Chapter 7

The Semantic Leibniz Hierarchy: Under the Bottom II

7.1 Introduction

In this chapter we continue the study of properties lying below properties in the bottom half of the classical Leibniz hierarchy [65, 89]. The underlying motivation is identical to that presented in the Introduction to, and governing the studies presented in, Chapter 6. Briefly, we note that, when one studies protoalgebraicity, no π-institution that is not almost inconsistent and does not have theorems can be considered. This is because such a π-institution has a theory family $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, i.e., with all its components nonempty, for which $\overline{\varnothing} \leq T$, whereas $\Omega(T) \leq \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$. Consequently, to incorporate nontrivial π-institutions without theorems in studies involving monotonicity properties of the Leibniz operator, one would have to devise ways to bypass, or otherwise suitably handle, theory families with one or more empty components. For properties involving reflectivity, which were handled in Chapter 6, this was done in the context of sentential logics in [90] (see, also, [97]). Here, we undertake a study similar to that presented in Chapter 6, but, instead of injectivity, reflectivity and complete reflectivity properties, we focus on monotonicity and complete monotonicity (c-monotonicity) properties.

In Section 7.2, we introduce some weakened versions of stability which serve in formalizing some of the properties studied later in the chapter. Recall from Section 3.2 that a π-institution $\mathcal{I}$ is *stable* if, for every theory family T of $\mathcal{I}$, $\Omega(\overleftarrow{T}) = \Omega(T)$. A first weakening is obtained by restricting the scope of the quantifier to theory families with all components nonempty. The ensuing property is termed *narrow stability*. A further weakening applies the condition only to those theory families T with all components nonempty which, in addition, satisfy that $\overleftarrow{T}$ has all its components nonempty. The resulting concept is termed *exclusive stability*. By definition, stability implies narrow stability, which implies exclusive stability and, as it turns out, both implications are actually strict.

In Section 7.3, we study *rough monotonicity properties*. These are the product of combining monotonicity properties with rough equivalence, introduced in Section 6.2. Rough equivalence formalizes an attempt at overcoming the hurdle imposed by theory families with empty components. Recall that two theory families are *roughly equivalent* if, whenever they differ at some signature Σ, one has Σ-component $\varnothing$ and the other $\mathrm{SEN}^\flat(\Sigma)$. Recall, also, that, given a theory family T, $\widetilde{T}$ denotes its *rough companion*, which results from T by replacing each of its empty Σ-components by $\mathrm{SEN}^\flat(\Sigma)$. Clearly $\widetilde{T}$ is roughly equivalent to T and, moreover, it is the largest theory family in the rough equivalence class $\widetilde{[T]}$ of T. All roughly equivalent theory families have identical Leibniz congruence systems. A π-institution $\mathcal{I}$ is called *roughly family monotone* if, for all theory families $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$. *Rough left monotonicity* results by replacing T and T' in the hypothesis by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. *Rough right monotonicity* is the result of the same replacement performed in the conclusion instead. *Rough system monotonicity* stipulates that $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$ hold for all theory systems T and T'. Rough left monotonicity implies both rough family and rough right monotonicity, and each of the latter two implies the system version. Additionally, rough left monotonicity is equivalent to the conjunction of rough system monotonicity and stability. Protoalgebraicity (which, recall from Section 3.3, names the equivalent notions of left and family monotonicity) implies rough left monotonicity. Prealgebraicity (naming the equivalent notions of right and system monotonicity), on the other hand, implies rough right monotonicity. But these interrelationships may be tied further, subject to some additional mild hypotheses. Namely, for non-almost inconsistent π-institutions, protoalgebraicity is equivalent to rough left or rough family monotonicity, coupled with availability of theorems. Moreover, for π-institutions possessing a theory family $T \neq \mathrm{SEN}^\flat$, with $\overleftarrow{T} \neq \overline{\varnothing}$, prealgebraicity is equivalent to rough right or rough system monotonicity, couple with availability of theorems. All four rough monotonicity properties transfer. E.g., a π-institution $\mathcal{I}$ is roughly right monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter families $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Finally, it is possible to recast rough family and rough system monotonicity in terms of the Leibniz operator viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$. The property one imposes is monotonicity, where, for rough equivalence classes $\widetilde{[T]}$, $\widetilde{[T']}$ in $\widetilde{\mathrm{ThFam}}(\mathcal{I})$, e.g., the order $\widetilde{[T]} \leq \widetilde{[T']}$ is the one induced by comparing the maximum elements $\widetilde{T} \leq \widetilde{T'}$ in the complete lattice of theory families of $\mathcal{I}$.

In Section 7.4, we look at *narrow monotonicity properties*. Narrowness is an alternative approach to roughness in dealing with theory families having one or more empty components. It literally bypasses theory families with empty components by altogether ignoring them and applying the relevant monotonicity conditions on the collections $\mathrm{ThFam}^{\natural}(\mathcal{I})$ and $\mathrm{ThSys}^{\natural}(\mathcal{I})$ of theory families and systems, respectively, all of whose components are nonempty. Accordingly, we say that a π-institution $\mathcal{I}$ is *narrowly family monotone* if, for all $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. In *narrow left monotonicity* T, T' in the hypothesis, are replaced by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, and the same substitution is applied in the conclusion, instead, for *narrow right monotonicity*. *Narrow system monotonicity* imposes the same condition as the family version, but restricts its scope to $T, T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$. Narrow left monotonicity implies narrow family monotonicity, which, in turn, implies narrow system monotonicity. The latter is also a consequence of narrow right monotonicity. The left version also implies exclusive stability, whereas the weakest version, i.e., narrow system monotonicity, supplemented by narrow systemicity, introduced in Section 6.3, implies both the left and right versions. Protoalgebraicity implies narrow left monotonicity and prealgebraicity implies narrow right monotonicity. As in the case of rough monotonicity properties, these connections may be strengthened under some fairly mild hypotheses. More precisely, for non almost inconsistent π-institutions, protoalgebraicity is equivalent to narrow left or narrow family monotonicity, augmented by existence of theorems. Similarly, for π-institutions possessing a theory system different from $\overline{\varnothing}$ and $\mathrm{SEN}^\flat$, prealgebraicity is equivalent to narrow right or narrow system monotonicity, coupled with existence of theorems. Of course, having introduced two seemingly different approaches to handling empty theory family components, it is of central importance to investigate the relations between rough monotonicity and narrow monotonicity classes. Narrow family monotonicity turns out to be equivalent to rough family monotonicity, whereas, with regards to the three remaining versions, each of the rough properties implies the corresponding narrow property. All four narrow monotonicity properties transfer. The section concludes with characterizations of narrow family and narrow system monotonicity in terms of the Leibniz operator viewed as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.5, we look at *rough complete monotonicity* (c-monotonicity) properties. These concepts, in analogy with the extension of monotonicity to the c-monotonicity properties of Section 3.4, extend rough monotonicity properties by allowing arbitrary unions on the right-hand side of the relevant inequalities. A π-institution $\mathcal{I}$ is called *roughly family c-monotone* if, for every collection $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$ implies $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. In *rough left c-monotonicity* the hypothesis is replaced by $\widetilde{\overleftarrow{T'}} \leq \bigcup_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}}$ and, in *rough right c-monotonicity*, the conclusion is replaced by $\Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. The *system version* imposes the same condition as the family version, but restricts it on collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$. Here, the only inclusions are those establishing that each of the rough left, family and right c-monotonicity classes form a subclass of the class of roughly system c-monotone π-institutions. Rough left c-monotonicity is equivalent to rough system c-monotonicity plus stability. Under stability, rough family c-monotonicity and rough right c-monotonicity are equivalent and, furthermore, under rough systemicity, the entire hierarchy collapses to a single class. From the definitions, it is obvious that each version of rough c-monotonicity implies the corresponding version of rough monotonicity, since, the definition of the latter specializes that of the former. Moreover, each version of c-monotonicity implies the corresponding version of rough c-monotonicity. As far as closer ties, analogous to those detailed for rough monotonicity classes in Section 7.3, for non almost inconsistent π-institutions, $\mathcal{I}$ is family (left, respectively) c-monotone if and only if it is rough;y family (left, respectively) c-monotone and has theorems. Along similar lines, for $\mathcal{I}$ having a theory family $T \neq \mathrm{SEN}^\flat$, such that $\overleftarrow{T} \neq \overline{\varnothing}$, $\mathcal{I}$ is system (right, respectively) c-monotone if and only if it is roughly system (right, respectively) c-monotone and has theorems. All four rough c-monotonicity properties transfer and, as was the case with rough monotonicity, the family and system versions have characterizations in terms of Ω seen as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

The same extension that led from rough monotonicity to rough c-monotonicity properties may be applied to narrow monotonicity properties and leads to *narrow c-monotonicity* properties, which constitute the objects of study in Section 7.6. A π-institution $\mathcal{I}$ is called *narrowly family c-monotone* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$, $T' \leq \bigcup_{T \in \mathcal{T}} T$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Once more, the *left version* results by replacing in the hypothesis all theory families by their arrow counterparts, and, similarly for the *right version*, except that the replacement is applied in the conclusion of the implication instead. The *system version* applies the same condition as the family version, but restricts its scope on collections of theory systems in $\mathrm{ThSys}^{\nleftrightarrow}(\mathcal{I})$. As was the case with rough c-monotonicity in Section 7.5, the only three implications assert that each of the narrow left, family and right c-monotonicity properties implies narrow system c-monotonicity. Each version of c-monotonicity implies its narrow c-monotonicity counterpart. It turns out that rough family c-monotonicity is equivalent to narrow family c-monotonicity. On the other hand, for the remaining three versions, each rough c-monotonicity variant implies the corresponding narrow c-monotonicity variant. Of course, due to the specializations in the relevant definitions, each narrow c-monotonicity property implies the corresponding narrow monotonicity property. All four narrow c-monotonicity properties transfer. Finally, it is the case here as well, that the family and the system versions can be characterized in terms of the Leibniz operator viewed as a mapping from $\mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleftrightarrow}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

7.2 Narrow and Exclusive Stability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is called *stable* if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \Omega(T).$$

Recall, also, that, in Section 6.5, we defined *narrow stability*, a concept that proved handy in demonstrating that the narrow right properties studied there implied the corresponding narrow family properties. We recall that definition and look at an additional concept weakening stability. These two notions aim at bypassing theory families with empty components.

Definition 497 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **narrowly stable** *if, for all $T \in \mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$,*

$$\Omega(\overleftarrow{T}) = \Omega(T);$$

- $\mathcal{I}$ *is* **exclusively stable** *if, for all* $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*, such that* $\overleftarrow{T} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,*
$$\Omega(\overleftarrow{T}) = \Omega(T).$$

It is clear that stability is the strongest of the three properties followed by narrow stability and exclusive stability, which is the weakest of the three.

Proposition 498 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is stable, then it is narrowly stable;*

(b) If $\mathcal{I}$ *is narrowly stable, then it is exclusively stable.*

Proof: It suffices to note that each property is a specialization of the one immediately dominating it in strength. ■

Thus, the following linear **stability hierarchy** is established.

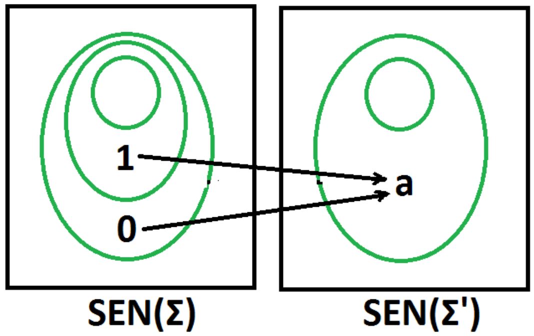
Stable
↓
Narrowly Stable
↓
Exclusively Stable

It is not difficult to see that all three classes are different. The following example provides a π-institution that is narrowly stable but not stable, showing that stable π-institutions form a proper subclass of the class consisting of the narrowly stable ones.

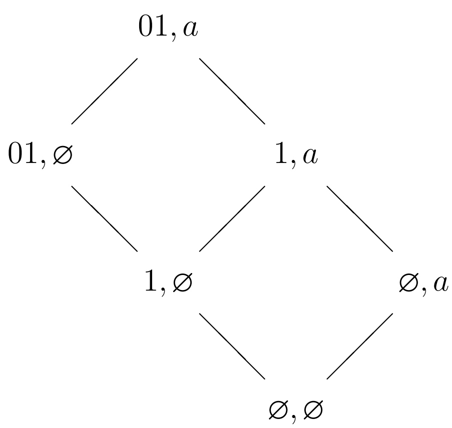
Example 499 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*
$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$



Clearly, there are six theory families in ThFam($\mathcal{I}$)*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\natural}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



Since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \{\{1,a\},\{01,a\}\}$ *and* $\overleftarrow{\{1,a\}} = \{1,a\}$ *and* $\overleftarrow{\{01,a\}} = \{01,a\}$*, we get that* $\mathcal{I}$ *is narrowly systemic and, hence, a fortiori, also narrowly stable. On the other hand, consider* $T = \{\{1\},\varnothing\}$*. We have*

$$\Omega(\overleftarrow{\{1,\varnothing\}}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \neq \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1,\varnothing\}),$$

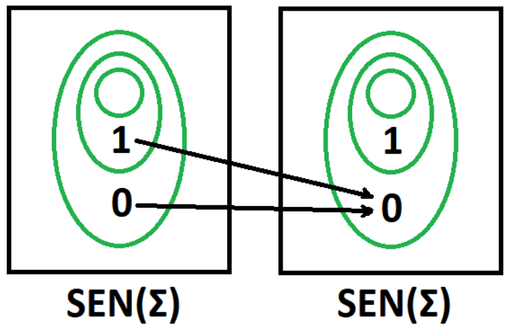
whence $\mathcal{I}$ *is not stable.*

Finally, we give an example of an exclusively stable π-institution which, however, fails to be narrowly stable. This shows that the inclusion of the class of narrowly stable π-institutions into the class of exclusively stable ones is also proper.

Example 500 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^{\flat}$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

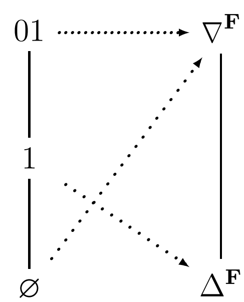
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$, $\{\{1\}\}$ *and* $\{\{0,1\}\}$, *but only two theory systems,* $\{\varnothing\}$ *and* $\{\{0,1\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



The only theory family $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, *such that* $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$ *is* $\{\{0,1\}\}$. *Moreover,* $\overleftarrow{\{0,1\}} = \{0,1\}$, *whence we get that* $\mathcal{I}$ *is exclusively stable. On the other hand, for* $\{\{1\}\} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, *we get*

$$\Omega(\overleftarrow{\{1\}}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \neq \Delta^{\mathbf{F}} = \Omega(\{1\}).$$

Therefore, $\mathcal{I}$ *is not narrowly stable.*

7.3 Rough Monotonicity

In this section we exploit the notion of rough equivalence, which was studied in some detail in Section 6.2, to introduce and study classes of π-institutions defined using monotonicity properties of the Leibniz operator applied on rough equivalence classes.

Definition 501 (Rough Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **roughly family monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T} \le \widetilde{T'} \quad \textit{implies} \quad \Omega(T) \le \Omega(T').$$
- $\mathcal{I}$ *is called* **roughly left monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}} \quad \textit{implies} \quad \Omega(T) \le \Omega(T').$$
- $\mathcal{I}$ *is called* **roughly right monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T} \le \widetilde{T'} \quad \textit{implies} \quad \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}).$$
- $\mathcal{I}$ *is called* **roughly system monotone** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,
$$\widetilde{T} \le \widetilde{T'} \quad \textit{implies} \quad \Omega(T) \le \Omega(T').$$

Next we look into establishing the *rough monotonicity hierarchy* of π-institutions. We show, first, that rough left monotonicity implies stability.

Lemma 502 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly left monotone, then it is stable.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone and let $T \in \mathrm{ThFam}(\mathcal{I})$. Since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we get that $\widetilde{\overleftarrow{\overleftarrow{T}}} = \widetilde{\overleftarrow{T}}$. Thus, by rough left monotonicity, $\Omega(\overleftarrow{T}) = \Omega(T)$. Hence, $\mathcal{I}$ is stable. ■

Lemma 502 leads to the conclusion that, under rough left monotonicity, the properties of rough family monotonicity and rough right monotonicity are equivalent.

Corollary 503 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly left monotone, then it is roughly family monotone if and only if it is roughly right monotone.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone. Then, by Lemma 502, it is stable. Now note that rough family monotonicity is equivalent to the condition that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T} \le \widetilde{T'} \quad \text{implies} \quad \Omega(T) \le \Omega(T'),$$
which, by stability, is equivalent to, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T} \le \widetilde{T'} \quad \text{implies} \quad \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}),$$
and this is equivalent, by definition, to rough right monotonicity. ■

Next we show that rough left monotonicity implies rough family monotonicity.

Proposition 504 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly left monotone, then it is roughly family monotone.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone, i.e., for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$ implies $\Omega(T) \leq \Omega(T')$. Let $X, Y \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{X} \leq \widetilde{Y}$. If $\widetilde{\overleftarrow{X}} \leq \widetilde{\overleftarrow{Y}}$, then, by rough left monotonicity, $\Omega(X) \leq \Omega(Y)$. So, assume that $\widetilde{\overleftarrow{X}} \nleq \widetilde{\overleftarrow{Y}}$, that is, that there exists $P \in |\mathbf{Sign}^\flat|$, such that $\widetilde{\overleftarrow{X}}_P \nsubseteq \widetilde{\overleftarrow{Y}}_P$. At the same time, since $\widetilde{X} \leq \widetilde{Y}$, we have that $\widetilde{X}_P \subseteq \widetilde{Y}_P$. This implies that $X_P \subseteq Y_P$ or $Y_P = \varnothing$. However, if $Y_P = \varnothing$, then, we would also have $\overleftarrow{Y}_P = \varnothing$, whence $\widetilde{\overleftarrow{X}}_P \subseteq \mathrm{SEN}^\flat(P) = \widetilde{\overleftarrow{Y}}_P$, contradicting our assumption. Hence, we conclude that $X_P \subseteq Y_P$. Now, based on $\widetilde{\overleftarrow{X}}_P \nsubseteq \widetilde{\overleftarrow{Y}}_P$, we distinguish two possibilities, $\overleftarrow{X}_P \nsubseteq \overleftarrow{Y}_P$ or $\overleftarrow{X}_P = \varnothing$.

- Suppose $X_P \subseteq Y_P$ and $\overleftarrow{X}_P \nsubseteq \overleftarrow{Y}_P$. Then, there exists $Q \in |\mathbf{Sign}^\flat|$ and $P \stackrel{f}{\to} Q$, such that $X_Q \nsubseteq Y_Q$. Since, however, $\widetilde{X}_Q \subseteq \widetilde{Y}_Q$, we would have $Y_Q = \varnothing$. This, combined with the fact that $\widetilde{\overleftarrow{X}}_P \nsubseteq \widetilde{\overleftarrow{Y}}_P$ implies that $\overleftarrow{Y}_P \neq \varnothing$, yield that there cannot exist $f : P \to Q$, a contradiction.

- So it must be the case that $X_P \subseteq Y_P$ and $\overleftarrow{X}_P = \varnothing$. Since $\widetilde{\overleftarrow{X}}_P \nsubseteq \widetilde{\overleftarrow{Y}}_P$, we must have $\overleftarrow{Y}_P \neq \varnothing$ and $\overleftarrow{Y}_P \neq \mathrm{SEN}^\flat(P)$. Note that it is not possible to have both $X_P = \overleftarrow{X}_P$ and $Y_P = \overleftarrow{Y}_P$. If that had been the case, we would have $X_P = \varnothing$ and $Y_P \neq \varnothing$ or $\mathrm{SEN}^\flat(P)$, whence $\widetilde{X}_P \nsubseteq \widetilde{Y}_P$, which contradicts the hypothesis. So, we must have $\varnothing = \overleftarrow{X}_P \subsetneq X_P$ or $\overleftarrow{Y}_P \subsetneq Y_P$.

  - Assume, first, that $\varnothing = \overleftarrow{X}_P \subsetneq X_P \subseteq Y_P \neq \mathrm{SEN}^\flat(P)$. Define $Z = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

    $$Z_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ X_P, & \text{if } \Sigma = P \end{cases}.$$

    Then, we have $Z \leq X$, whence $\overleftarrow{Z} \leq \overleftarrow{X}$ and, hence, $\overleftarrow{Z} = \overline{\varnothing} = \overleftarrow{\overline{\varnothing}}$. So, whereas $\widetilde{\overleftarrow{Z}} = \widetilde{\overleftarrow{\overline{\varnothing}}}$, $\Omega(Z) \neq \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$. This contradicts rough right monotonicity.

  - Suppose, next, that $\varnothing = \overleftarrow{X}_P = X_P$ and $\overleftarrow{Y}_P \subsetneq Y_P$. We already know that $\overleftarrow{Y}_P \neq \varnothing$. Moreover, since $\widetilde{X}_P \subseteq \widetilde{Y}_P$, we must have $Y_P = \mathrm{SEN}^\flat(P)$. Now we define $Z = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting

    $$Z_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ \overleftarrow{Y}_P, & \text{if } \Sigma = P \end{cases}.$$

If there had been no morphism of the form $P \xrightarrow{f} Q$, with $Q \neq P$, in $\mathbf{Sign}^\flat$, then, since $Y_P = \mathrm{SEN}^\flat(P)$, we would have $\overleftarrow{Y}_P = \mathrm{SEN}^\flat(P)$, contradicting our assumption. The existence of such a morphism implies that $\overleftarrow{Z} = \overline{\varnothing} = \overleftarrow{\overline{\varnothing}}$. However, $\Omega(Z) \neq \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$, which contradicts rough left monotonicity.

We conclude that $\mathcal{I}$ must be roughly family monotone. ■

We now have a picture of the rough monotonicity hierarchy.

Proposition 505 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly left monotone, then it is both roughly family and roughly right monotone;*

(b) If $\mathcal{I}$ *is roughly family or roughly right monotone, then it is roughly system monotone.*

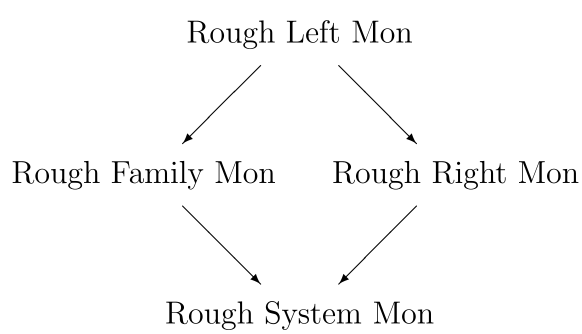
Proof:

(a) Suppose $\mathcal{I}$ is roughly left monotone. By Proposition 504, $\mathcal{I}$ is roughly family monotone. Therefore, by Corollary 503, it is also roughly right monotone.

(b) If $\mathcal{I}$ is roughly family monotone, then it is, a fortiori, roughly system monotone, since the condition defining the latter notion is a specialization of that defining the former. So, suppose $\mathcal{I}$ is roughly right monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \leq \widetilde{T'}$. Then, by rough right monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since T, T' are theory systems, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $\Omega(T) \leq \Omega(T')$ and, hence, $\mathcal{I}$ is roughly system monotone.

■

We have now established the following **rough monotonicity hierarchy** of π-institutions.



It is not difficult to see that being roughly left monotone is equivalent to being roughly system monotone and stable.

Proposition 506 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left monotone if and only if it is roughly system monotone and stable.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left monotone. Then, by Proposition 505, it is roughly system monotone. Moreover, by Lemma 502, $\mathcal{I}$ is stable.

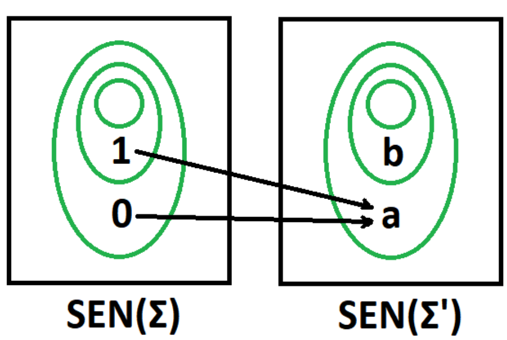
Assume, conversely, that $\mathcal{I}$ is stable and roughly system monotone. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$. Then, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we get, by rough system monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Thus, by stability, $\Omega(T) \leq \Omega(T')$. We conclude that $\mathcal{I}$ is roughly left monotone. ■

By Proposition 506, under stability, the rough monotonicity hierarchy collapses to a single class. Moreover, by Lemma 383, the same happens, a fortiori, under rough systemicity.

We present two examples to show that all four rough monotonicity classes depicted in the diagram above are different. The first example gives a roughly family monotone π-institution that is not roughly right monotone. It shows that the inclusions represented in the diagram by the two southwest pointing arrows are proper inclusions.

Example 507 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



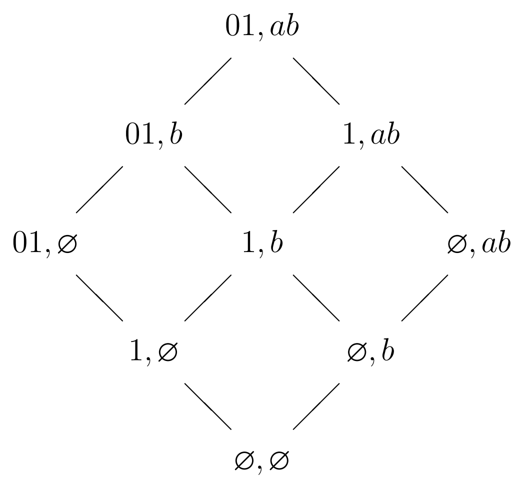
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing,\varnothing$	$\varnothing,\varnothing$	$\varnothing,ab$	$\varnothing,ab$
$1,\varnothing$	$\varnothing,\varnothing$	$01,b$	$\varnothing,b$
$\varnothing,b$	$\varnothing,b$	$1,ab$	$1,ab$
$01,\varnothing$	$\varnothing,\varnothing$	$01,ab$	$01,ab$
$1,b$	$\varnothing,b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family monotone. To this end, suppose* $\widetilde{T} \le \widetilde{T'}$.

- *If* $\widetilde{T'} = \{01,ab\}$, *then* $T' = \{\varnothing,\varnothing\}$ *or* $\{01,\varnothing\}$ *or* $\{\varnothing,ab\}$ *or* $\{01,ab\}$. *In all cases* $\Omega(T) \le \nabla^{\mathbf{F}} = \Omega(T')$;
- *If* $\widetilde{T'} = \{01,b\}$, *then* $T' = \{\varnothing,b\}$ *or* $\{01,b\}$ *and* $\widetilde{T} = \widetilde{T'}$ *or* $\widetilde{T} = \{1,b\} = T$, *whence* $\Omega(T) \le \{\nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_{\Sigma'}\} = \Omega(T')$;
- *If* $\widetilde{T'} = \{1,ab\}$, *then* $T' = \{1,\varnothing\}$ *or* $\{1,ab\}$ *and* $\widetilde{T} = \widetilde{T'}$ *or* $\widetilde{T} = \{1,b\} = T$, *whence* $\Omega(T) \le \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(T')$;
- *If* $\widetilde{T'} = \{1,b\}$, *then* $\widetilde{T} = \{1,b\}$, *whence* $T = T' = \{1,b\}$ *and* $\Omega(T) = \Omega(T')$.

Therefore, $\mathcal{I}$ *is indeed roughly family monotone.*

On the other hand, we have $\widetilde{\{1,b\}} = \{1,b\} \le \{1,ab\} = \widetilde{\{1,ab\}}$, *whereas*

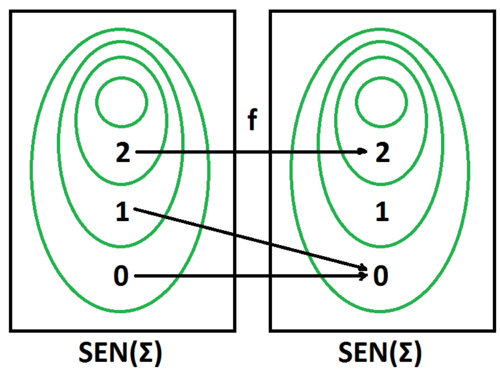
$$\Omega(\overleftarrow{\{1,b\}}) = \Omega(\{\varnothing,b\}) = \{\nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_{\Sigma'}\} \nleq \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1,ab\}) = \Omega(\overleftarrow{\{1,ab\}}).$$

Therefore, $\mathcal{I}$ *is not roughly right monotone.*

The second example shows that there exists a roughly right monotone π-institution that is not roughly family monotone. This has the effect of establishing that the inclusions represented by the two southeast arrows in the hierarchy diagram are also proper inclusions.

Example 508 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

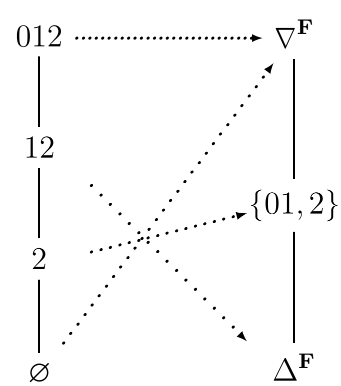
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*,* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems, namely* $\overline{\varnothing}$*,* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$*. The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:*



We show that $\mathcal{I}$ *is roughly right monotone. Suppose* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*, such that* $\widetilde{T} \leq \widetilde{T'}$*.*

- *If* $T' = \{\varnothing\}$ *or* $T' = \{\{0,1,2\}\}$, *i.e., if* $\widetilde{T'} = \{\{0,1,2\}\}$, $\Omega(\overleftarrow{T}) \le \nabla^{\mathbf{F}} = \Omega(\overleftarrow{T'})$*;*
- *If* $T' = \{\{1,2\}\}$, *then* $T = \{\{2\}\}$ *or* $T = \{\{1,2\}\}$. *So* $\Omega(\overleftarrow{T}) = \Omega(\{\{2\}\}) = \Omega(\overleftarrow{T'})$*;*
- *If* $T' = \{\{2\}\}$, *i.e., if* $\widetilde{T'} = \{1\}$, *then* $T = \{\{2\}\}$, *and the conclusion is trivial.*

Thus, $\mathcal{I}$ *is indeed roughly right monotone.*

On the other hand, setting $T = \{\{2\}\}$ *and* $T' = \{\{1,2\}\}$, *we get* $\widetilde{T} \le \widetilde{T'}$, *but* $\Omega(T) = \{\{0,1\},\{2\}\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. *Therefore,* $\mathcal{I}$ *is not roughly family monotone.*

We conclude, after these two examples, that the structure of the rough monotonicity hierarchy is, in fact, exactly as depicted in the diagram and no two classes are identical.

We look, next, at the connections between rough monotonicity and monotonicity classes. It turns out that protoalgebraicity (i.e., family/left monotonicity, by Proposition 171) is strong enough to ensure membership in all classes of the rough monotonicity hierarchy, whereas prealgebraicity (i.e., system/right monotonicity, by Proposition 173) is only sufficiently strong to yield corresponding rough monotonicity properties, i.e., implies rough right monotonicity and, a fortiori, rough system monotonicity.

Theorem 509 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is protoalgebraic, then it is roughly left monotone;*

(b) If $\mathcal{I}$ *is prealgebraic, then it is roughly right monotone.*

Proof:

(a) Suppose that $\mathcal{I}$ is protoalgebraic. By Lemma 170, this implies that $\mathcal{I}$ is stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T}} \le \overleftarrow{\overleftarrow{T'}}$. Then, by protoalgebraicity, $\Omega(\overleftarrow{\overleftarrow{T}}) \le \Omega(\overleftarrow{\overleftarrow{T'}})$. Hence, by Proposition 369, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Thus, by stability, $\Omega(T) \le \Omega(T')$. Therefore, $\mathcal{I}$ is roughly left monotone.

(b) Suppose that $\mathcal{I}$ is prealgebraic. If $\mathrm{ThSys}(\mathcal{I})$ consists of a single rough equivalence class, then $\mathcal{I}$ is trivially roughly right monotone. Otherwise, since $\mathcal{I}$ is prealgebraic and $\Omega(\overline{\varnothing}) = \Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$, $\mathcal{I}$ must have theorems. Therefore, rough equivalence is the identity relation on $\mathrm{ThFam}(\mathcal{I})$. Thus, for $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T} \le \widetilde{T'}$, we get $T \le T'$, whence, by Lemma 1, $\overleftarrow{T} \le \overleftarrow{T'}$. Thus, by prealgebraicity, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$, showing that $\mathcal{I}$ is roughly right monotone.

■

Moreover, the following additional relations hold.

Theorem 510 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if it has theorems and is roughly family or roughly left monotone.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic. Since, by hypothesis, it is not almost inconsistent, it must have theorems. Moreover, by Theorem 509 and Proposition 505, it is both roughly left and roughly family monotone.

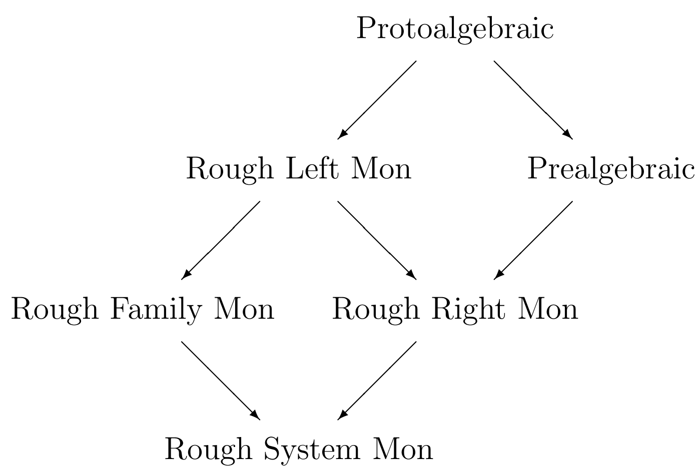
Assume, conversely, that $\mathcal{I}$ is roughly family or roughly left monotone and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get both $\widetilde{T} \leq \widetilde{T'}$ and $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$. Using either rough family or rough left monotonicity, as the case requires, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is protoalgebraic. ■

Theorem 511 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory family* $T \neq \mathrm{SEN}^\flat$ *such that* $\overleftarrow{T} \neq \overline{\varnothing}$*.* $\mathcal{I}$ *is prealgebraic if and only if it has theorems and it is roughly right or roughly system monotone.*

Proof: Suppose $\mathcal{I}$ is prealgebraic. Since, by hypothesis, it has a theory system $\overleftarrow{T} \neq \mathrm{SEN}^\flat, \overline{\varnothing}$, it must have theorems. Moreover, by Theorem 509, it is roughly right monotone and, hence, by Proposition 505, it is roughly system monotone.

Assume, conversely, that $\mathcal{I}$ is roughly right or roughly system monotone and has theorems. By Proposition 505, it is roughly system monotone and has theorems. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get $\widetilde{T} \leq \widetilde{T'}$. By rough system monotonicity, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is prealgebraic. ■

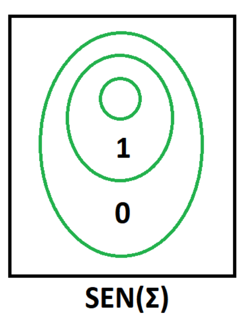
Theorem 509, together with Theorem 175 and Proposition 505, establish the mixed monotonicity and rough monotonicity hierarchy depicted in the diagram.



To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four rough monotonicity properties, which is not, however, prealgebraic and, therefore, a fortiori, it is not protoalgebraic either.

Example 512 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial clone.*

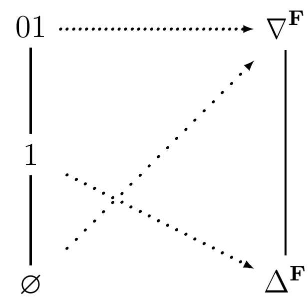


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



$\mathcal{I}$ belongs to all four classes of the rough monotonicity hierarchy. Indeed, since it is systemic, all four rough monotonicity conditions boil down to checking that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$.

- *If $\widetilde{T'} = \{\{0,1\}\}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1\}\}$, whence $\Omega(T) \leq \nabla^{\mathbf{F}} = \Omega(T')$;*
- *If $\widetilde{T'} = \{\{1\}\}$, then $\widetilde{T} = \{\{1\}\}$ and, hence, $T = T' = \{\{1\}\}$. Thus, the implication holds trivially.*

On the other hand, we have $\{\varnothing\} \leq \{\{1\}\}$, whereas $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$, whence $\mathcal{I}$ is not prealgebraic.

The rough monotonicity properties transfer from the theory families/ systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 513 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is roughly family monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;

(b) $\mathcal{I}$ is roughly left monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;

(c) $\mathcal{I}$ is roughly right monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$;

(d) $\mathcal{I}$ is roughly system monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly family monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T} \leq \widetilde{T'}$. Then $\alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$. By Theorem 377, $\widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Since, by Lemma 51, both $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are theory families of $\mathcal{I}$, we get, by rough family monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. Then $\alpha^{-1}(\widetilde{\overleftarrow{T}}) \leq \alpha^{-1}(\widetilde{\overleftarrow{T'}})$. By Theorem 377, $\widetilde{\alpha^{-1}(\overleftarrow{T})} \leq \widetilde{\alpha^{-1}(\overleftarrow{T'})}$. Hence, by Lemma 6, $\overleftarrow{\widetilde{\alpha^{-1}(T)}} \leq \overleftarrow{\widetilde{\alpha^{-1}(T')}}$. Since, by Lemma 51, $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are theory families, we get, by rough left monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$, whence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by the surjectivity of $\langle F, \alpha\rangle$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly right monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T} \leq \widetilde{T'}$. Then $\alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$ and, hence, by Theorem 377, $\widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Since, by Lemma 51, $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are theory families, we get, by rough right monotonicity, $\Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Thus, by Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Now, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$, whence, by the surjectivity of $\langle F, \alpha\rangle$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$.

(d) Similar to Part (a). ■

Finally, we may recast the rough monotonicity classes in terms of the monotonicity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Recall the orderings of the collections $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$: For all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{[T]} \leq \widetilde{[T']} \quad \text{iff} \quad \widetilde{T} \leq \widetilde{T'}$$

and, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{[T]} \leq \widetilde{[T']} \quad \text{iff} \quad \widetilde{T} \leq \widetilde{T'}$$

and the notation $\widetilde{\mathbf{ThFam}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThFam}}(\mathcal{I}), \leq\rangle$ and $\widetilde{\mathbf{ThSys}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThSys}}(\mathcal{I}), \leq\rangle$ for the corresponding ordered sets.

Proposition 514 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family monotone;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for rough system monotonicity, we have

Proposition 515 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system monotone;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

7.4 Narrow Monotonicity

We now introduce and study classes of π-institutions defined using, once more, monotonicity properties of the Leibniz operator, but applied only on theory families with all components nonempty. This is one of the ways used already in Chapter 6 to bypass theory families with empty components that may cause lack of monotonicity.

Definition 516 (Narrow Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **narrowly family monotone** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$*,*
$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **narrowly left monotone** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$*,*
$$\overleftarrow{T} \leq \overleftarrow{T'} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **narrowly right monotone** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$*,*
$$T \leq T' \quad \textit{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

- $\mathcal{I}$ *is called* **narrowly system monotone** *if, for all* $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$,

$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

We establish now the *narrow monotonicity hierarchy* of π-institutions.

Proposition 517 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is narrowly left monotone, then it is narrowly family monotone;*

(b) If $\mathcal{I}$ *is narrowly family monotone, then it is narrowly system monotone;*

(c) If $\mathcal{I}$ *is narrowly right monotone, then it is narrowly system monotone.*

Proof:

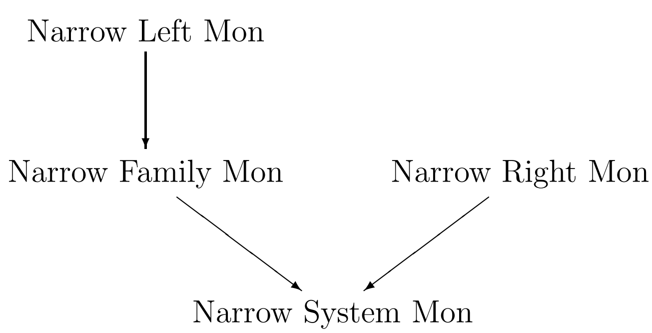
(a) Suppose that $\mathcal{I}$ is narrowly left monotone and let $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$, whence, by narrow left monotonicity, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is narrow family monotone.

(b) Suppose $\mathcal{I}$ is narrow family monotone. Then it is a fortiori narrow system monotone, since the condition defining the latter is a specialization of the one defining the former.

(c) Suppose $\mathcal{I}$ is narrowly right monotone and let $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, such that $T \leq T'$. Then, by narrow right monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since T, T' are theory systems, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $\Omega(T) \leq \Omega(T')$ and, hence, $\mathcal{I}$ is narrowly system monotone.

■

We have now established the following **narrow monotonicity hierarchy** of π-institutions.



Some additional relationships may be established between the narrow monotonicity classes. More precisely, we show that narrow left monotonicity implies exclusive stability, whereas narrow system monotonicity together with narrow systemicity, yield both narrow left and narrow right monotonicity.

Proposition 518 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly left monotone, then it is exclusively stable.*

Proof: Suppose that $\mathcal{I}$ is narrowly left monotone and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$ and $T, \overleftarrow{T} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow left monotonicity, $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, $\mathcal{I}$ is exclusively stable. ■

Proposition 519 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system monotone and narrowly systemic, then it is both narrowly left and narrowly right monotone.*

Proof: Suppose that $\mathcal{I}$ is narrowly system monotone and narrowly systemic and let $T, T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$.

- Assume that $\overleftarrow{T} \leq \overleftarrow{T'}$. By narrow systemicity, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Thus, by narrow system monotonicity, $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is narrowly left monotone.

- Assume that $T \leq T'$. Again, by narrow systemicity, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, which yields that $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Hence, by narrow system monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$, showing that $\mathcal{I}$ is narrowly right monotone.

■

By Propositions 517 and 519, under narrow systemicity, the narrow monotonicity hierarchy collapses to a single class.

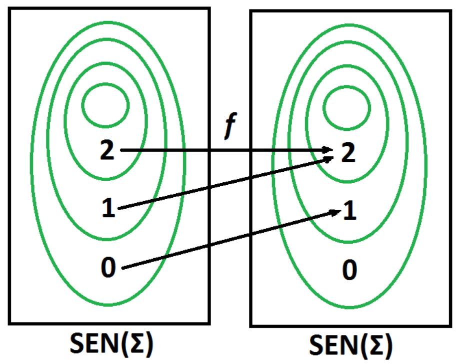
We present three examples to show that all four narrow monotonicity classes depicted in the diagram above are different. The first example gives a narrowly family monotone π-institution which is not narrowly left monotone. Thus, it shows that the class of narrowly left monotone π-institutions is properly contained in the class of narrowly family monotone π-institutions.

Example 520 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, z, o, t : \Sigma \to \Sigma$*, whose composition table is the following:*

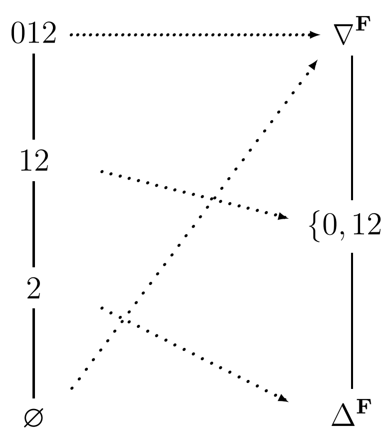
$\circ$	f	z	o	t
f	t	o	t	t
z	z	z	z	z
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*, with*
$$\mathrm{SEN}^\flat(f)(0) = 1, \quad \mathrm{SEN}^\flat(f)(1) = 2, \quad \mathrm{SEN}^\flat(f)(2) = 2,$$
whereas $\mathrm{SEN}^\flat(z)(x) = 0$*,* $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$*, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*
$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families* $\varnothing$*,* $\{\{2\}\}$*,* $\{\{1,2\}\}$ *and* $\{\{0,1,2\}\}$*, but only two theory systems,* $\varnothing$ *and* $\{\{0,1,2\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \mathrm{SEN}^\flat\}$*, it is clear that* $\mathcal{I}$ *is narrowly family monotone.*

On the other hand, for $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$*, we get* $\overleftarrow{T} = \overline{\varnothing} = \overleftarrow{T'}$*, whereas* $\Omega(T) = \{0,12\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$*. Therefore,* $\mathcal{I}$ *is not narrowly left monotone.*

The second example shows that there exists a narrowly family monotone π-institution that is not narrowly right monotone, thus showing, on the one hand, that the class of narrowly right monotone π institutions is properly included in the class of narrowly system monotone π-institutions and, on the other, that narrowly family monotone π-institutions do not form a subclass of narrowly right monotone π-institutions.

Example 521 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, g, o, t : \Sigma \to \Sigma$, *whose composition table is the following:*

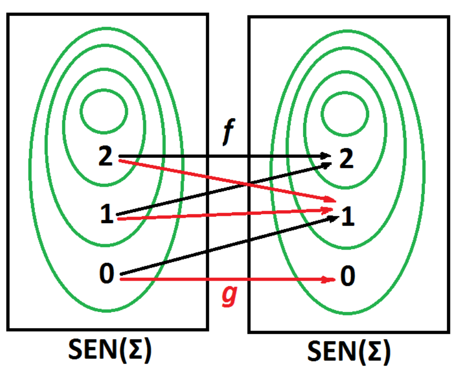
$\circ$	f	g	o	t
f	t	f	t	t
g	o	g	o	o
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$, *with*

$$\begin{array}{lll} \mathrm{SEN}^\flat(f)(0) = 1, & \mathrm{SEN}^\flat(f)(1) = 2, & \mathrm{SEN}^\flat(f)(2) = 2; \\ \mathrm{SEN}^\flat(g)(0) = 0, & \mathrm{SEN}^\flat(g)(1) = 1, & \mathrm{SEN}^\flat(g)(2) = 1, \end{array}$$

  whereas $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*;*

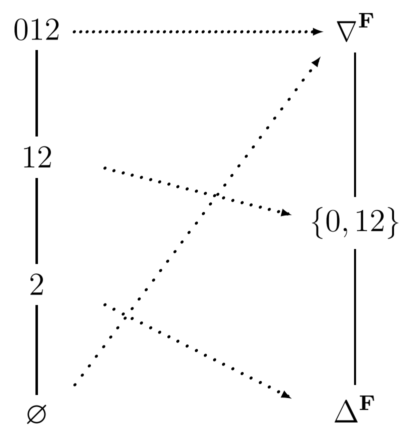
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has four theory families* $\varnothing$, $\{\{2\}\}$, $\{\{1, 2\}\}$ *and* $\{\{0, 1, 2\}\}$, *but only three theory systems,* $\varnothing$, $\{\{1, 2\}\}$ *and* $\{\{0, 1, 2\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \mathrm{SEN}^\flat\}$*, it is clear that* $\mathcal{I}$ *is narrowly family monotone.*

On the other hand, for $T = \{\{2\}\}$ *and* $T' = \{\{1,2\}\}$*, we get* $T \leq T'$*, whereas* $\Omega(\overleftarrow{T}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \nleq \{0,12\} = \Omega(T') = \Omega(\overleftarrow{T'})$*. Therefore,* $\mathcal{I}$ *is not narrowly right monotone.*

The third example shows that there exists a narrowly right monotone π-institution that is not narrowly family monotone. Combined with the preceding examples, it has the effect of establishing the following facts:

- The classes of narrowly family monotone and narrowly right monotone π-institutions are pairwise incomparable.
- The class of narrowly family monotone π-institutions is properly contained in the class of narrowly system monotone π-institutions.

Example 522 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

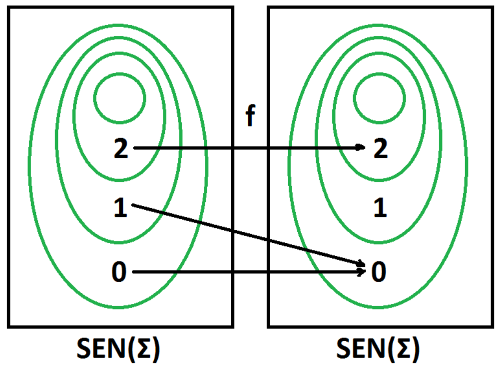
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*,* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

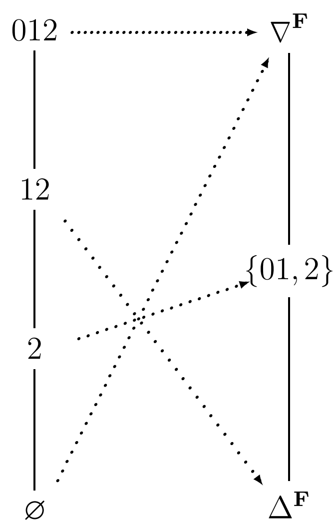
$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems, namely* $\overline{\varnothing}$*,* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$*. Moreover, clearly,*

$$\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \{\{0,1,2\}\}\}.$$



The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



We have

$$\begin{array}{rcl}\Omega(\overleftarrow{2}) & = & \Omega(2) = \{01,2\};\\ \Omega(\overleftarrow{12}) & = & \Omega(2) = \{01,2\};\\ \Omega(\overleftarrow{012}) & = & \Omega(012) = \nabla^{\mathbf{F}}.\end{array}$$

Thus, we get $\Omega(\overleftarrow{2}) \leq \Omega(\overleftarrow{12}) \leq \Omega(\overleftarrow{012})$ *and, therefore,* $\mathcal{I}$ *is narrowly right monotone.*

On the other hand, for $T = \{\{2\}\}$ *and* $T' = \{\{1,2\}\}$*, we get* $T \leq T'$*, whereas* $\Omega(T) = \{01,2\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$*. Thus,* $\mathcal{I}$ *is not narrowly family monotone.*

We conclude that the structure of the narrow monotonicity hierarchy is, in fact, exactly as depicted in the diagram and no two classes are identical.

We look, next, at the connections between narrow monotonicity and monotonicity classes. Once more, as was the case with monotonicity and rough monotonicity in Section 7.3, protoalgebraicity (i.e., family/left monotonicity, by Proposition 171) is strong enough to ensure membership in all classes of the narrow monotonicity hierarchy, whereas prealgebraicity (i.e., system/right monotonicity, by Proposition 173) is only sufficiently strong to yield corresponding narrow monotonicity properties, i.e., implies narrow right monotonicity and, a fortiori, narrow system monotonicity.

Theorem 523 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is protoalgebraic, then it is narrowly left monotone;*

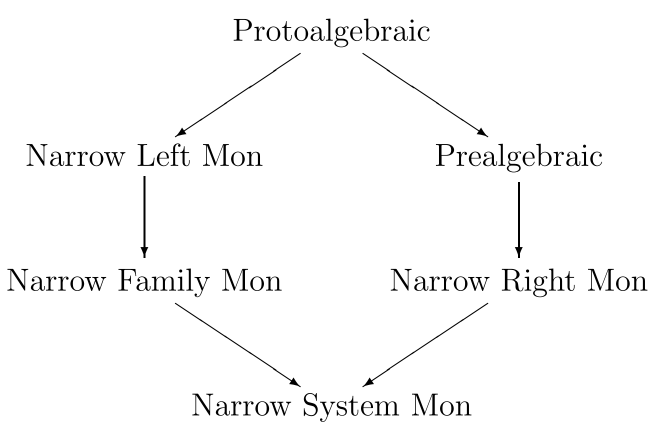
(b) If $\mathcal{I}$ *is prealgebraic, then it is narrowly right monotone.*

Proof:

(a) Suppose that $\mathcal{I}$ is protoalgebraic. By Proposition 171, it is left monotone, whence, it is, a fortiori, narrowly left monotone, since the condition defining the latter is a specialization of that defining the former.

(b) Suppose that $\mathcal{I}$ is prealgebraic. By Proposition 173, it is right monotone, whence, it is, a fortiori, narrowly right monotone, since the condition defining the latter is a specialization of that defining the former. ■

Thus, the following mixed monotonicity and narrow monotonicity hierarchy emerges.



We also have the following additional relations, paralleling the ones established between monotonicity and rough monotonicity classes in Theorems 510 and 511.

Theorem 524 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if it has theorems and is narrowly left or narrowly family monotone.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic. Since, by hypothesis, it is not almost inconsistent, it must have theorems. Moreover, by Theorem 523 and Proposition 517, it is both narrowly left and narrowly family monotone.

Assume, conversely, that $\mathcal{I}$ is narrowly left or narrowly family monotone and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, since $\mathcal{I}$ has theorems, $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$ and, moreover, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Using either narrow family or narrow left monotonicity, as the case requires, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is protoalgebraic. ∎

Theorem 525 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, that has a theory system $T \neq \overline{\varnothing}, \mathrm{SEN}^\flat$. $\mathcal{I}$ is prealgebraic if and only if it has theorems and it is narrowly right or narrowly system monotone.*

Proof: Suppose $\mathcal{I}$ is prealgebraic. Since, by hypothesis, it has a theory system $T \neq \overline{\varnothing}, \mathrm{SEN}^\flat$, it must have theorems. Moreover, by Theorem 523, it is narrowly right monotone and, hence, by Proposition 517, it is narrowly system monotone.

Assume, conversely, that $\mathcal{I}$ is narrowly right or narrowly system monotone and has theorems. By Proposition 517, it is narrowly system monotone and has theorems. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Since $\mathcal{I}$ has theorems, $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$. By narrow system monotonicity, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is prealgebraic. ∎

To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four narrow monotonicity properties, which is not, however, prealgebraic and, therefore, a fortiori, it is not protoalgebraic either.

Example 526 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

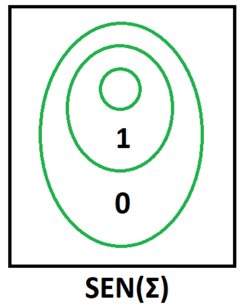
- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial clone.*

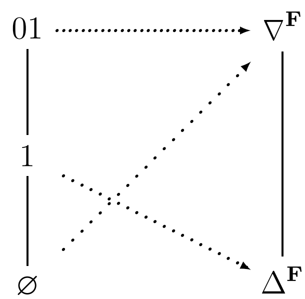
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$ and $\{\{1\}\}$ and $\{\{0, 1\}\}$, all of which are theory systems.

The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.





$\mathcal{I}$ belongs to all four classes of the narrow monotonicity hierarchy. Indeed, since it is systemic, all four narrow monotonicity conditions boil down to checking that, for all $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. This is obvious, since the only $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, with $T \lneq T'$, are $T = \{\{1\}\}$ and $T' = \{\{0,1\}\}$ and $\Omega(T) = \Delta^{\mathbf{F}} \leq \nabla^{\mathbf{F}} = \Omega(T')$.

On the other hand, we have $\{\varnothing\} \leq \{\{1\}\}$, whereas $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$, whence $\mathcal{I}$ is not prealgebraic.

We look, next, at relationships between narrow monotonicity and rough monotonicity classes. We show that the two family versions coincide and that, for the remaining three properties, each of the rough versions implies the corresponding narrow version.

Theorem 527 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ if roughly family monotone iff it is narrowly family monotone;

(b) If $\mathcal{I}$ is roughly left monotone, then it is narrowly left monotone;

(c) If $\mathcal{I}$ is roughly right monotone, then it is narrowly right monotone;

(d) If $\mathcal{I}$ is roughly system monotone, then it is narrowly system monotone.

Proof:

(a) Suppose that $\mathcal{I}$ is roughly family monotone and let $T, T' \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, such that $T \le T'$. Since $T, T' \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, $\widetilde{T} = T$ and $\widetilde{T'} = T'$, whence, by hypothesis, $\widetilde{T} \le \widetilde{T'}$. Thus, by rough family monotonicity, $\Omega(T) \le \Omega(T')$ and, therefore, $\mathcal{I}$ is narrowly family monotone. Suppose, conversely, that $\mathcal{I}$ is narrowly family monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T} \le \widetilde{T'}$. Since $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, we get, by narrow family monotonicity, $\Omega(\widetilde{T}) \le \Omega(\widetilde{T'})$. Therefore, by Proposition 369, $\Omega(T) \le \Omega(T')$ and, hence, $\mathcal{I}$ is roughly family monotone.

(b) Suppose that $\mathcal{I}$ is roughly left monotone, i.e., that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$ implies $\Omega(T) \le \Omega(T')$. Assume, for the sake of obtaining a contradiction, that $\mathcal{I}$ is not narrowly left monotone. Then, there exist $X, Y \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, such that $\overleftarrow{X} \le \overleftarrow{Y}$ and $\Omega(X) \nleq \Omega(Y)$.

First, observe that, if there existed $Z \in \mathrm{ThFam}(\mathcal{I})$ and $P \in |\mathbf{Sign}^\flat|$, such that $Z_P \ne \varnothing$ and $\overleftarrow{Z}_P = \varnothing$, then, setting $Z' = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$Z'_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \ne P \\ Z_P, & \text{if } \Sigma = P \end{cases},$$

we would have $\widetilde{\overleftarrow{Z'}} = \widetilde{\overleftarrow{\varnothing}}$, but $\Omega(Z') \ne \Omega(\overline{\varnothing})$, which contradicts rough left monotonicity. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma \ne \varnothing$ implies $\overleftarrow{T}_\Sigma \ne \varnothing$.

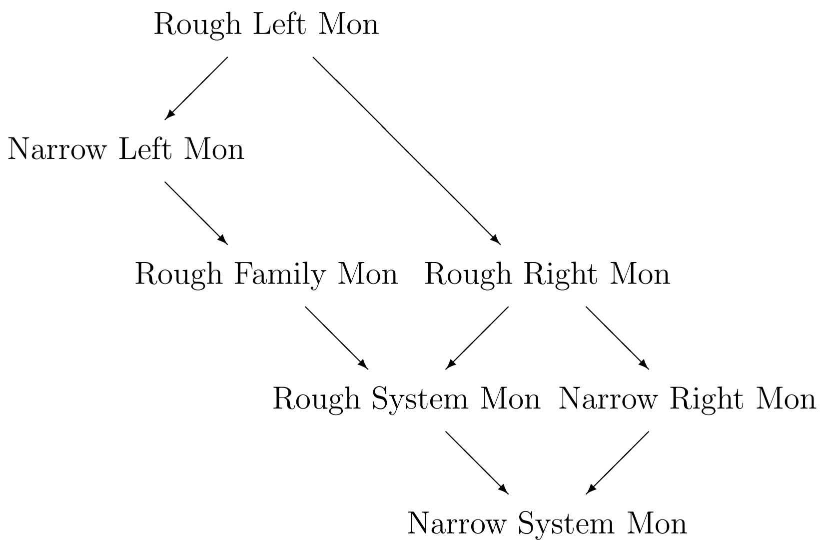
Continuing with the proof, by hypothesis, $\overleftarrow{X} \le \overleftarrow{Y}$ and $\Omega(X) \nleq \Omega(Y)$. Hence, by rough left monotonicity, $\widetilde{\overleftarrow{X}} \nleq \widetilde{\overleftarrow{Y}}$. Thus, there exists $P \in |\mathbf{Sign}^\flat|$, such that $\widetilde{\overleftarrow{X}}_P \nsubseteq \widetilde{\overleftarrow{Y}}_P$, whereas $\overleftarrow{X}_P \subseteq \overleftarrow{Y}_P$. But this gives $\overleftarrow{X}_P = \varnothing$, whence, by the preceding observation, $X_P = \varnothing$, which contradicts $X \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$. Therefore, $\mathcal{I}$ must be narrowly left monotone.

(c) Suppose that $\mathcal{I}$ is roughly right monotone and let $T, T' \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, such that $T \le T'$. Since $T, T' \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, we get $\widetilde{T} = T$ and $\widetilde{T'} = T'$, whence, by hypothesis, $\widetilde{T} \le \widetilde{T'}$. By rough right monotonicity, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$, whence $\mathcal{I}$ is narrowly right monotone.

(d) Suppose that $\mathcal{I}$ is roughly system monotone and let $T, T' \in \mathrm{ThSys}^{\nparallel}(\mathcal{I})$, such that $T \le T'$. Since $T, T' \in \mathrm{ThSys}^{\nparallel}(\mathcal{I})$, $\widetilde{T} = T$ and $\widetilde{T'} = T'$, whence, by hypothesis, $\widetilde{T} \le \widetilde{T'}$. Thus, by rough system monotonicity, $\Omega(T) \le \Omega(T')$ and, therefore, $\mathcal{I}$ is narrowly system monotone.

■

Thus, the following mixed rough monotonicity and narrow monotonicity hierarchy emerges.



To see that all classes in the hierarchy are different, we must find examples that separate the class of rough monotone from the class of narrow monotone π-institutions for each of the three allegedly distinct types, subject to the inclusions established in Theorem 527.

First, we provide an example of a narrowly left monotone π-institution that is not roughly left monotone. This proves that the class of roughly left monotone π-institutions is a proper subclass of the class of narrowly left monotone ones.

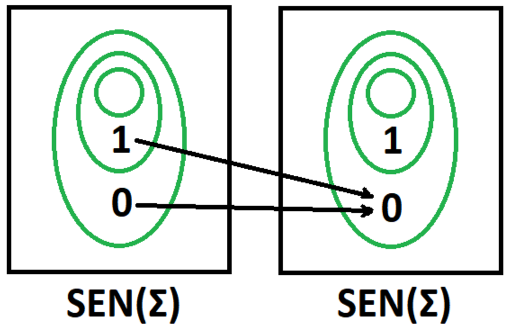
Example 528 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

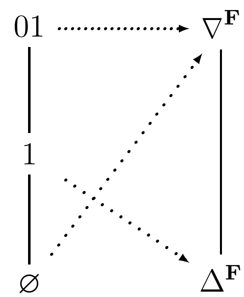
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$*,* $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, but only two theory systems,* $\{\varnothing\}$ *and* $\{\{0,1\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*





To see that $\mathcal{I}$ is narrowly left monotone, note that the only two different theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$ *are* $\{\{1\}\}$ *and* $\{\{0,1\}\}$ *and we have*

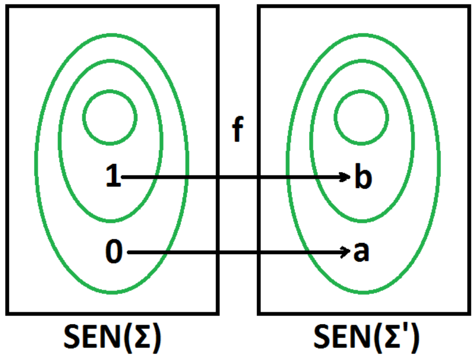
$$\overleftarrow{\{\{1\}\}} = \{\varnothing\} \le \{\{0,1\}\} = \overleftarrow{\{\{0,1\}\}}$$
$$\text{and} \quad \Omega(\{\{1\}\}) = \Delta^{\mathbf{F}} \le \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}).$$

On the other hand, $\mathcal{I}$ is not roughly left monotone, since $\widetilde{\overleftarrow{\{\varnothing\}}} = \{\{0,1\}\} = \widetilde{\overleftarrow{\{\{1\}\}}}$*, but* $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$*.*

Next we exhibit a narrowly right monotone but not roughly right monotone π-institution, showing that the class of roughly right monotone π-institutions is a proper subclass of that of the narrowly right monotone ones.

Example 529 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

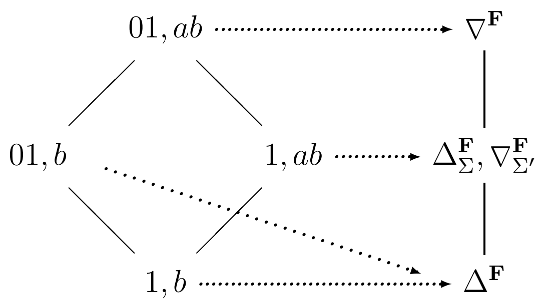
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are only four theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$*, all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



To see that $\mathcal{I}$ *is narrowly right monotone, we check all cases comparing theory families in* $\mathrm{ThFam}^{\natural}(\mathcal{I})$*:*

$$\begin{array}{ll}
\{1,b\} \le \{01,b\}, & \Omega(\overleftarrow{\{1,b\}}) = \Omega(\{1,b\}) = \Omega(\overleftarrow{\{01,b\}});\\
\{1,b\} \le \{1,ab\}, & \Omega(\overleftarrow{\{1,b\}}) = \Delta^{\mathbf{F}} \le \Omega(\overleftarrow{\{1,ab\}});\\
\{01,b\} \le \{01,ab\}, & \Omega(\overleftarrow{\{01,b\}}) \le \nabla^{\mathbf{F}} = \Omega(\overleftarrow{\{01,ab\}});\\
\{1,ab\} \le \{01,ab\}, & \Omega(\overleftarrow{\{1,ab\}}) \le \nabla^{\mathbf{F}} = \Omega(\overleftarrow{\{01,ab\}}).
\end{array}$$

On the other hand, since $\widetilde{\{1,\varnothing\}} \le \widetilde{\{1,ab\}}$*, but*

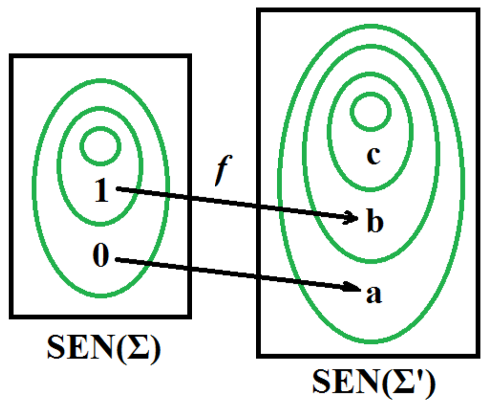
$$\Omega(\overleftarrow{\{1,\varnothing\}}) = \Omega(\{\varnothing,\varnothing\}) \nleq \Omega(\{1,ab\}) = \Omega(\overleftarrow{\{1,ab\}}),$$

$\mathcal{I}$ *is not roughly right monotone.*

Finally, we present an example of a narrowly system monotone π-institution which fails to be roughly system monotone, thereby establishing that the class of roughly system monotone π-institutions is properly contained in the class of narrowly system monotone ones.

Example 530 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two object* Σ, Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



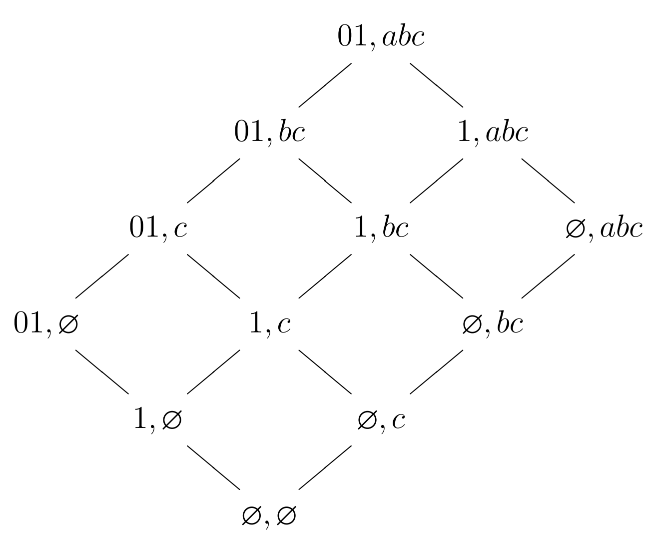
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{c\}, \{b,c\}, \{a,b,c\}\}.$$

$\mathcal{I}$ *has twelve theory families, but only seven theory systems. These are*

$$\overline{\varnothing}, \{\varnothing, c\}, \{\varnothing, bc\}, \{\varnothing, abc\}, \{1, bc\}, \{1, abc\}, \{01, abc\}.$$

The following diagram shows the structure of the lattice of theory families.



To see that $\mathcal{I}$ is narrow system monotone, note that there are only three theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$*, namely,* $\{1,bc\}$*,* $\{1,abc\}$ *and* $\{01,abc\}$ *and we have* $\{1,bc\} \leq \{1,abc\} \leq \{01,abc\}$ *and, also,*

$$\begin{aligned}\Omega(\{1,bc\}) = \{\Delta^{\mathbf{F}}_{\Sigma},\{a,bc\}\} & \\ \leq \Omega(\{1,abc\}) &= \{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\} \\ \leq \Omega(\{01,abc\}) &= \nabla^{\mathbf{F}}.\end{aligned}$$

On the other hand, setting $T = \{\varnothing,c\}$ *and* $T' = \{\varnothing,bc\}$*, which are both theory systems, we get*

$$\widetilde{T} = \{01,c\} \leq \{01,bc\} = \widetilde{T}',$$

whereas

$$\Omega(T) = \{\nabla^{\mathbf{F}}_{\Sigma},\{ab,c\}\} \nleq \{\Delta^{\mathbf{F}}_{\Sigma},\{a,bc\}\} = \Omega(T').$$

Therefore, $\mathcal{I}$ *is not roughly system monotone.*

The narrow monotonicity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. Recall the notations $\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$ and $\mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A})$ for the collections of $\mathcal{I}$-filter families and $\mathcal{I}$-filter systems, respectively, of an $\mathbf{F}$-algebraic system $\mathcal{A}$ all of whose components are nonempty.

Theorem 531 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly family monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T,T' \in \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*

(b) $\mathcal{I}$ *is narrowly left monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T,T' \in \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $\overleftarrow{T} \leq \overleftarrow{T'}$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*

(c) $\mathcal{I}$ *is narrowly right monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T,T' \in \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$*;*

(d) $\mathcal{I}$ *is narrowly system monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T,T' \in \mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*.*

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is narrowly family monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, such that $T \leq T'$. Then $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. Since, by Lemmas 51 and 376, both $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow family monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is narrowly left monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, such that $\overleftarrow{T} \leq \overleftarrow{T'}$. Then $\alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. By Lemma 6, $\overleftarrow{\alpha^{-1}(T)} \leq \overleftarrow{\alpha^{-1}(T')}$. Since, by Lemmas 51 and 376, $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow left monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$, whence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by the surjectivity of $\langle F, \alpha \rangle$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

Parts (c) and (d) may be proved similarly. ■

Finally, in analogs of Propositions 514 and 515, we recast the narrow monotonicity properties in terms of the monotonicity of mappings from posets of theory or filter families/systems into posets of congruence systems.

Proposition 532 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family monotone;*

(b) $\Omega : \mathbf{ThFam}^{\nmid}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for narrow system monotonicity, we have

Proposition 533 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system monotone;*

(b) $\Omega : \mathbf{ThSys}^{\nmid}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiSys}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

7.5 Rough Complete Monotonicity

In this section we study classes of π-institutions defined using complete monotonicity properties of the Leibniz operator applied on rough equivalence classes.

Definition 534 (Rough c-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **roughly family completely monotone**, *or* **roughly family c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is called* **roughly left completely monotone**, *or* **roughly left c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is called* **roughly right completely monotone**, *or* **roughly right c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$
- $\mathcal{I}$ *is called* **roughly system completely monotone**, *or* **roughly system c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

We start with an analog of Corollary 503 applying to rough complete monotonicity properties.

Corollary 535 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly left monotone, then it is roughly family c-monotone if and only if it is roughly right c-monotone.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone. Then, by Lemma 502, it is stable. Now note that rough family c-monotonicity is equivalent to the condition that, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \text{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T),$$

which, by stability, is equivalent to, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \text{implies} \quad \Omega(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}),$$

and this is equivalent, by definition, to rough right c-monotonicity. ∎

We establish a rough complete monotonicity hierarchy analogous to the one obtained in Proposition 505 for rough monotonicity.

Proposition 536 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly family c-monotone, then it is roughly system c-monotone;*

(b) If $\mathcal{I}$ *is roughly left c-monotone, then it is roughly system c-monotone;*

(c) If $\mathcal{I}$ *is roughly right c-monotone, then it is roughly system c-monotone.*

Proof:

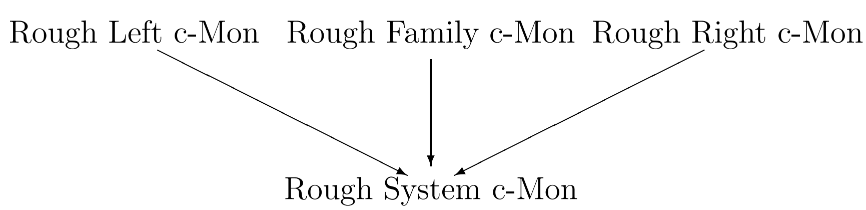
(a) The definition of rough system c-monotonicity is a specialization of that of rough family c-monotonicity, in which the universal quantification is restricted over theory systems.

(b) Suppose $\mathcal{I}$ is roughly left c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T'}$. Since $\mathcal{T} \cup \{T'\}$ consists of theory systems, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$. Hence, we get $\widetilde{\overleftarrow{T'}} \le \bigcup_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}}$. Thus, by rough left monotonicity, we get $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is roughly system c-monotone.

(c) Suppose $\mathcal{I}$ is roughly right monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Then, by rough right monotonicity, $\Omega(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Since $\mathcal{T} \cup \{T'\}$ consists of theory systems, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$, whence $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$ and, hence, $\mathcal{I}$ is roughly system c-monotone.

∎

We have now established the following **rough c-monotonicity hierarchy** of π-institutions.



In an analog of Proposition 506, it is shown that being roughly left c-monotone is equivalent to being roughly system c-monotone and stable.

Proposition 537 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left c-monotone if and only if it is roughly system c-monotone and stable.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left c-monotone. Then, by Proposition 536, it is roughly system c-monotone. Moreover, it is, a fortiori, roughly left monotone and, hence, by Lemma 502, it is stable.

Assume, conversely, that $\mathcal{I}$ is stable and roughly system c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\overleftarrow{T}}$. Then, since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, we get, by rough system c-monotonicity, $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Thus, by stability, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is roughly left c-monotone. ■

We show, next, that the rough complete monotonicity hierarchy collapses to two classes under stability and to a single class under rough systemicity.

Proposition 538 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is stable and roughly system c-monotone, then it is roughly left c-monotone.*

(b) If $\mathcal{I}$ *is stable, then it is roughly family c-monotone if and only if it is roughly right c-monotone.*

Proof:

(a) By Proposition 537.

(b) Suppose that $\mathcal{I}$ is stable. Then, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$, $T \in \mathcal{T}$, and $\Omega(\overleftarrow{T'}) = \Omega(T')$, whence the conditions defining rough family c-monotonicity and rough right c-monotonicity become identical. Therefore, under stability, roughly family and roughly right c-monotone π-institutions coincide.

■

By Proposition 538, under stability, we get the reduced hierarchy

Rough Family/Right c-Monotonicity

$\downarrow$

Rough System/Left c-Monotonicity

We also get that rough systemicity causes the further collapse of the entire hierarchy into a single class.

Proposition 539 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For roughly systemic* π*-institutions, all four rough c-monotonicity classes coincide.*

Proof: Since rough systemicity implies stability, by Proposition 538, it suffices to show that, if $\mathcal{I}$ is roughly system c-monotone and roughly systemic, then it is roughly family c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. By rough systemicity, $\overleftarrow{\widetilde{T}} = \widetilde{T}$, for all $T \in \mathcal{T}$, and $\overleftarrow{\widetilde{T'}} = \widetilde{T'}$. Therefore, $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}}$. Thus, since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\}$ consists of theory systems, by rough system c-monotonicity, $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Since rough systemicity implies stability, $\Omega(\overleftarrow{T}) = \Omega(T)$, for all $T \in \mathcal{T}$, and $\Omega(\overleftarrow{T'}) = \Omega(T')$. Thus, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is roughly family c-monotone. ■

We present several examples to show that all four rough complete monotonicity classes depicted in the diagram above are different and that the hierarchy is exactly as shown.

The first example gives a roughly left c-monotone π-institution which is not roughly family c-monotone. Thus, it shows that the class of roughly family c-monotone π-institutions is properly contained in the class of roughly system c-monotone π-institutions and that, moreover, the class of roughly left c-monotone π-institutions is not a subclass of the class of roughly family c-monotone π-institutions.

Example 540 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

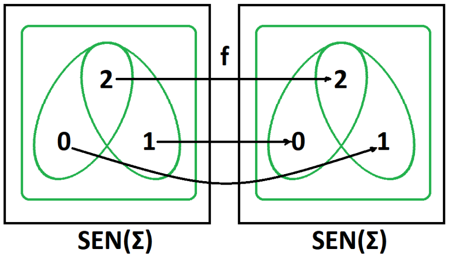
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$*, in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$*:*

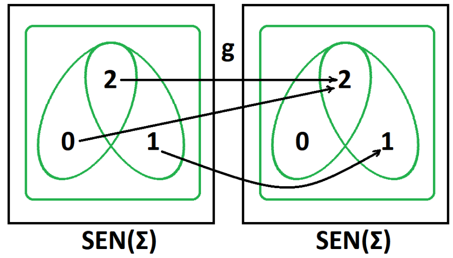
$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

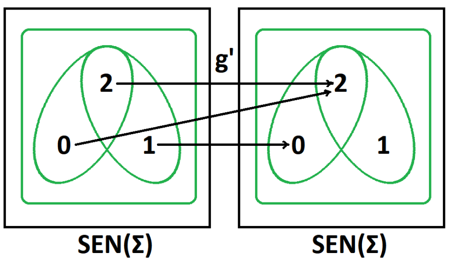
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

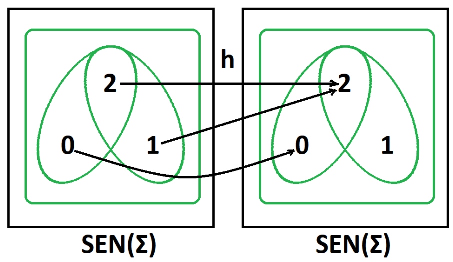
x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

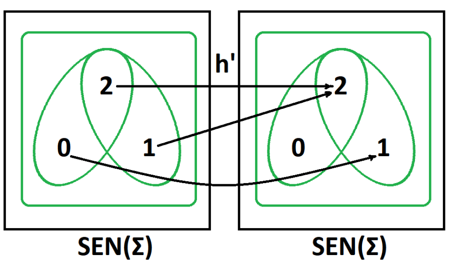
- *$N^\flat$ is the trivial clone.*

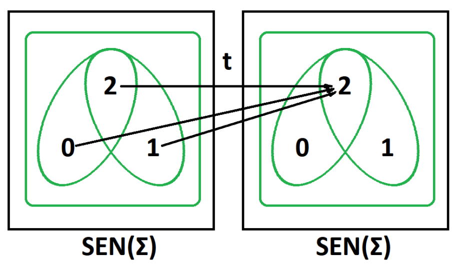












Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

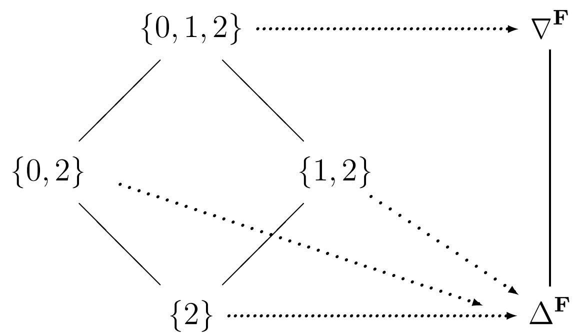
$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$*.*

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ has only two theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$*, and* $\mathrm{SEN} = \{\{0,1,2\}\}$*.*

To show that $\mathcal{I}$ is (roughly) left c-monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \le \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$.

- *If $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*
$$\Omega(T') \le \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \le \bigcup_{T \in \mathcal{T}} \Omega(T);$$
- *If $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \ne \{\{0,1,2\}\}$, whence*
$$\Omega(T') = \Delta^{\mathbf{F}} \le \bigcup_{T \in \mathcal{T}} \Omega(T).$$

Thus, in any case, $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$ and $\mathcal{I}$ is roughly left c-monotone.

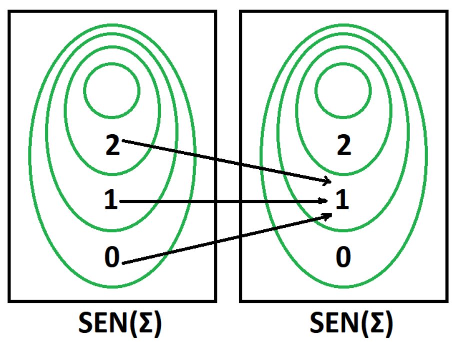
On the other hand, we have
$$\{\{0,1,2\}\} \le \{\{0,2\}\} \cup \{\{1,2\}\},$$
whereas
$$\Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\}).$$
Therefore, $\mathcal{I}$ is not roughly family c-monotone.

An additional conclusion obtained from Example 540, combined with the statement of Corollary 535, is that the class of roughly left c-monotone π-institutions is not a subclass of the class of roughly right c-monotone π-institutions either, since that inclusion would imply that the former is also a subclass of the class of roughly family c-monotone π-institutions, contradicting Example 540.

The second example shows that there exists a roughly family c-monotone π-institution that is not roughly right c-monotone, thus showing, on the one hand, that the class of roughly right c-monotone π-institutions is properly included in the class of roughly system c-monotone π-institutions and, on the other, that roughly family c-monotone π-institutions do not form a subclass of roughly right c-monotone π-institutions.

Example 541 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = \mathrm{SEN}^\flat(f)(2) = 1$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 2$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 2$.



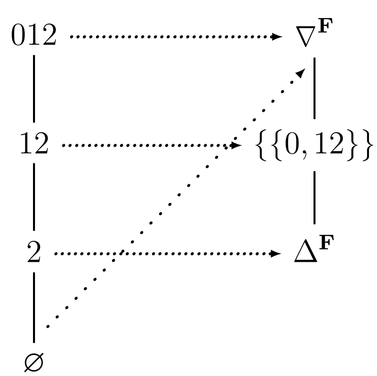
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

There are four theory families, but only three theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$
$\{\varnothing\}$	$\{\varnothing\}$
$\{2\}$	$\{\varnothing\}$
$\{12\}$	$\{12\}$
$\{012\}$	$\{012\}$

The lattice of theory families of $\mathcal{I}$ *together with the Leibniz congruence systems are shown in the diagram.*



We show that $\mathcal{I}$ is roughly family c-monotone. To this end, suppose $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$.

- *If $\widetilde{T'} = \{012\}$, then we must have $\{\varnothing\} \in \mathcal{T}$ or $\{012\} \in \mathcal{T}$. Hence, we get $\Omega(T') = \nabla^{\mathbf{F}} = \bigcup_{T \in \mathcal{T}} \Omega(T)$;*
- *If $\widetilde{T'} = \{12\}$, then $\mathcal{T}$ must include one of $\{12\}$, $\{\varnothing\}$ or $\{012\}$. Hence, we get $\Omega(T') = \{\{0,12\}\} \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$;*
- *If $\widetilde{T'} = \{2\}$, then $T' = \{2\}$ and, hence, $\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$.*

Therefore, $\mathcal{I}$ is indeed roughly family c-monotone.

On the other hand, we have $\widetilde{\{2\}} = \{2\} \leq \{12\} = \widetilde{\{12\}}$, whereas

$$\Omega(\overleftarrow{\{2\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} \nleq \{\{0,12\}\} = \Omega(\{12\}) = \Omega(\overleftarrow{\{12\}}).$$

Therefore, $\mathcal{I}$ is not roughly right c-monotone.

An additional conclusion obtained from Example 541, combined with the statement of Corollary 535, is that the class of roughly family c-monotone π-institutions is not a subclass of the class of roughly left c-monotone π-institutions. Otherwise, by Corollary 535, rough family c-monotonicity would imply rough right c-monotonicity, contradicting Example 541.

The third example shows that there exists a roughly right c-monotone π-institution that is not roughly left c-monotone. It establishes that the class of roughly left c-monotone π-institutions is properly contained in the class of roughly system c-monotone π-institutions and, also, that the class of roughly right c-monotone π-institutions does not form a subclass of the class consisting of the roughly left c-monotone ones.

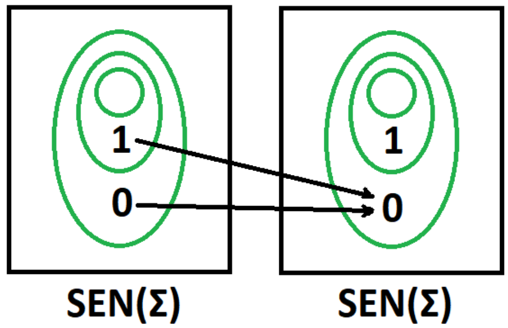
Example 542 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

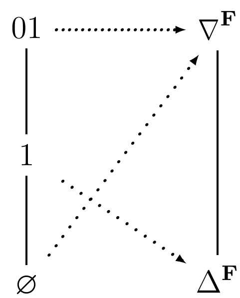
- *$\mathbf{Sign}^\flat$ is the category with the single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;*
- *$N^\flat$ is the trivial clone, consisting of the projections only.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\varnothing$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\varnothing$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.





We show that $\mathcal{I}$ is roughly right c-monotone. Suppose $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Since, for all $T \in \mathcal{T}$, we have $\overleftarrow{T} = \{\varnothing\}$ or $\overleftarrow{T} = \{01\}$, we have, trivially,

$$\Omega(\overleftarrow{T'}) \le \nabla^{\mathbf{F}} = \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$

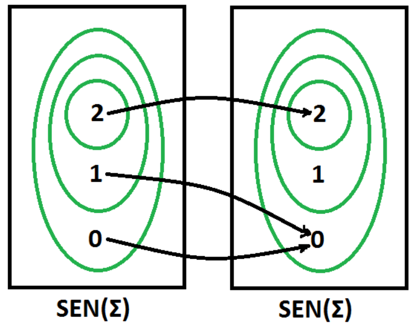
Thus, $\mathcal{I}$ is indeed roughly right monotone.

On the other hand, we have $\widetilde{\overleftarrow{\{\varnothing\}}} = \{01\} = \widetilde{\{\varnothing\}} = \widetilde{\overleftarrow{\{1\}}}$, but $\Omega(\{\varnothing\}) \nleq \Omega(\{1\})$. Therefore, $\mathcal{I}$ is not roughly left c-monotone.

The last example in this series depicts a roughly right c-monotone π-institution, which is not roughly family c-monotone. This shows that the class of roughly right c-monotone π-institutions does not form a subclass of the class of roughly family c-monotone ones.

Example 543 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is given by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ and $\mathrm{SEN}^\flat(f)(2) = 2$;*
- *$N^\flat$ is the trivial clone.*

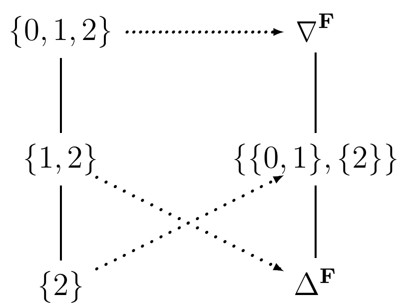


Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}$. Since $\mathcal{I}$ has theorems, rough equivalence on ThFam$(\mathcal{I})$ *coincides with the identity relation.*

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



We show that $\mathcal{I}$ is roughly right c-monotone. Suppose $\mathcal{T} \cup \{T'\} \subseteq$ ThFam$(\mathcal{I})$, *such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$.*

- *If $T' = \{012\}$, then $T' \in \mathcal{T}$, whence we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$;*
- *If $T' = \{12\}$, then $\{12\} \in \mathcal{T}$ or $\{012\} \in \mathcal{T}$. In either case $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$;*
- *If $T' = \{2\}$, then $\mathcal{T} \neq \varnothing$ and, since $\overleftarrow{\{12\}} = \{2\}$, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$.*

Therefore, $\mathcal{I}$ is roughly right c-monotone. On the other hand, since $\{2\} \leq \{12\}$, but $\Omega(\{2\}) = \{\{01, 2\}\} \nleq \Delta^{\mathbf{F}} = \Omega(\{12\})$, $\mathcal{I}$ is not roughly family c-monotone.

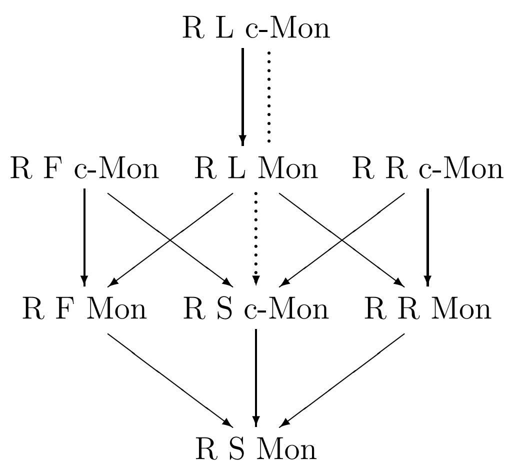
We conclude, after these four examples, that the structure of the rough complete monotonicity hierarchy is exactly as depicted in the diagram and no two classes are identical.

We look, next at the connections between the classes in the rough monotonicity and rough complete monotonicity hierarchies. We have a straightforward

Proposition 544 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly family (respectively, left, right, system) c-monotone, then it is roughly family (respectively, left, right, system) monotone.*

Proof: The condition defining a rough monotonicity class is a special case of the condition defining the respective rough c-monotonicity class, where the collection $\mathcal{T}$, in that definition, is taken to be a singleton. ■

Proposition 544, in view of Propositions 505 and 536, establishes the hierarchy depicted in the diagram (the dotted line and arrow represent jointly a single arrow signifying the inclusion of the class of roughly left c-monotone into the class of roughly system c-monotone π-institutions).



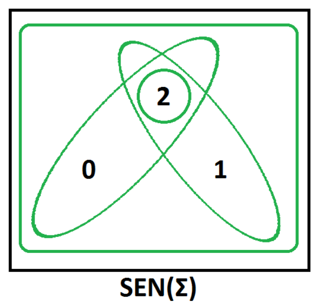
We present an example to show that the two hierarchies are separated. It shows a π-institution, which belongs to all steps of the rough monotonicity hierarchy but to none of the four rough complete monotonicity classes.

Example 545 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*

- *$N^\flat$ is the clone generated by the unary natural transformation $\sigma^\flat$: $\mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, given by*

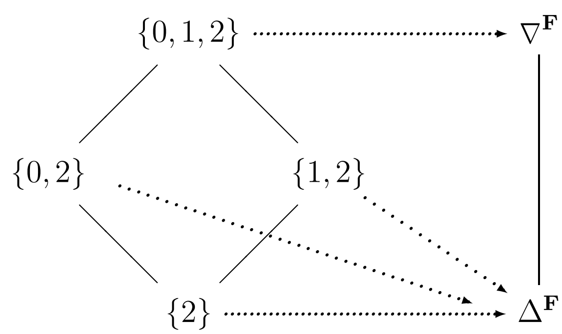
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



Since $\mathbf{Sign}^\flat$ is trivial, $\mathcal{I}$ is systemic and, since $\mathcal{I}$ has theorems, rough equivalence is the identity relation on $\mathrm{FiFam}(\mathcal{I})$. *We conclude that all four rough monotonicity properties for $\mathcal{I}$ coincide and, moreover, they are identical with both monotonicity properties, which they also coincide, due to systemicity. The same holds for c-monotonicity. All four rough c-monotonicity properties coincide and they, in turn, are identical with all c-monotonicity conditions.*

From the diagram one can verify immediately that $\mathcal{I}$ is (roughly left, right and family) monotone, On the other hand, we have $\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\}$, but, obviously, $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be roughly system c-monotone.

We turn, next, our attention to the relations between the classes in the rough c-monotonicity hierarchy and those in the c-monotonicity hierarchy. We start by showing that possessing any type of c-monotonicity forces a π-institution to either have theorems or, else, to have only one theory system rough equivalence class.

Proposition 546 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left c-monotone without theorems, then* $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$*.*

(b) If $\mathcal{I}$ *is system c-monotone without theorems, then* $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$*.*

Proof:

(a) Suppose that $\mathcal{I}$ is left c-monotone and does not have theorems. If $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| > 1$, then there exists $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T} \neq \mathrm{SEN}^\flat$. Thus, we get

$$\overleftarrow{\overline{\varnothing}} = \overline{\varnothing} \leq \overleftarrow{T} \quad \text{and} \quad \Omega(\overline{\varnothing}) \nleq \Omega(T).$$

Therefore, $\mathcal{I}$ is not left c-monotone, a contradiction. Thus, we must have $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$, as claimed.

(b) Suppose that $\mathcal{I}$ is system c-monotone and does not have theorems. If $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| > 1$, then there exists $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \neq \mathrm{SEN}^\flat$. Thus, we get

$$\overline{\varnothing} < T \quad \text{and} \quad \Omega(\overline{\varnothing}) \nleq \Omega(T).$$

Therefore, $\mathcal{I}$ is not system c-monotone, a contradiction. Thus, we must have $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$, as claimed.

■

We can establish the following relations between c-monotonicity and rough c-monotonicity classes.

Proposition 547 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left c-monotone, then it is roughly left c-monotone;*

(b) If $\mathcal{I}$ *is family c-monotone, then it is roughly family c-monotone;*

(c) If $\mathcal{I}$ *is right c-monotone, then it is roughly right c-monotone;*

(d) If $\mathcal{I}$ *is system c-monotone, then it is roughly system c-monotone.*

Proof:

(a) Suppose that $\mathcal{I}$ is left c-monotone. Let $\mathcal{T}\cup\{T'\}\subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\overleftarrow{T'}} \le \bigcup_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}}$. If $\mathcal{I}$ has theorems, then $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$, for all $T\in\mathcal{T}$, and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$, whence $\overleftarrow{T'} \le \bigcup_{t\in\mathcal{T}} \overleftarrow{T}$. Thus, by left c-monotonicity, $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Proposition 546, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(T') = \bigcup_{T\in\mathcal{T}} \Omega(T)$.

(b) Suppose that $\mathcal{I}$ is family c-monotone. Let $\mathcal{T}\cup\{T'\}\subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T\in\mathcal{T}} \widetilde{T}$. If $\mathcal{I}$ has theorems, then we get $\widetilde{T} = T$, for all $T\in\mathcal{T}$, and $\widetilde{T'} = T'$. Thus, $T' \le \bigcup_{T\in\mathcal{T}} T$. By family c-monotonicity, we now get $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Propositions 186 and 546, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(T') = \bigcup_{T\in\mathcal{T}} \Omega(T)$.

(c) Suppose that $\mathcal{I}$ is right c-monotone. Let $\mathcal{T}\cup\{T'\}\subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T\in\mathcal{T}} \widetilde{T}$. If $\mathcal{I}$ has theorems, then we get $\widetilde{T} = T$, for all $T\in\mathcal{T}$, and $\widetilde{T'} = T'$. Thus, $T' \le \bigcup_{T\in\mathcal{T}} T$. By right c-monotonicity, we now get $\Omega(\overleftarrow{T'}) \le \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Propositions 187 and 546, $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(\overleftarrow{T'}) = \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$.

(d) Suppose that $\mathcal{I}$ is system c-monotone. Let $\mathcal{T}\cup\{T'\}\subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T\in\mathcal{T}} \widetilde{T}$. If $\mathcal{I}$ has theorems, then we get $\widetilde{T} = T$, for all $T\in\mathcal{T}$, and $\widetilde{T'} = T'$. Thus, $T' \le \bigcup_{T\in\mathcal{T}} T$. By system c-monotonicity, we now get $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Proposition 546, $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(T') = \bigcup_{T\in\mathcal{T}} \Omega(T)$.

■

We can now prove the following additional, and more precise, relations.

Theorem 548 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (left, respectively) c-monotone if and only if it is roughly family (left, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is family or left c-monotone. Since, by hypothesis, it is not almost inconsistent, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ has theorems. Moreover, by Theorem 547, it is roughly family or left c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is roughly family (or left c-monotone) and has theorems. Let $\mathcal{T}\cup\{T'\}\subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \le \bigcup_{T\in\mathcal{T}} T$ (or $\overleftarrow{T'} \le \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$, respectively). Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get $\widetilde{T'} \le \bigcup_{T\in\mathcal{T}} \widetilde{T}$ (or $\widetilde{\overleftarrow{T'}} \le \bigcup_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}}$). Using rough family (or left, respectively) c-monotonicity, we obtain $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is family (or left) c-monotone. ■

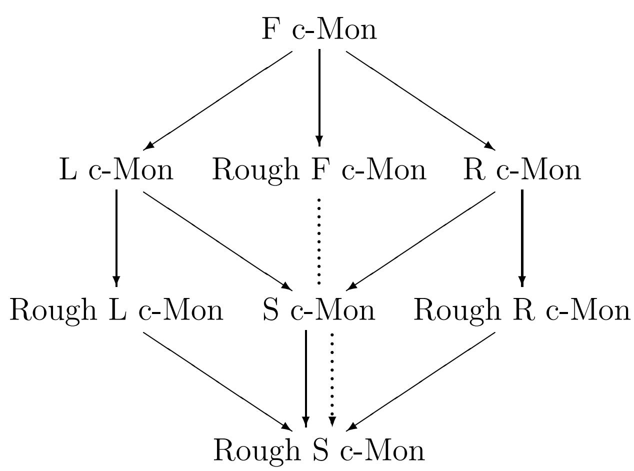
Analogously, for the cases of system and right c-monotonicity, we get the following

Theorem 549 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory family* $T \neq \mathrm{SEN}^\flat$ *such that* $\overleftarrow{T} \neq \overline{\varnothing}$*.* $\mathcal{I}$ *is system (right, respectively) c-monotone if and only if it roughly system (right, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is system or right c-monotone. Since, by hypothesis, it has a theory system $\overleftarrow{T} \neq \mathrm{SEN}^\flat, \overline{\varnothing}$, we get $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ must have theorems. Moreover, by Theorem 547, it is roughly system or right c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is roughly system (or right) c-monotone and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$ (or $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, respectively), such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. By rough system (or right, respectively) c-monotonicity, we obtain $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ (or $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$, respectively). Therefore, $\mathcal{I}$ is system (or right) c-monotone. ■

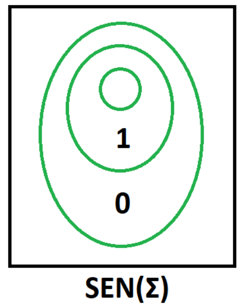
The preceding propositions allow us to draw the following hierarchical diagram concerning the complete monotonicity and the rough complete monotonicity classes.



To see that the rough c-monotonicity classes are separated from the c-monotonicity classes, we give an example. A π-institution is presented which belongs to all four rough c-monotonicity classes, but fails to be system c-monotone and, therefore, belongs to none of the four c-monotonicity classes. The secret lies, of course, in the absence of theorems.

Example 550 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$;
- $N^\flat$ *is the trivial clone.*

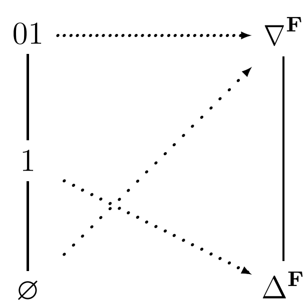


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



$\mathcal{I}$ *belongs to all four classes of the rough c-monotonicity hierarchy. Indeed, since it is systemic, all four rough monotonicity conditions boil down to checking that, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$.

- *If* $\bigcup_{T \in \mathcal{T}} \widetilde{T} = \{01\}$*, then* $\mathcal{T}$ *must include* $\{\varnothing\}$ *or* $\{01\}$*, whence* $\Omega(T') \leq \nabla^{\mathbf{F}} = \bigcup_{T \in \mathcal{T}} \Omega(T)$;
- *If* $\bigcup_{T \in \mathcal{T}} \widetilde{T} = \{1\}$*, then* $\widetilde{T'} = \{1\}$ *and, hence,* $\mathcal{T} = \{\{1\}\}$ *and* $T' = \{1\}$*. Thus, the implication holds trivially.*

Since $\bigcup_{T \in \mathcal{T}} \widetilde{T} = \{\varnothing\}$ *cannot occur, we get that* $\mathcal{I}$ *is roughly family c-monotone.*

On the other hand, we have $\{\varnothing\} \leq \{1\}$*, whereas* $\Omega(\{\varnothing\}) \nleq \Omega(\{1\})$*, whence* $\mathcal{I}$ *is not system c-monotone.*

Next, we turn to transfer theorems for the rough c-monotonicity classes.

Theorem 551 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is roughly family c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$;

(b) $\mathcal{I}$ *is roughly left c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\overleftarrow{T'}} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$;

(c) $\mathcal{I}$ *is roughly right c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$;

(d) $\mathcal{I}$ *is roughly system c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

Proof:

(a) The "if" results by applying the hypothesis to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

For the "only if", suppose that $\mathcal{I}$ is roughly family c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Then we get $\alpha^{-1}(\widetilde{T'}) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \widetilde{T})$, whence $\alpha^{-1}(\widetilde{T'}) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{T})$. Thus, by Theorem 377, $\widetilde{\alpha^{-1}(T')} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(T)}$. Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by rough family c-monotonicity, $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(b) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\overleftarrow{T'}} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}}$. Then we get $\alpha^{-1}(\widetilde{\overleftarrow{T'}}) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}})$, whence $\alpha^{-1}(\widetilde{\overleftarrow{T'}}) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{\overleftarrow{T}})$. Thus, by Theorem 377, $\widetilde{\alpha^{-1}(\overleftarrow{T'})} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(\overleftarrow{T})}$. Hence, by Lemma 6, we get $\widetilde{\overleftarrow{\alpha^{-1}(T')}} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\overleftarrow{\alpha^{-1}(T)}}$. Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by rough left c-monotonicity, $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq$

$\alpha^{-1}(\bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(T)$.

(c) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly right c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T}\cup\{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}}\widetilde{T}$. Then we get $\alpha^{-1}(\widetilde{T'}) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}}\widetilde{T})$, whence $\alpha^{-1}(\widetilde{T'}) \leq \bigcup_{T\in\mathcal{T}}\alpha^{-1}(\widetilde{T})$. Thus, by Theorem 377, $\widetilde{\alpha^{-1}(T')} \leq \bigcup_{T\in\mathcal{T}}\widetilde{\alpha^{-1}(T)}$. Since, by Lemma 51, $\{\alpha^{-1}(T) : T\in\mathcal{T}\}\cup\{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by rough right c-monotonicity, $\Omega(\overleftarrow{\alpha^{-1}(T')}) \leq \bigcup_{T\in\mathcal{T}}\Omega(\overleftarrow{\alpha^{-1}(T)})$. Thus, by Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}}\Omega(\alpha^{-1}(\overleftarrow{T}))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}}\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T}))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(\overleftarrow{T}))$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(\overleftarrow{T})$.

(d) Similar to Part (a).

■

We close this section by giving two characterizations concerning the rough family and rough system c-monotonicity classes, based on mappings between posets satisfying the complete monotonicity property.

Proposition 552 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family c-monotone;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proposition 553 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system c-monotone;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

7.6 Narrow Complete Monotonicity

In this section we revisit classes of π-institutions defined using complete monotonicity properties of the Leibniz operator. However, complete monotonicity is only applied on theory systems/families all of whose components are nonempty.

Definition 554 (Narrow c-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **narrowly family completely monotone***, or* **narrowly family c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*

$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

- $\mathcal{I}$ *is called* **narrowly left completely monotone***, or* **narrowly left c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*

$$\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

- $\mathcal{I}$ *is called* **narrowly right completely monotone***, or* **narrowly right c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*

$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$

- $\mathcal{I}$ *is called* **narrowly system completely monotone***, or* **narrowly system c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$*,*

$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

We establish a narrow complete monotonicity hierarchy analogous to the one obtained in Proposition 536 for rough complete monotonicity.

Proposition 555 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

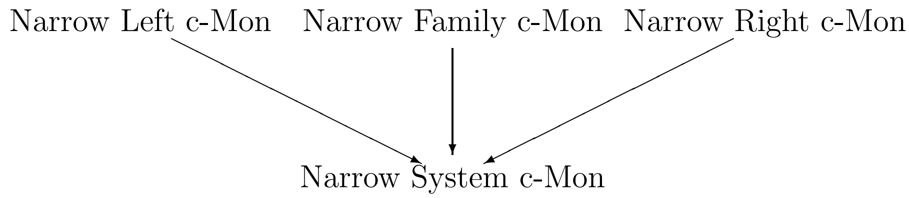
(a) If $\mathcal{I}$ *is narrowly left c-monotone, then it is narrowly system c-monotone;*

(b) If $\mathcal{I}$ *is narrowly family c-monotone, then it is narrowly system c-monotone;*

(c) If $\mathcal{I}$ *is narrowly right c-monotone, then it is narrowly system c-monotone.*

Proof: We sketch a proof that works for all three cases. Suppose that $\mathcal{I}$ is narrowly left (family or right) c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\ell}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Since $\mathcal{T} \cup \{T'\}$ consists of theory systems, we have $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$. Thus, by hypothesis, $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. Now we apply narrow left (narrow family or narrow right) c-monotonicity to get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ ($\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ or $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$). However, in all three cases, we conclude that $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is narrowly system c-monotone. ■

We have now established the following **narrow c-monotonicity hierarchy** of π-institutions.



We may establish some additional relationships between those classes once various types of stability and monotonicity are allowed into the mix. First, we show that narrow left c-monotonicity implies exclusive stability and that, under narrow stability, narrow family c-monotonicity and narrow right c-monotonicity coincide.

Proposition 556 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is narrowly left c-monotone, then it is exclusively stable.

(b) If $\mathcal{I}$ is narrowly stable, then it is narrowly family c-monotone if and only if it is narrowly right c-monotone.

Proof:

(a) Suppose $\mathcal{I}$ is narrowly left c-monotone. Let $T \in \mathrm{ThFam}^{\ell}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\ell}(\mathcal{I})$. Then, since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we get, by applying narrow left c-monotonicity, $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, $\mathcal{I}$ is exclusively stable.

(b) Suppose $\mathcal{I}$ is narrowly stable. Then, for all $T \in \mathrm{ThFam}^{\ell}(\mathcal{I})$, we have $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\ell}(\mathcal{I})$, the condition $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ is equivalent to the condition $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Therefore, the condition defining narrow family c-monotonicity is identical to that defining narrow right c-monotonicity.

■

Finally, under narrow systemicity, all four narrow complete monotonicity classes collapse into a single class.

Proposition 557 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly systemic and narrowly system c-monotone, then it has all four narrow c-monotonicity properties.*

Proof: Suppose that $\mathcal{I}$ is narrowly systemic and narrowly system c-monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\nsim(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. By narrow systemicity, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$, whence $T' \leq \bigcup_{T \in \mathcal{T}} T$ and, moreover, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^\nsim(\mathcal{I})$. Thus, by narrow system c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. This proves that $\mathcal{I}$ is narrowly left c-monotone.
- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\nsim(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. By narrow systemicity, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^\nsim(\mathcal{I})$. Thus, by narrow system c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. This proves that $\mathcal{I}$ is narrowly family c-monotone.
- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\nsim(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. By narrow systemicity, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$, and, thus, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^\nsim(\mathcal{I})$. Thus, by narrow system c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. But, then, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$, whence $\mathcal{I}$ is also narrowly right c-monotone.

■

We reuse some examples to show that all four rough complete monotonicity classes depicted in the diagram above are different and that no inclusions hold among the three top classes.

The first example gives a narrowly left c-monotone π-institution which is neither narrowly family nor narrowly right c-monotone. Thus, it shows that the classes of narrowly family and of narrowly right c-monotone π-institutions are properly contained in the class of narrowly system c-monotone π-institutions and that, moreover, the class of narrowly left c-monotone π-institutions is not a subclass of either the class of narrowly family or the class of narrowly right c-monotone π-institutions.

Example 558 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

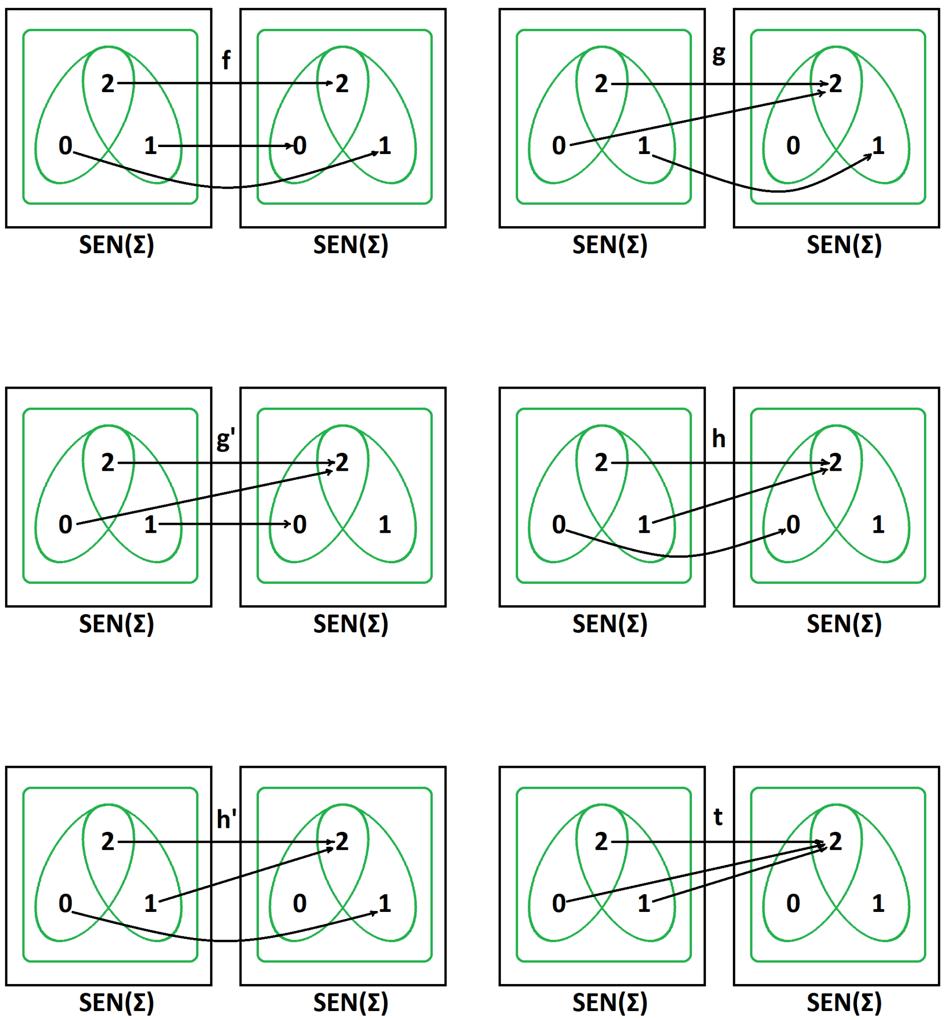
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$*, in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$*:*

$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- $N^\flat$ *is the trivial clone.*



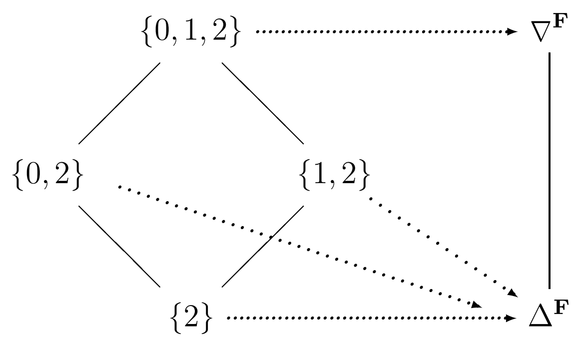
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$C_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ *has only two theory systems,* $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$, *and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

Since $\mathcal{I}$ has theorems, narrow left c-monotonicity coincides with left c-monotonicity. To show that $\mathcal{I}$ is left c-monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$.

- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*
$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigcup_{T\in\mathcal{T}} \Omega(T);$$
- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \neq \{\{0,1,2\}\}$, whence*
$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$

Thus, in any case, $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$ and $\mathcal{I}$ is left completely monotone.

On the other hand, we have

$$\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\},$$

whereas

$$\begin{array}{rcl}\Omega(\overleftarrow{\{\{0,1,2\}\}}) & = & \Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \\ & \nleq & \Delta^{\mathbf{F}} \\ & = & \Omega(\{\{2\}\}) \cup \Omega(\{\{2\}\}) \\ & = & \Omega(\overleftarrow{\{\{0,2\}\}}) \cup \Omega(\overleftarrow{\{\{1,2\}\}}).\end{array}$$

Therefore, $\mathcal{I}$ is not (narrowly) right c-monotone. Using the same theory families, we also get $\{\{0,1,2\}\} \le \{\{0,2\}\} \cup \{\{1,2\}\}$, *whereas* $\Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$, *whence $\mathcal{I}$ is not (narrowly) family c-monotone either.*

The second example shows that there exists a narrowly family c-monotone π-institution that is not narrowly right c-monotone, thus showing that narrowly family c-monotone π-institutions do not form a subclass of the class of narrowly right c-monotone π-institutions.

Example 559 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, g, o, t : \Sigma \to \Sigma$, *whose composition table is the following:*

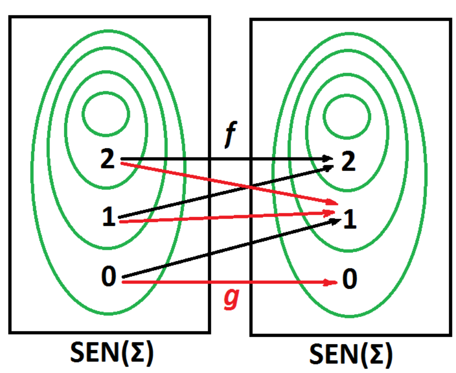
$\circ$	f	g	o	t
f	t	f	t	t
g	o	g	o	o
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, *with*

$$\begin{array}{lll} \mathrm{SEN}^\flat(f)(0) = 1, & \mathrm{SEN}^\flat(f)(1) = 2, & \mathrm{SEN}^\flat(f)(2) = 2; \\ \mathrm{SEN}^\flat(g)(0) = 0, & \mathrm{SEN}^\flat(g)(1) = 1, & \mathrm{SEN}^\flat(g)(2) = 1, \end{array}$$

  whereas $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$;

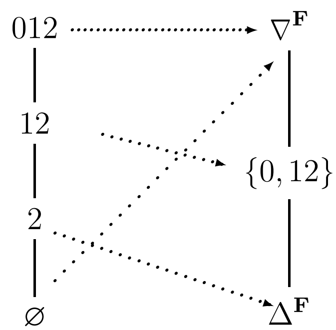
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families $\varnothing$, $\{\{2\}\}$, $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$, but only three theory systems, $\varnothing$, $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



Since, as shown in the diagram, $\Omega : \mathrm{ThFam}^{\nleq}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, $\mathcal{I}$ is narrowly family c-monotone.

On the other hand, for $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we get $T \leq T'$, whereas $\Omega(\overleftarrow{T}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \nleq \{0,12\} = \Omega(T') = \Omega(\overleftarrow{T'})$. Therefore, $\mathcal{I}$ is not narrowly right c-monotone.

The third example gives a narrowly right c-monotone π-institution which is neither narrowly family nor narrowly left c-monotone. Thus, it shows that the classes of narrowly family and of narrowly left c-monotone π-institutions are properly contained in the class of narrowly system c-monotone π-institutions and that, moreover, the class of narrowly right c-monotone π-institutions is not a subclass of either the class of narrowly family or the class of narrowly left c-monotone π-institutions.

Example 560 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

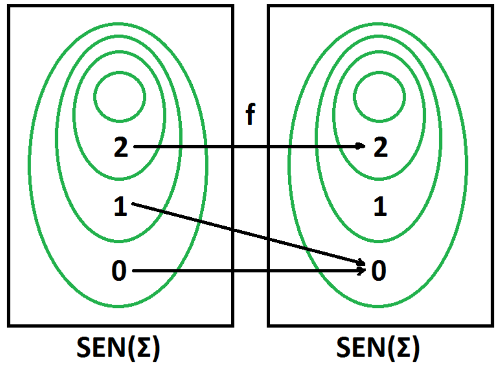
- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ and $\mathrm{SEN}^\flat(f)(2) = 2$;*
- *$N^\flat$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

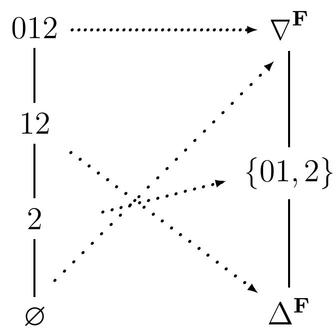
$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems, namely $\overline{\varnothing}$, $\{\{2\}\}$ and $\{\{0,1,2\}\}$. Moreover, clearly,

$$\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \{\{0,1,2\}\}\}.$$



The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



To see that $\mathcal{I}$ is narrowly right c-monotone, suppose $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Then,

- *if $T' = \{\{0,1,2\}\}$, we must have $\{\{0,1,2\}\} \in \mathcal{T}$, whence $\Omega(\overleftarrow{T'}) = \nabla^{\mathbf{F}} = \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$;*
- *if $T' \neq \{\{0,1,2\}\}$, then $\Omega(\overleftarrow{T'}) \leq \{\{01,2\}\} \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$.*

Therefore, $\mathcal{I}$ is narrowly right c-monotone.

On the other hand, for $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we get $T \leq T'$, whereas $\Omega(T) = \{01,2\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. Thus, $\mathcal{I}$ is not narrowly family c-monotone. Moreover, for the same theory families, $\overleftarrow{T} = \{\{2\}\} = \overleftarrow{T'}$, whereas $\Omega(T) = \{01,2\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. Thus, $\mathcal{I}$ is not narrowly left c-monotone.

The last example shows that there exists a narrowly family c-monotone π-institution that is not narrowly left c-monotone, thus showing that narrowly family c-monotone π-institutions do not form a subclass of the class of narrowly left c-monotone ones.

Example 561 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, z, o, t : \Sigma \to \Sigma$, *whose composition table is the following:*

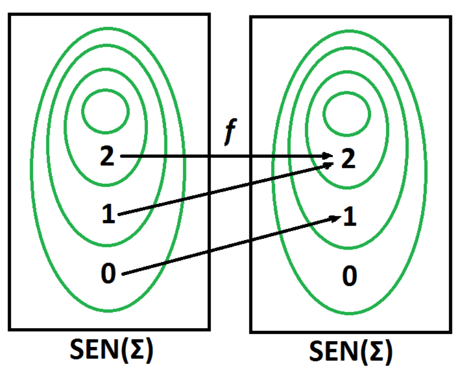
$\circ$	f	z	o	t
f	t	o	t	t
z	z	z	z	z
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$, *with*

$$\mathrm{SEN}^\flat(f)(0) = 1, \quad \mathrm{SEN}^\flat(f)(1) = 2, \quad \mathrm{SEN}^\flat(f)(2) = 2,$$

  whereas $\mathrm{SEN}^\flat(z)(x) = 0$, $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$;

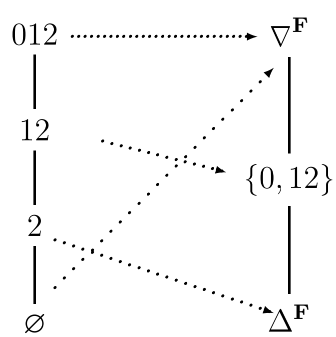
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

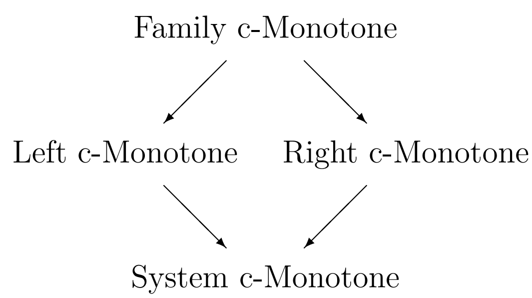
$\mathcal{I}$ *has four theory families* $\varnothing$, $\{\{2\}\}$, $\{\{1, 2\}\}$ *and* $\{\{0, 1, 2\}\}$, *but only two theory systems,* $\varnothing$ *and* $\{\{0, 1, 2\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since $\Omega : \mathrm{ThFam}^{\lightning}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism,* $\mathcal{I}$ *is narrowly family c-monotone. On the other hand, for* $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$*, we get* $\overleftarrow{T} = \overline{\varnothing} = \overleftarrow{T'}$*, whereas* $\Omega(T) = \{0,12\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$*. Therefore,* $\mathcal{I}$ *is not narrowly left c-monotone.*

We conclude, after these examples, that the structure of the narrow complete monotonicity hierarchy is, in fact, exactly as depicted in the diagram and no two classes are identical.

Recall from Chapter 3 that we have the following complete monotonicity hierarchy of π-institutions.



We establish now a combined c-monotonicity and narrow c-monotonicity hierarchy. It is not difficult to see that a c-monotonicity property implies the corresponding narrow c-monotonicity property. Alternatively, these relations can be derived by the relationships governing rough and narrow c-monotonicity classes, on the one hand, and the ones governing rough c-monotonicity and c-monotonicity classes on the other.

Proposition 562 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family (left, right, system, respectively) c-monotone, then it is narrowly family (narrowly left, narrow right, narrowly system, respectively) c-monotone.*

Proof: If $\mathcal{I}$ has a certain type of c-monotonicity, then it has, a fortiori, the same type of narrow c-monotonicity, since the condition defining the latter is a specialization of that defining the former, in which $\mathcal{T} \cup \{T'\}$ are only allowed to range over theory families or systems, as the case may be, in $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ or $\mathrm{ThSys}^{\lightning}(\mathcal{I})$, respectively. (An alternative way is to combine Proposition 547 with Theorem 566 that follows.) ■

Analogously to Theorems 548 and 549, we also get more precise relationships between c-monotonicity and narrow c-monotonicity classes.

Theorem 563 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (left, respectively) c-monotone if and only if it is narrowly family (left, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is family or left c-monotone. Since, by hypothesis, it is not almost inconsistent, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ has theorems. Moreover, by Proposition 562, it is narrowly family or left c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is narrowly family (or left c-monotone) and has theorems. Then, since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$, the condition defining narrow family (left) c-monotonicity coincides with the one defining family (left, respectively) c-monotonicity. ■

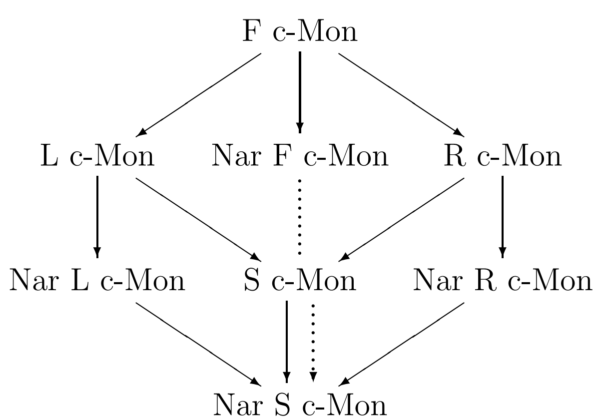
Analogously, for the cases of system and right c-monotonicity, we get the following

Theorem 564 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory family* $T \neq \mathrm{SEN}^\flat$ *such that* $\overleftarrow{T} \neq \overline{\varnothing}$*.* $\mathcal{I}$ *is system (right, respectively) c-monotone if and only if it roughly system (right, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is system or right c-monotone. Since, by hypothesis, it has a theory system $\overleftarrow{T} \neq \mathrm{SEN}^\flat, \overline{\varnothing}$, we get $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ must have theorems. Moreover, by Proposition 562, it is narrowly system or right c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is narrowly system (or right) c-monotone and has theorems. Then, since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\natural}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, the condition defining narrow right (system) c-monotonicity coincides with the one defining right (system, respectively) c-monotonicity. ■

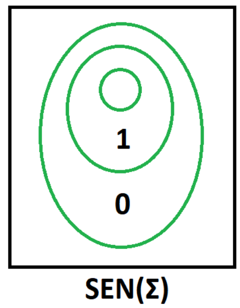
Thus, we have the following mixed hierarchy of c-monotonicity and narrow c-monotonicity properties.



We provide an example of a π-institution which has all four types of narrow c-monotonicity but fails to be system c-monotone and, as a consequence, does not belong to any of the four c-monotonicity classes.

Example 565 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial clone.*

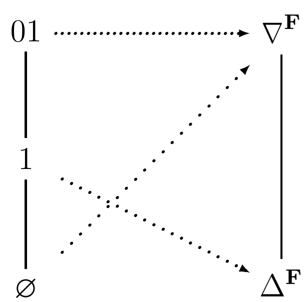


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



$\mathcal{I}$ *belongs to all four classes of the narrow c-monotonicity hierarchy. Indeed, since it is systemic, all four narrow c-monotonicity conditions boil down to checking that, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\sharp(\mathcal{I})$*,* $T' \le \bigcup_{T\in\mathcal{T}} T$ *implies* $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$*.*

- *If* $T' = \{\{1\}\}$*, then* $\Omega(T') = \Delta^{\mathbf{F}} \le \bigcup_{T\in\mathcal{T}} \Omega(T)$*;*
- *If* $T' = \mathrm{SEN}^\flat$*, then* $\mathrm{SEN}^\flat \in \mathcal{T}$*, whence* $\Omega(T') \le \nabla^{\mathbf{F}} = \bigcup_{T\in\mathcal{T}} \Omega(T)$*.*

On the other hand, we have $\{\varnothing\} \le \{\{1\}\}$*, whereas* $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$*, whence* $\mathcal{I}$ *is not system c-monotone.*

As far as connections between the rough c-monotonicity and narrow c-monotonicity classes are concerned, we get the following analog of Theorem 527.

Theorem 566 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is roughly family c-monotone if and only if it is narrowly family c-monotone;*

(b) $\mathcal{I}$ *is roughly left c-monotone, then it is narrowly left c-monotone;*

(c) If $\mathcal{I}$ *is roughly right c-monotone, then it is narrowly right c-monotone;*

(d) If $\mathcal{I}$ *is roughly system c-monotone, then it is narrowly system c-monotone.*

Proof:

(a) Suppose $\mathcal{I}$ is roughly family c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Then, by hypothesis, $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$, whence, by rough family c-monotonicity, $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. Thus $\mathcal{I}$ is narrowly family c-monotone.

Assume, conversely, that $\mathcal{I}$ is narrowly family c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$. Since $\{\widetilde{T} : T \in \mathcal{T}\} \cup \{\widetilde{T'}\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow family c-monotonicity, $\Omega(\widetilde{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\widetilde{T})$. Thus, by Proposition 369, $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$, showing that $\mathcal{I}$ is roughly family c-monotone.

(b) Suppose that $\mathcal{I}$ is roughly left c-monotone, i.e., that, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{\widetilde{T}}$ implies $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T')$. Assume, for the sake of obtaining a contradiction, that $\mathcal{I}$ is not narrowly left c-monotone. Then, there exist $\mathcal{X} \cup \{Y\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{Y} \leq \bigcup_{X\in\mathcal{X}} \overleftarrow{X}$ and $\Omega(Y) \nleq \bigcup_{X\in\mathcal{X}} \Omega(X)$.

First, observe that, if there existed $Z \in \mathrm{ThFam}(\mathcal{I})$ and $P \in |\mathbf{Sign}^\flat|$, such that $Z_P \neq \varnothing$ and $\overleftarrow{Z}_P = \varnothing$, then, setting $Z' = \{Z_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, with

$$Z'_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ Z_P, & \text{if } \Sigma = P \end{cases},$$

we would have $\overleftarrow{\widetilde{Z'}} = \overleftarrow{\widetilde{\varnothing}}$, but $\Omega(Z') \neq \Omega(\overline{\varnothing})$, which contradicts rough left c-monotonicity. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma \neq \varnothing$ implies $\overleftarrow{T}_\Sigma \neq \varnothing$.

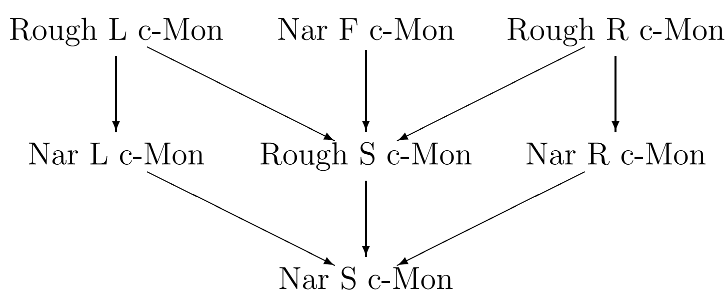
Continuing with the proof, by hypothesis, $\overleftarrow{Y} \le \bigcup_{X\in\mathcal{X}}\overleftarrow{X}$ and $\Omega(Y) \nleq \bigcup_{X\in\mathcal{X}}\Omega(X)$. Hence, by rough left c-monotonicity, $\widetilde{\overleftarrow{Y}} \nleq \bigcup_{X\in\mathcal{X}}\widetilde{\overleftarrow{X}}$. Thus, there exists $P \in |\mathbf{Sign}^\flat|$, such that $\widetilde{\overleftarrow{Y}}_P \nsubseteq \bigcup_{X\in\mathcal{X}}\widetilde{\overleftarrow{X}}_P$, whereas $\overleftarrow{Y}_P \subseteq \bigcup_{X\in\mathcal{X}}\overleftarrow{X}_P$. But this gives $\overleftarrow{Y}_P = \varnothing$, whence, by the preceding observation, $Y_P = \varnothing$, which contradicts $Y \in \mathrm{ThFam}^{\natural}(\mathcal{I})$. Therefore, $\mathcal{I}$ must be narrowly left c-monotone.

(c) Suppose $\mathcal{I}$ is roughly right c-monotone and let $\mathcal{T}\cup\{T'\} \subseteq \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $T' \le \bigcup_{T\in\mathcal{T}} T$. By hypothesis, $\widetilde{T'} \le \bigcup_{T\in\mathcal{T}}\widetilde{T}$. Thus, by rough right c-monotonicity, $\Omega(\overleftarrow{T'}) \le \bigcup_{T\in\mathcal{T}}\Omega(\overleftarrow{T})$. Thus, $\mathcal{I}$ is narrowly right c-monotone.

(d) Suppose $\mathcal{I}$ is roughly system c-monotone and let $\mathcal{T}\cup\{T'\} \subseteq \mathrm{ThSys}^{\natural}(\mathcal{I})$, such that $T' \le \bigcup_{T\in\mathcal{T}} T$. Then, by hypothesis, $\widetilde{T'} \le \bigcup_{T\in\mathcal{T}}\widetilde{T}$, whence, by rough system c-monotonicity, $\Omega(T') \le \bigcup_{T\in\mathcal{T}}\Omega(T)$. Thus $\mathcal{I}$ is narrowly system c-monotone.

■

Theorem 566 gives rise to the following mixed rough and narrow c-monotonicity hierarchy.



We insert, again, some examples to show that each of the three rough c-monotonicity classes is different from its narrow counterpart.

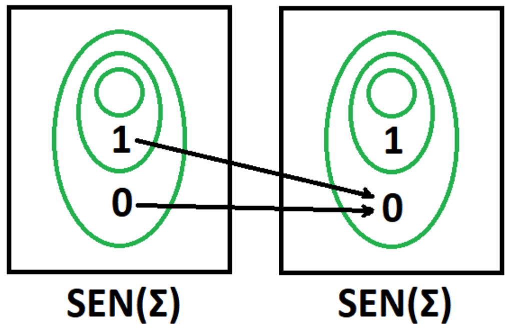
The first example gives a narrowly left c-monotone π-institution which is not roughly left c-monotone.

Example 567 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$*;*

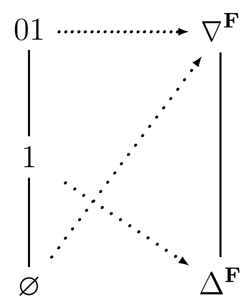
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



To see that $\mathcal{I}$ is narrowly left c-monotone, note that the only two different theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$ *are $\{\{1\}\}$ and $\{\{0,1\}\}$ and we have*

$$\overleftarrow{\{\{1\}\}} = \{\varnothing\} \leq \{\{0,1\}\} = \overleftarrow{\{\{0,1\}\}}$$
$$\text{and} \quad \Omega(\{\{1\}\}) = \Delta^{\mathbf{F}} \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}).$$

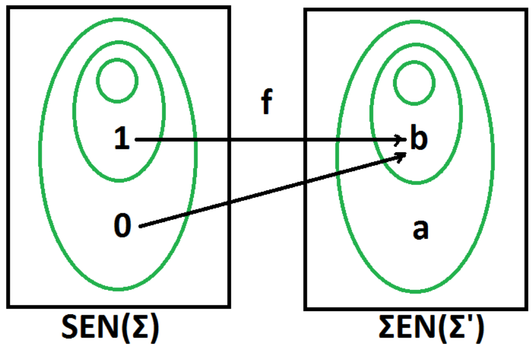
On the other hand, $\mathcal{I}$ is not roughly left c-monotone, since $\overleftarrow{\widetilde{\{\varnothing\}}} = \{\{0,1\}\} = \overleftarrow{\widetilde{\{\{1\}\}}}$, but $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$.

The second example shows that there exists a narrowly right c-monotone π-institution that is not roughly right c-monotone.

Example 568 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique morphism $f : \Sigma \to \Sigma'$;*

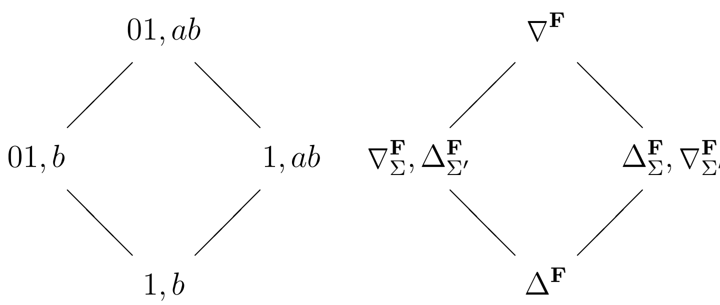
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

Clearly, there are only four theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



From this diagram and the fact that all theory families depicted are theory systems, we can see that, for all $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$,

$$T \le T' \quad \textit{iff} \quad \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}).$$

Therefore, $\mathcal{I}$ *is indeed narrowly right c-monotone.*

On the other hand, consider $T = \{1, \varnothing\}$ *and* $T' = \{1, ab\}$*. Then we have* $\widetilde{T} = \{1, ab\} = \widetilde{T'}$*, whereas*

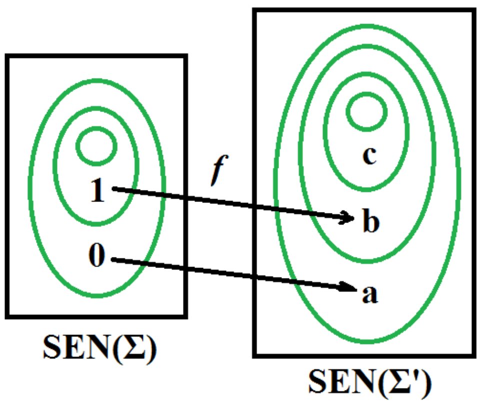
$$\Omega(\overleftarrow{T}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \not\le \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1, ab\}) = \Omega(\overleftarrow{T'}).$$

This shows that $\mathcal{I}$ *is not roughly right c-monotone.*

The last example gives a narrowly system c-monotone π-institution which is not roughly system c-monotone.

Example 569 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two object* Σ, Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



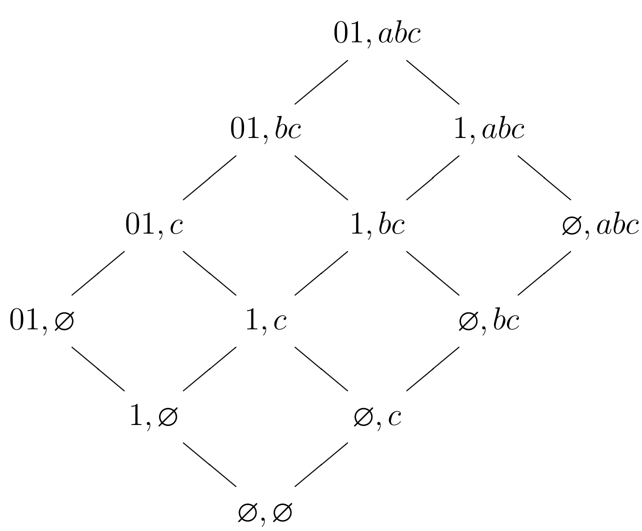
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$C_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad and \quad C_{\Sigma'} = \{\varnothing, \{c\}, \{b,c\}, \{a,b,c\}\}.$$

$\mathcal{I}$ *has twelve theory families, but only seven theory systems. These are*

$$\overline{\varnothing}, \{\varnothing, c\}, \{\varnothing, bc\}, \{\varnothing, abc\}, \{1, bc\}, \{1, abc\}, \{01, abc\}.$$

The following diagram shows the structure of the lattice of theory families.



To see that $\mathcal{I}$ *is narrow system c-monotone, note that there are only three theory systems in* $\mathrm{ThSys}^{\nleq}(\mathcal{I})$*, namely,* $\{1, bc\}$*,* $\{1, abc\}$ *and* $\{01, abc\}$ *and we have* $\{1, bc\} \leq \{1, abc\} \leq \{01, abc\}$ *and, also,*

$$\begin{aligned} \Omega(\{1, bc\}) &= \{\Delta^{\mathbf{F}}_{\Sigma}, \{a, bc\}\} \\ &\leq \Omega(\{1, abc\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} \\ &\leq \Omega(\{01, abc\}) = \nabla^{\mathbf{F}}. \end{aligned}$$

On the other hand, setting $T = \{\varnothing, c\}$ *and* $T' = \{\varnothing, bc\}$*, which are both theory systems, we get*

$$\widetilde{T} = \{01, c\} \leq \{01, bc\} = \widetilde{T}',$$

whereas

$$\Omega(T) = \{\nabla^{\mathbf{F}}_{\Sigma}, \{ab, c\}\} \nleq \{\Delta^{\mathbf{F}}_{\Sigma}, \{a, bc\}\} = \Omega(T').$$

Therefore, $\mathcal{I}$ *is not roughly system c-monotone.*

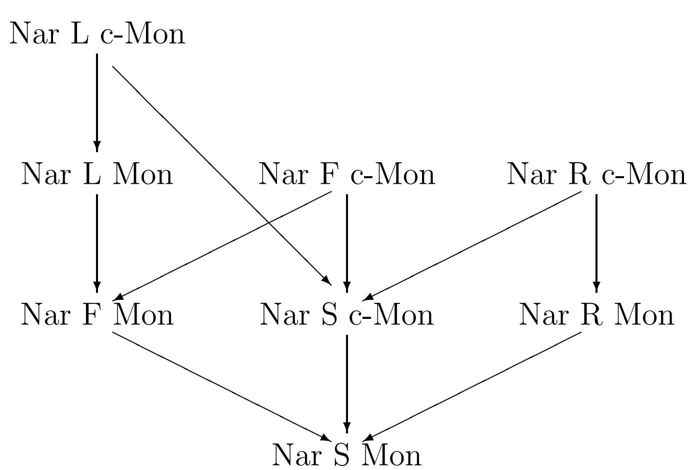
We conclude, after these examples, that the structure of the joint rough and narrow c-monotonicity hierarchy is as depicted in the diagram following Theorem 566, with no two classes being identical.

Finally, we look at some straightforward connections between the classes in the narrow monotonicity and narrow complete monotonicity hierarchies. These follow directly by the definitions involved.

Proposition 570 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly family (respectively, left, right, system) c-monotone, then it is narrowly family (respectively, left, right, system) monotone.*

Proof: The condition defining a narrow monotonicity class is a special case of the condition defining the corresponding narrow c-monotonicity class, where the collection $\mathcal{T}$, in that definition, is taken to be a singleton. ∎

Proposition 570, in view of Propositions 517 and 555, establishes the hierarchy depicted in the diagram.

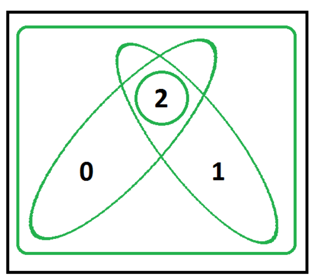


We present an example to show that the two hierarchies are separated. The showcased π-institution belongs to all steps of the narrow monotonicity hierarchy but to none of the four narrow c-monotonicity classes.

Example 571 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0

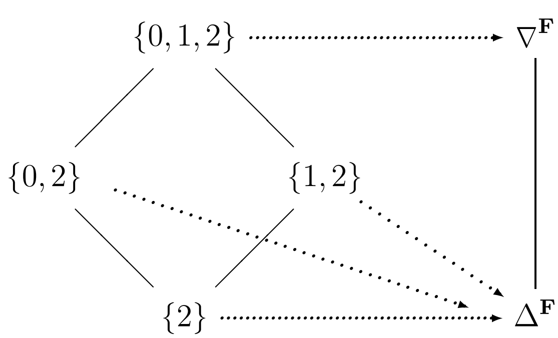


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



Since $\mathbf{Sign}^\flat$ *is trivial,* $\mathcal{I}$ *is systemic and, since* $\mathcal{I}$ *has theorems,* $\mathrm{FiFam}^{\not\flat}(\mathcal{I}) = \mathrm{FiFam}(\mathcal{I})$*. We conclude that all four narrow monotonicity properties for*

$\mathcal{I}$ coincide and, moreover, they are identical with both monotonicity properties, which they also coincide, due to systemicity. The same holds for c-monotonicity. All four narrow c-monotonicity properties coincide and they, in turn, are identical with all c-monotonicity conditions.

From the diagram one can verify immediately that $\mathcal{I}$ is (narrowly left, right and family) monotone, On the other hand, we have

$$\{\{0,1,2\}\} \le \{\{0,2\}\} \cup \{\{1,2\}\},$$

but, obviously, $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be narrowly system c-monotone.

Next, we turn to transfer theorems for the various narrow c-monotonicity properties.

Theorem 572 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is narrowly family c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $T' \le \bigcup_{T\in\mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(T') \le \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$;

(b) $\mathcal{I}$ is narrowly left c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $\overleftarrow{T'} \le \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$ implies $\Omega^{\mathcal{A}}(T') \le \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$;

(c) $\mathcal{I}$ is narrowly right c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $T' \le \bigcup_{T\in\mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \le \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$;

(d) $\mathcal{I}$ is narrowly system c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}\natural}(\mathcal{A})$, $T' \le \bigcup_{T\in\mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(T') \le \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

Proof:

(a) The "if" results by applying the hypothesis to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

For the "only if", suppose that $\mathcal{I}$ is narrowly family c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\natural}(\mathcal{I})$, such that $T' \le \bigcup_{T\in\mathcal{T}} T$. Then we get $\alpha^{-1}(T') \le \alpha^{-1}(\bigcup_{T\in\mathcal{T}} T)$, whence $\alpha^{-1}(T') \le \bigcup_{T\in\mathcal{T}} \alpha^{-1}(T)$. Since, by Lemmas 51 and 376, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\natural}(\mathcal{I})$, we get, by narrow family c-monotonicity, $\Omega(\alpha^{-1}(T')) \le \bigcup_{T\in\mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \le \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \le \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \le \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(b) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is narrowly left c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$. Then we get $\alpha^{-1}(\overleftarrow{T'}) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \overleftarrow{T})$, whence $\alpha^{-1}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\overleftarrow{T})$. By Lemma 6, we get $\overleftarrow{\alpha^{-1}(T')} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{\alpha^{-1}(T)}$. Since, by Lemmas 51 and 376, it holds $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow left c-monotonicity, that $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T\in\mathcal{T}} \Omega(\alpha^{-1}(T))$. Thus, by Proposition 24, we now get $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F, \alpha\rangle$, we obtain $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(c) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is narrowly right c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\nmid}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Then we get $\alpha^{-1}(T') \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} T)$, whence $\alpha^{-1}(T') \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(T)$. Since, by Lemmas 51 and 376, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow right c-monotonicity, $\Omega(\overleftarrow{\alpha^{-1}(T')}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)})$. Thus, by Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T}))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T}))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}))$. Taking into account the surjectivity of $\langle F, \alpha\rangle$, we obtain $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$.

(d) Similar to Part (a). ■

We close this section by giving two characterizations concerning the narrow family and narrow system c-monotonicity classes, based on mappings between posets satisfying the complete monotonicity property.

Proposition 573 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family c-monotone;*

(b) $\Omega : \mathbf{ThFam}^{\nmid}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proposition 574 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system c-monotone;*

(b) $\Omega : \mathbf{ThSys}^{\nmid}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiSys}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*