Chapter 15

The Syntactic Leibniz Hierarchy: Basement II

15.1 Introduction

This chapter deals with the syntactic rough/narrow monotonicity hierarchy, which forms the syntactic analog of the rough/narrow monotonicity hierarchy, studied in Chapter 7. We look at the classes of syntactically roughly/narrowly family monotone, syntactically roughly/narrowly system monotone and syntactically narrowly right monotone π-institutions.

In Section 15.2, we study *syntactic narrow family monotonicity*, the syntactic analog of rough/narrow family monotonicity. To introduce this class, one needs, first, to either be able to apply the constituent properties of natural transformations modulo rough representatives of theory families or relativize them to the collection $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ of theory families without empty components. Let us take, for instance, reflexivity. Consider a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ is *roughly family reflexive* if, for every theory family T, all signatures Σ and all Σ-sentences ϕ,

$$I^\flat_\Sigma[\phi,\phi] \leq \widetilde{T}.$$

We say that $I^\flat$ is *narrowly family reflexive* if, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$I^\flat_\Sigma[\phi,\phi] \leq T.$$

It is shown that these two properties are identical. Similar results are proven for rough/narrow family symmetry, rough/narrow family transitivity, rough/narrow family compatibility and rough/narrow family modus ponens, which are all defined analogously. We say that a π-institution $\mathcal{I}$ is *syntactically roughly/narrowly family monotone* if there exists a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies narrow family reflexivity, narrow family transitivity, narrow family compatibility and narrow family modus ponens. If $\mathcal{I}$ is syntactically narrow family monotone, with witnessing transformations $I^\flat$, then, for every $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\overleftarrow{\overleftrightarrow{I}}{}^\flat(T)$ is a congruence system on $\mathbf{F}$ compatible with T. This allows showing that $\overleftarrow{\overleftrightarrow{I}}{}^\flat(T) = \Omega(T)$, that is, $I^\flat$ narrowly defines Leibniz congruence systems of theory families in $\mathcal{I}$. It follows that syntactic narrow family monotonicity implies (semantic) narrow family monotonicity.

To provide an intrinsic characterization of syntactic narrow family monotonicity, we introduce the *rough/narrow reflexive core* of $\mathcal{I}$. It may be defined in two apparently different ways, one from the rough and one from the narrow point of view, which, however, turn out to be equivalent. The *rough reflexive core* $\widetilde{R}^{\mathcal{I}}$ consists of all those natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ, $\rho^\flat_\Sigma[\phi,\phi] \leq \widetilde{T}$. The *narrow reflexive core* $R^{\mathcal{I}\lightning}$ is defined similarly, but quantifying T over $\mathrm{ThFam}^{\nshortmid}(\mathcal{I})$ and replacing the inequality by $\rho^{\flat}_{\Sigma}[\phi,\phi] \leq T$. We have that $\widetilde{R}^{\mathcal{I}} = R^{\mathcal{I}\nshortmid}$. We show that, for all $T \in \mathrm{ThFam}^{\nshortmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\nshortmid}}(T)$ is a reflexive, symmetric relation system on $\mathbf{F}$ that has the compatibility property. Thus, $\overleftarrow{R^{\mathcal{I}\nshortmid}}(T)$ only falls short of becoming a congruence system compatible with T because of $R^{\mathcal{I}\nshortmid}$ possibly lacking the narrow family transitivity and the narrow family modus ponens properties. If $\mathcal{I}$ happens to be syntactically narrow family monotone, then $R^{\mathcal{I}\nshortmid}$ has the narrow family modus ponens and, as a consequence, it is also narrow family transitive. Conversely, if $R^{\mathcal{I}\nshortmid}$ has the narrow family modus ponens, then the π-institution $\mathcal{I}$ is syntactically family narrow monotone, with witnessing transformations $R^{\mathcal{I}\nshortmid}$. That is, the narrow family modus ponens property of $R^{\mathcal{I}\nshortmid}$ intrinsically characterizes syntactic narrow family monotonicity. Additionally, $\mathcal{I}$ is syntactically narrowly family monotone if and only if $R^{\mathcal{I}\nshortmid}$ narrowly defines Leibniz congruence systems of theory families, in the sense that, for all $T \in \mathrm{ThFam}^{\nshortmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\nshortmid}}(T) = \Omega(T)$.

The second part of Section 15.2 deals with devising a characterization of syntactic narrow family monotone π-institutions as a subclass of narrow family monotone π-institutions. Recalling the case of syntactic protoalgebraicity versus protoalgebraicity, where the separation was attained by the Leibniz property of the reflexive core $R^{\mathcal{I}}$ of the π-institution $\mathcal{I}$, we must devise here an analogous property for the narrow reflexive core $R^{\mathcal{I}\nshortmid}$. Due to the fact that we are dealing with theory families with nonempty components, this analog involves additional complications. First, we define, for every signature Σ and all Σ-sentences ϕ,ψ, the collection $[R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$ of all theory families $T \in \mathrm{ThFam}^{\nshortmid}(\mathcal{I})$, which contain $R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi]$. We then set $\min[R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$ for the family of all minimal elements in $[R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$. Note that, if $C(R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi]) \in \mathrm{ThFam}^{\nshortmid}(\mathcal{I})$, then $\min[R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi]) = \{C(R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])\}$. Thus, this construct is meant to handle the difficulty encountered in the special case in which $C(R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$ happens to have some empty components. We say that the narrow reflexive core $R^{\mathcal{I}\nshortmid}$ is *Leibniz* if, for all signatures Σ, all Σ-sentences ϕ,ψ and all $T \in \min[R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_{\Sigma}(T)$. It turns out that if $R^{\mathcal{I}\nshortmid}$ has the narrow family modus ponens, then it is Leibniz. Ideally, to have the prototypical example of syntactic protoalgebraicity versus protoalgebraicity applicable without changes, we would like to have that, under narrow family monotonicity, $R^{\mathcal{I}\nshortmid}$ being Leibniz implies that it has the narrow family modus ponens. This presents some obstacles and we are only able to advance under an additional hypothesis on the π-institution $\mathcal{I}$, thus obtaining only a partial result. Given a signature Σ and Σ-sentences ϕ,ψ, $\mathcal{I}$ is called $\langle\Sigma,\phi,\psi\rangle$-*reflexively covered* if, for all $T \in [R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$, there exists $T' \in \min[R^{\mathcal{I}\nshortmid}_{\Sigma}[\phi,\psi])$, such that $T' \leq T$. Moreover, we say that $\mathcal{I}$ is *reflexively covered* if, for all signatures Σ and all Σ-sentences ϕ,ψ, $\mathcal{I}$ is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered. We do show that, if $\mathcal{I}$ is reflexively covered, and narrowly family monotone, then $R^{\mathcal{I}\not\flat}$ being Leibniz implies that $R^{\mathcal{I}\not\flat}$ has the narrow family modus ponens. This enables us to show that, for reflexively covered π-institutions $\mathcal{I}$, $\mathcal{I}$ is syntactically narrowly family monotone if and only if it is narrowly family monotone and has a Leibniz narrow reflexive core.

In Section 15.3, we study *syntactic narrow system monotonicity*, a syntactic analog of narrow system monotonicity. We start by relativizing system reflexivity, system symmetry, system transitivity, system compatibility and system modus ponens of a collection $I^\flat$ of natural transformations to the collection $\mathrm{ThSys}^{\not\flat}(\mathcal{I})$ of theory systems without empty components, thus obtaining the corresponding narrow versions of these properties. A π-institution $\mathcal{I}$ is called *syntactically narrowly system monotone* if there exists a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, which is narrowly system reflexive, narrowly system transitive, has the narrow system compatibility and satisfies the narrow system modus ponens. If $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, $\overleftarrow{\overleftrightarrow{I^\flat}}(T) = \Omega(T)$, that is, $I^\flat$ narrowly defines Leibniz congruence systems of theory systems in $\mathcal{I}$. This yields that syntactic narrow system monotonicity implies narrow system monotonicity, its semantic counterpart.

The introduction of the *narrow system reflexive core* serves, among other things, to intrinsically characterize syntactic narrow system monotonicity. The *narrow system reflexive core* $R^{\mathcal{I}s}$ of $\mathcal{I}$ consists of all natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,

$$\rho^\flat_\Sigma[\phi, \phi] \leq T.$$

It turns out that, for all $T \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a reflexive and symmetric relation system on $\mathbf{F}$ and satisfies the compatibility property. Moreover, if $\mathcal{I}$ happens to be syntactically narrowly system monotone, then $R^{\mathcal{I}s}$ also satisfies the narrow system modus ponens and, as a result, it is narrowly system transitive as well. Conversely, if $R^{\mathcal{I}s}$ has the narrow system modus ponens, then $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $R^{\mathcal{I}s}$. Thus, $R^{\mathcal{I}s}$ having the narrow system modus ponens intrinsically characterizes syntactic narrow system monotonicity.

In the second part of Section 15.3, we use a Leibniz like property of the narrow system reflexive core in order to provide a partial characterization of syntactically narrowly system monotone π-institutions inside the class of narrowly system monotone π-institutions. Since the steps are similar to those employed in Section 15.2, which were described in some detail above, we only give a rough outline. We again define, for every signature Σ and all Σ-sentences ϕ, ψ, the collection

$$[R^{\mathcal{I}s}_\Sigma[\phi, \psi]) = \{T \in \mathrm{ThSys}^{\not\flat}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T\}.$$

We then set $\min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$ to be the subcollection of its minimal elements. We say that the narrow system reflexive core $R^{\mathcal{I}s}$ is *Leibniz* if, for all signatures Σ, all Σ-sentences ϕ,ψ and all $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. We show that, if $R^{\mathcal{I}s}$ has the narrow system modus ponens, then it is Leibniz. Ideally, we would have liked to be able to show that, provided that $\mathcal{I}$ is narrowly system monotone, the converse implication also holds. However, we are able to show this only for a certain subclass of π-institutions. For a fixed signature Σ and fixed Σ-sentences ϕ,ψ, we say that $\mathcal{I}$ is *$\langle\Sigma,\phi,\psi\rangle$-system reflexively covered* if, for all $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, there exists $T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, such that $T' \le T$. $\mathcal{I}$ is called *system reflexively covered* if it is $\langle\Sigma,\phi,\psi\rangle$-system reflexively covered, for all signatures Σ and all Σ-sentences ϕ,ψ. We do show that, for system reflexively covered π-institutions $\mathcal{I}$, if $\mathcal{I}$ is narrowly system monotone and $R^{\mathcal{I}s}$ is Leibniz, then $R^{\mathcal{I}s}$ has the narrow system modus ponens in $\mathcal{I}$. As a consequence, we get that, for system reflexively covered π-institutions $\mathcal{I}$, $\mathcal{I}$ is syntactically narrowly system monotone if and only if it is narrowly system monotone and has a Leibniz narrow system reflexive core.

In Section 15.4, we turn to a third type of syntactic narrow monotonicity. We say that a π-institution $\mathcal{I}$ is *syntactically narrowly right monotone* if there exists a collection $I^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, that satisfies narrow right reflexivity, narrow right transitivity, narrow right compatibility and narrow right modus ponens. The meaning of "right" in these terms is that, in any of them, the theory family T, universally quantified over $\mathrm{ThFam}^{\natural}(\mathcal{I})$, is replaced in the relevant condition by $\overleftarrow{T}$. E.g., $I^\flat$ is *narrowly right transitive* if, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,ψ,χ, $I^\flat_\Sigma[\phi,\psi] \le \overleftarrow{T}$ and $I^\flat_\Sigma[\psi,\chi] \le \overleftarrow{T}$ imply $I^\flat_\Sigma[\phi,\chi] \le \overleftarrow{T}$. *Narrow right reflexivity*, *narrow right symmetry* and *narrow right compatibility* are defined similarly. On the other hand, *narrow right modus ponens* is defined by stipulating that, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ,ψ, $\phi \in \overleftarrow{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le \overleftarrow{T}$ imply $\psi \in \overleftarrow{T}_\Sigma$. It turns out that all properties above, besides narrow right modus ponens, are equivalent to their narrow family counterparts. Hence, the only feature that differentiates syntactic narrow right monotonicity from syntactic narrow family monotonicity is the use of narrow right modus ponens instead of narrow family modus ponens.

If $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $I^\flat$, then $I^\flat$ narrowly defines Leibniz congruence systems of theory families up to arrow, in the sense that, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T) = \Omega(\overleftarrow{T})$. This yields that syntactic narrow right monotonicity implies (semantic) narrow right monotonicity, its semantic counterpart. Next, we reuse the narrow reflexive core $R^{\mathcal{I}\natural}$ of $\mathcal{I}$, introduced in Section 15.2, to intrinsically characterize syntactic narrow right monotonicity. We show that, if $\mathcal{I}$ is syntactically narrowly right monotone, then $R^{\mathcal{I}\nleq}$ has the narrow right modus ponens. Moreover, if $R^{\mathcal{I}\nleq}$ has the narrow right modus ponens, then it is narrowly right transitive. This allows showing that, if $R^{\mathcal{I}\nleq}$ has the narrow right modus ponens, then $\mathcal{I}$ is syntactically narrowly right monotone. Thus, $R^{\mathcal{I}\nleq}$ having the narrow right modus ponens characterizes syntactic narrow right monotonicity. The section ends with a characterization, also using the narrow reflexive core, of those narrowly right monotone π-institutions that are syntactically narrowly right monotone. Here the deciding condition is that the narrow reflexive core be *right Leibniz*. We omit the details of this characterization, since they parallel those described previously for both syntactic narrow family and syntactic narrow system monotonicity.

15.2 Syntactic Narrow Family Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 7 that $\mathcal{I}$ is *roughly/narrowly family monotone* if, for all $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

In this section we introduce and study a syntactic analog of this concept.

First, we relativize family reflexivity, family symmetry, family transitivity, family compatibility and family modus ponens to $\mathrm{ThFam}^{\nleq}(\mathcal{I})$.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family reflexive** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family reflexive** if, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \leq T.$$

As the following lemma establishes rough and narrow family reflexivity are identical properties.

Lemma 1213 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat \subseteq N^\flat$ *a family of natural transformations in* $N^\flat$*, with two distinguished arguments.* $I^\flat$ *is roughly family reflexive if and only if it is narrowly family reflexive.*

Proof: Suppose, first, that $I^\flat$ is roughly family reflexive and consider $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, we have $\widetilde{T} = T$, whence, by rough family reflexivity, $I^\flat_\Sigma[\phi,\phi] \leq \widetilde{T} = T$. Thus, $I^\flat$ is narrowly family reflexive.

Suppose, conversely, that $I^\flat$ is narrowly family reflexive and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\widetilde{T} \in \mathrm{ThFam}^\nsim(\mathcal{I})$, we get, by narrow family reflexivity, $I^\flat_\Sigma[\phi,\phi] \leq \widetilde{T}$. Thus, $I^\flat$ is roughly family reflexive. ■

Let, again, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family symmetric** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family symmetric** if, for all $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \leq T.$$

Similarly to rough and narrow family reflexivity, rough and narrow family symmetry coincide.

Lemma 1214 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family symmetric if and only if it is narrowly family symmetric.*

Proof: Suppose, first, that $I^\flat$ is roughly family symmetric and consider $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq T$. Since $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, we have $\widetilde{T} = T$, whence, by hypothesis, $I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T}$. Applying rough family symmetry, we get $I^\flat_\Sigma[\psi,\phi] \leq \widetilde{T} = T$. Thus, $I^\flat$ is narrowly family symmetric.

Suppose, conversely, that $I^\flat$ is narrowly family symmetric and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T}$. Since $\widetilde{T} \in \mathrm{ThFam}^\nsim(\mathcal{I})$, we get, by narrow family symmetry, $I^\flat_\Sigma[\psi,\phi] \leq \widetilde{T}$. Thus, $I^\flat$ is roughly family symmetric. ■

Let, once more, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family transitive** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \leq \widetilde{T};$$

- $I^\flat$ is **narrowly family transitive** if, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \leq T.$$

Rough and narrow family transitivity also coincide.

Lemma 1215 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family transitive if and only if it is narrowly family transitive.*

Proof: Similar to the proof of Lemma 1214. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family compatible** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i,\psi_i] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})] \leq \widetilde{T};$$

- $I^\flat$ is **narrowly family compatible** if, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i,\psi_i] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})] \leq T.$$

Rough and narrow family transitivity also coincide.

Lemma 1216 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family compatible if and only if it is narrowly family compatible.*

Proof: Similar to the proof of Lemma 1214. ■

Finally, we define the property of possessing the rough and the narrow family modus ponens. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ has the **rough family MP** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in \widetilde{T}_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi,\psi] \le \widetilde{T} \quad \text{imply} \quad \psi \in \widetilde{T}_\Sigma;$$
- $I^\flat$ has the **narrow family MP** if, for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi,\psi] \le T \quad \text{imply} \quad \psi \in T_\Sigma.$$

As with all preceding properties, the rough and narrow family MP turn out to be identical properties.

Lemma 1217 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ has the rough family MP if and only if it has the narrow family MP.*

Proof: The proof again follows along the lines of the proof of Lemma 1214, but we describe it also in detail.

Suppose, first, that $I^\flat$ has the rough family MP and let $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le T$. Again, by hypothesis, $\widetilde{T} = T$, whence, we get $\phi \in \widetilde{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le \widetilde{T}$. Thus, by rough family MP, we get that $\psi \in \widetilde{T}_\Sigma$, i.e., $\psi \in T_\Sigma$. Thus, $I^\flat$ has the narrow family MP.

Assume, conversely, that $I^\flat$ has the narrow family MP and consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le \widetilde{T}$. Since $\widetilde{T} \in \mathrm{ThFam}^\sharp(\mathcal{I})$, we may apply narrow family MP to conclude that $\psi \in \widetilde{T}_\Sigma$. This proves that $I^\flat$ has the rough family MP. ∎

We say that $\mathcal{I}$ is **syntactically roughly/narrowly family monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow family reflexivity;
- narrow family transitivity;
- narrow family compatibility; and
- narrow family MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic rough/narrow family monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly family monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$. This forms a "narrow" analog of Proposition 790.

Proposition 1218 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,* $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: The proof follows along the lines of the proof of Proposition 790. So we give an outline. Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow family reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$. The fact that $\overleftrightarrow{I}^\flat$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$. The narrow family transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow family compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi}) \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$. Thus, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is compatible with T. Assume that $\phi \in T_\Sigma$ and $\langle \phi, \psi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat_\Sigma(T)$. Since $I^\flat \subseteq \overleftrightarrow{I}^\flat$, we get, by the narrow family MP of $I^\flat$, that $\psi \in T_\Sigma$. Thus, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is also compatible with T. ∎

Proposition 1218 shows that $\overleftrightarrow{I}^\flat$ defines Leibniz congruence systems of theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$. Following similar terminology adopted in Chapter 14, we say that $I^\flat$ **roughly** or **narrowly defines Leibniz congruence systems** of theory families in $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\overleftarrow{\overleftrightarrow{I}}^\flat(T) = \Omega(T).$$

Then, in what is an analog of Corollary 791, we obtain

Corollary 1219 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory families in* $\mathcal{I}$*.*

Proof: By Proposition 1218 and Corollary 98. ∎

Corollary 1219 allows establishing the fact that syntactic narrow family monotonicity implies (semantic) narrow family monotonicity. This forms an analog of Theorem 792.

Theorem 1220 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, then it is narrowly family monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly family monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll} \Omega(T) & = & \overleftarrow{\overleftrightarrow{I^\flat}}(T) & \text{(by Corollary 1219)} \\ & \leq & \overleftarrow{\overleftrightarrow{I^\flat}}(T') & \text{(by Lemma 94)} \\ & = & \Omega(T'). & \text{(by Corollary 1219)} \end{array}$$

Thus, $\mathcal{I}$ is narrowly family monotone. ■

We now introduce the notion of the rough/narrow reflexive core of a π-institution $\mathcal{I}$ in a way analogous to the reflexive core, which was introduced in Chapter 11. Its introduction will enable us to provide a characterization of the syntactical narrow family monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- The **rough reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} \widetilde{R}^{\mathcal{I}} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \leq \widetilde{T})\}; \end{array}$$

- The **narrow reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} R^{\mathcal{I}\natural} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^\natural(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \leq T)\}. \end{array}$$

These two notions are identical, as shown in the following proposition, and this justifies the usage of the terms rough and narrow reflexive core interchangeably in this context.

Proposition 1221 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\widetilde{R}^{\mathcal{I}} = R^{\mathcal{I}\natural}$*.*

Proof: On the one hand, if $\rho^\flat \in \widetilde{R}^{\mathcal{I}}$, $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, by the definition of the rough reflexive core, $\rho^\flat_\Sigma[\phi, \phi] \leq \widetilde{T} = T$, where the equality follows from the assumption that $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. This shows that $\rho^\flat \in R^{\mathcal{I}\natural}$. On the other hand, if $\rho^\flat \in R^{\mathcal{I}\natural}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, since $\widetilde{T} \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get by the definition of $R^{\mathcal{I}\natural}$, $\rho^\flat_\Sigma[\phi, \phi] \leq \widetilde{T}$. This shows that $\rho^\flat \in \widetilde{R}^{\mathcal{I}}$. ■

Given any theory family in $\mathrm{ThFam}^\natural(\mathcal{I})$, the relation system $\overleftarrow{R^{\mathcal{I}\natural}}(T)$ is a reflexive relation system on $\mathbf{F}$. This forms an analog of Lemma 773.

Lemma 1222 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ *is a reflexive relation system on* $\mathbf{F}$.

Proof: Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. By Lemma 93, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ is a relation system on $\mathbf{F}$. For reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the narrow reflexive core, $R_\Sigma^{\mathcal{I}\nmid}[\phi, \phi] \le T$. Thus, $\langle \phi, \phi \rangle \in \overleftarrow{R^{\mathcal{I}\nmid}}_\Sigma(T)$ and, therefore, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ is reflexive. ∎

As in Lemma 775, it may also be established that $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ is a symmetric relation system on $\mathbf{F}$, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$.

Lemma 1223 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ *is a symmetric relation system on* $\mathbf{F}$.

Proof: Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. Again, Lemma 93 shows that $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ is a relation system. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \overleftarrow{R^{\mathcal{I}\nmid}}_\Sigma(T)$. Equivalently, $R_\Sigma^{\mathcal{I}\nmid}[\phi, \psi] \le T$. Consider any $\rho^\flat \in R^{\mathcal{I}\nmid}$. By the definition of $R^{\mathcal{I}\nmid}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}\nmid}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi, \psi] \le T$. But this gives $\rho_\Sigma^\flat[\psi, \phi] \le T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}\nmid}$, we conclude that $R_\Sigma^{\mathcal{I}\nmid}[\psi, \phi] \le T$. Hence, $\langle \psi, \phi \rangle \in \overleftarrow{R^{\mathcal{I}\nmid}}_\Sigma(T)$. Therefore, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ is a symmetric relation system on $\mathbf{F}$. ∎

Continuing the study of sequence of properties of $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$, we show that, for all theory families $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ has the compatibility property in $\mathbf{F}$.

Lemma 1224 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ *has the compatibility property in* $\mathbf{F}$.

Proof: Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. We rely on Corollary 12. Let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \overleftarrow{R^{\mathcal{I}\nmid}}_\Sigma(T)$ or, equivalently, $R_\Sigma^{\mathcal{I}\nmid}[\phi, \psi] \le T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}\nmid}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta, \eta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta, \eta, \vec{\chi}, \vec{\xi}) = \rho_\Sigma^\flat(\sigma_\Sigma^\flat(\zeta, \vec{\chi}), \sigma_\Sigma^\flat(\eta, \vec{\chi}), \vec{\xi}).$$

Note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{aligned}\rho'^\flat \;=\; & \rho^\flat \circ \langle \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2}, \ldots, p^{n+k,k} \rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2}, \ldots, p^{n+k,k} \rangle, \\ & p^{n+k,k+1}, \ldots, p^{n+k,n+k-1} \rangle,\end{aligned}$$

we get that $\rho'^\flat \in N^\flat$. Moreover, for all $T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\rho'^\flat_\Sigma(\zeta,\zeta,\vec{\chi},\vec{\xi}) & = & \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\vec{\xi}) \quad \text{(by definition of } \rho'^\flat\text{)} \\ & \in & T'_\Sigma. \quad \text{(since } \rho^\flat \in R^{\mathcal{I}\nmid}\text{)}.\end{array}$$

Thus, by the definition of the narrow reflexive core, we get that $\rho'^\flat \in R^{\mathcal{I}\nmid}$.

Now since $\rho'^\flat \in R^{\mathcal{I}\nmid}$ and, by hypothesis, $R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi] \leq T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi,\vec{\chi})),\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi,\vec{\chi})),\vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \leq T.$$

Since this holds for all $\rho^\flat \in R^{\mathcal{I}\nmid}$, we get that $R^{\mathcal{I}\nmid}_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \leq T$ or, equivalently, $\langle\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})\rangle \in \overleftarrow{R^{\mathcal{I}\nmid}}_\Sigma(T)$. Therefore, $\overleftarrow{R^{\mathcal{I}\nmid}}(T)$ has the congruence compatibility property in $\mathbf{F}$. ■

We now show, in an analog of Theorem 799, that possession of the narrow family modus ponens by the narrow reflexive core intrinsically characterizes syntactic narrow family monotonicity. We start by showing that possession of the narrow family MP by the narrow reflexive core is necessary for syntactic narrow family monotonicity. This forms an analog of Theorem 796.

Theorem 1225 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, then* $R^{\mathcal{I}\nmid}$ *has the narrow family MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly family monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly family reflexive, we get, by definition of $R^{\mathcal{I}\nmid}$, $I^\flat \subseteq R^{\mathcal{I}\nmid}$. Thus, since $I^\flat$ has narrow family MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}\nmid}$ also satisfies the narrow family MP. ■

Possession of narrow family MP by $R^{\mathcal{I}\nmid}$ implies that $R^{\mathcal{I}\nmid}$ has the narrow family transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 797.

Proposition 1226 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nmid}$ *has the narrow family MP, then it also has the narrow family transitivity in* $\mathcal{I}$*.*

Proof: Suppose that $R^{\mathcal{I}\nleq}$ has the narrow family MP and let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle, \langle\psi,\chi\rangle \in \overleftarrow{R^{\mathcal{I}\nleq}}_\Sigma(T)$. This means that $R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi] \leq T$ and $R^{\mathcal{I}\nleq}_\Sigma[\psi,\chi] \leq T$. Then, by Lemma 1224, we get that, for all $\rho^\flat \in R^{\mathcal{I}\nleq}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{array}{l} R^{\mathcal{I}\nleq}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \qquad\qquad\qquad \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T. \end{array}$$

But, by hypothesis, $R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi] \leq T$ and $R^{\mathcal{I}\nleq}$ has the narrow family MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}\nleq}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}\nleq}_\Sigma[\phi,\chi] \leq T$. This shows $\langle\phi,\chi\rangle \in \overleftarrow{R^{\mathcal{I}\nleq}}_\Sigma(T)$ and, hence, $R^{\mathcal{I}\nleq}$ is narrowly family transitive in $\mathcal{I}$. ∎

We are now ready to show a converse of Theorem 1225, i.e., that possession of the narrow family MP by $R^{\mathcal{I}\nleq}$ suffices to establish the syntactic narrow family monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}\nleq}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 798.

Theorem 1227 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nleq}$ *has the narrow family MP, then* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $R^{\mathcal{I}\nleq}$*.*

Proof: By Lemma 1222, $R^{\mathcal{I}\nleq}$ is narrowly family reflexive in $\mathcal{I}$. By Lemma 1223, $R^{\mathcal{I}\nleq}$ is narrowly family symmetric in $\mathcal{I}$. By hypothesis and Proposition 1226, it is narrowly family transitive in $\mathcal{I}$. By Lemma 1224, it has the narrow family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow family MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $R^{\mathcal{I}\nleq}$. ∎

Theorems 1225 and 1227 provide the promised characterization of syntactic narrow family monotonicity in terms of the narrow family MP of the narrow reflexive core.

$$\begin{array}{c} \mathcal{I} \text{ is Syntactically Narrow} \\ \text{Family Monotone} \end{array} \longleftrightarrow \begin{array}{c} R^{\mathcal{I}\nleq} \text{ has Narrow Family} \\ \text{Modus Ponens} \end{array}.$$

Theorem 1228 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly family monotone if and only if* $R^{\mathcal{I}\nleq}$ *has the narrow family MP in* $\mathcal{I}$*.*

Proof: Theorem 1225 gives the "only if" and the "if" is by Theorem 1227. ■

A related alternative characterization asserts that syntactic narrow family monotonicity amounts to the narrow definability of Leibniz congruence systems of theory families by the narrow reflexive core. This result forms an analog of Theorem 801.

$\mathcal{I}$ is Syntactically Narrow Family Monotone	$\longleftrightarrow$	$R^{\mathcal{I}\natural}$ Narrowly Defines Leibniz Congruence Systems of Theory Families.

Theorem 1229 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly family monotone if and only if, for all* $T \in \mathrm{ThFam}^\natural(\mathcal{I})$,

$$\Omega(T) = \overleftarrow{R^{\mathcal{I}\natural}}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly family monotone, then, by Theorem 1225, $R^{\mathcal{I}\natural}$ has the narrow family MP in $\mathcal{I}$. Thus, by Theorem 1227, $R^{\mathcal{I}\natural}$ is a family of witnessing transformations for the syntactic narrow family monotonicity of $\mathcal{I}$. Thus, by Corollary 1219, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Omega(T) = \overleftarrow{R^{\mathcal{I}\natural}}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}\natural}$ is narrowly family reflexive, narrowly family transitive and has the narrow family compatibility property and the narrow family MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly family monotone. ■

In the case of syntactic protoalgebraicity, in Chapter 11, it was shown that the property that separates syntactic protoalgebraicity from protoalgebraicity is the Leibniz compatibility property with respect to the theory family generated by the reflexive core, i.e., the property that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

The task of characterizing those π-institutions that are syntactically narrowly family monotone among those that are narrowly family monotone is more involved. The additional complications arise from the fact that the class of theory families $\mathrm{ThFam}^\natural(\mathcal{I})$ may not be, in general, closed under (signature-wise) intersections and, hence, may not possess a least element. Therefore, to pinpoint syntactic narrow family monotonicity inside the class of narrow family monotone π-institutions, we need to devise a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive core.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we interject a small discussion. Recall that a π-institution $\mathcal{I}$ is *protoalgebraic* if its Leibniz operator is monotone on theory families. Recall, also, that its reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

If a π-institution is protoalgebraic and has a Leibniz reflexive core, then it satisfies the global family modus ponens. This was shown in Chapter 11 using the following method. Considering $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, we get:

- $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$ first, by applying the Leibniz property;
- $\Omega(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])) \leq \Omega(T)$, by applying the hypothesis that $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$ and the postulated protoalgebraicity of $\mathcal{I}$.

However, in case of narrow family monotonicity, the plausibility of $R^{\mathcal{I}\natural}_\Sigma[\phi, \psi]$ having some empty components makes it likely that, in the second stage, narrow family monotonicity may not be applicable to ensure the inclusion $\Omega(C(R^{\mathcal{I}\natural}_\Sigma[\phi, \psi])) \leq \Omega(T)$.

An obvious remedy is to restrict attention to those π-institutions in which $C(R^{\mathcal{I}\natural}_\Sigma[\phi, \psi]) \in \mathrm{ThFam}^\natural(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, and leave the Leibniz property unaltered. A more relaxed approach is to assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, the poset

$$[R^{\mathcal{I}\natural}_\Sigma[\phi, \psi]) := \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi, \psi] \leq T\}$$

satisfies the descending chain condition and to postulate that every minimal element $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi, \psi])$ satisfies $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, define

$$[R^{\mathcal{I}\natural}_\Sigma[\phi, \psi]) := \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi, \psi] \leq T\};$$

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{I}$ is called $\langle \Sigma, \phi, \psi \rangle$-**reflexively covered** if, for every theory family $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi, \psi])$, there exists minimal $T' \in [R^{\mathcal{I}\natural}_\Sigma[\phi, \psi])$, such that $T' \leq T$;
- $\mathcal{I}$ is called **reflexively covered** if it is $\langle \Sigma, \phi, \psi \rangle$-reflexively covered, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

Given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we write

$$\min\,[R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi])$$

for the collection of minimal elements in $[R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive core $R^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min\,[R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi])$,

$$\langle \phi, \psi\rangle \in \Omega_\Sigma(T).$$

We show, in an analog of Proposition 785, that, if $R^{\mathcal{I}\nmid}$ has the narrow family MP, then it is Leibniz. In fact, the proof demonstrates that, under the narrow family MP, a stronger property than that of being Leibniz holds; more precisely, we get that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi])$,

$$\langle \phi, \psi\rangle \in \Omega_\Sigma(T).$$

Proposition 1230 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nmid}$ *has the narrow family MP, then for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ *and all* $T \in [R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi])$*,* $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$.

Proof: Suppose $R^{\mathcal{I}\nmid}$ has the narrow family MP and let $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}\nmid}_\Sigma[\phi,\psi] \leq T$. To verify that $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$. Since $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, by Lemma 1224,

$$R^{\mathcal{I}\nmid}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}\nmid}$ has the narrow family MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$ and, therefore, $R^{\mathcal{I}\nmid}$ is Leibniz. ■

Corollary 1231 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nmid}$ *has the narrow family MP, then it is Leibniz.*

Proof: Directly by Proposition 1230. ■

In the opposite direction, when dealing with reflexively covered π-institutions, we may show that narrow family monotonicity combined with the Leibniz property of the narrow reflexive core imply that the narrow reflexive core has the narrow family modus ponens in $\mathcal{I}$.

Proposition 1232 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered, narrowly family monotone* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nmid}$ *is Leibniz, then it has the narrow family MP in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly family monotone and that $R^{\mathcal{I}\nmid}$ is Leibniz. Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi] \leq T$. Since $\mathcal{I}$ is reflexively covered, there exists $T' \in \min[R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi])$, such that $T' \leq T$. Now we have

$$\begin{array}{rcll} \langle \phi, \psi \rangle & \in & \Omega_\Sigma(T') & (\text{since } R^{\mathcal{I}\nmid} \text{ is Leibniz and } T' \in \min[R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi])) \\ & \subseteq & \Omega_\Sigma(T). & (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly family monotone}) \end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}\nmid}$ has the narrow family MP in $\mathcal{I}$. ■

Thus, at least for reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly monotone ones inside the class of the narrowly monotone ones can be characterized exactly by the Leibniz property of the narrow reflexive core. This forms a partial analog of Theorem 805 in the narrow context.

Theorem 1233 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly family monotone if and only if it is narrowly family monotone and has a Leibniz narrow reflexive core.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly family monotone. By Theorem 1220, it is narrowly family monotone. Moreover, by Theorem 1225, its narrow reflexive core has the narrow family MP. Hence, by Corollary 1231, its narrow reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly family monotone with a Leibniz narrow reflexive core. Then, by Proposition 1232, its narrow reflexive core has the narrow family MP and, therefore, by Theorem 1227, $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $R^{\mathcal{I}\nmid}$. ■

We leave here as an open problem discovering a characterization along the lines of Theorem 1233, without the restriction that the π-institution in question be reflexively covered.

15.3 Syntactic Narrow System Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 7 that $\mathcal{I}$ is *narrowly system monotone* if, for all $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$T \le T' \quad \text{implies} \quad \Omega(T) \le \Omega(T').$$

In this section, in analogy with Section 15.2, we introduce and study a syntactic analog of this concept.

First, the concepts of narrow system reflexivity, narrow system symmetry, narrow system transitivity, narrow system compatibility and narrow system modus ponens can all be relativized to $\mathrm{ThSys}^{\nmid}(\mathcal{I})$.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **narrowly system reflexive** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \phi] \le T;$$

- $I^\flat$ is **narrowly system symmetric** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\psi, \phi] \le T;$$

- $I^\flat$ is **narrowly system transitive** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \cup I^\flat_\Sigma[\psi, \chi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\phi, \chi] \le T;$$

- $I^\flat$ is **narrowly system compatible** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \le T \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \le T;$$

- $I^\flat$ has the **narrow system MP** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \le T \quad \text{imply} \quad \psi \in T_\Sigma.$$

We say that $\mathcal{I}$ is **syntactically narrowly system monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow system reflexivity;
- narrow system transitivity;
- narrow system compatibility; and
- narrow system MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic narrow system monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly system monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. This forms a system analog of Proposition 1218.

Proposition 1234 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a congruence system on $\mathbf{F}$ compatible with T.*

Proof: The proof is similar to that of Proposition 1218. Let $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow system reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$. The fact that $\overleftrightarrow{I}^\flat$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$. The narrow system transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow system compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi}) \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$. Thus, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is compatible with T. Assume that $\phi \in T_\Sigma$ and $\langle \phi, \psi \rangle \in \overleftarrow{\overleftrightarrow{I}}^\flat{}_\Sigma(T)$. Since $I^\flat \subseteq \overleftrightarrow{I}^\flat$, we get, by the narrow system MP of $I^\flat$, that $\psi \in T_\Sigma$. Thus, $\overleftarrow{\overleftrightarrow{I}}^\flat(T)$ is also compatible with T. ■

Proposition 1234 shows that $\overleftrightarrow{I}^\flat$ defines Leibniz congruence systems of theory systems in $\mathrm{ThSys}^{\lightning}(\mathcal{I})$. Again, following terminology adopted in Section

15.2, we say that $I^\flat$ **narrowly defines Leibniz congruence systems** of theory systems in $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}^\lightning(\mathcal{I})$,

$$\overleftarrow{\overleftrightarrow{I^\flat}}(T) = \Omega(T).$$

Then, in what is an analog of Corollary 1219, we obtain

Corollary 1235 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory systems in* $\mathcal{I}$*.*

Proof: By Proposition 1234 and Corollary 98. ∎

Corollary 1235 shows that syntactic narrow system monotonicity implies (semantic) narrow system monotonicity. This forms an analog of Theorem 1220.

Theorem 1236 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, then it is narrowly system monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThSys}^\lightning(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll}
\Omega(T) & = & \overleftarrow{\overleftrightarrow{I^\flat}}(T) & \text{(by Corollary 1235)} \\
 & \leq & \overleftarrow{\overleftrightarrow{I^\flat}}(T') & \text{(by Lemma 94)} \\
 & = & \Omega(T'). & \text{(by Corollary 1235)}
\end{array}$$

Thus, $\mathcal{I}$ is narrowly system monotone. ∎

We now introduce the notion of the *narrow system reflexive core* of a π-institution $\mathcal{I}$ in a way analogous to the narrow reflexive core, which was introduced in Section 15.2. Its introduction will enable us to provide a characterization of the syntactic narrow system monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **narrow system reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}
R^{\mathcal{I}s} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^\lightning(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\
 & & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \leq T)\}.
\end{array}$$

Given any theory system in $\mathrm{ThSys}^\lightning(\mathcal{I})$, the relation system $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a reflexive relation system on $\mathbf{F}$. This forms an analog of Lemma 1222.

Lemma 1237 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,* $\overleftarrow{R^{\mathcal{I}s}}(T)$ *is a reflexive relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. By Lemma 93, $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a relation system on $\mathbf{F}$. For reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the narrow system reflexive core, $R^{\mathcal{I}s}_\Sigma[\phi, \phi] \leq T$. Thus, $\langle \phi, \phi \rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$ and, therefore, $\overleftarrow{R^{\mathcal{I}s}}(T)$ is reflexive. ∎

As in Lemma 1223, we establish that $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a symmetric relation system on $\mathbf{F}$, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$.

Lemma 1238 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,* $\overleftarrow{R^{\mathcal{I}s}}(T)$ *is a symmetric relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Again, Lemma 93 shows that $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a relation system. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$. Consider any $\rho^\flat \in R^{\mathcal{I}s}$. By the definition of $R^{\mathcal{I}s}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}s}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi, \psi] \leq T$. But this gives $\rho^\flat_\Sigma[\psi, \phi] \leq T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}s}$, we conclude that $R^{\mathcal{I}s}_\Sigma[\psi, \phi] \leq T$. Hence, $\langle \psi, \phi \rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$. Therefore, $\overleftarrow{R^{\mathcal{I}s}}(T)$ is a symmetric relation system on $\mathbf{F}$. ∎

We now show that, for all theory systems $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $\overleftarrow{R^{\mathcal{I}s}}(T)$ has the compatibility property in $\mathbf{F}$. This forms an analog of Lemma 1224.

Lemma 1239 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,* $\overleftarrow{R^{\mathcal{I}s}}(T)$ *has the compatibility property in* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. We rely on Corollary 12. Let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}s}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta, \eta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta, \eta, \vec{\chi}, \vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\eta, \vec{\chi}), \vec{\xi}).$$

Note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{aligned} \rho'^\flat \quad = \quad & \rho^\flat \circ \langle \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2}, \ldots, p^{n+k,k} \rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2}, \ldots, p^{n+k,k} \rangle, \\ & p^{n+k,k+1}, \ldots, p^{n+k,n+k-1} \rangle, \end{aligned}$$

we get that $\rho'^\flat \in N^\flat$. Moreover, for all $T' \in \mathrm{ThSys}^\sharp(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcl}\rho'^\flat_\Sigma(\zeta,\zeta,\vec{\chi},\vec{\xi}) & = & \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\vec{\xi}) \quad \text{(by definition of } \rho'^\flat) \\ & \in & T'_\Sigma. \quad (\text{since } \rho^\flat \in R^{\mathcal{I}s}).\end{array}$$

Thus, by the definition of the narrow system reflexive core, we get that $\rho'^\flat \in R^{\mathcal{I}s}$.

Now since $\rho'^\flat \in R^{\mathcal{I}s}$ and, by hypothesis, $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \le T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi,\vec{\chi})),\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi,\vec{\chi})),\vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \le T.$$

Since this holds for all $\rho^\flat \in R^{\mathcal{I}s}$, we get that $R^{\mathcal{I}s}_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \le T$ or, equivalently, $\langle\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})\rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$. Therefore, $\overleftarrow{R^{\mathcal{I}s}}(T)$ has the congruence compatibility property in $\mathbf{F}$. ∎

We now show, in an analog of Theorem 1228, that possession of the narrow system modus ponens by the narrow system reflexive core intrinsically characterizes syntactic narrow system monotonicity. We start by showing that possession of the narrow system MP by the narrow system reflexive core is necessary for syntactic narrow system monotonicity. This forms an analog of Theorem 1225.

Theorem 1240 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically narrowly system monotone, then $R^{\mathcal{I}s}$ has the narrow system MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly system reflexive, we get, by definition of $R^{\mathcal{I}s}$, $I^\flat \subseteq R^{\mathcal{I}s}$. Thus, since $I^\flat$ has the narrow system MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}s}$ also satisfies the narrow system MP. ∎

If $R^{\mathcal{I}s}$ has the narrow system MP, then it has the narrow system transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 1226.

Proposition 1241 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}s}$ has the narrow system MP, then it also has the narrow system transitivity in $\mathcal{I}$.*

Proof: Suppose that $R^{\mathcal{I}s}$ has the narrow system MP and let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$. This means that $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}s}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 1239, we get that, for all $\rho^\flat \in R^{\mathcal{I}s}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}s}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

But, by hypothesis, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}s}$ has the narrow system MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}s}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}s}_\Sigma[\phi, \chi] \leq T$. This shows $\langle \phi, \chi \rangle \in \overleftarrow{R^{\mathcal{I}s}}_\Sigma(T)$ and, hence, $R^{\mathcal{I}s}$ is narrowly system transitive in $\mathcal{I}$. ■

We are now ready to show a converse of Theorem 1240, i.e., that possession of the narrow system MP by $R^{\mathcal{I}s}$ suffices to establish the syntactic narrow system monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}s}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 1227.

Theorem 1242 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $R^{\mathcal{I}s}$*.*

Proof: By Lemma 1237, $R^{\mathcal{I}s}$ is narrowly system reflexive in $\mathcal{I}$. By Lemma 1238, $R^{\mathcal{I}s}$ is narrowly system symmetric in $\mathcal{I}$. By hypothesis and Proposition 1241, it is narrowly system transitive in $\mathcal{I}$. By Lemma 1239, it has the narrow system compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow system MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $R^{\mathcal{I}s}$. ■

Theorems 1240 and 1242 provide the promised characterization of syntactic narrow system monotonicity in terms of the narrow system MP of the narrow system reflexive core.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow} \\ \text{System Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}s} \text{ has Narrow System} \\ \text{Modus Ponens}\end{array}.$$

Theorem 1243 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly system monotone if and only if* $R^{\mathcal{I}s}$ *has the narrow system MP in* $\mathcal{I}$*.*

Proof: Theorem 1240 gives the "only if" and the "if" is by Theorem 1242. ∎

A related alternative characterization asserts that syntactic narrow system monotonicity amounts to the narrow definability of Leibniz congruence systems of theory systems by the narrow system reflexive core. This result forms an analog of Theorem 1229.

$\mathcal{I}$ is Syntactically Narrow System Monotone	$\longleftrightarrow$	$R^{\mathcal{I}s}$ Narrowly Defines Leibniz Congruence Systems of Theory Systems

Theorem 1244 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly system monotone if and only if, for all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$\Omega(T) = \overleftarrow{R^{\mathcal{I}s}}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly system monotone, then, by Theorem 1240, $R^{\mathcal{I}s}$ has the narrow system MP in $\mathcal{I}$. Thus, by Theorem 1242, $R^{\mathcal{I}s}$ is a family of witnessing transformations for the syntactic narrow system monotonicity of $\mathcal{I}$. Thus, by Corollary 1235, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Omega(T) = \overleftarrow{R^{\mathcal{I}s}}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}s}$ is narrowly system reflexive, narrowly system transitive and has the narrow system compatibility property and the narrow system MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly system monotone. ∎

To prove an analog of Theorem 1233, which, in a certain restricted sense characterizes syntactic narrow family monotonicity inside the class of narrow family monotone π-institutions, we create a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow system reflexive core. Once more, the difficulty in this case, similarly with that described in some detail in Section 15.2, arises from the fact that $\mathrm{ThSys}^{\nmid}(\mathcal{I})$ may not be, in general, closed under signature-wise intersections.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we elaborate further on the relevant discussion initiated in Section 15.2. Recall that a π-institution $\mathcal{I}$ is *prealgebraic* if its Leibniz operator is monotone on theory systems. Recall, also, once more, that its reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

If a π-institution is prealgebraic and has a Leibniz reflexive core, then it satisfies the global system modus ponens. This was shown in Chapter 11 using the following method. Considering $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$, we get:

- $\langle \phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi]))$ first, by applying the Leibniz property;
- $\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) \leq \Omega(T)$, by applying the hypothesis that $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$ and the postulated prealgebraicity of $\mathcal{I}$ and observing at the same time that $C(R^{\mathcal{I}}_\Sigma[\phi,\psi]) \in \mathrm{ThSys}(\mathcal{I})$, since $R^{\mathcal{I}}_\Sigma[\phi,\psi]$ is a sentence system.

However, in case of narrow system monotonicity, the plausibility of $R^{\mathcal{I}s}_\Sigma[\phi,\psi]$ having some empty components makes it likely that, in the second stage, narrow system monotonicity may not be applicable to ensure the inclusion $\Omega(C(R^{\mathcal{I}s}_\Sigma[\phi,\psi])) \leq \Omega(T)$. To deal with this plausibility, we assume, in a similar way as before, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, the poset

$$[R^{\mathcal{I}s}_\Sigma[\phi,\psi]) := \{T \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T\}$$

satisfies the descending chain condition and to postulate that every minimal element $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, define
$$[R^{\mathcal{I}s}_\Sigma[\phi,\psi]) := \{T \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T\};$$
- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{I}$ is called $\langle\Sigma,\phi,\psi\rangle$-**system reflexively covered** if, for every theory system $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, such that $T' \leq T$;
- $\mathcal{I}$ is called **system reflexively covered** if it is $\langle\Sigma,\phi,\psi\rangle$-system reflexively covered, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$.

Given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, we write

$$\min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$$

for the collection of minimal elements in $[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow system reflexive core $R^{\mathcal{I}s}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

We show, in an analog of Proposition 1230, that, if $R^{\mathcal{I}s}$ has the narrow system MP, then it is Leibniz. In fact, the proof demonstrates that, under the narrow system MP, a stronger property than that of being Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$ (and not only for $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$),

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

Proposition 1245 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ *and all* $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$*,* $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$*.*

Proof: Suppose $R^{\mathcal{I}s}$ has the narrow system MP and let $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T$. To verify that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$. Since $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, by Lemma 1239,

$$R^{\mathcal{I}s}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}s}$ has the narrow system MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. ∎

Corollary 1246 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then it is Leibniz.*

Proof: Directly by Proposition 1245. ∎

In the opposite direction, when dealing with system reflexively covered π-institutions, we may show that narrow system monotonicity combined with the Leibniz property of the narrow system reflexive core imply that the narrow system reflexive core has the narrow system modus ponens in $\mathcal{I}$. The following proposition forms an analog of Proposition 1232 in the system context.

Proposition 1247 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a system reflexively covered, narrowly system monotone* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *is Leibniz, then it has the narrow system MP in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a system reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly system monotone and that $R^{\mathcal{I}s}$ is Leibniz. Let $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T$. Since $\mathcal{I}$ is system reflexively covered, there exists $T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, such that $T' \leq T$. Now we have

$$\begin{array}{rcll} \langle \phi, \psi \rangle & \in & \Omega_\Sigma(T') & (\text{since } R^{\mathcal{I}s} \text{ is Leibniz and } T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])) \\ & \subseteq & \Omega_\Sigma(T). & (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly system monotone}) \end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}s}$ has the narrow system MP in $\mathcal{I}$. ■

Thus, at least for system reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly system monotone ones inside the class of the narrowly system monotone ones can be characterized exactly by the Leibniz property of the narrow system reflexive core. This forms a partial analog of Theorem 1233.

Theorem 1248 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a system reflexively covered* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly system monotone if and only if it is narrowly system monotone and has a Leibniz narrow system reflexive core.*

Proof: Let $\mathcal{I}$ be a system reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly system monotone. By Theorem 1236, it is narrowly system monotone. Moreover, by Theorem 1240, its narrow system reflexive core has the narrow system MP. Hence, by Corollary 1246, its narrow system reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly system monotone with a Leibniz narrow system reflexive core. Then, by Proposition 1247, its narrow system reflexive core has the narrow system MP and, therefore, by Theorem 1242, $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $R^{\mathcal{I}s}$. ■

We leave here as an open problem generalizing Theorem 1233 so that it is applicable to arbitrary π-institutions and not merely to those that are system reflexively covered.

15.4 Syntactic Narrow Right Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments. Recall from Proposition 99, that, for all $T \in \mathrm{SenFam}(\mathbf{F})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T}. \tag{15.1}$$

Let, now, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We may attempt to define "syntactic narrow left monotonicity" as the existence of a collection $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

Because of the the preceding remark, however, this condition would amount exactly to defining syntactic narrow family monotonicity. On the other hand, syntactic narrow system monotonicity is equivalent, again based on the remark above, to asserting the existence of $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, with $\overleftarrow{T} \in \mathrm{ThSys}^\nsim(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad \langle \phi, \psi\rangle \in \Omega_\Sigma(\overleftarrow{T}).$$

If we drop the restriction that $\overleftarrow{T}$ be in $\mathrm{ThSys}^\nsim(\mathcal{I})$, thus allowing the condition above to be imposed on the wider class of all $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, we obtain a concept slightly more general that syntactic narrow system monotonicity, which we term *syntactic narrow right monotonicity*. We study this notion in more detail in this section, following the study of syntactic narrow family (and system) monotonicity, carried out in Section 15.2 (and 15.3, respectively) of the chapter.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 7 that $\mathcal{I}$ is *narrowly right monotone* if, for all $T, T' \in \mathrm{ThFam}^\nsim(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

In this section, following the work on syntactic narrow family monotonicity of Section 15.2, we introduce and study a syntactic analog of narrow right monotonicity.

First, the concepts of narrow system reflexivity, narrow system symmetry, narrow system transitivity, narrow system compatibility and narrow system modus ponens are recast to accommodate theory systems that arise by applying the arrow operator $\overleftarrow{\ }$ on theory families in $\mathrm{ThFam}^\nsim(\mathcal{I})$. Note that such theory systems include, of course, all theory systems in $\mathrm{ThSys}^\nsim(\mathcal{I})$, since these arise by applying the arrow operator on themselves.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **narrowly right reflexive** if, for all $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\phi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right symmetric** if, for all $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right transitive** if, for all $T \in \mathrm{ThFam}^\nsim(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right compatible** if, for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq \overleftarrow{T};$$

- $I^\flat$ has the **narrow right MP** if, for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T} \quad \text{imply} \quad \psi \in \overleftarrow{T}_\Sigma.$$

Note that, because of Equivalence (15.1), narrow right reflexivity, narrow right symmetry, narrow right transitivity and narrow right compatibility are equivalent, respectively, to narrow family reflexivity, narrow family symmetry, narrow family transitivity and narrow family compatibility. They are simply recast involving the arrow operator, but the change is inessential. On the other hand, narrow right modus ponens is an essentially different property than narrow family modus ponens and it is the critical property that differentiates syntactic narrow right monotonicity from syntactic narrow family monotonicity.

Note, also, that, based on Equivalence (15.1), for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$,

$$\overleftarrow{I^\flat}(T) = \overleftarrow{I^\flat}(\overleftarrow{T}).$$

We say that $\mathcal{I}$ is **syntactically narrowly right monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow right reflexivity;
- narrow right transitivity;
- narrow right compatibility; and
- narrow right MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic narrow right monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly right monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ $(:= \overleftrightarrow{I^\flat}(\overleftarrow{T}))$ is a congruence system on $\mathbf{F}$ compatible with $\overleftarrow{T}$, for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$. This forms a "right" analog of Proposition 1218.

Proposition 1249 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$*,* $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* $\overleftarrow{T}$*.*

Proof: The proof is similar to that of Proposition 1218. Let $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow right reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$. The fact that $\overleftrightarrow{I}^\flat$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$ implies that $\langle \psi, \phi \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$. The narrow right transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$ imply $\langle \phi, \chi \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow right compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi}) \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$. Thus, $\overleftarrow{\overleftrightarrow{I}}^{\flat}(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftarrow{\overleftrightarrow{I}}^{\flat}(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftarrow{\overleftrightarrow{I}}^{\flat}(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftarrow{\overleftrightarrow{I}}^{\flat}(T)$ is compatible with $\overleftarrow{T}$. Assume that $\phi \in \overleftarrow{T}_\Sigma$ and $\langle \phi, \psi \rangle \in \overleftarrow{\overleftrightarrow{I}}^{\flat}_{\Sigma}(T)$. Since $I^\flat \subseteq \overleftrightarrow{I}^\flat$, we get, by the narrow right MP of $I^\flat$, that $\psi \in \overleftarrow{T}_\Sigma$. Thus, $\overleftarrow{\overleftrightarrow{I}}^{\flat}(T)$ is also compatible with $\overleftarrow{T}$. ■

Proposition 1249 shows that $\overleftrightarrow{I}^\flat$ defines Leibniz congruence systems of those theory systems of the form $\overleftarrow{T}$, for $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$. We say that $I^\flat$ **narrowly defines Leibniz congruence systems** of theory families in $\mathcal{I}$ **up to arrow** if, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$,

$$\overleftarrow{\overleftrightarrow{I}}^{\flat}(T) = \Omega(\overleftarrow{T}).$$

Then, in what is an analog of Corollary 1219, we obtain

Corollary 1250 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory families in* $\mathcal{I}$ *up to arrow.*

Proof: By Proposition 1234 and Corollary 98. ■

This corollary has as immediate consequence the fact that syntactic narrow right monotonicity implies (semantic) narrow right monotonicity. This forms an analog of Theorem 1220.

Theorem 1251 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, then it is narrowly right monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly right monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll}
\Omega(\overleftarrow{T}) & = & \overleftarrow{\overleftrightarrow{I^\flat}}(T) & \text{(by Corollary 1250)} \\
& \leq & \overleftarrow{\overleftrightarrow{I^\flat}}(T') & \text{(by Lemma 94)} \\
& = & \Omega(\overleftarrow{T'}). & \text{(by Corollary 1250)}
\end{array}$$

Thus, $\mathcal{I}$ is narrowly right monotone. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Section 15.2 that the *narrow reflexive core* of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}
R^{\mathcal{I}\natural} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\natural}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\
& & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \leq T)\}.
\end{array}$$

Recall, also, from Lemmas 1222, 1223 and 1224, that, given any theory family in $\mathrm{ThFam}^{\natural}(\mathcal{I})$, the relation system $\overleftarrow{R^{\mathcal{I}\natural}}(T)$ is a reflexive and symmetric relation system on $\mathbf{F}$ that has the (congruence) compatibility property in $\mathbf{F}$.

We now show, in an analog of Theorem 1228, that possession of the narrow right modus ponens by the narrow reflexive core intrinsically characterizes syntactic narrow right monotonicity. We start by showing that possession of the narrow right MP by the narrow reflexive core is necessary for syntactic narrow right monotonicity. This forms an analog of Theorem 1225.

Theorem 1252 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically narrowly right monotone, then $R^{\mathcal{I}\natural}$ has the narrow right MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly right monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly right reflexive, which is equivalent to being narrowly family reflexive, we get, by definition of $R^{\mathcal{I}\natural}$, $I^\flat \subseteq R^{\mathcal{I}\natural}$. Thus, since $I^\flat$ has the narrow right MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}\natural}$ also satisfies the narrow right MP. ■

If $R^{\mathcal{I}\natural}$ has the narrow right MP, then it has the narrow right transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 1226.

Proposition 1253 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\natural}$ has the narrow right MP, then it also has the narrow right transitivity in $\mathcal{I}$.*

Proof: Suppose that $R^{\mathcal{I}\natural}$ has the narrow right MP and let $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftarrow{R^{\mathcal{I}\natural}}_\Sigma(T)$. This means that $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T$ and $R^{\mathcal{I}\natural}_\Sigma[\psi,\chi] \le T$. Then, by Lemma 1239, we get that, for all $\rho^\flat \in R^{\mathcal{I}\natural}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}\natural}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\psi),\vec{\xi}), \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\chi),\vec{\xi})] \le T.$$

But, by hypothesis, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T$ and $R^{\mathcal{I}\natural}$ has the narrow right MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}\natural}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\chi),\vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}\natural}_\Sigma[\phi,\chi] \le T$. This shows $\langle\phi,\chi\rangle \in \overleftarrow{R^{\mathcal{I}\natural}}_\Sigma(T)$ and, hence, $R^{\mathcal{I}\natural}$ is narrowly right transitive in $\mathcal{I}$. ■

We are now ready to show a converse of Theorem 1252, i.e., that possession of the narrow right MP by $R^{\mathcal{I}\natural}$ suffices to establish the syntactic narrow right monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}\natural}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 1227.

Theorem 1254 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\natural}$ *has the narrow right MP, then* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $R^{\mathcal{I}\natural}$*.*

Proof: By Lemma 1222, $R^{\mathcal{I}\natural}$ is narrowly right reflexive in $\mathcal{I}$. By Lemma 1223, $R^{\mathcal{I}\natural}$ is narrowly right symmetric in $\mathcal{I}$. By hypothesis and Proposition 1253, it is narrowly right transitive in $\mathcal{I}$. By Lemma 1224 it has the narrow right compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow right MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $R^{\mathcal{I}\natural}$. ■

Theorems 1252 and 1254 provide the promised characterization of syntactic narrow right monotonicity in terms of the narrow right MP of the narrow reflexive core.

$\mathcal{I}$ is Syntactically Narrow Right Monotone	$\longleftrightarrow$	$R^{\mathcal{I}\natural}$ has Narrow Right Modus Ponens.

Theorem 1255 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if* $R^{\mathcal{I}\natural}$ *has the narrow right MP in* $\mathcal{I}$*.*

Proof: Theorem 1252 gives the "only if" and the "if" is by Theorem 1254. ■

A related alternative characterization asserts that syntactic narrow right monotonicity amounts to the narrow definability of Leibniz congruence systems of theory families up to arrow by the narrow reflexive core. This result forms an analog of Theorem 1229.

$\mathcal{I}$ is Syntactically Narrow Right Monotone	$\longleftrightarrow$	$R^{\mathcal{I}\natural}$ Narrowly Defines Leibniz Congruence Systems of Theory Families up to Arrow

Theorem 1256 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly right monotone if and only if, for all* $T \in \mathrm{ThFam}^\natural(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \overleftarrow{R^{\mathcal{I}\natural}}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly right monotone, then, by Theorem 1252, $R^{\mathcal{I}\natural}$ has the narrow right MP in $\mathcal{I}$. Thus, by Theorem 1254, $R^{\mathcal{I}\natural}$ is a family of witnessing transformations for the syntactic narrow right monotonicity of $\mathcal{I}$. Thus, by Corollary 1250, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \overleftarrow{R^{\mathcal{I}\natural}}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}\natural}$ is narrowly right reflexive, narrowly right transitive and has the narrow right compatibility property and the narrow right MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly right monotone. ■

To prove an analog of Theorem 1233, which, in a sense analogous to that seen for syntactic narrow family monotonicity, characterizes syntactic narrow right monotonicity inside the class of narrow right monotone π-institutions, we create a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive core. Once more, the difficulty in this case, similarly with that described in some detail in Section 15.2, arises from the fact that $\mathrm{ThFam}^\natural(\mathcal{I})$ may not be, in general, closed under signature-wise intersections.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we reembark, once more, on a discussion initiated in Section 15.2 and revisit some of the points with relevance in treating the "right" case.

Recall, again, the definition of protoalgebraicity and the Leibniz property of the reflexive core of a π-institution. Also recall the method employed to show that, if a π-institution is protoalgebraic and has a Leibniz reflexive core, then it satisfies the global modus ponens, which is done by first applying the Leibniz property and then protoalgebraicity. However, in case of narrow right monotonicity, the plausibility of $R^{\mathcal{I}\natural}_\Sigma[\phi, \psi]$ having some empty components makes it likely that, when one attempts to apply narrow right monotonicity in place of protoalgebraicity in the second stage of the argument outlined above, its application in order to derive the inclusion $\Omega(\overleftarrow{C(R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])}) \leq \Omega(\overleftarrow{T})$ may not be possible. To deal with this plausibility, we assume, in a similar way as before, that the π-institution under consideration is reflexively covered and postulate that every minimal element $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, recall the notation

$$[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) := \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T\}.$$

Recall, also, that $\mathcal{I}$ is said to be *reflexively covered* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, it is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered, i.e., for every theory family $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, such that $T' \leq T$. Recall, furthermore, that, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, we write $\min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ for the collection of minimal elements in $[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive core $R^{\mathcal{I}\natural}$ of $\mathcal{I}$ is **right Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T}).$$

We show, in an analog of Proposition 1230, that, if $R^{\mathcal{I}\natural}$ has the narrow right MP, then it is right Leibniz. In fact, the proof demonstrates that, under the narrow right MP, a stronger property than that of being right Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ (not only for $T \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$), $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.

Proposition 1257 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\natural}$ *has the narrow right MP, then for all* $\Sigma \in |\mathbf{Sign}^\flat|$, *all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ *and all* $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.

Proof: Suppose $R^{\mathcal{I}\natural}$ has the narrow right MP and let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$. To verify that $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$. Since $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, by Lemma 1224,

$$R^{\mathcal{I}\natural}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}\natural}$ has the narrow right MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in \overleftarrow{T}_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. ■

Corollary 1258 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ has the narrow right MP, then it is right Leibniz.*

Proof: Directly by Proposition 1257. ■

To prove a converse, we restrict attention to reflexively covered π-institutions. Inside this class, we may show that narrow right monotonicity combined with the right Leibniz property of the narrow reflexive core imply that the narrow reflexive core has the narrow right modus ponens in $\mathcal{I}$. The following proposition forms an analog of Propositions 1232 and 1247 in the "right" context.

Proposition 1259 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a reflexively covered, narrowly right monotone π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ is right Leibniz, then it has the narrow right MP in $\mathcal{I}$.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly right monotone and that $R^{\mathcal{I}\nleq}$ is right Leibniz. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$ and $R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi] \leq T$. Since $\mathcal{I}$ is reflexively covered, there exists $T' \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi]]$, such that $T' \leq T$. Now we have

$$\begin{array}{rcll} \langle \phi, \psi \rangle & \in & \Omega_\Sigma(\overleftarrow{T'}) & (\text{since } R^{\mathcal{I}\nleq} \text{ is right Leibniz and } T' \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi]]) \\ & \subseteq & \Omega_\Sigma(\overleftarrow{T}). & (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly right monotone}) \end{array}$$

Therefore, since $\phi \in \overleftarrow{T}_\Sigma$, we get, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, that $\psi \in \overleftarrow{T}_\Sigma$. We conclude that $R^{\mathcal{I}\nleq}$ has the narrow right MP in $\mathcal{I}$. ■

Thus, at least for reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly right monotone ones inside the class of the narrowly right monotone ones can be characterized exactly by the right Leibniz property of the narrow reflexive core. This forms a partial analog of Theorems 1233 and 1248.

Theorem 1260 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a reflexively covered π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically narrowly right monotone if and only if it is narrowly right monotone and has a right Leibniz narrow reflexive core.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly right monotone. By Theorem 1251, it is narrowly right monotone. By Theorem 1252, its narrow reflexive core has the narrow right MP. Hence, by Corollary 1258, its narrow reflexive core is right Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly right monotone with a right Leibniz narrow reflexive core. Then, by Proposition 1259, its narrow reflexive core has the narrow right MP and, therefore, by Theorem 1254, $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $R^{\mathcal{I}\lightning}$. ∎

An open problem, once more, is to provide a general intrinsic characterization, along the lines of Theorem 1260, applicable to arbitrary π-institutions and not only to those that are reflexively covered.

We conclude the section (and the chapter) by establishing some inclusions between the classes of π-institutions that were introduced in this chapter. More precisely, we show that each of syntactic narrow family monotonicity and syntactic narrow right monotonicity implies syntactic narrow system monotonicity.

Proposition 1261 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is syntactically narrowly family monotone, then it is syntactically narrowly system monotone;*

(b) If $\mathcal{I}$ *is syntactically narrowly right monotone, then it is syntactically narrowly system monotone.*

Proof: We prove Part (a) in two different ways. This is instructive, since one proof is simply based on the relevant definitions and the other makes use of some of the results obtained in the various sections. Then we prove Part (b) in only one way, using the definitions.

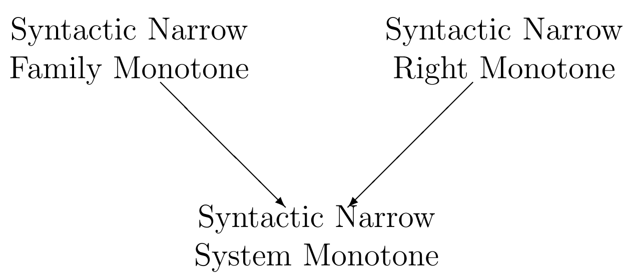
Suppose $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $I^\flat$. Then $I^\flat$ is narrow family reflexive, narrow family transitive, has narrow family compatibility and satisfies the narrow family modus ponens. In particular $I^\flat$ satisfies narrow system reflexivity, narrow system transitivity, narrow system compatibility and the narrow system modus ponens. Thus, by definition, $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $I^\flat$.

We now give the second (indirect) proof. Suppose $\mathcal{I}$ is syntactically narrowly family monotone. Then, by Theorem 1228, the narrow reflexive core $R^{\mathcal{I}\lightning}$ has the narrow family modus ponens in $\mathcal{I}$. But, by definition, we have $R^{\mathcal{I}\lightning} \subseteq R^{\mathcal{I}s}$. Thus, we get that the narrow system reflexive core has the narrow family modus ponens in $\mathcal{I}$. Hence, a fortiori, the narrow system reflexive core $R^{\mathcal{I}s}$ has the narrow system modus ponens in $\mathcal{I}$. Therefore, by Theorem

1243, $\mathcal{I}$ is syntactically narrowly system monotone. This proof shows how one may use the two characterization theorems, Theorems 1228 and 1243, to proceed with statements concerning the two classes related.

For an elementary proof for Part (b), assume that $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $I^\flat$. Then $I^\flat$ has the narrow right reflexivity, narrow right transitivity, narrow right compatibility and narrow right modus ponens. The first three properties are equivalent to narrow family reflexivity, narrow family transitivity and narrow family compatibility, respectively, and, consequently, imply narrow system reflexivity, narrow system transitivity and narrow system compatibility, respectively. The narrow right modus ponens asserts that, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ, ψ, $\phi \in \overleftarrow{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T}$ imply $\psi \in \overleftarrow{T}_\Sigma$. Since, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\overleftarrow{T} = T$, this implies that, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, all signatures Σ and all Σ-sentences ϕ, ψ, $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \leq T$ imply $\psi \in T_\Sigma$. That is, $I^\flat$ also satisfies the narrow system modus ponens. Hence, $I^\flat$ is a collection of witnessing transformations for the syntactic narrow system monotonicity of $\mathcal{I}$ as well. ■

Proposition 1261 gives rise to the following syntactic narrow monotonicity subhierarchy.



Two examples are now in order to show that none of the arrows in the diagram collapse and, therefore, the hierarchy is exactly as shown. The first example gives a π-institution which is syntactically narrowly family monotone, but fails to be syntactically narrowly right monotone.

Example 1262 EXAMPLE NOT FOUND YET!

The second example gives a π-institution which is syntactically narrowly right monotone, but fails to be syntactically narrowly family monotone.

Example 1263 EXAMPLE NOT FOUND YET!