Chapter 20

Full Adequacy

20.1 Gentzen π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\Sigma \in |\mathbf{Sign}^\flat|$. A Σ-**sequent** is a pair

$$\langle \Phi, \phi \rangle,$$

where $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$ (with Φ possibly empty). Sometimes we write

$$\Phi \rhd_\Sigma \phi \quad \text{or} \quad \Phi \vdash_\Sigma \phi$$

to denote the Σ-sequent $\langle \Phi, \phi \rangle$. The set Φ is called **set of antecedents of** $\langle \Phi, \phi \rangle$ and ϕ is called the **consequent of** $\langle \Phi, \phi \rangle$.

The collection of Σ-sequents is denoted by $\mathrm{Seq}_\Sigma(\mathbf{F})$ and the set of all Σ-sequents with nonempty set of antecedents is denoted by $\mathrm{Seq}^0_\Sigma(\mathbf{F})$. We then set

$$\mathrm{Seq}(\mathbf{F}) = \{\mathrm{Seq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|} \quad \text{and} \quad \mathrm{Seq}^0(\mathbf{F}) = \{\mathrm{Seq}^0_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}.$$

We sometimes use boldface Greek letters such as $\boldsymbol{\gamma}, \boldsymbol{\delta}, \ldots$ to denote Σ-sequents and boldface capital Greek letters such as $\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \ldots$ for sets of Σ-sequents. Moreover, we write $\Gamma \vdash_\Sigma \Phi$ to stand for the set $\{\Gamma \vdash_\Sigma \phi : \phi \in \Phi\}$ of Σ-sequents.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **Gentzen π-institution based on F of type 1** (**of type 0**, respectively) is a pair

$$\mathfrak{G} = \langle \mathbf{F}, G \rangle,$$

where $G : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ ($G : \mathcal{P}(\mathrm{Seq}^0(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}^0(\mathbf{F}))$, respectively) is a closure system on $\mathrm{Seq}(\mathbf{F})$ ($\mathrm{Seq}^0(\mathbf{F})$, respectively) that, in addition, satisfies the following **structural rules**, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$:

$$\begin{array}{rl} \text{(Axiom)} & \phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing); \\ \text{(Weakening)} & \Phi, \Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi); \\ \text{(Cut)} & \Phi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi). \end{array}$$

If $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ we call $\Phi \vdash_\Sigma \phi$ a **Σ-theorem** or a **derivable Σ-sequent** of $\mathfrak{G}$.

Each Gentzen π-institution based on an algberaic system $\mathbf{F}$ defines in a natural way a π-institution based on $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. The **π-institution $\mathcal{I}^{\mathfrak{G}} = \langle F, C^{\mathfrak{G}} \rangle$ defined** or **determined by $\mathfrak{G}$** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C^{\mathfrak{G}}_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing).$$

Proposition 1516 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*.* $C^{\mathfrak{G}} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ *is a closure system on* $\mathrm{SEN}^\flat$ *and, hence,* $\mathcal{I}^{\mathfrak{G}} = \langle \mathbf{F}, C^{\mathfrak{G}} \rangle$ *is a* π*-institution.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

If $\phi \in \Phi$, then, by (Axiom) $\phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and by (Weakening) $\Phi \vdash_\Sigma \phi \in G_\Sigma(\phi \vdash_\Sigma \phi)$, whence $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Therefore $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$.

If $\Phi \subseteq \Psi$ and $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$, then $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and, by (Weakening), $\Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi)$, whence $\Psi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$, giving $\phi \in C^{\mathfrak{G}}_\Sigma(\Psi)$.

If $\phi \in C^{\mathfrak{G}}_\Sigma(C^{\mathfrak{G}}_\Sigma(\Phi))$, then $C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and, by (Weakening),

$$\Phi, C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi \in G_\Sigma(C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi) \subseteq G_\Sigma(\varnothing).$$

Moreover, by definition $\Phi \vdash_\Sigma C^{\mathfrak{G}}_\Sigma(\Phi) \subseteq G_\Sigma(\varnothing)$, whence, by (Cut),

$$\Phi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma C^{\mathfrak{G}}_\Sigma(\Phi),\ \Phi, C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi) \subseteq G_\Sigma(\varnothing).$$

Therefore, $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$.

Finally, suppose $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$, $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then, by definition, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and, by structurality,

$$\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in G_{\Sigma'}(\varnothing).$$

This shows that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathfrak{G}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$ and, therefore, $C^{\mathfrak{G}}$ is a closure system on $\mathrm{SEN}^\flat$, as was to be shown. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution, also based on $\mathbf{F}$. We say that $\mathfrak{G}$ is **adequate for** $\mathcal{I}$ if $C = C^{\mathfrak{G}}$ and, moreover,

- $\mathfrak{G}$ is of type 1 if $\mathcal{I}$ has theorems and
- $\mathfrak{G}$ is of type 0 if $\mathcal{I}$ does not have theorems.

The following proposition clarifies the distinction imposed on the type, since it reveals the fact that, if $\mathcal{I}$ has no theorems, then it is sufficient to assume that a Gentzen π-institution adequate for $\mathcal{I}$ has type 0.

Proposition 1517 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathfrak{G}$ *is of type 0, then* $\mathcal{I}^{\mathfrak{G}}$ *does not have theorems.*

(b) If $\mathfrak{G}$ *is of type 1, then its restriction* $\mathfrak{G}^0 = \langle \mathbf{F}, G^0 \rangle$ *to* $\mathrm{Seq}^0(\mathbf{F})$ *is a Gentzen* π*-institution of type 0.*

(c) If $\mathfrak{G}$ *is of type 1 and* $\mathcal{I}^{\mathfrak{G}}$ *has no theorems, then* $\mathcal{I}^{\mathfrak{G}} = \mathcal{I}^{\mathfrak{G}^0}$*.*

Proof:

(a) Suppose $\mathcal{I}^{\mathfrak{G}}$ has theorems. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathfrak{G}}_\Sigma(\varnothing)$. Thus, by definition, $\varnothing \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Therefore, $\mathfrak{G}$ cannot be of type 0 (since it admits a sequent with an empty set of antecedents).

(b) Suppose $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ is of type 1. Consider $\mathfrak{G}^0 = \langle \mathbf{F}, G^0\rangle$. We must show that $G^0 : \mathcal{P}(\mathrm{Seq}^0(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}^0(\mathbf{F}))$ is a closure system on $\mathrm{Seq}^0(\mathbf{F})$ that satisfies the structural rules.

- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \mathbf{\Gamma}$. Then $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Gamma})$ and, hence, $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \mathbf{\Delta} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$. Then, by definition, $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$, whence $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Delta})$. So $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Delta})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in G^0_\Sigma(G^0_\Sigma(\mathbf{\Gamma}))$. Then $\boldsymbol{\gamma} \in G_\Sigma(G_\Sigma(\mathbf{\Gamma})) = G_\Sigma(\mathbf{\Gamma})$. As $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, it follows that $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$, $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Gamma})$, whence $\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) \in G_{\Sigma'}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$. Observing that, if $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, then $\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}) \cup \{\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma})\} \subseteq \mathrm{Seq}^0_{\Sigma'}(\mathbf{F})$, we conclude that
$$\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) \in G^0_{\Sigma'}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma})).$$

Next, for the structural rules:

(Axiom) For $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \vdash_\Sigma \phi \in \mathrm{Seq}^0_\Sigma(\mathbf{F})$, whence, since, by (Axiom), $\phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$, $\phi \vdash_\Sigma \phi \in G^0_\Sigma(\varnothing)$.

(Weakening) Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \neq \varnothing$. Then, since $\Phi \cup \Psi \neq \varnothing$ and since, by (Weakening), $\Phi, \Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi)$, we conclude that $\Phi, \Psi \vdash_\Sigma \phi \in G^0_\Sigma(\Phi \vdash_\Sigma \phi)$.

(Cut) Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$. Then $\Phi \cup \Psi \neq \varnothing$ and, since, by (Cut), $\Phi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi)$, we get $\Phi \vdash_\Sigma \phi \in G^0_\Sigma(\Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi)$.

(c) Suppose $\mathfrak{G}$ is of type 1 and $\mathcal{I}^{\mathfrak{G}}$ has no theorems. Clearly, $G^0 \leq G$ so that $C^{\mathfrak{G}^0} \leq C^{\mathfrak{G}}$. On the other hand, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$. Since $\mathcal{I}^{\mathfrak{G}}$ has no theorems, $\Gamma \neq \varnothing$. Moreover, by definition, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Thus, $\Phi \vdash_\Sigma \phi \in G^0_\Sigma(\varnothing)$. We conclude that $\phi \in C^{\mathfrak{G}^0}_\Sigma(\Phi)$. So $C^{\mathfrak{G}} \leq C^{\mathfrak{G}^0}$ and equality follows.

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Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\Sigma \in |\mathbf{Sign}^\flat|$.

- A **Gentzen Σ-axiom** is a Σ-sequent $\boldsymbol{\gamma} \in \mathrm{Seq}_\Sigma(\mathbf{F})$;
- A **Gentzen Σ-rule** is a pair $\langle \boldsymbol{\Gamma}, \boldsymbol{\gamma} \rangle$, where $\boldsymbol{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$.

A **Gentzen axiom system** is a collection $\mathrm{Ax} = \{\mathrm{Ax}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where Ax_Σ is a set of Gentzen Σ-axioms, which is $\mathbf{Sign}^\flat$-invariant.

A **Gentzen rule system** is a collection $\mathrm{Ir} = \{\mathrm{Ir}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where Ir_Σ is a set of Gentzen Σ-rules, which is also $\mathbf{Sign}^\flat$-invariant. Set

$$R = \mathrm{Ax} \cup \mathrm{Ir}.$$

The **Gentzen closure system** $G^R \subseteq \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ **generated by** R is the least closure system on $\mathrm{Seq}(\mathbf{F})$, satisfying the structural rules, that contains R, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

- $\boldsymbol{\gamma} \in G^R_\Sigma(\varnothing)$, for all $\boldsymbol{\gamma} \in \mathrm{Ax}_\Sigma$, and
- $\boldsymbol{\gamma} \in G^R_\Sigma(\boldsymbol{\Gamma})$, for all $\langle \boldsymbol{\Gamma}, \boldsymbol{\gamma} \rangle \in \mathrm{Ir}_\Sigma$.

We denote by $\mathfrak{G}^R = \langle \mathbf{F}, G^R \rangle$ the corresponding Gentzen π-institution, called the **Gentzen π-institution generated by** R.

A Finitary Parenthesis

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $R = \mathrm{Ax} \cup \mathrm{Ir}$ a set of finitary Gentzen axioms and rules of inference (i.e., such that the set of antecedents of all sequents involved is finite and the set of hypotheses of each rule of inference is also finite), and $\boldsymbol{\Gamma} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$ a set of Σ-sequents. We define a family

$$\Xi^R_\Sigma(\boldsymbol{\Gamma}) = \bigcup \{\Xi^{R,n}_\Sigma(\boldsymbol{\Gamma}) : n < \omega\},$$

where $\Xi^{R,n}_\Sigma(\boldsymbol{\Gamma})$ is defined by induction on $n < \omega$ as follows:

- $\Xi^{R,0}_\Sigma(\boldsymbol{\Gamma}) = \{\phi \vdash_\Sigma \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \cup \mathrm{Ax}_\Sigma \cup \boldsymbol{\Gamma}$;
- For all $n \geq 0$, $\Phi \cup \Psi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcl} \Xi^{R,n+1}_\Sigma(\boldsymbol{\Gamma}) & = & \{\Phi, \Psi \vdash_\Sigma \phi : \Phi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\boldsymbol{\Gamma})\} \\ & & \cup \{\Phi \vdash_\Sigma \phi : \Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\boldsymbol{\Gamma})\} \\ & & \cup \{\Phi \vdash_\Sigma \phi : \langle \boldsymbol{\Delta}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma, \boldsymbol{\Delta} \subseteq \Xi^{R,n}_\Sigma(\boldsymbol{\Gamma})\}. \end{array}$$

We define $\Xi^R : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$, by letting $\Xi^R := \{\Xi^R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $\Xi^R_\Sigma : \mathcal{P}(\mathrm{Seq}_\Sigma(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}_\Sigma(\mathbf{F}))$ as defined above.

We show, next, that this is closure system on $\mathrm{Seq}(\mathbf{F})$, which satisfies the structural rules, includes Ax and is closed under Ir.

Lemma 1518 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $R = \mathrm{Ax} \cup \mathrm{Ir}$ *a collection of finitary axioms and rules of inference.* $\Xi^R : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ *is a closure system on* $\mathrm{Seq}(\mathbf{F})$*, satisfying the structural rules and including* R*.*

Proof: We show, first, that Ξ^R is a closure system on $\mathrm{Seq}(\mathbf{F})$.

- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\mathbf{\Gamma} \cup \{\gamma\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\gamma \in \mathbf{\Gamma}$. Then $\gamma \in \Xi^{R,0}_\Sigma(\mathbf{\Gamma})$ and, hence, $\gamma \in \Xi^R_\Sigma(\mathbf{\Gamma})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \mathbf{\Delta} \cup \{\gamma\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\gamma \in \Xi^R_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$. Then, for some $n < \omega$, $\gamma \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$. We show by induction on n, that then $\gamma \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
  - If $n = 0$, then the conclusion follows directly from the inclusion $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$.
  - Now suppose that $n > 0$ and that the conclusion holds for $n-1$.
    * If $\gamma = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Delta})$, whence, it follows that $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
    * If $\gamma = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, we get, by the induction hypothesis, $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Delta})$, whence $\Phi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
    * If $\gamma = \Phi \vdash_\Sigma \phi$, with $\langle \mathbf{\Upsilon}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma$ and $\mathbf{\Upsilon} \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\mathbf{\Upsilon} \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Delta})$, whence, again, $\Phi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\gamma\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\gamma \in \Xi^R_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$. Then, for some $n < \omega$, $\gamma \in \Xi^{R,n}_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$. We show by induction on n, that then $\gamma \in \Xi^R_\Sigma(\mathbf{\Gamma})$.
  - If $n = 0$, then γ is of the form $\phi \vdash_\Sigma \phi$ or is in Ax_Σ or in $\Xi^R_\Sigma(\mathbf{\Gamma})$. In the first two cases, it is in $\Xi^{R,0}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$ and in the last in $\Xi^R_\Sigma(\mathbf{\Gamma})$.
  - Suppose $n > 0$ and the conclusion holds for $n-1$.
    * If $\gamma = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$, then, by the induction hypothesis, $\Phi \vdash_\Sigma \phi \in \Xi^R_\Sigma(\mathbf{\Gamma})$, i.e., $\Phi \vdash_\Sigma \phi \in \Xi^{R,m}_\Sigma(\mathbf{\Gamma})$, for some $m < \omega$. Thus $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,m+1}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$.
    * If $\gamma = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$, we get, by the induction hypothesis, $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^R_\Sigma(\mathbf{\Gamma})$. Since Ψ is finite, there exists $m > 0$, such that $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,m}_\Sigma(\mathbf{\Gamma})$. Thus, $\Phi \vdash_\Sigma \phi \in \Xi^{R,m+1}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$.

  * If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\langle \boldsymbol{\Delta}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma$ and $\boldsymbol{\Delta} \subseteq \Xi_\Sigma^{R,n-1}(\Xi_\Sigma^R(\boldsymbol{\Gamma}))$, then, by the induction hypothesis, $\boldsymbol{\Delta} \subseteq \Xi_\Sigma^R(\boldsymbol{\Gamma})$, whence, again, since $\boldsymbol{\Delta}$ is finite, there exists $m > 0$, such that $\boldsymbol{\Delta} \subseteq \Xi_\Sigma^{R,m}(\boldsymbol{\Gamma})$. Therefore, $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,m+1}(\boldsymbol{\Gamma}) \subseteq \Xi_\Sigma^R(\boldsymbol{\Gamma})$.

- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \Xi_\Sigma^R(\boldsymbol{\Gamma})$, and let $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then, for some $n < \omega$, $\boldsymbol{\gamma} \in \Xi_\Sigma^{R,n}(\boldsymbol{\Gamma})$. We show by induction on n, that then $\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma}))$.

  - If $n = 0$, then $\boldsymbol{\gamma}$ is of the form $\phi \vdash_\Sigma \phi$ or in Ax_Σ or in $\boldsymbol{\Gamma}$. In the first case, $\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) = \mathrm{SEN}^\flat(f)(\phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,0}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma}))$, by definition. In the second case, the conclusion holds by the postulated invariance of Ax under $\mathbf{Sign}^\flat$. In the last case, it holds because, by definition, $\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) \in \Xi_{\Sigma'}^{R,0}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma}))$.
  - Suppose $n > 0$ and the conclusion holds for $n - 1$.
    * If $\boldsymbol{\gamma} = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n-1}(\boldsymbol{\Gamma})$, then, by the induction hypothesis,

$$\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n-1}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma})),$$

      whence, by definition. $\mathrm{SEN}^\flat(f)(\Phi \cup \Psi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma}))$.
    * If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n-1}(\boldsymbol{\Gamma})$, we get, by the induction hypothesis, $\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\Psi)$, $\mathrm{SEN}^\flat(f)(\Phi \cup \Psi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n-1}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma}))$. So, again by definition,

$$\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma})).$$

    * If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\langle \boldsymbol{\Delta}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma$ and $\boldsymbol{\Delta} \subseteq \Xi_\Sigma^{R,n-1}(\boldsymbol{\Gamma})$, then, by the induction hypothesis,

$$\mathrm{SEN}^\flat(f)(\boldsymbol{\Delta}) \subseteq \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma})),$$

      whence, since Ir is invariant under $\mathbf{Sign}^\flat$, we get, by definition, $\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Gamma}))$.

We have concluded the proof that Ξ^R is a closure system on $\mathrm{Seq}(\mathbf{F})$.

Next, we show that it satisfies the structural rules.

- For (Axiom), if $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, by definition, $\phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,0}(\varnothing) \subseteq \Xi_\Sigma^R(\varnothing)$.

- For (Weakening), if $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^R(\boldsymbol{\Gamma})$, then, there exists $n < \omega$, such that $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n}(\boldsymbol{\Gamma})$. Therefore, by definition, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n+1}(\boldsymbol{\Gamma}) \subseteq \Xi_\Sigma^R(\boldsymbol{\Gamma})$.

- For (Cut), if $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^R_\Sigma(\mathbf{\Gamma})$, then, since $\Psi \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, there exists $n < \omega$, such that $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$. Thus, by definition, $\Phi \vdash_\Sigma \phi \in \Xi^{R,n+1}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$.

So Ξ^R does satisfy all three structural rules.

Finally, it does include all rules in R:

- For $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\gamma} \in \mathrm{Ax}_\Sigma$, we have $\boldsymbol{\gamma} \in \mathrm{Ax}_\Sigma \subseteq \Xi^{R,0}_\Sigma(\varnothing) \subseteq \Xi^R_\Sigma(\varnothing)$.
- For $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle \mathbf{\Gamma}, \boldsymbol{\gamma} \rangle \in \mathrm{Ir}_\Sigma$, such that $\mathbf{\Gamma} \subseteq \Xi^R_\Sigma(\mathbf{\Delta})$, since $\mathbf{\Gamma}$ is finite, there exists $n < \omega$, such that $\mathbf{\Gamma} \subseteq \Xi^{R,n}_\Sigma(\mathbf{\Delta})$. Thus, by definition, $\boldsymbol{\gamma} \in \Xi^{R,n+1}_\Sigma(\mathbf{\Delta}) \subseteq \Xi^R_\Sigma(\mathbf{\Delta})$.

This concludes the proof of the statement. ■

We show that, given a system R of finitary Gentzen axioms and rules of inference, the closure $G^R_\Sigma(\mathbf{\Gamma})$ of a set $\mathbf{\Gamma}$ of Σ-sequents in the least Gentzen π-institution generated by R is exactly $\Xi^R_\Sigma(\mathbf{\Gamma})$.

Proposition 1519 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $R = \mathrm{Ax} \cup \mathrm{Ir}$ *a set of finitary Gentzen axioms and rules of inference. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\mathbf{\Gamma} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$,

$$G^R_\Sigma(\mathbf{\Gamma}) = \Xi^R_\Sigma(\mathbf{\Gamma}).$$

Proof: Suppose, first, that $\boldsymbol{\gamma} \in \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \Xi^R_\Sigma(\mathbf{\Gamma})$. Then, $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$, for some $n < \omega$. We show by induction on $n < \omega$ that, if $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$, then $\boldsymbol{\gamma} \in G^R_\Sigma(\mathbf{\Gamma})$.

- The conclusion is obvious for $n = 0$, since, by definition, $G^R_\Sigma(\mathbf{\Gamma})$ satisfies the structural rules, contains Ax_Σ and, clearly, includes $\mathbf{\Gamma}$;
- A similar clause applies for the induction step:
  - If $\boldsymbol{\gamma} = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\Phi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$ and, since G^R satisfies the structural rules, $\Phi, \Psi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$ also.
  - If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$ and $\Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, again by the induction hypothesis, $\Phi \vdash_\Sigma \Psi \subseteq G^R_\Sigma(\mathbf{\Gamma})$ and $\Psi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$, whence, since G^R satisfies the structural rules, $\Phi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$.
  - If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\langle \mathbf{\Delta}, \boldsymbol{\gamma} \rangle \in \mathrm{Ir}_\Sigma$ and $\mathbf{\Delta} \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\mathbf{\Delta} \subseteq G^R_\Sigma(\mathbf{\Gamma})$ and, since G^R is closed under the rules of inference, we get that $\Phi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$.

We conclude that $\Xi^{R,n}_\Sigma(\mathbf{\Gamma}) \subseteq G^R_\Sigma(\mathbf{\Gamma})$, for all $n < \omega$, and, therefore, $\Xi^R_\Sigma(\mathbf{\Gamma}) \subseteq G^R_\Sigma(\mathbf{\Gamma})$.

Conversely, since, by Lemma 1518, $\Xi^R : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ is a closure system on $\mathrm{Seq}(\mathbf{F})$, which satisfies the structural rules, contains Ax and is closed under Ir, we conclude by the minimality of G^R, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Gamma} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, $G^R_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$. From this, the conclusion follows. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Define a Gentzen π-institution $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$, as follows:

1. If $\mathcal{I}$ has theorems, $\mathfrak{G}^{\mathcal{I}}$ is of type 1 and if $\mathcal{I}$ does not have theorems, then $\mathfrak{G}^{\mathcal{I}}$ is of type 0;

2. Set $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\}.$$

   Let $R^{\mathcal{I}} := \mathrm{Ax}^{\mathcal{I}}$. Then set $\mathfrak{G}^{\mathcal{I}} := \mathfrak{G}^{R^{\mathcal{I}}}$.

Of course, $\mathfrak{G}^{\mathcal{I}}$ is a Gentzen π-institution. Moreover, it turns out that, if $\mathcal{I}$ is finitary, then $\mathfrak{G}^{\mathcal{I}}$ is adequate for the π-institution $\mathcal{I}$.

Lemma 1520 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. Then $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$ is a Gentzen π-institution adequate for $\mathcal{I}$.*

Proof: Note that, by hypothesis and Proposition 1519, $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, \Xi^{R^{\mathcal{I}}} \rangle$.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing).$$

First, if $\phi \in C_\Sigma(\Phi)$, then, by definition $\Phi \vdash_\Sigma \phi \in \mathrm{Ax}^{\mathcal{I}}_\Sigma$. Therefore, since $\mathrm{Ax}^{\mathcal{I}}_\Sigma \subseteq \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing)$, we get $\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing)$.

Conversely, we must show that, if $\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing)$, then $\phi \in C_\Sigma(\Phi)$. We do this by showing, using induction on $n < \omega$, that

$$\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}},n}_\Sigma(\varnothing) \quad \text{implies} \quad \phi \in C_\Sigma(\Phi).$$

- If $\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}},0}_\Sigma(\varnothing)$, then it is either of the form $\phi \vdash_\Sigma \phi$ or in $\mathrm{Ax}^{\mathcal{I}}_\Sigma$. In the first case, the conclusion follows by the inflationarity of C and, in the second, by the definition of $\mathrm{Ax}^{\mathcal{I}}$.

- Suppose $n > 0$ and that the conclusion holds for $n - 1$. Then, since $\mathrm{Ir}^{\mathcal{I}} = \varnothing$, there are only two cases to consider.

- If $\Phi \vdash_\Sigma \phi$ is of the form $\Phi_1, \Phi_2 \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n}(\varnothing)$, with $\Phi_1 \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n-1}(\varnothing)$, then, by the induction hypothesis, $\phi \in C_\Sigma(\Phi_1)$ and, hence, by the monotonicity of C, $\phi \in C_\Sigma(\Phi_1, \Phi_2)$.
- If $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n}(\varnothing)$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n-1}(\varnothing)$, then, by the induction hypothesis, $\Psi \subseteq C_\Sigma(\Phi)$ and $\phi \in C_\Sigma(\Phi, \Psi)$, whence

$$\begin{array}{lll} \phi & \in & C_\Sigma(\Phi, \Psi) \quad \text{(hypothesis)} \\ & \subseteq & C_\Sigma(\Phi, C_\Sigma(\Phi)) \quad \text{(monotonicity)} \\ & \subseteq & C_\Sigma(C_\Sigma(\Phi)) \quad \text{(monotonicity)} \\ & = & C_\Sigma(\Phi). \quad \text{(idempotency)} \end{array}$$

This finishes the induction and concludes the proof. ■

End of the Finitary Parenthesis

20.2 $\mathfrak{G}$-Structures and $\mathfrak{G}$-Algebraic Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. $\mathbb{L}$ is a **$\mathfrak{G}$-structure** or a **model of $\mathfrak{G}$** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$,

$$\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\}) \text{ and } \alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \\ \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

In relation to $\mathfrak{G}$-structures, we use the following notation:

- $D \in \mathrm{ClFam}^{\mathfrak{G}}(\mathcal{A})$ if $\langle \mathcal{A}, D \rangle$ is a $\mathfrak{G}$-structure;
- $\mathrm{Str}(\mathfrak{G})$ is the collection of all $\mathfrak{G}$-structures;
- $\mathrm{Str}^{\mathfrak{G}}(\mathcal{A})$ is the collection of all $\mathfrak{G}$-structures on the $\mathbf{F}$-algebraic system $\mathcal{A}$.

Let, again, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathbf{\Gamma} = \{\mathbf{\Gamma}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} \in \mathrm{ThFam}(\mathfrak{G})$. Define $D^{\mathbf{\Gamma}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$D_\Sigma^{\mathbf{\Gamma}}(\Phi) = \{\phi \in \mathrm{SEN}^\flat(\Sigma) : \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma\}.$$

We show that $D^{\mathbf{\Gamma}}$, thus defined, is a closure family on $\mathrm{SEN}^\flat$ and, therefore, $\langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is an $\mathbf{F}$-structure. In fact, $\langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is a $\mathfrak{G}$-structure.

Lemma 1521 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$. Then $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, C^{\mathbf{\Gamma}} \rangle$ is a $\mathfrak{G}$-structure.*

Proof: We show, first, that $D^{\mathbf{\Gamma}}$ is a closure family on $\mathcal{F}$.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \Phi$. Then, by (Axiom) $\phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. By (Weakening), $\Phi \vdash_\Sigma \phi \in G_\Sigma(\phi \vdash_\Sigma \phi)$. Therefore, by (Cut), $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Therefore, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$ and, hence, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$.
- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$ and $\Phi \subseteq \Psi$. By definition, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$. Hence, by (Weakening) $\Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi) \subseteq G_\Sigma(\mathbf{\Gamma}_\Sigma) = \mathbf{\Gamma}_\Sigma$. We conclude that $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Psi)$.
- Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi))$. Then, by definition, $D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$ and $D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \subseteq \mathbf{\Gamma}_\Sigma$. Now we get

$$\begin{array}{rcl} D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma \phi & \in & G_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi), D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \vdash_\Sigma \phi, \\ & & \qquad D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi))) \\ & \subseteq & G_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \vdash_\Sigma \phi, \\ & & \qquad D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi))) \\ & \subseteq & G_\Sigma(\mathbf{\Gamma}_\Sigma) \\ & = & \mathbf{\Gamma}_\Sigma. \end{array}$$

  Therefore, by definition, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$.

We conclude that $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is an $\mathbf{F}$-structure. We show, next, that $\mathbb{L}^{\mathbf{\Gamma}}$ is a $\mathfrak{G}$-structure. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that

- $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ and
- $\phi_i \in D^{\mathbf{\Gamma}}_\Sigma(\Phi_i)$, for all $i \in I$.

Then, by definition, $\Phi_i \vdash_\Sigma \phi_i \in \mathbf{\Gamma}_\Sigma$, for all $i \in I$. Since $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$, we get $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$. Thus, by definition, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$. So $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is a $\mathfrak{G}$-structure. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, and $\mathbb{L} = \langle \mathcal{F}, D \rangle$ a $\mathfrak{G}$-structure. We define $\mathbf{\Gamma}^{\mathbb{L}} = \{\mathbf{\Gamma}^{\mathbb{L}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathbf{\Gamma}^{\mathbb{L}}_\Sigma = \{\Phi \vdash_\Sigma \phi \in \mathrm{Seq}_\Sigma(\mathbf{F}) : \phi \in D_\Sigma(\Phi)\}.$$

We show that $\mathbf{\Gamma}^{\mathbb{L}}$, thus defined, is a theory family of the Gentzen π-institution $\mathfrak{G}$.

Lemma 1522 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, and $\mathbb{L} = \langle \mathcal{F}, D \rangle$ a $\mathfrak{G}$-structure. Then $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that

- $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ and
- $\Phi_i \vdash_\Sigma \phi_i \in \mathbf{\Gamma}^{\mathbb{L}}_\Sigma$, for all $i \in I$.

Then, by definition, $\phi_i \in D_\Sigma(\Phi_i)$, for all $i \in I$. Thus, since $\mathbb{L}$ is a $\mathfrak{G}$-structure, $\phi \in D_\Sigma(\Phi)$. Therefore, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}^{\mathbb{L}}_\Sigma$. We conclude $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$. ■

We show next that the two preceding constructions, of a $\mathfrak{G}$-structure $\mathbb{L}^{\mathbf{\Gamma}}$ out of a given theory family $\mathbf{\Gamma}$ of $\mathfrak{G}$ and of a theory family $\mathbf{\Gamma}^{\mathbb{L}}$ out of a given $\mathfrak{G}$-structure $\mathbb{L} = \langle \mathcal{F}, D\rangle$ are inverses of one another.

Proposition 1523 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathbb{L} = \langle \mathcal{F}, D\rangle$ an $\mathbf{F}$-structure.*

(a) $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$ if and only if $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$ and $\mathbb{L} = \mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}}$;

(b) $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$ if and only if $\mathbb{L}^{\mathbf{\Gamma}} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}$;

(c) $\mathcal{I}^{\mathfrak{G}} = \langle \mathbf{F}, C^{\mathfrak{G}}\rangle$ is the smallest $\mathfrak{G}$-structure on $\mathcal{F}$ and $C^{\mathfrak{G}} = D^{\mathrm{Thm}(\mathfrak{G})}$.

Proof:

(a) Suppose, first, that $\mathbb{L} = \langle \mathcal{F}, D\rangle \in \mathrm{Str}(\mathfrak{G})$. Then, by Lemma 1522, $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rll} \phi \in D^{\mathbf{\Gamma}^{\mathbb{L}}}_\Sigma(\Phi) & \text{iff} & \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}^{\mathbb{L}}_\Sigma \\ & \text{iff} & \phi \in D_\Sigma(\Phi). \end{array}$$

So $D = D^{\mathbf{\Gamma}^{\mathbb{L}}}$.

Assume, conversely, that $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$ and $\mathbb{L} = \mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}}$. By Lemma 1521, $\mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}} \in \mathrm{Str}(\mathfrak{G})$. Thus, $\mathbb{L} = \mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}} \in \mathrm{Str}(\mathfrak{G})$.

(b) Suppose, first, that $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$. Then, by Lemma 1521, $\mathbb{L}^{\mathbf{\Gamma}} \in \mathrm{Str}(\mathfrak{G})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}_\Sigma & \text{iff} & \phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi) \\ & \text{iff} & \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma. \end{array}$$

So $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}$.

Suppose, conversely, that $\mathbb{L}^{\mathbf{\Gamma}} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}$. Then, by Lemma 1522, $\mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}} \in \mathrm{ThFam}(\mathfrak{G})$ and, hence, $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$.

(c) By Parts (a) and (b),

$$\mathbb{L} \longmapsto \mathbf{\Gamma}^{\mathbb{L}}$$
$$\mathbb{L}^{\mathbf{\Gamma}} \longleftarrow\!\!\dashv \mathbf{\Gamma}$$

are mutually inverse mappings between $\mathrm{ThFam}(\mathfrak{G})$ and $\mathrm{Str}^{\mathfrak{G}}(\mathcal{F})$ and both are clearly order-preserving. Thus $\mathbb{L}^{\mathrm{Thm}(\mathfrak{G})} = \mathcal{I}^{\mathfrak{G}}$ is the least $\mathfrak{G}$-structure on $\mathcal{F}$.

■

The next result shows that a Gentzen π-institution is complete with respect to class of all $\mathfrak{G}$-structures.

Proposition 1524 (Completeness Theorem) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ if and only if, for every $\mathfrak{G}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,*

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \textit{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$.

Suppose, first, that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ and let $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathfrak{G})$, such that $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, for all $i \in I$. Then, by the definition of a $\mathfrak{G}$-structure, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

Assume, conversely, that the displayed condition in the statement holds. Let $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$, such that $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \subseteq \mathbf{\Gamma}_\Sigma$. Then, by definition, $\phi_i \in D^{\mathbf{\Gamma}}_\Sigma(\Phi_i)$, for all $\in I$. Since, by Lemma 1521, $\mathbb{L}^{\mathbf{\Gamma}}$ is a $\mathfrak{G}$-structure, we get, by hypothesis, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$. Therefore, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$. We conclude that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$. ■

Next, we show that the property of being a model of a Gentzen π-institution is preserved under bilogical morphisms.

Proposition 1525 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$ two $\mathbf{F}$-algebraic systems, $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$ two $\mathbf{F}$-structures and $\langle H, \gamma\rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism. $\mathbb{L}$ is a $\mathfrak{G}$-structure if and only if $\mathbb{L}'$ is a $\mathfrak{G}$-structure.*

Proof: For the proof, it suffices to notice that, since $\langle H, \gamma\rangle$ is a bilogical morphism, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)) & \text{iff} & \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in D'_{H(F(\Sigma))}(\gamma_{F(\Sigma)}(\alpha_\Sigma(\Phi)))\\ & \text{iff} & \alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi)).\end{array}$$

The rest of the argument is straightforward: If $\mathbb{L}$ is a $\mathfrak{G}$-structure, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash \phi_i : i \in I\})$ and $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, for all $i \in I$, we get $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, for all $i \in I$, whence, by hypothesis, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, which gives $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi))$. We conclude that $\mathbb{L}'$ is also a $\mathfrak{G}$-structure. If, conversely, $\mathbb{L}'$ is a $\mathfrak{G}$-structure, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash \phi_i : i \in I\})$ and $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, for all $i \in I$, we get $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, for all $i \in I$, whence, by hypothesis, $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi))$, which gives $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. We conclude that $\mathbb{L}$ is also a $\mathfrak{G}$-structure. ∎

In particular, we obtain

Corollary 1526 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. An $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a $\mathfrak{G}$-structure if and only if its reduction $\mathbb{L}^*$ is a $\mathfrak{G}$-structure.*

Proof: This follows directly from Proposition 1525, since the quotient morphism $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. $\mathcal{A}$ is a **$\mathfrak{G}$-algebraic system** if it is the underlying algebraic system of a reduced $\mathfrak{G}$-structure. We denote the class of all $\mathfrak{G}$-algebraic systems by $\mathrm{AlgSys}(\mathfrak{G})$, i.e., we have

$$\mathrm{AlgSys}(\mathfrak{G}) = \{\mathcal{A} : (\exists D \in \mathrm{ClFam}^{\mathfrak{G}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}})\}.$$

We show that, if a Gentzen π-institution $\mathfrak{G}$ happens to be adequate for a π-institution $\mathcal{I}$, then every $\mathfrak{G}$-algebraic system is also an $\mathcal{I}$-algebraic system.

Lemma 1527 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ that is adequate for $\mathcal{I}$. Then:*

(a) Every $\mathfrak{G}$-structure is an $\mathcal{I}$-structure;

(b) $\mathrm{AlgSys}(\mathfrak{G}) \subseteq \mathrm{AlgSys}(\mathcal{I})$.

Proof:

(a) Let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G})$. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. By the adequacy of $\mathfrak{G}$ for $\mathcal{I}$, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Since $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Thus, by Lemma 50, $\mathbb{L}$ is an $\mathcal{I}$-structure.

(b) Assume that $\mathfrak{G}$ is adequate for $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathfrak{G})$. Then, there exists a $\mathfrak{G}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, such that $\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. To conclude that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, it suffices, by Proposition 1436, to show that $\mathbb{L} \in \mathrm{Str}^{\mathcal{I}}(\mathcal{A})$. But this was done in Part (a).

■

20.3 Fully Adequate Gentzen π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, such that $\mathfrak{G}$ is adequate for $\mathcal{I}$. Then, with $\mathcal{I}$ may be associated two classes of $\mathbf{F}$-algebraic systems and two classes of $\mathcal{I}$-structures:

- $\mathcal{I}$-algebraic systems and full $\mathcal{I}$-structures;
- $\mathfrak{G}$-algebraic systems and $\mathfrak{G}$-structures.

We devise certain conditions that, when possible to enforce, would guarantee that a Gentzen π-institution $\mathfrak{G}$ adequate for $\mathcal{I}$ can be picked in such a way as to have $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathcal{I})$ and $\mathrm{Str}(\mathfrak{G}) = \mathrm{FStr}(\mathcal{I})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. $\mathfrak{G}$ is said to be **fully adequate for** $\mathcal{I}$ if one of the following two conditions holds:

- $\mathcal{I}$ has theorems, $\mathfrak{G}$ is of type 1 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$;
- $\mathcal{I}$ does not have theorems, $\mathfrak{G}$ is of type 0 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbb{L}$ does not have theorems.

We show that, if $\mathfrak{G}$ is fully adequate for $\mathcal{I}$, then it is also adequate for $\mathcal{I}$.

Proposition 1528 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is fully adequate for* $\mathcal{I}$, *then* $\mathfrak{G}$ *is adequate for* $\mathcal{I}$.

Proof: Assume that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

If $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$, then, by definition, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Since, by Corollary 1428, $\langle \mathcal{F}, C \rangle \in \mathrm{FStr}(\mathcal{I})$, we get, by hypothesis, $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G})$. Therefore, $\phi \in C_\Sigma(\Phi)$.

Assume, conversely, that $\phi \in C_\Sigma(\Phi)$. Since, by Proposition 1523, $\langle \mathcal{F}, C^{\mathfrak{G}} \rangle \in \mathrm{Str}(\mathfrak{G})$, which, additionally, does not have theorems, if $\mathcal{I}$ has no theorems, we get, by hypothesis, $\langle \mathcal{F}, C^{\mathfrak{G}} \rangle \in \mathrm{FStr}(\mathcal{I})$. But, by Corollary 1428, $\langle \mathcal{F}, C \rangle$ is the weakest full $\mathcal{I}$-structure on $\mathcal{F}$. Therefore, since $\phi \in C_\Sigma(\Phi)$, we get that $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$. ∎

We provide next a characterization of full adequacy, which also showcases its features and hints at why it is a useful notion in trying to connect π-institutions with Gentzen π-institutions.

Proposition 1529 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is fully adequate for* $\mathcal{I}$ *if and only if*

1. $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathcal{I})$*;*
2. *For all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is the only reduced* $\mathfrak{G}$*-structure on* $\mathcal{A}$ *(without theorems if* $\mathcal{I}$ *does not have any);*
3. $\mathcal{I}$ *has theorems and* $\mathfrak{G}$ *is of type 1 or* $\mathcal{I}$ *does not have theorems and* $\mathfrak{G}$ *is of type 0.*

Proof: Assume $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. Note that Condition 3 holds by definition. By Proposition 1528, $\mathcal{I}$ is adequate for $\mathcal{I}$. By Lemma 1527, $\mathrm{AlgSys}(\mathfrak{G}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. If, on the other hand, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced full $\mathcal{I}$-structure. Thus, by hypothesis, it is a reduced $\mathfrak{G}$-structure. It follows that $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$. This shows that Condition 1 also holds. It remains now to prove Condition 2. To this end, suppose $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced full $\mathcal{I}$-structure. By hypothesis, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced $\mathfrak{G}$-structure. By the Isomorphism Theorem 1445, it is the only full $\mathcal{I}$-structure on $\mathcal{A}$ that is reduced. Hence, by hypothesis, it is the only reduced $\mathfrak{G}$-structure on $\mathcal{A}$. This proves Condition 2 and concludes the "only if".

Assume, conversely, that Conditions 1-3 hold. Then, for all $\mathbf{F}$-structures $\mathbb{L} = \langle \mathcal{A}, D\rangle$,

$$\begin{array}{lll}\mathbb{L} \in \mathrm{FStr}(\mathcal{I}) & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I}) \text{ and } \mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\\ & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathfrak{G}) \text{ and } \langle \mathcal{A}^*, \mathcal{D}^*\rangle \in \mathrm{Str}(\mathfrak{G})\\ & & \text{(w/o theorems if } \mathcal{I} \text{ does not have any)}\\ & \text{iff} & \langle \mathcal{A}, \mathcal{D}\rangle \in \mathrm{Str}(\mathfrak{G})\\ & & \text{(w/o theorems if } \mathcal{I} \text{ does not have any).}\end{array}$$

This, combined with Condition 3, gives that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. ∎

If a π-institution $\mathcal{I}$ has a fully adequate Gentzen π-institution, then that Gentzen π-institution is unique.

Proposition 1530 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *two Gentzen* π*-institutions based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are fully adequate for* $\mathcal{I}$*, then* $\mathfrak{G} = \mathfrak{G}'$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$. Then, we get $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ if and only if, by Proposition 1524, for every $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathfrak{G})$,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$$

if and only if, by full adequacy, for all $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$$

if and only if, by full adequacy, for every $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathfrak{G}')$,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$$

if and only if, by Proposition 1524, $\Phi \vdash_\Sigma \phi \in G'_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$. Therefore, $G = G'$ and, hence, $\mathfrak{G} = \mathfrak{G}'$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. Recall the notation for the family of $\mathbf{F}$-equations $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2$. Let K be a class of $\mathbf{F}$-algebraic systems and recall the relative equational consequence of K

$$C^{\mathsf{K}} = \{C^{\mathsf{K}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} : \mathcal{P}(\mathrm{Eq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}(\mathbf{F}))$$

given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, by

$$\begin{array}{lll} \phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K}, \\ & & E \subseteq \mathrm{Ker}_\Sigma(\mathcal{A}) \text{ implies } \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}). \end{array}$$

We show that the structure $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}^2, C^{\mathsf{K}}\rangle$ is a π-structure.

Lemma 1531 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and let* K *be a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}^2, C^{\mathsf{K}}\rangle$ *is a* π*-structure.*

Proof: By Lemma **??**. ∎

Recall from Proposition 115, that $\mathcal{Q}^{\mathsf{K}}$ satisfies the properties of reflexivity, symmetry, transitivity, congruence and invariance. So we have

Corollary 1532 *et* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and let* K *be a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}^2, C^{\mathsf{K}}\rangle$ *is an equational* π*-structure.*

Proof: By Lemma 1531 and Proposition 115. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic systems. According to the framework developed in Chapter 12, we say that $\mathfrak{G}$ is **equivalent to** $\mathcal{Q}^{\mathsf{K}}$ if there exists a conjugate pair of translations $(t,s):\mathfrak{G}\rightleftarrows\mathcal{Q}^{\mathsf{K}}$, where

$$t:\quad \mathfrak{G}\longrightarrow\mathcal{Q}^{\mathsf{K}}$$
$$\mathfrak{G}\longleftarrow\mathcal{Q}^{\mathsf{K}}\quad :s$$

We will focus specifically on the case in which the translation $sq:\mathrm{Eq}(\mathbf{F})\to\mathrm{SenFam}(\mathfrak{G})$ is natural and given by the natural transformation $\kappa:\mathrm{Eq}(\mathbf{F})\to\mathcal{P}(\mathrm{Seq}(\mathbf{F}))$, determined, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, by

$$\kappa_\Sigma(\phi,\psi)=\{\phi\vdash_\Sigma\psi,\psi\vdash_\Sigma\phi\}.$$

Recall that, in this case, since κ does not have any parameters, we have that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$,

$$sq_\Sigma[\phi\approx\psi]=\{sq_{\Sigma,\Sigma'}[\phi\approx\psi]\}_{\Sigma'\in|\mathbf{Sign}^\flat|},$$

where

$$sq_{\Sigma,\Sigma'}[\phi\approx\psi]=\{\mathrm{SEN}^\flat(f)(\phi\vdash_\Sigma\psi),\mathrm{SEN}^\flat(\psi\vdash_\Sigma\phi):f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')\}.$$

Finally, we say that the Gentzen π-institution $\mathfrak{G}$ **has** or **satisfies Congruence** if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi_i,\psi_i\in\mathrm{SEN}^\flat(\Sigma)$, $i<k$,

$$\sigma^\flat_\Sigma(\vec{\phi})\vdash_\Sigma\sigma^\flat_\Sigma(\vec{\psi})\in G_\Sigma(\bigcup_{i<k}sq_\Sigma[\phi_i\approx\psi_i]).$$

We show that the equivalence of a Gentzen π-institution $\mathfrak{G}$ with an equational π-institution $\mathcal{Q}^{\mathsf{K}}$ implies that $\mathfrak{G}$ satisfies Congruence and, moreover, that it has interesting consequences for any π-institution for which $\mathfrak{G}$ happens to be adequate. More precisely, such a π-institution must be self extensional and the variety generated by its Lindenbaum-Tarski $\mathbf{F}$-algebraic system must coincide with the variety generated by the class K.

Proposition 1533 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathfrak{G}=\langle\mathbf{F},G\rangle$ a Gentzen π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic systems. If $\mathfrak{G}$ is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via a conjugate pair $(t,sq):\mathfrak{G}\rightleftarrows\mathcal{Q}^{\mathsf{K}}$, then $\mathfrak{G}$ satisfies Congruence. If, in addition, $\mathfrak{G}$ is adequate for a π-institution $\mathcal{I}=\langle\mathbf{F},C\rangle$, then $\mathcal{I}$ is self extensional and $\mathsf{Q}(\mathsf{K})=\mathsf{K}^{\mathcal{I}}$.*

Proof: Suppose $\mathfrak{G}$ is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via a conjugate pair $(t,sq):\mathfrak{G}\rightleftarrows\mathcal{Q}^{\mathsf{K}}$ and let $\sigma^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi_i,\psi_i\in\mathrm{SEN}^\flat(\Sigma)$, $i<k$. By Proposition 115,

$$\sigma^\flat_\Sigma(\vec{\phi})\approx\sigma^\flat_\Sigma(\vec{\psi})\in C^{\mathsf{K}}_\Sigma(\{\phi_i\approx\psi_i:i<k\}).$$

Thus, by the hypothesis,

$$\sigma^\flat_\Sigma(\vec{\phi})\vdash_\Sigma\sigma^\flat_\Sigma(\vec{\psi})\in G_\Sigma(\bigcup_{i<k}sq_\Sigma[\phi_i\approx\psi_i]).$$

Thus, $\mathfrak{G}$ satisfies Congruence.

Suppose, next, that $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a π-institution, for which $\mathfrak{G}$ is adequate, and let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$, such that $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$. By structurality, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Then, by adequacy,

$$\mathrm{SEN}^\flat(f)(\phi_i \vdash_\Sigma \psi_i), \mathrm{SEN}^\flat(f)(\psi_i \vdash_\Sigma \phi_i) \in G_{\Sigma'}(\varnothing).$$

Since $\mathfrak{G}$ has Congruence, we get

$$\begin{array}{rcl}\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}), \sigma^\flat_\Sigma(\vec{\psi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\phi}) & \in & G_\Sigma(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]) \\ & \subseteq & G_\Sigma(G_\Sigma(\varnothing)) \\ & = & G_\Sigma(\varnothing)\end{array}.$$

Again using adequacy, $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$. Therefore, $\widetilde{\lambda}(\mathcal{I})$ is a congruence system on $\mathbf{F}$ and, by Proposition 1464, $\mathcal{I}$ is self extensional.

For the last claim, recall that

$$\begin{array}{rcl}\mathsf{Q}(\mathsf{K}) & = & \{\mathcal{A} : \mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})\}; \\ \mathsf{K}^\mathcal{I} = \mathsf{Q}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I})) & = & \{\mathcal{A} : \widetilde{\Omega}(\mathcal{I}) \leq \mathrm{Ker}(\mathcal{A})\}.\end{array}$$

Moreover, note that $\mathrm{Ker}(\mathsf{K}) = \mathrm{Thm}(\mathcal{Q}^\mathsf{K})$. Therefore, to see that the claim holds, it suffices to show that $\mathrm{Thm}(\mathcal{Q}^\mathsf{K}) = \widetilde{\Omega}(\mathcal{I})$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll}\phi \approx \psi \in D^\mathsf{K}_\Sigma(\varnothing) & \text{iff} & sq_\Sigma[\phi \approx \psi] \leq G(\varnothing) \quad \text{(by hypothesis)} \\ & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\psi) \quad \text{(by adequacy)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(by definition)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}). \quad \text{(by self extensionality)}\end{array}$$

Thus, we have $\mathsf{Q}(\mathsf{K}) = \mathsf{K}^\mathcal{I}$, as claimed. ■

In closing the section, we show that, given a π-institution $\mathcal{I}$ that has an adequate finitary Gentzen π-institution $\mathfrak{G}$, satisfying Congruence, the equational consequence based on the variety $\mathsf{K}^\mathcal{I}$ is translated into the consequence of the Gentzen π-institution via sq.

Proposition 1534 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a finitary Gentzen π-institution, having Congruence, that is adequate for $\mathcal{I}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \approx \psi \in D^{\mathsf{K}^\mathcal{I}}_\Sigma(E) \quad \textit{implies} \quad sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E]).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in D^{\mathsf{K}^\mathcal{I}}_\Sigma(E)$. By Theorem 119, we have $D^{\mathsf{K}^\mathcal{I}} = \Xi^{\mathrm{Ker}(\mathsf{K}^\mathcal{I})} = \Xi^{\widetilde{\Omega}(\mathcal{I})}$. So, we get $\phi \approx \psi \in \Xi^{\widetilde{\Omega}(\mathcal{I})}_\Sigma(E)$. We show by induction on $n < \omega$, that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{\widetilde{\Omega}(\mathcal{I}),n}_\Sigma(E) \quad \text{implies} \quad sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E]).$$

- For $n = 0$, we must have $\phi = \psi$ or $\langle\phi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$ or $\phi \approx \psi \in E$.

  In the first case the conclusion follows by (Axiom).

  In the second case, we have that $C_\Sigma(\phi) = C_\Sigma(\psi)$, whence, by adequacy, $sq_\Sigma[\phi \approx \psi] \leq G(\varnothing) \leq G(sq_\Sigma[E])$.

  In the last case, the conclusion follows by the inflationarity of G.

- Suppose, now, that the implication holds for $n > 0$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n+1}(E)$.

  If $\psi \approx \phi \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n}(E)$, then, by the induction hypothesis, $sq_\Sigma[\psi \approx \phi] \leq G(sq_\Sigma[E])$. Since $sq_\Sigma[\phi \approx \psi] = sq_\Sigma[\psi \approx \phi]$, we conclude that $sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E])$.

  If $\phi \approx \chi, \chi \approx \psi \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n}(E)$, then, by the induction hypothesis,

  $$sq_\Sigma[\phi \approx \chi], sq_\Sigma[\chi \approx \psi] \leq G(sq_\Sigma[E]).$$

  Using (Cut) and monotonicity, we get

  $$\begin{array}{rcl} sq_\Sigma[\phi \approx \psi] & \leq & G(sq_\Sigma[\phi \approx \chi], sq_\Sigma[\chi \approx \psi]) \\ & \leq & G(sq_\Sigma[E]). \end{array}$$

  If $\phi \approx \psi$ is of the form $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi})$, with $\phi_i \approx \psi_i \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n}(E)$, $i < k$, then, by the induction hypothesis, $sq_\Sigma[\phi_i \approx \psi_i] \leq G(sq_\Sigma[E])$ $i < k$. Then, since $\mathfrak{G}$ has Congruence, we conclude

  $$\begin{array}{rcl} sq_\Sigma[\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi})] & \leq & G(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]) \\ & \leq & G(sq_\Sigma[E]). \end{array}$$

  Last, assume that $\phi \approx \psi$ has the form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, for some $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma',\Sigma)$, such that $\phi' \approx \psi' \in \Xi_{\Sigma'}^{\widetilde{\Omega}(\mathcal{I}),n}(E)$. Then, by the induction hypothesis, $sq_{\Sigma'}[\phi' \approx \psi'] \leq G(sq_\Sigma[E])$. But, note that $sq_\Sigma[\phi \approx \psi] = sq_\Sigma[\mathrm{SEN}^\flat(f)(\phi' \approx \psi')] \leq sq_{\Sigma'}[\phi' \approx \psi']$. Thus, we get $sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E])$.

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, $\phi \approx \psi \in D_\Sigma^{\mathsf{K}^\mathcal{I}}(E)$ implies that $sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E])$. ■

20.4 Smoothness and Finitary Adaptations

In this section we define smooth Gentzen π-institutions and we also adapt some of the preceding results to the case of finitary π-institutions. This work is meant to pave the way for upcoming results on self extensionality and conjunction, presented in the next section, and on self extensionality and the deduction detachment theorem, which follow in the section after that.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. We say that $\mathfrak{G}$ is **smooth** if G operates on finite sequents and it is systemic, i.e., by Proposition 149, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(\Phi \vdash_\Sigma \phi) \in G_{\Sigma'}(\Phi \vdash_\Sigma \phi).$$

In the case of smooth Gentzen systems, the equivalence of the Gentzen system with an algebraic π-structure may be simplified as follows.

Proposition 1535 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a smooth Gentzen π-institution and K a class of $\mathbf{F}$-algebraic systems. Then $\mathfrak{G}$ is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via the conjugate pair $(t, sq) : \mathfrak{G} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$ if and only if its is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via the conjugate pair $(t, \kappa) : \mathfrak{G} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$.*

Proof: By Lemma 892, it is enough to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$G(sq_\Sigma[\phi \approx \psi]) = G(\kappa_\Sigma(\phi \approx \psi)).$$

This is, however, a consequence of smoothness. ■

Moreover, for a smooth Gentzen π-institution $\mathfrak{G}$, satisfying Congruence is equivalent to an apparently simpler condition.

Proposition 1536 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a smooth Gentzen π-institution. $\mathfrak{G}$ satisfies Congruence if and only if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,*

$$\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \in G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}).$$

Proof: Assume, first, that $\mathfrak{G}$ satisfies Congruence. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,

$$\begin{array}{lll}\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) & \in & G_\Sigma(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]) \\ & & \text{(by Congruence)} \\ & \subseteq & G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}). \\ & & \text{(by Smoothness)}\end{array}$$

Assume, conversely, that the given condition holds. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,

$$\begin{array}{lll}\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) & \in & G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}) \\ & & \text{(Hypothesis)} \\ & \subseteq & G_\Sigma(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]). \\ & & \text{(by Monotonicity)}\end{array}$$

We conclude that $\mathfrak{G}$ has Congruence. ■

If a π-institution $\mathcal{I}$ is finitary, any $\mathcal{I}$-structure must also be finitary. Therefore, for any Gentzen π-institution $\mathfrak{G}$, no infinitary $\mathfrak{G}$-structure can be a full $\mathcal{I}$-structure. It is this observation that leads to the following modification of the definition of full adequacy for finitary π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. $\mathfrak{G}$ is said to be **fully adequate for** $\mathcal{I}$ if one of the following two conditions holds:

- $\mathcal{I}$ has theorems, $\mathfrak{G}$ is of type 1 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L}$ is finitary and $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$;
- $\mathcal{I}$ does not have theorems, $\mathfrak{G}$ is of type 0 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbb{L}$ is finitary without theorems.

For the sequel we need a finitary adaptation of Proposition 1529. This is a characterization of full adequacy of a Gentzen system for a finitary π-institution $\mathcal{I}$.

Proposition 1537 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is fully adequate for* $\mathcal{I}$ *if and only if*

1. $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathcal{I})$*;*
2. *For all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ *is the only finitary and reduced* $\mathfrak{G}$*-structure on* $\mathcal{A}$ *(without theorems if* $\mathcal{I}$ *does not have any);*
3. $\mathcal{I}$ *has theorems and* $\mathfrak{G}$ *is of type 1 or* $\mathcal{I}$ *does not have theorems and* $\mathfrak{G}$ *is of type 0.*

Proof: Assume $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. Then, by Proposition 1529, Conditions 1-3 hold, where in Condition 2 $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is finitary by Proposition 114. Thus, the "only if" holds.

Assume, conversely, that Conditions 1-3 hold. Then, for all $\mathbf{F}$-structures $\mathbb{L} = \langle \mathcal{A}, D \rangle$,

$$\begin{array}{lll} \mathbb{L} \in \mathrm{FStr}(\mathcal{I}) & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I}), \mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \\ & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathfrak{G}) \text{ and } \langle \mathcal{A}^*, \mathcal{D}^* \rangle \in \mathrm{Str}(\mathfrak{G}) \\ & & \text{finitary (w/o theorems if } \mathcal{I} \text{ does not have any)} \\ & \text{iff} & \langle \mathcal{A}, \mathcal{D} \rangle \in \mathrm{Str}(\mathfrak{G}) \text{ finitary} \\ & & \text{(w/o theorems if } \mathcal{I} \text{ does not have any).} \end{array}$$

This, combined with Condition 3, gives that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. ■

20.5 IsoFull Adequacy and the DD Theorem

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and, for all $n < \omega$, $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a collection of natural transformations in $N^\flat$, with $n+1$ distinguished arguments. Set

$$\Delta = \{\Delta^n : n < \omega\}.$$

Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, and $T \in \mathrm{ThFam}(\mathcal{I})$, Δ is a **Parameterized Graded Deduction Detachment** (**PGDD**) **system for** $\mathcal{I}$ **over** T if, for all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0, \ldots, \phi_{n-1}, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(T_\Sigma, \phi_0, \ldots, \phi_{n-1}) \quad \text{iff} \quad \Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi] \leq T.$$

The left-to-right implication is the **Graded Deduction Property over** T and the right-to-left implication is the **Graded Detachment Property over** T.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\Delta = \{\Delta^n : n < \omega\}$ in $N^\flat$. Define a family $r^{\Delta^n} = \{r^{\Delta^n}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of Gentzen $\mathbf{F}$-rules by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$r^{\Delta^n}_\Sigma = \{\langle \{\phi_0, \ldots, \phi_{n-1} \vdash_\Sigma \psi\}, \vdash_\Sigma \Delta^n_\Sigma(\phi_0, \ldots, \phi_{n-1}, \psi, \vec{\chi}) \rangle : \vec{\phi}, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Existence of a PGDD system Δ over a theory family T guarantees that the $\mathcal{I}$-structure $\langle \mathcal{F}, C^T \rangle$ satisfies all Gentzen $\mathbf{F}$-rules in r^{Δ^n}, $n < \omega$.

Lemma 1538 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $T \in \mathrm{ThFam}(\mathcal{I})$ *and* $\Delta = \{\Delta^n : n < \omega\}$ *a PGDD system for* $\mathcal{I}$ *over* T*. Then, for all* $n < \omega$*,*

$$\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T \rangle \vDash r^{\Delta^n}.$$

Proof: Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi}, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \in C^T_\Sigma(\phi_0, \ldots, \phi_{n-1})$. Equivalently, we get $\psi \in C_\Sigma(T_\Sigma, \phi_0, \ldots, \phi_{n-1})$. By hypothesis, since Δ is a PGDD system for $\mathcal{I}$ over T, we get $\Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi] \leq T$. In particular, we get, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\Delta^n_\Sigma(\phi_0, \ldots, \phi_{n-1}, \psi, \vec{\chi}) \subseteq T_\Sigma$. Equivalently, $\vdash_\Sigma \Delta^n_\Sigma(\phi_0, \ldots, \phi_{n-1}, \psi, \vec{\chi}) \in C^T_\Sigma(\varnothing)$. Thus, $\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T \rangle \vDash r^{\Delta^n}$. ■

We show, next, that all Gentzen $\mathbf{F}$-rules are preserved by bilogical morphisms between $\mathbf{F}$-structures.

Lemma 1539 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures and* $\langle H, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, every* $\mathbf{F}$*-sequent* $\Psi \vdash_\Sigma \psi$ *and every Gentzen* $\mathbf{F}$*-rule* $r := \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$*,* *(a)* $\mathbb{L} \vDash_\Sigma \Psi \vdash_\Sigma \psi$ *if and only if* $\mathbb{L}' \vDash_\Sigma \Psi \vdash_\Sigma \psi$*;*

(b) $\mathbb{L} \vDash_\Sigma r$ *if and only if* $\mathbb{L}' \vDash_\Sigma r$*.*

Proof:

(a) We have

$$\begin{array}{lll}\mathbb{L} \vDash_\Sigma \Psi \vdash_\Sigma \psi & \text{iff} & \alpha_\Sigma(\psi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi)) \\ & \text{iff} & \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi)) \in D'_{H(F(\Sigma))}(\gamma_{H(F(\Sigma))}(\alpha_\Sigma(\Psi))) \\ & \text{iff} & \alpha'_\Sigma(\psi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Psi)) \\ & \text{iff} & \mathbb{L}' \vDash_\Sigma \Psi \vdash_\Sigma \psi.\end{array}$$

(b) This part follows easily from Part (a).

($\Rightarrow$) If $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, $i \in I$, then $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, $i \in I$, whence $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. So $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi))$.

($\Leftarrow$) If $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, $i \in I$, then $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, $i \in I$, whence $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, $i \in I$, and, therefore, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

■

Some of the elements of the discussion that follows will be revisited in Chapter 21 on $\mathcal{I}$-operators in a more general context. We give a preview of a few results here, as needed, restricting the discussion mostly to protoalgebraic π-institutions. This restriction will be lifted in Chapter 21, where the concepts will be revisited in full generality.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and an $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we let

$$[T] = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)\},$$

the **equi-Leibniz class of** T. If $[T]$ has a smallest member, it is denoted by T^*. T is called a **Leibniz filter** if $T = T^*$, i.e., if it is the smallest filter in its equi-Leibniz class. We denote by $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ the collection of all Leibniz $\mathcal{I}$-filter families of $\mathcal{A}$.

We show that Leibniz filter families are preserved under inverse surjective morphisms with isomorphic functor components. For a more general result, see Corollary 1612 in Chapter 21.

Lemma 1540 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism. Then*

$$\gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Let $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})$ and $T = \gamma^{-1}(T')$. Then $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'))$. Consider $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $X \in [T]$, i.e., such that $\Omega^{\mathcal{A}}(X) = \Omega^{\mathcal{A}}(T)$. Then, since $\Omega^{\mathcal{A}}(T) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'))$, $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with X. Hence $\gamma(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Furthermore,

$$\gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(X))) = \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(X))) = \Omega^{\mathcal{A}}(X) = \Omega^{\mathcal{A}}(T) = \gamma^{-1}(\Omega^{\mathcal{B}}(T')).$$

So $\Omega^{\mathcal{B}}(\gamma(X)) = \Omega^{\mathcal{B}}(T')$. Thus, since $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})$, $T' \le \gamma(X)$. Now we get, taking again into account the compatibility of $\mathrm{Ker}(\langle H,\gamma\rangle)$ with X, $T = \gamma^{-1}(T') \le \gamma^{-1}(\gamma(X)) = X$. This proves that $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

As a corollary, we obtain

Corollary 1541 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems and* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism. Then, for all* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*, such that* T'^* *exists,*

$$\gamma^{-1}(T'^*) = \gamma^{-1}(T')^*.$$

Proof: Suppose $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that T'^* exists. Then $\gamma^{-1}(T'^*) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, by Lemma 1540. Hence, we have

$$\Omega^{\mathcal{A}}(\gamma^{-1}(T'^*)) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'^*)) = \gamma^{-1}(\Omega^{\mathcal{B}}(T')) = \Omega^{\mathcal{A}}(\gamma^{-1}(T')).$$

Thus, since $\gamma^{-1}(T'^*)$ is Leibniz, we get $\gamma^{-1}(T'^*) = \gamma^{-1}(T')^*$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. An $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, is called **isofull** if it is full and F is an isomorphism.

We show, next, that, if $\Delta = \{\Delta^n : n < \omega\}$ is a PGDD system for $\mathcal{I}$ over every $\mathcal{I}$-theory family, then every isofull $\mathcal{I}$-structure satisfies the Gentzen $\mathbf{F}$-rules r^{Δ^n}.

Lemma 1542 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$ *and* $\Delta = \{\Delta^n : n < \omega\}$ *a PGDD system for* $\mathcal{I}$ *over every Leibniz* $\mathcal{I}$*-theory family. Then every isofull* $\mathcal{I}$*-structure satisfies* r^{Δ^n}*, for all* $n < \omega$*.*

Proof: By Lemma 1539, it suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with F an isomorphism, and every $n < \omega$,

$$\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \vDash r^{\Delta^n}.$$

Let $T = C^{\mathcal{I},\mathcal{A}}(\varnothing)$ be the smallest $\mathcal{I}$-filter family on $\mathcal{A}$. By the Correspondence Theorem for protoalgebraic π-institutions, we have $\alpha^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) =$

$\mathrm{ThFam}(\mathcal{I})^{\alpha^{-1}(T)}$. Since T is least among all $\mathcal{I}$-filter families of $\mathcal{A}$, we have $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Therefore, by Lemma 1540, $\alpha^{-1}(T) \in \mathrm{ThFam}^*(\mathcal{I})$. Thus, by the hypothesis and Lemma 1538, we get that $\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^{\alpha^{-1}(T)}\rangle \vDash r^{\Delta^n}$. However, $\langle F, \alpha\rangle : \langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^{\alpha^{-1}(T)}\rangle \vdash \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a bilogical morphism, whence, by Lemma 1539, we get $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \vDash r^{\Delta^n}$. ■

In the next lemma, it is shown that, in case the π-institution $\mathcal{I}$ is syntactically protoalgebraic, the witnessing transformations may be used to generate Leibniz filter families.

Lemma 1543 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then*

$$T^* = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)\}).$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We set

$$\widetilde{T} = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)\}).$$

Our goal is to show that $T^* = \widetilde{T}$. First, let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T' \in [T]$, i.e., $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Then, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rll} \langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T) & \text{iff} & \langle \phi,\psi\rangle \Omega^{\mathcal{A}}_\Sigma(T') \\ & \text{iff} & \overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] \leq T'. \end{array}$$

We conclude that $\widetilde{T} \leq T^*$ and, by protoalgebraicity, $\Omega^{\mathcal{A}}(\widetilde{T}) \leq \Omega^{\mathcal{A}}(T^*)$. On the other hand, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$, $\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] \leq \widetilde{T}$. Thus, $\Omega^{\mathcal{A}}(T^*) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(\widetilde{T})$. Therefore, $\Omega^{\mathcal{A}}(\widetilde{T}) = \Omega^{\mathcal{A}}(T^*)$ and, since we showed that $\widetilde{T} \leq T^*$, we get by the minimality property of T^* in $[T]$, $T^* = \widetilde{T}$. ■

Corollary 1544 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then* $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ *if and only if, there exists* $X \leq \mathrm{SEN}^2$*, such that*

$$T = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle \phi,\psi\rangle \in X_\Sigma\}).$$

Proof: For the left-to-right implication, assume $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Take $X = \Omega^{\mathcal{A}}(T)$. Then we have, using the hypothesis and Lemma 1543, $T = T^* = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle \phi,\psi\rangle \in X_\Sigma\})$.

Suppose, conversely, that $T = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in X_\Sigma\})$, for some $X \leq \mathrm{SEN}$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, $\langle\phi,\psi\rangle \in X_\Sigma$ implies $\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] \leq T$. Thus, $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$. Therefore, by Lemma 1543, $T \leq C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in X_\Sigma\}) = T^*$. Since, it is always the case that $T^* \leq T$, we get that $T = T^*$ and $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. A collection $\Delta = \{\Delta^n : n < \omega\}$, where $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}$ in $N^\flat$, with $n+1$ distinguished arguments, is called **Leibniz generating over** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi},\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\Delta^n_\Sigma[\vec{\phi},\psi]) \in \mathrm{ThFam}^*(\mathcal{I}),$$

for all $n < \omega$.

We show that, for a syntactically protoalgebraic π-institution $\mathcal{I}$, the property of being Leibniz generating over $\mathcal{I}$, transfers, in certain sense, to the filter families over arbitrary $\mathbf{F}$-algebraic systems.

Lemma 1545 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\Delta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a Leibniz generating collection in* $N^\flat$*, with* $n+1$ *distinguished arguments. Then for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\Sigma \in |\mathbf{Sign}|$ *and* $\vec{\phi},\psi \in \mathrm{SEN}(\Sigma)$*,*

$$C^{\mathcal{I},\mathcal{A}}(\Delta^{\mathcal{A}}_\Sigma[\vec{\phi},\psi]) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: By hypothesis, Δ is Leibniz generating. Hence, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi},\psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\Delta_\Sigma[\vec{\phi},\psi]) \in \mathrm{ThFam}^*(\mathcal{I})$. Thus, by Corollary 1544, there exists $X \leq (\mathrm{SEN}^\flat)^2$, such that

$$C(\Delta_\Sigma[\vec{\phi},\psi]) = C(\bigcup\{\overset{\leftrightarrow\flat}{I}_{\Sigma'}[\phi',\psi'] : \Sigma' \in |\mathbf{Sign}^\flat|, \langle\phi',\psi'\rangle \in X_{\Sigma'}\}).$$

Now we get

$$\begin{array}{l}C^{\mathcal{I},\mathcal{A}}(\Delta^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\vec{\phi}),\alpha_\Sigma(\psi)])\\ = C^{\mathcal{I},\mathcal{A}}(\alpha(\Delta_\Sigma[\phi,\psi]))\\ = C^{\mathcal{I},\mathcal{A}}(\alpha(C(\bigcup\{\overset{\leftrightarrow\flat}{I}_{\Sigma'}[\phi',\psi'] : \Sigma' \in |\mathbf{Sign}^\flat|, \langle\phi',\psi'\rangle \in X_{\Sigma'}\})))\\ = C^{\mathcal{I},\mathcal{A}}(\alpha(\bigcup\{\overset{\leftrightarrow\flat}{I}_{\Sigma'}[\phi',\psi'] : \Sigma' \in |\mathbf{Sign}^\flat|, \langle\phi',\psi'\rangle \in X_{\Sigma'}\}))\\ = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_{F(\Sigma')}[\alpha_{\Sigma'}(\phi'),\alpha_{\Sigma'}(\psi')] : \Sigma' \in |\mathbf{Sign}^\flat|,\\ \qquad\qquad \langle\alpha_{\Sigma'}(\phi'),\alpha_{\Sigma'}(\psi')\rangle \in \alpha_{\Sigma'}(X_{\Sigma'})\}).\end{array}$$

Thus, taking into account the surjectivity of $\langle F,\alpha\rangle$, we obtain, using again Corollary 1544, that for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi},\psi \in \mathrm{SEN}(\Sigma)$, $C^{\mathcal{I},\mathcal{A}}(\Delta^{\mathcal{A}}_\Sigma[\vec{\phi},\psi]) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. A PGDD system $\Delta = \{\Delta^n : n < \omega\}$ for $\mathcal{I}$ is called **Leibniz generating** if Δ^n is Leibniz generating, for every $n < \omega$.

It is not difficult to see that Leibniz generating PGDD systems have a graded Modus Ponens property, in the sense detailed in the following

Lemma 1546 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\Delta = \{\Delta^n : n < \omega\}$ is a PGDD system for $\mathcal{I}$ over every Leibniz theory family, then, for all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\psi \in C_\Sigma(\Delta^n_\Sigma[\vec{\phi}, \psi], \vec{\phi}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and set $T = C(\Delta^n_\Sigma[\vec{\phi}, \psi])$. By hypothesis, $T \in \mathrm{ThFam}^*(\mathcal{I})$. Since Δ is a PGDD system for $\mathcal{I}$ over every Leibniz theory family, we get

$$\psi \in C_\Sigma(T_\Sigma, \vec{\phi}) \quad \text{iff} \quad \Delta^n_\Sigma[\vec{\phi}, \psi] \leq C(T) = T.$$

Thus, since the right hand side of the equivalence holds, we obtain $\psi \in C_\Sigma(T_\Sigma, \vec{\phi}) = C_\Sigma(C_\Sigma(\Delta^n_\Sigma[\vec{\phi}, \psi]), \vec{\phi}) = C_\Sigma(\Delta^n_\Sigma[\vec{\phi}, \psi], \vec{\phi})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\Delta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with a single distinguished argument. We say that Δ **isodefines Leibniz filter families over** $\mathcal{I}$ if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with F an isomorphism, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

$$T^*_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : \Delta^{\mathcal{A}}_\Sigma[\phi] \leq T\}.$$

We show, next, that, in a syntactically protoalgebraic π-institution $\mathcal{I}$, which has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over every Leibniz theory family, the 0-th component Δ^0 does isodefine Leibniz filter families over $\mathcal{I}$.

A couple of lemmas are needed first.

Lemma 1547 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \leq T'$ implies $T^* \leq T'^*$.*

Proof: Suppose $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. By protoalgebraicity of $\mathcal{I}$, we get

$$\Omega^{\mathcal{A}}(T \cap T'^*) = \Omega^{\mathcal{A}}(T) \cap \Omega^{\mathcal{A}}(T'^*) = \Omega^{\mathcal{A}}(T) \cap \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T).$$

Thus $T^* \leq T \cap T'^* \leq T'^*$. ∎

Lemma 1548 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I}) \quad \textit{iff} \quad T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have, using Theorem 1432 and protoalgebraicity, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$ if and only if

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

Since, under protoalgebraicity, it always holds that

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\},$$

it suffices to show that

$$\begin{array}{lll} T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) & \text{iff} & \text{for all } T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \\ & & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } T \leq T'. \end{array}$$

The right to left implication is trivial, since the condition on the right implies that T is smallest among all filter families sharing the same Leibniz congruence system with T. For the converse, suppose T is a Leibniz filter family of $\mathcal{A}$ and that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then, using protoalgebraicity, we get $\Omega(T \cap T') = \Omega^{\mathcal{A}}(T) \cap \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, we conclude that $T = T^* \leq T \cap T' \leq T'$. ∎

Theorem 1549 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with* $\Delta = \{\Delta^n : n < \omega\}$ *a Leibniz generating PGDD system for* $\mathcal{I}$ *over every Leibniz theory family. Then* Δ^0 *isodefines Leibniz filter families over* $\mathcal{I}$*.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, with F an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$.

Suppose, first, that $\Delta^0_\Sigma[\phi] \leq T$. Let $T' = C^{\mathcal{I},\mathcal{A}}(\Delta^0_\Sigma[\phi])$. By hypothesis and Lemma 1545, $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Since $T' \leq T$, by Lemma 1547, $T' \leq T^*$. Hence, $\Delta^0_\Sigma[\phi] \leq T^*$. Therefore, by Lemma 1546, $\phi \in T^*_\Sigma$.

Suppose, conversely, that $\phi \in T^*_\Sigma$. Then, by definition, the $\mathcal{I}$-structure $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*} \rangle$ satisfies $\vdash_\Sigma \phi$. By Lemma 1548, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*} \rangle \in \mathrm{FStr}(\mathcal{I})$. Thus, by Lemma 1542, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})^{T^*} \rangle$ satisfies r^{Δ^0}. Hence, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$, $\Delta^0_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T^*_{\Sigma'} \subseteq T_{\Sigma'}$. We conclude that $\Delta^0_\Sigma[\phi] \leq T$. ∎

We are ready now to prove one half of the main result of this section. We would like to show that, for a syntactically protoalgebraic finitary π-institution, the existence of a Leibniz generating PGDD system over all

Leibniz theory families implies the existenc of an isofully adequate Gentzen π-institution.

We define that institution, first, preceding the statement of the theorem that involves it in its proof.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\Delta = \{\Delta^n : n < \omega\}$ in $N^\flat$, where Δ^n has $n+1$ distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Define:

- $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\};$$
- $\mathrm{Ir}^{\mathcal{I}} = \{\mathrm{Ir}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ir}^{\mathcal{I}}_\Sigma = \{r^{\Delta^n}_\Sigma : n < \omega\}.$$

Set $R^{\mathcal{I}} = \mathrm{Ax}^{\mathcal{I}} \cup \mathrm{Ir}^{\mathcal{I}}$ and let
$$\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$$
be the Gentzen π-institution generated by $R^{\mathcal{I}}$ (recall that $G^{\mathcal{I}}$ is required to be a structural closed system on $\mathrm{Seq}(\mathbf{F})$ and, therefore, it is assumed to satisfy, by default, (Axiom), (Weakening) and (Cut)).

Theorem 1550 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over all Leibniz theory families, then it has an isofully adequate Gentzen π-institution, namely the Gentzen π-institution $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$.*

Proof: We must show that, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with F an isomorphism, we have
$$\mathbb{L} \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}}) \quad \text{iff} \quad \mathbb{L} \in \mathrm{FStr}(\mathcal{I}).$$

Suppose, first, that $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$. Then $\mathbb{L}$ is, in particular, an $\mathcal{I}$-structure. Therefore, it satisfies $\mathrm{Ax}^{\mathcal{I}}$. Moreover, by Lemma 1542, $\mathbb{L}$ satisfies r^{Δ^n}, for all $n < \omega$. Hence, it also satisfies $\mathrm{Ir}^{\mathcal{I}}$. We conclude that $\mathbb{L} \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$.

Suppose, conversely, that $\mathbb{L} \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. Clearly, $\mathbb{L} \in \mathrm{Str}(\mathcal{I})$, since it satisfies $\mathrm{Ax}^{\mathcal{I}}$. So it suffices to show that it is also full. Assume, to the contrary, that $\mathbb{L}$ is not full and let $T = D(\varnothing)$. Since $\mathcal{I}$ is protoalgebraic and $\mathbb{L}$ is not full, we have, using Lemma 1548, $\mathcal{D} \subsetneq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$. Consider $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*} - \mathcal{D}$. Then we get $D(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$ and $T^* \leq T' \lneq D(T')$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(T') - T'_\Sigma$. Then, there exists $\Phi \subseteq_f T'_\Sigma$, such that $\phi \in D_\Sigma(\Phi)$. Since $\mathbb{L}$ satisfies $\mathrm{Ir}^{\mathcal{I}}$, we get $\Delta^n_\Sigma[\Phi, \phi] \leq T$. But Δ is also Leibniz generating, whence, by Lemma 1545,

$D(\Delta^n_\Sigma[\Phi,\phi]) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Therefore, by Lemma 1547, $\Delta^n_\Sigma[\Phi,\phi] \leq T^*$. Now we get $\phi \in D_\Sigma(T'_\Sigma,\Phi) \subseteq D_\Sigma(T'_\Sigma) = T'_\Sigma$, which contradicts our assumption. Therefore, $\mathbb{L}$ is also full, as was to be shown. ■

Suppose, now, that $\mathcal{I}$ is a syntactically protoalgebraic, finitary π-institution with an isofully adequate Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$. Then, for all $\langle \mathcal{F}, D\rangle \in \mathrm{Str}(\mathfrak{G})$, we must have$\langle \mathcal{F}, D\rangle \in \mathrm{FStr}(\mathcal{I})$ and, therefore, taking into account Lemma 1548, we get that $\mathcal{D} = \mathrm{ThFam}(\mathcal{I})^T$, where $T \in \mathrm{ThFam}^*(\mathcal{I})$. We denote by

$$h^{\mathfrak{G}} : \mathrm{Str}^{\mathfrak{G}}(\mathcal{F}) \to \mathrm{ThFam}^*(\mathcal{I})$$

the bijection that is established by this association, which is, in addiction an order isomorphism.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$. Given a theory family $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$, recall the $\mathbf{F}$-structure $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, D^{\mathbf{\Gamma}}\rangle$, which was shown in Lemma 1521 to be a $\mathfrak{G}$-structure. For notational purposes, given $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_0,\ldots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$, let us write

$$\mathfrak{G}[\phi_0,\ldots,\phi_{n-1} \vdash_\Sigma \psi] \coloneqq \mathbb{L}^{G(\phi_0,\ldots,\phi_{n-1}\vdash_\Sigma\psi)},$$

where, as usual, $G(\phi_0,\ldots,\phi_{n-1} \vdash_\Sigma \psi)$ denotes the least theory family of $\mathfrak{G}$ including the $\mathbf{F}$-sequent $\phi_0,\ldots,\phi_{n-1} \vdash_\Sigma \psi$.

We call the Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ **transformational** if, for all all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0,\ldots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\ldots,\phi_{n-1} \vdash_\Sigma \psi]) = C(\Delta^n_\Sigma[\phi_0,\ldots,\phi_{n-1},\psi]),$$

for some $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $n+1$ distinguished arguments.

We can show that the isofully adequate Gentzen π-institution $\mathfrak{G}^{\mathcal{I}}$ associated with a syntactically protoalgebraic finitary π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ that has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over all Leibniz theory families, as in Theorem 1550, is, in fact, a transformational Gentzen π-institution.

Theorem 1551 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically protoalgebraic finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over all Leibniz theory families, then the isofully adequate Gentzen π-institution $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}}\rangle$ for $\mathcal{I}$ is transformational.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_0,\ldots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$ and denote

$$T \coloneqq C(\Delta^n_\Sigma[\phi_0,\ldots,\phi_{n-1},\psi]).$$

By hypothesis, we have $T \in \mathrm{ThFam}^*(\mathcal{I})$. Since T is a Leibniz $\mathcal{I}$-theory family and Δ is a PGDD system over all Leibniz theory families, T is closed under all axioms and rules of $\mathfrak{G}^{\mathcal{I}}$. Moreover, if $\phi_0,\dots,\phi_{n-1} \in T_\Sigma$, then, since $\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi] \le T$, we get, by Lemma 1546, $\psi \in T_\Sigma$. Hence, T is a theory family of the $\mathbf{F}$-structure $\mathfrak{G}^{\mathcal{I}}[\phi_0,\dots,\phi_{n-1},\psi]$. Since, by definition, $h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi])$ is its least theory family, we get that $h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]) \le C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi])$.

On the other hand, by definition,

$$\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi \in G^{\mathcal{I}}_\Sigma(\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi),$$

whence, writing $\mathfrak{G}^{\mathcal{I}}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi] := \langle \mathcal{F}, D\rangle$, we get $\psi \in D_\Sigma(\phi_0,\dots,\phi_{n-1})$. Recalling that every full model is structural, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_0),\dots,\mathrm{SEN}^\flat(f)(\phi_{n-1})).$$

Thus, since $\mathfrak{G}^{\mathcal{I}}$ satisfies r^{Δ^n}, we get, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\Delta_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_0),\dots,\mathrm{SEN}^\flat(f)(\phi_{n-1}),\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \subseteq D_\Sigma(\varnothing),$$

i.e., that $\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi] \le h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi])$. We now conclude that

$$C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]) \le h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]).$$

Therefore, for all all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]) = C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]),$$

showing that $\mathfrak{G}^{\mathcal{I}}$ is transformational. ■

Finally, we show that, for a syntactically protoalgebraic, finitary π-institution $\mathcal{I}$, the existence of an isofully adequate, transformational Gentzen π-institution $\mathfrak{G}$ for $\mathcal{I}$ implies that $\mathcal{I}$ has a Leibniz generating PGDD system over every Leibniz theory family.

Theorem 1552 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically protoalgebraic, finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has an isofully adequate transformational Gentzen π-institution, then $\mathcal{I}$ has a Leibniz generating PGDD system over every Leibniz theory family.*

Proof: Suppose that $\mathcal{I}$ has an isofully adequate transformational Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$. Thus, by definition, for all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$, there exists a collection $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $n+1$ distinguished arguments, such that

$$h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1},\psi]) = C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]).$$

By the fact that $h^{\mathfrak{G}}$ maps, by hypothesis and Lemma 1548, into $\mathrm{ThFam}^*(\mathcal{I})$, ensures that $\Delta = \{\Delta^n : n < \omega\}$ is Leibniz generating. So it suffices to show that Δ is a PGDD system for $\mathcal{I}$ over every Leibniz theory family. To this end, assume that $T \in \mathrm{ThFam}^*(\mathcal{I})$, $n < \omega$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi_0, \ldots, \phi_{n-1}, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\psi \in C_\Sigma(T_\Sigma, \phi_0, \ldots, \phi_{n-1}) \quad \text{iff} \quad \Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi] \le T.$$

We have

$$\begin{array}{rll}
\psi \in C_\Sigma(T_\Sigma, \phi_0, \ldots, \phi_{n-1}) & \text{iff} & \psi \in C^T_\Sigma(\phi_0, \ldots, \phi_{n-1}) \\
& \text{iff} & \mathfrak{G}[\phi_0, \ldots, \phi_{n-1} \vdash_\Sigma \psi] \le \mathcal{C}^T \\
& \text{iff} & h^{\mathfrak{G}}(\mathfrak{G}(\phi_0, \ldots, \phi_{n-1} \vdash_\Sigma \psi]) \le T \\
& \text{iff} & C(\Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi]) \le T \\
& \text{iff} & \Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi] \le T.
\end{array}$$

We conclude that Δ is indeed Leibniz generating PGDD system for $\mathcal{I}$ over every Leibniz theory family. ∎

In conclusion, we have

Theorem 1553 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. A syntactically protoalgebraic, finitary* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *based on* $\mathbf{F}$ *has an isofully adequate transformational Gentzen* π*-institution if and only if it has a Leibniz generating PGDD system over every Leibniz theory family.*

Proof: The "if" was proven in Theorems 1550 and 1551. The "only if" is by Theorem 1552. ∎