrow family compatibility and narrow family modus ponens. If $\mathcal{I}$ is syntactically narrow family monotone, with witnessing transformations $I^\flat$, then, for every $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, $\overleftarrow{\overleftrightarrow{I^\flat}}(T)$ is a congruence system on $\mathbf{F}$ compatible with T. This allows showing that $\overleftarrow{\overleftrightarrow{I^\flat}}(T) = \Omega(T)$, that is, $I^\flat$ narrowly defines Leibniz congruence systems of theory families in $\mathcal{I}$. It follows that syntactic narrow family monotonicity implies (semantic) narrow family monotonicity.

To provide an intrinsic characterization of syntactic narrow family monotonicity, we introduce the *rough/narrow reflexive core* of $\mathcal{I}$. It may be defined in two apparently different ways, one from the rough and one from the narrow point of view, which, however, turn out to be equivalent. The *rough reflexive core* $\widetilde{R}^{\mathcal{I}}$ consists of all those natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ, $\rho^\flat_\Sigma[\phi,\phi] \le \widetilde{T}$. The *narrow reflexive core* $R^{\mathcal{I}\not\,}$ is defined similarly, but quantifying T over $\mathrm{ThFam}^{\not\,}(\mathcal{I})$ and replacing the inequality by $\rho^\flat_\Sigma[\phi,\phi] \le T$. We have that $\widetilde{R}^{\mathcal{I}} = R^{\mathcal{I}\not\,}$. We show that, for all $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\not\,}}(T)$ is a reflexive, symmetric relation system on $\mathbf{F}$ that has the compatibility property. Thus, $\overleftarrow{R^{\mathcal{I}\not\,}}(T)$ only falls short of becoming a congruence system compatible with T because of $R^{\mathcal{I}\not\,}$ possibly lacking the narrow family transitivity and the narrow family modus ponens properties. If $\mathcal{I}$ happens to be syntactically narrow family monotone, then $R^{\mathcal{I}\not\,}$ has the narrow family modus ponens and, as a consequence, it is also narrow family transitive. Conversely, if $R^{\mathcal{I}\not\,}$ has the narrow family modus ponens, then the π-institution $\mathcal{I}$ is syntactically family narrow monotone, with witnessing transformations $R^{\mathcal{I}\lightning}$. That is, the narrow family modus ponens property of $R^{\mathcal{I}\lightning}$ intrinsically characterizes syntactic narrow family monotonicity. Additionally, $\mathcal{I}$ is syntactically narrowly family monotone if and only if $R^{\mathcal{I}\lightning}$ narrowly defines Leibniz congruence systems of theory families, in the sense that, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\lightning}}(T) = \Omega(T)$.

The second part of Section 15.2 deals with devising a characterization of syntactic narrow family monotone π-institutions as a subclass of narrow family monotone π-institutions. Recalling the case of syntactic protoalgebraicity versus protoalgebraicity, where the separation was attained by the Leibniz property of the reflexive core $R^{\mathcal{I}}$ of the π-institution $\mathcal{I}$, we must devise here an analogous property for the narrow reflexive core $R^{\mathcal{I}\lightning}$. Due to the fact that we are dealing with theory families with nonempty components, this analog involves additional complications. First, we define, for every signature Σ and all Σ-sentences ϕ, ψ, the collection $[R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$ of all theory families $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, which contain $R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi]$. We then set $\min[R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$ for the family of all minimal elements in $[R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$. Note that, if $C(R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi]) \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, then $\min[R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi]) = \{C(R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])\}$. Thus, this construct is meant to handle the difficulty encountered in the special case in which $C(R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$ happens to have some empty components. We say that the narrow reflexive core $R^{\mathcal{I}\lightning}$ is *Leibniz* if, for all signatures Σ, all Σ-sentences ϕ, ψ and all $T \in \min[R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. It turns out that if $R^{\mathcal{I}\lightning}$ has the narrow family modus ponens, then it is Leibniz. Ideally, to have the prototypical example of syntactic protoalgebraicity versus protoalgebraicity applicable without changes, we would like to have that, under narrow family monotonicity, $R^{\mathcal{I}\lightning}$ being Leibniz implies that it has the narrow family modus ponens. This presents some obstacles and we are only able to advance under an additional hypothesis on the π-institution $\mathcal{I}$, thus obtaining only a partial result. Given a signature Σ and Σ-sentences ϕ, ψ, $\mathcal{I}$ is called *$\langle\Sigma,\phi,\psi\rangle$-reflexively covered* if, for all $T \in [R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$, there exists $T' \in \min[R^{\mathcal{I}\lightning}_\Sigma[\phi,\psi])$, such that $T' \le T$. Moreover, we say that $\mathcal{I}$ is *reflexively covered* if, for all signatures Σ and all Σ-sentences ϕ, ψ, $\mathcal{I}$ is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered. We do show that, if $\mathcal{I}$ is reflexively covered, and narrowly family monotone, then $R^{\mathcal{I}\lightning}$ being Leibniz implies that $R^{\mathcal{I}\lightning}$ has the narrow family modus ponens. This enables us to show that, for reflexively covered π-institutions $\mathcal{I}$, $\mathcal{I}$ is syntactically narrowly family monotone if and only if it is narrowly family monotone and has a Leibniz narrow reflexive core.

In Section 15.3, we study *syntactic narrow system monotonicity*, a syntactic analog of narrow system monotonicity. We start by relativizing system reflexivity, system symmetry, system transitivity, system compatibility and system modus ponens of a collection $I^\flat$ of natural transformations to the collection $\mathrm{ThSys}^{\lightning}(\mathcal{I})$ of theory systems without empty components, thus obtaining the corresponding narrow versions of these properties. A π-institution $\mathcal{I}$ is called *syntactically narrowly system monotone* if there exists a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, which is narrowly system reflexive, narrowly system transitive, has the narrow system compatibility and satisfies the narrow system modus ponens. If $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T) = \Omega(T)$, that is, $I^\flat$ narrowly defines Leibniz congruence systems of theory systems in $\mathcal{I}$. This yields that syntactic narrow system monotonicity implies narrow system monotonicity, its semantic counterpart.

The introduction of the *narrow system reflexive core* serves, among other things, to intrinsically characterize syntactic narrow system monotonicity. The *narrow system reflexive core* $R^{\mathcal{I}s}$ of $\mathcal{I}$ consists of all natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$\rho^\flat_\Sigma[\phi,\phi] \le T.$$

It turns out that, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a reflexive and symmetric relation system on $\mathbf{F}$ and satisfies the compatibility property. Moreover, if $\mathcal{I}$ happens to be syntactically narrowly system monotone, then $R^{\mathcal{I}s}$ also satisfies the narrow system modus ponens and, as a result, it is narrowly system transitive as well. Conversely, if $R^{\mathcal{I}s}$ has the narrow system modus ponens, then $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $R^{\mathcal{I}s}$. Thus, $R^{\mathcal{I}s}$ having the narrow system modus ponens intrinsically characterizes syntactic narrow system monotonicity.

In the second part of Section 15.3, we use a Leibniz like property of the narrow system reflexive core in order to provide a partial characterization of syntactically narrowly system monotone π-institutions inside the class of narrowly system monotone π-institutions. Since the steps are similar to those employed in Section 15.2, which were described in some detail above, we only give a rough outline. We again define, for every signature Σ and all Σ-sentences ϕ,ψ, the collection

$$[R^{\mathcal{I}s}_\Sigma[\phi,\psi]) = \{T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi,\psi] \le T\}.$$

We then set $\min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$ to be the subcollection of its minimal elements. We say that the narrow system reflexive core $R^{\mathcal{I}s}$ is *Leibniz* if, for all signatures Σ, all Σ-sentences ϕ,ψ and all $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. We show that, if $R^{\mathcal{I}s}$ has the narrow system modus ponens, then it is Leibniz. Ideally, we would have liked to be able to show that, provided that $\mathcal{I}$ is narrowly system monotone, the converse implication also holds. However, we are able to show this only for a certain subclass of π-institutions. For a fixed signature Σ and fixed Σ-sentences ϕ,ψ, we say that $\mathcal{I}$ is $\langle\Sigma,\phi,\psi\rangle$-*system reflexively covered* if, for all $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, there exists $T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, such that $T' \leq T$. $\mathcal{I}$ is called *system reflexively covered* if it is $\langle \Sigma, \phi, \psi \rangle$-system reflexively covered, for all signatures Σ and all Σ-sentences ϕ, ψ. We do show that, for system reflexively covered π-institutions $\mathcal{I}$, if $\mathcal{I}$ is narrowly system monotone and $R^{\mathcal{I}s}$ is Leibniz, then $R^{\mathcal{I}s}$ has the narrow system modus ponens in $\mathcal{I}$. As a consequence, we get that, for system reflexively covered π-institutions $\mathcal{I}$, $\mathcal{I}$ is syntactically narrowly system monotone if and only if it is narrowly system monotone and has a Leibniz narrow system reflexive core.

In Section 15.4, we turn to a third type of syntactic narrow monotonicity. We say that a π-institution $\mathcal{I}$ is *syntactically narrowly right monotone* if there exists a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, that satisfies narrow right reflexivity, narrow right transitivity, narrow right compatibility and narrow right modus ponens. The meaning of "right" in these terms is that, in any of them, the theory family T, universally quantified over $\mathrm{ThFam}^{\lightning}(\mathcal{I})$, is replaced in the relevant condition by $\overleftarrow{T}$. E.g., $I^\flat$ is *narrowly right transitive* if, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ, ψ, χ, $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}$ and $I^\flat_\Sigma[\psi, \chi] \leq \overleftarrow{T}$ imply $I^\flat_\Sigma[\phi, \chi] \leq \overleftarrow{T}$. *Narrow right reflexivity*, *narrow right symmetry* and *narrow right compatibility* are defined similarly. On the other hand, *narrow right modus ponens* is defined by stipulating that, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ, ψ, $\phi \in \overleftarrow{T}_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}$ imply $\psi \in \overleftarrow{T}_\Sigma$. It turns out that all properties above, besides narrow right modus ponens, are equivalent to their narrow family counterparts. Hence, the only feature that differentiates syntactic narrow right monotonicity from syntactic narrow family monotonicity is the use of narrow right modus ponens instead of narrow family modus ponens.

If $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $I^\flat$, then $I^\flat$ narrowly defines Leibniz congruence systems of theory families up to arrow, in the sense that, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T) = \Omega(\overleftarrow{T})$. This yields that syntactic narrow right monotonicity implies (semantic) narrow right monotonicity, its semantic counterpart. Next, we reuse the narrow reflexive core $R^{\mathcal{I}\lightning}$ of $\mathcal{I}$, introduced in Section 15.2, to intrinsically characterize syntactic narrow right monotonicity. We show that, if $\mathcal{I}$ is syntactically narrowly right monotone, then $R^{\mathcal{I}\lightning}$ has the narrow right modus ponens. Moreover, if $R^{\mathcal{I}\lightning}$ has the narrow right modus ponens, then it is narrowly right transitive. This allows showing that, if $R^{\mathcal{I}\lightning}$ has the narrow right modus ponens, then $\mathcal{I}$ is syntactically narrowly right monotone. Thus, $R^{\mathcal{I}\lightning}$ having the narrow right modus ponens characterizes syntactic narrow right monotonicity. The section ends with a characterization, also using the narrow reflexive core, of those narrowly right monotone π-institutions that are syntactically narrowly right monotone. Here the deciding condition is that the narrow reflexive core be *right Leibniz*. We omit the details of this characterization, since they parallel those described previously for both syntactic narrow family and syntactic narrow system monotonicity.

1.3.15 Chapter 16

In Section 3.3, we studied the classes of prealgebraic and of protoalgebraic π-institutions. In Sections 11.2 and 11.3, we studied their syntactic counterparts, syntactically prealgebraic and syntactically protoalgebraic, π-institutions, respectively. In Section 5.4, we studied strengthenings of the semantic properties, obtaining the classes of preequivalential and equivalential π-institutions, whereas in Sections 13.3 and 13.4, we looked at their syntactic counterparts, syntactically preequivalential and syntactically equivalential π-institutions. On the other hand, in Section 8.2, we studied semantic regularity and in Section 8.3, we studied assertionality. Section 16.2 will introduce the corresponding syntactic notion of regularity and Section 16.5 will deal with syntactic assertionality. In Sections 8.4, 8.5, 8.6 and 8.7, combining the basic semantic notions listed above, we obtained four classes close to the very top of the semantic algebraic hierarchy, namely those of regularly weakly prealgebraizable and regularly weakly algebraizable π-institutions and those of regularly prealgebraizable and of regularly algebraizable π-institutions. Using the syntactic analogs of those properties, we shall define syntactic analogs of these four classes. In Section 16.6, we get syntactically regularly weakly prealgebraizable π-institutions, in Section 16.7, syntactically regularly weakly algebraizable π-institutions and in Section 16.8, syntactically regularly prealgebraizable and syntactically regularly algebraizable π-institutions.

In Section 16.2, we study various versions of the regularity property of a collection $I^\flat$ of natural transformations, with two distinguished arguments, in a π-institution $\mathcal{I}$. There are four apparently different ones, family, left, right and system regularity. E.g., we say that $I^\flat$ is *family regular* in $\mathcal{I}$ if, for every theory family T, all signatures Σ and all Σ-sentences ϕ,ψ, $\phi,\psi \in T_\Sigma$ imply $I^\flat_\Sigma[\phi,\psi] \leq T$. It is shown that family and right regularity coincide, as do left and system regularity. So we fix the terms *family regularity* and *system regularity* for the two genuinely different regularity properties. Family regularity implies system regularity and these two coincide if the π-institution happens to be systemic. Finally, it is proven that these properties transfer from a π-institution $\mathcal{I}$ to all models of the form $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$.

In Section 16.3, we combine the properties of syntactic pre/protoalgebraicity with those of regularity of the witnessing transformations to get the classes of syntactically regularly pre/protoalgebraic π-institutions. Since we have two versions of syntactic monotonicity (pre- and protoalgebraicity) and two versions of regularity of natural transformations, we obtain, a priori, four versions of syntactic regular pre/protoalgebraicity. These are termed *syntactic family regular protoalgebraicity*, *syntactic system regular protoal-*

gebraicity, *syntactic family regular prealgebraicity* and *syntactic system regular prealgebraicity*. E.g., we say that a π-institution $\mathcal{I}$ is *syntactically family regularly protoalgebraic* if it is syntactically protoalgebraic, with witnessing collection of natural transformations $I^\flat$, which is family regular in $\mathcal{I}$. Now recall from Section 8.2 the semantic regularity hierarchy, which consists of four properties, which are, enumerated from strongest to weakest, family regularity, right regularity, left regularity and system regularity. Connecting them to the present context, we show that, if a π-institution $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $I^\flat$, then $\mathcal{I}$ is family regular if and only if $I^\flat$ is family regular in $\mathcal{I}$ and it is left regular if and only if $I^\flat$ is system regular in $\mathcal{I}$. On the other hand, provided that $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, then $\mathcal{I}$ is right regular if and only if $I^\flat$ is family regular in $\mathcal{I}$ and it is system regular if and only if $I^\flat$ is system regular in $\mathcal{I}$. These equivalences, together with the forms of the subhierarchies of syntactic pre/protoalgebraicity, family/system regularity of natural transformations and family/right/left/system regularity of π-institutions lead to the subhierarchy of syntactic regular pre/protoalgebraicity properties. This turns out to be a rhombus, with syntactic family regular protoalgebraicity at the top, syntactic family regular prealgebraicity and syntactic system regular protoalgebraicity below it and syntactic system regular prealgebraicity at the bottom. Besides identifying its shape, the section also provides some relationships between classes of the hierarchy and those syntactic classes lying right below it and, also, some relationships with their semantic counterparts. We then recall, also from Section 8.2, the three characterizations of regularity properties involving the Suszko operator. For a π-institution $\mathcal{I}$, $\mathcal{I}$ is family regular if and only if, for all signatures Σ and all Σ-sentences ϕ,ψ, $\langle\phi,\psi\rangle\in\widetilde{\Omega}^{\mathcal{I}}(C(\phi,\psi))$. For left regularity, the condition is replaced by $\langle\phi,\psi\rangle\in\widetilde{\Omega}^{\mathcal{I}}(\overrightarrow{C}(\phi,\psi))$, whereas for system regularity by $\langle\phi,\psi\rangle\in\widehat{\Omega}^{\mathcal{I}}(\overrightarrow{C}(\phi,\psi))$. These characterizations enable us to obtain corresponding ones for family regularity of a collection of witnessing transformations in case of syntactic protoalgebraicity and for system regularity of witnessing transformations in case of syntactic prealgebraicity. E.g., if $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $I^\flat$, then $I^\flat$ is family regular if and only if, for all signatures Σ and all Σ-sentences ϕ,ψ, $\langle\phi,\psi\rangle\in\widetilde{\Omega}^{\mathcal{I}}(C(\phi,\psi))$. The section ends with a transfer theorem asserting that all four versions of syntactic regular pre/protoalgebraicity transfer from a π-institution to all $\mathcal{I}$-models of the form $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$.

In Section 16.4, we upgrade syntactic pre- and protoalgebraicity to syntactic preequivalentiality and equivalentiality, respectively, and combining them, once more, with regularity properties of witnessing transformations, we get syntactic regular preequivalentiality and syntactic regular equivalentiality properties, respectively. We again obtain, a priori, four versions, *syntactic family regular equivalentiality*, *syntactic system regular equivalential-*

ity, *syntactic family regular preequivalentiality* and *syntactic system regular preequivalentiality*. E.g., we say that a π-institution $\mathcal{I}$ is *syntactically family regularly equivalential* if it is syntactically equivalential, with witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$. If $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat$, then $\mathcal{I}$ is family regular if and only if $I^\flat$ is family regular in $\mathcal{I}$ and $\mathcal{I}$ is left regular is and only if $I^\flat$ is system regular in $\mathcal{I}$. Similarly, if $\mathcal{I}$ is syntactically preequivalential, then $\mathcal{I}$ is right regular if and only if $I^\flat$ is family regular in $\mathcal{I}$ and $\mathcal{I}$ is system regular if and only if $I^\flat$ is system regular in $\mathcal{I}$. The four classes of the hierarchy are stratified in a rhombus, with syntactic family regular equivalentiality at the apex, syntactic family regular preequivalentiality and syntactic system regular equivalentiality below it and syntactic system regular preequivalentiality at the bottom. We detail how this subhierarchy is related to other syntactic properties lying right below it in the syntactic hierarchy. Further, we show that the rhombus of syntactic regular (pre)equivalentiality properties lies above (property by property) the rhombus of syntactic regular pre/protoalgebraicity properties, studied in Section 16.3. Similarly, with the case of syntactic pre/protoalgebraicity, we get that, under syntactic equivalentiality, the parameter free collection $I^\flat$ of witnessing transformations is family regular in $\mathcal{I}$ if and only if, for all signatures Σ and all Σ-sentences ϕ, ψ, $\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}(C(\phi, \psi))$, while, under syntactic preequivalentiality, the parameter free collection of witnessing transformations $I^\flat$ is system regular in $\mathcal{I}$ if and only if, for all signatures Σ and all Σ-sentences ϕ, ψ, $\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}(\overrightarrow{C}(\phi, \psi))$. This section also concludes with a theorem asserting that all four versions of syntactic regular (pre)equivalentiality properties transfer.

In Section 8.2, we introduced the four (semantic) regularity properties, ranging, in order from strongest to weakest, from family to right to left to system regularity. In Section 8.3, to those regularity properties was added the property of having theorems, which resulted into three classes of assertionality, family/right, left and system assertionality. These three classes were shown to dominate each the corresponding version of the complete reflectivity hierarchy. In Section 16.5, we replace existence of theorems by the stronger property of having natural theorems, studied in Section 14.7, thus obtaining the *syntactic assertionality* hierarchy. In fact, adding to (semantic) regularity the property of possessing natural theorems, allows replacing c-reflectivity by truth equationality, its syntactic counterpart, and this forms a strong motivation and justification for this move.

Since there are four regularity properties, we get, a priori, four classes of syntactic assertionality, syntactic family, left, right and system assertionality. However, as was the case with (semantic) assertionality, syntactic family and syntactic right assertionality coincide. Among the three different classes, syntactic family/right assertionality implies syntactic left assertionality, which, in turn, implies syntactic system assertionality. Additionally, syntactic left and syntactic system assertionality are identified if applied only to stable π-institutions and the whole hierarchy collapses to a single class if only systemic π-institutions are taken into account. By definition and the fact that possessing natural theorems implies having theorems, we get that each syntactic assertionality class dominates the corresponding semantic assertionality class. Based on previously obtained characterizations of the semantic assertionality classes, we obtain analogous characterizations of syntactic classes involving the Leibniz operator. More precisely, we get that a π-institution $\mathcal{I}$ is syntactically family assertional if and only if, for every theory family T, $T = \tau/\Omega(T)$, where τ is a natural theorem. In the corresponding characterization for syntactic left assertionality, the condition is replaced by $\overleftarrow{T} = \tau/\Omega(T)$, whereas for syntactic system assertionality, the first condition involves quantification over theory systems rather than theory families. It is also shown that each of these three syntactic assertionality classes lies above the corresponding truth equationality class, where, in each case, the witnessing equation of truth equationality is $\iota \approx \tau$, with ι the identity natural transformation and τ a (unary) natural theorem. The section closes with a transfer theorem to the effect that all syntactic assertionality properties transfer from a π-institution $\mathcal{I}$ to all models of the form $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$.

in Section 16.6, we combine syntactic prealgebraicity with the three versions of syntactic assertionality to obtain *syntactic regular weak prealgebraizability* classes, abbreviated to syntactic RW prealgebraizability. We say that a π-institution $\mathcal{I}$ is *syntactically regularly weakly family prealgebraizable* (syntactically RWF prealgebraizable) if it is syntactically prealgebraic and syntactically family assertional. It is *syntactically regularly weakly left prealgebraizable* (syntactically RWL prealgebraizable) if it is syntactically prealgebraic and syntactically left assertional and it is *syntactically regularly weakly system prealgebraizable* (syntactically RWS prealgebraizable) if it is syntactically prealgebraic and syntactically system assertional. Because of the implications governing the syntactic assertionality properties, described in Section 16.5, we obtain that syntactic RWF prealgebraizability implies syntactic RWL prealgebraizability, which implies syntactic RWS prealgebraizability.

As far as relationships with classes further down in the syntactic hierarchy go, we have that syntactic RWF prealgebraizability implies syntactic family regular protoalgebraicity and syntactic RWS prealgebraizability implies syntactic system regular prealgebraicity. Concerning relationships with classes in the syntactic assertionality hierarchy, we get that, if $\mathcal{I}$ is syntactically RW family (left, system) prealgebraizable, then it is syntactically family (left, system, respectively) assertional. Still staying with relationships with other syntactically defined classes, we also have that each class in the syntactic RW prealgebraizability hierarchy dominates the corresponding class in the syntactic weak prealgebraizability hierarchy. As regards relationships with semantically defined classes, we obtain that each syntactic RW prealgebraizability class is contained in the corresponding regularly weak prealgebraizability class. Having, at least partially, placed the syntactic RW prealgebraizability subhierarchy relative to its surroundings, we show that all its three versions transfer. We end the section by providing some characterization/transfer theorems in terms of isomorphisms between lattices of theory families and of congruence systems and of definability of theory families via their Leibniz congruence systems. With the intent of merely giving a flavor, we describe the result for syntactic RWF prealgebraizability. It is equivalent to $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ being an order isomorphism and $\mathcal{I}$ having a Leibniz reflexive core and a natural theorem τ, such that, for all theory families T, $T = \tau/\Omega(T)$.

If we switch from syntactic prealgebraicity to syntactic protoalgebraicity and add versions of syntactic assertionality, we pass from syntactic RW prealgebraizability to *syntactic regular weak algebraizability* (syntactic RW algebraizability). This constitutes the focus of Section 16.7. However, as will be described shortly, this section may be seen as an addendum to Section 16.6, since its end result is adding one more class between the top and middle classes of the three class hierarchy studied in Section 16.6. We say that a π-institution is *syntactically regularly weakly family algebraizable* (syntactically RWF algebraizable) if it is syntactically protoalgebraic and syntactically family assertional. It is *syntactically regularly weakly left algebraizable* (syntactically RWL algebraizable) if it is syntactically protoalgebraic and syntactically left assertional and it is *syntactically regularly weakly system algebraizable* (syntactically RWS algebraizable) if it is syntactically protoalgebraic and syntactically system assertional. Family assertionality implies systemicity and, under systemicity, syntactic prealgebraicity coincides with syntactic protoalgebraicity. So we get that syntactic RWF prealgebraizability and syntactic RWF algebraizability are identical properties. Thus, the top two classes in the hierarchies of Sections 16.6 and 16.7 coincide. Even more, due to the fact that protoalgebraicity implies stability, syntactic RWL algebraizability turns out to be identical with syntactic RWS algebraizability. Hence the syntactic regular weak algebraizability hierarchy consists only of two classes and, when merged with the three class syntactic regular weak prealgebraizability hierarchy of Section 16.6, gives a total of four linearly ordered classes.

Returning to the relations between different subhierarchies from Section 16.6, and in light of this new form, we show that each of the four classes lies above the corresponding syntactic regular pre/protoalgebraicity class. A similar picture emerges in relation to syntactic assertionality properties. Moving to relations with classes up higher in the syntactic hierarchy, we see that each of the four syntactic regular weak (pre)algebraizability properties implies the corresponding syntactic weak (pre)algebraizability property. Turning to relationships of the four classes with corresponding classes in the semantic regular weak (pre)algebraizability hierarchy, we see that the latter are dominated by the corresponding ones in the former. This section also concludes, as does Section 16.6, with a transfer theorem for the new class, as well as with a characterization theorem involving mappings between lattices of theory families and of congruence systems. These results complete the picture already appearing at the end of Section 16.6 for three of the four classes in the combined hierarchy of the two sections.

In Sections 16.6 and 16.7, we established the syntactic regular weak (pre)algebraizability hierarchy, which arises by combining syntactic pre/protoalgebraicity with syntactic assertionality properties. If, instead of syntactic pre/protoalgebraicity, syntactic (pre)equivalentiality is used, we get a hierarchy of four stronger classes, constituting the *syntactic regular (pre)algebraizability* subhierarchy. This is the main topic of Section 16.8. More precisely, if syntactic preequivalentiality is combined with syntactic family, syntactic left or syntactic system assertionality, we obtain *syntactic regular family*, *left* or *system prealgebraizability*, respectively. If, on the other hand, syntactic equivalentiality is combined with each of the three syntactic assertionality properties, we get the classes of *syntactically regularly family*, *left* or *system algebraizable* π-institutions. As with syntactic regular weak (pre)algebraizability, this apparent six class hierarchy collapses to a linear hierarchy of four properties, in order, from strongest to weakest, syntactic regular family algebraizability, syntactic regular system algebraizability, syntactic regular left prealgebraizability and syntactic regular system prealgebraizability. Each of these classes dominates a corresponding syntactic regular (pre)equivalentiality class. It is not difficult to see that each of the four classes lies above the corresponding syntactic regular weak (pre)algebraizability class and this yields, also, that it lies above the corresponding syntactic assertionality class. Further, each of the four classes lies above the corresponding syntactic (pre)algebraizability class. Compared with semantically defined classes, syntactic regular (pre)algebraizability classes dominate the corresponding regular (pre)algebraizability classes. The section ends with a transfer theorem and with characterization/transfer theorems using mappings between lattices of theory families and of congruence systems.

1.3.16 Chapter 17

Chapter 17 deals with the issue of finitarity for syntactically defined classes in the hierarchy. An analogous study from the semantic point of view was carried out in Chapter 9.

In Section 17.2, we revisit the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$, this time in relation to classes near the top of the syntactic hierarchy. Our goal is to discover conditions under which syntactic properties are inherited by $\mathcal{I}$ from $\mathcal{I}^f$ and vice versa. Recall that the finitary companion $\mathcal{I}^f$ of

$\mathcal{I} = \langle \mathbf{F}, C \rangle$ is the π-institution $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$, where, for all signatures Σ and all subsets Φ of Σ-sentences,

$$C^f_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

$\subseteq_f$ denoting the finite subset relation. $\mathcal{I}^f$ is the largest finitary π-institution lying below $\mathcal{I}$ in the $\leq$ ordering. This implies that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Due to this inclusion, many of the key properties related to the algebraic hierarchy that $\mathcal{I}^f$ possesses are inherited by $\mathcal{I}$. E.g., we show that, if $\mathcal{I}^f$ happens to be syntactically protoalgebraic, then so is $\mathcal{I}$. Similarly, if $\mathcal{I}^f$ happens to be truth equational, then so is $\mathcal{I}$. Moreover, under protoalgebraicity of $\mathcal{I}^f$, the Leibniz property is inherited from the reflexive core $R^{\mathcal{I}^f}$ of $\mathcal{I}^f$ by the reflexive core $R^{\mathcal{I}}$ of $\mathcal{I}$. Analogously, under c-reflectivity of $\mathcal{I}^f$, adequacy of the Suszko core $S^{\mathcal{I}^f}$ of $\mathcal{I}^f$ implies adequacy of the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$. These properties enable us to show that, if $\mathcal{I}^f$ is syntactically weakly family algebraizable, then so is $\mathcal{I}$.

In the second part of Section 17.2, we turn to some properties that transfer from $\mathcal{I}$ to $\mathcal{I}^f$ rather than in the opposite direction. In Chapter 9 it was shown that continuity of the Leibniz operator implies protoalgebraicity. Moreover, under this strengthening of protoalgebraicity, and the additional hypothesis that the category of signatures is finite, it was shown that $\mathcal{I}^f$ is also protoalgebraic. On the syntactic side, following along similar lines, we show that, if the category of signatures is finite and $\mathcal{I}$ is syntactically protoalgebraic, with a finite collection of parameter free witnessing transformations, then the Leibniz operator is continuous. We obtain that, under these hypotheses, $\mathcal{I}^f$ is also syntactically protoalgebraic with the same collection of witnessing transformations. To establish a similar result regarding truth equationality, instead of syntactic protoalgebraicity, we need, first, to ensure that the inverse $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ is well defined. This happens when, e.g., $\mathcal{I}$ is weakly family algebraizable. We use this hypothesis, together with the finiteness of the category of signatures and the truth equationality of $\mathcal{I}$ via a finite, parameter free collection of witnessing equations, to get that Ω^{-1} is continuous. This also yields that, under the same hypotheses, $\mathcal{I}^f$ is also family truth equational, with the same collection of witnessing equations. These two results combined allow us to show that, if $\mathcal{I}$ has a finite signature category and is syntactically strongly family algebraizable, via a conjugate pair of finite collections of natural transformations, then $\mathcal{I}^f$ is also syntactically strongly family algebraizable via the same conjugate pair of transformations.

In Section 17.3, we look at *natural finitarity*. A π-institution is finitary if $\mathcal{I} = \mathcal{I}^f$, that is, if, for every signature Σ and all subsets Φ of Σ-sentences,

$$C_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\}.$$

Equivalently, for all signatures Σ and all subsets $\Phi \cup \{\phi\}$ of Σ-sentences, $\phi \in C_\Sigma(\Phi)$ implies that $\phi \in C_\Sigma(\Phi')$, for some $\Phi' \subseteq_f \Phi$. On the other hand,

$\mathcal{I}$ is *naturally finitary* if, for all collections μ, ν of natural transformations in $N^\flat$, with $|\mu|$ finite,

$$\mu \leq C(\nu) \quad \text{implies} \quad \mu \leq C(\nu'), \text{ for finite } \nu' \subseteq \nu,$$

where the notation $\mu \leq C(\nu)$ means that, for all signatures Σ and all Σ-sentences $\vec{\phi}$, $\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}])$. If a π-institution $\mathcal{I}$ is naturally finitary and syntactically family algebraizable via a finite witnessing family $I^\flat$ of natural transformations, then every such family possesses a finite witnessing subfamily. Dually, if $\mathcal{I}$ is syntactically strongly family algebraizable, with a naturally finitary equivalent equational π-structure $\mathcal{Q}$ via a finite witnessing family $\tau^\flat$ of equations, then every witnessing family of equations possesses a finite witnessing subfamily. These results allow showing that, for a naturally finitary, syntactically strongly algebraizable π-institution $\mathcal{I}$, every witnessing collection of equations contains a witnessing finite subcollection and that, dually, if $\mathcal{I}$ is syntactically strongly family algebraizable, with naturally finitary equational counterpart $\mathcal{Q}$, then every witnessing collection of natural transformations contains a finite witnessing subcollection. Subject to $\mathcal{I}$ having a finite category of signatures, being finitary and syntactically family algebraizable, if $\mathcal{I}$ has a finite, parameter free witnessing set of transformations, then its equational counterpart $\mathcal{Q}$ may be taken to be finitary. Similarly, if finitarity is replaced by natural finitarity. Dually, under the hypotheses that $\mathcal{I}$ has a finite category of signatures, is syntactically strongly family algebraizable, with a finitary equational counterpart, via a finite, parameter free collection of equations, then $\mathcal{I}$ is finitary. And, similarly, with natural finitarity in lieu of finitarity. The section concludes with a theorem summarizing the results obtained and an accompanying diagram providing a schematic summary. In a nutshell, one deals with a π-institution, with a finite category of signatures, that is syntactically strongly family algebraizable via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$. In case $\tau^\flat, I^\flat$ are finite, then $\mathcal{I}$ is naturally finitary if and only if $\mathcal{Q}$ is. If $\mathcal{I}$ is naturally finitary, then $\mathcal{Q}$ is naturally finitary if and only if $I^\flat$ can be taken finite and, dually, if $\mathcal{Q}$ is naturally finitary, then $\mathcal{I}$ is naturally finitary if and only if $\tau^\flat$ can be taken finite. And in each of these three biconditionals, if the equivalent alternatives hold, then all four "finitarity" conditions hold.

1.4 A Very Concise Summary of Contents

In Chapter 2, we introduce the basic definitions and fundamental results of algebra and logic and some indispensable notions and results pertaining to their interaction. These form the necessary background and the prerequisites for the general theory of algebraization of logics formalized as π-institutions that is presented in the monograph.

In Chapter 3, we introduce fundamental classes of the semantic Leibniz hierarchy. The term semantic alludes to the fact that they are defined purely by properties of the Leibniz operator on the complete lattices of the theory families or theory systems of π-institutions. Very central to our studies throughout, partly because they equip us with indispensable terminology regarding crucial properties, are the classes of systemic and stable π-institutions. At the bottom center of the hierarchy lie the loyalty properties. These simultaneously abstract monotonicity properties, on the one side, and reflectivity properties, on the other side. On the monotonicity side, we study monotonicity and two kinds of complete monotonicity, complete $\bigcup$-monotonicity, using the union operation, and complete $\bigvee$-monotonicity, using the join operation. In crossing over to the reflectivity side, we pass through, and study, injectivity properties. On the other side, we look, first, at reflectivity and, finally, at complete reflectivity properties. In Chapter 3, we not only define various flavors of each of these properties and compare their various strengths, but we also investigate the relations across those different kinds of properties. On the way, we also present many concrete examples, some of which are reused throughout the monograph to illustrate concepts, but, also - and mainly - to separate classes in the various hierarchies.

In Chapter 4, we study weak prealgebraizability and weak algebraizability properties. Weak prealgebraizability arises by combining prealgebraicity (system monotonicity) with one of the ten possible versions of injectivity, reflectivity or complete reflectivity. On the other hand, weal algebraizability results when combining protoalgebraiciy (family monotonicity) with one of those ten versions. Taking into account the combined hierarchy of injectivity, reflectivity and complete reflectivity properties, established in Chapter 3, we obtain a hierarchy of ten potentially different classes of weak prealgebraizability and a similar one consisting of ten potentially different classes of weak algebraizability. However, it is shown that the weak prealgebraizability hierarchy collapses down to six classes, whereas the one of weak algebraizability down to only two. Moreover, the top classes in the two hierarchies are identical. Therefore, when the two hierarchies are merged, a combined hierarchy consisting of seven distinct classes is obtained. The chapter includes, inter alia, characterizations of these seven classes using the Leibniz operator perceived as a mapping from the lattice of filter families to the poset of congruence systems over arbitrary algebraic systems.

In Chapter 5, we study the hierarchies of prealgebraizable and of algebraizable π-institutions. We look, first, at the property of extensionality and the seemingly weaker property of 2-extensionality and show that they are equivalent. Roughly speaking, extensionality relates Leibniz congruence systems of theories of an institution with those of corresponding theories of subinstitutions. Then, we look at the closely related properties of (Leibniz) commutativity and inverse (Leibniz) commutativity. These two properties are equivalent under monotonicity and, moreover, inverse commutativity is equivalent to extensionality. By combining monotonicity with extensionality properties, we build the hierarchy of equivalential π-institutions. Depending on which of the available versions of monotonicity or extensionality are imposed, three versions of equivalentiality arise, namely, equivalentiality, family preequivalentiality and (system) preequivalentiality in decreasing strength. By combining versions of preequivalentiality with injectivity, reflectivity or complete reflectivity properties, the ten classes of the prealgebraizability hierarchy are obtained. Similarly, by combining equivalentiality with injectivity properties (which are, in the presence of equivalentiality, equivalent to corresponding reflectivity or complete reflectivity properties), we get the two classes of algebraizable π-institutions.

In Chapter 6, we look at classes of the Leibniz hierarchy lying below the classes of injective, reflective and completely reflective π-institutions, which were introduced in Chapter 3. The motivating observation is that, if a π-institution satisfies injectivity or, a fortiori, reflectivity or complete reflectivity, then it must possess theorems. Thus, π-institutions without theorems are automatically excluded from consideration in contexts where these properties are postulated or studied. To bypass this hurdle, we define and study weakened versions of injectivity, reflectivity and complete reflectivity that can accommodate absence of theorems, but are equivalent to injectivity, reflectivity and complete reflectivity, respectively, in the presence of theorems. For each of those three properties, we study the rough versions and the narrow versions and carefully compare them to the original versions, as well as to each other, to obtain the hierarchies of injectivity, rough injectivity, narrow injectivity, reflectivity, rough reflectivity and narrow reflectivity, and c-reflectivity, rough c-reflectivity and narrow c-reflectivity classes of π-institutions. Roughly speaking, roughness identifies two theory families if their Σ-components are either equal or one is $\varnothing$ and the other is $\mathrm{SEN}^\flat(\Sigma)$. Those turn out to have identical Leibniz congruence systems. On the other hand, narrowness excludes from consideration altogether theory families with at least one empty component.

In Chapter 7, we continue the study of properties of π-institutions obtained by combining properties lying at the bottom of the Leibniz hierarchy with rough equivalence, on the one hand, and with narrowness, on the other. As opposed to Chapter 6, which considered properties lying below injectivity, reflectivity and complete reflectivity, this chapter undertakes the study of properties lying below monotonicity and complete monotonicity (c-monotonicity) properties. In a nutshell, roughly monotone and roughly c-monotone π-institutions form super classes, respectively, of the classes of monotone and c-monotone π-institutions. Additionally, narrowly monotone and narrow c-monotone π-institutions encompass respectively, roughly monotone and roughly c-monotone ones. By studying all four versions of each of these properties, we obtain a mixed hierarchy of rough and narrow monotonicity and rough and narrow c-monotonicity properties.

In Chapter 8, we study properties obtained by combining pre- or protoalgebraicity or (pre)equivalentiality, on the one hand, with assertionality, on the other. The latter, a property that strengthen complete reflectivity asserts, roughly speaking, that a π-institution has theorems and, in addition, each of its theory families is determined by its associated Leibniz congruence system as the equivalence class of a theorem. The chapter starts with the study of regularity, a property similar to assertionality, except that it does not require existence of theorems. It holds when any two sentences belonging to a theory family are identified modulo the Leibniz congruence system relative to that theory family. Assertionality properties are formalized next. The hierarchy they form and its interrelationships with the classes of the regularity hierarchy are explored in detail. Prealgebraicity, coupled with asserrtionality, gives rise to regular weak prealgebraizability, strengthening the classes of weak prealgebraizability properties. Protoalgebraicity, together with assertionality, leads to regular weak algebraizability properties. This hierarchy strengthens both regular weak prealgebraizability and weak algebraizability properties. Preequivalentiality and assertionality give rise to regular prealgebraizability, which strengthens both regular weak prealgebraizability and prealgebraizability. The chapter concludes with the study of regular algebraizability, which combines equivalentiality with assertionality. The classes of this hierarchy form subclasses of both those consisting of regularly prealgebraizable and those consisting of algebraizable π-institutions.

Chapter 9 starts with the introduction of the finitary companion of a π-institution. It is the largest finitary π-institution below the given one in the $\leq$ ordering of π-institutions based on the same algebraic system. The focus is on those properties defining weak family algebraizability, namely protoalgebraicity and family reflectivity. We investigate under which conditions, if any, those properties are passed from a π-institution to its finitary companion and vice-versa. In the second part, the focus shifts to the study of finitarity properties of weakly family algebraizable π-institutions. This class of π-institutions is chosen because, on its members, the Leibniz operator is an isomorphism and, hence, it makes sense to consider the inverse Leibniz operator. The four finitarity properties under investigation are the finitarity of the π-institution itself, the finitarity of its algebraic counterpart and the continuity of the Leibniz operator and of the inverse Leibniz operator. The implications holding between these properties give rise to the finitarity hierarchy of weakly family algebraizable π-institutions. The chapter also revisits some examples of sentential logics and formalizes them as π-institutions. The latter are then used to separate the various classes in the finitarity hierarchy. The three examples are Łukasiewicz's infinite valued logic, Dellunde's logic and a logic due to Raftery.

In Chapter 10, the focus shifts from semantical properties of π-institutions, that is, properties based on the behavior of the Leibniz operator applied on their theory families, to syntactical properties. These are properties that involve natural transformations on the sentence functors. Those natural transformations are perceived as corresponding to formulas in the traditional framework, based on a formalism owing to Lawvere's algebraic theories [10]. Chapter 10 studies properties of those sets of natural transformations themselves, modulo the theories of a π-institution, whereas their impact on classifying π-institutions, in complete analogy with the semantics studies of Chapters 3-9, is presented in Chapters 11-17, respectively. The properties, which a collection of natural transformations may have in a π-institution, that we study here are reflexivity, symmetry, transitivity and equivalence, based on the previous three; Antisymmetry and order, based of reflexivity, antisymmetry and transitivity. Compatibility and congruence, based on equivalence and compatibility; Modus ponens and syntactic protoalgebraicity, based on congruence and modus ponens; invertibility and syntactic algebraizability, based on syntactic protoalgebraicity and invertibility; Regularity and syntactic regularity, based on syntactic protoalgebraicity and regularity; Modus fortis and the Rasiowa property, based on syntactic protoalgebraicity and modus fortis. We study several versions of most of those properties. These versions arise by considering two fundamental dualities. The first is the theory family/system duality, that is whether one wants to take into account all theory families of the π-institution or restrict attention only to theory systems. The second is the local/global duality, that is, whether one is allowed to use signature morphisms to switch signatures (global perspective) or only local behavior at each signature is considered (local perspective). The various versions of each property are ordered in terms of their relative strength, thus forming a subhierarchy for that property. Some hierarchies, however, involve versions of a few of the properties considered simultaneously. The chapter also considers the question of whether the properties considered transfer from the π-institution to its models.

In Chapter 11, we undertake the study of syntactic properties corresponding to some of the most important semantic properties studied in Chapter 3. On the one hand, we study syntactic prealgebraicity and syntactic protoalgebraicity, which correspond to (semantic) prealgebraicity and (semantic) protoalgebraicity, respectively. We use a collection of natural transformations intrinsic to the π-institution under study, called the reflexive core, to provide characterizations of these properties. We show that syntactic prealgebraicity is equivalent to the reflexive core having the global system modus ponens and syntactic protoalgebraicity to the reflexive core having the global family modus ponens. We also use the reflexive core to relate the semantic with the syntactic notions. Along these lines we show that a π-institution $\mathcal{I}$ is syntactically prealgebraic if and only if it is prealgebraic and its reflexive core is Leibniz and that it is syntactically protoalgebraic if and only if it is protoalgebraic and its reflexive core is Leibniz. On the other hand, we study family, left and system truth equationality, syntactic properties that correspond to family, left and system c-reflectivity, respectively. In order to characterize these properties, we also use some collections of natural transformations intrinsic to the π-institution, namely, the Suszko core, the left Suszko core and the system core. It is shown that $\mathcal{I}$ is family truth equational if and only if it has a soluble Suszko core, is left truth equational if and only if it has a left soluble left Suszko core and system truth equational if and only if it has a soluble system core. We use the same three constructs, albeit with different associated properties, to relate the syntactic with the semantic counterparts. We show that $\mathcal{I}$ is family truth equational if and only if it is family c-reflective and has an adequate Suszko core, it is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core and, finally, that it is system truth equational if and only if it is system c-reflective and has an adequate system core.

In Chapter 12, we consider those syntactically defined classes of π-institutions that correspond to the semantic classes introduced in Chapter 4, occupying positions near the top of the algebraic hierarchy of π-institutions. From strongest to weakest, these are syntactically weakly family algebraizable, syntactically weakly algebraizable, syntactically weakly left c-reflectively prealgebraizable and syntactically weakly system prealgebraizable π-institutions. These correspond, respectively, to weakly family algebraizable, weakly algebraizable, weakly c-reflectively prealgebraizable and weakly system prealgebraizable π-institutions, all introduced and studied in Chapter 4. For each of these classes, we detail its relationship with its semantic and syntactic constituent properties and with its semantic counterpart. Furthermore, we provide a characterization based on correspondences between lattices of theory families and of congruence systems. Additionally, we delve into characterizations based on equivalences between logical and equational π-structures and on the relationships that these equivalences entail between the theory families of the logical structures and the theory families, i.e., the congruence systems, of the associated equational structures. We note that equational π-structures as well as the machinery to establish these equivalences between logical and equational π-structures, namely, translations, interpretations and conjugate pairs of interpretations, are developed in the first two preparatory sections of the chapter. In the sentential logic framework, these form the cornerstones of the seminal, groundbreaking theory of Algebraizable Logics of Blok and Pigozzi [36].

In Chapter 13, we study classes of π-institutions defined syntactically via the use of collections of natural transformations without parameters. After an introduction and some preliminaries, we focus on the properties of syntactic preequivalentiality and syntactic equivalentiality, which are the syntactic analogs of preequivalentiality and equivalentiality, respectively. They are determined using parameter free collections of natural transformations defining Leibniz congruence systems of theory systems and of arbitrary theory families, respectively. Then, we turn to the study of strong (family) truth equationality, strong left truth equationality and strong system truth equationality, which are also counterparts of semantic properties, namely, family c-reflectivity, left c-reflectivity and system c-reflectivity. These involve the parameter free definability of theory families/systems via their Leibniz congruence systems. In the second part of the chapter, we use these basic properties to build the hierarchies of syntactically prealgebraizable and syntactically algebraizable π-institutions, consisting each of six classes. Those classes constitute the syntactic analogs of the hierarchies of prealgebraizable and algebraizable π-institutions that were studied in Chapter 5.

Chapter 14 introduces a subhierarchy of rough and narrow truth equationality properties, which lies below the truth equationality hierarchy and forms a syntactic analog of the rough and narrow c-reflectivity subhierarchy. The hierarchy includes rough/narrow (family) truth equationality, rough left and narrow left truth equationality and rough system and narrow system truth equationality. In both rough and narrow properties, the goal is to avoid the deficit caused in relation to truth equationality by lack of theorems. The rough properties deal with the issue by replacing theory families with empty components by their rough equivalence representatives, whereas the narrow properties attain this by ignoring theory families with empty components altogether. For all classes, we provide intrinsic characterizations in terms of a solubility like property of a modified Suszko, left Suszko of system core associated with the π-institution in question. Moreover, we pinpoint the exact relationships of these classes with their semantic counterparts, the rough and narrow c-reflectivity properties, via adequacy type properties of the same associated cores. in the last section of the chapter we study the property of having natural theorems and establish its exact connection with the weaker property of having theorems using another core intrinsically associated to a given π-institution, termed the Lindenbaum core.

Chapter 15 introduces three versions of syntactic narrow monotonicity, which form syntactic analogs of the narrow monotonicity properties presented in Chapter 7. These are syntactic narrow family, syntactic narrow system and syntactic narrow right monotonicity. For each class, we provide an intrinsic characterization using a collection of natural transformations intrinsically associated with a π-institution, such as the narrow reflexive core and the narrow system reflexive core. Each of the three properties is stronger than its semantic counterpart. Thus, another task undertaken is pinpointing exact conditions that separate the semantic and the syntactic counterparts. In other words, we seek properties that characterize those members in the semantic narrow monotonicity class that are also in the corresponding syntactic narrow monotonicty class. In this task, where the narrow reflexive core also plays a crucial role, we are only partially successful.

In preceding chapters, we have introduced the classes of prealgebraic and protoalgebraic π-institutions, their strengthenings of preequivalential and equivalential π-institutions and the semantic regularity and assertionality properties. These properties have been used in various combinations to obtain the classes of regularly weakly prealgebraizable, regularly weakly algebraizable, regularly prealgebraizable and regularly algebraizable π-institutions, which constitute semantically defined classes at the very top tiers of the algebraic hierarchy. Chapter 16 introduces corresponding syntactically defined classes. Some of the constituent syntactic properties have been defined in preceding chapters, e.g., we have covered syntactic prealgebraicity and syntactic protoalgebraicity, as well as syntactic preequivalentiality and syntactic equivalentiality. This chapter introduces syntactic regularity and syntactic assertionality. Using these syntactic properties, we define syntactic analogs of the four aforementioned semantic classes, namely, syntactically regularly weakly prealgebraizable, syntactically regularly weakly algebraizable, syntactically regularly prealgebraizable and syntactically regularly algebraizable π-institutions. For each class, we provide its relationship with other syntactically and semantically defined classes further down in the hierarchy and we prove some characterization and transfer theorems.

Chapter 17 deals with issues of finitarity in syntactically defined classes of the algebraic hierarchy of π-institutions. The first part focuses on syntactic properties that transfer from a π-institution to its finitary companion and vice-versa. In the second part, we restrict attention to syntactically strongly family algebraizable π-institutions and look at the natural finitarity property of both the π-institution itself and of its equational counterpart and the finiteness of the witnessing collections of transformations and of equations. We provide several implications and equivalences involving combinations of those properties that clarify the landscape. The semantic side of this work is presented in Chapter 9.

1.5 Further Reading

This is the first attempt to systematize the body of knowledge gathered over the years concerning the algebraization of logics formalized as π-institutions. However, for the readers interested in learning much more about the origins, history, concepts, results and developments in algebraic logic as applied to deductive systems, i.e., "abstract algebraic logic", there are a few excellent sources available that have served well over the years in educating the second and third generations of "abstract algebraic logicians".

Starting tangentially to the subject, but of interest, since they provide a comprehensive study of logical calculi and of institutions, respectively, the latter being the precursors of π-institutions used here, are the monographs by Wójcicki [35] and Diaconescu [80].

Two of the first sources that played a critical role in establishing and solidifying the discipline in its present form were the seminal "Memoirs" monograph of Blok and Pigozzi [36], in which algebraizable logics were introduced, and the pioneering monograph of Font and Jansana [53], in which generalized matrices were studied in a systematic way and the notion of Tarski congruence and accompanying reduced class of generalized matrices and underlying class of algebras were defined and studied in detail.

More at the textbook, rather than at the research, level, are the books of Czelakowski [65] and the more recent textbook by Font [89]. These are the only two books, to my knowledge, that are focused on systematically treating and presenting the most important results in the abstract setting. It goes, of course, without saying, that they both contain a plethora of concrete examples that have been studied in the literature, showcasing various aspects of the general theory and exemplifying the wide reach of its applicability.

Apart from research monographs and books, a few surveys have also appeared that provide overviews of, and/or details on, significant parts of the theory. Among them are [41], [69], [70, 81] and [98].

Finally, there have been a few, as far as I am aware, Ph.D. Dissertations which have dealt, either in their introductions or in their main corpus, with expositions and/or overviews of significant parts of the theory. Among them, some that have helped my own understanding and enhanced and/or diversified my point of view of various aspects of the theory are, in chronological order, those of Herrmann [44], Elgueta [48], Rebagliato [50], Dellunde [52], Gyuris [61], Martins [71], Russo [79], Albuquerque [87] and Moraschini [90].

The algebraization of logics formalized as π-institutions may be said to have started with the Ph.D. Dissertation by the author [106] (see, also, [107]), under the influence of preceding unpublished work by Zinovy Diskin [47] (see, also, [52]), which had been communicated to Professor Don Pigozzi, the author's Ph.D. Dissertation advisor, and used with Zinovy's kind permission and encouragement.