Chapter 12

The Syntactic Leibniz Hierarchy: Edifice

12.1   Introduction

In this chapter, we develop the syntactic machinery of translations, interpretations and equivalences in order to characterize syntactic analogs of the various types of prealgebraizable and algebraizable π-institutions studied in Chapter 4. The roots of these investigations can be traced back to the famous "Memoir monograph" of Blok and Pigozzi [36]. Of course various advancements, refinements and generalizations have been presented in the literature since then, e.g., in [41, 53, 50, 74, 82] and in the categorical setting [106]. The monograph [65] and the textbook [89] give a more comprehensive list of more recent approaches pertaining to the sentential logic case.

Section 12.2 introduces π-*structures*, a variant of π-institutions that may lack structurality. By analogy with the terminology concerning theory families and theory systems, their closures are termed *closure families* rather than closure systems. Introduced, also, are *transformations* between algebraic systems and *interpretations* between π-structures. Let us consider, for concreteness, two algebraic systems $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$. A *translation* $\alpha : \mathbf{K} \to \mathbf{K}'$ from $\mathbf{K}$ to $\mathbf{K}'$ is a collection $\alpha = \{\alpha_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of mappings

$$\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

from Σ-sentences of $\mathbf{K}$ to sentence families of $\mathbf{K}'$. Now, given two π-structures $\mathcal{K} = \langle \mathbf{K}, D \rangle$ based on $\mathbf{K}$ and $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ based on $\mathbf{K}'$, a translation $\alpha : \mathbf{K} \to \mathbf{K}'$ is an *interpretation* $\alpha : \mathcal{K} \to \mathcal{K}'$ from $\mathcal{K}$ to $\mathcal{K}'$ if, for every signature Σ of $\mathcal{K}$ and all Σ-sentences $\Phi \cup \{\phi\}$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma[\phi] \leq D'(\alpha_\Sigma[\Phi]).$$

If such an α exists, then $\mathcal{K}$ is said to be *interpretable in* $\mathcal{K}'$. Two interpretations $\alpha : \mathcal{K} \to \mathcal{K}'$ and $\beta : \mathcal{K}' \to \mathcal{K}$, acting in opposite directions, are called *inverses* and $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ a *conjugate pair* if, for all $\Sigma \in |\mathbf{Sign}|$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\psi \in \mathrm{SEN}'(\Sigma')$,

$$D(\phi) = D(\beta[\alpha_\Sigma[\phi]]) \quad \text{and} \quad D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]]).$$

Two π-structures are called *equivalent* if there exist inverse interpretations in both directions. As in the classical case of sentential logics, not all four conditions required for equivalence are independent. The first of the interpretation conditions and the second invertibility condition imply the other two and vive versa. This was originally pointed out by Blok and Pigozzi in Corollary 2.9 of [36] in the context of interpretations between a sentential logic and an equational logic over the same algebraic signature.

Given a translation $\alpha : \mathbf{K} \to \mathbf{K}'$, one may define a mapping

$$\alpha^* : \mathrm{SenFam}(\mathbf{K}') \to \mathrm{SenFam}(\mathbf{K}),$$

called the *residual* of α. Such a mapping, when applied to interpretations $\alpha :$ $\mathcal{K} \to \mathcal{K}'$ between π-structures $\mathcal{K}$ and $\mathcal{K}'$, yields a mapping $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to$ $\mathrm{ThFam}(\mathcal{K})$. Additionally, if $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$, then $\mathrm{ThFam}(\mathcal{K}') \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} \mathrm{ThFam}(\mathcal{K})$ is a pair of inverse mappings. These establish an order isomorphism between the complete lattice of theory families of $\mathcal{K}'$ and the complete lattice of theory families of $\mathcal{K}$. The process by which a conjugate pair of interpretations α and β gives rise to the order isomorphisms α^* and β^* can be reversed. This direction occupies the second part of Section 12.2.

Suppose $h : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ is an order isomorphism. We define translations $\overrightarrow{h} = \{\overrightarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} : \mathbf{K} \to \mathbf{K}'$ and $\overleftarrow{h} = \{\overleftarrow{h}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|} : \mathbf{K}' \to \mathbf{K}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$, all $\phi \in \mathrm{SEN}(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\psi \in \mathrm{SEN}'(\Sigma')$,

$$\overrightarrow{h}_\Sigma(\phi) = h^{-1}(D(\phi)) \quad \text{and} \quad \overleftarrow{h}_{\Sigma'}(\psi) = h(D'(\psi)).$$

It is shown that $\overrightarrow{h} : \mathcal{K} \to \mathcal{K}'$ and $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ are interpretations and, moreover, that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ forms a conjugate pair. Finally, it is shown that the residuals $\overrightarrow{h}^*$ and $\overleftarrow{h}^*$ coincide with h and h^{-1}, respectively.

In Section 12.3, we single out a special type of translation, which is closely connected to the syntactic apparatus of a given algebraic system, in particular of one that is the base algebraic system on which a π-institution is built. More precisely, instead of considering arbitrary maps from a sentence functor into the sentence families over another sentence functor, we specialize to the case in which the mappings occur between different powers of the same sentence functor, akin to the framework of k-deductive systems of Blok and Pigozzi [41]. In this special case, arbitrary translations may be replaced by *transformations*, i.e., sets of natural transformations, possibly with parameters. Those are said to be *natural*, i.e., are called *natural transformations*, if they are parameter free. Inherited by the special case of transformations from the general case of translations from Section 12.2 are the notions of interpretability and of equivalence between π-structures. Virtually all results involving arbitrary translations and interpretations are specialized here to the case of transformations and natural transformations and of interpretations induced by transformations. Accompanying those is the notion of a *transformational order isomorphism* between lattices of theory families which reflects the fact that those isomorphisms are induced by transformations rather than by arbitrary translations. It is shown that, if there exists a transformational order isomorphism between lattices of theory families, then the pair of transformations that induce the isomorphism do form a conjugate pair between the corresponding π-structures and, consequently, these two π-structures are equivalent. Similar results apply if arbitrary transformations are replaced by natural transformations.

In the sentential logic framework, the passage from arbitrary interpretations to ones between k-deductive systems serves in transitioning from general equivalences to equivalences between a deductive system and an equational deductive system, which defines algebraizability (albeit the development of this passage occurred in reverse chronological order). Similarly, here, the passage from arbitrary translations and interpretations to transformations and interpretations based on transformations serves an analogous purpose.

The second part of Section 12.3 introduces *equational π-structures*. Those, roughly speaking, are the ones in which reflexivity, symmetry, transitivity, compatibility (or congruence) and invariance under signature morphisms hold. In a characterization theorem governing these structures, it is shown that a π-structure is equational if and only if its closure systems are congruence systems on the underlying algebraic system of the π-structure if and only if it is one induced on the underlying algebraic system $\mathbf{F}$ by a certain class of $\mathbf{F}$-algebraic systems. The notion of an equational π-structure facilitates connecting the framework of equivalence between π-structures with that of syntactic protoalgebraicity and of (family) truth equationality, studied in Sections 11.3 and 11.6, respectively, of Chapter 11. Namely, it is shown that, if a π-institution is equivalent to an equational π-structure over the same underlying algebraic system via a conjugate pair of transformations, then it is syntactically protoalgebraic and truth equational with witnessing transformations and equations those constituting the conjugate pair. Finally, in an analog of a well known result of Blk and Pigozzi (Theorem 2.15 of [36]), it is shown that equivalence of a π-institution with an equational π-structure via transformations can only occur in an essentially unique way in the sense that any two such π-structures must be identical and, furthermore, the transformations serving the purpose must be unique up to deductive closure.

In Chapter 4, we defined the class of weakly family algebraizable (WF algebraizable) π-institutions. Weak family algebraizability is characterized by protoalgebraicity (family monotonicity) and family injectivity of the Leibniz operator. In Section 12.4, the goal is to define a syntactic counterpart of this class. Critical here are the results obtained in Chapter 11 which connect syntactic protoalgebraicity and protoalgebraicity via the Leibniz property of the reflexive core $R^{\mathcal{I}}$ and, also, truth equationality and family c-reflectivity via the adequacy property of the Suszko core $S^{\mathcal{I}}$. Starting from these, we say that a π-institution $\mathcal{I}$ is $R^{\mathcal{I}}S^{\mathcal{I}}$-*fortified* if it has a Leibniz reflexive core and an adequate Suszko core and, further, declare that a π-institution is *syntactically weakly family algebraizable* (*syntactically WF algebraizable*) if it is protoalgebraic, family injective and $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified. This definition seems to be the "right" one for a syntactic counterpart of weakly family algebraizable π-institutions. According to this, e.g., syntactically WF algebraizable π-institutions are characterized as those which are, at the same time, syntactically protoalgebraic and family truth equational. Moreover, they are exactly those that are WF algebraizable and $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified. Another char-

acterization obtained from this and one for WF algebraizability asserts that $\mathcal{I}$ is syntactically WF algebraizable if and only if it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. Yet another characterization uses the notion of equivalence with an equational π-structure introduced in Section 12.3. Recall from Chapter 2, Section 2.7, that $\mathrm{AlgSys}^{*}(\mathcal{I})$ is the class of all $\mathbf{F}$-algebraic systems that are algebraic system reducts of reduced $\mathcal{I}$-matrix families. We define the collection $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ of congruence systems on $\mathcal{A}$ whose quotients are algebraic systems in $\mathrm{AlgSys}^{*}(\mathcal{I})$. An alternative characterization of $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is that it consists of all those congruence systems on $\mathcal{A}$ that are Leibniz congruence systems of some $\mathcal{I}$-filter family of $\mathcal{A}$. In general, the collection of all $\mathcal{I}^{*}$-congruence systems on an $\mathbf{F}$-algebraic system $\mathcal{A}$ is not closed under signature-wise intersections and, therefore, does not form a closure family on $\mathcal{A}^{2}$. However, this turns out to be the case if the π-institution happens to be protoalgebraic. Specializing to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, we get that, provided $\mathcal{I}$ is protoalgebraic, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ is closed under arbitrary intersections. This enables us to define an equational π-structure $\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^{2}, D^{\mathcal{I}*} \rangle$, where $D^{\mathcal{I}*}$ is the closure operator associated with the closure family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$. The first indication that this definition serves well our purposes is the result asserting that, if a π-institution is syntactically WF algebraizable, then, with $I^{\flat} : (\mathrm{SEN}^{\flat})^{\omega} \to \mathrm{SEN}^{\flat}$, with two distinguished arguments, being the collection of witnessing transformations of the syntactic protoalgebraicity of $\mathcal{I}$ and with $\tau^{\flat} : (\mathrm{SEN}^{\flat})^{\omega} \to (\mathrm{SEN}^{\flat})^{2}$ with a single distinguished argument, being the collection of witnessing equations of the truth equationality of $\mathcal{I}$, $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via the conjugate pair $(\tau^{\flat}, \overleftrightarrow{I^{\flat}}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. Putting many preceding results together, we obtain a fundamental characterization of syntactic WF algebraizability, tracing its origins to the seminal "Memoirs monograph" [36] of Blok and Pigozzi. Namely, a π-institution $\mathcal{I}$ is syntactically WF algebraizable if and only if it is equivalent to its associated equational π-struture $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^{\flat}, I^{\flat}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ of transformations. More generally, $\mathcal{I}$ is syntactically WF algebraizable if and only if it is equivalent to an equational π-structure via a conjugate pair of transformations. Finally, this generalized version implies that $\mathcal{I}$ is syntactically WF algebraizable if and only if there exists a transformational isomorphism between its lattice of theory families and the lattice of theory families of an equational π-structure.

Syntactic weak family algebraizability encompasses family reflectivity, which implies systemicity. Thus, if a π-institution is syntactically WF algebraizable, it must necessarily be systemic. This leads to the idea of considering weaker types of syntactic algebraizability admitting also non-systemic π-institutions. This endeavor is pursued in Sections 12.5 and 12.6. Section 12.5 considers *syntactic weak algebraizability*. The development parallels that of Section 12.4.

We say that a π-institution is $R^{\mathcal{I}}L^{\mathcal{I}}$-*fortified* if it has a Leibniz reflexive core $R^{\mathcal{I}}$ and a left adequate left Suszko core $L^{\mathcal{I}}$. $\mathcal{I}$ is *syntactically weakly algebraizable* (*syntactically W algebraizable*) if it is protoalgebraic, system injective and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. Syntactic weak algebraizability is characterized as the conjunction of syntactic protoalgebraicity and either system or left truth equationality. Moreover, $\mathcal{I}$ is syntactically W algebraizable if and only if it is weakly algebraizable and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. Yet another characterization using isomorphisms asserts that $\mathcal{I}$ is syntactically W algebraizable if and only if it is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified, stable and, for every **F**-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$: $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

In the second part of Section 12.5, we give a characterization of syntactic W algebraizability using the equational counterpart $\mathcal{Q}^{\mathcal{I}*}$ of a π-institution $\mathcal{I}$. This mimics in style the fundamental characterization of syntactic weak family algebraizability, presented in Section 12.4. First, given a π-institution $\mathcal{I}$, we define a π-structure $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$, called the *systemic skeleton* of $\mathcal{I}$, by taking $K^{\mathcal{I}}$ to be the closure operator whose theory families are the theory systems of $\mathcal{I}$. This π-structure is not a π-institution in general, since $K^{\mathcal{I}}$ may fail to be structural. We show, nevertheless, that, if a π-institution $\mathcal{I}$ is syntactically W algebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its equational counterpart $\mathcal{Q}^{\mathcal{I}*}$ via the conjugate pair $(\tau^{\flat}, \overleftrightarrow{I^{\flat}})$ of transformations, where $I^{\flat}$ witnesses syntactic protoalgebraicity and $\tau^{\flat}$ witnesses left truth equationality. Conversely, if $\mathcal{I}$ is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to the equational counterpart $\mathcal{Q}^{\mathcal{I}*}$ of $\mathcal{I}$ via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically weakly algebraizable. More generally, we show that syntactic weak algebraizability is characterized by stability plus the existence of an equivalence between its systemic skeleton and an equational π-structure via a conjugate pair of transformations. This characterization yields also that $\mathcal{I}$ is syntactically W algebraizable if and only if it is stable and there exists a transformational order isomorphism between the lattice of its theory systems and that of the theory families of an equational π-institution.

Syntactic weak algebraizability is broad enough to accommodate non-systemic π-institutions, but it is still too narrow to allow non-stable π-institutions to be considered. So in Section 12.6, we further weaken the required conditions so as to be able to also include non-stable π-institutions. We say that a π-institution $\mathcal{I}$ is $R^{\mathcal{I}}Z^{\mathcal{I}}$-*fortified* if its reflexive core $R^{\mathcal{I}}$ is Leibniz and its system core $Z^{\mathcal{I}}$ is adequate. Then $\mathcal{I}$ is called *syntactically weakly system prealgebraizable* (*syntactically WS prealgebraizable*) if it is prealgebraic, system injective and $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified. According to this definition, $\mathcal{I}$ is syntactically WS prealgebraizable if and only if it is syntactically prealgebraic and system truth equational. Moreover, $\mathcal{I}$ is syntactically WS prealgebraizable if and only if it is weakly system prealgebraizable and $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified. In terms of isomorphisms between theory and congruence lattices, we get that

$\mathcal{I}$ is syntactically WS prealgebraizable if and only if it is $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding.

To characterize syntactic WS prealgebraizability via an equivalence between π-structures, we use again the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ of $\mathcal{I}$, but we need to introduce an equational π-structure induced by $\mathcal{I}$ different from the equational π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with $\mathcal{I}$. To this end, instead of the collection $\mathrm{AlgSys}^{*}(\mathcal{I})$ of all algebraic system reducts of reduced $\mathcal{I}$-filter families, we use the collection $\mathrm{AlgSys}^{\bullet}(\mathcal{I})$ of all algebraic system reducts of reduced $\mathcal{I}$-filter systems. Then we define the class of $\mathcal{I}^{\bullet}$-congruence systems $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ on an $\mathbf{F}$-algebraic system $\mathcal{A}$ as the collection of all those congruence systems θ on $\mathcal{A}$, such that $\mathcal{A}/\theta \in \mathrm{AlgSys}^{\bullet}(\mathcal{I})$. Alternatively, $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ may be characterized as the collection of all congruence systems on $\mathcal{A}$ which are images under $\Omega^{\mathcal{A}}$ of $\mathcal{I}$-filter systems on $\mathcal{A}$. Assuming prealgebraicity of a π-institution $\mathcal{I}$, we can show that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ is closed under arbitrary signature-wise intersections. In particular, this holds for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, whence one may consider the *systemic equational π-structure* $\mathcal{Q}^{\mathcal{I}\bullet} = \langle \mathbf{F}^2, D^{\mathcal{I}\bullet} \rangle$ *associated with* $\mathcal{I}$, where $D^{\mathcal{I}\bullet}$ is the closure operator corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$. With this machinery at hand, it is shown that, if $\mathcal{I}$ is syntactically WS prealgebraizable, then $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via the conjugate pair $(\tau^{\flat}, \overleftrightarrow{I^{\flat}})$ of transformations, where $I^{\flat}$ witnesses the syntactic prealgebraicity of $\mathcal{I}$ and $\tau^{\flat}$ witnesses the system truth equationality of $\mathcal{I}$. Conversely, if the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ of $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically WS prealgebraizable. More generally, $\mathcal{I}$ is syntactically WS prealgebraizable if and only if its systemic skeleton is equivalent to an equational π-structure via a conjugate pair of transformations. This characterization enables us to show that syntactic WS prealgebraizability is also characterized by the existence of a transformational order isomorphism between the lattice of theory systems of $\mathcal{I}$ and that of the theory families of an equational π-structure.

In Section 12.5, we studied syntactically weakly algebraizable π-institutions, one of the motivations being opening syntactic weak algebraizability to non-systemic π-institutions. Subsequently, in Section 12.6, we studied syntactically weakly system prealgebraizable π-institutions, weaker than syntactically weakly algebraizable ones, able to also encompass some non-stable π-institutions. Section 12.7 aims to bridge a gap between syntactic WS prealgebraizability and syntactic W algebraizability, which becomes apparent when one compares with the corresponding semantic hierarchy of Chapter 4. Recall that a π-institution $\mathcal{I}$ is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified if its reflexive core $R^{\mathcal{I}}$ is Leibniz and its left Suszko core $L^{\mathcal{I}}$ is left adequate. We call $\mathcal{I}$ *syntactically weakly left c-reflectively prealgebraizable (syntactically WLC prealgebraizable)* if it is prealgebraic, left c-reflective and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. Based on previously

obtained characterizations of syntactic prealgebraicity and of left truth equationality, we get that syntactic WLC prealgebraizability is the conjunction of syntactic prealgebraicity and of left truth equationality. Moreover, $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is WLC prealgebraizable and $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified. In terms of morphisms between lattices of theory systems and of congruence systems, $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rightarrow \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left completely order reflecting surjection, restricting to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \rightarrow \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Further, in terms of equivalences, syntactic WLC prealgebraizability is characterized by left truth equationality plus an equivalence of the systemic skeleton of the π-institution with an equational π-structure via a conjugate pair of transformations. In terms of isomorphisms between lattices of theory families, we have that a π-institution $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism between the lattice of its theory systems and that of an equational π-structure. In concluding the section and the chapter, a complete subhierarchy of the various (semantic) prealgebraizability and syntactic prealgebraizability classes is presented. This is reproduced here in lieu of a summary and to provide an overview of the location of the syntactic classes in the overall hierarchy.
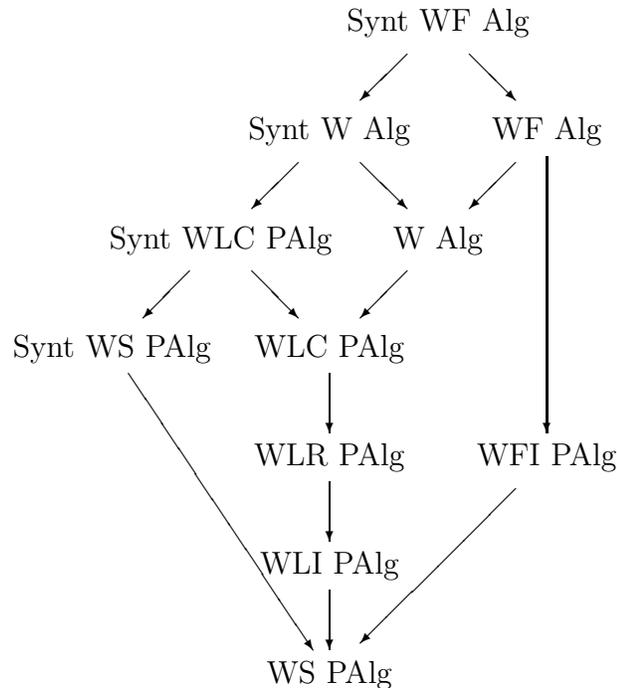
12.2 Translations

In this section we discuss translations, interpretations and equivalence that will be used later in the context of algebraizable π-institutions. In the context of algebraizability, the algebraic counterparts of π-institutions may consist of algebraic closure families that lack the property of structurality, i.e., they are not closure systems, as introduced previously. Since these closure families are not structural in general, the corresponding algebraic structures do not constitute π-institutions. To accommodate these, we deal with more general structures that include all π-institutions, but also pairs of algebraic systems and closure families that are non-structural. We call these π-*structures*.

Definition 891 *A* π-**structure** $\mathcal{K} = \langle \mathbf{K}, D \rangle$ *is a pair consisting of:*

- *an algebraic system* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$;

- *a* $|\mathbf{Sign}|$-*indexed family* $D = \{D_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ *of closure operators* $D_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$, $\Sigma \in |\mathbf{Sign}|$.

Such a family D is called a **closure family** *on* $\mathbf{K}$.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be two algebraic systems. A **translation** $\alpha : \mathbf{K} \to \mathbf{K}'$ is a collection

$$\alpha = \{\alpha_\Sigma\}_{\Sigma \in |\mathbf{Sign}|},$$

where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

assigns to each Σ-sentence ϕ of $\mathbf{K}$ a sentence family

$$\alpha_\Sigma[\phi] = \{\alpha_{\Sigma,\Sigma'}[\phi]\}_{\Sigma' \in |\mathbf{Sign}'|}.$$

For $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we set

$$\alpha_\Sigma[\Phi] = \bigcup\{\alpha_\Sigma[\phi] : \phi \in \Phi\},$$

where the union is, as usual, taken signature-wise and, hence, $\alpha_\Sigma[\Phi] \in \mathrm{SenFam}(\mathbf{K}')$. More generally, for $T \in \mathrm{SenFam}(\mathbf{K})$, we set

$$\alpha[T] = \bigcup\{\alpha_\Sigma[T_\Sigma] : \Sigma \in |\mathbf{Sign}|\}.$$

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ be π-structures based on $\mathbf{K}, \mathbf{K}'$, respectively. An

interpretation $\alpha : \mathcal{K} \to \mathcal{K}'$ is a translation $\alpha : \mathbf{K} \to \mathbf{K}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma[\phi] \leq D'(\alpha_\Sigma[\Phi]).$$

If such an interpretation exists, then $\mathcal{K}$ is said to be **interpretable in** $\mathcal{K}'$.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively. Let, also,

$$\alpha : \mathcal{K} \to \mathcal{K}' \quad \text{and} \quad \beta : \mathcal{K}' \to \mathcal{K}$$

be interpretations from $\mathcal{K}$ to $\mathcal{K}'$ and from $\mathcal{K}'$ to $\mathcal{K}$, respectively. α and β are said to be **inverses** of each other and the pair $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is referred to as a **conjugate pair** if:

- for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$D(\phi) = D(\beta[\alpha_\Sigma[\phi]]);$$

- for all $\Sigma' \in |\mathbf{Sign}'|$ and all $\psi \in \mathrm{SEN}'(\Sigma')$,

$$D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]]).$$

The π-structures $\mathcal{K}$ and $\mathcal{K}'$ are called **equivalent** if there exists a conjugate pair $\mathcal{K} \overset{(\alpha,\beta)}{\rightleftarrows} \mathcal{K}'$.

Lemma 892 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $\alpha : \mathbf{K} \to \mathbf{K}'$, $\beta : \mathbf{K}' \to \mathbf{K}$ *translations. The following are equivalent:*

(i) *$\alpha : \mathcal{K} \to \mathcal{K}'$ is an interpretation and, for all $\Sigma' \in |\mathbf{Sign}'|$, $\psi \in \mathrm{SEN}'(\Sigma')$, $D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]])$;*

(ii) *$\beta : \mathcal{K}' \to \mathcal{K}$ is an interpretation and, for all $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, $D(\phi) = D(\beta[\alpha_\Sigma[\phi]])$.*

Proof: By symmetry, it suffices to show (i)$\Rightarrow$(ii).

Suppose, first, that $\Sigma' \in |\mathbf{Sign}'|$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$. Then, we have

$$
\begin{aligned}
\psi \in D'_{\Sigma'}(\Psi) \quad &\text{iff} \quad D'(\psi) \leq D'(\Psi) \quad \text{(Property of } D') \\
&\text{iff} \quad D'(\alpha[\beta_{\Sigma'}[\psi]]) \leq D'(\alpha[\beta_{\Sigma'}[\Psi]]) \quad \text{(Hypothesis)} \\
&\text{iff} \quad \alpha[\beta_{\Sigma'}[\psi]] \leq D'(\alpha[\beta_{\Sigma'}[\Psi]]) \quad \text{(Property of } D') \\
&\text{iff} \quad \beta_{\Sigma'}[\psi] \leq D(\beta_{\Sigma'}[\Psi]). \quad \text{(Hypothesis)}
\end{aligned}
$$

So $\beta : \mathcal{K}' \to \mathcal{K}$ is an interpretation.

Assume, next, that $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, by the hypothesis applied to $\alpha_\Sigma[\phi] \in \mathrm{SenFam}(\mathbf{K}')$, we have

$$D'(\alpha[\beta[\alpha_\Sigma[\phi]]]) = D'(\alpha_\Sigma[\phi]).$$

Hence, we get that

$$\alpha_\Sigma[\phi] \leq D'(\alpha[\beta[\alpha_\Sigma[\phi]]]) \quad \text{and} \quad \alpha[\beta[\alpha_\Sigma[\phi]]] \leq D'(\alpha_\Sigma[\phi]).$$

Therefore, by the fact that α is an interpretation,

$$\phi \in D_\Sigma(\beta[\alpha_\Sigma[\phi]]) \quad \text{and} \quad \beta[\alpha_\Sigma[\phi]] \leq D(\phi).$$

So we conclude that $D(\phi) = D(\beta[\alpha_\Sigma[\phi]])$. ∎

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems and $\alpha : \mathbf{K} \to \mathbf{K}'$ a translation. Define the **residual α^* of the translation** α,

$$\alpha^* : \mathrm{SenFam}(\mathbf{K}') \to \mathrm{SenFam}(\mathbf{K})$$

by letting, for all $T' \in \mathrm{SenFam}(\mathbf{K}')$,

$$\alpha^*(T') = \{\alpha_\Sigma^*(T')\}_{\Sigma \in |\mathbf{Sign}|}$$

be given, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\alpha_\Sigma^*(T') = \{\phi \in \mathrm{SEN}(\Sigma) : \alpha_\Sigma[\phi] \leq T'\}.$$

The following proposition shows that, when applied to interpretations between π-structures the star operator restricts to mappings from theory families to theory families.

Proposition 893 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $\alpha : \mathcal{K} \to \mathcal{K}'$ *an interpretation. Then, for all* $T' \in \mathrm{ThFam}(\mathcal{K}')$, $\alpha^*(T') \in \mathrm{ThFam}(\mathcal{K})$.

Proof: Suppose $T' \in \mathrm{ThFam}(\mathcal{K}')$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\alpha_\Sigma^*(T'))$. Then, since $\alpha : \mathcal{K} \to \mathcal{K}'$ is an interpretation, we have

$$\alpha_\Sigma[\phi] \leq D'(\alpha[\alpha_\Sigma^*(T')]) \leq D'(T') = T'.$$

Hence $\phi \in \alpha_\Sigma^*(T')$. Since $\Sigma \in |\mathbf{Sign}|$ was arbitrary, we conclude that $\alpha_\Sigma^*(T') \in \mathrm{ThFam}(\mathcal{K})$. ∎

In addition, we show that, when $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair, then $\beta^* : \mathrm{ThFam}(\mathcal{K}) \to \mathrm{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ are inverse mappings.

Lemma 894 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then, for all* $T \in \mathrm{ThFam}(\mathcal{K})$,

$$\alpha^*(\beta^*(T)) = T.$$

Proof: Suppose $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair, $T \in \mathrm{ThFam}(\mathcal{K})$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$
\begin{aligned}
\phi \in \alpha^*_\Sigma(\beta^*(T)) \quad &\text{iff} \quad \alpha_\Sigma[\phi] \leq \beta^*(T) \quad (\text{Definition of } \alpha^*) \\
&\text{iff} \quad \beta[\alpha_\Sigma[\phi]] \leq T \quad (\text{Definition of } \beta^*) \\
&\text{iff} \quad D(\beta[\alpha_\Sigma[\phi]]) \leq T \quad (T \in \mathrm{ThFam}(\mathcal{K})) \\
&\text{iff} \quad D_\Sigma(\phi) \leq T_\Sigma \quad ((\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}') \\
&\text{iff} \quad \phi \in T_\Sigma. \quad (T \in \mathrm{ThFam}(\mathcal{K}))
\end{aligned}
$$

Thus, we conclude that $\alpha^*(\beta^*(T)) = T$. ∎

Based on Lemma 894, we can show that $\beta^* : \mathrm{ThFam}(\mathcal{K}) \to \mathrm{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ are bijections.

Lemma 895 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then* $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ *is a bijection.*

Proof: Let $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ be a conjugate pair. First, by Proposition 893, $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ is well-defined. To see that it is surjective, let $T \in \mathrm{ThFam}(\mathcal{K})$. Then, by Proposition 893, $\beta^*(T) \in \mathrm{ThFam}(\mathcal{K}')$ and, by Lemma 894, $\alpha^*(\beta^*(T)) = T$. Thus, α^* is indeed surjective. For injectivity, assume $S', T' \in \mathrm{ThFam}(\mathcal{K}')$, such that $\alpha^*(S') = \alpha^*(T')$. Then, by surjectivity, there exist $S, T \in \mathrm{ThFam}(\mathcal{K})$, such that $\beta^*(S) = S'$ and $\beta^*(T) = T'$. Therefore, we get

$$S = \alpha^*(\beta^*(S)) = \alpha^*(S') = \alpha^*(T') = \alpha^*(\beta^*(T)) = T.$$

But then we get $S' = \beta^*(S) = \beta^*(T) = T'$. we conclude that α^* is also injective and, hence, it is a bijection. ∎

In the main theorem of this section, it is shown that if $\mathcal{K}$ and $\mathcal{K}'$ are equivalent π-structures via a conjugate pair $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$, then $\beta^* : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ form a pair of mutually inverse order isomorphisms between the complete lattices of the corresponding theory families.

Recall that, given a π-institution $\mathcal{I}$, we denote by

$$\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$$

the complete lattice of theory families of $\mathcal{I}$ ordered by signature-wise inclusion. We extend the notation to the collections of theory families of π-structures. Thus, given a π-structure $\mathcal{K} = \langle \mathbf{K}, D \rangle$, we define

$$\mathbf{ThFam}(\mathcal{K}) = \langle \mathrm{ThFam}(\mathcal{K}), \leq \rangle.$$

Theorem 896 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be* π-*structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then*

$$\beta^* : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}') \quad and \quad \alpha^* : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$$

are mutually inverse order isomorphisms.

Proof: We know, by Lemma 895, that β^* and α^* are mutually inverse bijections. Moreover, by definition, they are both order preserving. Thus, each is also order-reflecting, since, e.g., for all $S', T' \in \mathrm{ThFam}(\mathcal{K}')$,

$$\alpha^*(S') \leq \alpha^*(T') \quad \text{implies} \quad \beta^*(\alpha^*(S')) \leq \beta^*(\alpha^*(T'))$$
$$\text{implies} \quad S' \leq T',$$

the latter implication following by Lemma 894. ∎

Conversely, it is true that given mutually inverse order isomorphisms between the complete lattices of two π-structures, one may define a conjugate pair between the two that establishes this order-isomorphism via the process that was described above. We provide, next, more details on this inverse process.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems, $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively, and

$$h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$$

an order isomorphism between the corresponding complete lattices of theory families.

Define $\overrightarrow{h} = \{ \overrightarrow{h}_{\Sigma} \}_{\Sigma \in |\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{h}_{\Sigma} : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

be given, for all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\overrightarrow{h}_{\Sigma}[\phi] = h^{-1}(D(\phi)).$$

Further, define $\overleftarrow{h} = \{ \overleftarrow{h}_{\Sigma'} \}_{\Sigma' \in |\mathbf{Sign}'|}$ by letting, for all $\Sigma' \in |\mathbf{Sign}'|$,

$$\overleftarrow{h}_{\Sigma'} : \mathrm{SEN}'(\Sigma') \to \mathrm{SenFam}(\mathbf{K})$$

be given, for all $\psi \in \mathrm{SEN}'(\Sigma')$, by

$$\overleftarrow{h}_{\Sigma'}[\psi] = h(D'(\psi)).$$

We show that, the two translations $\overrightarrow{h} : \mathbf{K} \to \mathbf{K}'$ and $\overleftarrow{h} : \mathbf{K}' \to \mathbf{K}$, defined above, constitute interpretations between the corresponding π-structures.

Lemma 897 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ *is an interpretation.*

Proof: Suppose $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is an order isomorphism and let $\Sigma' \in |\mathbf{Sign}'|$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$. Then we have

$$
\begin{aligned}
\psi \in D'_{\Sigma'}(\Psi) \quad &\text{iff} \quad D'(\psi) \leq D'(\Psi) \quad \text{(Property of } D') \\
&\text{iff} \quad h(D'(\psi)) \leq h(D'(\Psi)) \quad (h \text{ order iso}) \\
&\text{iff} \quad h(D'(\psi)) \leq h(\bigvee^{\mathcal{K}'}\{D'(\chi) : \chi \in \Psi\}) \quad \text{(Property of } \vee^{\mathcal{K}'}) \\
&\text{iff} \quad h(D'(\psi)) \leq \bigvee^{\mathcal{K}}\{h(D'(\chi)) : \chi \in \Psi\} \quad (h \text{ order iso}) \\
&\text{iff} \quad \overleftarrow{h}_{\Sigma'}[\psi] \leq \bigvee^{\mathcal{K}}\{\overleftarrow{h}_{\Sigma'}[\chi] : \chi \in \Psi\} \quad \text{(Definition of } \overleftarrow{h}) \\
&\text{iff} \quad \overleftarrow{h}_{\Sigma'}[\psi] \leq D(\overleftarrow{h}_{\Sigma'}[\Psi]). \quad \text{(Property of } \vee^{\mathcal{K}})
\end{aligned}
$$

Thus, $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is indeed an interpretation.   ∎

We now know (by symmetry, based on Lemma 897) that $\overrightarrow{h} : \mathcal{K} \to \mathcal{K}'$ and $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ are interpretations. It is, in fact, the case that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ forms a conjugate pair, as is shown next.

Lemma 898 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair.*

Proof: By Lemma 892, it suffices to show that $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is an interpretation and that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$D(\phi) = D(\overleftarrow{h}[\overrightarrow{h}_{\Sigma}[\phi]]).$$

The former has been shown in Lemma 897. So it suffices to show the latter.

To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$
\begin{aligned}
D(\overleftarrow{h}[\overrightarrow{h}_\Sigma[\phi]]) \;&=\; D(\overleftarrow{h}[h^{-1}(D(\phi))]) \quad (\text{Definition of } \overrightarrow{h}\,) \\
&=\; D(\bigcup\{\overleftarrow{h}[\chi] : \chi \in h^{-1}(D(\phi))\}) \\
&=\; \bigvee^{\mathcal{K}}\{\overleftarrow{h}[\chi] : \chi \in h^{-1}(D(\phi))\} \quad (\text{Property of } \vee^{\mathcal{K}}) \\
&=\; \bigvee^{\mathcal{K}}\{h(D'(\chi)) : \chi \in h^{-1}(D(\phi))\} \quad (\text{Definition of } \overleftarrow{h}\,) \\
&=\; h(\bigvee^{\mathcal{K}'}\{D'(\chi) : \chi \in h^{-1}(D(\phi))\}) \quad (h \text{ order iso}) \\
&=\; h(D'(h^{-1}(D(\phi)))) \quad (\text{Property of } \vee^{\mathcal{K}'}) \\
&=\; h(h^{-1}(D(\phi))) \quad (h^{-1}(D(\phi)) \text{ theory family}) \\
&=\; D(\phi).
\end{aligned}
$$

We conclude that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair.   ∎

Based on Lemma 898, we can now formulate one of the main theorems of this section to the effect that every order isomorphism between the complete lattices of theory families of two π-structures gives rise to a conjugate pair of interpretations that induce the isomorphism via the star construction.

Theorem 899 *Let* $\mathbf{K} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle\mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle\mathbf{K}, D\rangle$, $\mathcal{K}' = \langle\mathbf{K}', D'\rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair, such that* $\overrightarrow{h}^* = h$ *and* $\overleftarrow{h}^* = h^{-1}$.

Proof: By Lemma 898, we know that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ forms a conjugate pair. We show that $\overrightarrow{h}^* = h$. The equality $\overleftarrow{h}^* = h^{-1}$ may be proved similarly. To this end, let $T' \in \mathrm{ThFam}(\mathcal{K}')$. Then we have

$$
\begin{aligned}
\overrightarrow{h}^*_\Sigma(T') \;&=\; \{\phi \in \mathrm{SEN}(\Sigma) : \overrightarrow{h}_\Sigma[\phi] \leq T'\} \quad (\text{Definition of } ^*) \\
&=\; \{\phi \in \mathrm{SEN}(\Sigma) : h^{-1}(D(\phi)) \leq T'\} \quad (\text{Definition of } \overrightarrow{h}\,) \\
&=\; D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : h^{-1}(D(\phi)) \leq T'\}) \\
&=\; D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : D(\phi) \leq h(T')\}) \quad (h \text{ order iso}) \\
&=\; D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : \phi \in h_\Sigma(T')\}) \quad (\text{Property of } D) \\
&=\; D_\Sigma(h_\Sigma(T')) \\
&=\; h_\Sigma(T'). \quad (h(T') \text{ theory family})
\end{aligned}
$$

Similarly, $\overleftarrow{h}^* = h^{-1}$.   ∎

12.3   Transformations

Let $\mathbf{K} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $k \geq 1$ be an integer. Then a **power algebraic system**

$$\mathbf{K}^k = \langle\mathbf{Sign}, \mathrm{SEN}^k, N^k\rangle$$

is the algebraic system whose sentence functor $\mathrm{SEN}^k : \mathbf{Sign} \to \mathbf{Set}$ is the k-th direct power of SEN and whose category N^k of natural transformations consists of k-tuples of natural transformations having the same arity in N.

Let $k, \ell \geq 1$ be integers. A translation $\alpha : \mathbf{K}^k \to \mathbf{K}^\ell$ is called a **transformation** if there exists a set

$$\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell,$$

in N, with k distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma[\vec{\phi}] = \tau_\Sigma[\vec{\phi}].$$

Moreover a translation $\alpha : \mathbf{K}^k \to \mathbf{K}^\ell$ is called a **natural transformation** if it is a parameter free transformation, i.e., if there exists $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ in N, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma[\vec{\phi}] = \tau_\Sigma[\vec{\phi}].$$

Based on the results obtained in Section 12.2, we may formulate some propositions concerning interpretability and equivalence based on transformations.

Proposition 900 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{F}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two π-structures. $\mathcal{K}$ is interpretable in $\mathcal{K}'$ via a transformation if and only if there exists a set $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$, with k distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$,*
$$\vec{\phi} \in D_\Sigma(\Phi) \quad iff \quad \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]).$$

If $\mathcal{K}$ is interpretable in $\mathcal{K}'$ as above, then it is equivalent to $\mathcal{K}'$ via a conjugate pair $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ of transformations if and only if, for all $\Sigma \in |\mathbf{Sign}|$, all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and all $\Psi \cup \{\vec{\psi}\} \subseteq \mathrm{SEN}(\Sigma)^\ell$,

- $\vec{\psi} \in D'_\Sigma(\Psi)$ *iff* $I_\Sigma[\vec{\psi}] \leq D(I_\Sigma[\Psi])$;

- $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;

- $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: This is a restatement of the definition of interpretability under the additional hypothesis that the corresponding interpretations are transformations. ∎

Proposition 901 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two π-structures. $\mathcal{K}$ is equivalent to $\mathcal{K}'$ via a conjugate pair $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ of transformations if and only if one of the following equivalent conditions hold:*

(a) $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\psi} \in$ $\mathrm{SEN}(\Sigma)^\ell$, $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;

(b) $I : \mathcal{K}' \to \mathcal{K}$ is an interpretation and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in$ $\mathrm{SEN}(\Sigma)^k$, $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: Directly by Lemma 892.   ∎

Taking the point of view of order isomorphisms between lattices of theory families, we would like to have a concept ensuring that such an isomorphism is induced not merely by a conjugate pair of translations, as is asserted by Theorem 899, but, more emphatically, by a conjugate pair of transformations. We focus on this task next.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ be two π-structures based on $\mathbf{K}^k$, $\mathbf{K}^\ell$, respectively. An order isomorphism $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is called **transformational** if there exist sets

- $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$ in N, with k distinguished arguments;

- $I : \mathrm{SEN}^\omega \to \mathrm{SEN}^k$ in N, with ℓ distinguished arguments,

such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$,

$$\overrightarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\vec{\psi}] = D(I_\Sigma[\vec{\psi}]).$$

These conditions are, by definition, equivalent, respectively, to the conditions

$$h^{-1}(D(\vec{\phi})) = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad h(D'(\vec{\psi})) = D(I_\Sigma[\vec{\psi}]).$$

In this case, we say that h is **induced by** $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. (Note that, since we will be able to show that (τ, I) is a conjugate pair of transformations, this notation makes sense.)

In fact, the defining conditions yield some crucial relations between theory families, as is shown in the following lemma.

Lemma 902 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ a transformational order isomorphism induced by $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. Then, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$ and all $\Psi \subseteq \mathrm{SEN}(\Sigma)^\ell$,*

$$h^{-1}(D(\Phi)) = D'(\tau_\Sigma[\Phi]) \quad \text{and} \quad h(D'(\Psi)) = D(I_\Sigma[\Psi]).$$

Proof: By symmetry, it suffices to show the first equation. We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,

$$
\begin{aligned}
h^{-1}(D(\Phi)) &= h^{-1}(\textstyle\bigvee_{\phi \in \Phi}^{\mathcal{K}} D(\phi)) \quad \text{(join in } \mathbf{ThFam}(\mathcal{K})) \\
&= \textstyle\bigvee_{\phi \in \Phi}^{\mathcal{K}'} h^{-1}(D(\phi)) \quad (h^{-1} \text{ order isomorphism}) \\
&= \textstyle\bigvee_{\phi \in \Phi}^{\mathcal{K}'} D'(\tau_\Sigma[\phi]) \quad (h^{-1}(D(\phi)) = \overrightarrow{h}_\Sigma[\phi]) \\
&= D'(\textstyle\bigcup_{\phi \in \Phi} \tau_\Sigma[\phi]) \quad \text{(join in } \mathbf{ThFam}(\mathcal{K}')) \\
&= D'(\tau_\Sigma[\Phi]). \quad \text{(by definition)}
\end{aligned}
$$

The second equation now follows by symmetry.   ∎

Now we are in a position to show that a transformational order isomorphism between the lattices of theory families of two π-structures is induced by a conjugate pair of transformations between the two π-structures and, as a consequence, gives rise to an equivalence $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ via a conjugate pair of transformations.

Theorem 903 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π-*structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair of transformations.*

Proof: We use Proposition 901. Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$. We then have:

$$
\begin{aligned}
\vec{\phi} \in D_\Sigma(\Phi) \quad &\text{iff} \quad D_\Sigma(\vec{\phi}) \leq D_\Sigma(\Phi) \quad (\text{Property of } D) \\
&\text{iff} \quad h^{-1}(D(\vec{\phi})) \leq h^{-1}(D(\Phi)) \quad (h \text{ order iso}) \\
&\text{iff} \quad D'(\tau_\Sigma[\vec{\phi}]) \leq D'(\tau_\Sigma[\Phi]) \quad (\text{Lemma } 902) \\
&\text{iff} \quad \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]). \quad (\text{Property of } D')
\end{aligned}
$$

Thus, $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation. Moreover, we have:

$$
\begin{aligned}
D'(\vec{\psi}) \quad &= \quad h^{-1}(h(D'(\vec{\psi}))) \quad (h \text{ order isomorphism}) \\
&= \quad h^{-1}(D(I_\Sigma[\vec{\psi}])) \quad (h \text{ transformational}) \\
&= \quad D'(\tau[I_\Sigma[\vec{\psi}]]). \quad (\text{Lemma } 902)
\end{aligned}
$$

We conclude by Proposition 901, that $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair of transformations.   ∎

As a consequence, we have the following

Theorem 904 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π-*structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then the* π-*structures* $\mathcal{K}$ *and* $\mathcal{K}'$ *are equivalent via the conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations.*

Proof: This follows directly by Theorem 903.   ∎

Similarly, for interpretability and equivalence based on natural transformations, we have the following corresponding propositions.

Proposition 905 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π-*structures.* $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *via a natural transformation if and only if there exists a set* $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ *in* N, *such that, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$,

$$
\vec{\phi} \in D_\Sigma(\Phi) \quad \text{iff} \quad \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]).
$$

*If $\mathcal{K}$ is interpretable in $\mathcal{K}'$ as above, then it is equivalent to $\mathcal{K}'$ via a conjugate pair $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ of natural transformations if and only if, for all $\Sigma \in$ |**Sign**|, all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and all $\Psi \cup \{\vec{\psi}\} \subseteq \mathrm{SEN}(\Sigma)^{\ell}$,*

- $\vec{\psi} \in D_{\Sigma}'(\Psi)$ *iff* $I_{\Sigma}[\vec{\psi}] \leq D(I_{\Sigma}[\Psi])$;

- $D'(\vec{\psi}) = D'(\tau[I_{\Sigma}[\vec{\psi}]])$;

- $D(\vec{\phi}) = D(I[\tau_{\Sigma}[\vec{\phi}]])$.

Proof: This is a restatement of the definition of interpretability under the additional hypothesis that the corresponding interpretations are natural transformations. ∎

Proposition 906 *Let* **K** $= \langle$**Sign**$, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^{\ell}, D'\rangle$ *be two π-structures. $\mathcal{K}$ is equivalent to $\mathcal{K}'$ via a conjugate pair $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ of natural transformations if and only if one of the following equivalent conditions hold:*

(a) *$\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation and, for all $\Sigma \in$ |**Sign**| and all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^{\ell}$, $D'(\vec{\psi}) = D'(\tau[I_{\Sigma}[\vec{\psi}]])$;*

(b) *$I : \mathcal{K}' \to \mathcal{K}$ is an interpretation and, for all $\Sigma \in$ |**Sign**| and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $D(\vec{\phi}) = D(I[\tau_{\Sigma}[\vec{\phi}]])$.*

Proof: Directly by Lemma 892. ∎

In terms of order isomorphisms between lattices of theory families, we have analogs of preceding results that allow us to work with isomorphisms that are induced by conjugate pairs of natural transformations.

Let **K** $= \langle$**Sign**$, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^{\ell}, D'\rangle$ be two π-structures based on $\mathbf{K}^k$, $\mathbf{K}^{\ell}$, respectively. An order isomorphism $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is called **natural** if there exist sets

- $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^{\ell}$ in N;

- $I : \mathrm{SEN}^{\ell} \to \mathrm{SEN}^k$ in N,

such that, for all $\Sigma \in$ |**Sign**| and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^{\ell}$,

$$\overrightarrow{h}_{\Sigma}[\vec{\phi}] = D'(\tau_{\Sigma}[\vec{\phi}]) \quad \text{and} \quad \overleftarrow{h}_{\Sigma}[\vec{\psi}] = D(I_{\Sigma}[\vec{\psi}]).$$

In this case, we say that h is **induced by** the pair of natural transformations $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$.

Similarly, with the case of a transformational isomorphism, we can show that a natural order isomorphism between the lattices of theory families of two π-structures is induced by a conjugate pair of natural transformations between the two π-structures.

Theorem 907 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a natural order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair of natural transformations.*

Proof: This follows from Theorem 903.   ∎

As a consequence, we have the following analog of Theorem 904.

Theorem 908 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a natural order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then the* π*-structures* $\mathcal{K}$ *and* $\mathcal{K}'$ *are equivalent via the conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of natural transformations.*

Proof: This follows directly by Theorem 907.   ∎

We now revert to the case of a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$. Our focus, in this standard context, will be on $\mathbf{F}$ itself, on the one hand, and on $\mathbf{F}^2$, on the other. In the context of $\mathbf{F}^2$, given $\Sigma \in |\mathbf{Sign}^\flat|$, we sometimes denote a pair $\langle \phi, \psi \rangle \in \mathrm{SEN}^\flat(\Sigma)^2$ in the equational form

$$\phi \approx \psi.$$

Given a π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$, we say that $\mathcal{Q}$ is **equational** if the following five axioms hold:

(R) $\phi \approx \phi \in D_\Sigma(\varnothing)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$;

(S) $\psi \approx \phi \in D_\Sigma(\phi \approx \psi)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$;

(T) $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$;

(C) $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi}) \in D_\Sigma(\{\phi_i \approx \psi_i : i < k\})$, for all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$;

(I) $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\phi \approx \psi)$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

Note that according to the relevant definitions introduced in Chapter 2, the meaning of (I) is that the Σ'-component of the least theory family including $\phi \approx \psi$ in its Σ-component includes $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi)$.

These properties are called **reflexivity**, **symmetry**, **transitivity**, **compatibility** and **invariance**, respectively. The first three ensure that, for all $E \in \mathrm{SenFam}(\mathbf{F}^2)$, $D(E)$ is an equivalence family. The fourth ensures that $D(E)$ is a congruence family and the last that it is a congruence system, i.e., invariant under the action of signature morphisms. In fact, the following characterization theorem holds, showing that a π-structure is equational if

and only if it is structural and all its closure families are congruence systems on $\mathbf{F}$ if and only if it is the equational π-structure relative to a class K of $\mathbf{F}$-algebraic systems according to the definition given in Section 2.17.

Theorem 909 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *a* π*-structure. The following statements are equivalent:*

(i) $\mathcal{Q}$ *is equational;*

(ii) *For all* $\theta \in \mathrm{SenFam}(\mathcal{Q})$, $D(\theta) \in \mathrm{ConSys}(\mathbf{F})$;

(iii) *For some class* K *of* $\mathbf{F}$*-algebraic systems,* $D = D^{\mathsf{K}}$.

Proof:

(i)$\Rightarrow$(ii) Suppose $\mathcal{Q}$ is equational and let $\theta \in \mathrm{SenFam}(\mathcal{Q})$. We must show that $D(\theta) = \{D_\Sigma(\theta)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a congruence system on $\mathbf{F}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational, we have $\phi \approx \phi \in D_\Sigma(\varnothing) \subseteq D_\Sigma(\theta)$. So $D_\Sigma(\theta)$ is reflexive. Suppose, next, that $\phi \approx \psi \in D_\Sigma(\theta)$. Since $\mathcal{Q}$ is equational, we get $\psi \approx \phi \in D_\Sigma(\phi \approx \psi) \subseteq D_\Sigma(\theta)$. Hence, $D_\Sigma(\theta)$ is also symmetric. Further, if $\phi \approx \psi, \psi \approx \chi \in D_\Sigma(\theta)$, then, since $\mathcal{Q}$ is equational, we get $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi) \subseteq D_\Sigma(\theta)$. Thus, $D_\Sigma(\theta)$ is also transitive and, hence, an equivalence relation on $\mathrm{SEN}^\flat(\Sigma)$.

Suppose, now, that $\sigma^\flat \in N^\flat$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, for $i < k$, such that $\phi_i \approx \psi_i \in D_\Sigma(\theta)$, for all $i < k$. Since $\mathcal{Q}$ is equational, we get $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi}) \in D_\Sigma(\{\phi_i \approx \psi_i : i < k\}) \subseteq D_\Sigma(\theta)$. Hence, $D_\Sigma(\theta)$ is a congruence family on $\mathbf{F}$. Finally, if $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in D_\Sigma(\theta)$, then, again based on the fact that $\mathcal{Q}$ is equational, we obtain $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\phi \approx \psi) \subseteq D_{\Sigma'}(\theta)$, whence $D(\theta)$ is a congruence system on $\mathbf{F}$, as was to be shown.

(ii)$\Rightarrow$(iii) Suppose D satisfies (ii). We construct a class K of $\mathbf{F}$-algebraic systems as follows. For $\theta \in \mathrm{SenFam}(\mathcal{Q})$, define

$$\mathcal{F}^\theta = \langle \mathbf{F}^\theta, \langle I, \pi^\theta \rangle \rangle := \langle \mathbf{F}/D(\theta), \langle I, \pi^{D(\theta)} \rangle \rangle$$

and set

$$\mathsf{K} = \{\mathcal{F}^\theta : \theta \in \mathrm{SenFam}(\mathcal{Q})\}.$$

Note that the definition of $\mathcal{F}^\theta$ makes sense, since, by hypothesis, $D(\theta) \in \mathrm{ConSys}(\mathbf{F})$, for all $\theta \in \mathrm{SenFam}(\mathcal{Q})$. Our task now is to show that $D = D^{\mathsf{K}}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\theta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$.

Suppose, first, that $\phi \approx \psi \in D_\Sigma(\theta)$ and let $\theta' \in \mathrm{SenFam}(\mathcal{Q})$, such that $\pi_\Sigma^{\theta'}(\theta) \subseteq \Delta_\Sigma^{\mathbf{F}/D(\theta')}$. This is equivalent to $\theta_\Sigma \subseteq D_\Sigma(\theta'_\Sigma)$. Hence, we obtain

$\phi \approx \psi \in D_\Sigma(\theta) \subseteq D_\Sigma(\theta')$. Thus, $\pi_\Sigma^{\theta'}(\phi) = \pi_\Sigma^{\theta'}(\psi)$. We conclude that $\phi \approx \psi \in D_\Sigma^{\mathsf{K}}(\theta)$. Hence, $D \leq D^{\mathsf{K}}$.

Assume, conversely, that $\phi \approx \psi \notin D_\Sigma(\theta)$. Then, clearly, for $\mathcal{F}^\theta \in \mathsf{K}$, we get $\pi_\Sigma^\theta(D_\Sigma(\theta)) \subseteq \Delta_\Sigma^{\mathbf{F}/D(\theta)}$, but $\pi_\Sigma^\theta(\phi) \neq \pi_\Sigma^\theta(\psi)$. Hence, $\phi \approx \psi \notin D_\Sigma^{\mathsf{K}}(\theta)$. Therefore, $D^{\mathsf{K}} \leq D$ and, hence, $D = D^{\mathsf{K}}$.

(iii)$\Rightarrow$(i) This implication was shown in Proposition 115, which was proven by appealing to the implication (iii)$\Rightarrow$(ii), which was, in turn, the content of Proposition 30.   ∎

We have the following useful technical lemma, where, for $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, we use the abbreviation

$$\vec{\phi} \approx \vec{\psi} = \{\phi_i \approx \psi_i : i < k\}.$$

Lemma 910 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. Then, for all* $\delta^\flat, \epsilon^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\delta_\Sigma^\flat(\vec{\psi}) \approx \epsilon_\Sigma^\flat(\vec{\psi}) \in D_\Sigma(\vec{\phi} \approx \vec{\psi}, \delta_\Sigma^\flat(\vec{\phi}) \approx \epsilon_\Sigma^\flat(\vec{\phi})).$$

Proof: We have, for all $\delta^\flat, \epsilon^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$
\begin{aligned}
\delta_\Sigma^\flat(\vec{\psi}) \approx \epsilon_\Sigma^\flat(\vec{\psi}) \quad &\in \quad D_\Sigma(\delta_\Sigma^\flat(\vec{\psi}) \approx \delta_\Sigma^\flat(\vec{\phi}), \delta_\Sigma^\flat(\vec{\phi}) \approx \epsilon_\Sigma^\flat(\vec{\phi}), \epsilon_\Sigma^\flat(\vec{\phi}) \approx \epsilon_\Sigma^\flat(\vec{\psi})) \\
&\qquad \text{(by transitivity)} \\
&\subseteq \quad D_\Sigma(\vec{\phi} \approx \vec{\psi}, \delta_\Sigma^\flat(\vec{\phi}) \approx \epsilon_\Sigma^\flat(\vec{\phi})). \\
&\qquad \text{(by symmetry and compatibility)}
\end{aligned}
$$

This proves the lemma.   ∎

Lemma 910 has the following corollary.

Corollary 911 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. Then, for all* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *with* k *distinguished arguments, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau_\Sigma^\flat[\vec{\psi}] \leq D(\vec{\phi} \approx \vec{\psi}, \tau_\Sigma^\flat[\vec{\phi}]).$$

Proof: This follows from Lemma 910, using the reflexivity and the invariance of the closure family D.   ∎

We next show that, if a π-institution $\mathcal{I}$, based on an algebraic system $\mathbf{F}$, happens to be equivalent to an equational π-structure $\mathcal{Q}$, based on $\mathbf{F}^2$, via a conjugate pair $(\tau, I) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is syntactically protoalgebraic with set of witnessing transformations I.

Theorem 912 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically protoalgebraic with witnessing transformations* $I^\flat$.

Proof: By definition, it suffices to show that $I^\flat : \mathrm{SEN}^\omega \to \mathrm{SEN}$, with two distinguished arguments, is reflexive, globally family transitive and has the global family compatibility and the global family modus ponens in $\mathcal{I}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, in turn:

- By reflexivity of $\mathcal{Q}$, $\phi \approx \phi \in D_\Sigma(\varnothing)$. Hence, by interpretability, we get $I^\flat_\Sigma[\phi, \phi] \leq C(\varnothing)$. Therefore, $I^\flat$ is reflexive in $\mathcal{I}$;

- By transitivity of $\mathcal{Q}$, $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$. Hence, by interpretability, we get $I^\flat_\Sigma[\phi, \chi] \leq C(I^\flat_\Sigma[\phi, \psi], I^\flat_\Sigma[\psi, \chi])$. Therefore, $I^\flat$ is globally family transitive in $\mathcal{I}$;

- By the reflexivity and compatibility of $\mathcal{Q}$, we have, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in N and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, that $\sigma^\flat_\Sigma(\phi, \vec{\chi}) \approx \sigma^\flat_\Sigma(\psi, \vec{\chi}) \in D_\Sigma(\phi \approx \psi)$. Hence, by interpretability,

$$I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq C(I^\flat_\Sigma[\phi, \psi]).$$

  Therefore, $I^\flat$ has the global family compatibility in $\mathcal{I}$;

- Finally, for global family MP, we have

$$\begin{aligned} C(\psi) &= C(I^\flat[\tau^\flat_\Sigma[\psi]]) \quad \text{(by equivalence)} \\ &\leq C(I^\flat_\Sigma[\phi, \psi], I^\flat[\tau^\flat_\Sigma[\phi]]) \\ &\quad \text{(by Lemma 910 and interpretability)} \\ &= C(I^\flat_\Sigma[\phi, \psi], \phi). \quad \text{(by equivalence)} \end{aligned}$$

  Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, if $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq T$, then $\psi \in T_\Sigma$, i.e., $I^\flat$ has the global family modus ponens in $\mathcal{I}$.

We conclude that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. ∎

Using Theorem 912, we can show that, if a π-institution $\mathcal{I}$, based on an algebraic system $\mathbf{F}$, happens to be equivalent to an equational π-structure $\mathcal{Q}$, based on $\mathbf{F}^2$, via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is family truth equational, with witnessing equations $\tau^\flat$.

Theorem 913 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is family truth equational, with witnessing equations* $\tau^\flat$.

Proof: By definition, it suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^\flat[\phi] \leq \Omega(T).$$

We, indeed, have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad I^\flat[\tau_\Sigma[\phi]] \leq T \quad ((\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q} \text{ an equivalence})$$
$$\text{iff} \quad \tau_\Sigma^\flat[\phi] \leq \Omega(T). \quad \text{(by Theorem 912 and Corollary 791)}$$

Therefore, $\mathcal{I}$ is family truth equational, with witnessing equations $\tau^\flat$.   ∎

We close the section by showing that equivalence between a given π-institution and an equational π-structure established via conjugate pairs of transformations is essentially unique in the sense that both the closure family on $\mathbf{F}^2$ must be unique and the closures of the translations used must be identical. More precisely, we have the following

Theorem 914 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. *Suppose that* $\mathcal{Q}^1 = \langle \mathbf{F}^2, D^1 \rangle$ *and* $\mathcal{Q}^2 = \langle \mathbf{F}^2, D^2 \rangle$ *are equational* π-*structures that are equivalent to* $\mathcal{I}$ *via the conjugate pairs* $\langle \tau^1, I^1 \rangle : \mathcal{I} \rightleftarrows \mathcal{Q}^1$ *and* $\langle \tau^2, I^2 \rangle : \mathcal{I} \rightleftarrows \mathcal{Q}^2$, *respectively, of transformations. Then, we have:*

(a) $D^1 = D^2$ (=: D) *and, hence,* $\mathcal{Q}^1 = \mathcal{Q}^2$ (=: $\mathcal{Q}$);

(b) *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(I_\Sigma^1[\phi, \psi]) = C(I_\Sigma^2[\phi, \psi])$;

(c) *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $D(\tau_\Sigma^1[\phi]) = D(\tau_\Sigma^2[\phi])$.

Proof: By Theorem 912, we know that both I^1 and I^2 are witnessing the syntactic protoalgebraicity of $\mathcal{I}$. Thus, by Corollary 791, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I_\Sigma^1[\phi, \psi] \leq T \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad I_\Sigma^2[\phi, \psi] \leq T.$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(I_\Sigma^1[\phi, \psi]) = C(I_\Sigma^2[\phi, \psi])$, which proves Part (b).

For Part (a), suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$. Then, we have

$$\phi \approx \psi \in D_\Sigma^1(E) \quad \text{iff} \quad I_\Sigma^1[\phi, \psi] \leq C(I_\Sigma^1[E]) \quad \text{(interpretability)}$$
$$\text{iff} \quad C(I_\Sigma^1[\phi, \psi]) \leq C(I_\Sigma^1[E]) \quad \text{(property of } C)$$
$$\text{iff} \quad C(I_\Sigma^2[\phi, \psi]) \leq C(I_\Sigma^2[E]) \quad \text{(Part (b))}$$
$$\text{iff} \quad I_\Sigma^2[\phi, \psi] \leq C(I_\Sigma^2[E]) \quad \text{(property of } C)$$
$$\text{iff} \quad \phi \approx \psi \in D_\Sigma^2(E). \quad \text{(interpretability)}$$

Therefore, we get that $D^1 = D^2$. This justifies using $D := D^1 = D^2$ and since the π-structures $\mathcal{Q}^1$ and $\mathcal{Q}^2$, which are both based on $\mathbf{F}^2$, have the same closure families, we obtain $\mathcal{Q} := \mathcal{Q}^1 = \mathcal{Q}^2$.

Finally, for Part (c), suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$
\begin{aligned}
D(\tau_\Sigma^1[\phi]) \le D(\tau_\Sigma^2[\phi]) \quad &\text{iff}\quad \tau_\Sigma^1[\phi] \le D(\tau_\Sigma^2[\phi]) \quad \text{(property of } D) \\
&\text{iff}\quad I^2[\tau_\Sigma^1[\phi]] \le C(I^2[\tau_\Sigma^2[\phi]]) \quad \text{(interpretability)} \\
&\text{iff}\quad I^2[\tau_\Sigma^1[\phi]] \le C(\phi) \quad \text{(equivalence)} \\
&\text{iff}\quad I^1[\tau_\Sigma^1[\phi]] \le C(\phi) \quad \text{(Part (b))} \\
&\text{iff}\quad \phi \in C_\Sigma(\phi). \quad \text{(equivalence)}
\end{aligned}
$$

By symmetry, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $D(\tau_\Sigma^1[\phi]) = D(\tau_\Sigma^2[\phi])$. This proves Part (c) and concludes the proof of the theorem.   ∎

12.4   Syntactic Weak Family Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 4, Section 4.3, that a π-institution is weakly family algebraizable (WF algebraizable) if it is protoalgebraic and family injective. We work now towards establishing a syntactic counterpart of this class of π-institutions.

We say that $\mathcal{I}$ is $R^{\mathcal{I}} S^{\mathcal{I}}$-(**syntactically**) **fortified** if:

- The reflexive core $R^{\mathcal{I}}$ is Leibniz;

- The Suszko core $S^{\mathcal{I}}$ is adequate.

Moreover, we say that $\mathcal{I}$ is **syntactically weakly family algebraizable** (abbreviated to **syntactically WF algebraizable**)   if:

- $\mathcal{I}$ is $R^{\mathcal{I}} S^{\mathcal{I}}$-fortified;

- $\mathcal{I}$ is protoalgebraic;

- $\mathcal{I}$ is family injective.

By Theorem 288, under protoalgebraicity, the properties of family injectivity, family reflectivity and family c-reflectivity coincide. This enables us to formulate the following alternative characterization of syntactic WF algebraizability.

Theorem 915 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically WF algebraizable if and only if it is syntactically protoalgebraic and family truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically WF algebraizable. Then, on the one hand, it is protoalgebraic and has a Leibniz reflexive core. Thus, by Theorem 805, it is syntactically protoalgebraic. On the other, it is, by Theorem 288, family c-reflective and has an adequate Suszko core. Therefore, by Theorem 848, it is family truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically protoalgebraic and family truth equational. Then, by Theorem 805, it is protoalgebraic and has a Leibniz reflexive core, and, by Theorem 848, it is family c-reflective and has an adequate Suszko core. Therefore, $\mathcal{I}$ is syntactically WF algebraizable. ∎

Directly from the definitions, we may derive the following relationship between the semantic and syntactic WF algebraizability classes of π-institutions.

Theorem 916 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically WF algebraizable if and only if $\mathcal{I}$ is WF algebraizable and $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified.*

Proof: $\mathcal{I}$ is syntactically WF algebraizable if and only if, by definition, it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, protoalgebraic and family injective, i.e., iff it is, by definition, $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and WF algebraizable. ∎

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak family algebraizability in terms of isomorphisms between complete lattices of theory families.

Theorem 917 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically WF algebraizable if and only if it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: We have that $\mathcal{I}$ is syntacticaly WF algebraizable if and only if, by Theorem 916, it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and WF algebraizable, if and only if, by Theorem 296, it it $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. ∎

Next, we show that syntactic WF algebraizability may also be characterized by the existence of an equivalence between the π-institution and its algebraic π-structure counterpart via a pair of conjugate transformations.

We embark on the path by defining first the algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with a given π-institution $\mathcal{I}$. We recall some concepts that we have already introduced previously which culminate in the definition of $\mathcal{Q}^{\mathcal{I}*}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall the definition of the class $\mathrm{AlgSys}^*(\mathcal{I})$ of all reduced $\mathbf{F}$-algebraic systems:

$$\mathrm{AlgSys}^*(\mathcal{I}) = \{ \mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}) \}.$$

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we define the class of $\mathcal{I}^*$-**congruence systems on** $\mathcal{A}$ by

$$\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \{ \theta \in \mathrm{ConSys}(\mathbf{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I}) \}.$$

It turns out that congruence systems in $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ have a straightforward characterization.

Proposition 918 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$-*algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \{ \theta \in \mathrm{ConSys}(\mathbf{A}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \theta) \}.$$

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system.

Suppose, first, that $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that

$$\Omega^{\mathcal{A}/\theta}(T') = \Delta^{\mathcal{A}/\theta}.$$

By applying the inverse of the quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}/\theta$, we get

$$(\pi^\theta)^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = (\pi^\theta)^{-1}(\Delta^{\mathcal{A}/\theta}).$$

Since $\langle I, \pi^\theta \rangle$ is surjective, we get by Proposition 24 and by Corollary 55, that $(\pi^\theta)^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and

$$\Omega^{\mathcal{A}}((\pi^\theta)^{-1}(T')) = \theta.$$

Therefore, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \theta$.

Suppose, conversely, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, with $\Omega^{\mathcal{A}}(T) = \theta$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, we have $\Omega^{\mathcal{A}/\theta}(T/\theta) = \Delta^{\mathcal{A}/\theta}$ and, therefore, by definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})$, implying that $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$.   ∎

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ of $\mathcal{I}^*$-congruence systems on $\mathcal{A}$ need not be closed under signature-wise intersections, i.e., may not form a closure family on $\mathbf{A}^2$. However, we can show that, if $\mathcal{I}$ is protoalgebraic, this is always the case.

Proposition 919 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$-*algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{A}^2$.

Proof: First, note that $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ has a top element $\nabla^{\mathcal{A}}$. To see this, observe that $\mathcal{A}/\nabla^{\mathcal{A}}$ is a trivial algebraic system, which is always a member of $\mathrm{AlgSys}^{*}(\mathcal{I})$.

It suffices now to show that $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is closed under arbitrary intersections. To this end, suppose $\theta^{i} \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$, for $i \in I$. By Proposition 918, for all $i \in I$, there exists $T^{i} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T^{i}) = \theta^{i}$. But, by Lemma 23 and protoalgebraicity, we get that

$$\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^{i}) = \bigcap_{i\in I}\Omega^{\mathcal{A}}(T^{i}) = \bigcap_{i\in I}\theta^{i}.$$

Now, again by Proposition 918, we conclude that $\bigcap_{i\in I}\theta^{i} \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ∎

Applying Proposition 919 to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get the following

Corollary 920 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$. *Then,* $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{F}^{2}$.

Proof: This is a special case of Proposition 919. ∎

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution. We define, in accordance with Corollary 920, the **equational** or **algebraic π-structure** $\mathcal{Q}^{\mathcal{I}*}$ **associated with** $\mathcal{I}$ to be the π-structure

$$\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^{2}, D^{\mathcal{I}*} \rangle,$$

where $D^{\mathcal{I}*}$ is the closure (operator) family corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$.

Our first result in connecting syntactic WF algebraizability with the associated equational π-structure shows that, if a π-institution is syntactically WF algebraizable, then it is equivalent to its associated equational π-structure via a conjugate pair of transformations.

Theorem 921 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically WF algebraizable π-institution based on* $\mathbf{F}$. *Then* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^{\flat}, \overleftrightarrow{I^{\flat}}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations. More precisely:*

- $I^{\flat} : (\mathrm{SEN}^{\flat})^{\omega} \to \mathrm{SEN}^{\flat}$ *in* $N^{\flat}$, *with two distinguished arguments, is a set of witnessing transformations of the syntactic protoalgebraicity of* $\mathcal{I}$;

- $\tau^{\flat} : (\mathrm{SEN}^{\flat})^{\omega} \to (\mathrm{SEN}^{\flat})^{2}$, *with a single distinguished argument, is a set of witnessing equations for the family truth equationality of* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}$ is syntactically WF algebraizable. Then, by Theorem 915, $\mathcal{I}$ is syntactically protoalgebraic and family truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic protoalgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing family truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$ and $\overleftrightarrow{I^\flat}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]$ are as required. Therefore, by Proposition 901, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, by family truth equationality,

$$\begin{aligned} \Phi \subseteq T_\Sigma &\quad \text{iff} \quad \tau^\flat_\Sigma[\Phi] \leq \Omega(T); \\ \phi \in T_\Sigma &\quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T). \end{aligned}$$

Therefore, $\phi \in C_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 918, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{aligned} \phi \approx \psi \in \Omega_\Sigma(T) \quad &\text{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T \quad \text{(Corollary 791)} \\ &\text{iff} \quad \tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]] \leq \Omega(T). \quad \text{(truth equationality)} \end{aligned}$$

Using again Proposition 918, we conclude that

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

Therefore $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via $(\tau^\flat, \overleftrightarrow{I^\flat}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. ∎

Putting together Theorems 912, 913 and 921, we get the following fundamental result to the effect that syntactic WF algebraizability boils down to the equivalence of a π-institution with its associated equational π-structure via a conjugate pair of transformations.

Theorem 922 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations.*

Proof: If $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations, then, by Theorem 912, it is syntactically protoalgebraic and, by Theorem 913, it is family truth equational. Therefore, by Theorem 915, it is syntactically WF algebraizable.

If, conversely, $\mathcal{I}$ is syntactically WF algebraizable, then, by Theorem 921, it is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. ∎

We close the section by slightly generalizing the preceding characterization. Namely, we show that existence of an equivalence with an equational π-structure induced by conjugate transformations is sufficient to yield syntactic WF algebraizability.

Theorem 923 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is equivalent to an equational* π-*structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically WF algebraizable, then the conclusion follows from Theorem 922. Conversely, if $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then it is syntactically protoalgebraic by Theorem 912 and family truth equational by Theorem 913, whence it is syntactically WF algebraizable. ∎

Taking into account Theorem 904, we have the following alternative characterization of syntactically WF algebraizable π-institutions.

Theorem 924 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if there is a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$, *where* $\mathcal{Q}$ *is an equational* π-*structure.*

Proof: The "only if" follows by Theorem 923 and Theorem 896. The "if" is given by Theorem 904 and Theorem 923. ∎

12.5   Syntactic Weak Algebraizability

Syntactic WF algebraizability determines one of the highest levels of the main algebraic hierarchy of π-institutions. Since every syntactically WF algebraizable π-institution is, in particular, family reflective, it follows that

every syntactically WF algebraizable π-institution is systemic. To avoid systemicity, one has to weaken the hypothesis of family reflectivity. In this section, we follow this line of thought by keeping the assumption of syntactic protoalgebraicity, but insisting only that the π-institution is system truth equational, rather than family truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

We say that $\mathcal{I}$ is $R^{\mathcal{I}} L^{\mathcal{I}}$-(**syntactically**) **fortified** if:

- The reflexive core $R^{\mathcal{I}}$ is Leibniz;

- The left Suszko core $L^{\mathcal{I}}$ is left adequate.

A π-institution $\mathcal{I}$ is **syntactically weakly algebraizable** (abbreviated to **syntactically W algebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified;

- $\mathcal{I}$ is protoalgebraic;

- $\mathcal{I}$ is system injective.

By Corollary 300, under protoalgebraicity, the six properties of system injectivity, left injectivity, system reflectivity, left reflectivity, system complete reflectivity and left complete reflectivity coincide. This enables us to formulate the following alternative characterization of syntactic weak algebraizability.

Theorem 925 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically weakly algebraizable if and only if it is syntactically protoalgebraic and system (or, equivalently, left) truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically weakly algebraizable. Then, on the one hand, it is protoalgebraic and has a Leibniz reflexive core. Thus, by Theorem 805, it is syntactically protoalgebraic. On the other, it is, by Theorem 300, left c-reflective and has a left adequate left Suszko core. Therefore, by Theorem **??**, it is left truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically protoalgebraic and left truth equational. Then, by Theorem 805, it is protoalgebraic and has a Leibniz reflexive core, and, by Theorem 871, it is left c-reflective and has a left adequate left Suszko core. Therefore, by definition, $\mathcal{I}$ is syntactically weakly algebraizable. ∎

Directly from the definitions, we may derive the following relationship between the semantic and syntactic weak algebraizability classes of π-institutions.

Theorem 926 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if* $\mathcal{I}$ *is weakly algebraizable and* $R^\mathcal{I} L^\mathcal{I}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically weakly algebraizable if and only if, by definition, it is $R^\mathcal{I} L^\mathcal{I}$-fortified, protoalgebraic and system injective, i.e., iff it is, by definition, $R^\mathcal{I} L^\mathcal{I}$-fortified and weakly algebraizable.   ∎

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak algebraizability in terms of isomorphisms between complete lattices of theory systems.

Theorem 927 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is* $R^\mathcal{I} L^\mathcal{I}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^\mathcal{A} : \mathrm{FiSys}^\mathcal{I}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: We have that $\mathcal{I}$ is syntactically weakly algebraizable if and only if, by Theorem 926, it is $R^\mathcal{I} L^\mathcal{I}$-fortified and weakly algebraizable, if and only if, by Theorem 298, it it $R^\mathcal{I} L^\mathcal{I}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^\mathcal{A} : \mathrm{FiSys}^\mathcal{I}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.   ∎

Next, we show that syntactic weak algebraizability may also be characterized by stability in conjunction with the existence of an equivalence between the systemic skeleton of a π-institution and its algebraic π-structure counterpart via a pair of conjugate transformations. To start, we define the *systemic skeleton* of a given π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathrm{ThSys}(\mathcal{I})$ forms a complete lattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq \rangle$ under signature wise inclusion. Therefore, we are justified in defining the π-structure

$$\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I} \rangle$$

of $\mathcal{I}$ by stipulating that $K^\mathcal{I} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is the closure family on $\mathbf{F}$ corresponding to the closed set family $\mathrm{ThSys}(\mathcal{I})$. We call $\mathcal{K}^\mathcal{I}$ the **systemic skeleton** of $\mathcal{I}$.

We give an example to show that, in general, $K^\mathcal{I}$ is not a π-institution, since $K^\mathcal{I} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ may not satisfy structurality.

Example 928 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be defined as follows:*

- **Sign**$^\flat$ *is the category with objects* Σ, Σ' *and, except the identities, a morphism* $f : \Sigma \to \Sigma$ *and two morphisms* $g, h : \Sigma \to \Sigma'$, *satisfying the following composition rules:*
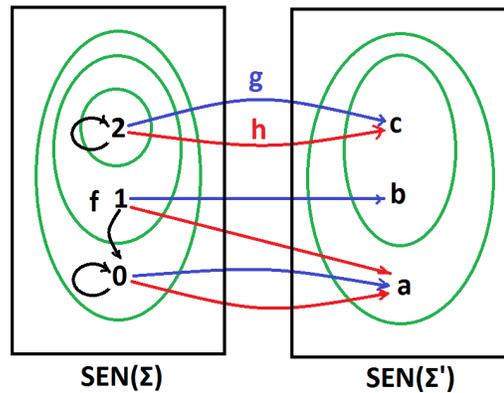
$$f \circ f = f, \quad gf = h, \quad hf = h.$$

- SEN$^\flat$: **Sign**$^\flat$ → **Set** *is defined by setting*

$$\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}, \quad \mathrm{SEN}^\flat(\Sigma') = \{a, b, c\}$$

  and

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\mathrm{SEN}^\flat(f)(x)$	$\mathrm{SEN}^\flat(g)(x)$	$\mathrm{SEN}^\flat(h)(x)$
0	0	a	a
1	0	b	a
2	2	c	c

- *Finally,* $\mathrm{N}^\flat$ *is the trivial category of natural transformations (consisting of the projections only).*

Next define the π-*institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\} \quad and \quad \mathcal{C}_{\Sigma'} = \{\{b, c\}, \{a, b, c\}\}.$$

This π-*institution has six theory families, having the lattice structure shown on the left below. It has, however, only three theory systems, whose lattice structure is given on the right.*

$$\text{SEN}^\flat \qquad\qquad\qquad\qquad \text{SEN}^\flat$$

12 abc   *012 bc*   *2 abc*

2 abc   *12 bc*   *2 bc*

2 bc
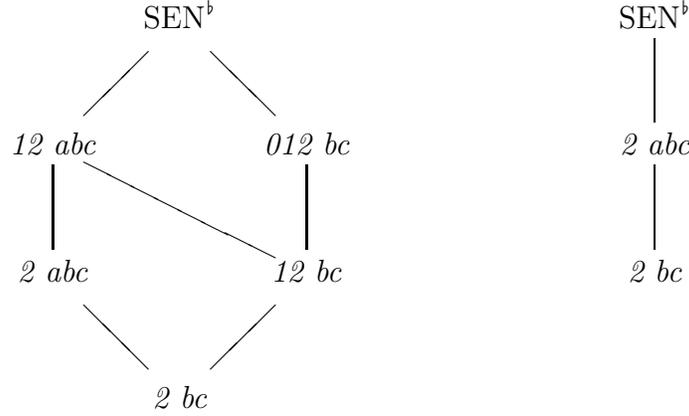
The theory systems of $\mathcal{I}$ are the theory families of the systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$. *We can see that $\mathcal{K}^{\mathcal{I}}$ is not a π-institution by considering* $\Phi = \{1\} \subseteq \text{SEN}^\flat(\Sigma)$. *We have*

$$
\begin{aligned}
\text{SEN}^\flat(g)(K_\Sigma^{\mathcal{I}}(\{1\})) &= \text{SEN}^\flat(g)(\bigcap\{T_\Sigma : \{\{1\}, \varnothing\} \le T \in \text{ThSys}(\mathcal{I})\}) \\
&= \text{SEN}^\flat(g)(\{0, 1, 2\}) \\
&= \{a, b, c\}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
K_{\Sigma'}^{\mathcal{I}}(\text{SEN}^\flat(g)(\{1\})) &= K_{\Sigma'}^{\mathcal{I}}(\{b\}) \\
&= \bigcap\{T_{\Sigma'} : \{\varnothing, \{b\}\} \le T \in \text{ThSys}(\mathcal{I})\} \\
&= \{b, c\}.
\end{aligned}
$$

Therefore

$$\text{SEN}^\flat(g)(K_\Sigma^{\mathcal{I}}(\{1\})) \nsubseteq K_{\Sigma'}^{\mathcal{I}}(\text{SEN}^\flat(g)(\{1\}))$$

showing that $K^{\mathcal{I}}$ is not structural and, hence, $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ is a π-structure, but not a π-institution.

We now resume our work on the characterization of syntactic weak algebraizability. We will again make use of the equational π-structure $\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^2, D^{\mathcal{I}*} \rangle$ associated with a protoalgebraic π-institution $\mathcal{I}$. Recall from Section 12.4 that this is the π-structure whose closure operator is the one corresponding to the closure family $\text{ConSys}^{\mathcal{I}*}(\mathcal{F})$.

Our first result connecting syntactic weak algebraizability of a π-institution with the associated equational π-structure shows that, if a π-institution is syntactically weakly algebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its associated equational π-structure $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations.

Theorem 929 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically weakly algebraizable π-institution based on $\mathbf{F}$. Then $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, \overleftrightarrow{I^\flat}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ of transformations. More precisely:*

- $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, is a set of witnessing transformations of the syntactic protoalgebraicity of* $\mathcal{I}$;

- $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, *with a single distinguished argument, is a set of witnessing equations for the left truth equationality of* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}$ is syntactically weakly algebraizable. Then, by Theorem 925, $\mathcal{I}$ is syntactically protoalgebraic and left truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic protoalgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing left truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$ and $\overset{\leftrightarrow}{I^\flat}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overset{\leftrightarrow}{I^\flat}_\Sigma[\phi, \psi]$ are as required. Therefore, by Proposition 901, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in K^\mathcal{I}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overset{\leftrightarrow}{I^\flat}_\Sigma[\phi, \psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have

$$\begin{aligned}\Phi \subseteq T_\Sigma \quad &\text{iff} \quad \Phi \subseteq \overset{\leftarrow}{T}_\Sigma \quad (T \in \mathrm{ThSys}(\mathcal{I})) \\ &\text{iff} \quad \tau^\flat_\Sigma[\Phi] \leq \Omega(T) \quad \text{(left truth equationality)}\end{aligned}$$

and, similarly,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Therefore, $\phi \in K^\mathcal{I}_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by stability, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 918, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{aligned}\phi \approx \psi \in \Omega_\Sigma(T) \quad &\text{iff} \quad \overset{\leftrightarrow}{I^\flat}_\Sigma[\phi, \psi] \leq T \quad \text{(Corollary 791)} \\ &\text{iff} \quad \overset{\leftrightarrow}{I^\flat}_\Sigma[\phi, \psi] \leq \overset{\leftarrow}{T} \quad (T \in \mathrm{ThSys}(\mathcal{I})) \\ &\text{iff} \quad \tau^\flat[\overset{\leftrightarrow}{I^\flat}_\Sigma[\phi, \psi]] \leq \Omega(T). \quad \text{(left truth equationality)}\end{aligned}$$

Using again Proposition 918 and stability, we conclude that

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

Therefore $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via $(\tau^\flat, \overrightarrow{I^\flat}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$.   ∎

Towards the converse, we show, first, that, if a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is such that there exists an equivalence $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$, via a conjugate pair of transformations, between its systemic skeleton and an equational π-structure $\mathcal{Q}$, then $I^\flat$ defines Leibniz congruence systems of theory systems of $\mathcal{I}$.

Recall that for a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, and a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, we define, for all $T \in \mathrm{SenFam}(\mathcal{I})$, $\overleftarrow{I^\flat}(T) = \{\overleftarrow{I^\flat}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\overleftarrow{I^\flat}_\Sigma(T) = \{\langle \phi, \psi \rangle \in \mathrm{SEN}^\flat(\Sigma)^2 : I^\flat_\Sigma[\phi, \psi] \leq T\}.$$

Proposition 930 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *is equivalent to an equational* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\Omega(T) = \overleftarrow{I^\flat}(T)$*.*

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. It suffices to show, by Corollary 98, that $\overleftarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T. We know by Lemma 93 that it is a relation system on $\mathbf{F}$.

Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ is equational, we have $\phi \approx \phi \in D_\Sigma(\varnothing)$. Therefore, by interpretability, $I^\flat_\Sigma[\phi, \phi] \leq K^{\mathcal{I}}(\varnothing) = C(\varnothing) \leq T$. Hence, $\langle \phi, \phi \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$ and $\overleftarrow{I^\flat}(T)$ is reflexive.

Suppose, now, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational, we have that $\psi \approx \phi \in D_\Sigma(\phi \approx \psi)$. Therefore, by interpretability, $I^\flat_\Sigma[\psi, \phi] \leq K^{\mathcal{I}}(I^\flat_\Sigma[\phi, \psi])$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if $I^\flat_\Sigma[\phi, \psi] \leq T$, then $I^\flat_\Sigma[\psi, \phi] \leq T$. In other words $\langle \phi, \psi \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$ implies $\langle \psi, \phi \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$. Therefore, $\overleftarrow{I^\flat}(T)$ is also symmetric.

Suppose, next, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational, we have that $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$. Therefore, by interpretability, $I^\flat_\Sigma[\phi, \chi] \leq K^{\mathcal{I}}(I^\flat_\Sigma[\phi, \psi], I^\flat_\Sigma[\psi, \chi])$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if $I^\flat_\Sigma[\phi, \psi], I^\flat_\Sigma[\psi, \chi] \leq T$, then $I^\flat_\Sigma[\phi, \chi] \leq T$. In other words, $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$. Therefore, $\overleftarrow{I^\flat}(T)$ is transitive.

We have now shown that $\overleftarrow{I^\flat}(T)$ is an equivalence system on $\mathbf{F}$. It remains to show that it satisfies the congruence property and that it is compatible with T.

Suppose that $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational, we have that $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi}) \in D_\Sigma(\vec{\phi} \approx \vec{\psi})$ (recall that $\vec{\phi} \approx \vec{\psi}$ means $\{\phi_i \approx \psi_i : i < k\}$). Therefore, by interpretability,

$$I_\Sigma^\flat[\sigma_\Sigma^\flat(\vec{\phi}), \sigma_\Sigma^\flat(\vec{\psi})] \leq K^{\mathcal{I}}(\bigcup\{I_\Sigma^\flat[\phi_i, \psi_i] : i < k\}).$$

Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if, for all $i < k$, $I_\Sigma^\flat[\phi_i, \psi_i] \leq T$, then $I_\Sigma^\flat[\sigma_\Sigma^\flat(\vec{\phi}), \sigma_\Sigma^\flat(\vec{\psi})] \leq T$. In other words $\langle \phi_i, \psi_i \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$, for all $i < k$, imply $\langle \sigma_\Sigma^\flat(\vec{\phi}), \sigma_\Sigma^\flat(\vec{\psi}) \rangle \in \overleftarrow{I^\flat}_\Sigma(T)$. Therefore, $\overleftarrow{I^\flat}(T)$ satisfies the congruence property.

Finally, to see that $\overleftarrow{I^\flat}(T)$ is compatible with T, suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational and $\tau^\flat \in N^\flat$, we have, by Lemma 910,

$$\tau_\Sigma^\flat[\psi] \leq D(\tau_\Sigma^\flat[\phi], \phi \approx \psi).$$

By interpretability, this yields

$$I^\flat[\tau_\Sigma^\flat[\psi]] \leq K^{\mathcal{I}}(I^\flat[\tau_\Sigma^\flat[\phi]], I_\Sigma^\flat[\phi, \psi]).$$

Since $(\tau^\flat, I^\flat)$ is a conjugate pair, the latter is equivalent to

$$\psi \in K_\Sigma^{\mathcal{I}}(\phi, I_\Sigma^\flat[\phi, \psi]).$$

In other words, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \text{and} \quad \langle \phi, \psi \rangle \in \overleftarrow{I^\flat}_\Sigma(T) \quad \text{imply} \quad \psi \in T_\Sigma.$$

Hence $\overleftarrow{I^\flat}(T)$ is compatible with T. ∎

Using Proposition 930, we can show that stability and the existence of an equivalence between the systemic skeleton and an equational π-structure ensure syntactic protoalgebraicity.

Theorem 931 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *is equivalent to an equational* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$.

Proof: Suppose that $\mathcal{I}$ is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ is equivalent to an equational π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{aligned} \Omega(T) &= \Omega(\overleftarrow{T}) &&\text{(by stability)} \\ &= \overleftarrow{I^\flat}(\overleftarrow{T}) &&\text{(by Proposition 930)} \\ &= \overleftarrow{I^\flat}(T). &&\text{(by Proposition 99)} \end{aligned}$$

Therefore, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$.   ∎

Finally, before the main theorem, we show that stability and the existence of a transformational equivalence between the systemic skeleton and an equational π-structure ensure left truth equationality.

Theorem 932 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is stable and its systemic skeleton $\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I} \rangle$ is equivalent to an equational π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat): \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$.*

Proof: We have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$
\begin{aligned}
\phi \in \overleftarrow{T}_\Sigma \quad &\text{iff} \quad I^\flat[\tau_\Sigma^\flat[\phi]] \le \overleftarrow{T} \quad ((\tau^\flat, I^\flat) \text{ an equivalence}) \\
&\text{iff} \quad I^\flat[\tau_\Sigma^\flat[\phi]] \le T \quad \text{(by Proposition 99)} \\
&\text{iff} \quad \tau_\Sigma^\flat[\phi] \le \Omega(T). \quad \text{(by Theorem 931)}
\end{aligned}
$$

Therefore, $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$.   ∎

Putting together Theorems 931, 932 and 929, we get the following fundamental result to the effect that syntactic weak algebraizability boils down to stability, together with the equivalence of the systemic skeleton of a π-institution with its associated equational π-structure via a conjugate pair of transformations.

Theorem 933 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically weakly algebraizable if and only if it is stable and its systemic skeleton $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, I^\flat): \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ of transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. Then, by Theorem 931, it is syntactically protoalgebraic and, by Theorem 932, it is left truth equational. Therefore, by Theorem 925, it is syntactically weakly algebraizable.

If, conversely, $\mathcal{I}$ is syntactically weakly algebraizable, then, on the one hand, it is protoalgebraic and, therefore, stable, and, on the other, by Theorem 929, it is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. ∎

Generalizing again, we show that stability together with the existence of an equivalence of the systemic skeleton with an equational π-structure, induced by conjugate transformations, is sufficient to yield syntactic weak algebraizability.

Theorem 934 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π-*structure via a conjugate pair of transformations.*
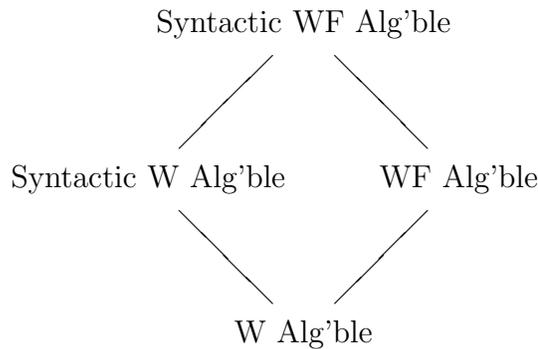
Proof: If $\mathcal{I}$ is syntactically weakly algebraizable, then the conclusion follows from Theorem 933. Conversely, if $\mathcal{K}^\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically protoalgebraic by Theorem 931 and left truth equational by Theorem 932, whence it is syntactically weakly algebraizable.   ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically weakly algebraizable π-institutions.

Theorem 935 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$, *where* $\mathcal{Q}$ *is an equational* π-*structure.*

Proof: The "only if" follows by Theorem 934 and Theorem 896. The "if" is given by Theorem 904 and Theorem 934.   ∎

Let us give, in closing the section, the picture of the **weak algebraizability hierarchy** that we have established, consisting of both semantic and syntactic classes of π-institutions.

$$
\begin{array}{ccc}
 & \text{Syntactic WF Alg'ble} & \\
 & \diagup \quad \diagdown & \\
\text{Syntactic W Alg'ble} & & \text{WF Alg'ble} \\
 & \diagdown \quad \diagup & \\
 & \text{W Alg'ble} &
\end{array}
$$

We give an example of a syntactically weakly algebraizable π-institution which is not syntactically weakly family algebraizable.

Example 936 EXAMPLE NOT FOUND YET!!

12.6   Syntactic WS PreAlgebraizability

Syntactic WS prealgebraizability, requires, like syntactic WF algebraizability, the monotonicity of the Leibniz operator on theory systems and the injectivity of the Leibniz operator on theory systems but, unlike WF algebraizability, it requires these two properties only on theory systems and not on the entire complete lattice of theory families. As a consequence of this weakened requirement, syntactic WS prealgebraizability implies neither systemicity (as does syntactic WF algebraizability) nor the even weaker condition of stability (as do both kinds of syntactic algebraizability). Thus, as other conditions that were under our scrutiny previously, it allows us to consider for membership π-institutions that are not necessarily stable.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

We say that $\mathcal{I}$ is $R^\mathcal{I}Z^\mathcal{I}$-(**syntactically**) **fortified** if:

- The reflexive core $R^\mathcal{I}$ is Leibniz;

- The system core $Z^\mathcal{I}$ is adequate.

$\mathcal{I}$ is **syntactically weakly system prealgebraizable** (abbreviated to **syntactically WS prealgebraizable**) if:

- $\mathcal{I}$ is $R^\mathcal{I}Z^\mathcal{I}$-fortified;

- $\mathcal{I}$ is prealgebraic;

- $\mathcal{I}$ is system injective.

By Theorem 248, under prealgebraicity, the properties of system injectivity, system reflectivity and system complete reflectivity coincide. As a result, we have the following alternative characterization of syntactic weak system prealgebraizability.

Theorem 937 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if it is syntactically prealgebraic and system truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically weakly system prealgebraizable. Then, on the one hand, it is prealgebraic and has a Leibniz reflexive core. Thus, by Theorem 788, it is syntactically prealgebraic. On the other, it is, by Theorem 248, system c-reflective and has an adequate system core. Therefore, by Theorem 889, it is system truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically prealgebraic and system truth equational. Then, by Theorem 788, it is prealgebraic and has a Leibniz reflexive core, and, by Theorem 889, it is system c-reflective and has an

adequate system core. Therefore, by definition, $\mathcal{I}$ is syntactically weakly system prealgebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic weak system prealgebraizability classes of π-institutions.

Theorem 938 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if* $\mathcal{I}$ *is weakly system prealgebraizable and* $R^\mathcal{I} Z^\mathcal{I}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if, by definition, it is $R^\mathcal{I} Z^\mathcal{I}$-fortified, prealgebraic and system injective, i.e., iff it is, by definition, $R^\mathcal{I} Z^\mathcal{I}$-fortified and weakly system prealgebraizable. ■

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak system prealgebraizability in terms of morphisms between complete lattices of theory systems.

Theorem 939 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if it is* $R^\mathcal{I} Z^\mathcal{I}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^\mathcal{A} : \mathrm{FiSys}^\mathcal{I}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding.

Proof: We have that $\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if, by Theorem 938, it is $R^\mathcal{I} Z^\mathcal{I}$-fortified and weakly system prealgebraizable, if and only if, by Theorem 256, it it $R^\mathcal{I} Z^\mathcal{I}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^\mathcal{A} : \mathrm{FiSys}^\mathcal{I}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding ■

Next, we show that syntactic weak system prealgebraizability may also be characterized by the existence of an equivalence between the systemic skeleton of a π-institution and an algebraic π-structure associated with the π-institution (different, in general, than $\mathcal{Q}^{\mathcal{I}*}$) via a pair of conjugate transformations.

We embark on the path by defining first the algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with a given π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall the definition of the class $\mathrm{AlgSys}^\bullet(\mathcal{I})$ of all $\mathbf{F}$-algebraic systems reduced with respect to $\mathcal{I}$-filter systems,

$$\mathrm{AlgSys}^\bullet(\mathcal{I}) = \{ \mathcal{A} : (\exists T \in \mathrm{FiSys}^\mathcal{I}(\mathcal{A}))(\Omega^\mathcal{A}(T) = \Delta^\mathcal{A}) \}.$$

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we define the class of $\mathcal{I}^\bullet$-**congruence systems on** $\mathcal{A}$ by

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})\}.$$

It turns out that congruence systems in $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ have a straightforward characterization.

Proposition 940 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π-*institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$-*algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : (\exists T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \theta)\}.$$

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system.

Suppose, first, that $\theta \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$. Thus, there exists $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that

$$\Omega^{\mathcal{A}/\theta}(T') = \Delta^{\mathcal{A}/\theta}.$$

By applying the inverse of the quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}/\theta$, we get

$$(\pi^\theta)^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = (\pi^\theta)^{-1}(\Delta^{\mathcal{A}/\theta}).$$

Since $\langle I, \pi^\theta \rangle$ is surjective, we get by Proposition 24 and Corollary 55, that $(\pi^\theta)^{-1}(T') \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and

$$\Omega^{\mathcal{A}}((\pi^\theta)^{-1}(T')) = \theta.$$

Therefore, there exists $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \theta$.

Suppose, conversely, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, with $\Omega^{\mathcal{A}}(T) = \theta$, for some $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Then, we have $\Omega^{\mathcal{A}/\theta}(T/\theta) = \Delta^{\mathcal{A}/\theta}$ and, therefore, by definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$, implying that $\theta \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. ∎

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ of systemic $\mathcal{I}$-congruence systems on $\mathcal{A}$ need not be closed under signature-wise intersections, i.e., may not form a closure family on $\mathbf{A}^2$. However, we can show that, if $\mathcal{I}$ is prealgebraic, this is always the case.

Proposition 941 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a prealgebraic* π-*institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$-*algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{A}^2$.

Proof: First, note that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ has a top element $\nabla^{\mathcal{A}}$. To see this, observe that $\mathcal{A}/\nabla^{\mathcal{A}}$ is a trivial algebraic system, which is always a member of $\mathrm{AlgSys}^{\bullet}(\mathcal{I})$.

It suffices now to show that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ is closed under arbitrary intersections. To this end, suppose $\theta^i \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$, for $i \in I$. By Proposition 940, for all $i \in I$, there exists $T^i \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}/\theta^i)$, such that $\Omega^{\mathcal{A}}(T^i) = \theta^i$. But, by Lemma 23 and prealgebraicity, we get that

$$\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) = \bigcap_{i\in I}\Omega^{\mathcal{A}}(T^i) = \bigcap_{i\in I}\theta^i.$$

Now, again by Proposition 940, we conclude that $\bigcap_{i\in I}\theta^i \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. ■

Applying Proposition 941 to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get the following

Corollary 942 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a prealgebraic π-institution based on* $\mathbf{F}$. *Then,* $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{F}^2$.

Proof: This is a special case of Proposition 941. ■

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a prealgebraic π-institution. We define, in accordance with Corollary 942, the **systemic equational π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with $\mathcal{I}$** to be the π-structure $\mathcal{Q}^{\mathcal{I}\bullet} = \langle \mathbf{F}^2, D^{\mathcal{I}\bullet}\rangle$, where $D^{\mathcal{I}\bullet}$ is the closure (operator) family corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$.

We recall, also, the definition of the systemic skeleton of $\mathcal{I}$, i.e., of the π-structure

$$\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle,$$

where $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is the closure operator on $\mathbf{F}$ corresponding to the closure family $\mathrm{ThSys}(\mathcal{I})$.

Now we have the components needed to resume work on the characterization of syntactic weak system prealgebraizability. Our first result connecting syntactic weak system prealgebraizability of a π-institution with the associated systemic equational π-structure shows that, if a π-institution is syntactically weakly system prealgebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its associated systemic equational π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations.

Theorem 943 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically weakly system prealgebraizable π-institution based on* $\mathbf{F}$. *Then* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^{\flat}, \overleftrightarrow{I^{\flat}}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations. More precisely:*

- $I^{\flat} : (\mathrm{SEN}^{\flat})^{\omega} \to \mathrm{SEN}^{\flat}$ *in* $N^{\flat}$, *with two distinguished arguments, is a set of witnessing transformations of the syntactic prealgebraicity of* $\mathcal{I}$;

- $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, *with a single distinguished argument, is a set of witnessing equations for the system truth equationality of $\mathcal{I}$.*

Proof: Suppose that $\mathcal{I}$ is syntactically weakly system prealgebraizable. Then, by Theorem 937, $\mathcal{I}$ is syntactically prealgebraic and system truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic prealgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing system truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$, and $\overleftrightarrow{I^\flat}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]$, are as required. Therefore, by Proposition 901, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in K^{\mathcal{I}}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le D^{\mathcal{I}\bullet}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}\bullet}(\phi \approx \psi) = D^{\mathcal{I}\bullet}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have, by system truth equationality,

$$\Phi \subseteq T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\Phi] \le \Omega(T)$$
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega(T).$$

Therefore, $\phi \in K^{\mathcal{I}}_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \le \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \le \Omega(T)$, if and only if, by Proposition 940, $\tau^\flat_\Sigma[\phi] \le D^{\mathcal{I}\bullet}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\phi \approx \psi \in \Omega_\Sigma(T) \quad \text{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T \quad \text{(Corollary 770)}$$
$$\text{iff} \quad \tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]] \le \Omega(T).$$
$$\text{(system truth equationality)}$$

Using again Proposition 940, we conclude that

$$D^{\mathcal{I}\bullet}(\phi \approx \psi) = D^{\mathcal{I}\bullet}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

Therefore $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via $(\tau^\flat, \overleftrightarrow{I^\flat}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$. ∎

We show, next that the existence of an equivalence between the systemic skeleton of a given π-institution and an equational π-structure ensures syntactic prealgebraicity.

Theorem 944 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$*. If the systemic skeleton* $\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I} \rangle$ *of* $\mathcal{I}$ *is equivalent to an equational* π-*structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat)$: $\mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $I^\flat$.

Proof: Suppose that $\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I} \rangle$ is equivalent to an equational π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations. Then, we have, by Proposition 930, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = \overleftarrow{I^\flat}(T)$. Therefore, $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$.   ∎

Finally, as a last step before the main theorem, we show that the existence of a transformational equivalence between the systemic skeleton of a given π-institution and an equational π-structure ensures system truth equationality.

Theorem 945 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$*. If the systemic skeleton* $\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I} \rangle$ *of* $\mathcal{I}$ *is equivalent to an equational* π-*structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat)$: $\mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is system truth equational, with witnessing equations* $\tau^\flat$.

Proof: We have, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned} \phi \in T_\Sigma \quad &\text{iff} \quad I^\flat[\tau_\Sigma^\flat[\phi]] \le T \quad ((\tau^\flat, I^\flat) \text{ an equivalence}) \\ &\text{iff} \quad \tau_\Sigma^\flat[\phi] \le \Omega(T). \quad (\text{by Theorem 944}) \end{aligned}$$

Therefore, $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$.   ∎

Putting together Theorems 944, 945 and 943, we get the following fundamental result to the effect that syntactic weak system prealgebraizability boils down to the existence of an equivalence of the systemic skeleton of a π-institution with its associated systemic equational π-structure via a conjugate pair of transformations.

Theorem 946 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if its systemic skeleton* $\mathcal{K}^\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. Then, by Theorem 944, it is syntactically prealgebraic and, by Theorem 945, it is system truth equational. Therefore, by Theorem 937, it is syntactically weakly system prealgebraizable. If, conversely, $\mathcal{I}$ is syntactically weakly system prealgebraizable, then, by Theorem 943, it is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations.   ∎

It turns out that the existence of an equivalence of the systemic skeleton with an equational π-structure, induced by conjugate transformations, is sufficient to yield syntactic weak system prealgebraizability.

Theorem 947 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if its systemic skeleton is equivalent to an equational π-structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically weakly system prealgebraizable, then the conclusion follows from Theorem 946. Conversely, if $\mathcal{K}^\mathcal{I}$ is equivalent to an equational π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically prealgebraic by Theorem 944 and system truth equational by Theorem 945, whence it is syntactically weakly system prealgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically weakly system prealgebraizable π-institutions.

Theorem 948 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$, *where $\mathcal{Q}$ is an equational π-structure.*

Proof: The "only if" follows by Theorem 947 and Theorem 896. The "if" is given by Theorem 904 and Theorem 947. ■

12.7   Syntactic WLC PreAlgebraizability

Between syntactic WS prealgebraizability and syntactic weak algebraizability we find the class of syntactic weakly left c-reflective prealgebraizability. This strengthens WS prealgebraizability by replacing system c-reflectivity by the stronger condition of left c-reflectivity. Alternatively, it weakens syntactic weak algebraizability by replacing ptotoalgebraicity by prealgebraicity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **syntactically weakly left c-reflectively prealgebraizable** (abbreviated to **syntactically WLC prealgebraizable**) if:

- $\mathcal{I}$ is $R^\mathcal{I} L^\mathcal{I}$-fortified;

- $\mathcal{I}$ is prealgebraic;

- $\mathcal{I}$ is left c-reflective.

We have the following alternative characterization of syntactic WLC prealgebraizability.

Theorem 949 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is syntactically WLC prealgebraizable if and only if it is syntactically prealgebraic and left truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically WLC prealgebraizable. Then, on the one hand, it is prealgebraic and has a Leibniz reflexive core. Thus, by Theorem 788, it is syntactically prealgebraic. On the other, it is left c-reflective and has a left adequate left Suszko core. Therefore, by Theorem 871, it is left truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically prealgebraic and left truth equational. Then, by Theorem 788, it is prealgebraic and has a Leibniz reflexive core, and, by Theorem 871, it is left c-reflective and has a left adequate left Suszko core. Therefore, by definition, $\mathcal{I}$ is syntactically WLC prealgebraizable. ∎

Directly from the definitions, we may derive the following relationship between the semantic and syntactic WLC prealgebraizability classes of π-institutions.

Theorem 950 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if* $\mathcal{I}$ *is WLC prealgebraizable and* $R^{\mathcal{I}} L^{\mathcal{I}}$-*fortified.*

Proof: $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if, by definition, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified, prealgebraic and left c-reflective, i.e., iff it is, by definition, $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and WLC prealgebraizable. ∎

For an alternative characterization of syntactic WLC prealgebraizability, we take advantage of the corresponding characterization of WLC prealgebraizability in terms of morphisms between complete lattices of theory systems.

Theorem 951 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π-*institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is* $R^{\mathcal{I}} L^{\mathcal{I}}$-*fortified and, for every* $\mathbf{F}$-*algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have that $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if, by Theorem 950, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and WLC prealgebraizable, if and only if, by Theorem 276, it it $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ∎

Recall that syntactic weak system prealgebraizability was characterized by the existence of an equivalence between the systemic skeleton $K^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ and the systemic equational π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with the π-institution, via a pair of conjugate transformations. To adapt this characterization to capture syntactic WLC prealgebraizability, we need to postulate alongside this equivalence the property of left truth equationality of the π-institution.

Theorem 952 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. Then, $\mathcal{I}$ is left truth equational and, by Theorem 944, it is syntactically prealgebraic. Therefore, by Theorem 949, it is syntactically WLC prealgebraizable. If, conversely, $\mathcal{I}$ is syntactically WLC prealgebraizable, then, by Theorem 949, it is left truth equational and it is weakly system prealgebraizable. Thus, by Theorem 943, it is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. ∎

Because of Theorem 947, left truth equationality and the existence of an equivalence of the systemic skeleton with an equational π-structure, induced by conjugate transformations, is sufficient to yield syntactic WLC prealgebraizability.

Theorem 953 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and its systemic skeleton is equivalent to an equational* π*-structure via a conjugate pair of transformations.*
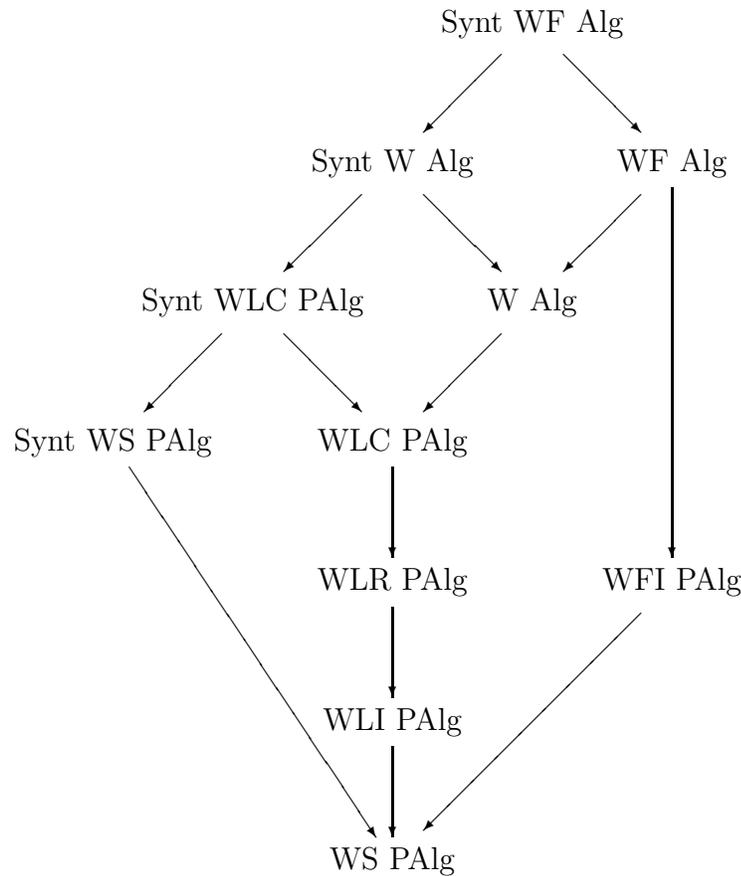
Proof: If $\mathcal{I}$ is syntactically WLC prealgebraizable, then the conclusion follows from Theorem 952. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an equational π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically prealgebraic by Theorem 944. Since, by hypothesis, it is also left truth equational, it is syntactically WLC prealgebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically WLC prealgebraizable π-institutions.

Theorem 954 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, *where* $\mathcal{Q}$ *is an equational* π*-structure.*

Proof: The "only if" follows by Theorem 953 and Theorem 896. The "if" is given by Theorem 904 and Theorem 953.   ∎

Let us give, in closing the section, the picture of the **weak prealgebraizability hierarchy** that we have established, consisting of both semantic and syntactic classes of π-institutions.

Synt WF Alg

Synt W Alg   WF Alg

Synt WLC PAlg   W Alg

Synt WS PAlg   WLC PAlg

WLR PAlg   WFI PAlg

WLI PAlg

WS PAlg

To separate syntactic WLC prealgebraizability from the class lying immediately above it in the hierarchy, we give an example of a syntactically weakly left c-reflective prealgebraizable π-institution which is not syntactically weakly algebraizable.

Example 955 EXAMPLE NOT FOUND YET!!

We also give an example separating syntactic WLC prealgebraizability from the class lying immediately below it in the hierarchy. That is, we present a syntactically weakly system prealgebraizable π-institution which is not syntactically weakly left c-reflective prealgebraizable.

Example 956 EXAMPLE NOT FOUND YET!!