Chapter 17

The Syntactic Leibniz Hierarchy: Attic II

17.1 Introduction

This chapter deals with the issue of finitarity for syntactically defined classes in the hierarchy. An analogous study from the semantic point of view was carried out in Chapter 9.

In Section 17.2, we revisit the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$, this time in relation to classes near the top of the syntactic hierarchy. Our goal is to discover conditions under which syntactic properties are inherited by $\mathcal{I}$ from $\mathcal{I}^f$ and vice versa. Recall that the finitary companion $\mathcal{I}^f$ of $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is the π-institution $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$, where, for all signatures Σ and all subsets Φ of Σ-sentences,

$$C^f_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

$\subseteq_f$ denoting the finite subset relation. $\mathcal{I}^f$ is the largest finitary π-institution lying below $\mathcal{I}$ in the $\leq$ ordering. This implies that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Due to this inclusion, many of the key properties related to the algebraic hierarchy that $\mathcal{I}^f$ possesses are inherited by $\mathcal{I}$. E.g., we show that, if $\mathcal{I}^f$ happens to be syntactically protoalgebraic, then so is $\mathcal{I}$. Similarly, if $\mathcal{I}^f$ happens to be truth equational, then so is $\mathcal{I}$. Moreover, under protoalgebraicity of $\mathcal{I}^f$, the Leibniz property is inherited from the reflexive core $R^{\mathcal{I}^f}$ of $\mathcal{I}^f$ by the reflexive core $R^{\mathcal{I}}$ of $\mathcal{I}$. Analogously, under c-reflectivity of $\mathcal{I}^f$, adequacy of the Suszko core $S^{\mathcal{I}^f}$ of $\mathcal{I}^f$ implies adequacy of the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$. These properties enable us to show that, if $\mathcal{I}^f$ is syntactically weakly family algebraizable, then so is $\mathcal{I}$.

In the second part of Section 17.2, we turn to some properties that transfer from $\mathcal{I}$ to $\mathcal{I}^f$ rather than in the opposite direction. In Chapter 9 it was shown that continuity of the Leibniz operator implies protoalgebraicity. Moreover, under this strengthening of protoalgebraicity, and the additional hypothesis that the category of signatures is finite, it was shown that $\mathcal{I}^f$ is also protoalgebraic. On the syntactic side, following along similar lines, we show that, if the category of signatures is finite and $\mathcal{I}$ is syntactically protoalgebraic, with a finite collection of parameter free witnessing transformations, then the Leibniz operator is continuous. We obtain that, under these hypotheses, $\mathcal{I}^f$ is also syntactically protoalgebraic with the same collection of witnessing transformations. To establish a similar result regarding truth equationality, instead of syntactic protoalgebraicity, we need, first, to ensure that the inverse $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ is well defined. This happens when, e.g., $\mathcal{I}$ is weakly family algebraizable. We use this hypothesis, together with the finiteness of the category of signatures and the truth equationality of $\mathcal{I}$ via a finite, parameter free collection of witnessing equations, to get that Ω^{-1} is continuous. This also yields that, under the same hypotheses, $\mathcal{I}^f$ is also family truth equational, with the same collection of witnessing equations. These two results combined allow us to show that, if $\mathcal{I}$ has a finite signature category and is syntactically strongly family algebraizable, via a conjugate pair of finite collections of natural transformations, then $\mathcal{I}^f$ is also syntactically strongly family algebraizable via the same conjugate pair of transformations.

In Section 17.3, we look at *natural finitarity*. A π-institution is finitary if $\mathcal{I} = \mathcal{I}^f$, that is, if, for every signature Σ and all subsets Φ of Σ-sentences,

$$C_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\}.$$

Equivalently, for all signatures Σ and all subsets $\Phi \cup \{\phi\}$ of Σ-sentences, $\phi \in C_\Sigma(\Phi)$ implies that $\phi \in C_\Sigma(\Phi')$, for some $\Phi' \subseteq_f \Phi$. On the other hand, $\mathcal{I}$ is *naturally finitary* if, for all collections μ, ν of natural transformations in $N^\flat$, with $|\mu|$ finite,

$$\mu \leq C(\nu) \quad \text{implies} \quad \mu \leq C(\nu'), \text{ for finite } \nu' \subseteq \nu,$$

where the notation $\mu \leq C(\nu)$ means that, for all signatures Σ and all Σ-sentences $\vec{\phi}$, $\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}])$. If a π-institution $\mathcal{I}$ is naturally finitary and syntactically family algebraizable via a finite witnessing family $I^\flat$ of natural transformations, then every such family possesses a finite witnessing subfamily. Dually, if $\mathcal{I}$ is syntactically strongly family algebraizable, with a naturally finitary equivalent equational π-structure $\mathcal{Q}$ via a finite witnessing family $\tau^\flat$ of equations, then every witnessing family of equations possesses a finite witnessing subfamily. These results allow showing that, for a naturally finitary, syntactically strongly algebraizable π-institution $\mathcal{I}$, every witnessing collection of equations contains a witnessing finite subcollection and that, dually, if $\mathcal{I}$ is syntactically strongly family algebraizable, with naturally finitary equational counterpart $\mathcal{Q}$, then every witnessing collection of natural transformations contains a finite witnessing subcollection. Subject to $\mathcal{I}$ having a finite category of signatures, being finitary and syntactically family algebraizable, if $\mathcal{I}$ has a finite, parameter free witnessing set of transformations, then its equational counterpart $\mathcal{Q}$ may be taken to be finitary. Similarly, if finitarity is replaced by natural finitarity. Dually, under the hypotheses that $\mathcal{I}$ has a finite category of signatures, is syntactically strongly family algebraizable, with a finitary equational counterpart, via a finite, parameter free collection of equations, then $\mathcal{I}$ is finitary. And, similarly, with natural finitarity in lieu of finitarity. The section concludes with a theorem summarizing the results obtained and an accompanying diagram providing a schematic summary. In a nutshell, one deals with a π-institution, with a finite category of signatures, that is syntactically strongly family algebraizable via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$. In case $\tau^\flat, I^\flat$ are finite, then $\mathcal{I}$ is naturally finitary if and only if $\mathcal{Q}$ is. If $\mathcal{I}$ is naturally finitary, then $\mathcal{Q}$ is naturally finitary if and only if $I^\flat$ can be taken finite and, dually, if $\mathcal{Q}$ is naturally finitary, then $\mathcal{I}$ is naturally finitary if and only if $\tau^\flat$ can be taken finite. And in each of these three biconditionals, if the equivalent alternatives hold, then all four "finitarity" conditions hold.

17.2 Finitary Companions Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 9 the construction of the *finitary companion* $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ of $\mathcal{I}$. It is defined, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C^f_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

where $\subseteq_f$ denotes the finite subset relation. It was shown in Corollary 653 that $\mathcal{I}^f$ is the largest finitary π-institution based on $\mathbf{F}$ that lies below $\mathcal{I}$ in the $\leq$ ordering. Furthermore, even though it is obvious, based on $\mathcal{I}^f \leq \mathcal{I}$, that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$, Proposition 655 provided a characterization of those sentence families of $\mathbf{F}$ that are theory families of $\mathcal{I}^f$. More concretely, it asserted that $T \in \mathrm{ThFam}(\mathcal{I}^f)$ if and only if it is the union of a directed locally finitely generated collection of theory families of $\mathcal{I}$.

Turning now to the Leibniz hierarchy, some of the semantic aspects of which, in relation to finitarity, were studied in some detail in Chapter 9, it was proven in Lemma 656 that protoalgebraicity is inherited by $\mathcal{I}$ from $\mathcal{I}^f$, i.e., if $\mathcal{I}^f$ is protoalgebraic, then so is $\mathcal{I}$ itself. This is a rather simple consequence of the fact that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$.

Recall from Chapter 11 the definition of the *reflexive core* $R^{\mathcal{I}}$ of a π-institution $\mathcal{I}$. It consists of all natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, having the property that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}).$$

It is not very difficult to show that the reflexive core of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ is included in that of $\mathcal{I}$.

Lemma 1341 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}.$$

Proof: Suppose $\rho^\flat \in R^{\mathcal{I}^f}$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{lll}\rho^\flat_\Sigma[\phi,\phi] & \leq & \mathrm{Thm}(\mathcal{I}^f) \quad (\rho^\flat \in R^{\mathcal{I}^f}) \\ & \leq & \mathrm{Thm}(\mathcal{I}). \quad (\mathrm{Thm}(\mathcal{I}) \in \mathrm{ThFam}(\mathcal{I}^f))\end{array}$$

Thus, by definition, $\rho^\flat \in R^{\mathcal{I}}$. It follows that $R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}$. ■

Recall that the reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])).$$

From the fact that $R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}$ it follows at once that, if $\mathcal{I}^f$ is protoalgebraic and $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$, then $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$.

Proposition 1342 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is protoalgebraic and* $R^{\mathcal{I}^f}$ *is Leibniz in* $\mathcal{I}^f$*, then so is* $R^{\mathcal{I}}$ *in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}^f$ is protoalgebraic and $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We then have

$$\begin{array}{rcll}\langle \phi, \psi \rangle & \in & \Omega_\Sigma(C^f(R^{\mathcal{I}^f}_\Sigma[\phi, \psi])) & (R^{\mathcal{I}^f} \text{ Leibniz in } \mathcal{I}^f) \\ & \subseteq & \Omega_\Sigma(C^f(R^{\mathcal{I}}_\Sigma[\phi, \psi])) & (\text{Lemma 1341 and hypothesis}) \\ & \subseteq & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])). & (\text{Corollary 653 and hypothesis})\end{array}$$

Therefore, $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$. ■

We can now show that syntactic protoalgebraicity is inherited by a π-institution $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$. This forms an analog in the syntactic context of Lemma 656.

Theorem 1343 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is syntactically protoalgebraic, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is syntactically protoalgebraic. By Theorem 805, it is protoalgebraic and its reflexive core $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$. Therefore, by Lemma 656, $\mathcal{I}$ is protoalgebraic and, by Proposition 1342, $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$. Therefore, again by Theorem 805, $\mathcal{I}$ is syntactically protoalgebraic. ■

Recalling Theorem 799, which characterizes syntactic protoalgebraicity in terms of the global family modus ponens property of the reflexive core, we derive the following

Corollary 1344 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}^f}$ *has the global family MP in* $\mathcal{I}^f$*, then* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$*.*

Proof: If $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$, then, by Theorem 799, $\mathcal{I}^f$ is syntactically protoalgebraic. Thus, by Theorem 1343, $\mathcal{I}$ is syntactically protoalgebraic, whence, again by Theorem 799, applied in the opposite direction, $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. ■

Alternatively, instead of deriving the implication in Corollary 1344 by applying Theorem 1343, we may prove it first and then use Theorem 799 to establish that syntactic protoalgebraicity of $\mathcal{I}^f$ implies the syntactic protoalgebraicity of $\mathcal{I}$. We outline this reasoning also, at the expense of having to repeat Corollary 1344 and Theorem 1343.

Lemma 1345 (Corollary 1344) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}^f}$ *has the global family MP in* $\mathcal{I}^f$*, then* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$*.*

Proof: Suppose $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in T_\Sigma \quad \text{and} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T.$$

By Lemma 1341, we get

$$\phi \in T_\Sigma \quad \text{and} \quad R^{\mathcal{I}^f}_\Sigma[\phi, \psi] \leq T.$$

But $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$ and $R^{\mathcal{I}^f}$ is assumed to have the global family MP in $\mathcal{I}^f$. Thus, $\psi \in T_\Sigma$. This proves that $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. ■

Corollary 1346 (Theorem 1343) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is syntactically protoalgebraic, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is syntactically protoalgebraic. Then, by Theorem 799, $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$. Thus, by Lemma 1345, $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. Hence, again by applying Theorem 799, only now in the reverse direction, $\mathcal{I}$ is syntactically protoalgebraic. ■

A similar work can be undertaken concerning truth equationality, based on an analog of Lemma 657, but referring to family c-reflectivity, which can be proved in a similar fashion as Lemma 657. We now provide the details.

It is straightforward to see, first of all, that family complete reflectivity is also inherited by $\mathcal{I}$ itself from its finitary companion.

Lemma 1347 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is family c-reflective, then so is* $\mathcal{I}$.

Proof: If $\mathcal{I}^f$ is family c-reflective, then, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I}^f)$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

In particular, the condition holds if quantification is restricted over the collection $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, $\mathcal{I}$ is family c-reflective. ■

It is not very hard either to see that the the Suszko core of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ is contained in the Suszko core of $\mathcal{I}$ itself, just as was the case with the reflexive core. Recall that the Suszko core $S^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ consists of those natural transformations $\sigma^\flat$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

This means, of course, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{implies} \quad \sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Lemma 1348 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$S^{\mathcal{I}^f} \subseteq S^{\mathcal{I}}.$$

Proof: Suppose that $\sigma^\flat \in S^{\mathcal{I}^f}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, since $\sigma^\flat \in S^{\mathcal{I}^f}$ and $T \in \mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$, we get $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}^f}(T) \leq \widetilde{\Omega}^{\mathcal{I}}(T)$, where the second inclusion follows from the fact that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, we conclude that $\sigma^\flat \in S^{\mathcal{I}}$. Hence, $S^{\mathcal{I}^f} \subseteq S^{\mathcal{I}}$. ■

With this result available, we can see that, if $\mathcal{I}^f$ is family c-reflective and its Suszko core is adequate, then the Suszko core of $\mathcal{I}$ is also adequate. Recall that adequacy of the Suszko core $S^{\mathcal{I}}$ means that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Recall also, that the right-to-left inclusion always holds. So the definition is tantamount to the assertion that the left-to-right inclusion also holds.

Proposition 1349 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is family c-reflective and* $S^{\mathcal{I}^f}$ *is adequate in* $\mathcal{I}^f$*, then so is* $S^{\mathcal{I}}$ *in* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is family c-reflective and that $S^{\mathcal{I}^f}$ is adequate. Then, by Theorem 848, $\mathcal{I}^f$ is truth equational, whence, by Theorem 841, for all $T \in \mathrm{ThFam}(\mathcal{I}^f)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T).$$

Consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\
& & \text{(Definition of } \widetilde{\Omega}^{\mathcal{I}}) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T)\} \\
& & (\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f) \text{ and displayed equivalence}) \\
& \leq & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \\
& & \text{(Lemma 1348)}
\end{array}$$

Thus, by definition, $S^{\mathcal{I}}$ is also adequate in $\mathcal{I}$. ■

We can now show that truth equationality is inherited by a π-institution $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$. This forms an analog of Lemma 1343.

Theorem 1350 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is truth equational, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is truth equational. By Theorem 848, it is family c-reflective and its Suszko core $S^{\mathcal{I}^f}$ is adequate in $\mathcal{I}^f$. Therefore, by Lemma 1347, $\mathcal{I}$ is family c-reflective and, by Proposition 1349, $S^{\mathcal{I}}$ is adequate in $\mathcal{I}$. Therefore, again by Theorem 848, $\mathcal{I}$ is truth equational. ∎

Theorem 841 characterized truth equationality in terms of the solubility property of the Suszko core. In fact, the solubility of the Suszko core is the condition asserting that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Since the reverse implication always holds, the condition is equivalent to the assertion that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Corollary 1351 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$, then $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$.*

Proof: If $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$, then, by Theorem 839, $\mathcal{I}^f$ is truth equational. Thus, by Theorem 1350, $\mathcal{I}$ is also truth equational, whence, again by Theorem 839, applied in the opposite direction, $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. ∎

Once more, as was the case with syntactic protoalgebraicity, instead of deriving the implication in Corollary 1351 by applying Theorem 1350, we may prove it first and then use Theorem 839 to establish that truth equationality of $\mathcal{I}^f$ implies truth equationality of $\mathcal{I}$.

Lemma 1352 (Corollary 1351) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$, then $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$.*

Proof: Suppose $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by Lemma 1348, we get $S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T)$. But $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$ and $S^{\mathcal{I}^f}$ is assumed to be soluble in $\mathcal{I}^f$. Thus, $\phi \in T_\Sigma$. This proves that $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. ∎

Corollary 1353 (Theorem 1350) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f$ is truth equational, then so is $\mathcal{I}$.*

Proof: Suppose $\mathcal{I}^f$ is truth equational. Then, by Theorem 839, $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$. Thus, by Lemma 1352, $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. Hence, again by applying Theorem 839, only now in the reverse direction, $\mathcal{I}$ is truth equational. ∎

We conclude the section by synthesizing Theorems 1343 and 1350. Recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is syntactically weakly family algebraizable if:

- $\mathcal{I}$ is protoalgebraic;
- $\mathcal{I}$ is family c-reflective;
- $\mathcal{I}$ is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, i.e., $\mathcal{I}$ has a Leibniz reflexive core and an adequate Suszko core.

By Theorem 915, $\mathcal{I}$ is syntactically weakly family algebraizable if and only if it is syntactically protoalgebraic and family truth equational. Thus, we get

Theorem 1354 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is syntactically weakly family algebraizable, then so is* $\mathcal{I}$*.*

Proof: If $\mathcal{I}^f$ is syntactically weakly family algebraizable, then, by Theorem 915, it is syntactically protoalgebraic and family truth equational. Hence, by Theorems 1343 and 1350, $\mathcal{I}$ possesses the same properties. Therefore, applying again Theorem 915 in the reverse direction, we conclude that $\mathcal{I}$ is also syntactically weakly family algebraizable. ∎

In Section 9.4, we saw that continuity of the Leibniz operator is one of the key properties when studying finitarity conditions. Lemma 660 showed that, if $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then $\mathcal{I}$ is protoalgebraic. That is asserting the continuity of the Leibniz operator strengthens protoalgebraicity. Additionally, it was proven in Lemma 661 that, if $\mathbf{Sign}^\flat$ is finite, then continuity of Ω also ensures that the finitary companion $\mathcal{I}^f$ of $\mathcal{I}$ is also protoalgebraic.

We begin, here, our parallel treatment on the syntactic side by showing that, maintaining the finiteness of $\mathbf{Sign}^\flat$, the condition that $\mathcal{I}$ be syntactically protoalgebraic, with a finite collection of parameter free witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, constitutes an additional strengthening on protoalgebraicity, on top of the continuity of Ω.

Proposition 1355 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, with a finite, parameter free collection* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of witnessing transformations, then* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution based on $\mathbf{F}$, with a finite, parameter free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations. Suppose $\{T^i : i \in I\}$ is a directed collection of theory families of $\mathcal{I}$, such that $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\langle \phi, \psi \rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i) & \text{iff} & I^\flat_\Sigma[\phi, \psi] \leq \bigcup_{i\in I} T^i \\ & \text{iff} & I^\flat_\Sigma[\phi, \psi] \leq T^i, \text{ some } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \Omega_\Sigma(T^i), \text{ some } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \bigcup_{i\in I} \Omega_\Sigma(T^i).\end{array}$$

Note that the second equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $I^\flat$ is finite and parameter free. Thus, $\Omega(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega(T^i)$ and, hence, Ω is indeed continuous. ■

We next see that this stronger condition than the continuity of the Leibniz operator suffices to ensure that $\mathcal{I}^f$ is also syntactically protoalgebraic, with the same collection of witnessing transformations. Thus, the following proposition may be viewed as a syntactic analog of Lemma 661.

Proposition 1356 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, with a finite and parameter free collection* $I^\flat$ *of witnessing transformations, then* $\mathcal{I}^f$ *is also syntactically protoalgebraic, with the same collection of witnessing transformations.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically protoalgebraic π-institution, with a finite and parameter free collection $I^\flat$ of witnessing transformations. Let $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Then, by Proposition 655, there exists a directed locally finitely generated collection $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T = \bigcup_{i\in I} T^i$. Now we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}
\langle \phi,\psi\rangle \in \Omega_\Sigma(T) & \text{iff} & \langle \phi,\psi\rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i)\\
& \text{iff} & \langle \phi,\psi\rangle \in \bigcup_{i\in I} \Omega_\Sigma(T^i) \quad \text{(Proposition 1355)}\\
& \text{iff} & \langle \phi,\psi\rangle \in \Omega_\Sigma(T^i),\ \text{some } i \in I,\\
& \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi,\psi] \leq T^i,\ \text{some } i \in I,\\
& \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi,\psi] \leq \bigcup_{i\in I} T^i\\
& \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi,\psi] \leq T.
\end{array}$$

Again, note that the one-before-the-last equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $I^\flat$ is finite and parameter free. Therefore, by Corollary 791, $\mathcal{I}^f$ is also syntactically protoalgebraic, with the same collection $I^\flat$ of witnessing transformations. ■

Suppose, now, that $\mathbf{Sign}^\flat$ is finite and $\mathcal{I}$ is weakly family algebraizable, so that $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ be defined. An analog of Proposition 1355 asserts that, if $\mathcal{I}$ is truth equational, with a finite and parameter free witnessing family $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of equations, then the inverse Leibniz operator Ω^{-1} is continuous. Thus, under these hypotheses, the truth equationality of $\mathcal{I}$ via a finite, parameter free collection of witnessing equations is stronger than the continuity of Ω^{-1}.

Proposition 1357 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, with a finite and parameter free collection* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of witnessing equations, then* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a weakly family algebraizable π-institution, which is, in addition, truth equational, with a finite, parameter free collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of witnessing equations. Let $\{\theta^i : i \in I\}$ be a directed collection of $\mathcal{I}^*$-congruence systems, such that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$. Now we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in \Omega_\Sigma^{-1}(\bigcup_{i \in I} \theta^i) & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \bigcup_{i \in I} \theta^i \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \theta^i, \text{ some } i \in I, \\ & \text{iff} & \phi \in \Omega_\Sigma^{-1}(\theta^i), \text{ some } i \in I, \\ & \text{iff} & \phi \in \bigcup_{i \in I} \Omega_\Sigma^{-1}(\theta^i). \end{array}$$

Thus, Ω^{-1} is indeed continuous. ∎

Recall from Theorem 663 that given a weakly family algebraizable π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F}$ over a finite category of signatures, the continuity of both $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ and $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ are sufficient to ensure that $\mathcal{I}^f$ is also weakly family algebraizable. In Propositions 1355 and 1357, by comparison, it was shown that the continuities of Ω and Ω^{-1} are strengthened by assuming, respectively, that $\mathcal{I}$ is syntactically protoalgebraic, with a finite, parameter free witnessing family of transformations, and that $\mathcal{I}$ is family truth equational, with a finite, parameter free witnessing family of equations. We show, next, in an analog of Theorem 663, that imposing these two stronger conditions on $\mathcal{I}$ suffices to ensure that syntactic strong algebraizability transfers from $\mathcal{I}$ to its finitary companion $\mathcal{I}^f$.

Proposition 1358 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution, with a finite, parameter free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations. If $\mathcal{I}$ is family truth equational, with a finite and parameter free collection $\tau^\flat$ of witnessing equations, then $\mathcal{I}^f$ is also family truth equational, with the same collection of witnessing equations.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution, with a finite, parameter free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations, which is family truth equational, with a finite and parameter free collection $\tau^\flat$ of witnessing equations. Let $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Then, by Proposition 655, there exists a directed locally finitely generated collection $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T = \bigcup_{i \in I} T^i$. Now we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all

$\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\phi \in T_\Sigma & \text{iff} & \phi \in \bigcup_{i\in I} T^i_\Sigma \\ & \text{iff} & \phi \in T^i_\Sigma, \text{ some } i \in I, \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T^i), \text{ some } i \in I, \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \bigcup_{i\in I} \Omega(T^i) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(\bigcup_{i\in I} T^i) \quad \text{(Proposition 1355)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T).\end{array}$$

Again, note that the fourth equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $\tau^\flat$ is finite and parameter free. We conclude that $\mathcal{I}^f$ is also family truth equational, with the same collection $\tau^\flat$ of witnessing equations. ■

Putting together Propositions we finally obtain the promised analog of Theorem 663.

Theorem 1359 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically strongly family algebraizable* π*-institution, via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$ *consisting of finite and parameter free collections of transformations. Then* $\mathcal{I}^f$ *is also syntactically strongly family algebraizable, via the same conjugate pair of transformations.*

Proof: We simply put together Propositions 1356 and 1358. ■

17.3 Natural Finitarity

This section deals with concepts analogous to those studied in Section 9.4, but in the syntactic, rather than in the semantic, context. In the semantic context, the four key ingredients of our study were the finitarity of the π-institutions involved as well as the continuity of the Leibniz operator and its inverse. Recall that for the inverse to be defined in the context under consideration, the general underlying hypothesis that the π-institution $\mathcal{I}$ be weakly family algebraizable was adhered to. In the present, syntactic, context, we assume that $\mathcal{I}$ is syntactically strongly family algebraizable, that is, syntactically family algebraizable via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$, where both $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ and $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ are parameter free witnessing collections of equations and of transformations, respectively. The four notions involved are the properties of $\mathcal{I}$ and $\mathcal{Q}^{\mathsf{K}}$ being naturally finitary, a strengthening of finitarity, and those of $\tau^\flat$ and $I^\flat$ being finite, also strengthening the continuity of the Leibniz operator and its inverse operator. But let us embark on the developments so as to clarify these introductory remarks and to make the concepts and the details involved precise.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is *finitary* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in C_\Sigma(\Phi')$. Equivalently, $\mathcal{I}$ is finitary if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\}.$$

We say that $\mathcal{I}$ is **naturally finitary** if it is finitary and, in addition, the following condition holds:

(NATFIN) If, for some collections $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, such that $|\mu| < \infty$, it holds that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}]),$$

then, there exists a finite subset $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}]).$$

It is not difficult to see that, if $\mathcal{I}$ is naturally finitary, the implication resulting from (NATFIN) by replacing the two inclusions by equalities of closure families also holds.

Lemma 1360 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is naturally finitary, then, for all* $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with* $|\mu| < \infty$*, such that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$*,* $C(\mu_\Sigma[\vec{\phi}]) = C(\nu_\Sigma[\vec{\phi}])$*, there exists a finite* $\nu' \subseteq \nu$*, such that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$*,* $C(\mu_\Sigma[\vec{\phi}]) = C(\nu'_\Sigma[\vec{\phi}])$*.*

Proof: Suppose $\mathcal{I}$ is naturally finitary and let $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}]) = C(\nu_\Sigma[\vec{\phi}])$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}])$. Thus, by natural finitarity, there exists a finite subset $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}])$. But, then, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\nu_\Sigma[\vec{\phi}]) = C(\mu_\Sigma[\vec{\phi}]) \leq C(\nu'_\Sigma[\vec{\phi}]) \leq C(\nu_\Sigma[\vec{\phi}]).$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}]) = C(\nu'_\Sigma[\vec{\phi}])$. ■

Starting to take advantage of natural finitarity, we show that it allows to draw the conclusion that, in case of syntactic family algebraizability, the existence of a finite witnessing family of transformations ensures that every witnessing family possesses a finite witnessing subfamily. More precisely, we have

Lemma 1361 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary* π*-institution based on* $\mathbf{F}$*. Suppose* $\mathcal{I}$ *is syntactically family algebraizable, with equivalent guasivariety* K*. If* $\mathcal{I}$ *has a finite witnessing family* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations, then every witnessing family* $J^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *possesses a finite witnessing subfamily* $J'^\flat$*.*

Proof: Suppose that $\mathcal{I}$ is naturally finitary and syntactically family algebraizable, with equivalent guasivariety K. Let $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ be a finite set of witnessing transformations and $J^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a family of witnessing transformations. By Theorem 914, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(I^\flat_\Sigma[\phi, \psi]) = C(J^\flat_\Sigma[\phi, \psi]).$$

Since $\mathcal{I}$ is naturally finitary and, by hypothesis, $|I^\flat| < \infty$, we get, by Lemma 1360, that there exists finite $J'^\flat \subseteq J^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(J'^\flat_\Sigma[\phi, \psi]) = C(I^\flat_\Sigma[\phi, \psi]).$$

Thus, applying Proposition 906, we conclude that $J'^\flat$ is also a witnessing family of transformations. ■

Dually, we may also prove a corresponding result concerning the witnessing equations for the truth equationality of $\mathcal{I}$.

Lemma 1362 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution based on* $\mathbf{F}$*, with an equivalent naturally finitary equational* π*-structure* $\mathcal{Q}$*, via a finite witnessing family* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations. Then every witnessing family* $\rho^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations possesses a finite witnessing subfamily* $\rho'^\flat$*.*

Proof: Follows along the lines of the proof of Lemma 1361. Suppose that $\mathcal{I}$ is syntactically strongly family algebraizable, with an equivalent naturally finitary equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$. Let $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ be a finite set of witnessing equations and $\rho^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ a family of witnessing equations. By Theorem 914, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D(\tau^\flat_\Sigma[\phi]) = D(\rho^\flat_\Sigma[\phi]).$$

Since $\mathcal{Q}$ is naturally finitary and, by hypothesis, $|\tau^\flat| < \infty$, we get, by Lemma 1360, that there exists finite $\rho'^\flat \subseteq \rho^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D(\rho'^\flat_\Sigma[\phi]) = D(\tau^\flat_\Sigma[\phi]).$$

Thus, applying Proposition 906, we conclude that $\rho'^\flat$ is also a witnessing family of equations. ■

We now establish a theorem to the effect that, under natural finitarity and syntactic strong family algebraizability, every witnessing family of equations contains a finite witnessing subfamily. This is the first main result in a series of finitarity results that we aim to prove in the present section, with the ultimate goal of obtaining a hierarchy on the syntactic side, analogous to that obtained on the semantic side at the end of Section 9.4.

Theorem 1363 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary, syntactically strongly family algebraizable* π*-institution, with equivalent equational* π*-structure* $\mathcal{Q}$*. Then every witnessing collection* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations contains a finite subcollection* $\tau'^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$*, which is also a witnessing collection.*

Proof: Suppose $\mathcal{I}$ is naturally finitary and syntactically strongly family algebraizable, with equivalent equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$. Let $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ be a collection of witnessing equations. Then, by definition, there exists a collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\iota_\Sigma[\phi]) = C(\phi) = C(I^\flat[\tau^\flat_\Sigma[\phi]]).$$

Since $\mathcal{I}$ is naturally finitary, there exist finite $I'^\flat \subseteq I^\flat$ and $\tau'^\flat \subseteq \tau^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(I'^\flat[\tau'^\flat_\Sigma[\phi]]).$$

Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(I^\flat[\tau'^\flat_\Sigma[\phi]]).$$

Thus, since, by the properties of $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$,

$$\phi \approx \psi \in D_\Sigma(E) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E]),$$

we get, by Proposition 906, that $\tau'^\flat$ is a witnessing family of equations. ∎

Dually, we may prove that, under syntactic strong family algebraizability and natural finitarity of the equational counterpart $\mathcal{Q}$, every witnessing family of transformations contains a finite witnessing subfamily.

Theorem 1364 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with a naturally finitary equivalent equational* π*-structure* $\mathcal{Q}$*. Then every witnessing collection* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations contains a finite subcollection* $I'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, which is also a witnessing collection.*

Proof: Suppose $\mathcal{I}$ is syntactically strongly family algebraizable, with a naturally finitary equivalent equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$. Let $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ be a collection of witnessing transformations. Then, by definition, there exists a collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D(\langle p^{2,0}, p^{2,1} \rangle_\Sigma[\phi, \psi]) = D(\phi \approx \psi) = D(\tau^\flat[I^\flat_\Sigma[\phi, \psi]]).$$

Since $\mathcal{I}$ is naturally finitary, there exist finite $I'^\flat \subseteq I^\flat$ and $\tau'^\flat \subseteq \tau^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D(\phi \approx \psi) = D(\tau'^\flat[I'^\flat_\Sigma[\phi, \psi]]).$$

Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D(\phi \approx \psi) = D(\tau^\flat[I'^\flat_\Sigma[\phi, \psi]]).$$

Thus, since, by the properties of $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D(\tau^\flat_\Sigma[\Phi]),$$

we get, by Proposition 906, that $I'^\flat$ is also a witnessing family of transformations. ∎

The following proposition asserts that, under similar hypotheses, but adding finiteness of the signature category, the finitarity of $\mathcal{I}$ and of the witnessing collection $I^\flat$ imply the finitarity of the equational counterpart $\mathcal{Q}$.

Proposition 1365 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary, syntactically family algebraizable π-institution, with equivalent equational π-structure $\mathcal{Q}$. If $\mathcal{I}$ has a finite witnessing set $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of transformations, then $\mathcal{Q}$ is also finitary.*

Proof: Suppose $\mathcal{I}$ is a finitary π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically family algebraizable, with equivalent equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$, and that it has a finite witnessing collection $I^\flat$ of transformations. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that

$$\phi \approx \psi \in D_\Sigma(E).$$

Since $I^\flat$ is a witnessing collection of transformations,

$$I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E]).$$

Since $\mathbf{Sign}^\flat$ is finite and $I^\flat$ is finite, we get, by the finitarity of $\mathcal{I}$, that there exists finite $E' \subseteq E$, such that $I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E'])$. Thus, again by the fact that $I^\flat$ is a set of witnessing transformations, we obtain $\phi \approx \psi \in D_\Sigma(E')$. Thus, $\mathcal{Q}$ is indeed finitary. ∎

A similar result can also be established when focus is shifted from finitarity to natural finitarity.

Proposition 1366 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary, syntactically family algebraizable* π*-institution, with equivalent equational* π*-structure* $\mathcal{Q}$*. If* $\mathcal{I}$ *has a finite witnessing set* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations, then* $\mathcal{Q}$ *is also naturally finitary.*

Proof: Suppose $\mathcal{I}$ is a naturally finitary π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically family algebraizable, with equivalent equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$, and that it has a finite witnessing collection $I^\flat$ of transformations. By Proposition 1365, we know that $\mathcal{Q}$ is finitary. Let $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ be collections of natural transformations in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq D(\nu_\Sigma[\vec{\phi}]).$$

Since $I^\flat$ is a witnessing collection of transformations, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat[\mu_\Sigma[\vec{\phi}]] \leq C(I^\flat[\nu_\Sigma[\vec{\phi}]]).$$

But both μ and $I^\flat$ are finite and, also, $\mathbf{Sign}^\flat$ is assumed to be finite. Hence, since $\mathcal{I}$ is naturally finitary, there exists finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat[\mu_\Sigma[\vec{\phi}]] \leq C(I^\flat[\nu'_\Sigma[\vec{\phi}]]).$$

Therefore, again by the fact that $I^\flat$ is a set of witnessing transformations, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq D(\nu'_\Sigma[\vec{\phi}]).$$

Thus, $\mathcal{Q}$ is indeed naturally finitary. ■

We turn, next, to results dual to those established in Propositions 1365 and 1366. We start with a dual to Proposition 1365 to the effect that, if $\mathcal{Q}$ is finitarty and $\mathcal{I}$ has a finite witnessing collection of equations, then $\mathcal{I}$ is itself finitary.

Proposition 1367 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent equational* π*-structure* $\mathcal{Q}$*. If* $\mathcal{Q}$ *is finitary and* $\mathcal{I}$ *has a finite witnessing set* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then* $\mathcal{I}$ *is also finitary.*

Proof: Suppose $\mathcal{I}$ is a π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$, such that

$\mathcal{Q}$ is finitary, and that it has a finite witnessing collection $\tau^\flat$ of equations. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\Phi).$$

Since $\tau^\flat$ is a witnessing collection of equations,

$$\tau^\flat_\Sigma[\phi] \leq D(\tau^\flat_\Sigma[\Phi]).$$

Since $\mathbf{Sign}^\flat$ is finite and $\tau^\flat$ is finite, we get, by the finitarity of $\mathcal{Q}$, that there exists finite $\Phi' \subseteq \Phi$, such that $\tau^\flat_\Sigma[\phi] \leq D(\tau^\flat_\Sigma[\Phi'])$. Thus, again by the fact that $\tau^\flat$ is a set of witnessing equations, we obtain $\phi \in C_\Sigma(\Phi')$. Thus, $\mathcal{I}$ is indeed finitary. ■

A dual of Proposition 1366 addresses the case of natural finitarity.

Proposition 1368 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent equational* π*-structure* $\mathcal{Q}$*. If* $\mathcal{Q}$ *is naturally finitary and* $\mathcal{I}$ *has a finite witnessing set* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then* $\mathcal{I}$ *is also naturally finitary.*

Proof: Suppose $\mathcal{I}$ is a π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent equational π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$, such that $\mathcal{Q}$ is naturally finitary, and that it has a finite witnessing collection $\tau^\flat$ of equations. Let $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ be collections of natural transformations in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}]).$$

Since $\tau^\flat$ is a witnessing collection of equations, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat[\mu_\Sigma[\vec{\phi}]] \leq D(\tau^\flat[\nu_\Sigma[\vec{\phi}]]).$$

But both μ and $\tau^\flat$ are finite and, also, $\mathbf{Sign}^\flat$ is assumed to be finite. Hence, since $\mathcal{Q}$ is naturally finitary, there exists finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat[\mu_\Sigma[\vec{\phi}]] \leq D(\tau^\flat[\nu'_\Sigma[\vec{\phi}]]).$$

Therefore, again by the fact that $\tau^\flat$ is a set of witnessing equations, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}]).$$

Thus, $\mathcal{I}$ is naturally finitary. ■

Finally, we present a syntactic analog of Corollary 668, which summarizes the conclusions drawn from the study of the various finitarity properties, at the center of the investigations carried out in the present chapter.

Corollary 1369 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, via the conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$*.*

(a) If both $\tau^\flat$ *and* $I^\flat$ *are finite, then* $\mathcal{I}$ *is naturally finitary if and only if* $\mathcal{Q}$ *is naturally finitary;*

(b) If $\mathcal{I}$ *is naturally finitary, then* $\mathcal{Q}$ *is naturally finitary if and only if* $I^\flat$ *can be taken to be finite;*

(c) If $\mathcal{Q}$ *is naturally finitary, then* $\mathcal{I}$ *is naturally finitary if and only if* $\tau^\flat$ *can be taken to be finite.*

In each case, if the equivalent alternatives hold, then all four "finitarity" conditions hold.

Proof:

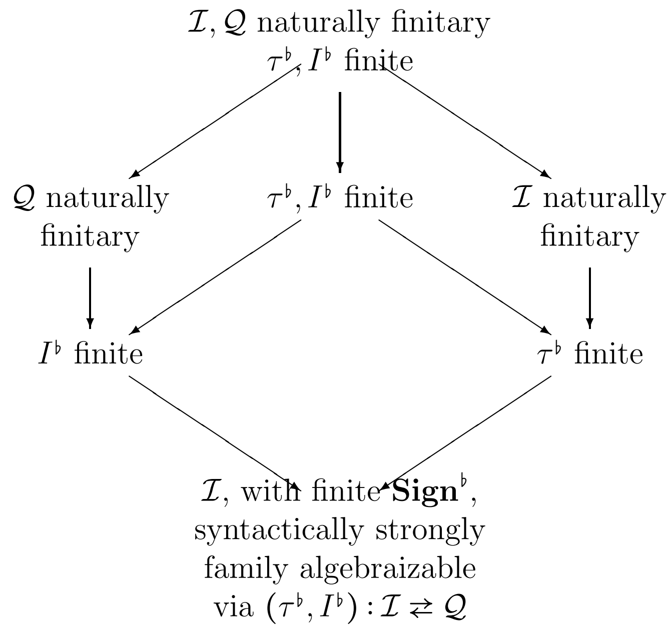
(a) Suppose both $\tau^\flat$ and $I^\flat$ are finite. If $\mathcal{I}$ is naturally finitary, then, by Proposition 1366, $\mathcal{Q}$ is also naturally finitary. If, on the other hand, $\mathcal{Q}$ is naturally finitary, then, by Proposition 1368, $\mathcal{I}$ is naturally finitary.

(b) Assume that $\mathcal{I}$ is naturally finitary. If $\mathcal{Q}$ is naturally finitary, then, by Theorem 1364, $I^\flat$ may be taken to be finite. If, on the other hand, $I^\flat$ can be taken to be finite, then, by Proposition 1366, $\mathcal{Q}$ is naturally finitary.

(c) Assume $\mathcal{Q}$ is naturally finitary. If $\mathcal{I}$ is naturally finitary, then, by Theorem 1363, $\tau^\flat$ may be taken to be finite. If, on the other hand, $\tau^\flat$ may be taken to be finite, then, by Proposition 1368, $\mathcal{I}$ is naturally finitary.

■

In summary, Corollary 1369 establishes the hierarchy depicted below, which parallels in the syntactic context the hierarchy pictured at the end of Chapter 9, concerning the semantic side.


$\mathcal{I}, \mathcal{Q}$ naturally finitary
$\tau^\flat, I^\flat$ finite
$\mathcal{Q}$ naturally finitary
$\tau^\flat, I^\flat$ finite
$\mathcal{I}$ naturally finitary
$I^\flat$ finite
$\tau^\flat$ finite
$\mathcal{I}$, with finite **Sign**$^\flat$, syntactically strongly family algebraizable via $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$