

Chapter 18

Properties of Selected Classes

18.1 Protoalgebraic π -Institutions

18.1.1 The Correspondence Theorem

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} .

Recall that \mathcal{I} is **protoalgebraic** if the Leibniz operator is monotone on theory families, i.e., if, for all $T, T' \in \text{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

Recall, also, that, by Theorem 175, every protoalgebraic π -institution is stable and that, moreover, by Theorem 179, \mathcal{I} is protoalgebraic if and only if, for all \mathbf{F} -algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T').$$

The π -institution \mathcal{I} has the **compatibility property** if, for every \mathbf{F} -algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $T, T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and all $\theta \in \text{ConSys}(\mathbf{A})$,

$$\theta \text{ compatible with } T \quad \text{implies} \quad \theta \text{ compatible with } T'.$$

The π -institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ has the **filter correspondence property** if, for all \mathbf{F} -algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, and surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \rightarrow \mathcal{B}$, with $H : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ an isomorphism,

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle F, \alpha \rangle \swarrow & & \searrow \langle G, \beta \rangle \\ \mathbf{A} & \xrightarrow{\langle H, \gamma \rangle} & \mathbf{B} \end{array}$$

and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\gamma^{-1}(\widehat{\gamma}(T) \vee T') = T \vee \gamma^{-1}(T'),$$

where $\widehat{\gamma}(T) = C^{\mathcal{I}, \mathcal{B}}(\gamma(T))$ is the least \mathcal{I} -filter family on \mathcal{B} that includes $\gamma(T)$.

Our goal is to show that both the compatibility property and the filter correspondence property characterize protoalgebraic π -institutions. We start with a lemma to the effect that, for every \mathcal{I} -filter family T of \mathcal{A} , if the kernel of $\langle H, \gamma \rangle$ happens to be compatible with T , then $\gamma(T)$ is already an \mathcal{I} -filter family of \mathcal{B} and, therefore, $\widehat{\gamma}(T) = \gamma(T)$.

Lemma 1370 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ \mathbf{F} -algebraic systems, and $\langle H, \gamma \rangle : \mathcal{A} \rightarrow \mathcal{B}$ a surjective morphism, with $H : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ an isomorphism. If $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\text{Ker}(\langle H, \gamma \rangle)$ is compatible with T , then $\widehat{\gamma}(T) = \gamma(T)$.*

Proof: By definition, $\gamma(T) \leq \widehat{\gamma}(T)$ always holds. To show the reverse inequality, it suffices to show that $\gamma(T)$, under the hypothesis of the compatibility of $\text{Ker}(\langle H, \gamma \rangle)$ with T , is an \mathcal{I} -filter family of \mathcal{B} . So assume $\Sigma \in |\mathbf{Sign}^b|$, $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, and let $\Sigma' \in |\mathbf{Sign}^b|$, $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$, such that

$$\beta_{\Sigma'}(\text{SEN}^b(f)(\Phi)) \subseteq \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

This gives

$$\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi))) \subseteq \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

By the postulated compatibility of $\text{Ker}(\langle H, \gamma \rangle)$ with T , we obtain

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi)) \subseteq T_{F(\Sigma')}.$$

Since $\phi \in C_\Sigma(\Phi)$ and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) \in T_{F(\Sigma')}.$$

Thus, $\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\text{SEN}^b(f)(\phi))) \in \gamma_{F(\Sigma')}(T_{F(\Sigma')})$, and, therefore,

$$\beta_{\Sigma'}(\text{SEN}^b(f)(\phi)) \in \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

This shows that $\gamma(T) \in \text{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\widehat{\gamma}(T) = \gamma(T)$. ■

Next, we give an equivalent formulation of the Filter Correspondence Property.

Proposition 1371 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} has the filter correspondence property iff*

for all \mathcal{I} -matrix families $\mathfrak{A} = \langle \mathcal{A}', T' \rangle$, $\mathfrak{A}'' = \langle \mathcal{A}'', T'' \rangle$ and strict surjective matrix morphism $\langle H, \gamma \rangle : \mathfrak{A}' \rightarrow \mathfrak{A}''$, with $H : \mathbf{Sign}' \rightarrow \mathbf{Sign}''$ an isomorphism,

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle F', \alpha' \rangle \swarrow & & \searrow \langle F'', \alpha'' \rangle \\ \mathbf{A}' & \xrightarrow{\langle H, \gamma \rangle} & \mathbf{A}'' \end{array}$$

$$T = \gamma^{-1}(\gamma(T)), \quad \text{for all } T' \leq T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}').$$

Proof: Suppose, first, that \mathcal{I} has the Filter Correspondence Property and consider \mathcal{I} -matrix families $\mathfrak{A} = \langle \mathcal{A}', T' \rangle$, $\mathfrak{A}'' = \langle \mathcal{A}'', T'' \rangle$, a strict surjective

matrix morphism $\langle H, \gamma \rangle : \mathfrak{A}' \rightarrow \mathfrak{A}''$, with $H : \mathbf{Sign}' \rightarrow \mathbf{Sign}''$ an isomorphism, and $T' \leq T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Then we have

$$\begin{aligned} \gamma^{-1}(\gamma(T)) &\leq \gamma^{-1}(\widehat{\gamma}(T) \vee T'') \quad (\gamma(T) \leq \widehat{\gamma}(T)) \\ &= T \vee \gamma^{-1}(T'') \quad (\text{Filter Correspondence}) \\ &= T \vee T' \quad (\langle H, \gamma \rangle \text{ strict}) \\ &= T. \quad (T' \leq T \text{ by hypothesis}) \end{aligned}$$

Thus, the displayed property holds. Assume, conversely, that the displayed property holds. We must show that \mathcal{I} has the Filter Correspondence Property. So suppose that $\mathcal{A} = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \langle F'', \alpha'' \rangle \rangle$ are \mathbf{F} -algebraic systems, $\langle H, \gamma \rangle : \mathcal{A}' \rightarrow \mathcal{A}''$ a surjective morphism, with $H : \mathbf{Sign}' \rightarrow \mathbf{Sign}''$ an isomorphism, and let $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}')$ and $T'' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}'')$. Our goal is to show that

$$\gamma^{-1}(\widehat{\gamma}(T') \vee T'') = T' \vee \gamma^{-1}(T'').$$

Notice that $\langle H, \gamma \rangle : \langle \mathcal{A}', \gamma^{-1}(T'') \rangle \rightarrow \langle \mathcal{A}'', T'' \rangle$ is a strict surjective morphism, with H an isomorphism, and $\gamma^{-1}(T'') \leq T' \vee \gamma^{-1}(T'')$. Thus, we fit the setup of the hypothesis, which allows us to conclude that

$$\gamma^{-1}(\gamma(T' \vee \gamma^{-1}(T''))) = T' \vee \gamma^{-1}(T'').$$

So, it suffices, in turn, to show that $\widehat{\gamma}(T') \vee T'' = \gamma(T' \vee \gamma^{-1}(T''))$ and, since, $\text{Ker}(\langle H, \gamma \rangle)$ is compatible with T' (having $\gamma^{-1}(\gamma(T')) = T'$, by hypothesis), it suffices, by Lemma 1370, to show that

$$\gamma(T') \vee T'' = \gamma(T' \vee \gamma^{-1}(T'')).$$

The left to right inclusion is obvious, since $\gamma(T'), T'' \leq \gamma(T' \vee \gamma^{-1}(T''))$. Conversely, note that, taking into account the hypothesis, $T', \gamma^{-1}(T'') \leq \gamma^{-1}(\gamma(T') \vee T'')$. Therefore, $T' \vee \gamma^{-1}(T'') \leq \gamma^{-1}(\gamma(T') \vee T'')$ and, therefore, $\gamma(T' \vee \gamma^{-1}(T'')) \leq \gamma(T') \vee T''$ and, hence, the right to left inclusion also holds. Thus, the Filter Correspondence Property holds. \blacksquare

Now we proceed with the formulation and proof of the main theorem.

Theorem 1372 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \text{SEN}^{\flat}, N^{\flat} \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . The following statements are equivalent:*

- (i) \mathcal{I} is protoalgebraic;
- (ii) \mathcal{I} has the compatibility property;
- (iii) \mathcal{I} has the filter correspondence property.

Proof:

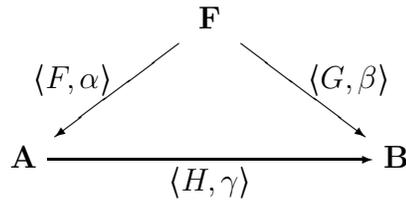
(i)⇒(ii) Suppose \mathcal{I} is protoalgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an \mathbf{F} -algebraic system, $T, T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and $\theta \in \text{ConSys}(\mathbf{A})$, such that θ is compatible with T . Then we have

$$\begin{aligned} \theta &\leq \Omega(T) && \text{(by the compatibility of } \theta \text{ with } T) \\ &\leq \Omega(T'). && \text{(by protoalgebraicity)} \end{aligned}$$

We conclude that θ is also compatible with T' and, hence, \mathcal{I} has the compatibility property.

(ii)⇒(i) Suppose that \mathcal{I} has the compatibility property and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an \mathbf{F} -algebraic system and $T, T' \in \text{ThFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then $\Omega(T) \in \text{ConSys}(\mathbf{A})$ and, by the definition of a Leibniz congruence system, it is compatible with T . Now it follows by the compatibility property, that $\Omega(T)$ is also compatible with T' . Hence $\Omega(T) \leq \Omega(T')$. We conclude that \mathcal{I} is protoalgebraic.

(ii)⇒(iii) Suppose that \mathcal{I} has the compatibility property and consider \mathbf{F} -algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, a commutative triangle



with $H : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ an isomorphism, and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{B})$. Note that it is always the case that

$$T \vee \gamma^{-1}(T') \leq \gamma^{-1}(\widehat{\gamma}(T) \vee T').$$

Thus, it suffices to show that, under the hypothesis of compatibility, the reverse inclusion also holds.

Consider, temporarily, $X \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\gamma^{-1}(T') \leq X$. Since $\text{Ker}(\langle H, \gamma \rangle)$ is compatible with $\gamma^{-1}(T')$, by the postulated compatibility property, it is also compatible with X . Thus, by Lemma 1370, $\widehat{\gamma}(X) = \gamma(X)$. Moreover, we have $T' \leq \gamma(X) = \widehat{\gamma}(X)$.

Now set $X = T \vee \gamma^{-1}(T')$ and reason as follows:

$$\begin{aligned} \gamma^{-1}(\widehat{\gamma}(T) \vee T') &\leq \gamma^{-1}(\widehat{\gamma}(X) \vee T') && (T \leq X) \\ &= \gamma^{-1}(\widehat{\gamma}(X)) && (T' \leq \widehat{\gamma}(X)) \\ &= \gamma^{-1}(\gamma(X)) && (\widehat{\gamma}(X) = \gamma(X)) \\ &= X. && (\text{Ker}(\langle H, \gamma \rangle) \text{ compatible with } X) \end{aligned}$$

So we get $\gamma^{-1}(\widehat{\gamma}(T) \vee T') = T \vee \gamma^{-1}(T')$ and \mathcal{I} has the correspondence property.

- (iii) \Rightarrow (ii) Suppose that \mathcal{I} has the correspondence property and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an \mathbf{F} -algebraic system, $T, T' \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and $\theta \in \mathbf{ConSys}(\mathbf{A})$, such that θ is compatible with T . We look at the commutative diagram

$$\begin{array}{ccc}
 & \mathbf{F} & \\
 \langle F, \alpha \rangle \swarrow & & \searrow \langle F, \pi^\theta \alpha \rangle \\
 \mathbf{A} & \xrightarrow{\langle I, \pi^\theta \rangle} & \mathbf{A}^\theta
 \end{array}$$

and calculate

$$\begin{aligned}
 (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T')) &= (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T') \vee \widehat{\pi^\theta}(T)) \quad (T \leq T') \\
 &= T' \vee (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T)) \quad (\text{correspondence property}) \\
 &\leq T' \vee T \quad (\theta \text{ compatible with } T) \\
 &= T'. \quad (T \leq T')
 \end{aligned}$$

Thus, θ is also compatible with T' and \mathcal{I} has the compatibility property. \blacksquare

As a consequence we obtain the *Correspondence Theorem*, which asserts that, under the same hypothesis, $\langle H, \gamma \rangle$ induces an order isomorphism between the principal filter of the lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ generated by $\gamma^{-1}(T')$ and the principal filter of the lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$ generated by T' .

Theorem 1373 (Correspondence Theorem) *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π -institution based on \mathbf{F} . Let, also, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ be \mathbf{F} -algebraic systems and $\langle H, \gamma \rangle : \mathcal{A} \rightarrow \mathcal{B}$ a surjective morphism, with $H : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ an isomorphism, and $T' \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$. Then, $Y \mapsto \gamma^{-1}(Y)$, $T' \leq Y \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$, defines an order isomorphism $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$.*

Proof: $\gamma^{-1} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \rightarrow \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$ is well defined by Corollary 55 and it is clearly monotone. Furthermore, $\widehat{\gamma} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \rightarrow \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is also well-defined and monotone. So it suffices to show that, for all $T' \leq Y \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$, $\widehat{\gamma}(\gamma^{-1}(Y)) = Y$ and that, for all $\gamma^{-1}(T') \leq X \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\gamma^{-1}(\widehat{\gamma}(X)) = X$.

First, for $T' \leq Y \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$, since $\langle H, \gamma \rangle$ is surjective, $\gamma(\gamma^{-1}(Y)) = Y$ and, therefore, $\widehat{\gamma}(\gamma^{-1}(Y)) = Y$, since $Y \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$. For the other equation, if $\gamma^{-1}(T') \leq X \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have,

$$\begin{aligned}
 \gamma^{-1}(\widehat{\gamma}(X)) &= \gamma^{-1}(\widehat{\gamma}(X) \vee T') \quad (\gamma^{-1}(T') \leq X \Rightarrow T' \leq \widehat{\gamma}(X)) \\
 &= X \vee \gamma^{-1}(T') \quad (\text{correspondence property}) \\
 &= X. \quad (\gamma^{-1}(T') \leq X)
 \end{aligned}$$

So $\gamma^{-1} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$. \blacksquare

18.1.2 The Homomorphism Theorem

We show that, in the case of protoalgebraic π -institutions \mathcal{I} , every surjective morphism of \mathcal{I} -matrix families gives rise to a corresponding surjective morphism between their reductions. This establishes a “reduction” functor and, moreover, gives rise to a version of the Homomorphism Theorem of Universal Algebra.

Recall that, given a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ and an \mathbf{F} -matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we denote by \mathfrak{A}^* the reduction of \mathfrak{A} , i.e.,

$$\mathfrak{A}^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T) \rangle.$$

Moreover, extending this notation, given $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathbf{SEN}(\Sigma)$, we set

$$\phi^* = \phi/\Omega_{\Sigma}^{\mathcal{A}}(T) \in \mathbf{SEN}^*(\Sigma).$$

Theorem 1374 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π -institution based on \mathbf{F} . Further, let $\mathbf{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathbf{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathbf{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathbf{SEN}', N' \rangle$, be \mathbf{F} -algebraic systems, $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ be \mathcal{I} -matrix families and $\langle H, \gamma \rangle : \mathfrak{A} \rightarrow \mathfrak{A}'$ a surjective morphism. Then, there exists a surjective morphism $\langle H, \gamma^* \rangle : \mathfrak{A}^* \rightarrow \mathfrak{A}'^*$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathbf{SEN}(\Sigma)$, by*

$$\gamma_{\Sigma}^*(\phi^*) = \gamma_{\Sigma}(\phi)^*.$$

Proof: First, we show that, for all $\Sigma \in |\mathbf{Sign}|$, $\gamma_{\Sigma}^* : \mathbf{SEN}^*(\Sigma) \rightarrow \mathbf{SEN}'^*(H(\Sigma))$ is well-defined. Indeed, suppose $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathbf{SEN}(\Sigma)$, such that $\phi^* = \psi^*$, i.e., $\langle \phi, \psi \rangle \in \Omega_{\Sigma}^{\mathcal{A}}(T)$. Then, since $T \leq \gamma^{-1}(T')$, we get, by protoalgebraicity, $\langle \phi, \psi \rangle \in \Omega_{\Sigma}^{\mathcal{A}}(\gamma^{-1}(T'))$, whence, by Proposition 24, $\langle \phi, \psi \rangle \in \gamma_{\Sigma}^{-1}(\Omega_{H(\Sigma)}^{\mathcal{A}'}(T'))$, and, hence, $\langle \gamma_{\Sigma}(\phi), \gamma_{\Sigma}(\psi) \rangle \in \Omega_{H(\Sigma)}^{\mathcal{A}'}(T')$, or, equivalently, $\gamma_{\Sigma}(\phi)^* = \gamma_{\Sigma}(\psi)^*$.

Next we see that $\gamma^* : \mathbf{SEN}^* \rightarrow \mathbf{SEN}'^* \circ H$ is a natural transformation. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathbf{SEN}(\Sigma)$. Then we have

$$\begin{array}{ccc} \mathbf{SEN}^*(\Sigma) & \xrightarrow{\gamma_{\Sigma}^*} & \mathbf{SEN}'^*(H(\Sigma)) \\ \mathbf{SEN}^*(f) \downarrow & & \downarrow \mathbf{SEN}'^*(H(f)) \\ \mathbf{SEN}^*(\Sigma') & \xrightarrow{\gamma_{\Sigma'}^*} & \mathbf{SEN}'^*(H(\Sigma')) \end{array}$$

$$\begin{aligned} \mathbf{SEN}'^*(H(f))(\gamma_{\Sigma}^*(\phi^*)) &= \mathbf{SEN}'^*(H(f))(\gamma_{\Sigma}(\phi)^*) \\ &= \mathbf{SEN}'(H(f))(\gamma_{\Sigma}(\phi))^* \\ &= \gamma_{\Sigma'}(\mathbf{SEN}(f)(\phi))^* \\ &= \gamma_{\Sigma'}^*(\mathbf{SEN}(f)(\phi)^*) \\ &= \gamma_{\Sigma'}^*(\mathbf{SEN}^*(f)(\phi^*)). \end{aligned}$$

Surjectivity of $\langle H, \gamma^* \rangle : \mathcal{A}^* \rightarrow \mathcal{A}'^*$ follows from the fact that $\langle H, \gamma \rangle : \mathcal{A} \rightarrow \mathcal{A}'$ is surjective. So it suffices to show that $\langle F', \pi\alpha' \rangle = \langle H, \gamma^* \rangle \circ \langle F, \pi, \alpha \rangle$ and that $\langle H, \gamma^* \rangle : \mathfrak{A}^* \rightarrow \mathfrak{A}'^*$ is a matrix family morphism. The first equation follows from the fact that the upper triangle of the diagram commutes by hypothesis and the rectangle commutes by the definition of $\langle H, \gamma^* \rangle$.

$$\begin{array}{ccc}
 & \mathbf{F} & \\
 \langle F, \alpha \rangle \swarrow & & \searrow \langle F', \alpha' \rangle \\
 \mathbf{A} & \xrightarrow{\langle H, \gamma \rangle} & \mathbf{A}' \\
 \langle I, \pi \rangle \downarrow & & \downarrow \langle I, \pi \rangle \\
 \mathbf{A}^* & \xrightarrow{\langle H, \gamma^* \rangle} & \mathbf{A}'^*
 \end{array}$$

To finish the proof, we calculate, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \text{SEN}(\Sigma)$, $\phi^* \in T_\Sigma / \Omega_\Sigma^{\mathbf{A}}(T)$ if and only if, by compatibility, $\phi \in T_\Sigma$ implies, by hypothesis, $\phi \in \gamma_\Sigma^{-1}(T'_{H(\Sigma)})$ if and only if $\gamma_\Sigma(\phi) \in T'_{H(\Sigma)}$ if and only if, by compatibility, $\gamma_\Sigma(\phi)^* \in T'_{H(\Sigma)} / \Omega_{H(\Sigma)}^{\mathbf{A}'}$ if and only if, by the definition of γ^* , $\gamma_\Sigma^*(\phi^*) \in T'_{H(\Sigma)} / \Omega_{H(\Sigma)}^{\mathbf{A}'}$ if and only if $\phi^* \in (\gamma_\Sigma^*)^{-1}(T'_{H(\Sigma)} / \Omega_{H(\Sigma)}^{\mathbf{A}'})$. ■

We also have the following construction.

Corollary 1375 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π -institution based on \mathbf{F} . Let, also $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \text{SEN}', N' \rangle$, be \mathbf{F} -algebraic systems, $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ an \mathcal{I} -matrix family and $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ a reduced \mathcal{I} -matrix family and $\langle H, \gamma \rangle : \mathfrak{A} \rightarrow \mathfrak{A}'$ a surjective morphism.*

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\langle I, \pi \rangle} & \mathfrak{A}^* \\
 & \searrow \langle H, \gamma \rangle & \downarrow \langle H, \gamma^* \rangle \\
 & & \mathfrak{A}'
 \end{array}$$

There exists a unique surjective morphism $\langle H, \gamma^ \rangle : \mathfrak{A}^* \rightarrow \mathfrak{A}'$ that makes the triangle commute.*

Proof: By Theorem 1374, there exists a surjective matrix morphism $\langle H, \gamma^* \rangle :$

$\mathfrak{A}^* \rightarrow \mathfrak{A}'^*$, such that the following rectangle commutes:

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\langle I, \pi \rangle} & \mathfrak{A}^* \\
 \langle H, \gamma \rangle \downarrow & & \downarrow \langle H, \gamma^* \rangle \\
 \mathfrak{A}' & \xrightarrow{\langle I, \pi \rangle} & \mathfrak{A}'^*
 \end{array}$$

But, by hypothesis, \mathfrak{A}' is reduced, whence $\mathfrak{A}'^* = \mathfrak{A}'$ and $\langle I, \pi \rangle = \langle I, \iota \rangle : \mathfrak{A}' \rightarrow \mathfrak{A}'^*$ is the identity morphism. We now obtain the triangle depicted in the diagram of the statement. ■

Let us denote by $\mathbf{MatFam}(\mathcal{I})$ the category of \mathcal{I} -matrix families with surjective matrix morphisms between them and, similarly, $\mathbf{MatFam}^*(\mathcal{I})$ the category of reduced \mathcal{I} -matrix families with surjective matrix morphisms between them. Then, based on Theorem 1374, we obtain the following functor.

Theorem 1376 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Then*

$$* : \mathbf{MatFam}(\mathcal{I}) \rightarrow \mathbf{MatFam}^*(\mathcal{I})$$

is a functor. The subcategory $\mathbf{MatFam}^(\mathcal{I})$ is a reflective subcategory of $\mathbf{MatFam}(\mathcal{I})$ with $*$ a reflector from $\mathbf{MatFam}(\mathcal{I})$ to $\mathbf{MatFam}^*(\mathcal{I})$.*

Proof: Given $\mathcal{A} \in \mathbf{MatFam}(\mathcal{I})$, it is easy to see that $\langle I, \iota^* \rangle : \mathfrak{A}^* \rightarrow \mathfrak{A}^*$ is the identity matrix morphism. For the composition property, assume $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}'' \in \mathbf{MatFam}(\mathcal{I})$, and $\langle G, \beta \rangle : \mathfrak{A} \rightarrow \mathfrak{A}'$, $\langle H, \gamma \rangle : \mathfrak{A}' \rightarrow \mathfrak{A}''$ be matrix morphisms. Then, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathbf{SEN}(\Sigma)$,

$$\begin{aligned}
 (\gamma_{G(\Sigma)} \circ \beta_{\Sigma})^*(\phi^*) &= \gamma_{G(\Sigma)}(\beta_{\Sigma}(\phi))^* \\
 &= \gamma_{G(\Sigma)}^*(\beta_{\Sigma}(\phi)^*) \\
 &= \gamma_{G(\Sigma)}^*(\beta_{\Sigma}^*(\phi^*)).
 \end{aligned}$$

Thus, $(\langle H, \gamma \rangle \circ \langle G, \beta \rangle)^* = \langle H, \gamma \rangle^* \circ \langle G, \beta \rangle^*$. Therefore, $* : \mathbf{MatFam}(\mathcal{I}) \rightarrow \mathbf{MatFam}^*(\mathcal{I})$ is a functor.

As far as reflectivity is concerned, for every $\mathfrak{A} \in \mathbf{MatFam}^*(\mathcal{I})$, we consider the natural quotient morphism $\langle I, \pi \rangle : \mathfrak{A} \rightarrow \mathfrak{A}^*$. Given reduced $\mathfrak{B} \in \mathbf{MatFam}^*(\mathcal{I})$ and a surjective morphism $\langle H, \gamma \rangle : \mathfrak{A} \rightarrow \mathfrak{B}$, the surjective morphism $\langle H, \gamma^* \rangle : \mathfrak{A}^* \rightarrow \mathfrak{B}$ of Corollary 1375 is the unique surjective morphism such that the following diagram commutes.

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\langle I, \pi \rangle} & \mathfrak{A}^* \\
 \searrow \langle H, \gamma \rangle & & \downarrow \langle H, \gamma^* \rangle \\
 & & \mathfrak{B}
 \end{array}$$

Thus, $\mathbf{MatFam}^*(\mathcal{I})$ is a reflective subcategory of $\mathbf{MatFam}(\mathcal{I})$ with $*$ a reflector from $\mathbf{MatFam}(\mathcal{I})$ to $\mathbf{MatFam}^*(\mathcal{I})$. \blacksquare

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π -institution based on \mathbf{F} . Given an \mathcal{I} -matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we denote by $\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A})$ the principal filter of the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ generated by the \mathcal{I} -filter family T :

$$\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \{T' \in \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) : T \leq T'\}.$$

Recall that this set is also denoted by $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, without explicit reference to the matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$.

The Correspondence Theorem allows us to prove the following.

Theorem 1377 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π -institution based on \mathbf{F} . For every \mathcal{I} -matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}) \cong \mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}^*).$$

Proof: By The Correspondence Theorem 1373, with $\mathcal{B} = \mathcal{A}/\Omega^{\mathcal{A}}(T)$, $\langle H, \gamma \rangle = \langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\Omega^{\mathcal{A}}(T)$ and $T' = T/\Omega^{\mathcal{A}}(T)$, we get

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))^{T/\Omega^{\mathcal{A}}(T)} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\pi^{-1}(T/\Omega^{\mathcal{A}}(T))}.$$

But this amounts to $\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}^*) \cong \mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A})$. \blacksquare

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Consider \mathbf{F} -matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$ and a surjective matrix morphism $\langle H, \gamma \rangle : \mathfrak{A} \rightarrow \mathfrak{B}$. By definition, we have $T \leq \gamma^{-1}(T')$. We call $\gamma^{-1}(T')$ the **filter kernel** of $\langle H, \gamma \rangle$. By the inclusion relation above, we can see that, if $\mathfrak{B} \in \mathbf{MatFam}(\mathcal{I})$, then $\gamma^{-1}(T') \in \mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A})$.

Given $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $X \in \mathbf{SenFam}(\mathcal{A})$, such that $T \leq X$, we define

$$\mathfrak{A}/X := \langle \mathcal{A}, X \rangle^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(X), X/\Omega^{\mathcal{A}}(X) \rangle.$$

We call \mathfrak{A}/X the **quotient of \mathfrak{A} by X** . We note that, if $\mathfrak{A} \in \mathbf{MatFam}(\mathcal{I})$, then

$$X \in \mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}) \quad \text{iff} \quad \mathfrak{A}/X \in \mathbf{MatFam}^*(\mathcal{I}).$$

The following is an analog in the context of \mathcal{I} -matrix families of the Homomorphism Theorem of Universal Algebra.

Theorem 1378 (Homomorphism Theorem) *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π -institution based on \mathbf{F} . Let also $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle \in \mathbf{MatFam}(\mathcal{I})$ and $\langle H, \gamma \rangle : \mathfrak{A} \rightarrow \mathfrak{A}'$ a surjective morphism.*

- (i) *There exists a strict surjective morphism $\langle H, \gamma' \rangle : \mathfrak{A}/\gamma^{-1}(T') \rightarrow \mathfrak{A}'^*$ with isomorphic components;*
- (ii) *If $X \in \text{FiFam}^{\mathcal{I}}(\mathfrak{A})$ and $X \leq \gamma^{-1}(T')$, then, there exists a surjective morphism $\langle H, \gamma^X \rangle : \mathfrak{A}/X \rightarrow \mathfrak{A}'^*$, such that*

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\langle H, \gamma \rangle} & \mathfrak{A}' \\
 \langle I, \pi^X \rangle \downarrow & & \downarrow \langle I, \pi \rangle \\
 \mathfrak{A}/X & \xrightarrow{\langle H, \gamma^X \rangle} & \mathfrak{A}'^*
 \end{array}$$

$$\langle H, \gamma^X \rangle \circ \langle I, \pi^X \rangle = \langle I, \pi \rangle \circ \langle H, \gamma \rangle.$$

Proof:

- (i) First, note that $\gamma^{-1}(T') \leq \gamma^{-1}(T')$, whence, $\langle H, \gamma \rangle : \langle \mathcal{A}, \gamma^{-1}(T') \rangle \rightarrow \mathfrak{A}'$ is also a surjective matrix morphism. Thus, taking into account that $T \leq \gamma^{-1}(T')$, we get, by Theorem 1374, a surjective matrix morphism $\langle H, \gamma^* \rangle : \mathfrak{A}/\gamma^{-1}(T') \rightarrow \mathfrak{A}'^*$, such that the following diagram commutes.

$$\begin{array}{ccc}
 \langle \mathcal{A}, \gamma^{-1}(T') \rangle & \xrightarrow{\langle H, \gamma \rangle} & \mathfrak{A}' \\
 \langle I, \pi \rangle \downarrow & & \downarrow \langle I, \pi \rangle \\
 \mathfrak{A}/\gamma^{-1}(T') & \xrightarrow{\langle H, \gamma^* \rangle} & \mathfrak{A}'^*
 \end{array}$$

It remains to show that, for every $\Sigma \in |\mathbf{Sign}|$,

$$\gamma_{\Sigma}^* : \text{SEN}^{\gamma^{-1}(T')}(\Sigma) \rightarrow \text{SEN}'^*(H(\Sigma))$$

is a bijection and that $\langle H, \gamma^* \rangle$ is strict. To see that γ_{Σ}^* is a bijection, let $\phi, \psi \in \text{SEN}(\Sigma)$, such that

$$\gamma_{\Sigma}^*(\phi/\Omega_{\Sigma}^{\mathcal{A}}(\gamma^{-1}(T'))) = \gamma_{\Sigma}^*(\psi/\Omega_{\Sigma}^{\mathcal{A}}(\gamma^{-1}(T'))).$$

Then, by the commutativity of the rectangle,

$$\gamma_{\Sigma}(\phi)/\Omega_{H(\Sigma)}^{\mathcal{A}'}(T') = \gamma_{\Sigma}(\psi)/\Omega_{H(\Sigma)}^{\mathcal{A}'}(T').$$

This gives that $\langle \phi, \psi \rangle \in \gamma_{\Sigma}^{-1}(\Omega_{H(\Sigma)}^{\mathcal{A}'}(T'))$. Thus, $\langle \phi, \psi \rangle \in \Omega_{\Sigma}^{\mathcal{A}}(\gamma^{-1}(T'))$ and, hence,

$$\phi/\Omega_{\Sigma}^{\mathcal{A}}(\gamma^{-1}(T')) = \psi/\Omega_{\Sigma}^{\mathcal{A}}(\gamma^{-1}(T')).$$

Therefore, $\gamma_{\Sigma}^* : \text{SEN}^{\gamma^{-1}(T')}(\Sigma) \rightarrow \text{SEN}'^*(H(\Sigma))$ is a bijection, for all $\Sigma \in |\mathbf{Sign}|$.

To prove strictness, assume $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \text{SEN}(\Sigma)$, such that $\phi/\Omega_{\Sigma}^A(\gamma^{-1}(T')) \in \gamma_{\Sigma}^{*-1}(T'_{H(\Sigma)}/\Omega_{H(\Sigma)}^{A'}(T'))$. Then $\gamma_{\Sigma}^*(\phi/\Omega_{\Sigma}^A(\gamma^{-1}(T'))) \in T'_{H(\Sigma)}/\Omega_{H(\Sigma)}^{A'}(T')$. Hence, by the definition of γ^* , we get $\gamma_{\Sigma}(\phi)^* \in T'_{H(\Sigma)}/\Omega_{H(\Sigma)}^{A'}(T')$. By compatibility, we obtain $\gamma_{\Sigma}(\phi) \in T'_{H(\Sigma)}$, whence $\phi \in \gamma_{\Sigma}^{-1}(T'_{H(\Sigma)})$. This, finally, yields

$$\phi/\Omega_{\Sigma}^A(\gamma^{-1}(T')) \in \gamma_{\Sigma}^{-1}(T'_{H(\Sigma)})/\Omega_{\Sigma}^A(\gamma^{-1}(T')),$$

proving strictness.

(ii) This part is proven by the following diagram chase:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle H, \gamma \rangle} & \mathfrak{A}' \\ \langle I, \pi^X \rangle \downarrow & & \downarrow \langle I, \pi \rangle \\ \mathfrak{A}/X & \xrightarrow{\langle I, \pi \rangle} \mathfrak{A}/\gamma^{-1}(T') \xrightarrow{\langle H, \gamma' \rangle} & \mathfrak{A}'^* \end{array}$$

where $\langle I, \pi \rangle : \mathfrak{A}/X \rightarrow \mathfrak{A}/\gamma^{-1}(T')$ is the canonical projection morphism, defined because of the hypothesis $X \leq \gamma^{-1}(T')$ and protoalgebraicity, and

$$\langle H, \gamma' \rangle : \mathfrak{A}/\gamma^{-1}(T') \rightarrow \mathfrak{A}'^*$$

is the morphism obtained in Part (i). ■

18.2 Pointed Classes of Algebraic Systems

Proposition 1379 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π -institution based on \mathbf{F} , having theorems. Then the following conditions are equivalent:*

- (i) \mathcal{I} is family regular;
- (ii) For all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^b|$ and all $\phi \in T_{\Sigma}$, $T_{\Sigma} = \phi/\Omega_{\Sigma}(T)$;
- (iii) For all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^b|$ and all $t \in \text{Thm}_{\Sigma}(\mathcal{I})$, $T_{\Sigma} = t/\Omega_{\Sigma}(T)$;
- (iv) For all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^b|$ and some $\phi \in T_{\Sigma}$, $T_{\Sigma} = \phi/\Omega_{\Sigma}(T)$;
- (v) For all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^b|$ and some $t \in \text{Thm}_{\Sigma}(\mathcal{I})$, $T_{\Sigma} = t/\Omega_{\Sigma}(T)$.

Proof:

(i) \Rightarrow (ii) Suppose \mathcal{I} is family regular and let $T \in \text{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^b|$ and $\phi \in T_\Sigma$. Then, we have, for all $\psi \in \text{SEN}^b(\Sigma)$,

$$\begin{array}{llll}
\psi \in T_\Sigma & \text{iff} & \phi, \psi \in T_\Sigma & (\phi \in T_\Sigma) \\
& & \text{iff} & C(\phi, \psi) \leq T & (\text{definition of } C(\phi, \psi)) \\
& & \text{implies} & \Omega_\Sigma(C(\phi, \psi)) \leq \Omega_\Sigma(T) & (\mathcal{I} \text{ protoalgebraic}) \\
& & \text{implies} & \langle \phi, \psi \rangle \in \Omega_\Sigma(T) & (\mathcal{I} \text{ family regular}) \\
& & \text{iff} & \psi \in \phi/\Omega_\Sigma(T). & (\text{definition of } \phi/\Omega_\Sigma(T))
\end{array}$$

On the other hand, if $\psi \in \phi/\Omega_\Sigma(T)$, then, since $\phi \in T_\Sigma$, by the compatibility of $\Omega(T)$ with T , $\psi \in T_\Sigma$. Thus, we conclude that $T_\Sigma = \phi/\Omega_\Sigma(T)$.

(ii) \Rightarrow (iii) Suppose (ii) holds and let $T \in \text{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^b|$ and $t \in \text{Thm}_\Sigma(\mathcal{I})$. Then, since $\text{Thm}(\mathcal{I}) \leq T$, we get that $t \in T_\Sigma$ and, hence, by hypothesis, $T_\Sigma = t/\Omega_\Sigma(T)$.

(iii) \Rightarrow (iv) Assume (iii) holds and let $T \in \text{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^b|$. Since \mathcal{I} has theorems, there exists $t \in \text{Thm}_\Sigma(\mathcal{I})$. Then, $t \in T_\Sigma$ and, by hypothesis, $T_\Sigma = t/\Omega_\Sigma(T)$.

(iv) \Rightarrow (v) Assume (iv) holds and let $T \in \text{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^b|$. Then, by hypothesis, there exists $\phi \in T_\Sigma$, such that $T_\Sigma = \phi/\Omega_\Sigma(T)$. Moreover, \mathcal{I} has theorems, whence, there exists $t \in \text{Thm}_\Sigma(\mathcal{I})$. Then, we have $t \in T_\Sigma = \phi/\Omega_\Sigma(T)$, whence $\langle \phi, t \rangle \in \Omega_\Sigma(T)$ and, therefore, $T_\Sigma = \phi/\Omega_\Sigma(T) = t/\Omega_\Sigma(T)$.

(v) \Rightarrow (i) Assume that (v) holds and let $T \in \text{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^b|$ and $\phi, \psi \in T_\Sigma$. By hypothesis, for some $t \in \text{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$. Hence, $\phi, \psi \in t/\Omega_\Sigma(T)$, i.e., $\langle \phi, t \rangle \in \Omega_\Sigma(T)$ and $\langle t, \psi \rangle \in \Omega_\Sigma(T)$. By transitivity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, \mathcal{I} is family regular. ■

We show now that a protoalgebraic family assertional π -institution \mathcal{I} is weakly family algebraizable.

Theorem 1380 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π -institution based on \mathbf{F} . If \mathcal{I} is family assertional, then it is weakly family algebraizable.*

Proof: By Definition 613, protoalgebraicity and family assertional are equivalent to regular weak family algebraizability. By Proposition 620, this entails weak family algebraizability.

More directly, assume \mathcal{I} is family assertional. Since it is protoalgebraic, by hypothesis, it suffices to show that \mathcal{I} is family injective. To this end, let $T, T' \in \text{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^b|$ and all $t \in \text{Thm}_\Sigma(\mathcal{I})$,

$$T_\Sigma = t/\Omega_\Sigma(T) = t/\Omega_\Sigma(T') = T'_\Sigma.$$

Therefore $T = T'$. Hence \mathcal{I} is family injective and, therefore, it is weakly family algebraizable. \blacksquare

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a class of \mathbf{F} -algebraic systems. We say that \mathbf{K} is τ^b -**pointed** if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathbf{K}$, with $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \text{SEN}(\Sigma)$,

$$\tau_{\Sigma}^{\mathcal{A}}(\vec{\phi}) = \tau_{\Sigma}^{\mathcal{A}}(\vec{\psi}).$$

\mathbf{K} is called **pointed** if it is τ^b -pointed with respect to some τ^b in N^b .

If a class \mathbf{K} is pointed, then, for every $\mathcal{A} \in \mathbf{K}$, we write $\tau^{\mathcal{A}} = \{\tau_{\Sigma}^{\mathcal{A}}\}_{\Sigma \in |\mathbf{Sign}|}$, where $\tau_{\Sigma}^{\mathcal{A}} := \tau_{\Sigma}^{\mathcal{A}}(\vec{\phi})$, for some $\vec{\phi} \in \text{SEN}(\Sigma)$, this value being independent of the choice of $\vec{\phi} \in \text{SEN}(\Sigma)$.

We focus now on protoalgebraic, family regular π -institutions that have natural theorems. Recall that this means that there exists a natural transformation τ^b in N^b , such that τ^b is evaluated to a theorem in every signature and at all tuples of sentences. Of course, by definition, all π -institutions that fit this description are regularly weakly family algebraizable. We show that for such π -institutions, the class $\text{AlgSys}^*(\mathcal{I})$ of their reduced algebraic systems is a pointed class of \mathbf{F} -algebraic systems, where any natural theorem may serve as the ‘‘point’’.

Proposition 1381 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic family regular π -institution based on \mathbf{F} , having natural theorems. Then, the class $\text{AlgSys}^*(\mathcal{I})$ is a pointed class of \mathbf{F} -algebraic systems.*

Proof: Suppose \mathcal{I} is protoalgebraic and family regular, with a natural theorem $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^b|$ and all $\vec{\phi} \in \text{SEN}^b(\Sigma)$, $\tau_{\Sigma}^b(\vec{\phi}) \in \text{Thm}_{\Sigma}(\mathcal{I})$. By family regularity, we have, for all $\Sigma \in |\mathbf{Sign}^b|$ and all $\vec{\phi}, \vec{\psi} \in \text{SEN}^b(\Sigma)$, $\langle \tau_{\Sigma}^b(\vec{\phi}), \tau_{\Sigma}^b(\vec{\psi}) \rangle \in \Omega_{\Sigma}(\text{Thm}(\mathcal{I}))$. Now, let $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Therefore, for all $\Sigma \in |\mathbf{Sign}^b|$ and all $\vec{\phi}, \vec{\psi} \in \text{SEN}^b(\Sigma)$, we get, by what was shown above and protoalgebraicity, and taking into account Lemma 51, $\langle \tau_{\Sigma}^b(\vec{\phi}), \tau_{\Sigma}^b(\vec{\psi}) \rangle \in \Omega_{\Sigma}(\alpha^{-1}(T))$, whence, by Proposition 24, $\langle \tau_{\Sigma}^b(\vec{\phi}), \tau_{\Sigma}^b(\vec{\psi}) \rangle \in \alpha_{\Sigma}^{-1}(\Omega_{F(\Sigma)}^{\mathcal{A}}(T))$. Thus,

$$\langle \alpha_{\Sigma}(\tau_{\Sigma}^b(\vec{\phi})), \alpha_{\Sigma}(\tau_{\Sigma}^b(\vec{\psi})) \rangle \in \Omega_{F(\Sigma)}^{\mathcal{A}}(T) = \Delta_{F(\Sigma)}^{\mathcal{A}},$$

i.e., since τ^b is a natural transformation, $\tau_{F(\Sigma)}^{\mathcal{A}}(\alpha_{\Sigma}(\vec{\phi})) = \tau_{F(\Sigma)}^{\mathcal{A}}(\alpha_{\Sigma}(\vec{\psi}))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\text{AlgSys}^*(\mathcal{I})$ is a pointed class of \mathbf{F} -algebraic systems, with any natural theorem serving as the ‘‘point’’ natural transformation. \blacksquare

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b and \mathbf{K} a τ^b -pointed class of \mathbf{F} -algebraic systems. We say that \mathbf{K}

is **relatively point regular** if, for every $\theta, \theta' \in \text{ConSys}^{\mathbf{K}}(\mathcal{F})$,

$$\tau^b/\theta = \tau^b/\theta' \quad \text{implies} \quad \theta = \theta'.$$

It is not difficult to show that the defining property transfers from \mathbf{K} -congruence systems on \mathcal{F} to \mathbf{K} -congruence systems on every \mathbf{F} -algebraic system, under the proviso that \mathbf{K} be an abstract class.

Lemma 1382 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ a natural transformation in N^b , and \mathbf{K} a τ^b -pointed abstract class of \mathbf{F} -algebraic systems. If \mathbf{K} is relatively point regular, then, for every \mathbf{F} -algebraic system \mathcal{A} and all $\theta, \theta' \in \text{ConSys}^{\mathbf{K}}(\mathcal{A})$,*

$$\tau^{\mathcal{A}}/\theta = \tau^{\mathcal{A}}/\theta' \quad \text{implies} \quad \theta = \theta'.$$

Proof: Suppose \mathcal{A} is an \mathbf{F} -algebraic system, $\theta, \theta' \in \text{ConSys}^{\mathbf{K}}(\mathcal{A})$, such that $\tau^{\mathcal{A}}/\theta = \tau^{\mathcal{A}}/\theta'$. Then, for all $\Sigma \in |\mathbf{Sign}^b|$ and $\phi \in \text{SEN}^b(\Sigma)$,

$$\begin{aligned} \langle \phi, \tau_{\Sigma}^b \rangle \in \alpha_{\Sigma}^{-1}(\theta_{F(\Sigma)}) & \quad \text{iff} \quad \langle \alpha_{\Sigma}(\phi), \alpha_{\Sigma}(\tau_{\Sigma}^b) \rangle \in \theta_{F(\Sigma)} \\ & \quad \text{iff} \quad \langle \alpha_{\Sigma}(\phi), \tau_{F(\Sigma)}^{\mathcal{A}} \rangle \in \theta_{F(\Sigma)} \\ & \quad \text{iff} \quad \langle \alpha_{\Sigma}(\phi), \tau_{F(\Sigma)}^{\mathcal{A}} \rangle \in \theta'_{F(\Sigma)} \\ & \quad \text{iff} \quad \langle \alpha_{\Sigma}(\phi), \alpha_{\Sigma}(\tau_{\Sigma}^b) \rangle \in \theta'_{F(\Sigma)} \\ & \quad \text{iff} \quad \langle \phi, \tau_{\Sigma}^b \rangle \in \alpha_{\Sigma}^{-1}(\theta'_{F(\Sigma)}). \end{aligned}$$

Thus, $\tau^b/\alpha^{-1}(\theta) = \tau^b/\alpha^{-1}(\theta')$. Since \mathbf{K} is abstract and $\mathcal{A}/\theta, \mathcal{A}/\theta' \in \mathbf{K}$, we get that $\mathcal{F}/\alpha^{-1}(\theta), \mathcal{F}/\alpha^{-1}(\theta') \in \mathbf{K}$. It follows that $\alpha^{-1}(\theta), \alpha^{-1}(\theta') \in \text{ConSys}^{\mathbf{K}}(\mathcal{F})$. Since \mathbf{K} is relatively point regular, by definition, $\alpha^{-1}(\theta) = \alpha^{-1}(\theta')$. Therefore, by surjectivity of $\langle F, \alpha \rangle$, $\theta = \theta'$. ■

Moreover, we can show that, for a protoalgebraic family regular π -institution \mathcal{I} , having natural theorems, the associated class $\text{AlgSys}^*(\mathcal{I})$ of its reduced algebraic systems is a relatively point regular class.

Proposition 1383 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic family regular π -institution based on \mathbf{F} , having natural theorems. Then, the class $\text{AlgSys}^*(\mathcal{I})$ is a relatively point regular class of \mathbf{F} -algebraic systems.*

Proof: We know, by Proposition 1381, that $\text{AlgSys}^*(\mathcal{I})$ is pointed, with any natural theorem τ^b serving as a “point”. Consider $\theta, \theta' \in \text{ConSys}^*(\mathcal{I})$, such that $\tau^b/\theta = \tau^b/\theta'$. Since $\theta, \theta' \in \text{ConSys}^*(\mathcal{I})$, there exist $T, T' \in \text{ThFam}(\mathcal{I})$, such that $\theta = \Omega(T)$ and $\theta' = \Omega(T')$. But then, since \mathcal{I} is protoalgebraic and family regular, with theorems, we get, by Proposition 1379,

$$\begin{aligned} \theta &= \Omega(T) \\ &= \Omega(\tau^b/\Omega(T)) \quad (\text{Proposition 1379}) \\ &= \Omega(\tau^b/\theta) \\ &= \Omega(\tau^b/\theta') \quad (\text{hypothesis}) \\ &= \Omega(\tau^b/\Omega(T')) \\ &= \Omega(T') \quad (\text{Proposition 1379}) \\ &= \theta'. \end{aligned}$$

Hence $\text{AlgSys}^*(\mathcal{I})$ is indeed relatively point regular. \blacksquare

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a τ^b -pointed class of \mathbf{F} -algebraic systems. Define on \mathbf{F} the family $C^{\mathbf{K}, \tau} = \{C_{\Sigma}^{\mathbf{K}, \tau}\}_{\Sigma \in |\mathbf{Sign}^b|}$, by letting, for all $\Sigma \in |\mathbf{Sign}^b|$,

$$C_{\Sigma}^{\mathbf{K}, \tau} : \mathcal{P}(\text{SEN}^b(\Sigma)) \rightarrow \mathcal{P}(\text{SEN}^b(\Sigma)),$$

be given, for all $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, by

$$\phi \in C_{\Sigma}^{\mathbf{K}, \tau}(\Phi) \quad \text{iff} \quad \phi \approx \tau_{\Sigma}^b \in C_{\Sigma}^{\mathbf{K}}(\Phi \approx \tau_{\Sigma}^b),$$

i.e., $\phi \in C_{\Sigma}^{\mathbf{K}, \tau}(\Phi)$ if and only if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathbf{K}$, all $\Sigma' \in |\mathbf{Sign}^b|$ and all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi)) \subseteq \{\tau_{F(\Sigma')}^{\mathcal{A}}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) = \tau_{F(\Sigma')}^{\mathcal{A}}.$$

In the next proposition, it is shown that $C^{\mathbf{K}, \tau}$ is a closure system on \mathbf{F} . In this way the pointed class \mathbf{K} of \mathbf{F} -algebraic systems defines a bona fide π -institution based on \mathbf{F} .

Proposition 1384 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a τ^b -pointed class of \mathbf{F} -algebraic systems. $C^{\mathbf{K}, \tau}$ is a closure system on \mathbf{F} .*

Proof: Let $\Sigma \in |\mathbf{Sign}^b|$. It is obvious from the definition that

$$C_{\Sigma}^{\mathbf{K}, \tau} : \mathcal{P}(\text{SEN}^b(\Sigma)) \rightarrow \mathcal{P}(\text{SEN}^b(\Sigma))$$

is inflationary and monotone. To show that it is also idempotent, let $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, such that $\phi \in C_{\Sigma}^{\mathbf{K}, \tau}(C_{\Sigma}^{\mathbf{K}, \tau}(\Phi))$. Thus, we have, by definition, for all $\mathcal{A} \in \mathbf{K}$, all $\Sigma' \in |\mathbf{Sign}^b|$ and all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(C_{\Sigma}^{\mathbf{K}, \tau}(\Phi))) \subseteq \{\tau_{F(\Sigma')}^{\mathcal{A}}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) = \tau_{F(\Sigma')}^{\mathcal{A}}.$$

But, also by definition, we have, for all $\Sigma' \in |\mathbf{Sign}^b|$ and all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi)) \subseteq \{\tau_{F(\Sigma')}^{\mathcal{A}}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}^b(f)(C_{\Sigma}^{\mathbf{K}, \tau}(\Phi))) \subseteq \{\tau_{F(\Sigma')}^{\mathcal{A}}\}.$$

Therefore, we get that, for all $\Sigma' \in |\mathbf{Sign}^b|$ and all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi)) \subseteq \{\tau_{F(\Sigma')}^{\mathcal{A}}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) = \tau_{F(\Sigma')}^{\mathcal{A}},$$

showing that $\phi \in C_{\Sigma}^{\mathbf{K}, \tau}(\Phi)$.

It remains, finally, to show that $C^{\mathbf{K}, \tau}$ is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^b|$, $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, such that $\phi \in C_{\Sigma}^{\mathbf{K}, \tau}(\Phi)$. Consider $\mathcal{A} \in \mathbf{K}$, such that, for all $\Sigma'' \in |\mathbf{Sign}^b|$ and all $g \in \mathbf{Sign}^b(\Sigma', \Sigma'')$,

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

$\alpha_{\Sigma''}(\text{SEN}^b(g)(\text{SEN}^b(f)(\Phi))) \subseteq \{\tau_{F(\Sigma'')}^A\}$. This gives $\alpha_{\Sigma''}(\text{SEN}^b(gf)(\Phi)) \subseteq \{\tau_{F(\Sigma'')}^A\}$, whence, by hypothesis, $\alpha_{\Sigma''}(\text{SEN}^b(gf)(\phi)) = \tau_{F(\Sigma'')}^A$. Thus, for all $\Sigma'' \in |\mathbf{Sign}^b|$ and all $g \in \mathbf{Sign}^b(\Sigma', \Sigma'')$, $\alpha_{\Sigma''}(\text{SEN}^b(g)(\text{SEN}^b(f)(\phi))) = \tau_{F(\Sigma'')}^A$. We conclude that $\text{SEN}^b(f)(\phi) \in C_{\Sigma'}^{\mathbf{K}, \tau}(\text{SEN}^b(f)(\Phi))$ and, therefore, $C^{\mathbf{K}, \tau}$ is also structural. ■

Based on Proposition 1384, it makes sense, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a τ^b -pointed class of \mathbf{F} -algebraic systems, to define the **assertional π -institution of \mathbf{K}** as the pair

$$\mathcal{I}^{\mathbf{K}, \tau} = \langle \mathbf{F}, C^{\mathbf{K}, \tau} \rangle.$$

We have seen in Proposition 1383 that, if $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a protoalgebraic and family regular π -institution, having natural theorems, then its class $\text{AlgSys}^*(\mathcal{I})$ of reduced \mathbf{F} -algebraic systems is a relatively point regular class. We show next, in a form of converse, that if \mathbf{K} is a relatively point regular quasivariety of \mathbf{F} -algebraic systems, then the assertional π -institution $\mathcal{I}^{\mathbf{K}, \tau}$, associated with \mathbf{K} , is a protoalgebraic family regular π -institution that has natural theorems.

First, we establish possession of natural theorems, under the assumption that \mathbf{K} is pointed.

Proposition 1385 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a pointed class of \mathbf{F} -algebraic systems. Then $\mathcal{I}^{\mathbf{K}, \tau}$ has natural theorems.*

Proof: Let \mathbf{K} be a pointed class of \mathbf{F} -algebraic systems. Since \mathbf{K} is pointed, there exists $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , such that, for all $\Sigma \in |\mathbf{Sign}^b|$ and all $\vec{\phi} \in \text{SEN}^b(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}^b|$ and all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$,

$$\tau_{F(\Sigma')}^A(\alpha_{\Sigma'}(\text{SEN}^b(f)(\vec{\phi}))) = \tau_{F(\Sigma')}^A.$$

This implies that $\alpha_{\Sigma'}(\tau_{\Sigma'}^b(\text{SEN}^b(f)(\vec{\phi}))) = \tau_{F(\Sigma')}^A$ and, hence, we obtain $\alpha_{\Sigma'}(\text{SEN}^b(f)(\tau_{\Sigma'}^b(\vec{\phi}))) = \tau_{F(\Sigma')}^A$. Thus, by definition, $\tau_{\Sigma'}^b(\vec{\phi}) \in C_{\Sigma'}^{\mathbf{K}, \tau}(\emptyset)$ and, therefore, τ^b is a natural theorem. ■

Next, we turn to proving family regularity, again under the assumption of pointedness.

Proposition 1386 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a pointed class of \mathbf{F} -algebraic systems. Then $\mathcal{I}^{\mathbf{K}, \tau}$ is a family regular π -institution.*

Proof: Let \mathbf{K} be a pointed class of \mathbf{F} -algebraic systems. We know, by Proposition 1385, that $\mathcal{I}^{\mathbf{K}, \tau}$ has a natural theorem τ^b , where τ^b is a point

in \mathbf{K} . We show that $\mathcal{I}^{\mathbf{K},\tau}$ is family regular. To this end, let $\Sigma \in |\mathbf{Sign}^b|$ and $\phi, \psi \in \text{SEN}^b(\Sigma)$. Then, for all $\Sigma' \in |\mathbf{Sign}^b|$ and $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$, we have

$$\text{SEN}^b(f)(\phi) \approx \tau_{\Sigma'}^b, \text{SEN}^b(f)(\psi) \approx \tau_{\Sigma'}^b \in C_{\Sigma'}^{\mathbf{K}}(\phi \approx \tau_{\Sigma}^b, \psi \approx \tau_{\Sigma}^b).$$

This implies that, for all σ^b in N^b and all $\bar{\chi} \in \text{SEN}^b(\Sigma')$,

$$\sigma_{\Sigma'}^b(\text{SEN}^b(f)(\phi), \bar{\chi}) \approx \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\psi), \bar{\chi}) \in C_{\Sigma'}^{\mathbf{K}}(\phi \approx \tau_{\Sigma}^b, \psi \approx \tau_{\Sigma}^b).$$

Now we get

$$\begin{aligned} \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\phi), \bar{\chi}) &\approx \tau_{\Sigma'}^b \in C_{\Sigma'}^{\mathbf{K}}(\phi \approx \tau_{\Sigma}^b, \psi \approx \tau_{\Sigma}^b) \\ \text{iff } \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\psi), \bar{\chi}) &\approx \tau_{\Sigma'}^b \in C_{\Sigma'}^{\mathbf{K}}(\phi \approx \tau_{\Sigma}^b, \psi \approx \tau_{\Sigma}^b). \end{aligned}$$

Hence, by definition,

$$\begin{aligned} \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\phi), \bar{\chi}) &\in C_{\Sigma'}^{\mathbf{K},\tau}(\phi, \psi) \\ \text{iff } \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\psi), \bar{\chi}) &\in C_{\Sigma'}^{\mathbf{K},\tau}(\phi, \psi). \end{aligned}$$

Therefore, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_{\Sigma}(C(\phi, \psi))$. ■

Before establishing protoalgebraicity, we need a couple of lemmas. We show, first, that, if \mathbf{K} is a pointed class, then all theory families of $\mathcal{I}^{\mathbf{K},\tau}$ are fully determined by the corresponding Leibniz class of the point.

Lemma 1387 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with a natural transformation $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a pointed class of \mathbf{F} -algebraic systems. Then, for all $T \in \text{ThFam}(\mathcal{I}^{\mathbf{K},\tau})$,*

$$T = \tau^b / \Omega(T).$$

Proof: Let \mathbf{K} be a pointed class of \mathbf{F} -algebraic systems, $T \in \text{ThFam}(\mathcal{I}^{\mathbf{K},\tau})$, $\Sigma \in |\mathbf{Sign}^b|$ and $\phi \in \text{SEN}^b(\Sigma)$.

- Suppose $\phi \in \tau_{\Sigma}^b / \Omega_{\Sigma}(T)$. This means that $\langle \phi, \tau_{\Sigma}^b \rangle \in \Omega_{\Sigma}(T)$. But, by definition, $\tau_{\Sigma}^b \in \text{Thm}_{\Sigma}(\mathcal{I}^{\mathbf{K},\tau}) \subseteq T_{\Sigma}$, whence, by the compatibility property of $\Omega(T)$ with T , we get that $\phi \in T_{\Sigma}$.
- Suppose $\phi \in T_{\Sigma}$. Then $\phi \approx \tau_{\Sigma}^b \in C_{\Sigma}^{\mathbf{K}}(T \approx \tau^b)$. This implies that, for all σ^b in N^b , all $\Sigma' \in |\mathbf{Sign}^b|$, all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$, and all $\bar{\chi} \in \text{SEN}^b(\Sigma')$,

$$\begin{aligned} \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\phi), \bar{\chi}) &\approx \tau_{\Sigma'}^b \in C_{\Sigma'}^{\mathbf{K}}(T \approx \tau^b) \\ \text{iff } \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\tau_{\Sigma}^b), \bar{\chi}) &\approx \tau_{\Sigma'}^b \in C_{\Sigma'}^{\mathbf{K}}(T \approx \tau^b). \end{aligned}$$

This is, by definition, equivalent to the statement that, for all σ^b in N^b , all $\Sigma' \in |\mathbf{Sign}^b|$, all $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$, and all $\bar{\chi} \in \text{SEN}^b(\Sigma')$,

$$\sigma_{\Sigma'}^b(\text{SEN}^b(f)(\phi), \bar{\chi}) \in C_{\Sigma'}^{\mathbf{K},\tau}(T) \quad \text{iff} \quad \sigma_{\Sigma'}^b(\text{SEN}^b(f)(\tau_{\Sigma}^b), \bar{\chi}) \in C_{\Sigma'}^{\mathbf{K},\tau}(T).$$

We conclude that $\langle \phi, \tau_{\Sigma}^b \rangle \in \Omega_{\Sigma}(T)$, i.e., that $\phi \in \tau_{\Sigma}^b / \Omega_{\Sigma}(T)$.

Thus, we get that $T = \tau^b/\Omega(T)$. \blacksquare

Next, we show that, if \mathbf{K} is a relatively point regular guasivariety, then, for every theory family of $\mathcal{I}^{\mathbf{K},\tau}$, the quotient of \mathcal{F} by the Leibniz congruence system of T , belongs to \mathbf{K} and, therefore, for every theory family T of $\mathcal{I}^{\mathbf{K},\tau}$, the Leibniz congruence system $\Omega(T)$ is a \mathbf{K} -congruence system on \mathcal{F} .

Lemma 1388 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system, with a natural transformation $\tau^b : (\mathbf{SEN}^b)^k \rightarrow \mathbf{SEN}^b$ in N^b , and \mathbf{K} a relatively point regular guasivariety of \mathbf{F} -algebraic systems. Then, for all $T \in \text{ThFam}(\mathcal{I}^{\mathbf{K},\tau})$, $\mathcal{F}/\Omega(T) \in \mathbf{K}$.*

Proof: Suppose that \mathbf{K} is a relatively point regular guasivariety of \mathbf{F} -algebraic systems and let $\Sigma \in |\mathbf{Sign}^b|$, $\phi_i, \psi_i \in \mathbf{SEN}^b(\Sigma)$, $i \in I$, $\phi, \psi \in \mathbf{SEN}^b(\Sigma)$, such that

$$\langle \{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi \rangle \in \text{GEq}_\Sigma(\mathbf{K}).$$

This is equivalent to the statement $\langle \phi, \psi \rangle \in \Theta_\Sigma^{\mathbf{K},\mathcal{F}}(\{\langle \phi_i, \psi_i \rangle : i \in I\})$. Since $\Theta^{\mathbf{K},\mathcal{F}}(\{\langle \phi_i, \psi_i \rangle : i \in I\}) \in \text{ConSys}^{\mathbf{K}}(\mathcal{F})$ and \mathbf{K} is relatively point regular, $\Theta^{\mathbf{K},\mathcal{F}}(\{\langle \phi_i, \psi_i \rangle : i \in I\})$ is completely determined by its τ^b -equivalence class. So it suffices to consider guasiequations of the form

$$\langle \{\phi_i \approx \tau_\Sigma^b : i \in I\}, \phi \approx \tau_\Sigma^b \rangle \in \text{GEq}_\Sigma(\mathbf{K}).$$

Now, let $T \in \text{ThFam}(\mathcal{I}^{\mathbf{K},\tau})$, such that $\langle \phi_i, \tau_\Sigma^b \rangle \in \Omega_\Sigma(T)$, for all $i \in I$. Then, taking into account Lemma 1387, $\phi_i \in \tau_\Sigma^b/\Omega_\Sigma(T) = T_\Sigma$, for all $i \in I$. Therefore, by definition, $\phi_i \approx \tau_\Sigma^b \in C_\Sigma^{\mathbf{K}}(T \approx \tau^b)$, for all $i \in I$. Since, by hypothesis, $\langle \{\phi_i \approx \tau_\Sigma^b : i \in I\}, \phi \approx \tau_\Sigma^b \rangle \in \text{GEq}_\Sigma(\mathbf{K})$, we get $\phi \approx \tau_\Sigma^b \in C_\Sigma^{\mathbf{K}}(T \approx \tau^b)$, i.e., $\phi \in C_\Sigma^{\mathbf{K},\tau}(T)$. Since $T \in \text{ThFam}(\mathcal{I}^{\mathbf{K},\tau})$, $\phi \in T_\Sigma = \tau_\Sigma^b/\Omega_\Sigma(T)$. Therefore, $\langle \phi, \tau_\Sigma^b \rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{F}/\Omega(T)$ satisfies all guasiequations of \mathbf{K} and, hence, since \mathbf{K} is a guasivariety, $\mathcal{F}/\Omega(T) \in \mathbf{K}$. \blacksquare

Finally, we establish protoalgebraicity of $\mathcal{I}^{\mathbf{K},\tau}$, under the hypotheses that \mathbf{K} is a relatively point regular guasivariety of \mathbf{F} -algebraic systems.

Proposition 1389 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\mathbf{SEN}^b)^k \rightarrow \mathbf{SEN}^b$ in N^b , and \mathbf{K} a relatively point regular guasivariety of \mathbf{F} -algebraic systems. Then $\mathcal{I}^{\mathbf{K},\tau}$ is a protoalgebraic π -institution.*

Proof: Let \mathbf{K} be a relatively point regular guasivariety of \mathbf{F} -algebraic systems. We know, by Proposition 1385, that $\mathcal{I}^{\mathbf{K},\tau}$ has a natural theorem τ^b , where τ^b is a point in \mathbf{K} , and, by Proposition 1386, that $\mathcal{I}^{\mathbf{K},\tau}$ is a family regular π -institution.

Now we show that $\mathcal{I}^{\mathbf{K},\tau}$ is protoalgebraic. Suppose that $T, T' \in \text{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1387, we get $\tau^b/\Omega(T) \leq \tau^b/\Omega(T')$. Since, by Lemma 1388, $\Omega(T)$ and $\Omega(T')$ are \mathbf{K} -congruence systems on \mathcal{F} and \mathbf{K} is

relatively point regular, they are completely determined (generated) by their τ^b -classes and, hence, we get $\Omega(T) \leq \Omega(T')$. Thus, $\mathcal{I}^{\mathbf{K}, \tau}$ is protoalgebraic. ■

We show, next, that, for a protoalgebraic family regular π -institution \mathcal{I} , having natural theorems, the assertional π -institution of its class $\text{AlgSys}^*(\mathcal{I})$ of reduced \mathbf{F} -algebraic systems coincides with \mathcal{I} .

Theorem 1390 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a family regular protoalgebraic π -institution based on \mathbf{F} , having a natural theorem τ . Then*

$$\mathcal{I}^{\text{AlgSys}^*(\mathcal{I}), \tau} = \mathcal{I}.$$

Proof: Set, for brevity in the course of this proof, $\mathbf{K} := \text{AlgSys}^*(\mathcal{I})$. Let $\Sigma \in |\mathbf{Sign}^b|$ and $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$. Then

$$\begin{aligned} \phi \in C_{\Sigma}^{\mathbf{K}, \tau}(\Phi) & \text{ iff } \phi \approx \tau_{\Sigma}^b \in C_{\Sigma}^{\mathbf{K}}(\Phi \approx \tau_{\Sigma}^b) \\ & \text{ iff for all } T \in \text{ThFam}(\mathcal{I}), \Sigma' \in |\mathbf{Sign}^b|, f \in \mathbf{Sign}^b(\Sigma, \Sigma'), \\ & \quad \text{SEN}^b(f)(\Phi) \approx \tau_{\Sigma'}^b \in \Omega_{\Sigma'}(T) \\ & \quad \text{implies } \text{SEN}^b(f)(\phi) \approx \tau_{\Sigma'}^b \in \Omega_{\Sigma'}(T) \\ & \text{ iff for all } T \in \text{ThFam}(\mathcal{I}), \Sigma' \in |\mathbf{Sign}^b|, f \in \mathbf{Sign}^b(\Sigma, \Sigma'), \\ & \quad \text{SEN}^b(f)(\Phi) \in T_{\Sigma'} \text{ implies } \text{SEN}^b(f)(\phi) \in T_{\Sigma'} \\ & \text{ iff } \phi \in C_{\Sigma}(\Phi). \end{aligned}$$

We conclude that $C^{\mathbf{K}, \tau} = C$ and, therefore, $\mathcal{I}^{\text{AlgSys}^*(\mathcal{I}), \tau} = \mathcal{I}$. ■

Moreover, starting with a relatively point regular quasivariety of \mathbf{F} -algebraic systems, the class of all reduced \mathbf{F} -algebraic systems of its assertional π -institution coincides with the original class.

Theorem 1391 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\tau^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b , and \mathbf{K} a relatively point regular quasivariety of \mathbf{F} -algebraic systems. Then*

$$\text{AlgSys}^*(\mathcal{I}^{\mathbf{K}, \tau}) = \mathbf{K}.$$

Proof: Let \mathbf{K} be a relatively point regular quasivariety of \mathbf{F} -algebraic systems. Assume that $\mathcal{A} \in \mathbf{K}$ and consider $\{\tau^{\mathcal{A}}\} := \{\tau_{\Sigma}^{\mathcal{A}}\}_{\Sigma \in |\mathbf{Sign}|} \in \text{SenFam}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}^b|$, all $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, such that $\phi \in C_{\Sigma}^{\mathbf{K}, \tau}(\Phi)$ and $\alpha_{\Sigma}(\Phi) \subseteq \{\tau_{F(\Sigma)}^{\mathcal{A}}\}$, we get, by the definition of $C^{\mathbf{K}, \tau}$, $\alpha_{\Sigma}(\phi) = \tau_{F(\Sigma)}^{\mathcal{A}}$. Therefore, $\{\tau^{\mathcal{A}}\} \in \text{ThFam}(\mathcal{I}^{\mathbf{K}, \tau})$. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \text{SEN}(\Sigma)$,

$$\begin{aligned} \langle \phi, \tau_{\Sigma}^{\mathcal{A}} \rangle \in \Omega_{\Sigma}^{\mathcal{A}}(\{\tau^{\mathcal{A}}\}) & \text{ iff } \phi = \tau_{\Sigma}^{\mathcal{A}} \quad (\text{Lemma 1387}) \\ & \text{ iff } \langle \phi, \tau_{\Sigma}^{\mathcal{A}} \rangle \in \Delta_{\Sigma}^{\mathcal{A}}. \end{aligned}$$

Thus, $\tau^{\mathcal{A}}/\Omega^{\mathcal{A}}(\{\tau^{\mathcal{A}}\}) = \tau^{\mathcal{A}}/\Delta^{\mathcal{A}}$. Therefore, by relative point regularity, we obtain $\Omega^{\mathcal{A}}(\{\tau^{\mathcal{A}}\}) = \Delta^{\mathcal{A}}$. This yields $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I}^{\mathbf{K}, \tau})$.

Assume, conversely, that $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I}^{\mathbf{K}, \top})$. Then, by definition, there exists $T \in \text{FiFam}^{\mathcal{I}^{\mathbf{K}, \top}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Suppose that $\Sigma \in |\mathbf{Sign}^b|$, $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, such that

$$\langle \Phi \approx \top_{\Sigma}^b, \phi \approx \top_{\Sigma}^b \rangle \in \text{GEq}_{\Sigma}(\mathbf{K})$$

and $\alpha_{\Sigma}(\Phi) \subseteq \{\top_{F(\Sigma)}^{\mathcal{A}}\}$. Then, since $T \in \text{FiFam}^{\mathcal{I}^{\mathbf{K}, \top}}(\mathcal{A})$, $\alpha_{\Sigma}(\Phi) \subseteq T_{F(\Sigma)}$. Hence, $\Phi \subseteq \alpha_{\Sigma}^{-1}(T_{F(\Sigma)})$. Since $T \in \text{ThFam}^{\mathcal{I}^{\mathbf{K}, \top}}(\mathcal{A})$, by Lemma 51, $\alpha^{-1}(T) \in \text{ThFam}(\mathcal{I}^{\mathbf{K}, \top})$, whence, by Lemma 1387, $\alpha^{-1}(T) = \top^b/\Omega(\alpha^{-1}(T))$. Thus, we get $\Phi \subseteq \top_{\Sigma}^b/\Omega_{\Sigma}(\alpha^{-1}(T))$. Hence, $\Phi \approx \top_{\Sigma}^b \in \Omega_{\Sigma}(\alpha^{-1}(T))$. By Lemma 1388, $\Omega(\alpha^{-1}(T)) \in \text{ConSys}^{\mathbf{K}}(\mathcal{F})$, whence, since $\langle \Phi \approx \top_{\Sigma}^b, \phi \approx \top_{\Sigma}^b \rangle \in \text{GEq}_{\Sigma}(\mathbf{K})$, $\phi \approx \top_{\Sigma}^b \in \Omega_{\Sigma}(\alpha^{-1}(T))$. By Proposition 24, $\phi \approx \top_{\Sigma}^b \in \alpha_{\Sigma}^{-1}(\Omega_{F(\Sigma)}^{\mathcal{A}}(T))$, i.e., $\alpha_{\Sigma}(\phi) \approx \top_{F(\Sigma)}^{\mathcal{A}} \in \Omega_{F(\Sigma)}^{\mathcal{A}}(T) = \Delta_{F(\Sigma)}^{\mathcal{A}}$. Thus, $\alpha_{\Sigma}(\phi) = \top_{F(\Sigma)}^{\mathcal{A}}$. We conclude that $\langle \Phi \approx \top_{\Sigma}^b, \phi \approx \top_{\Sigma}^b \rangle \in \text{GEq}_{\Sigma}(\mathcal{A})$. Since \mathcal{A} satisfies all guasiequations in $\text{GEq}(\mathbf{K})$ and \mathbf{K} is, by hypothesis, a guasivariety, we get that $\mathcal{A} \in \mathbf{K}$. Therefore, $\text{AlgSys}^*(\mathcal{I}^{\mathbf{K}, \top}) = \mathbf{K}$. ■

Now we can formulate the main theorems of the section.

Theorem 1392 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} is protoalgebraic family regular, with natural theorems, if and only if it is the assertional π -institution of a relatively point regular guasivariety of \mathbf{F} -algebraic systems.*

More precisely, \mathcal{I} is protoalgebraic family regular, with natural theorems, if and only if $\text{AlgSys}^(\mathcal{I})$ is a relatively point regular guasivariety and $\mathcal{I} = \mathcal{I}^{\text{AlgSys}^*(\mathcal{I}), \top}$, where \top^b is any natural theorem.*

Proof: Suppose \mathcal{I} is protoalgebraic family regular, with natural theorems. Then, by Proposition 1383, $\text{AlgSys}^*(\mathcal{I})$ is a relatively point regular class of \mathbf{F} -algebraic systems and, by protoalgebraicity, Proposition 68 and Theorem ??, it is a guasivariety. Moreover, by Theorem 1390, $\mathcal{I} = \mathcal{I}^{\text{AlgSys}^*(\mathcal{I}), \top}$.

Assume, conversely, that $\mathcal{I}^{\mathbf{K}, \top}$ is the assertional π -institution of a relatively point regular guasivariety \mathbf{K} of \mathbf{F} -algebraic systems. Then, by Proposition 1386, it is family regular, by Proposition 1389, it is protoalgebraic and, by Proposition 1385, it has natural theorems. ■

Theorem 1393 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, with $\top^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$ in N^b . Then, there exists a one to one correspondence between relatively point regular guasivarieties, with point \top^b , and family regular protoalgebraic π -institutions, with a natural theorem \top^b .*

Every relatively point regular guasivariety with point \top^b determines a unique family regular protoalgebraic π -institution with natural theorems, its assertional π -institution.

Every family regular protoalgebraic π -institution with natural theorems is the assertional π -institution of a unique relatively point regular quasivariety, the quasivariety $\text{AlgSys}^(\mathcal{I})$ of all its reduced \mathbf{F} -algebraic systems.*

For each family regular protoalgebraic π -institution, with a natural theorem \top^b , we have $\mathcal{I} = \mathcal{I}^{\text{AlgSys}^(\mathcal{I}), \top}$ and, conversely, for every relatively point regular quasivariety \mathbf{K} , with point \top^b , we have $\mathbf{K} = \text{AlgSys}^*(\mathcal{I}^{\mathbf{K}, \top})$.*

Proof: This is a recap of Theorems 1390 and 1391. ■