Chapter 21

Operators on π-Institutions

21.1 $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

An **$\mathcal{I}$-operator on $\mathcal{A}$** is a map

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{I}) \to \mathrm{EqvFam}(\mathcal{A}),$$

where $\mathrm{EqvFam}(\mathcal{A})$ is the collection of equivalence families on $\mathcal{A}$.

Given an $\mathcal{I}$-operator $O^{\mathcal{A}}$ on $\mathcal{A}$, we define three derived operators (functions) as follows:

- The **lifting of** $O^{\mathcal{A}}$, $\widetilde{O}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{EqvFam}(\mathcal{A})$, is given, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by
$$\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \bigcap \{ O^{\mathcal{A}}(T) : T \in \mathcal{T} \};$$
- The **relativization of** $O^{\mathcal{A}}$ **to** $\mathcal{I}$, $\widetilde{O}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$, is given, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by
$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \bigcap \{ O^{\mathcal{A}}(T') : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \} = \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T);$$
- $O^{\mathcal{A}^{-1}} : \mathrm{EqvFam}(\mathcal{A}) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is given, for all $\theta \in \mathrm{EqvFam}(\mathcal{A})$, by
$$O^{\mathcal{A}^{-1}}(\theta) = \{ T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \le O^{\mathcal{A}}(T) \}.$$

Note that the lifting of the Leibniz operator $\Omega^{\mathcal{A}}$ on $\mathcal{A}$ is the Tarski operator $\widetilde{\Omega}^{\mathcal{A}}$ on $\mathcal{A}$, whereas the relativization of the Leibniz operator on $\mathcal{A}$ is the Suszko operator $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ on $\mathcal{A}$.

Immediately from the definitions, we obtain the following:

Lemma 1554 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ an $\mathcal{I}$-operator on $\mathcal{A}$.*

(a) $\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \le O^{\mathcal{A}}(T)$*, for all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(b) $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \le O^{\mathcal{A}}(T)$*, for all* $T \in \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: Obvious from the definitions. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, let there be given an $\mathcal{I}$-operator $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$. We write

$$O = \{ O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) \}$$

and refer to this family as a **family of $\mathcal{I}$-operators**.

Since $\mathcal{I}$-operators are meant to abstract the operators of abstract algebraic logic, those properties that were studied in preceding chapters concerning the Leibniz operator play also a significant role when it comes to $\mathcal{I}$-operators.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

an $\mathcal{I}$-operator on $\mathcal{A}$.

- $O^{\mathcal{A}}$ is **order-preserving** or **monotone** if, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,
$$T \le T' \quad \text{implies} \quad O^{\mathcal{A}}(T) \le O^{\mathcal{A}}(T');$$
- $O^{\mathcal{A}}$ is **order-reflecting** or **reflective** if, for all $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$,
$$O^{\mathcal{A}}(T) \le O^{\mathcal{A}}(T') \quad \text{implies} \quad T \le T';$$
- $O^{\mathcal{A}}$ is **completely order reflecting** or **c-reflective** if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,
$$\bigcap_{T \in \mathcal{T}} O^{\mathcal{A}}(T) \le O^{\mathcal{A}}(T') \quad \text{implies} \quad \bigcap \mathcal{T} \le T'.$$

Some important characterizations are related to these properties.

Lemma 1555 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ an $\mathcal{I}$-operator on $\mathcal{A}$.*

$$O^{\mathcal{A}} \text{ is monotone if and only if } O^{\mathcal{A}} = \widetilde{O}^{\mathcal{I},\mathcal{A}}.$$

Proof: Suppose, first, that $O^{\mathcal{A}}$ is monotone. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & \bigcap\{O^{\mathcal{A}}(T') : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T). \end{array}$$

If, conversely, $O^{\mathcal{A}} = \widetilde{O}^{\mathcal{I},\mathcal{A}}$, then, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$, we get

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \\ & = & \bigcap\{O^{\mathcal{A}}(T'') : T \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & \le & \bigcap\{O^{\mathcal{A}}(T'') : T' \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T') \\ & = & O^{\mathcal{A}}(T'). \end{array}$$

Therefore, $O^{\mathcal{A}}$ is monotone. ■

Lemma 1556 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.* $O^{\mathcal{A}}$ *is c-reflective if and only if, for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \leq O^{\mathcal{A}}(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $O^{\mathcal{A}}$ is c-reflective and let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \leq O^{\mathcal{A}}(T')$. Then, by definition, $\bigcap\{O^{\mathcal{A}}(T'') : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq O^{\mathcal{A}}(T')$. By c-reflectivity, $\bigcap\{T'' : T \leq T'' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})\} \leq T'$, i.e., $T \leq T'$.

Suppose, conversely, that the displayed condition holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T')$. Then we get

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(\bigcap \mathcal{T}) \leq \bigcap_{T \in \mathcal{T}} O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T').$$

Hence, by the hypothesis, $\bigcap \mathcal{T} \leq T'$ and $O^{\mathcal{A}}$ is c-reflective. ■

We now show that the operators $\widetilde{O}^{\mathcal{A}}$ and $O^{\mathcal{A}^{-1}}$, associated with a given $\mathcal{I}$-operator $O^{\mathcal{A}}$, establish a Galois connection between the class $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ of bundles of $\mathcal{I}$-filters on $\mathcal{A}$ and the class $\mathrm{EqvFam}(\mathcal{A})$ of equivalence families on $\mathcal{A}$. This will yield several important consequences.

Proposition 1557 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*. The maps*

$$\begin{array}{rcl} \widetilde{O}^{\mathcal{A}} : \ \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & \longrightarrow & \mathrm{EqvFam}(\mathcal{A}) \\ \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & \longleftarrow & \mathrm{EqvFam}(\mathcal{A}) \quad : O^{\mathcal{A}^{-1}} \end{array}$$

establish a Galois connection, where $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ *is ordered under the subclass relation and* $\mathrm{EqvFam}(\mathcal{A})$ *under signature-wise inclusion.*

Proof: We must show that, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\theta \in \mathrm{EqvFam}(\mathcal{A})$,

$$\mathcal{T} \subseteq O^{\mathcal{A}^{-1}}(\theta) \quad \text{iff} \quad \widetilde{O}^{\mathcal{A}}(\mathcal{T}) \geq \theta.$$

In fact, we have

$$\begin{array}{rll} \mathcal{T} \subseteq O^{\mathcal{A}^{-1}}(\theta) & \text{iff} & \theta \leq O^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \theta \leq \bigcap\{O^{\mathcal{A}}(T) : T \in \mathcal{T}\} \\ & \text{iff} & \theta \leq \widetilde{O}^{\mathcal{A}}(\mathcal{T}). \end{array}$$

Thus $(\widetilde{O}^{\mathcal{A}}, O^{\mathcal{A}^{-1}}) : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \rightleftarrows \mathrm{EqvFam}(\mathcal{A})$ is, in fact, a Galois connection. ■

Corollary 1558 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

(a) The operators

$$\widetilde{O}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{EqvFam}(\mathcal{A})$$
$$O^{\mathcal{A}^{-1}} : \mathrm{EqvFam}(\mathcal{A}) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$$

are order reversing;

(b) The operators

$$O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$$
$$\widetilde{O}^{\mathcal{A}} \circ O^{\mathcal{A}^{-1}} : \mathrm{EqvFam}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

are closure operators;

(c) The collection of fixed-points of $O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}}$ *is the range of* $O^{\mathcal{A}^{-1}}$ *and the collection of fixed-points of* $\widetilde{O}^{\mathcal{A}} \circ O^{\mathcal{A}^{-1}}$ *is the range of* $\widetilde{O}^{\mathcal{A}}$*;*

(d) $\widetilde{O}^{\mathcal{A}}$ *and* $O^{\mathcal{A}^{-1}}$ *restrict to mutually inverse order isomorphisms between the collections of fixed-points of* $O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}}$ *and of fixed-points of* $\widetilde{O}^{\mathcal{A}} \circ O^{\mathcal{A}^{-1}}$*.*

Proof: Known facts about Galois connections. ■

We capture the elements described in Part (c) of Corollary 1558, by making the following definitions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

an $\mathcal{I}$-operator on $\mathcal{A}$.

- A family $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is called $O^{\mathcal{A}}$-**full** if $\mathcal{T} = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$ if and only if $\mathcal{T} \in \mathrm{Ran}(O^{\mathcal{A}^{-1}})$;
- An equivalence family $\theta \in \mathrm{EqvFam}(\mathcal{A})$ is $O^{\mathcal{A}}$-**full** if $\theta = \widetilde{O}^{\mathcal{A}}(O^{\mathcal{A}^{-1}}(\theta))$ if and only if $\theta \in \mathrm{Ran}(\widetilde{O}^{\mathcal{A}})$.

The following statements provide a justification of the terminology used.

Proposition 1559 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

(a) *A collection* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is* $O^{\mathcal{A}}$*-full if and only if it is the largest* $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{O}^{\mathcal{A}}(\mathcal{D}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$*;*

(b) *An equivalence family* $\theta \in \mathrm{EqvFam}(\mathcal{A})$ *is* $O^{\mathcal{A}}$*-full if and only if it is the largest* $\eta \in \mathrm{EqvFam}(\mathcal{A})$*, such that* $O^{\mathcal{A}^{-1}}(\eta) = O^{\mathcal{A}^{-1}}(\theta)$*.*

Proof: We do Part (a). Part (b) can be proved analogously. Suppose, first, that $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is $O^{\mathcal{A}}$-full and let $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{D}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. Then, we have

$$\mathcal{D} \subseteq O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{D})) = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \mathcal{T}.$$

Suppose, conversely, that $\mathcal{T}$ is the largest among $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{D}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$ and let $T \in O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$. Then, by definition, $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \leq O^{\mathcal{A}}(T)$. Hence, $\widetilde{O}^{\mathcal{A}}(\mathcal{T} \cup \{T\}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. By the maximality of $\mathcal{T}$, we conclude that $T \in \mathcal{T}$. This shows that $O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) \subseteq \mathcal{T}$. Since the opposite inclusion always holds, we conclude that $\mathcal{T}$ is a fixed point of $O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}}$ and, hence, it is $O^{\mathcal{A}}$-full. ■

We have the following consequences:

Corollary 1560 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

(a) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is* $O^{\mathcal{A}}$*-full;*

(b) $\nabla^{\mathcal{A}}$ *is* $O^{\mathcal{A}}$*-full;*

(c) *If* $O^{\mathcal{A}}$ *is monotone and* $\mathcal{T}$ *is* $O^{\mathcal{A}}$*-full, then* $\mathcal{T}$ *is an upset in the poset* $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$*.*

Proof: All three statements are direct consequences of Proposition 1559. ■

21.2 Congruential $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. An $\mathcal{I}$-operator $O^{\mathcal{A}}$: $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ is called **congruential** if, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $O^{\mathcal{A}}(T) \in \mathrm{ConSys}(\mathcal{A})$. Thus a congruential $\mathcal{I}$-operator is an operator $O^{\mathcal{A}}$: $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$.

Proposition 1561 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system,* $\theta \in \mathrm{ConSys}(\mathcal{A})$ *and* $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$ *the quotient natural transformation.*

(a) $\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$ *and* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\Omega^{\mathcal{A}^{-1}}(\theta))$*;*

(b) *The mappings*

$$\begin{array}{l}\pi : \mathrm{SenFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{SenFam}(\mathcal{A}/\theta)\\ \pi^{-1} : \mathrm{SenFam}(\mathcal{A}/\theta) \to \mathrm{SenFam}(\mathcal{A})\end{array}$$

restrict to mutually inverse order isomorphisms between $\Omega^{\mathcal{A}^{-1}}(\theta)$ *and* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$.

Proof:

(a) Suppose $T \in \Omega^{\mathcal{A}^{-1}}(\theta)$. Then $\theta \le \Omega^{\mathcal{A}}(T)$. Hence θ is compatible with T, which implies, by Corollary 57, $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$. Suppose, conversely, that $T \in \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$. Then $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$ and, hence, by Corollary 57, $\pi^{-1}(\pi(T)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, T is compatible with θ, showing that $\theta \le \Omega^{\mathcal{A}}(T)$. This gives $T \in \Omega^{\mathcal{A}^{-1}}(\theta)$.

The second equality of Part (a) is obtained from the first, using the surjectivity of $\langle I, \pi\rangle$.

(b) By Part (a), the mappings

$$\begin{array}{l}\pi \restriction_{\Omega^{\mathcal{A}^{-1}}(\theta)} : \Omega^{\mathcal{A}^{-1}}(\theta) \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\\ \pi^{-1} \restriction_{\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \Omega^{\mathcal{A}^{-1}}(\theta)\end{array}$$

are well-defined. Moreover, they are clearly inverses of one another and order preserving. Thus, they establish an order isomorphism between $\Omega^{\mathcal{A}^{-1}}(\theta)$ and $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$.

■

21.3 O-Classes and O-Filter Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

an $\mathcal{I}$-operator on $\mathcal{A}$. For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the O-**class of** T, denoted $[\![T]\!]^O$, is the collection

$$[\![T]\!]^O = \Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : O^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')\}.$$

It turns out that this class forms a closure family on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proposition 1562 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $[\![T]\!]^O$ *is a closure family on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: First, observe that $\Omega^{\mathcal{A}}(\mathrm{SEN}) = \nabla^{\mathcal{A}}$, whence $\mathrm{SEN} \in [\![T]\!]^O$. Next, let $\{T^i : i \in I\} \subseteq [\![T]\!]^O$. Then, we have

$$O^{\mathcal{A}}(T) \leq \bigcap_{i\in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i).$$

So $\bigcap_{i\in I} T^i \in [\![T]\!]^O$ and $[\![T]\!]^O$ is a closure family on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ∎

Something even stronger is true in case $O^{\mathcal{A}}$ happens to be a congruential $\mathcal{I}$-operator. In that case, the pair $\langle \mathcal{A}, [\![T]\!]^O \rangle$ turns out to be a full $\mathcal{I}$-structure.

Proposition 1563 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *a congruential* $\mathcal{I}$*-operator on* $\mathcal{A}$*. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\langle \mathcal{A}, [\![T]\!]^O \rangle$ *is a full* $\mathcal{I}$*-structure.*

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then $[\![T]\!]^O = \Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T))$. By hypothesis, $O^{\mathcal{A}}(T) \in \mathrm{ConSys}(\mathcal{A})$. Thus, by Proposition 1561,

$$[\![T]\!]^O = \Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T)) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))),$$

where $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ is the quotient natural transformation. Thus, by definition, $\langle \mathcal{A}, [\![T]\!]^O \rangle$ is a full $\mathcal{I}$-structure. ∎

As a corollary, we obtain the fact that $[\![T]\!]^O$ is a closure system on $\mathcal{A}$ and, therefore, $\langle \mathcal{A}, [\![T]\!]^O \rangle$ is a π-institution and not merely a π-structure.

Corollary 1564 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *a congruential* $\mathcal{I}$*-operator on* $\mathcal{A}$*. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $[\![T]\!]^O$ *is a closure system on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: By Propositions 1563 and 1426. ∎

Corollary 1564 justifies the following definition.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$$

a congruential $\mathcal{I}$-operator on $\mathcal{A}$. For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the least element of the O-class of T is denoted by T^O:

$$T^O = \bigcap [\![T]\!]^O.$$

A $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is called an O-**filter family** if $T = T^O$. Note that, by Corollary 1564, an O-filter family must be an $\mathcal{I}$-filter system.

The collection of all O-filter systems of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$.

Proposition 1565 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^\mathcal{A} : \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.* $O^\mathcal{A}$ *is reflective (and, hence, injective) on* $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$*.*

Proof: Let $T, T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$, such that $O^\mathcal{A}(T) \le O^\mathcal{A}(T')$. Then, $[\![T']\!]^O \subseteq [\![T]\!]^O$. Therefore,

$$\begin{array}{rcll} T & = & T^O & (T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})) \\ & = & \bigcap [\![T]\!]^O & (\text{definition}) \\ & \le & \bigcap [\![T']\!]^O & ([\![T']\!]^O \subseteq [\![T]\!]^O) \\ & = & T'^O & (\text{definition}) \\ & = & T'. & (T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})) \end{array}$$

We conclude that $O^\mathcal{A}$ is reflective. ∎

Proposition 1566 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^\mathcal{A} : \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *a monotone* $\mathcal{I}$*-operator on* $\mathcal{A}$*. Then the mapping* $T \mapsto T^O$ *is monotone, i.e., for all* $T, T' \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$*,*

$$T \le T' \quad \textit{implies} \quad T^O \le T'^O.$$

Proof: We have, for all $T, T' \in \mathrm{ThFam}^\mathcal{I}(\mathcal{A})$,

$$\begin{array}{rll} T \le T' & \text{implies} & O^\mathcal{A}(T) \le O^\mathcal{A}(T') \quad (\text{hypothesis}) \\ & \text{implies} & [\![T']\!]^O \subseteq [\![T]\!]^O \quad (\text{definitions of } [\![T]\!]^O, [\![T']\!]^O) \\ & \text{implies} & T^O \le T'^O. \quad (\text{definitions of } T^O, T'^O) \end{array}$$

So $T \mapsto T^O$ is a monotone mapping. ∎

Proposition 1567 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system,* $O^\mathcal{A} : \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *a congruential* $\mathcal{I}$*-operator on* $\mathcal{A}$ *and* $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$*.* $T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$ *if and only if* $T/O^\mathcal{A}(T)$ *is the least* $\mathcal{I}$*-filter family of* $\mathcal{A}/O^\mathcal{A}(T)$*.*

Proof: By hypothesis, $O^\mathcal{A}(T) \in \mathrm{ConSys}(\mathcal{A})$. Consider the quotient natural transformation

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/O^\mathcal{A}(T).$$

Since $\Omega^{\mathcal{A}^{-1}}(O^\mathcal{A}(T)) = [\![T]\!]^O$, we get, by Proposition 1561, that

$$\pi : [\![T]\!]^O \to \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/O^\mathcal{A}(T))$$

is an order isomorphism. Thus, $T^O/O^\mathcal{A}(T)$ is the least $\mathcal{I}$-filter family on $\mathcal{A}/O^\mathcal{A}(T)$. ∎

21.4 Compatibility $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ an $\mathcal{I}$-operator on $\mathcal{A}$. $O^{\mathcal{A}}$ is called a **compatibility $\mathcal{I}$-operator** if, for all $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$,

$$O^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T).$$

Clearly, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ is the largest compatibility $\mathcal{I}$-operator on $\mathcal{A}$. If one assumes monotonicity, then this role is played by the Suszko operator instead:

Lemma 1568 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Suszko operator $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ is the largest monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$.*

Proof: Suppose that $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ is a monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{rcll} O^{\mathcal{A}}(T) & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T) & \text{(by Lemma 1555)} \\ & = & \bigcap\{O^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} & \text{(by Definition)} \\ & \leq & \bigcap\{\Omega^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} & \text{(by Compatibility)} \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T). & \text{(by Definition)} \end{array}$$

So $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is the largest monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. ∎

For compatibility $\mathcal{I}$-operators, we have the following properties pertaining to O-classes and O-filter systems.

Lemma 1569 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ a compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \in [\![T]\!]^O \quad \textit{and} \quad T^O \leq T.$$

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $O^{\mathcal{A}}$ is a compatibility $\mathcal{I}$-operator, $O^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$. Thus, by definition of $[\![T]\!]^O$, we get $T \in [\![T]\!]^O$. Moreover, since $T \in [\![T]\!]^O$, we now get $T^O = \bigcap [\![T]\!]^O \leq T$. ∎

For monotone compatibility $\mathcal{I}$-operators, we have the following properties.

Lemma 1570 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ a monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$:*

(a) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^O$;

(b) $[\![T]\!]^O = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *iff* $T = T^O$ *iff* $T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$.

Proof:

(a) Suppose $T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$\begin{array}{rcll} O^{\mathcal{A}}(T) & \leq & O^{\mathcal{A}}(T') & \text{(by Monotonicity)} \\ & \leq & \Omega^{\mathcal{A}}(T'). & \text{(by Compatibility)} \end{array}$$

So, by definition of $[\![T]\!]^O$, $T' \in [\![T]\!]^O$.

(b) The second equivalence if simply the definition of $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. So it suffices to prove the first equivalence. Assume, first, that $[\![T]\!]^O = \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. Then, we have $T^O = \bigcap [\![T]\!]^O = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = T$.

Assume, conversely, that $T = T^O$. Then, if $T' \in [\![T]\!]^O$, we get $T = T^O = \bigcap [\![T]\!]^O \leq T'$. Thus, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since, by Part (a), the converse always holds, we get $[\![T]\!]^O = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$.

■

In the case of compatibility $\mathcal{I}$-operators, there are also close relationships between their classes and their filter families and those associated to the Leibniz operator. More precisely, we get:

Lemma 1571 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *a compatibility* $\mathcal{I}$*-operator on* $\mathcal{A}$*. Then, for every* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*:*

(a) $[\![T]\!]^\Omega \subseteq [\![T]\!]^O$;

(b) $T^O \leq T^\Omega$;

(c) $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})$.

Proof:

(a) Let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{array}{rll} T' \in [\![T]\!]^\Omega & \text{implies} & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \text{(by Definition of } [\![T]\!]^\Omega) \\ & \text{implies} & O^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \text{(by Compatibility)} \\ & \text{implies} & T' \in [\![T]\!]^O. \quad \text{(by Definition of } [\![T]\!]^O) \end{array}$$

Thus, $[\![T]\!]^\Omega \subseteq [\![T]\!]^O$.

(b) Using Part (a), we get $T^O = \bigcap [\![T]\!]^O \leq \bigcap [\![T]\!]^\Omega = T^\Omega$.

(c) Assume $T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. Then, by definition, $T'^O = T'$. Thus, by Part (b), $T' \leq T'^\Omega$. Since, by Lemma 1569, $T'^\Omega \leq T'$, we get $T'^\Omega = T'$ and, therefore, $T' \in \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})$. We conclude that $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})$.

■

For monotone compatibility $\mathcal{I}$-operators, we have similar relationships between their classes and their filter families and those associated to the Suszko operator.

Lemma 1572 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^\mathcal{A} : \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ a monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for every $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$:*

(a) $[\![T]\!]^{\widetilde{\Omega}^\mathcal{I}} \leq [\![T]\!]^O$;

(b) $T^O \leq T^{\widetilde{\Omega}^\mathcal{I}}$;

(c) $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^\mathcal{I}}(\mathcal{A})$.

Proof:

(a) Let $T' \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$. Then we have

$$\begin{array}{lll} T' \in [\![T]\!]^{\widetilde{\Omega}^\mathcal{I}} & \text{implies} & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^\mathcal{A}(T') \quad (\text{by Definition of } [\![T]\!]^{\widetilde{\Omega}^\mathcal{I}}) \\ & \text{implies} & O^\mathcal{A}(T) \leq \Omega^\mathcal{A}(T') \quad (\text{by Lemma 1568}) \\ & \text{implies} & T' \in [\![T]\!]^O. \quad (\text{by Definition of } [\![T]\!]^O) \end{array}$$

Thus, $[\![T]\!]^{\widetilde{\Omega}^\mathcal{I}} \subseteq [\![T]\!]^O$.

(b) Using Part (a), we get $T^O = \bigcap [\![T]\!]^O \leq \bigcap [\![T]\!]^{\widetilde{\Omega}^\mathcal{I}} = T^{\widetilde{\Omega}^\mathcal{I}}$.

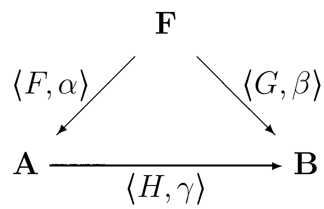
(c) Assume $T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. Then, by definition, $T'^O = T'$. Thus, by Part (b), $T' \leq T'^{\widetilde{\Omega}^\mathcal{I}}$. Since, by Lemma 1569, $T'^{\widetilde{\Omega}^\mathcal{I}} \leq T'$, we get $T'^{\widetilde{\Omega}^\mathcal{I}} = T'$ and, therefore, $T' \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^\mathcal{I}}(\mathcal{A})$. We conclude that $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^\mathcal{I}}(\mathcal{A})$.

■

21.5 Commuting $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$-algebraic systems and $\langle H, \gamma \rangle :$

$\mathcal{A} \to \mathcal{B}$ a surjective morphism.



Let, also $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ and $O^{\mathcal{B}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}) \to \mathrm{EqvFam}(\mathcal{B})$ be $\mathcal{I}$-operators on $\mathcal{A}$ and on $\mathcal{B}$, respectively. We say that the pair $(O^{\mathcal{A}}, O^{\mathcal{B}})$ is **commuting** if, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$O^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(O^{\mathcal{B}}(T')).$$

More generally, let $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ be a family of $\mathcal{I}$-operators. We say that O is a **commuting family** if, for every pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ of $\mathbf{F}$-algebraic systems, and every surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, the pair $(O^{\mathcal{A}}, O^{\mathcal{B}})$ is commuting.

A slightly more relaxed version, which will be of use to us later, is that of semi-commutation. We say that a family of $\mathcal{I}$-operators $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ is a **semi-commuting family** if, for every pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ of $\mathbf{F}$-algebraic systems, and every surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, the pair $(O^{\mathcal{A}}, O^{\mathcal{B}})$ is commuting.

It turns out that semi-commutation is too restrictive when applied to compatibility $\mathcal{I}$-operators, since there is only one semi-commuting family of compatibility $\mathcal{I}$-operators, namely, the Leibniz operator.

Theorem 1573 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a semi-commuting family of compatibility $\mathcal{I}$-operators. Then $O = \Omega$.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

We get, using Compatibility,

$$O^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) \le \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}.$$

So, we get $O^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}$. Since, by hypothesis, O is a semi-commuting family, we now get

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & O^{\mathcal{A}}(\pi^{-1}(T/\Omega^{\mathcal{A}}(T))) \\ & = & \pi^{-1}(O^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T))) \\ & = & \pi^{-1}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & = & \Omega^{\mathcal{A}}(T). \end{array}$$

We conclude that $O = \Omega$. ■

In particular, we have

Corollary 1574 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The Suszko operator* $\widetilde{\Omega}^{\mathcal{I}}$ *is semi-commuting if and only if* $\widetilde{\Omega}^{\mathcal{I}} = \Omega$*.*

Proof: If $\widetilde{\Omega}^{\mathcal{I}} = \Omega$, then, by Proposition 24, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting and, hence,semi-commuting. If conversely, $\widetilde{\Omega}^{\mathcal{I}}$ is semi-commuting, then, by Theorem 1573, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. ■

21.6 Coherent $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of $\mathcal{I}$-operators. Moreover, let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

- The morphism $\langle H, \gamma \rangle$ is said to be O-**compatible with** T if
$$\mathrm{Ker}(\langle H, \gamma \rangle) \leq O^{\mathcal{A}}(T);$$
- The morphism $\langle H, \gamma \rangle$ is said to be O-**compatible with** $\mathcal{T}$ if
$$\mathrm{Ker}(\langle H, \gamma \rangle) \leq O^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T},$$
i.e., if and only if
$$\mathrm{Ket}(\langle H, \gamma \rangle) \leq \widetilde{O}^{\mathcal{A}}(\mathcal{T}).$$

For the Leibniz operator, we have

Corollary 1575 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}, \mathcal{B}$ $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism.* $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T *if and only if* $\mathrm{Ker}(\langle H, \gamma \rangle)$ *is compatible with* T*.*

Proof: We have $\langle H, \gamma \rangle$ is Ω-compatible with T if and only if, by definition, $\mathrm{Ker}(\langle H, \gamma \rangle) \leq \Omega^{\mathcal{A}}(T)$ if and only if, by the compatibility of $\Omega^{\mathcal{A}}(T)$ with T, $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T. ■

Moreover, for a family O of compatibility $\mathcal{I}$-operators, we get

Corollary 1576 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a family of compatibility* $\mathcal{I}$*-operators,* $\mathcal{A}, \mathcal{B}$ $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H, \gamma \rangle$ *is* O*-compatible with* T*, then:*

(a) $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T*;*

(b) *If H is an isomorphism, $T = \gamma^{-1}(\gamma(T))$;*

(c) *If H is an isomorphism, $O^{\mathcal{A}}(T) = \gamma^{-1}(\gamma(O^{\mathcal{A}}(T))$.*

Proof:

(a) We have

$$\begin{array}{rcll}\mathrm{Ker}(\langle H,\gamma\rangle) & \leq & O^{\mathcal{A}}(T) & \text{(hypothesis)}\\ & \leq & \Omega^{\mathcal{A}}(T). & \text{(by Compatibility)}\end{array}$$

Thus, $\langle H,\gamma\rangle$ is Ω-compatible with T.

(b) Suppose H is an isomorphism. By Part (a) and Corollary 1576, we get $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T. Therefore, $\gamma^{-1}(\gamma(T)) \leq T$. Since the reverse inclusion is always satisfied, we get $T = \gamma^{-1}(\gamma(T))$.

(c) Again the inclusion $O^{\mathcal{A}}(T) \leq \gamma^{-1}(\gamma(O^{\mathcal{A}}(T)))$ is always satisfied. So it suffices to show the reverse inclusion. So assume $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \gamma_\Sigma^{-1}(\gamma_\Sigma(O_\Sigma^{\mathcal{A}}(T)))$. Thus, by definition, $\langle\gamma_\Sigma(\phi),\gamma_\Sigma(\psi)\rangle \in \gamma_\Sigma(O_\Sigma^{\mathcal{A}}(T))$. Therefore, there exist $\phi',\psi' \in \mathrm{SEN}(\Sigma)$, with $\langle\phi',\psi'\rangle \in O_\Sigma^{\mathcal{A}}(T)$, such that $\langle\gamma_\Sigma(\phi),\gamma_\Sigma(\psi)\rangle = \langle\gamma_\Sigma(\phi'),\gamma_\Sigma(\psi')\rangle$. This shows that

$$\langle\phi,\phi'\rangle,\langle\psi,\psi'\rangle \in \mathrm{Ker}_\Sigma(\langle H,\gamma\rangle) \leq O_\Sigma^{\mathcal{A}}(T).$$

Since $\langle\phi',\psi'\rangle \in O_\Sigma^{\mathcal{A}}(T)$ and $O^{\mathcal{A}}(T)$ is an equivalence family, we get, using symmetry and transitivity, that $\langle\phi,\psi\rangle \in O_\Sigma^{\mathcal{A}}(T)$. We conclude that $\gamma^{-1}(\gamma(O^{\mathcal{A}}(T))) \leq O^{\mathcal{A}}(T)$.

■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of $\mathcal{I}$-operators.

- O is called **coherent** if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$, every surjective morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ and all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\begin{array}{l}\langle H,\gamma\rangle\ O\text{-compatible with } \gamma^{-1}(T')\\ \qquad\qquad\text{implies } O^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(O^{\mathcal{B}}(T')).\end{array}$$

- O is called **semi-coherent** if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$, every surjective morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\begin{array}{l}\langle H,\gamma\rangle\ O\text{-compatible with } \gamma^{-1}(T')\\ \qquad\qquad\text{implies } O^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(O^{\mathcal{B}}(T')).\end{array}$$

Clearly, if O is coherent, then it is also semi-coherent.

We define the **identity** $\mathcal{I}$-operator

$$I = \{I^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\},$$

by letting, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $I^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$, be given, for all $T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A})$, by

$$I^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

Lemma 1577 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The identity I is a coherent family of compatibility $\mathcal{I}$-operators.*

Proof: It is clear that $I^{\mathcal{A}}$ is a compatibility $\mathcal{I}$-operator, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. So it suffices to prove coherence. To this end, let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathrm{Ker}(\langle H, \gamma \rangle) \leq I^{\mathcal{A}}(\gamma^{-1}(T')) = \Delta^{\mathcal{A}}$. Thus, we have $\mathrm{Ker}(\langle H, \gamma \rangle) = \Delta^{\mathcal{A}}$. Now we get

$$\gamma^{-1}(I^{\mathcal{B}}(T')) = \gamma^{-1}(\Delta^{\mathcal{B}}) = \mathrm{Ker}(\langle H, \gamma \rangle) = \Delta^{\mathcal{A}} = I^{\mathcal{A}}(\gamma^{-1}(T)).$$

Thus, I is a coherent family of compatibility $\mathcal{I}$-operators. ■

Another example of a coherent $\mathcal{I}$-operator is the Leibniz operator.

Lemma 1578 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The Leibniz operator Ω is a coherent family of compatibility operators.*

Proof: By definition Ω is a family of compatibility $\mathcal{I}$-operators. For coherence, assume that $\mathcal{A}$, $\mathcal{B}$ are $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Since, by Proposition 24, $\Omega^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'))$, we get that the coherence implication is trivially satisfied and, hence Ω is a coherent family of compatibility $\mathcal{I}$-operators. ■

For semi-coherence of compatibility $\mathcal{I}$-operators, we get the following characterization.

Lemma 1579 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of compatibility $\mathcal{I}$-operators. O is semi-coherent if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$, all surjective morphisms $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H, \gamma \rangle$ is O-compatible with T, then $\gamma(O^{\mathcal{A}}(T)) = O^{\mathcal{B}}(\gamma(T))$.*

Proof: Suppose, first, that O is semi-coherent and let $\mathcal{A}$, $\mathcal{B}$ be **F**-algebraic systems, $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$ a surjective morphism, with H an isomorphism and $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that

$$\mathrm{Ker}(\langle H,\gamma\rangle)\le O^{\mathcal{A}}(T).$$

Then, by Corollary 1576, $\mathrm{Ker}(\langle H,\gamma\rangle)\le O^{\mathcal{A}}(\gamma^{-1}(\gamma(T)))$. Applying semi-coherence gives

$$\begin{array}{rcll}\gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) & = & O^{\mathcal{A}}(\gamma^{-1}(\gamma(T))) & \text{(by semi-coherence)}\\ & = & O^{\mathcal{A}}(T). & \text{(by Corollary 1576)}\end{array}$$

By the surjectivity of $\langle H,\gamma\rangle$, $O^{\mathcal{B}}(\gamma(T))=\gamma(O^{\mathcal{A}}(T))$.

Assume, conversely, that the condition in the statement holds. Let $\mathcal{A}$, $\mathcal{B}$ be **F**-algebraic systems, $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$ a surjective morphism, with H an isomorphism, and $T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that

$$\mathrm{Ker}(\langle H,\gamma\rangle)\le O^{\mathcal{A}}(\gamma^{-1}(T')).$$

Then, we get

$$\begin{array}{rcll}O^{\mathcal{A}}(\gamma^{-1}(T')) & = & \gamma^{-1}(\gamma(O^{\mathcal{A}}(\gamma^{-1}(T')))) & \text{(by Corollary 1576)}\\ & = & \gamma^{-1}(O^{\mathcal{B}}(\gamma(\gamma^{-1}(T')))) & \text{(by hypothesis)}\\ & = & \gamma^{-1}(O^{\mathcal{B}}(T')). & \text{(by surjectivity of } \langle H,\gamma\rangle)\end{array}$$

So O is a semi-coherent family of compatibility $\mathcal{I}$-operators. ■

We also have the following alternative characterization for semi-coherence of compatibility $\mathcal{I}$-operators.

Lemma 1580 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$ and $O=\{O^{\mathcal{A}}:\mathcal{A}\in\mathrm{AlgSys}(\mathbf{F})\}$ a family of compatibility $\mathcal{I}$-operators. O is semi-coherent if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$ and all surjective morphisms $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$, with H an isomorphism,*

$$O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))=\{T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}):\gamma^{-1}(O^{\mathcal{B}}(\gamma(T)))=O^{\mathcal{A}}(T)\}.$$

Proof: Suppose O is semi-coherent and let $\mathcal{A}$, $\mathcal{B}$ be **F**-algebraic systems, $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$ a surjective morphism, with H an isomorphism, and $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

- If $T\in O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$, then, by definition, $\mathrm{Ker}(\langle H,\gamma\rangle)\le O^{\mathcal{A}}(T)$. Thus, by Lemma 1579, $\gamma(O^{\mathcal{A}}(T))=O^{\mathcal{B}}(\gamma(T))$. Hence,

  $$\gamma^{-1}(\gamma(O^{\mathcal{A}}(T)))=\gamma^{-1}(O^{\mathcal{B}}(\gamma(T))).$$

  Thus, by Corollary 1576, $O^{\mathcal{A}}(T)=\gamma^{-1}(O^{\mathcal{B}}(\gamma(T)))$.

- If $\gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)$, then we get $\mathrm{Ker}(\langle H,\gamma\rangle) = \gamma^{-1}(\Delta^{\mathcal{B}}) \leq \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)$. Thus, $T \in O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$.

We conclude that $O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle)) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)\}$.

Assume, conversely, that the condition of the statement holds. Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathrm{Ker}(\langle H,\gamma\rangle) \leq O^{\mathcal{A}}(\gamma^{-1}(T'))$. Then $\gamma^{-1}(T') \in O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$, whence, by hypothesis,

$$\gamma^{-1}(O^{\mathcal{B}}(\gamma(\gamma^{-1}(T')))) = O^{\mathcal{A}}(\gamma^{-1}(T')).$$

By surjectivity of $\langle H,\gamma\rangle$, $\gamma^{-1}(O^{\mathcal{B}}(T')) = O^{\mathcal{A}}(\gamma^{-1}(T'))$ and, hence, O is a semi-coherent family of compatibility $\mathcal{I}$-operators. ∎

Next we show that semi-coherence of compatibility $\mathcal{I}$-operators is preserved under relativization.

Proposition 1581 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a semi-coherent family of compatibility $\mathcal{I}$-operators. Then*

$$\widetilde{O}^{\mathcal{I}} = \{\widetilde{O}^{\mathcal{I},\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$$

is also a semi-coherent family of compatibility $\mathcal{I}$-operators.

Proof: It is easy to see that $\widetilde{O}^{\mathcal{I}}$ is also a compatibility operator. We have, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{lll}\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) & = & \bigcap\{O^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad \text{(definition)}\\ & \leq & \bigcap\{\Omega^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad \text{(compatibility)}\\ & \leq & \Omega^{\mathcal{A}}(T). \quad \text{(set theory)}\end{array}$$

Thus, $\widetilde{O}^{\mathcal{I}}$ is indeed a compatibility $\mathcal{I}$-operator.

For semi-coherence, assume $\mathcal{A}$, $\mathcal{B}$ are $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathrm{Ker}(\langle H,\gamma\rangle) \leq \widetilde{O}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$. We must show that

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\widetilde{O}^{\mathcal{I},\mathcal{B}}(T')).$$

Claim: We have

$$\{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(T') \leq T\} = \{\gamma^{-1}(T'') : T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\}.$$

- Suppose $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\gamma^{-1}(T') \leq T$. Then, by Corollary 1576, $T = \gamma^{-1}(\gamma(T))$, where, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Moreover, by hypothesis and the surjectivity of $\langle H,\gamma\rangle$, $T' = \gamma(\gamma^{-1}(T')) \leq \gamma(T)$. This proves the left-to-right inclusion.

- Let $T' \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Then, by Corollary 55, we obtain $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by hypothesis, $\gamma^{-1}(T') \le \gamma^{-1}(T'')$. This shows that the right-to-left inclusion also holds.

This proves the Claim. Now, based on the Claim, we reason as follows:

$$\begin{array}{rcl}\gamma^{-1}(\widetilde{O}^{\mathcal{I},\mathcal{B}}(T')) & = & \gamma^{-1}(\bigcap\{O^{\mathcal{B}}(T'') : T' \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\}) \\ & = & \bigcap\{\gamma^{-1}(O^{\mathcal{B}}(T'')) : T' \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\} \\ & = & \bigcap\{O^{\mathcal{A}}(\gamma^{-1}(T'')) : T' \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\} \\ & & \text{(by Semi-Coherence of } O) \\ & = & \bigcap\{O^{\mathcal{A}}(T) : \gamma^{-1}(T') \le T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & & \text{(by the Claim)} \\ & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')).\end{array}$$

Thus, $\widetilde{O}^{\mathcal{I}}$ is indeed semi-coherent. ∎

Proposition 1582 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a coherent family of compatibility* $\mathcal{I}$*-operators. Let, also,* $\mathcal{A}$*,* $\mathcal{B}$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism. For all* $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}')$*, such that* $\langle H, \gamma \rangle$ *is* O*-compatible with* $\gamma^{-1}(\mathcal{T}')$*,*

$$\widetilde{O}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}')) = \gamma^{-1}(\widetilde{O}^{\mathcal{B}}(\mathcal{T}')).$$

Proof: We have

$$\begin{array}{rcl}\gamma^{-1}(\widetilde{O}^{\mathcal{B}}(\mathcal{T}')) & = & \gamma^{-1}(\bigcap\{O^{\mathcal{B}}(T') : T' \in \mathcal{T}'\}) \\ & = & \bigcap\{\gamma^{-1}(O^{\mathcal{B}}(T')) : T' \in \mathcal{T}'\} \\ & = & \bigcap\{O^{\mathcal{A}}(\gamma^{-1}(T')) : T' \in \mathcal{T}'\} \\ & & \text{(hypothesis and coherence)} \\ & = & \bigcap\{O^{\mathcal{A}}(T) : T \in \gamma^{-1}(\mathcal{T}')\} \\ & = & \widetilde{O}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}')).\end{array}$$

∎

Proposition 1583 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Let, also,* $\mathcal{A}$*,* $\mathcal{B}$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism. For all* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle H, \gamma \rangle$ *is* O*-compatible with* $\mathcal{T}$*,* $\gamma(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \widetilde{O}^{\mathcal{B}}(\gamma(\mathcal{T}))$*.*

Proof: By the hypothesis and Corollary 1576, we get that $\mathcal{T} = \gamma^{-1}(\gamma(\mathcal{T}))$. So exploiting Proposition 1582, we get

$$\begin{array}{rcl}\gamma(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) & = & \gamma(\widetilde{O}^{\mathcal{A}}(\gamma^{-1}(\gamma(\mathcal{T})))) \\ & = & \gamma(\gamma^{-1}(\widetilde{O}^{\mathcal{B}}(\gamma(\mathcal{T})))) \\ & = & \widetilde{O}^{\mathcal{B}}(\gamma(\mathcal{T})).\end{array}$$

∎

21.7 Semi-Coherence and Full Objects

We start by providing a characterization of the inverse operator associated with a semi-coherent family of compatibility $\mathcal{I}$-operators.

Proposition 1584 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\theta \in \mathrm{ConSys}(\mathcal{A})$*,*

$$\begin{array}{rcl} O^{\mathcal{A}^{-1}}(\theta) & = & \{T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) : \pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T)\} \\ & = & \pi^{-1}(\{T' \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^\mathcal{A}(\pi^{-1}(T'))\}), \end{array}$$

where $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ *denotes the quotient morphism.*

Proof: We have by hypothesis and Lemma 1580,

$$O^{\mathcal{A}^{-1}}(\theta) = \{T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) : \pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T)\}.$$

For the second equality, if $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$, such that $\pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T)$, then $T = \pi^{-1}(T/\theta)$ and, also,

$$\pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T) = O^\mathcal{A}(\pi^{-1}(T/\theta)).$$

This proves the left-to-right inclusion, since, by Corollary 57, we have $T/\theta \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/\theta)$.

Assume, conversely, that $T' \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/\theta)$, such that $\pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^\mathcal{A}(\pi^{-1}(T'))$. Then, by Corollary, 57, $\pi^{-1}(T') \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$ and, moreover,

$$\pi^{-1}(O^{\mathcal{A}/\theta}(\pi^{-1}(T')/\theta)) = \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^\mathcal{A}(\pi^{-1}(T')).$$

This proves the right-to-left-inclusion. ■

We now give a characterization of O-full $\mathcal{I}$-classes for semi-coherent families of congruential compatibility $\mathcal{I}$-operators.

Corollary 1585 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$*.* $\mathcal{T}$ *is* $O^\mathcal{A}$*-full if and only if, for some surjective* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, which may be taken to be the quotient morphism* $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{O}(\mathcal{T})$*,*

$$\mathcal{T} = \{T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) : \gamma^{-1}(O^\mathcal{B}(\gamma(T))) = O^\mathcal{A}(T)\}.$$

Proof: Suppose $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is $O^{\mathcal{A}}$-full. By definition, $\mathcal{T} = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$. Let $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})$ be the quotient morphism. Then we have $\mathcal{T} = O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle I,\pi\rangle))$ whence, by Proposition 1584,

$$\mathcal{T} = \{T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}) : \pi^{-1}(O^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\pi(T))) = O^{\mathcal{A}}(T)\}.$$

Assume, conversely, that $\mathcal{T} = \{T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)\}$, for some surjective $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism. By Proposition 1584, $\mathcal{T} = O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$, whence $\mathcal{T} \in \mathrm{Ran}(O^{\mathcal{A}^{-1}})$, showing that $\mathcal{T}$ is $O^{\mathcal{A}}$-full. ∎

Corollary 1586 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a semi-coherent family of congruential compatibility $\mathcal{I}$-operators, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. $\mathcal{T}$ is $O^{\mathcal{A}}$-full if and only if, for some $\theta \in \mathrm{ConSys}(\mathcal{I})$, which can be taken to be $\widetilde{O}^{\mathcal{A}}(\mathcal{T})$, and with $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ the corresponding quotient morphism,*

$$\mathcal{T} = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

Proof: Assume, first, that $\mathcal{T}$ is $O^{\mathcal{A}}$-full. Then, by definition, we have $\mathcal{T} = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$. Take $\theta = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. Then $\mathcal{T} = O^{\mathcal{A}^{-1}}(\theta)$, whence, by Proposition 1584,

$$\mathcal{T} = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

Assume, conversely, that $\mathcal{T}$ is given by the displayed expression above, for some $\theta \in \mathrm{ConSys}(\mathcal{A})$ and $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ the quotient morphism. Then, by Proposition 1584, $\mathcal{T} = O^{\mathcal{A}^{-1}}(\theta) \in \mathrm{Ran}(O^{\mathcal{A}^{-1}})$ and, therefore, $\mathcal{T}$ is $O^{\mathcal{A}}$-full, by definition. ∎

Turning, next, to the full congruence systems, we obtain the following characterization.

Proposition 1587 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a semi-coherent family of compatibility $\mathcal{I}$-operators, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $\theta \in \mathrm{ConSys}(\mathcal{A})$. θ is $O^{\mathcal{A}}$-full if and only if*

$$\widetilde{O}^{\mathcal{A}/\theta}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}) = \Delta^{\mathcal{A}/\theta},$$

where $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ is the quotient morphism.

Proof: Let $\mathcal{T}' = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}$. Then $\langle I,\pi\rangle$ is compatible with $\pi^{-1}(\mathcal{T}')$, since, for all $T' \in \mathcal{T}'$,

$$\mathrm{Ker}(\langle I,\pi\rangle) = \pi^{-1}(\Delta^{\mathcal{A}/\theta}) \leq \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T')).$$

Thus, by Propositions 1584 and 1582, θ is $O^{\mathcal{A}}$-full if and only if

$$\theta = \widetilde{O}^{\mathcal{A}}(O^{\mathcal{A}^{-1}}(\theta)) = \widetilde{O}^{\mathcal{A}}(\pi^{-1}(\mathcal{T}')) = \pi^{-1}(\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}')).$$

Now, if θ is $O^{\mathcal{A}}$-full, then we get, using the surjectivity of the quotient morphism,

$$\begin{array}{rcl}\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}') & = & \pi(\pi^{-1}(\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}')))\\ & = & \pi(\theta) = \Delta^{\mathcal{A}/\theta}.\end{array}$$

If, conversely, $\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}') = \Delta^{\mathcal{A}/\theta}$, then $\theta = \pi^{-1}(\Delta^{\mathcal{A}/\theta}) = \pi^{-1}(\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}'))$. Hence, by the equivalence detailed above, θ is $O^{\mathcal{A}}$-full. ■

Since Ω is a semi-coherent family of compatibility $\mathcal{I}$-operators, we now get

Corollary 1588 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*. Then*

$$\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)),$$

where $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ *is the quotient morphism.*

Proof: By Proposition 1584,

$$\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta') : \pi^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

But, by Proposition 24, Ω is commuting and, hence,

$$\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta') : \pi^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))\} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta).$$

Therefore, $\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$. ■

21.8 The General Correspondence Theorem

Theorem 1589 (General Correspondence Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$*, all surjective morphisms* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if* $\langle H, \gamma\rangle$ *is* O*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{O^{\mathcal{A}}}$ *onto* $[\![\gamma(T)]\!]^{O^{\mathcal{B}}}$*, with inverse* γ^{-1}*.*

Proof: Assume that $\langle H, \gamma\rangle$ is O-compatible with T. By Corollary 1576, $\langle H, \gamma\rangle$ is Ω-compatible with T. By the same Corollary and by Corollary 56, $T = \gamma^{-1}(\gamma(T))$ and $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.

Suppose, next, that $T' \in [\![T]\!]^{O^{\mathcal{A}}}$. Then, $\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Again, based on Corollaries 1576 and 56, we get $\gamma^{-1}(\gamma(T')) = T'$ and $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Moreover, we get

$$\begin{array}{rcll} O^{\mathcal{B}}(\gamma(T)) & = & \gamma(O^{\mathcal{A}}(T)) & \text{(by Lemma 1579)}\\ & \le & \gamma(\Omega^{\mathcal{A}}(T')) & \\ & = & \Omega^{\mathcal{B}}(\gamma(T')). & \text{(by Lemma 1579)} \end{array}$$

Thus, $\gamma(T') \in [\![\gamma(T)]\!]^{O^{\mathcal{B}}}$.

Suppose, next, that $T'' \in [\![\gamma(T)]\!]^{O^{\mathcal{B}}}$. Then, we get $O^{\mathcal{B}}(\gamma(T)) \le \Omega^{\mathcal{B}}(T')$. By Corollary 55, $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and by surjectivity, $\gamma(\gamma^{-1}(T'')) = T''$. Since $\langle H,\gamma\rangle$ is O-compatible with $T = \gamma^{-1}(\gamma(T))$, we get, using semi-coherence,

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & O^{\mathcal{A}}(\gamma^{-1}(\gamma(T))\\ & = & \gamma^{-1}(O^{\mathcal{B}}(\gamma(T)))\\ & \le & \gamma^{-1}(\Omega^{\mathcal{B}}(T'))\\ & = & \Omega^{\mathcal{A}}(\gamma^{-1}(T')). \end{array}$$

Hence, $\gamma^{-1}(T'') \in [\![T]\!]^{O^{\mathcal{A}}}$. We conclude that γ is a bijection from $[\![T]\!]^{O^{\mathcal{A}}}$ onto $[\![\gamma(T)]\!]^{O^{\mathcal{B}}}$, with inverse γ^{-1}. But, clearly, both γ and γ^{-1} are order preserving functions, whence they establish an order isomorphism between these two ordered sets. ■

The General Correspondence Theorem has the following consequence concerning O-filter systems on different $\mathbf{F}$-algebraic systems.

Corollary 1590 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$, *all surjective morphisms* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, *with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *if* $\langle H,\gamma\rangle$ *is* O*-compatible with* T, *then*

$$T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \quad \textit{iff} \quad \gamma(T) \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{B}).$$

Proof: We have the following chain of equivalences:

$$\begin{array}{rll} T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) & \text{iff} & T = T^O\\ & \text{iff} & T = \bigcap [\![T]\!]^{O^{\mathcal{A}}}\\ & \text{iff} & \gamma(T) = \bigcap [\![\gamma(T)]\!]^{O^{\mathcal{B}}} \quad \text{(by Theorem 1589)}\\ & \text{iff} & \gamma(T) = \gamma(T)^O\\ & \text{iff} & \gamma(T) \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{B}). \end{array}$$

Thus, the claim is established. ■

For semi-coherent congruential compatibility $\mathcal{I}$-operators, we obtain a relation between the O-filter systems on an algebraic system and those on the quotient algebraic system.

Corollary 1591 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, and all* $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$*,*

$$T^O / O^\mathcal{A}(T) = (T / O^\mathcal{A}(T))^O$$

and it is the least $\mathcal{I}$*-filter family on* $\mathcal{A}/O^\mathcal{A}(T)$*.*

Proof: Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/O^\mathcal{A}(T)$. $\langle I, \pi \rangle$ is surjective, with I an isomorphism, and it is O-compatible with T. By Theorem 1589, $\pi : [\![T]\!]^{O^\mathcal{A}} \to [\![T/O^\mathcal{A}(T)]\!]^{O^{\mathcal{A}/O^\mathcal{A}(T)}}$ is an order isomorphism with inverse π^{-1}. Since T^O is the least $\mathcal{I}$-filter family of $[\![T]\!]^{O^\mathcal{A}}$, it follows that $T^O/O^\mathcal{A}(T)$ must be the least $\mathcal{I}$-filter family of $[\![T/O^\mathcal{A}(T)]\!]^{O^{\mathcal{A}/O^\mathcal{A}(T)}}$, which is, by definition, $(T/O^\mathcal{A}(T))^O$. Finally, since $O^{\mathcal{A}/O^\mathcal{A}(T)}(T/O^\mathcal{A}(T)) = \Delta^{\mathcal{A}/O^\mathcal{A}(T)}$, it follows that $[\![T/O^\mathcal{A}(T)]\!]^{O^{\mathcal{A}/O^\mathcal{A}(T)}} = \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/O^\mathcal{A}(T))$. Thus, $(T/O^\mathcal{A}(T))^O$ is the least $\mathcal{I}$-filter family on $\mathcal{A}/O^\mathcal{A}(T)$. ■

Finally, applying the General Correspondence Theorem to the relativization of an operator, we obtain the following:

Theorem 1592 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all surjective morphisms* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$*, if* $\langle H, \gamma \rangle$ *is* $\widetilde{O}^\mathcal{I}$*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{\widetilde{O}^{\mathcal{I},\mathcal{A}}}$ *onto* $[\![\gamma(T)]\!]^{\widetilde{O}^{\mathcal{I},\mathcal{B}}}$*, with inverse* γ^{-1}*.*

Proof: It is clear that if O is a compatibility $\mathcal{I}$-operator, the same holds for $\widetilde{O}^\mathcal{I}$. Moreover, by Proposition 1581, if O is a semi-coherent family, then $\widetilde{O}^\mathcal{I}$ is also semi-coherent. Therefore, under the given hypotheses, we can apply Theorem 1589 with $\widetilde{O}^\mathcal{I}$ in place of O and the result immediately follows. ■

21.9 Algebraic Systems of $\mathcal{I}$-Operators

With a given family of congruential operators, there are associated several classes of algebraic systems, which it is the purpose of this section to study closely, in analogy to the various classes ensued from applications of the Leibniz operator, and to explore their interrelationships.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of congruential $\mathcal{I}$-operators. We define the following classes of $\mathbf{F}$-algebraic systems associated with O (assuming closure under isomorphisms):

- $\mathrm{AlgSys}^O(\mathcal{I}) = \{\mathcal{A}/O^{\mathcal{A}}(T) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;
- $\mathrm{AlgSys}_O(\mathcal{I}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(O^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}$;
- $\mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \{\mathcal{A}/\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;
- $\mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}$;
- $\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;
- $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}})\}$.

Names corresponding to these classes go as follows:

- $\mathrm{AlgSys}^O(\mathcal{I})$ is the class of O-**reduced F-algebraic systems**;
- $\mathrm{AlgSys}_O(\mathcal{I})$ is the class of O-**reductions of F-algebraic systems**;
- $\mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I})$ is the class of $\widetilde{O}^{\mathcal{I}}$-**reduced F-algebraic systems**;
- $\mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I})$ is the class of $\widetilde{O}^{\mathcal{I}}$-**reductions of F-algebraic systems**;
- $\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$ is the class of $\widetilde{O}$-**reduced F-algebraic systems**;
- $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$ is the class of $\widetilde{O}$-**reductions of F-algebraic systems**.

We provide some alternative characterizations for the classes associated with the lifting $\widetilde{O}$ of the operator O.

Lemma 1593 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a family of congruential* $\mathcal{I}$*-operators.*

(a) $\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), \mathcal{T}\ O^{\mathcal{A}}\text{-}\mathit{full}\}$*;*

(b) $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A} : \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}\}$*;*

(c) $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A} : (\exists \mathcal{T}\ O^{\mathcal{A}}\text{-}\mathit{full})(\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}})\}$.

Proof: Note that, for all three equalities claimed, the right-to-left inclusions are trivial, given the definitions of the corresponding classes on the left. Therefore, in working out the various parts, it suffices to show the left-to-right inclusions.

(a) Suppose that $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$, for some $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})$. Since $\widetilde{O}^{\mathcal{A}}(\mathcal{T})$ is by definition, an O-full congruence system on $\mathcal{A}$, there exists, by Corollary 1558, an O-full $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{T}') = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. Thus, we get $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}') \in \{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T\ O^{\mathcal{A}}\text{-full}\}$.

(b) Assume $\mathcal{A} \in \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. Then, by definition, there exists a collection $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. Therefore,

$$\widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})) \le \widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}.$$

Thus, $\widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. We get that $\mathcal{A} \in \{\mathcal{A} : \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}\}$.

(c) This follows directly from Part (b) and Corollary 1560. ■

We now show that the three pairs of classes of reduced - classes of reductions, associated with the same operator, consist of identical classes of **F**-algebraic systems. This is due to the fact that the reduction of an **F**-algebraic system results in a reduced **F**-algebraic system, taken with respect to the same operator.

Lemma 1594 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\theta \in \mathrm{ConSys}(\mathcal{A})$*,*

$$\theta \le O^{\mathcal{A}}(T) \quad \textit{implies} \quad O^{\mathcal{A}/\theta}(T/\theta) = O^{\mathcal{A}}(T)/\theta.$$

In particular, $O^{\mathcal{A}/O^{\mathcal{A}}(T)}(T/O^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/O^{\mathcal{A}}(T)}$*.*

Proof: Let $\mathcal{A}$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\theta \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \le O^{\mathcal{A}}(T)$. Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$. It is surjective and, by hypothesis, O-compatible with T. By the assumption of semi-coherence and Lemma 1579, we get

$$O^{\mathcal{A}/\theta}(T/\theta) = O^{\mathcal{A}/\theta}(\pi(T)) = \pi(O^{\mathcal{A}}(T)) = O^{\mathcal{A}}(T)/\theta.$$

The last assertion in the statement is the specialization of what was just proven for $\theta = O^{\mathcal{A}}(T)$. ■

Proposition 1595 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{O}(\mathcal{I}) = \mathrm{AlgSys}_{O}(\mathcal{I}).$$

Proof: Suppose $\mathcal{A} \in \mathrm{AlgSys}_{O}(\mathcal{I})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $O^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. But then $\mathcal{A} \cong \mathcal{A}/\Delta^{\mathcal{A}} = \mathcal{A}/O^{\mathcal{A}}(T) \in \mathrm{AlgSys}^{O}(\mathcal{I})$.

On the other hand, if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ so that $\mathcal{A}/O^{\mathcal{A}}(T) \in \mathrm{AlgSys}^{O}(\mathcal{I})$, then, for $T/O^{\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/O^{\mathcal{A}}(T))$, we get, by Lemma 1594,

$$O^{\mathcal{A}/O^{\mathcal{A}}(T)}(T/O^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/O^{\mathcal{A}}(T)},$$

whence, by definition $\mathcal{A}/O^{\mathcal{A}}(T) \in \mathrm{AlgSys}_{O}(\mathcal{I})$. ■

Corollary 1596 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}).$$

Proof: By Proposition 1581 together with Proposition 1595. ∎

Lemma 1597 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$

$$\widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \Delta^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})$. It is surjective and O-compatible with $\mathcal{T}$. By the assumption of semi-coherence and Proposition 1583, we get

$$\begin{array}{rcl}\widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})) & = & \widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\pi(\mathcal{T})) \\ & = & \pi(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) \\ & = & \widetilde{O}^{\mathcal{A}}(\mathcal{T})/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \\ & = & \Delta^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}.\end{array}$$

This concludes the proof of the statement. ∎

Proposition 1598 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}).$$

Proof: Suppose $\mathcal{A} \in \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. Then, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. But then $\mathcal{A} \cong \mathcal{A}/\Delta^{\mathcal{A}} = \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$.

On the other hand, if $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ so that $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$, then, for $\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$, we get, by Lemma 1597,

$$\widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \Delta^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})},$$

whence, by definition $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. ∎

Proposition 1599 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}).$$

Proof: For every **F**-algebraic system and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. This equality gives that

$$\mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) \quad \text{and} \quad \mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}).$$

Assume, conversely, in the first case, that $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. By Lemma 1593, $\widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. Let $T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we get

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \widetilde{O}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

This shows that $\mathcal{A} \in \mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I})$. Due to Corollary 1596 and Proposition 1598 the equality just proven suffices to guarantee the conclusion. ∎

21.10 Leibniz Operator as an $\mathcal{I}$-Operator

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on **F**. We consider in this section the Leibniz operator

$$\Omega = \{\Omega^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\},$$

which is a coherent, congruential, compatibility $\mathcal{I}$-operator. We saw that its lifting is the Tarski operator $\widetilde{\Omega}$ and its relativization is the Suszko operator $\widetilde{\Omega}^{\mathcal{I}}$. Using the definition, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\theta \in \mathrm{ConSys}(\mathcal{A})$, we have

$$\begin{aligned} \Omega^{\mathcal{A}^{-1}}(\theta) &= \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \leq \Omega^{\mathcal{A}}(T)\} \\ &= \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \text{ is compatible with } T\}. \end{aligned}$$

We have the following characterizations of Ω-full objects:

Proposition 1600 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. $\mathcal{T}$ is Ω-full if and only if $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure.*

Proof: We have

$$\begin{aligned} \mathcal{T} \text{ is } \Omega\text{-full} \quad &\text{iff} \quad \mathcal{T} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) \\ &\qquad (\text{by definition of } \Omega\text{-full}) \\ &\text{iff} \quad \mathcal{T} = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &\qquad (\text{by definition of } \Omega^{\mathcal{A}^{-1}}) \\ &\text{iff} \quad \langle \mathcal{A}, \mathcal{T} \rangle \text{ is a full } \mathcal{I}\text{-structure.} \\ &\qquad (\text{by Theorem 1432}) \end{aligned}$$

∎

Recall that $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ denotes the collection of all $\mathrm{AlgSys}(\mathcal{I})$-congruence systems on an **F**-algebraic system $\mathcal{A}$, i.e., those congruence systems θ on $\mathcal{A}$, such that $\mathcal{A}/\theta \in \mathrm{ConSys}(\mathcal{I})$. For Ω-full congruence systems, we get

Proposition 1601 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*.* θ *is* Ω*-full if and only if* $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: We have

$$\begin{array}{rll}\theta \text{ is } \Omega\text{-full} & \text{iff} & \widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}/\theta} \\ & & \text{(by Proposition 1587)} \\ & \text{iff} & \mathcal{A}/\theta \in \mathrm{AlgSys}(\mathcal{I}) \\ & & \text{(by Proposition 1436)} \\ & \text{iff} & \theta \in \mathrm{ConSys}(\mathcal{A}). \\ & & \text{(by definition).}\end{array}$$

■

As a corollary of these two characterizations, we can derive from our work on Galois connections (more precisely Corollary 1558) the Isomorphism Theorem 1445 between full $\mathcal{I}$-structures and $\mathcal{I}$-congruence systems.

Corollary 1602 (Isomorphism Theorem 1445) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. The operators* $\widetilde{\Omega}^{\mathcal{A}}$ *and* $\Omega^{\mathcal{A}^{-1}}$ *establish a Galois connection between* $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ *and* $\mathrm{EqvFam}(\mathcal{A})$*, which restricts to mutually inverse isomorphisms between* $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ *and* $\langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$*.*

Proof: By Corollary 1558 and Propositions 1600 and 1601, noting that the order on $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ is the converse from that inherited by $\langle \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})), \subseteq \rangle$.
■

By applying Proposition 1559 to the Leibniz operator, we get a characterization of full $\mathcal{I}$-structures.

Proposition 1603 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $\langle \mathcal{A}, \mathcal{T} \rangle$ *is a full* $\mathcal{I}$*-structure if and only if* $\mathcal{T}$ *is the largest collection* $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$*.*

Proof: By instantiating Proposition 1559 to the Leibniz operator. ■

Moreover, directly from Lemma 1555, we get:

Proposition 1604 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if* $\widetilde{\Omega}^{\mathcal{I}} = \Omega$*.*

Proof: By instantiating Lemma 1555 to the Leibniz operator. ■

We turn now to Ω-classes and Ω-filter families. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

The Ω-**class of** T or **Leibniz class of** T is

$$[\![T]\!]^* := \Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

The **Leibniz filter family of** T is the $\mathcal{I}$-filter family

$$T^* = \bigcap [\![T]\!]^*.$$

We say that T is a **Leibniz filter family** if $T^* = T$. The collection of all Leibniz filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$.

We further denote by $[T]$ the **equi-Leibniz class of** T, i.e., the collection of all $\mathcal{I}$-filter families of $\mathcal{A}$ that share the same Leibniz congruence system with T:

$$[T] = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)\} \subseteq [\![T]\!]^*.$$

Some basic properties involving these concepts follow.

Lemma 1605 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $T^* \leq \bigcap [T] \leq T$*;*

(b) If $T^* = T$*, then* $T = \bigcap [T]$*;*

(c) If $\mathcal{I}$ *is protoalgebraic, then* $T = T^*$ *if and only if* $T = \bigcap [T]$*.*

Proof:

(a) We have $T^* = \bigcap [\![T]\!]^* \leq \bigcap [T] \leq T$.

(b) If $T^* = T$, then, by Part (a), $T = \bigcap [T]$.

(c) Suppose that $\mathcal{I}$ is protoalgebraic. The necessity is given by Part (b). For the sufficiency, assume that $T = \bigcap [T]$. Since, by Part (a), $T^* \leq T$, by protoalgebraicity, $\Omega^{\mathcal{A}}(T^*) \leq \Omega^{\mathcal{A}}(T)$. Since $T^* \in [\![T]\!]^*$, we get, by definition, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$. Hence, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*)$ and, therefore, $T^* \in [T]$. Now we conclude that $T = \bigcap [T] \leq T^*$. By Part (a), the reverse inclusion always holds.

■

Proposition 1606 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $\langle \mathcal{A}, [\![T]\!]^*\rangle \in \mathrm{FStr}(\mathcal{I})$*;*

(b) $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*) = \Omega^{\mathcal{A}}(T)$*.*

Proof: By Proposition 1600, $\langle\mathcal{A},[\![T]\!]^*\rangle$ is a full $\mathcal{I}$-structure. Since $T\in[\![T]\!]^*$, it follows that $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*)\le\Omega^{\mathcal{A}}(T)$. On the other hand, for all $T'\in[\![T]\!]^*$, $\Omega^{\mathcal{A}}(T)\le\Omega^{\mathcal{A}}(T')$. Thus, $\Omega^{\mathcal{A}}(T)\le\bigcap_{T'\in[\![T]\!]^*}\Omega^{\mathcal{A}}(T')=\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*)$. ∎

It turns out that, for every theory family, its Leibniz counterpart is in fact a Leibniz theory family.

Proposition 1607 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then* $T^*\in\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$*.*

Proof: By Lemma 1605, we have $(T^*)^*\le T^*$. On the other hand, $T^*\in[\![T]\!]^*$. So, by definition $\Omega^{\mathcal{A}}(T)\le\Omega^{\mathcal{A}}(T^*)$. This shows that $[\![T^*]\!]^*\subseteq[\![T]\!]^*$. This, in turn, yields $T^*=\bigcap[\![T]\!]^*\le\bigcap[\![T^*]\!]^*=(T^*)^*$. We conclude that $(T^*)^*=T^*$ and, hence, $T^*\in\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. ∎

We also have a characterization of Leibniz filter families in terms of full structures.

Proposition 1608 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T\in\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$ *if and only if, there exists* $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$*, such that* $T=\bigcap\mathcal{T}$*.*

Proof: Suppose, first, that $T\in\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Then $T=T^*=\bigcap[\![T]\!]^*$ and, by Proposition 1606, $\langle\mathcal{A},[\![T]\!]^*\rangle$ is a full $\mathcal{I}$-structure.

Assume, conversely, that $T=\bigcap\mathcal{T}$, with $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}(\mathcal{I})$. Since $T=\bigcap\mathcal{T}\in\mathcal{T}$, we get $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})\le\Omega^{\mathcal{A}}(T)$. Thus, we get

$$[\![T]\!]^*=\Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T))\subseteq\Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(T))=\mathcal{T}.$$

So $T=\bigcap\mathcal{T}\le\bigcap[\![T]\!]^*=T^*$. Since, by Lemma 1605, $T^*\le T$, we conclude that $T^*=T$ and, hence, $T\in\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. ∎

Corollary 1591, applied to the Leibniz operator, gives another characterization of Leibniz filter families.

Proposition 1609 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T\in\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$ *if and only if* $T/\Omega^{\mathcal{A}}(T)$ *is the least filter family in* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$*.*

Proof: By specializing Corollary 1591 to the Leibniz operator. ∎

Leibniz filter families may also be used in characterizing the reflectivity of the Leibniz operator, which characterizes family reflective π-institutions.

Proposition 1610 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* Ω *is reflective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: We have that Ω is reflective if and only if, by definition, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \text{implies} \quad T \leq T',$$

if and only if, by definition of $[\![T]\!]^*$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in [\![T]\!]^*$ implies $T \leq T'$, if and only if, since $T^* = \min [\![T]\!]^*$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = T^*$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ∎

Surjective morphisms between algebraic systems, with isomorphic signature components, that satisfy a compatibility condition, induce order isomorphisms between Leibniz classes, which restrict to order isomorphisms between equi-Leibniz classes.

Theorem 1611 (Correspondence Theorem for Leibniz Classes) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^*$ *onto* $[\![\gamma(T)]\!]^*$*, with inverse* γ^{-1}*. In addition, for all* $T' \in [\![T]\!]^*$*,* γ *induces an order isomorphism from* $[T']$ *onto* $[\gamma(T')]$.

Proof: The first statement follows from the General Correspondence Theorem 1589 by instantiation to the Leibniz operator. So we undertake the proof of the additional statement. Suppose that $T', T'' \in [\![T]\!]^*$. Since $T' \in [\![T]\!]^*$, we get $[T'] \subseteq [\![T']\!]^* \subseteq [\![T]\!]^*$. Thus, by the first statement, $\gamma^{-1}(\gamma(T')) = T'$ and $\gamma^{-1}(\gamma(T'')) = T''$. Thus, we get

$$\begin{array}{rll}\Omega^{\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T') & \text{iff} & \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(T''))) = \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(T'))) \\ & \text{iff} & \gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(T''))) = \gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(T'))) \\ & \text{iff} & \Omega^{\mathcal{B}}(\gamma(T'')) = \Omega^{\mathcal{B}}(\gamma(T')).\end{array}$$

So $T'' \in [T']$ if and only if $\gamma(T'') \in [\gamma(T')]$. Thus, the order isomorphism $\gamma : [\![T]\!]^* \to [\![\gamma(T)]\!]^*$ restricts to an order isomorphism $\gamma : [T] \to [\gamma(T)]$. ∎

As a consequence of Correspondence Theorem, we get a correspondence between Leibniz filter families.

Corollary 1612 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T*, then*

$$T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \textit{iff} \quad \gamma(T) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B}).$$

Proof: By Theorem 1611, under the isomorphism $\gamma : [\![T]\!]^* \to [\![\gamma(T)]\!]^*$, the least theory family T^* of $[\![T]\!]^*$ corresponds to the least theory family $\gamma(T)^*$ of $[\![\gamma(T)]\!]^*$. Therefore, $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ if and only if $T = T^*$ if and only if $\gamma(T) = \gamma(T)^*$ if and only if $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})$. ∎

Rephrased in terms of strict surjective morphisms Corollary 1612 yields

Corollary 1613 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{B})$ *and* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ *a strict surjective morphism, with* H *an isomorphism. Then*

$$T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \textit{iff} \quad T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B}).$$

Proof: It suffices to show that $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is Ω-compatible with T. If that is the case, then, since $T = \gamma^{-1}(T')$, we get, $T' = \gamma(\gamma^{-1}(T')) = \gamma(T)$, and the statement follows by applying Corollary 1612. We have, indeed

$$\begin{array}{rcl}\mathrm{Ker}(\langle H, \gamma \rangle) & = & \gamma^{-1}(\Delta^{\mathcal{B}}) \\ & \leq & \gamma^{-1}(\Omega^{\mathcal{B}}(T')) \\ & = & \Omega^{\mathcal{A}}(\gamma^{-1}(T')) \\ & = & \Omega^{\mathcal{A}}(T).\end{array}$$

Therefore, $\langle H, \gamma \rangle$ is indeed compatible with T. ∎

The Correspondence Theorem 1611 allows us to formulate a Correspondence Theorem for the special case of protoalgebraic π-institutions that, as it turns out, provides an additional characterization of protoalgebraicity.

Theorem 1614 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ *and every strict surjective morphism* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$*, with* H *an isomorphism,* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$*, with inverse* γ^{-1}*.*

Proof: Suppose, first, that $\mathcal{I}$ is protoalgebraic and let $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ be a strict surjective morphism, with H an isomorphism. Then, we get $T = \gamma^{-1}(T')$ and $T' = \gamma(\gamma^{-1}(T')) = \gamma(T)$. So $T = \gamma^{-1}(\gamma(T))$. This implies that $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is compatible with T. By the Correspondence Theorem for Leibniz Classes 1611, γ induces an order isomorphism $\gamma : [\![T]\!]^* \to [\![T']\!]^*$, with inverse γ^{-1}. But, by protoalgebraicity, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ and $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ are upsets of $[\![T]\!]^*$ and $[\![T']\!]^*$, respectively and T corresponds to T' under γ. Therefore, γ restricts to an order isomorphism $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$, with inverse γ^{-1}.

Suppose, conversely, that the given condition holds. Let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Consider the quotient morphism $\langle I,\pi\rangle : \mathcal{F} \to \mathcal{F}/\Omega(T)$. It gives a strict surjective morphism

$$\langle I,\pi\rangle : \langle \mathcal{F},T\rangle \to \langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle.$$

Since, by hypothesis, $\pi : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\Omega(T))^{T/\Omega(T)}$ is an order isomorphism. with inverse π^{-1} and, clearly, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})^T$, we get that $\pi(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\Omega(T))^{T/\Omega(T)}$ and $T' = \pi^{-1}(\pi(T'))$. Now we get

$$\begin{array}{rcl}\Omega(T) & = & \mathrm{Ker}(\langle I,\pi\rangle)\\ & = & \pi^{-1}(\Delta^{\mathcal{F}/\Omega(T)})\\ & \le & \pi^{-1}(\Omega^{\mathcal{F}/\Omega(T)}(\pi(T')))\\ & = & \Omega(\pi^{-1}(\pi(T')))\\ & = & \Omega(T').\end{array}$$

Since Ω is monotone, we conclude that $\mathcal{I}$ is a protoalgebraic π-institution. ■

Now we get a characterization of those full $\mathcal{I}$-structures whose closure families are Leibniz classes.

Proposition 1615 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle \mathcal{A},\mathcal{T}\rangle$ *is a full* $\mathcal{I}$*-structure. Then* $\mathcal{T} = [\![T]\!]^*$*, for some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if and only if* $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: Suppose, first, that $\mathcal{T} = [\![T]\!]^*$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, using Proposition 1606, we get

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*) = \Omega^{\mathcal{A}}(T).$$

Therefore, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$.

Assume, conversely, that $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^*(\mathcal{I})$. By definition, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))$, such that $\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}(T) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}$. This equality implies that $[\![T]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))$. Now consider the quotient morphism $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$. Since, by hypothesis $\langle \mathcal{A},\mathcal{T}\rangle$ is a full $\mathcal{I}$-structure, we get

$$\mathcal{T} = \pi^{-1}(\mathcal{T}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))) = \pi^{-1}([\![T]\!]^*).$$

Moreover,

$$\mathrm{Ker}(\langle I,\pi\rangle) = \pi^{-1}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}) = \pi^{-1}(\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}(T)) = \Omega^{\mathcal{A}}(\pi^{-1}(T)).$$

So $\langle I,\pi\rangle$ is Ω-compatible with $\pi^{-1}(T)$. By the Correspondence Theorem for Leibniz Classes 1611, we get an order isomorphism $\pi : [\![\pi^{-1}(T)]\!]^* \to [\![T]\!]^*$. This gives $\mathcal{T} = \pi^{-1}([\![T]\!]^*) = [\![\pi^{-1}(T)]\!]^*$. ■

We get, as a consequence, a characterization of those π-institutions for which all full $\mathcal{I}$-structures are determined by Leibniz classes of $\mathcal{I}$-filter families.

Proposition 1616 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, [\![T]\!]^* \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$ *if and only if* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$.

Proof: Suppose, first, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, [\![T]\!]^* \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$. Since $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ holds in general, it suffices to show the reverse inclusion. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Since $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, we get, by hypothesis, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} = [\![T']\!]^*$. Now notice the following:

- $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}}$, whence $\Omega^{\mathcal{A}}(T') \leq \bigcap_{T \leq T''} \Omega^{\mathcal{A}}(T'') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$;
- $T' \in [\![T']\!]^*$ implies $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

We conclude that $\Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Hence, we have $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$.

Assume, conversely, that $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. Since, by Proposition 1606, we have, in general, $\{\langle \mathcal{A}, [\![T]\!]^* \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, it suffices to show the reverse inclusion. To this end, let $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}(\mathcal{I})$. By hypothesis, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^*(\mathcal{I})$. Therefore, by Proposition 1615, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = [\![T]\!]^*$. ■

So, one way to characterize π-institutions $\mathcal{I}$ for which $\mathcal{I}$-algebraic systems and $\mathcal{I}^*$-algebraic systems coincide is to look at the form of full $\mathcal{I}$-structures. An alternative characterization uses the Leibniz and Suszko operators.

Proposition 1617 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *there exists* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *such that* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$.

Proof: Assume $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, so that $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$. By Proposition 1616, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} = [\![T']\!]^*$. We now get

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}}) = \widetilde{\Omega}^{\mathcal{A}}([\![T']\!]^*) = \Omega^{\mathcal{A}}(T').$$

Assume, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. By hypothesis, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. We conclude that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. The reverse inclusion always holds. Therefore, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. ■

Proposition 1617, gives the following feature of protoalgebraic π-institutions.

Corollary 1618 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is protoalgebraic, then every full* $\mathcal{I}$*-structure is of the form* $\langle \mathcal{A}, [\![T]\!]^* \rangle$*, for some* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: We know that, if $\mathcal{I}$ is protoalgebraic and $\mathcal{A}$ is an $\mathbf{F}$-algebraic system, then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T)$. Therefore, by Proposition 1617 and Proposition 1616, every full $\mathcal{I}$-structure has the form claimed in the statement. ∎

This property of the full $\mathcal{I}$-structures in a more precise form, yields a characterization of protoalgebraicity.

Theorem 1619 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) Every full $\mathcal{I}$*-structure is of the form* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$*, for some* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) Every full $\mathcal{I}$*-structure is of the form* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$*, for some* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and some* $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*;*

(iv) $[\![T]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose that $\mathcal{I}$ is protoalgebraic and let $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1600, $\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)\}$. By protoalgebraicity, $\mathcal{T}$ is an upset in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, $\mathcal{T}$ has a least element, $T = \bigcap \mathcal{T}$. Thus, we have $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\cap \mathcal{T}}$.

(ii)⇒(iii) Assume (ii) holds and let $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. By Proposition 1608, $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$.

(iii)⇒(iv) Assume (iii) holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1606, $\langle \mathcal{A}, [\![T]\!]^* \rangle \in \mathrm{FStr}(\mathcal{I})$. By Proposition 1607, $T^* \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ and, by definition $T^* = \bigcap [\![T]\!]^*$. Thus, by (iii), we get $[\![T]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$.

(iv)⇒(i) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, by Lemma 1605, $T^* \leq T \leq T'$. By hypothesis, $T' \in [\![T]\!]^*$. So we get, by definition of $[\![T]\!]^*$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Since Ω is monotone on every $\mathbf{F}$-algebraic system, we get that $\mathcal{I}$ is protoalgebraic. ∎

21.11 Suszko Operator as an $\mathcal{I}$-Operator

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

The $\widetilde{\Omega}^{\mathcal{I}}$**-class of** T or **Suszko class of** T is

$$[\![T]\!]^{\mathsf{Su}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')\}.$$

The **Suszko filter family of** T is

$$T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}}.$$

T is a **Suszko filter family** if $T^{\mathsf{Su}} = T$. The collection of all Suszko filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

The following lemma gives some of the basic properties of Suszko classes and Suszko theory families.

Lemma 1620 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(a) $T^{\mathsf{Su}} \le T^* \le T$;

(b) $T^{\mathsf{Su}} = T$ *implies* $T^* = T$;

(c) If $T \le T'$*, then* $[\![T']\!]^{\mathsf{Su}} \subseteq [\![T]\!]^{\mathsf{Su}}$ *and* $T^{\mathsf{Su}} \le T'^{\mathsf{Su}}$;

(d) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^{\mathsf{Su}} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^{\mathsf{Su}}}$;

(e) $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *if and only if* $T^{\mathsf{Su}} = T$.

Proof:

(a) We have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T)$. Hence $[\![T]\!]^* \subseteq [\![T]\!]^{\mathsf{Su}}$. This gives

$$T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}} \le [\![T]\!]^* = T^*.$$

The last inequality is by Lemma 1605.

(b) If $T = T^{\mathsf{Su}}$, then, by Part (a), $T = T^*$.

(c) If $T \le T'$, by the monotonicity of the Suszko operator, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Thus, we get $[\![T']\!]^{\mathsf{Su}} \subseteq [\![T]\!]^{\mathsf{Su}}$. Finally, $T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}} \le \bigcap [\![T']\!]^{\mathsf{Su}} = T'^{\mathsf{Su}}$.

(d) Suppose $T \le T'$. Then $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \le \Omega^{\mathcal{A}}(T')$. So $T' \in [\![T]\!]^{\mathsf{Su}}$. Moreover, if $T' \in [\![T]\!]^{\mathsf{Su}}$, then $T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}} \le T'$.

(e) By specializing Lemma 1570. ∎

For the Suszko classes, we get an analogous result to Proposition 1606, to the effect that they are closure families of full $\mathcal{I}$-structures and their Tarski congruence systems equal the Suszko congruence system of their generating theory family.

Proposition 1621 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $\langle \mathcal{A}, [\![T]\!]^{\mathsf{Su}} \rangle \in \mathrm{FStr}(\mathcal{I})$*;*

(b) $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$*.*

Proof: Part (a) is a specialization of Proposition 1563.

Since, by Lemma 1620, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^{\mathsf{Su}}$, we get $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. On the other hand, if $T' \in [\![T]\!]^{\mathsf{Su}}$, then $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Hence $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \bigcap_{T' \in [\![T]\!]^{\mathsf{Su}}} \Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}})$. Equality now follows. ∎

The mapping $T \mapsto T^{\mathsf{Su}}$ is monotone.

Lemma 1622 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T \le T' \quad \text{implies} \quad T^{\mathsf{Su}} \le T'^{\mathsf{Su}}.$$

Proof: By Proposition 1566. ∎

Moreover, even though T^{Su} is not necessarily a Suszko theory family, in case it happens to be, it is the largest such below T.

Lemma 1623 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. For all* $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*, such that* $T' \le T$*, we have* $T' \le T^{\mathsf{Su}}$*.*

Proof: Suppose $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $T' \le T$. Then, by the hypothesis and Lemma 1622, $T' = T'^{\mathsf{Su}} \le T^{\mathsf{Su}}$. ∎

As far as characterizing Suszko theory families, we have the following

Proposition 1624 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ *if and only if* $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ *is the least* $\mathcal{I}$*-filter family of* $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$*.*

Proof: By Proposition 1567. ■

It turns out that the collection of Suszko theory families of a π-institution forms a join complete subsemilattice of the lattice of all theory families.

Lemma 1625 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ *is a join complete subsemilattice of* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: Suppose $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. By Lemma 1622, we get, for all $i \in I$,

$$T^i = (T^i)^{\mathsf{Su}} \leq (\bigvee_{i \in I} T)^{\mathsf{Su}}.$$

This gives $\bigvee_{i \in I} T^i \leq (\bigvee_{i \in I} T^i)^{\mathsf{Su}}$. But, by Lemma 1620, $(\bigvee_{i \in I} T^i)^{\mathsf{Su}} \leq \bigvee_{i \in I} T^i$. Hence, we conclude that $\bigvee_{i \in I} T^i \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. ■

For an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, it turns out that T is a Suszko $\mathcal{I}$-filter family exactly when it is the least filter family of a full $\mathcal{I}$-structure, whose closure family consists of the upset in the lattice of $\mathcal{I}$-theory families generated by T.

Theorem 1626 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. The following conditions are equivalent:*

(i) $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*;*

(ii) $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$*;*

(iii) $T = \bigcap \mathcal{T}$*, where* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is an upset and* $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})$*.*

Proof:

(i)⇒(ii) Assume that $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Then, by Lemma 1620, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = [\![T]\!]^{\mathsf{Su}}$ and, moreover, by Proposition 1621, $\langle \mathcal{A}, [\![T]\!]^{\mathsf{Su}} \rangle \in \mathrm{FStr}(\mathcal{I})$.

(ii)⇒(iii) Assume (ii) holds and set $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Then, $T = \bigcap \mathcal{T}$, $\mathcal{T}$ is an upset in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by hypothesis, $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})$.

(iii)⇒(i) Suppose, finally, that $T = \bigcap \mathcal{T}$, where $\mathcal{T}$ is an upset in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{T})$ and $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure. We then have $T = \bigcap \mathcal{T} \in \mathcal{T}$, since $\mathcal{T}$ is a closure family. Hence, since $\mathcal{T}$ is an upset, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq \mathcal{T}$. But, by hypothesis $T = \bigcap \mathcal{T}$, whence $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Thus, we get that $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure, we have, by Theorem 1432,

$$\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \Omega^{\mathcal{A}}(T')\}.$$

But $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$, whence $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = [\![T]\!]^{\mathsf{Su}}$. Now we get $T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = \bigcap [\![T]\!]^{\mathsf{Su}} = T^{\mathsf{Su}}$ and $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

■

It turns out that requiring that all $\mathcal{I}$-filter families on all $\mathbf{F}$-algebraic systems be Suszko filter families is tantamount to $\mathcal{I}$ being family completely reflective.

Theorem 1627 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof:

(i)⇒(iii) Assume that $\mathcal{I}$ is family c-reflective and let $\mathcal{A}$ be an $\mathcal{I}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $T' \in [\![T]\!]^{\mathsf{Su}}$, we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. By family c-reflectivity and Lemma 827, we get $T \leq T'$, Thus, $T \leq \bigcap [\![T]\!]^{\mathsf{Su}} = T^{\mathsf{Su}}$. Since, by Lemma 1620, $T^{\mathsf{Su}} \leq T$, we get $T = T^{\mathsf{Su}}$, i.e., $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. We conclude that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(iii)⇒(ii) Suppose that (iii) holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Set $T' \in \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. Since $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is compatible with T, by Corollary 57, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. Thus, by definition, $T' \leq T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, we get

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T') \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}.$$

By hypothesis, since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$, we get that

$$T', T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

By Proposition 1565, $T' = T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the least $\mathcal{I}$-theory family on $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Therefore, by Proposition 1624, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

(ii)⇒(i) Assume (ii) and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. By definition, $T' \in [\![T]\!]^{\mathsf{Su}}$. Since $T = T^{\mathsf{Su}}$, we get that $T = \bigcap [\![T]\!]^{\mathsf{Su}} \leq T'$. By Lemma 827, $\mathcal{I}$ is family c-reflective. ■

Using Theorem 1627, we get additional characterizations of family c-reflectivity.

Corollary 1628 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Assume $\mathcal{I}$ is family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Theorem 1627, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Thus, by Theorem 1626, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$.

(ii)⇒(iii) Assume (ii). Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By hypothesis, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$. Thus, by Theorem 1626, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Therefore, by Lemma 1620, $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$.

(iii)⇒(i) Assume (iii). Then, by Lemma 1620, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Therefore, by Theorem 1627, $\mathcal{I}$ is family c-reflective.

■

The condition that all full $\mathcal{I}$-structures are of the form given in Part (ii) of Theorem 1628 is tantamount to weak family algebraizability.

Corollary 1629 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof: By definition $\mathcal{I}$ is WF algebraizable if and only if it is protoalgebraic and family c-reflective, if and only if, by Theorem 1619 and by Corollary 1628, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \subseteq \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$ and $\{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, if and only if $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. ■

Moreover, as far as characterizations of WF algberaizability we obtain one that involves both Suszko classes and Suszko filter families.

Proposition 1630 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ *and, for every* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})$*, there exists* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$*.*

Proof: Suppose that $\mathcal{I}$ is weakly family algebraizable. Since it is protoalgebraic, by Theorem 1619, for every **F**-algebraic system $\mathcal{A}$, if $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}(\mathcal{I})$, then $\mathcal{T}=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, for some $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\mathcal{I}$ is family c-reflective, by Corollary 1628, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T=[\![T]\!]^{\mathsf{Su}}$. Hence, if $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}(\mathcal{I})$, then $\mathcal{T}=[\![T]\!]^{\mathsf{Su}}$, for some $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Finally, by Theorem 1627, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})=\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

Suppose, conversely, that the given property holds. Since, for every **F**-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})=\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, by Theorem 1627, $\mathcal{I}$ is family c-reflective. By, Corollary 1628, for all $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $[\![T]\!]^{\mathsf{Su}}=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. By hypothesis, for all $\mathcal{T}\subseteq\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}(\mathcal{I})$, there exists $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T}=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Hence, by Theorem 1619, $\mathcal{I}$ is also protoalgebraic. We conclude that $\mathcal{I}$ is WF algebraizable. ■

As far as Suszko classes go, we have a special correspondence theorem that follows from the General Correspondence Theorem 1589 for O-classes.

Theorem 1631 (Correspondence Theorem for Suszko Classes) *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*,* $\mathcal{B}=\langle\mathbf{B},\langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems,* $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H,\gamma\rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{\mathsf{Su}}$ *to* $[\![\gamma(T)]\!]^{\mathsf{Su}}$*, with inverse* γ^{-1}*.*

Proof: By Proposition 1581, $\widetilde{\Omega}^{\mathcal{I}}$ is a semi-coherent family of compatibility $\mathcal{I}$-operators, whence, by Theorem 1589, we get the conclusion. ■

Under the hypotheses of Theorem 1631, we also obtain a correspondence between Suszko filter families:

Corollary 1632 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*,* $\mathcal{B}=\langle\mathbf{B},\langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems,* $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle H,\gamma\rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*. Then*

$$T\in\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\quad\textit{iff}\quad\gamma(T)\in\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{B}).$$

Proof: We have

$$\begin{array}{lll}T\in\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) & \text{iff} & T=T^{\mathsf{Su}}\\ & \text{iff} & T=\bigcap[\![T]\!]^{\mathsf{Su}}\\ & \text{iff} & \gamma(T)=\bigcap[\![\gamma(T)]\!]^{\mathsf{Su}}\quad\text{(by Theorem 1631)}\\ & \text{iff} & \gamma(T)=\gamma(T)^{\mathsf{Su}}\\ & \text{iff} & \gamma(T)\in\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{B}).\end{array}$$

■

Analogously to Theorem 1614, characterizing protoalgebraicity via a correspondence between posets of filter families of $\mathcal{F}$-algebraic systems related via surjective strict morphisms, we get a correspondence theorem characterizing family c-reflectivity.

Theorem 1633 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*, and all strict surjective* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$*, with* H *an isomorphism, such that* $\langle H, \gamma \rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*,* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'^{\mathsf{Su}}}$*, with inverse* γ^{-1}*.*

Proof: Suppose, first, that $\mathcal{I}$ is family c-reflective. Let $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ be a strict surjective morphism, with H an isomorphism, such that $\langle H, \gamma \rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. By Theorem 1631, $\gamma : [\![T]\!]^{\mathsf{Su}} \to [\![T']\!]^{\mathsf{Su}}$ is an order isomorphism with inverse γ^{-1}. By Corollary 1628, $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^T$ and $[\![T']\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$. Moreover, by Theorem 1627, $T' = T'^{\mathsf{Su}}$. Thus, we get the conclusion.

Assume, conversely, that the given condition holds. It suffices, by Theorem 1627, to show that every $\mathcal{I}$-filter family on every $\mathbf{F}$-algebraic system is a Suszko $\mathcal{I}$-filter family. So let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Then $\langle I, \pi \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T), T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \rangle$ is a strict surjective morphism, with I an isomorphism and it is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. By hypothesis,

$$\pi : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{\mathsf{Su}}}$$

is an order isomorphism with inverse π^{-1}.

- By Lemma 1594, we get $\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}$. Thus, by the definition of a Suszko class,

  $$[\![T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)),$$

  whence $(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{\mathsf{Su}} = \bigcap \mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$ and, therefore,

  $$\mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{\mathsf{Su}}} = \mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

- By Theorem 1631, $\pi : [\![T]\!]^{\mathsf{Su}} \to [\![T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)]\!]^{\mathsf{Su}}$ is an order isomorphism with inverse π^{-1}.

We conclude that $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. By Lemma 1620, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Thus, every $\mathcal{I}$-filter family on every $\mathbf{F}$-algebraic system is a Suszko $\mathcal{I}$-filter family and, by Theorem 1627, $\mathcal{I}$ is family c-reflective. ∎

Along similar lines, for weakly family algebraizable π-institutions, we get the following characterization, which consists of strengthening the condition in Theorem 1633 by requiring that it holds for all strict surjective morphisms with isomorphic signature components, without additional compatibility requirements.

Theorem 1634 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*, and all strict surjective morphisms* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$*, with* H *an isomorphism,* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'^{\mathsf{Su}}}$*, with inverse* γ^{-1}*.*

Proof: Suppose, first, that $\mathcal{I}$ is weakly family algebraizable. On the one hand, it is protoalgberaic, whence, by Theorem 1614, $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is an order isomorphism with inverse γ^{-1}. On the other hand, it is family c-reflective, whence by Theorem 1627, $T' = T'^{\mathsf{Su}}$. This establishes the conclusion.

Assume, conversely, that the property in the statement holds. Then, by Theorem 1633, $\mathcal{I}$ is family c-reflective. Thus, by Theorem 1627, $T'^{\mathsf{Su}} = T'$. So $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is an order isomorphism with inverse γ^{-1}. Hence, by Theorem 1614, $\mathcal{I}$ is also protoalgebraic. Therefore, $\mathcal{I}$, being both protoalgebraic and family c-reflective, is weakly family algebraizable. ∎

Next, in analogy with Proposition 1616, we give a characterization of those π-institutions $\mathcal{I}$ all of whose $\mathcal{I}$-structures correspond to closure families consisting of Suszko classes.

Proposition 1635 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, [\![T]\!]^{\mathsf{Su}} \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(ii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is surjective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(ii) Suppose (i) holds. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Then, by Corollary 1602, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle \mathcal{A}, \mathcal{T} \rangle \in$

$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \theta$. By hypothesis, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$. Now we get, using Proposition 1621,

$$\theta = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is indeed surjective.

(ii)⇒(i) Assume that (ii) holds. Since, by Proposition 1621, the right-to-left inclusion in (i) always holds, it suffices to show the reverse inclusion. To this end, let $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}(\mathcal{I})$, which gives that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{ConSys}^{\mathcal{A}}(\mathcal{A})$. Therefore, by hypothesis, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Since, by Proposition 1621, $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ and, by Proposition 1621, $\langle\mathcal{A}.[\![T]\!]^{\mathsf{Su}}\rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, we get, by the isomorphism established in Corollary 1602, that $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$.

■

The next proposition provides a characterization of weakly family algebraizable π-institution inside the class of family c-reflective ones, based on the form of their full $\mathcal{I}$-structures.

Proposition 1636 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a family c-reflective* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle\mathcal{A},[\![T]\!]^{\mathsf{Su}}\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Suppose, first, that $\mathcal{I}$ is weakly family algebraizable. Since this implies that $\mathcal{I}$ is protoalgebraic, we get that $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. Thus, by Proposition 1616, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle\mathcal{A},[\![T]\!]^{\mathsf{Su}}\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Suppose, conversely, that the condition given in the statement holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, by hypothesis and Proposition 1635, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Hence, $[\![T]\!]^* = [\![T']\!]^{\mathsf{Su}}$. Since $\mathcal{I}$ is family c-reflective, by Theorem 1627 and Lemma 1620, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, $T = T^* = T'^{\mathsf{Su}} = T'$. We conclude that $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. By Proposition 1604, we conclude that $\mathcal{I}$ is protoalgebraic. Since, by hypothesis, it is family c-reflective, we conclude that $\mathcal{I}$ is weakly family algebraizable. ■

We see, next, that family c-reflectivity is characterized by the property that all principal filters in the lattice of filter families are Suszko full classes and, also, by the reflectivity of the Suszko operator on every $\mathbf{F}$-algebraic system.

Proposition 1637 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *is Suszko full for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(iii) Suppose that $\mathcal{I}$ is family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then we get

$$\bigcap\{\Omega^{\mathcal{A}}(T'') : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T').$$

By hypothesis and Lemma 827, $\bigcap\{T'' : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq T'$, i.e., $T \leq T'$. Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is order reflecting.

(iii)⇒(i) If $\widetilde{\Omega}^{\mathcal{I}}$ is order reflecting, then it is a fortiori injective. Thus, by Theorem 828, $\mathcal{I}$ is family c-reflective.

(ii)⇒(iii) Assume (ii) holds. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then, by hypothesis,

$$\begin{aligned}\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T &= \widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)) \\ &= \widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\ &= \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'')\}.\end{aligned}$$

Similarly, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'} = \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'')\}$. Therefore, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, i.e., $T \leq T'$ and $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is order reflecting.

(iii)⇒(ii) Assume $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is order reflecting for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Then

$$\begin{aligned}&\widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)) \\ &\quad = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\ &\quad = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')\} \\ &\quad = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T.\end{aligned}$$

Hence $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ is Suszko full.

■

Finally, we conclude the section with a characterization of protoalgebraicity in terms of the form of full $\mathcal{I}$-structures and, also, by the coincidence of Leibniz and Suszko classes.

Proposition 1638 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) $\mathrm{FStr}(\mathcal{I}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}, \widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A}) \rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$*;*

(iii) $[\![T]\!]^* = [\![T]\!]^{\mathsf{Su}}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is protoalgebraic. Then, by Theorem 1619,

$$\mathrm{FStr}(\mathcal{I}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}, \Omega}(\mathcal{A}) \rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}.$$

But, by Lemma 1555, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$, whence, the conclusion follows.

(i)⇒(iii) If $\mathcal{I}$ is protoalgebraic, then, by Lemma 1555, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. Therefore, $[\![T]\!]^* = [\![T]\!]^{\mathsf{Su}}$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(ii)⇒(i) Suppose (ii) holds. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}, \widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A})\}.$$

So by Theorem 1619, $\mathcal{I}$ is protoalgebraic.

(iii)⇒(i) Assume (iii). Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. By Lemma 1620,

$$T' \in [\![T']\!]^{\mathsf{Su}} \subseteq [\![T]\!]^{\mathsf{Su}} = [\![T]\!]^*.$$

So $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Thus, Ω is monotone and, therefore, $\mathcal{I}$ is protoalgebraic.

■

21.12 Frege Operator as an $\mathcal{I}$-Operator

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

Recall that $\lambda^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ is given, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by setting $\lambda^{\mathcal{A}}(T) = \{\lambda^{\mathcal{A}}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\lambda^{\mathcal{A}}_\Sigma(T) = \{\langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : \phi \in T_\Sigma \text{ iff } \psi \in T_\Sigma\}.$$

Its lifting is the operator $\widetilde{\lambda}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{EqvFam}(\mathcal{A})$, given, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\widetilde{\lambda}^{\mathcal{A}}(\mathcal{T}) = \bigcap \{\lambda^{\mathcal{A}}(T') : T' \in \mathcal{T}\}.$$

Its relativization is the operator $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$, given, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \bigcap\{\lambda^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the $\widetilde{\lambda}^{\mathcal{I}}$**-class of** T or **Frege class of** T is

$$[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

Since λ is not a compatibility $\mathcal{I}$-operator, $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ may not be the closure family of a full $\mathcal{I}$-structure. But, nevertheless, it is still a closure family on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proposition 1639 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then* $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ *is a closure family on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: This is specialization of Proposition 1562. ■

Given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, based on Proposition 1639, we denote by $T^{\widetilde{\lambda}^{\mathcal{I}}}$ the least $\mathcal{I}$-filter family of $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$, i.e.,

$$T^{\widetilde{\lambda}^{\mathcal{I}}} = \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}.$$

Moreover, we say that T is a **Frege filter family** if $T = T^{\widetilde{\lambda}^{\mathcal{I}}}$. The collection of all Frege $\mathcal{I}$-filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$.

We give, now, a characterization of Frege filter families for π-institutions with theorems.

Lemma 1640 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*, having theorems,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

$$T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A}) \quad \text{iff} \quad \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T) \quad \text{iff} \quad T \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}.$$

Proof: The last equivalence is by the definition of $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. So it suffices to show the first equivalence.

Suppose, first, that $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$. Then, we have

$$\begin{array}{rcl} T & = & T^{\widetilde{\lambda}^{\mathcal{I}}} \\ & = & \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \\ & = & \bigcap\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}. \end{array}$$

Thus, taking into account Proposition 1639, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$.

Suppose, conversely, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$. Let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T' \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$, i.e., $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Let $\Sigma \in |\mathbf{Sign}|$ and $t \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$, which exists, since $\mathcal{I}$ is assumed to have theorems. Then, if $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$, we get $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\phi) = T_\Sigma = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,t)$. Thus, $\langle\phi,t\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T) \subseteq \Omega^{\mathcal{A}}_\Sigma(T')$. Since $t \in T'_\Sigma$, by compatibility, $\phi \in T'_\Sigma$. Therefore, $T \leq T'$. Now we have

$$\begin{array}{rcl} T & \leq & \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \quad (T \leq T', \text{ for all } T' \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}) \\ & = & T^{\widetilde{\lambda}^{\mathcal{I}}} \quad (\text{by definition}) \\ & \leq & T. \quad (\text{since } T \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}) \end{array}$$

Hence, we conclude that $T = T^{\widetilde{\lambda}^{\mathcal{I}}}$ and $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$. ■

Assuming that the π-institution $\mathcal{I}$ is protoalgebraic, gives the following characterization of Frege filter families.

Corollary 1641 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *be a protoalgebraic* π*-institution based on* $\mathbf{F}$*, having theorems,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

$$T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A}) \quad \textit{iff} \quad \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T).$$

Proof: If $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$, then

$$\begin{array}{rcl} \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) & \leq & \Omega^{\mathcal{A}}(T) \quad (\text{by Lemma 1640}) \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \quad (\text{by protoalgebraicity}) \\ & \leq & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T). \quad (\text{by compatibility}) \end{array}$$

Thus, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T)$. The converse is by Lemma 1640. ■

Each component of any $\mathcal{I}$-filter family is determined by any of its elements modulo the Frege operator.

Proposition 1642 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*, having theorems,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in T_\Sigma$*,* $T_\Sigma = \phi/\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$*.*

Proof: Suppose that $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}|$, $\phi \in T_\Sigma$.

- Let $\psi \in T_\Sigma$. Then, we have $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\phi) = T_\Sigma = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\psi)$. Thus, $\langle\phi,\psi\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, i.e., $\psi \in \phi/\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$.
- Conversely, if $\psi \in \phi/\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, then $\langle\phi,\psi\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, which gives $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\psi)$. Since $\phi \in T_\Sigma$, we get $\psi \in T_\Sigma$.

We conclude that $T_\Sigma = \phi/\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$. ■

Every Frege filter family is also a Leibniz filter family.

Lemma 1643 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, having theorems. Then*

$$\mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$. Then, by Lemma 1640, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$. Since $T^* \in [\![T]\!]^*$, we also have $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$. Therefore, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$. Thus, by definition, $T^* \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Now we have

$$T = T^{\lambda^{\mathcal{I}}} = \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \leq T^*$$

and, since, by Lemma 1605, $T^* \leq T$ always holds, we get $T = T^*$, i.e., $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

We saw that, in general, the Leibniz and Suszko filter families of a given filter family T are included in T, i.e., $T^*, T^{\mathsf{Su}} \leq T$. On the other hand, for Frege filter families, we have the reverse inclusion.

Lemma 1644 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, having theorems, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \leq T^{\widetilde{\lambda}^{\mathcal{I}}}$.*

Proof: By Proposition 1639, $T^{\lambda^{\mathcal{I}}} \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Thus, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\lambda^{\mathcal{I}}})$. Let $\Sigma \in |\mathbf{Sign}|$ and $t \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$ and assume $\phi \in T_\Sigma$. Then, we have $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, t) = T_\Sigma = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \phi)$, i.e., $\langle t, \phi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$. By the preceding inequality, $\langle t, \phi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T^{\widetilde{\lambda}^{\mathcal{I}}})$. But $t \in T^{\widetilde{\lambda}^{\mathcal{I}}}_\Sigma$, whence, by compatibility, $\phi \in T^{\lambda^{\mathcal{I}}}_\Sigma$. We conclude that $T \leq T^{\widetilde{\lambda}^{\mathcal{I}}}$. ■

Strong Fregeanity is characterized by compatibility of the Frege operator on theory families and, similarly, full strong Fregeanity by the compatibility of the Frege operator on filter families of arbitrary algebraic systems.

Proposition 1645 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is strongly Fregean if and only if $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}$ is a compatibility $\mathcal{I}$-operator on $\mathcal{F}$;

(b) $\mathcal{I}$ is fully strongly Fregean if and only if $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ is a compatibility $\mathcal{I}$-operator on every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Proof: We only prove Part (a) in detail. Part (b) may be proved similarly, by working on an arbitrary **F**-algebraic system $\mathcal{A}$ instead of on $\mathcal{F}$.

Suppose $\mathcal{I}$ is strongly Fregean. Then, by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. So $\lambda^{\mathcal{I},\mathcal{F}}$ is a compatibility $\mathcal{I}$-operator on $\mathcal{F}$.

Suppose, conversely, that $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}$ is a compatibility $\mathcal{I}$-operator on $\mathcal{F}$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. Therefore,

$$\begin{array}{rcll}\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) & = & \bigcap\{\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\} & (\text{monotonicity of } \widetilde{\lambda}^{\mathcal{I},\mathcal{F}}) \\ & \le & \bigcap\{\Omega^{\mathcal{F}}(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\} & (\text{by the hypothesis}) \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{F}}(T). \quad (\text{by definition}) & \end{array}$$

Since, by compatibility, $\widetilde{\Omega}^{\mathcal{I},\mathcal{F}}(T) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T)$ always holds, we conclude that $\mathcal{I}$ is strongly Fregean. ∎

The characterizations of Proposition 1645 imply that a π-institution if strongly Fregean if and only if every theory family is Frege and that it is fully strongly Fregean if and only if every filter family of any algebraic system is a Frege filter family.

Corollary 1646 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* **F***, having theorems.* $\mathcal{I}$ *is strongly Fregean if and only if* $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThFam}^{\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{I})$.

Proof: Suppose $\mathcal{I}$ is strongly Fregean and let $T \in \mathrm{ThFam}(\mathcal{I})$. By Proposition 1645, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. Thus, by Lemma 1640, $T \in \mathrm{ThFam}^{\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{I})$.

Assume, conversely, that every theory family of $\mathcal{I}$ is Frege. Then, by Lemma 1640, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. Thus, by Proposition 1645, $\mathcal{I}$ is strongly Fregean. ∎

Corollary 1647 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* **F***, having theorems.* $\mathcal{I}$ *is fully strongly Fregean if and only if* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$*, for every* **F***-algebraic system* $\mathcal{A}$.

Proof: The proof follows along the same lines as that of Corollary 1646, using Proposition 1645 and Lemma 1640, but applied over an arbitrary **F**-algebraic system $\mathcal{A}$ instead of over $\mathcal{F}$. ∎

Our next goal is to show that the Frege operator $\widetilde{\lambda}^{\mathcal{I}}$ is a semi-coherent family of $\mathcal{I}$-operators. But, first, we need to have available an isomorphism theorem involving this operator. So we embark on some preparatory work.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on **F**, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ **F**-algebraic systems, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. A surjective morphism $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ is called **deductive** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi) \quad \text{implies} \quad C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma, \phi) = C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma, \psi).$$

Equivalently, $\langle H,\gamma\rangle$ is deductive if and only if

$$\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T),$$

i.e., if and only if $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T.

We now show that for a surjective morphism, with an isomorphic signature component, compatibility properties and deductive morphisms are very closely interrelated.

Lemma 1648 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. For a surjective morphism* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, the following statements are equivalent:*

(i) $\langle H,\gamma\rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*;*

(ii) $\langle H,\gamma\rangle$ *is* $\widetilde{\lambda}^{\mathcal{I}}$*-compatible with* T*;*

(iii) $\langle H,\gamma\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{B}, \gamma(T)\rangle$ *is deductive.*

Proof:

(i)⇒(ii) Suppose $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Since, by compatibility, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$, we get that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. Thus $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T.

(ii)⇒(iii) Suppose $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. This implies that $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T. Indeed, if $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$ and $\phi \in T_\Sigma$, then, by the hypothesis, $\langle \phi,\psi\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, i.e., $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\psi)$. Since $\phi \in T_\Sigma$, we get that $\psi \in T_\Sigma$. Thus, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Moreover, by hypothesis and the comments preceding the lemma, $\langle H,\gamma\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{B}, \gamma(T)\rangle$ is a deductive morphism.

(iii)⇒(i) Suppose that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. Then, since, by Corollary 17, $\mathrm{Ker}(\langle H,\gamma\rangle)$ is a congruence system on $\mathcal{A}$ and, by Proposition 1457, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the largest congruence system on $\mathcal{A}$ included in $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$, we conclude that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, $\langle H,\gamma\rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T.

■

We now show that each deductive morphism, with an isomorphic signature component, induces an order isomorphism between the principal filter of the lattice of filter families generated by the domain and the principal filter of the lattice of theory families generate by its codomain.

Theorem 1649 (Correspondence Theorem for Deductive Morphisms) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ *and* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ *a deductive morphism, with* H *an isomorphism. Then* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$*, with inverse* γ^{-1}*.*

Proof: By Lemma 1648, $\langle H, \gamma \rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. This implies that, for every $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$, $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T''. It follows by Corollary 56 that, for all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$, $\gamma(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$. Moreover, the same compatibility property implies that $\gamma^{-1}(\gamma(T'')) = T''$, for all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$. Finally, by surjectivity of $\langle H, \gamma \rangle$, we get, for all $T''' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$, $\gamma(\gamma^{-1}(T''')) = T'''$. Therefore, $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is a bijection and, since, clearly, both γ and γ^{-1} are order preserving, they are mutually inverse order isomorphisms, as claimed. ■

Now we are ready to return to the main line of work and establish that $\widetilde{\lambda}^{\mathcal{I}}$ constitutes a semi-coherent family of $\mathcal{I}$-operators.

Theorem 1650 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The Frege operator* $\widetilde{\lambda}^{\mathcal{I}}$ *is a semi-coherent family of* $\mathcal{I}$*-operators.*

Proof: Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\langle H, \gamma \rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with $\gamma^{-1}(T')$. Then, by Lemma 1648 and Theorem 1649, $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is an order isomorphism with inverse γ^{-1}. Thus, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ and $\gamma^{-1}(\gamma(T)) = T$. Now we get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl} C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\phi)) & = & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(\gamma_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)})), \gamma_\Sigma(\phi)) \\ & = & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(\gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)}), \phi))) \\ & = & \gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)}), \phi)). \end{array}$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ll} \multicolumn{2}{l}{\langle \phi, \psi \rangle \in \gamma_\Sigma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'))} \\ \text{iff} & \langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T') \\ \text{iff} & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\phi)) = C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\psi)) \\ \text{iff} & \gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)}), \phi)) = \gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)}), \psi)) \\ \text{iff} & C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)}), \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma_\Sigma^{-1}(T'_{H(\Sigma)}), \psi) \\ \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}(T')). \end{array}$$

We conclude that $\widetilde{\lambda}^{\mathcal{I}}$ is semi-coherent. ■

It turns out that if we strengthen the semi-coherence condition by requiring that $\widetilde{\lambda}^{\mathcal{I}}$ be commuting over all morphisms, with isomorphic signature components (regardless of compatibility), then we get a characterization of protoalgebraicity.

Theorem 1651 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if* $\widetilde{\lambda}^{\mathcal{I}}$ *is a semi-commuting family of* $\mathcal{I}$*-operators.*

Proof: Suppose, first, that $\mathcal{I}$ is protoalgebraic and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Then, by Corollary 55, $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Then

$$\begin{array}{ll} \langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_{\Sigma}(\gamma^{-1}(T')) & \\ \quad \text{iff} & C^{\mathcal{I},\mathcal{A}}_{\Sigma}(\gamma^{-1}_{\Sigma}(T'_{H(\Sigma)}), \phi) = C^{\mathcal{I},\mathcal{A}}_{\Sigma}(\gamma^{-1}_{\Sigma}(T'_{H(\Sigma)}), \psi) \\ \quad \text{iff} & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\phi)) = C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\psi)) \\ \quad \text{iff} & \langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T') \\ \quad \text{iff} & \langle \phi, \psi \rangle \in \gamma^{-1}_{\Sigma}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T')). \end{array}$$

Therefore, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T'))$ and $\widetilde{\lambda}^{\mathcal{I}}$ is a semi-commuting family of $\mathcal{I}$-operators.

Assume, conversely, that $\widetilde{\lambda}^{\mathcal{I}}$ is semi-commuting and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Since $\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T') \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T')$, we get

$$\gamma^{-1}(\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T')) \leq \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T')) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')).$$

By Corollary 17, $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T'))$ is a congruence system on $\mathcal{A}$. By Proposition 1457, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$ is the largest congruence system below $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$. Therefore, we get $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T')) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$. Since the converse inclusion always holds, $\widetilde{\Omega}^{\mathcal{I}}$ is semi-commuting. Thus, by Corollary 1574, we get that $\widetilde{\Omega}^{\mathcal{I}} = \Omega$ and, therefore, by Lemma 1555, $\mathcal{I}$ is protoalgebraic. ■

We also get a commutativity property with direct images.

Lemma 1652 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H, \gamma \rangle$ *is* $\widetilde{\lambda}^{\mathcal{I}}$*-compatible with* T*, then*

$$\gamma(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) = \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)).$$

Proof: By hypothesis and Lemma 1648, $\langle H, \gamma \rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. Therefore, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, also, $T = \gamma^{-1}(\gamma(T))$. Since, by Theorem 1650, $\widetilde{\lambda}^{\mathcal{I}}$ is semi-coherent, we get

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(\gamma(T))) = \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T))).$$

Hence, by the surjectivity of $\langle H,\gamma\rangle$, we get $\gamma(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) = \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T))$. ∎

In analogy with previous correspondence theorems we have the following one regarding correspondence between Frege classes.

Theorem 1653 (Correspondence Theorem for Frege Classes) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems,* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H,\gamma\rangle$ *is* $\widetilde{\lambda}^{\mathcal{I}}$*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ *onto* $[\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$*, with inverse* γ^{-1}*.*

Proof: Since $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T, we get, by Lemma 1648, that $\langle H,\gamma\rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. Therefore, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, also, $T = \gamma^{-1}(\gamma(T))$.

Now, let $T' \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Then $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. As a consequence, we get $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\gamma^{-1}(\gamma(T')) = T'$. Now we get

$$\begin{array}{rcll}
\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)) & = & \gamma(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) & \text{(by Lemma 1652)}\\
& \le & \gamma(\Omega^{\mathcal{A}}(T')) & (\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T'))\\
& = & \Omega^{\mathcal{B}}(\gamma(T')). & \text{(by Lemma 1579).}
\end{array}$$

We conclude that $\gamma(T') \in [\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$.

Suppose, conversely, that $T' \in [\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Then $\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)) \le \Omega^{\mathcal{B}}(T')$. By Corollary 55, $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by surjectivity, $\gamma(\gamma^{-1}(T')) = T'$. Moreover, $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with $\gamma^{-1}(\gamma(T)) = T$. Hence, we have

$$\begin{array}{rcll}
\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(\gamma(T))) & \\
& = & \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T)) & \text{(by Theorem 1650)}\\
& \le & \gamma^{-1}(\Omega^{\mathcal{B}}(T')) & (\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)) \le \Omega^{\mathcal{B}}(T'))\\
& = & \Omega^{\mathcal{A}}(\gamma^{-1}(T')). & \text{(by Proposition 24)}
\end{array}$$

Hence, $\gamma^{-1}(T') \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$.

Thus, $\gamma : [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \to [\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ is a bijection, with inverse γ^{-1}. Since both mappings are order-preserving, we conclude that they form a pair of mutually inverse order isomorphisms. ∎

This correspondence theorem allows us to provide characterizations of full self extensionality and full strong Fregeanity in the following two corollaries.

Corollary 1654 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is fully self extensional if and only if, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$*.*

Proof: Suppose, first, that $\mathcal{I}$ is fully self extensional and let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then we have

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \\ & & \text{(by full self extensionality)} \\ & = & \Delta^{\mathcal{A}}. \quad (\text{since } \mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}))\end{array}$$

Suppose, conversely, that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$, and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. Set, for notational convenience and brevity, $\mathcal{B} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, and consider the quotient morphism

$$\langle I,\pi\rangle : \mathcal{A} \to \mathcal{B}.$$

Then $\mathrm{Ker}(\langle I,\pi\rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, whence $\mathrm{Ker}(\langle I,\pi\rangle)$ is compatible with $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence, we get, by Corollary 56, $\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\pi^{-1}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is the least $\mathcal{I}$-family of $\mathcal{A}$, it must be a Suszko $\mathcal{I}$-filter family. Hence, by Corollary 1591, $\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is the least $\mathcal{I}$-filter family on $\mathcal{B}$, i.e., we have

$$\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}).$$

Since $\mathrm{Ker}(\langle I,\pi\rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, $\langle I,\pi\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with $\pi^{-1}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, since $\mathcal{B} \in \mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis, $\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})) = \Delta^{\mathcal{B}}$. Now we get

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & = & \mathrm{Ker}(\langle I,\pi\rangle) \quad (\text{definition of } \langle I,\pi\rangle) \\ & = & \pi^{-1}(\Delta^{\mathcal{B}}) \quad (\text{definition of kernel}) \\ & = & \pi^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})))) \\ & & (\text{shown above}) \\ & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\pi^{-1}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})))) \\ & & (\text{Theorem 1651}) \\ & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).\end{array}$$

We conclude that $\mathcal{I}$ is fully self extensional. ■

Corollary 1655 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is fully strongly Fregean if and only if, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$.

Proof: The left-to-right implication follows directly by the definition of full strong Fregeanity. Assume, conversely, that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Let $\mathcal{A}$ be an arbitrary $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

$\mathrm{Ker}(\langle I,\pi\rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is compatible with T. Hence $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$ and $\pi^{-1}(\pi(T)) = T$. Moreover, $\mathrm{Ker}(\langle I,\pi\rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. Thus, $\langle I,\pi\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T. Since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis,

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T)) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}.$$

Now we have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) & = & \mathrm{Ker}(\langle I,\pi\rangle) \quad \text{(definition of } \langle I,\pi\rangle) \\ & = & \pi^{-1}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}) \quad \text{(definition of kernel)} \\ & = & \pi^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T))) \quad \text{(shown above)} \\ & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\pi^{-1}(\pi(T))) \quad \text{(Theorem 1651)} \\ & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T).\end{array}$$

We conclude that $\mathcal{I}$ is fully strongly Fregean. ■

On the other hand, strong Fregeanity, combined with the existence of natural theorems, implies assertionality.

Corollary 1656 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly Fregean and has natural theorems, then it is syntactically family assertional.*

Proof: Assume $\mathcal{I}$ is strongly Fregean and has natural theorems. Let $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be a natural theorem. Then define $\tau^\flat : (\mathrm{SEN}^\flat)^{k+1} \to \mathrm{SEN}^\flat$ by

$$\tau^\flat := \{p^{k+1,0} \approx \vartheta^\flat \circ \langle p^{k+1,1},\dots,p^{k+1,k}\rangle\}.$$

We show, first, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi},\vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \vartheta^\flat_\Sigma(\vec{\chi}), \vartheta^\flat_\Sigma(\vec{\chi}'\rangle \in \Omega_\Sigma(T)$. Since, $\vartheta^\flat$ is a natural theorem, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi},\vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma)$, $\vartheta^\flat_\Sigma(\vec{\chi}), \vartheta^\flat_\Sigma(\vec{\chi}') \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Therefore, we get $\langle \vartheta^\flat_\Sigma(\vec{\chi}), \vartheta^\flat_\Sigma(\vec{\chi}')\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)$. Thus, by strong Fregeanity, $\langle \vartheta^\flat_\Sigma(\vec{\chi}), \vartheta^\flat_\Sigma(\vec{\chi}')\rangle \in \Omega^{\mathcal{I}}_\Sigma(T) \le \Omega_\Sigma(T)$.

We show, next, that $\mathcal{I}$ is systemic. To this end, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Let $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, we have $\langle \phi, t\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T) = \widetilde{\Omega}^{\mathcal{I}}_\Sigma(T)$. Hence, since $\widetilde{\Omega}^{\mathcal{I}}(T)$ is a congruence system, we get $\langle \mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(t)\rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T) = \widetilde{\lambda}^{\mathcal{I}}_{\Sigma'}(T)$. But $\mathrm{SEN}^\flat(f)(t) \in \mathrm{Thm}_{\Sigma'}(\mathcal{I}) \subseteq T_{\Sigma'}$ and, therefore, by compatibility, $\mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}$. Hence, $T \in \mathrm{ThSys}(\mathcal{I})$ and $\mathcal{I}$ is systemic.

Finally, we show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega_\Sigma(T).$$

Assume, first $\phi \in T_\Sigma$. By systemicity, for every $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}$. Therefore, for all $\vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \mathrm{SEN}^\flat(f)(\phi), \vartheta^\flat_{\Sigma'}(\vec{\chi}')\rangle \in \widetilde{\lambda}^{\mathcal{I}}_{\Sigma'}(T) = \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T) \le \Omega_{\Sigma'}(T).$$

If, conversely, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma')$, we have $\langle \mathrm{SEN}^\flat(\phi), \vartheta^\flat_{\Sigma'}(\vec{\chi}')\rangle \in \Omega_{\Sigma'}(T)$, then, in particular for $f = i_\Sigma$, we get, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \phi, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \Omega_\Sigma(T)$. Since $\vartheta^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$, we get, by compatibility, that $\phi \in T_\Sigma$. ■

Finally, combining this work with previously obtained results, we get the following corollary comparing the injectivity of the various $\mathcal{I}$-operators we have studied.

Corollary 1657 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$.

(a) If $\widetilde{\Omega}^{\mathcal{I}}$ *is injective, then* $\widetilde{\Omega}$ *is injective.*

(b) If Ω *is injective, then* $\widetilde{\lambda}^{\mathcal{I}}$ *is injective.*

Proof:

(a) If $\widetilde{\Omega}^{\mathcal{I}}$ is injective, then, by Theorem 828, Ω is c-reflective. Thus, it is, a fortiori, injective.

(b) If Ω is injective, then, necessarily, $\mathcal{I}$ has theorems. Therefore, by Theorem 495, we get that $\widetilde{\lambda}^{\mathcal{I}}$ is injective.

■

21.13 Leibniz Hierarchy Revisited

Using the Isomorphism Theorem 1445 between full $\mathcal{I}$-structures and $\mathcal{I}$-congruence systems, we obtain, in the case of protoalgebraic π-institutions, the following special isomorphism theorem between Leibniz $\mathcal{I}$-filter families and $\mathcal{I}^*$-congruence systems.

Proposition 1658 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$-*algebraic system* $\mathcal{A}$, *the Leibniz operator* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ *is an order isomorphism.*

Proof: By Theorem 1445, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\widetilde{\Omega}^{\mathcal{A}} : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism. By protoalgebraicity and Theorem 1619,

$$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\rangle : T \in \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})\}.$$

Moreover, by protoalgebraicity, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \Omega^{\mathcal{A}}(T)$ and, also, $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. Therefore, we get that $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism. ■

We now show that $\Omega^{\mathcal{A}}$, as a mapping from Leibniz $\mathcal{I}$-filter families to $\mathcal{I}^*$-congruence systems on $\mathcal{I}$-algebraic systems, being an order isomorphism is sufficient to establish that the same mapping is an order isomorphism for all $\mathbf{F}$-algebraic systems.

Proposition 1659 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, the following conditions are equivalent:*

(i) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ *is an order isomorphism;*

(ii) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ *is an order isomorphism.*

Proof: Since (i)⇒(ii) is trivial, assume (ii) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an arbitrary $\mathbf{F}$-algebraic system.

By Proposition 1565, $\Omega^{\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$.

To show surjectivity, assume $\theta \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. By definition, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\theta = \Omega^{\mathcal{A}}(T)$. Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$. Since $\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T)$ is compatible with T, by Corollary 56, $\pi(T) \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Since $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis, that there exists $T' \in \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$, such that $\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)) = \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T')$. Now we have

$$\begin{array}{lll} \Omega^{\mathcal{A}}(T) & = & \Omega^{\mathcal{A}}(\pi^{-1}(\pi(T))) \quad (\mathrm{Ker}(\langle I, \pi \rangle) \text{ compatible with } T) \\ & = & \pi^{-1}(\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T))) \\ & = & \pi^{-1}(\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T')) \\ & = & \Omega^{\mathcal{A}}(\pi^{-1}(T')). \end{array}$$

Since $\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))$, we get that $\mathrm{Ker}(\langle I, \pi \rangle)$ is compatible with $\pi^{-1}(T')$, and, hence, $\pi(\pi^{-1}(T')) = T' \in \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. by Corollary 1612, $\pi^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. We showed that $\theta = \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))$, with $\pi^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is surjective.

Next, we turn to monotonicity. To this end, let $T, T' \in \mathrm{Fifam}^{\mathcal{I}^*}(\mathcal{A})$, such that $T \leq T'$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

We have $\mathrm{Ker}(\langle I, \pi \rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$ and, also, $\mathrm{Ker}(\langle I, \pi \rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T')$. Thus, by Corollary 56,

$$\pi(T), \pi(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

Since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis, $\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T)) \leq \Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T'))$. Therefore,

$$\begin{array}{rcll} \Omega^{\mathcal{A}}(T) & = & \Omega^{\mathcal{A}}(\pi^{-1}(\pi(T))) & \text{(compatibility)} \\ & = & \pi^{-1}(\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T))) & \\ & \leq & \pi^{-1}(\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T'))) & \\ & = & \Omega^{\mathcal{A}}(\pi^{-1}(\pi(T'))) & \\ & = & \Omega^{\mathcal{A}}(T'). & \end{array}$$

Hence $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is monotone. Finally, by Proposition 1565, $\Omega^{\mathcal{A}}$ is reflective on $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, we conclude that $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. ■

Next, we show that, if $\Omega^{\mathcal{A}}$ from the Leibniz filter families onto the $\mathcal{I}^*$-congruence systems happens to be an order isomorphism on every $\mathcal{I}$-algebraic system, then the class of $\mathcal{I}$-algebraic systems coincides with the class of $\mathcal{I}^*$-algebraic systems.

Lemma 1660 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, then $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: By Corollary 1442, we know that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ always holds. So it suffices to show the reverse inclusion. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and let $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, we have, by the hypothesis, $\Omega^{\mathcal{A}}(T^m) \leq \Omega^{\mathcal{A}}(T)$, which yields $[\![T]\!]^* \subseteq [\![T^m]\!]^*$.

Now let $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$. By hypothesis, there exists $T \in \mathrm{ThFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, we get $T' \in [\![T']\!]^* = [\![T]\!]^* \subseteq [\![T^m]\!]^*$. Since this holds for every $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we conclude that $[\![T^m]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1615, we get $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, as, by hypothesis, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and, hence, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$, we get $\mathcal{A} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}^*(\mathcal{I})$. We conclude that $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathrm{AlgSys}^*(\mathcal{I})$ and, therefore, the two classes of algebraic systems coincide. ■

The same conclusion may be drawn if we assume that $\Omega^{\mathcal{A}}$ is an order isomorphism from the collection of Suszko $\mathcal{I}$-filter families to the collection of all $\mathcal{I}^*$-congruence systems and, in fact, the proof follows along very similar lines.

Lemma 1661 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, then $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: By Corollary 1442, we know that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ always holds. So it suffices to show the reverse inclusion. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and let $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, we have, by the hypothesis, $\Omega^{\mathcal{A}}(T^m) \le \Omega^{\mathcal{A}}(T)$, which yields $[\![T]\!]^* \subseteq [\![T^m]\!]^*$.

Now let $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$. By hypothesis, there exists $T \in \mathrm{ThFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, we get $T' \in [\![T']\!]^* = [\![T]\!]^* \subseteq [\![T^m]\!]^*$. Since this holds for every $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we conclude that $[\![T^m]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1615, we get $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, as, by hypothesis, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and, hence, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$, we get $\mathcal{A} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}^*(\mathcal{I})$. We conclude that $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathrm{AlgSys}^*(\mathcal{I})$ and, therefore, the two classes of algebraic systems coincide. ∎

We know that, under protoalgebraicity, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. The converse is true when family c-reflectivity is also assumed.

Proposition 1662 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a family completely reflective* π*-institution based on* $\mathbf{F}$*. If* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$*, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: Assume that $\mathcal{I}$ is family c-reflective and that $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. By Proposition 1616, every full $\mathcal{I}$-structure on an $\mathbf{F}$-algebraic system $\mathcal{A}$ has the form $\langle \mathcal{A}, [\![T]\!]^* \rangle$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1621, then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\mathsf{Su}} = [\![T']\!]^*$. Hence, $T^{\mathsf{Su}} = T'^*$. Now we have

$$\begin{array}{rcll} T & = & T^{\mathsf{Su}} & \text{(by Theorem 1627)} \\ & = & T'^* & \text{(shown above)} \\ & = & T'. & \text{(by Lemma 1620)} \end{array}$$

We conclude that $[\![T]\!]^{\mathsf{Su}} = [\![T]\!]^*$. Since this holds, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A})$, we conclude, by Proposition 1638, that $\mathcal{I}$ is protoalgebraic. ∎

By Lemma 1660, we may replace equality of the two classes of algebraic system in Proposition 1662 by the condition that the Leibniz operator be an order isomorphism.

Proposition 1663 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a family completely reflective* π*-institution based on* $\mathbf{F}$*. If, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: By Proposition 1662 and Lemma 1660. ∎

We also get a characterization of weak family algebraizability.

Corollary 1664 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is weakly family algebraizable if and only if it is family c-reflective and* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$.

Proof: If $\mathcal{I}$ is weakly family algebraizable, then it is, by definition, family c-reflective and protoalgebraic. By protoalgebraicity, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. On the other hand, if $\mathcal{I}$ is family c-reflective and $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$, then it is family c-reflective and, by Proposition 1662, it is also protoalgebraic. Hence, $\mathcal{I}$ is weakly family algebraizable. ■

If, in Proposition 1663, we drop the hypothesis of $\mathcal{I}$ being family c-reflective, but compensate by assuming that $\Omega^{\mathcal{A}}$ is an order isomorphism between the collection of Suszko filter families and $\mathcal{I}^*$-congruence systems on all $\mathbf{F}$-algebraic systems, then we can still infer the protoalgebraicity of $\mathcal{I}$.

Theorem 1665 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ *is an order isomorphism.*

Proof: By Proposition 1658, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism. By protoalgebraicity and Theorem 1638, $\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{I})$ is an order isomorphism.

Conversely, assume that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism. By Lemma 1555, it suffices to show that the Leibniz and Suszko operators on an arbitrary $\mathbf{F}$-algebraic system coincide. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

- Note that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1661 and the hypothesis, there exists $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Hence, $[\![T]\!]^{\mathsf{Su}} = [\![T']\!]^*$ and, therefore, by Lemma 1620, $T^{\mathsf{Su}} = T'^* = T'$. We conclude that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathsf{Su}})$.

- Note that $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. Thus, there exists, by hypothesis, $T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'')$. So $[\![T]\!]^* = [\![T'']\!]^*$. Hence, by Lemma 1620, $T^* = T''^* = T''$. This gives $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*)$. Since $T^* = T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, $(T^*)^{\mathsf{Su}} = T^*$. But, by Lemma 1605, $T^* \leq T$. Hence, $[\![T]\!]^{\mathsf{Su}} \subseteq [\![T^*]\!]^{\mathsf{Su}}$ and, thus, $T^* = (T^*)^{\mathsf{Su}} \leq T^{\mathsf{Su}}$. By Lemma 1620, the reverse inclusion always holds, whence $T^* = T^{\mathsf{Su}}$. Now we get $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathsf{Su}})$.

Since, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathsf{Su}}) = \Omega^{\mathcal{A}}(T)$, we get that $\mathcal{I}$ is a protoalgebraic π-institution. ■

Theorem 1665 allows us to give a related characterization of equivalential π-institutions.

Corollary 1666 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family equivalential if and only if it is family commuting and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism.*

Proof: Suppose, first, that $\mathcal{I}$ is family equivalential. Then, by definition, $\mathcal{I}$ is family extensional and protoalgebraic. Thus, by Theorem 327, it is family inverse commuting and, by Theorem 325, it is family commuting. Moreover, by Theorem 1665, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

Assume, conversely, that $\mathcal{I}$ is family commuting and that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. Then, by Theorem 1665, it is protoalgebraic. Therefore, by Theorem 325, it is family inverse commuting and, by Theorem 327, it is family extensional. Being protoalgebraic and family extensional, it is, by definition, family equivalential. ■

We turn now to establishing some characterizations of semantic classes in the Leibniz hierarchy via the use of the Suszko operator. First, we show that the family c-reflectivity of the Leibniz operator is equivalent with the universal injectivity of the Suszko operator and, in turn, a sufficient (and, trivially, necessary) condition fr it is the injectivity of the Suszko operator on an $\mathcal{I}$-algebraic systems.

Theorem 1667 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ *is injective, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$*;*

(iii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ *is injective, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*.*

Proof:

(i)⇒(ii) By Proposition 1565, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. By hypothesis and Theorem 1627, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is injective, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) We use again Theorem 1627, showing that for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. To this end. let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. We have $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$ and, by Corollary 56, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$.

Hence, by assumption, $T^m \le T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. By monotonicity of the Suszko operator, Proposition 1581 and Lemma 1594,

$$\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T^m) \le \widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I}<\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}.$$

Hence, by hypothesis, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = T^m$. Therefore, by Proposition 1624, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. This proves that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by Theorem 1627, yields that $\mathcal{I}$ is family c-reflective.

■

As regards protoalgebraicity, we have the following characterization.

Theorem 1668 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is commuting.*

Proof: If $\mathcal{I}$ is protoalgebraic, then, by Lemma 1555, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$ and, by Proposition 24, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting. If, conversely, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting, then, by Corollary 1574, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. Therefore, by Lemma 1555, $\mathcal{I}$ is protoalgebraic. ■

We also get characterizations for equivalential, weakly algebraizable and algebraizable π-institutions.

Theorem 1669 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is family equivalential if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is commuting and family extensional.*

(b) $\mathcal{I}$ *is weakly family algebraizable if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is injective and commuting.*

(c) $\mathcal{I}$ *is family algebraizable if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is injective and commuting and family extensional.*

Proof:

(a) Suppose $\mathcal{I}$ is equivalential. Then, by Theorem 334, Ω is monotone and family extensional. By Lemma 1555, $\Omega = \widetilde{\Omega}^{\mathcal{I}}$. Thus, by Proposition 24, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting and family extensional. Conversely, if $\widetilde{\Omega}^{\mathcal{I}}$ is commuting and family extensional, then, by Corollary 1574, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. Thus, Ω is monotone and family extensional. By Theorem 334, $\mathcal{I}$ is family equivalential.

(b) By Theorems 1667 and 1668.

(c) By Part (a) and Theorem 1667.

■

In general, it is not hard to show that the Suszko operator on an **F**-algebraic system is an order embedding from the collection of Suszko $\mathcal{I}$-filter families into the family of $\mathcal{I}$-congruence systems.

Proposition 1670 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**. *For every* **F***-algebraic system* $\mathcal{A}$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding.

Proof: The Suszko operator $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is always into $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. It is monotone by definition, and it is order-reflecting on $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ by Proposition 1565. Therefore, it is an order embedding, as claimed. ■

Requiring the preceding embedding to be an order isomorphism turns out to be equivalent to the protoalgebraicity of $\mathcal{I}$.

Theorem 1671 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**. *The following conditions are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism, for every* **F***-algebraic system* $\mathcal{A}$*;*

(iii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is surjective, for every* **F***-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(ii) By hypothesis and Lemma 1555, $\Omega^{\mathcal{A}} = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$. Thus, by Proposition 1617, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. It follows that $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. Now, by Theorem 1665, we get that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Assume (iii). We show that the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone on the $\mathcal{I}$-filter families of every **F**-algebraic system $\mathcal{A}$. To this end, let $\mathcal{A}$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, there exists, by hypothesis, $T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T)$. Thus, we have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}}([\![T'']\!]^{\mathsf{Su}}) & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'') \\ & = & \Omega^{\mathcal{A}}(T) \\ & = & \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*).\end{array}$$

Since $\langle\mathcal{A},[\![T'']\!]^{\mathsf{Su}}\rangle$, $\langle\mathcal{A},[\![T]\!]^*\rangle \in \mathrm{FStr}(\mathcal{I}|)$, by Theorem 1445, $[\![T'']\!]^{\mathsf{Su}} = [\![T]\!]^*$. Moreover, since $T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, by Lemma 1620, we obtain $[\![T'']\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T''}$. Since $T \in [\![T]\!]^* = [\![T'']\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T''}$, we get $T'' \le T \le T'$. Thus, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T''} = [\![T'']\!]^{\mathsf{Su}} = [\![T]\!]^*$. In other words, $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. We conclude that $\Omega^{\mathcal{A}}$ is monotone on every $\mathcal{A}$, whence $\mathcal{I}$ is protoalgebraic.

■

In closing the section, we exploit Theorem 1671 to provide characterizations of some of the classes of the semantic Leibniz hierarchy.

Theorem 1672 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism;*

(b) *$\mathcal{I}$ is family c-reflective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order embedding;*

(c) *$\mathcal{I}$ is weakly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism;*

(d) *$\mathcal{I}$ is family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism and* $\mathcal{I}$ *is family extensional.*

Proof:

(a) By Theorem 1671.

(b) By Proposition 1670, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is always an order embedding. By Theorem 1627, family c-reflectivity implies $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We conclude that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding. If, conversely, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding, then it is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Theorem 1667, $\mathcal{I}$ is family c-reflective.

(c) Assume, first, that $\mathcal{I}$ is weakly family algebraizable. By Theorem 1665, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. By Theorem 1627, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. Finally, by protoalgebraicity and Lemma 1555, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} = \Omega^{\mathcal{A}}$, and by protoalgebraicity and Proposition 1617, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism.

If, conversely, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism, then, by Theorem 1667, $\mathcal{I}$ is family c-reflective, whence, by

Theorem 1627, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, hence, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is onto $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Thus, by Theorem 1671, $\mathcal{I}$ is protoalgebraic. We conclude that $\mathcal{I}$ is weakly family algebraizable.

(d) By Part (c) and the definition of family algebraizability.

■

21.14 Suszko Operator and Truth Equationality

Recall that by Proposition 68 and Proposition 28, it makes sense, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, to consider the relative congruence system $\Theta^{\mathcal{I},\mathcal{A}}(R) := \Theta^{\mathrm{AlgSys}(\mathcal{I}),\mathcal{A}}(R)$ on $\mathcal{A}$ generated by a relation family $R \in \mathrm{RelFam}(\mathcal{A})$.

Lemma 1673 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $X \in \mathrm{SenFam}(\mathcal{A})$*, if* $\theta = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X])$ *and* $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$ *is the quotient morphism, then*

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X) \text{ and } \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X.$$

Proof: Let us set

$$\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \le \Omega^{\mathcal{A}}(T)\}.$$

By Proposition 1561, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\mathcal{T})$ and $\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \mathcal{T}$. But we also have

$$\begin{array}{rcl}
\mathcal{T} & = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \le \Omega^{\mathcal{A}}(T)\} \\
& & \text{(definition of } \mathcal{T}) \\
& = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \tau^{\mathcal{A}}[X] \le \Omega^{\mathcal{A}}(T)\} \\
& & \text{(since } \theta = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \text{ and } \Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})) \\
& = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : X \le T\} \\
& & \text{(by family truth equationality)} \\
& = & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X. \quad \text{(definition of } \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X)
\end{array}$$

The conclusion follows. ■

We show, next that, under the same hypotheses, the Suszko congruence system of an $\mathcal{I}$-filter family generated by a sentence family X equals the least $\mathcal{I}$-congruence system on $\mathcal{A}$ generated by the relation family $\tau^{\mathcal{A}}[X]$.

Proposition 1674 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $X \in \mathrm{SenFam}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]).$$

In particular, if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[T])$*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $X \in \mathrm{SenFam}(\mathcal{A})$ and set $\theta = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X])$. Since $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, we have $\mathcal{A}/\theta \in \mathrm{AlgSys}(\mathcal{I})$. Therefore,

$$\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \Delta^{\mathcal{A}/\theta}.$$

Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta.$$

We have

$$\begin{array}{rcll}
\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) & = & \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X) & \\
& = & \widetilde{\Omega}^{\mathcal{A}}(\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) & \text{(Lemma 1673)} \\
& = & \pi^{-1}(\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) & \\
& = & \pi^{-1}(\Delta^{\mathcal{A}/\theta}) & \\
& = & \theta. &
\end{array}$$

Therefore, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X])$, as was to be shown. ∎

Proposition 1675 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \Theta^{\mathcal{I}}(\tau^\flat_\Sigma[\phi]).$$

Proof: Directly from Proposition 1674, letting $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $X_\Sigma = \{\phi\}$ and $X_{\Sigma'} = \varnothing$, for all $\Sigma' \neq \Sigma$. ∎

Another property is that the Suszko congruence family of the $\mathcal{I}$-filter family generated by a sentence family X can be obtained as the join of the Suszko congruence families of the $\mathcal{I}$-filter families generated by each singleton in X.

Proposition 1676 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $X \in \mathrm{SenFam}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) = \bigvee\{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\phi)) : \phi \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}.$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Then, we have, for every **F**-algebraic system $\mathcal{A}$ and all $X \in \mathrm{SenFam}(\mathcal{A})$,

$$\begin{aligned}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) &= \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \\ &\quad \text{(by Proposition 1674)} \\ &= \bigvee\{\Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}_\Sigma[\phi]) : \phi \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\} \\ &\quad \text{(by Proposition 35)} \\ &= \bigvee\{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\phi)) : \phi \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}. \\ &\quad \text{(by Proposition 1674)}\end{aligned}$$

This proves the statement. ∎

More generally, we have

Proposition 1677 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigvee_{i\in I}^{\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})} T^i) = \bigvee_{i\in I}^{\mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})} \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T^i).$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and let $\mathcal{A}$ be an **F**-algebraic system and $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, we have

$$\begin{aligned}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\textstyle\bigvee_{i\in I} T^i) &= \widetilde{\Omega}(C^{\mathcal{I},\mathcal{A}}(\textstyle\bigcup_{i\in I} T^i)) \quad \text{(joins in } \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})) \\ &= \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[\textstyle\bigcup_{i\in I} T^i]) \quad \text{(Proposition 1674)} \\ &= \Theta^{\mathcal{I},\mathcal{A}}(\textstyle\bigcup_{i\in I} \tau^{\mathcal{A}}[T^i]) \\ &= \textstyle\bigvee_{i\in I} \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[T^i]) \quad \text{(joins in } \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})) \\ &= \textstyle\bigvee_{i\in I} \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T^i). \quad \text{(Proposition 1674)}\end{aligned}$$

This proves the statement. ∎

Another property is the commutativity of the Suszko operator with surjective morphisms with isomorphic functor components.

Proposition 1678 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*. For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*,* $\mathcal{B}$*, all surjective morphisms* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(C^{\mathcal{I},\mathcal{B}}(\gamma(T))) = \Theta^{\mathcal{I},\mathcal{B}}(\gamma(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))).$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and let $\mathcal{A}$, $\mathcal{B}$ be **F**-algebraic systems, $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We now get

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(C^{\mathcal{I},\mathcal{B}}(\gamma(T))) & = & \Theta^{\mathcal{I},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(T)]) & \text{(by Proposition 1674)}\\ & = & \Theta^{\mathcal{I},\mathcal{B}}(\gamma(\Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[T]))) & \text{(by Proposition 34)}\\ & = & \Theta^{\mathcal{I},\mathcal{B}}(\gamma(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))). & \text{(by Proposition 1674)}\end{array}$$

This proves the equality in the statement. ∎

We now build a little further on our work of Section 12.3 in order to give another characterization of family truth equationality.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D\rangle$ and $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be two π-structures based on $\mathbf{K}$ and $\mathbf{K}'$, respectively. Consider an order embedding

$$h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}').$$

Recall that $\overleftarrow{h} = \{\overleftarrow{h}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$, by letting

$$\overleftarrow{h}_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

be given, for all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\overleftarrow{h}_\Sigma[\phi] = h(D(\phi)).$$

Then we have the following analog of Lemma 897.

Lemma 1679 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ *an order embedding, which preserves suprema. Then* $\overleftarrow{h} : \mathcal{K} \to \mathcal{K}'$ *is an interpretation.*

Proof: Suppose $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is an order embedding and let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{rll}\phi \in D_\Sigma(\Phi) & \text{iff} & D(\phi) \le D(\Phi)\\ & \text{iff} & h(D(\phi)) \le h(D(\Phi))\\ & \text{iff} & h(D(\phi)) \le h(\bigvee\{D(\chi) : \chi \in \Phi\})\\ & \text{iff} & h(D(\phi)) \le \bigvee\{h(D(\chi)) : \chi \in \Phi\}\\ & \text{iff} & \overleftarrow{h}_\Sigma[\phi] \le \bigvee\{\overleftarrow{h}_\Sigma[\chi] : \chi \in \Phi\}\\ & \text{iff} & \overleftarrow{h}_\Sigma[\phi] \le D'(\overleftarrow{h}_\Sigma[\Phi]).\end{array}$$

Thus, $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is indeed an interpretation. ∎

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$ and $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures based on $\mathbf{K}^k$ and $\mathbf{K}^\ell$, respectively. Consider a suprema preserving order embedding

$$h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}').$$

We say that the order embedding $h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ is **transformational** if there exists $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$, with k distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\overleftarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]).$$

We have the following analog of Lemma 902.

Lemma 1680 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ a transformational suprema preserving order embedding induced by $\tau : \mathbf{K}^k \to \mathbf{K}^\ell$. Then, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,*

$$h(D(\Phi)) = D'(\tau_\Sigma[\Phi]).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\begin{array}{rcll} h(D(\Phi)) & = & h(\bigvee_{\phi\in\Phi} D(\phi)) & \text{(join in } \mathbf{ThFam}(\mathcal{K})) \\ & = & \bigvee_{\phi\in\Phi} h(D(\phi)) & (h \text{ suprema preserving}) \\ & = & \bigvee_{\phi\in\Phi} D'(\tau_\Sigma[\phi]) & (\overleftarrow{h}_\Sigma[\phi] = D'(\tau_\Sigma[\phi])) \\ & = & D'(\bigcup_{\phi\in\Phi} \tau_\Sigma[\phi]) & \text{(join in } \mathbf{ThFam}(\mathcal{K}')) \\ & = & D'(\tau_\Sigma[\Phi]). & \text{(by definition)} \end{array}$$

This proves the equality of the statement. ■

Furthermore, we have an analog of Theorem 903:

Theorem 1681 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ a transformational suprema preserving order embedding induced by $\tau : \mathbf{K}^k \to \mathbf{K}^\ell$. Then $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)^k$. We then have:

$$\begin{array}{rll} \phi \in D_\Sigma(\Phi) & \text{iff} & D_\Sigma(\phi) \leq D_\Sigma(\Phi) \\ & \text{iff} & h(D(\phi)) \leq h(D(\Phi)) \quad (h \text{ order embedding}) \\ & \text{iff} & D'(\tau_\Sigma[\phi]) \leq D'(\tau_\Sigma[\Phi]) \quad \text{(Lemma 1680)} \\ & \text{iff} & \tau_\Sigma[\phi] \leq D'(\tau_\Sigma[\Phi]). \end{array}$$

Thus, $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation. ■

Now, we obtain the following theorem characterizing family truth equationality in terms of transformational suprema preserving order embeddings.

Theorem 1682 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family truth equational if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is a transformational suprema preserving order embedding.*

Proof: Suppose, first, that $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$. Then, it is, a fortiori, family c-reflective, whence, by Theorem 1672, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. By Proposition 1674, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is transformational and, by Proposition 1677, it is suprema preserving.

Assume, conversely, that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is a transformational suprema preserving order embedding. Then, on the one hand, by Theorem 1672, $\mathcal{I}$ is family c-reflective, and, on the other, by definition, there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, such that $\widetilde{\Omega}^{\mathcal{I}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{F})$ is induced by $\tau^\flat$. Thus, by Theorem 1681, $\tau^\flat : \mathcal{I} \to \mathcal{Q}^{\mathrm{AlgSys}(\mathcal{I})}$ is an interpretation. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. we have

$$\begin{array}{rcl}\phi \in T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Theta^{\mathcal{I},\mathcal{F}}(\tau^\flat_\Sigma[T_\Sigma]) \quad (\tau^\flat \text{ an interpretation}) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \quad (\text{by Lemma 1680}) \\ & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T))\end{array}$$

If, conversely, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, then $\Theta^{\mathcal{I},\mathcal{F}}(\tau^\flat_\Sigma[\phi]) \leq \Omega(T)$, whence, by Lemma 1680, $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. Thus, by family c-reflectivity and Lemma 1556, $\phi \in T_\Sigma$. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega_\Sigma(T).$$

We conclude that $\mathcal{I}$ is family truth-equational, with witnessing transformations $\tau^\flat$. ■

21.15 Relations With Algebraic Semantics

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Given a class K of $\mathbf{F}$-algebraic systems, recall the definition oof the closure system $C^{\mathsf{K},\tau} = \{C^{\mathsf{K},\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C^{\mathsf{K},\tau}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ is given, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in C^{\mathsf{K},\tau}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]).$$

Define the class $\mathsf{K}(\mathcal{I},\tau)$ of $\mathbf{F}$-algebraic systems by

$$\mathsf{K}(\mathcal{I},\tau) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : C \leq C^{\mathcal{A},\tau}\}.$$

The following proposition gives a characterization of this class.

Proposition 1683 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$\mathsf{K}(\mathcal{I}, \tau) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Suppose, first, that $\mathcal{A} \in \mathsf{K}(\mathcal{I}, \tau)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. Then, by definition, $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}}$. This implies $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Since, by hypothesis, $\phi \in C_\Sigma(\Phi)$ and $C \leq C^{\mathcal{A},\tau}$, we get $\alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. Equivalently, $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$, i.e., $\alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. We conclude that $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. This proves the left-to-right inclusion.

Assume, conversely, that $\mathcal{A}$ is an $\mathbf{F}$-algebraic system, such that $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Then $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}}$, i.e., $\alpha_\Sigma(\Phi) \subseteq \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. Since, by hypothesis, $\phi \in C_\Sigma(\Phi)$ and $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$, whence $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$ or, equivalently, $\alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. We conclude that $\phi \in C^{\mathcal{A},\tau}_\Sigma(\Phi)$ and, hence, $\mathcal{A} \in \mathsf{K}(\mathcal{I}, \tau)$. ■

It is readily inferred from the definition that, provided $\mathcal{I}$ has a $\tau^\flat$-algebraic semantics, then the class $\mathsf{K}(\mathcal{I}, \tau)$ is the largest such.

Corollary 1684 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *has a* $\tau^\flat$*-algebraic semantics, then* $\mathsf{K}(\mathcal{I}, \tau)$ *is its largest* $\tau^\flat$*-algebraic semantics.*

Proof: Suppose K is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$ and let $\mathcal{A} \in \mathsf{K}$. Then, by the definition of $\tau^\flat$-algebraic semantics and taking into account the membership $\mathcal{A} \in \mathsf{K}$, we get $C = C^{\mathsf{K},\tau} \leq C^{\mathcal{A},\tau}$. Therefore, by definition of $\mathsf{K}(\mathcal{I}, \tau)$, $\mathcal{A} \in \mathsf{K}(\mathcal{I}, \tau)$. We conclude that $\mathsf{K} \subseteq \mathsf{K}(\mathcal{I}, \tau)$. ■

we can also show that, if $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat$, then $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$.

Proposition 1685 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is family truth equational, with witnessing transformations* $\tau^\flat$, *then* $\mathrm{AlgSys}(\mathcal{I})$ *is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$.

Proof: We must show that $C = C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$.

Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Since $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, there exists $\mathcal{T} \subseteq$

$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. Hence, we get

$$\begin{array}{rcl}
\alpha(\tau^\flat_\Sigma[\Phi]) \le \Delta^{\mathcal{A}} & \text{iff} & \alpha(\tau^\flat_\Sigma[\Phi]) \le \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \\
& \text{iff} & \alpha(\tau^\flat_\Sigma[\Phi]) \le \Omega^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}, \\
& \text{iff} & \tau^\flat_\Sigma[\Phi] \le \Omega(\alpha^{-1}(T)), \text{ for all } T \in \mathcal{T}, \\
& \text{iff} & \Phi \subseteq \alpha^{-1}_\Sigma(T), \text{ for all } T \in \mathcal{T}, \\
& \text{implies} & \phi \subseteq \alpha^{-1}_\Sigma(T), \text{ for all } T \in \mathcal{T}, \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(\alpha^{-1}(T)), \text{ for all } T \in \mathcal{T}, \\
& \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \le \Omega^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}, \\
& \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \le \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \\
& \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \le \Delta^{\mathcal{A}}.
\end{array}$$

We conclude that $\phi \in C^{\mathrm{AlgSys}(\mathcal{I}),\tau}_\Sigma(\Phi)$. Therefore, $C \le C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathrm{AlgSys}(\mathcal{I}),\tau}_\Sigma(\Phi)$ and $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Phi \subseteq T_\Sigma$. Then, we have $\tau^\flat_\Sigma[\Phi] \le \Omega(T)$, i.e., $\tau^{\mathcal{F}/\Omega(T)}_\Sigma[\Phi/\Omega_\Sigma(T)] \le \Delta^{\mathcal{F}/\Omega(T)}$. But $\mathcal{F}/\Omega(T) \in \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Therefore, since $\phi \in C^{\mathrm{AlgSys}(\mathcal{I}),\tau}_\Sigma(\Phi)$, we get that $\tau^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T)] \le \Delta^{\mathcal{F}/\Omega(T)}$, whence $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Therefore, $\phi \in T_\Sigma$ and we conclude that $\phi \in C_\Sigma(\Phi)$. This proves that $C^{\mathrm{AlgSys}(\mathcal{I}),\tau} \le C$ and, as a result, equality follows.

We have now shown that $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$. ■

Corollary 1686 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family truth equational, with witnessing transformations* $\tau^\flat$*, then* $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}(\mathcal{I},\tau)$.

Proof: By Proposition 1685 and Corollary 1684. ■

For family truth equational π-institutions we have the following characterization of the least $\mathcal{I}$-filter families on arbitrary algebraic systems and on $\mathcal{I}$-algebraic systems.

Lemma 1687 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$.

(a) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$*,* $C^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$*;*

(b) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$.

Proof:

(a) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. We have $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \le \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\varnothing)))$. By family truth equationality, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \le C^{\mathcal{I},\mathcal{A}}(\varnothing)$. It follows that $C^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$.

(b) Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then, by Part (a), $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq C^{\mathcal{I},\mathcal{A}}(\varnothing)$. Assume, conversely, that $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in C_{\Sigma}^{\mathcal{I},\mathcal{A}}(\varnothing)$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\phi \in T_{\Sigma} = \tau_{\Sigma}^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$, i.e., for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\tau_{\Sigma}^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T)$. We conclude that $\tau_{\Sigma}^{\mathcal{A}}[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\varnothing)) = \Delta^{\mathcal{A}}$. This shows that $\phi \in \tau_{\Sigma}^{\mathcal{A}}(\Delta^{\mathcal{A}})$. Thus, $C^{\mathcal{I},\mathcal{A}}(\varnothing) \leq \tau^{\mathcal{A}}(\Delta^{\mathcal{A}})$. Equality now follows. ∎

So in the case of family truth equational π-institutions, we may strengthen the characterization of the class $\mathsf{K}(\mathcal{I},\tau)$ given in Proposition 1683.

Proposition 1688 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. Then*

$$\mathsf{K}(\mathcal{I},\tau) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)\}.$$

Proof: Note that, taking into account Proposition 1683,

$$\begin{aligned}\{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)\}\\ \subseteq \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}\\ = \mathsf{K}(\mathcal{I},\tau).\end{aligned}$$

Assume, conversely, that $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. Then, by Proposition 1683, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by definition of $\mathsf{K}(\mathcal{I},\tau)$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq C^{\mathcal{I},\mathcal{A}}(\varnothing)$. Hence $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$. ∎

Proposition 1688 has some interesting consequences. First, any two sets of witnessing transformations for truth equationality are, roughly speaking, deductively equivalent over any $\mathcal{I}$-algebraic system.

Corollary 1689 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat \tau'^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family truth equational, with witnessing transformations* $\tau^\flat$ *and* $\tau'^\flat$*, then, for every* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C^{\mathcal{A}}(\tau_{\Sigma}^\flat[\phi]) = C^{\mathcal{A}}(\tau_{\Sigma}'^\flat[\phi]).$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat$ and $\tau'^\flat$ and let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha(\tau_{\Sigma}'^\flat[\phi]) \leq \Delta^{\mathcal{A}}$. This is equivalent to $\tau'^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$, i.e., $\alpha_\Sigma(\phi) \in \tau'^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. By Proposition 1688, $\alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing)$. Again by Proposition 1688, $\alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. Thus, $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$. Hence, $\alpha_\Sigma(\tau_{\Sigma}^\flat[\phi]) \leq \Delta^{\mathcal{A}}$. This shows that $\tau_{\Sigma}^\flat[\phi] \leq C^{\mathcal{A}}(\tau_{\Sigma}'^\flat[\phi])$. By symmetry, we conclude that $C^{\mathcal{A}}(\tau_{\Sigma}^\flat[\phi]) = C^{\mathcal{A}}(\tau_{\Sigma}'^\flat[\phi])$. ∎

Finally, the Suszko core $S^\flat$ has a special position among all witnessing transformations. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} =$

$\langle\mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$. Recall that the Suszko core of $\mathcal{I}$ is the collection

$$S^{\mathcal{I}}=\{\sigma^\flat\in N^\flat:(\forall T\in\mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T]\le\widetilde{\Omega}(T)\}.$$

Recall, also, that, by Lemma 836, if $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat\subseteq N^\flat$, then $\tau^\flat\subseteq S^{\mathcal{I}}$.

Corollary 1690 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. Then*

$$\mathsf{K}(\mathcal{I},S^{\mathcal{I}})\subseteq\mathsf{K}(\mathcal{I},\tau).$$

Proof: Suppose $\mathcal{A}\in\mathsf{K}(\mathcal{I},S^{\mathcal{I}})$. By hypothesis and Lemma 836, $\tau^\flat\subseteq S^{\mathcal{I}}$. Hence $S^{\mathcal{I}}(\Delta^{\mathcal{A}})\le\tau^\flat(\Delta^{\mathcal{A}})$. But, by hypothesis, Theorem 841 and Proposition 1688, $S^{\mathcal{I}}(\Delta^{\mathcal{A}})=C^{\mathcal{I},\mathcal{A}}(\varnothing)$ and, by hypothesis and Lemma 1687, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})\le C^{\mathcal{I},\mathcal{A}}(\varnothing)$. Hence, we have

$$C^{\mathcal{I},\mathcal{A}}(\varnothing)=S^{\mathcal{I}}(\Delta^{\mathcal{A}})\le\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})\le C^{\mathcal{I},\mathcal{A}}(\varnothing).$$

Therefore, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})=C^{\mathcal{I},\mathcal{A}}(\varnothing)$ and, thus, by Proposition 1688, $\mathcal{A}\in\mathsf{K}(\mathcal{I},\tau)$. We conclude that $\mathsf{K}(\mathcal{I},S^{\mathcal{I}})\subseteq\mathsf{K}(\mathcal{I},\tau)$. ■

Corollary 1691 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. Then* $\mathsf{K}(\mathcal{I},S^{\mathcal{I}})$ *is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$*.*

Proof: Observe that we have

$$\begin{array}{rcll}\mathrm{AlgSys}(\mathcal{I}) & \subseteq & \mathsf{K}(\mathcal{I},S^{\mathcal{I}}) & \text{(by Theorem 841 and Corollary 1686)}\\ & \subseteq & \mathsf{K}(\mathcal{I},\tau). & \text{(by Corollary 1690)}\end{array}$$

Since, by Proposition 1685, $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$ and, by Corollary 1684, $\mathsf{K}(\mathcal{I},\tau)$ is also a $\tau^\flat$-algebraic semantics for $\mathcal{I}$, we conclude that $\mathsf{K}(\mathcal{I},S^{\mathcal{I}})$ is one also. ■

21.16 The $\mathcal{I}$-Operator $\Psi^{\mathsf{K},\tau}$

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I})\subseteq\mathsf{K}$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$, we define the operator

$$\Psi^{\mathsf{K},\tau,\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{EqvFam}(\mathcal{A})$$

by setting, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) = \Theta^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]).$$

Note that, by the hypotheses and Proposition 28, $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ is well-defined, since $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$-congruence systems on $\mathcal{A}$ form a closure system on $\mathcal{A}^2$.

It is the case that if a class K of $\mathbf{F}$-algebraic systems is a $\tau^\flat$-algebraic semantics for a π-institution $\mathcal{I}$, then so is the larger class $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$.

Proposition 1692 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If a class* K *of* $\mathbf{F}$*-algebraic systems is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$*, then so is* $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$*.*

Proof: First, observe that $\mathsf{K} \subseteq \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$, whence $C^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\tau} \le C^{\mathsf{K},\tau} = C$. To show the converse, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Let

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$, and assume that $\alpha(\tau^\flat_\Sigma[\Phi]) \le \Delta^{\mathcal{A}}$. Since, by definition $\Delta^{\mathcal{A}} = \bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle)$, we get that $\alpha(\tau^\flat_\Sigma[\Phi]) \le \mathrm{Ker}(\langle H^i, \gamma^i \rangle)$, for all $i \in I$, i.e., $\gamma^i(\alpha(\tau^\flat_\Sigma[\Phi])) \le \Delta^{\mathcal{A}^i}$, $i \in I$, or, equivalently, $\alpha^i(\tau^\flat_\Sigma[\Phi]) \le \Delta^{\mathcal{A}^i}$, $i \in I$. Since $\phi \in C_\Sigma(\Phi)$, $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$ and K is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$, we get $\alpha^i(\tau^\flat_\Sigma[\phi]) \le \Delta^{\mathcal{A}^i}$, for all $i \in I$. We now reverse the steps above. We get $\gamma^i(\alpha(\tau^\flat_\Sigma[\phi])) \subseteq \Delta^{\mathcal{A}^i}$, $i \in I$, then $\alpha(\tau^\flat_\Sigma[\phi]) \le \mathrm{Ker}(\langle H^i, \gamma^i \rangle)$, $i \in I$, and, finally, $\alpha(\tau^\flat_\Sigma[\phi]) \le \Delta^{\mathcal{A}}$. Thus, $\phi \in C^{\mathcal{A},\tau}_\Sigma(\Phi)$. Since, for all $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$, $C \le C^{\mathcal{A},\tau}$, we conclude that $C \le C^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\tau}$. Therefore, $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$ is also a $\tau^\flat$-algebraic semantics for $\mathcal{I}$. ■

Tying the operator $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ with our preceding work in this Chapter, we show that it is a congruential monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$.

Proposition 1693 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, with a* $\tau^\flat$*-algebraic semantics* K*, such that* $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ *is a congruential, monotone, compatibility* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

Proof: $\Psi^{\mathsf{K},\tau,mcA}$ is, by definition, an $\mathcal{I}$-operator on $\mathcal{A}$. It is congruential, since, again by definition, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Theta^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) \in \mathrm{ConSys}(\mathcal{A})$. It is monotone, since, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \le T'$, we get $\tau^{\mathcal{A}}[T] \le \tau^{\mathcal{A}}[T']$ and, therefore, $\Theta^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) \le \Theta^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T'])$.

To see that it is also a compatibility $\mathcal{I}$-operator, consider $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Note, first, that

$$\begin{array}{rcll}\mathrm{AlgSys}(\mathcal{I}) & = & \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathrm{AlgSys}^*(\mathcal{I})) & \text{(by Theorem 1441)}\\ & \subseteq & \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}). & \text{(since } \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K})\end{array}$$

Thus, we get $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})}(\mathcal{A})$. Since, by Corollary 825, $\tau^{\mathcal{A}}[T] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$, we get

$$\Theta^{\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T).$$

Therefore, $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ is also a compatibility $\mathcal{I}$-operator on $\mathcal{A}$. ∎

It turns out that $\Psi^{\mathsf{K},\tau} = \{\Psi^{\mathsf{K},\tau,\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ is also semi-coherent. To show this, we formulate two technical lemmas on the way.

Lemma 1694 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$. *For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$, $\mathcal{B}$, *every surjective morphism* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$, *with* H *an isomorphism, and all* $T' \in \mathrm{SenFam}(\mathcal{B})$,

(a) $\tau^{\mathcal{B}}[T'] = \gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')])$;

(b) $\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \gamma^{-1}(\tau^{\mathcal{B}}[T'])$.

Proof: First, note that, for all $T \in \mathrm{SenFam}(\mathcal{A})$, we have, taking into account the fact that $\langle H, \gamma\rangle$ is a surjective morphism, $\gamma(\tau^{\mathcal{A}}[T]) = \tau^{\mathcal{B}}[\gamma(T)]$. Now, we set $T = \gamma^{-1}(T')$. This gives

$$\gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')]) = \tau^{\mathcal{B}}[\gamma(\gamma^{-1}(T'))] = \tau^{\mathcal{B}}[T'],$$

which conclude the proof of Part (a). For Part (b), we have, using Part (a),

$$\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \gamma^{-1}(\gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')])) = \gamma^{-1}(\tau^{\mathcal{B}}[T']).$$

∎

Lemma 1695 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *and* K *a class of* $\mathbf{F}$*-algebraic systems, such that* $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}) \subseteq \mathsf{K}$. *For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$, $\mathcal{B}$, *every surjective morphism* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$, *with* H *an isomorphism, and all* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\begin{array}{l}\{\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) : \mathrm{Ker}(\langle H, \gamma\rangle) \leq \theta \text{ and } \tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta\}\\ \quad = \{\gamma^{-1}(\theta') : \theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B}) \text{ and } \tau^{\mathcal{B}}[T'] \leq \theta'\}.\end{array}$$

Proof: Suppose K is closed under subdirect intersections and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{SenFam}(\mathcal{B})$.

(⊆) Let $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, such that $\mathrm{Ker}(\langle H,\gamma\rangle) \leq \theta$ and $\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta$. By Lemma 1694, $\tau^{\mathcal{B}}[T'] = \gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')]) \leq \gamma(\theta)$. By Proposition 33, $\gamma(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$. Finally, by Lemma 25, $\theta = \gamma^{-1}(\gamma(\theta))$. Hence, we get

$$\theta = \gamma^{-1}(\gamma(\theta)) \in \{\gamma^{-1}(\theta') : \theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B}) \text{ and } \tau^{\mathcal{B}}[T'] \leq \theta'\}.$$

(⊇) Suppose, now, $\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$, such that $\tau^{\mathcal{B}}[T'] \leq \theta'$. By Lemma 1694, $\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \gamma^{-1}(\tau^{\mathcal{B}}[T']) \leq \gamma^{-1}(\theta')$. Finally, $\mathrm{Ker}(\langle H,\gamma\rangle) = \gamma^{-1}(\Delta^{\mathcal{B}}) \leq \gamma^{-1}(\theta')$. So we get

$$\gamma^{-1}(\theta') \in \{\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) : \mathrm{Ker}(\langle H,\gamma\rangle) \leq \theta \text{ and } \tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta\}.$$ ■

Now, for the main theorem to the effect that $\Psi^{\mathsf{K},\tau}$ is a semi-coherent family of congruential monotone compatibility $\mathcal{I}$-operators.

Theorem 1696 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$. $\Psi^{\mathsf{K},\tau}$ is a semi-coherent family of congruential monotone compatibility $\mathcal{I}$-operators.*

Proof: Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\langle H,\gamma\rangle$ is $\Psi^{\mathsf{K},\tau}$-compatible with $\gamma^{-1}(T')$. Then, by definition,

$$\mathrm{Ker}(\langle H,\gamma\rangle) \leq \Psi^{\mathsf{K},\tau,\mathcal{A}}(\gamma^{-1}(T')) = \Theta^{\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[\gamma^{-1}(T')]).$$

So we have

$$\begin{array}{rcl}\Psi^{\mathsf{K},\tau,\mathcal{A}}(\gamma^{-1}(T')) & = & \Theta^{\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[\gamma^{-1}(T')]) \\ & = & \bigcap\{\theta \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})}(\mathcal{A}) : \\ & & \qquad \mathrm{Ker}(\langle H,\gamma\rangle) \leq \theta \text{ and } \tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta\} \\ & = & \bigcap\{\gamma^{-1}(\theta') : \theta' \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})}(\mathcal{B}) \text{ and } \tau^{\mathcal{B}}[T'] \leq \theta'\} \\ & = & \gamma^{-1}(\bigcap\{\theta' \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})}(\mathcal{B}) : \tau^{\mathcal{B}}[T'] \leq \theta'\}) \\ & = & \gamma^{-1}(\Theta^{\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}),\mathcal{B}}(\tau^{\mathcal{B}}[T'])) \\ & = & \gamma^{-1}(\Psi^{\mathsf{K},\tau,\mathcal{B}}(T')).\end{array}$$

This proves that $\Psi^{\mathsf{K},\tau}$ is also semi-coherent (the remaining properties having been demonstrated in Proposition 1693). ■

Since, by Proposition 1685, every family truth equational π-institution $\mathcal{I}$ has $\mathrm{AlgSys}(\mathcal{I})$ as a $\tau^\flat$-algebraic semantics and $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$, setting $\mathsf{K} := \mathrm{AlgSys}(\mathcal{I})$, we get that $\Psi^{\mathsf{K},\tau}$ is a semi-coherent family of monotone congruential compatibility $\mathcal{I}$-operators.

Our last result shows that the classes of **F**-algebraic systems associated with $\Psi^{\mathsf{K},\tau}$ (which are equal by Proposition 1595) coincide with $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$.

First, however, we show that, for any π-institution $\mathcal{I}$, with $\tau^\flat$ in $N^\flat$, the class $\mathsf{K}(\mathcal{I},\tau)$ is closed under subdirect intersections.

Lemma 1697 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}(\mathcal{I},\tau)) \subseteq \mathsf{K}(\mathcal{I},\tau).$$

Proof: Let $\mathcal{A}^i \in \mathsf{K}(\mathcal{I},\tau)$, for all $i \in I$, and

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection. Then, we have, by definition of subdirect intersection, $\bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle) = \Delta^{\mathcal{A}}$ and, by Proposition 1683, $\tau^{\mathcal{A}^i}(\Delta^{\mathcal{A}^i}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, for all $i \in I$. These give

$$\begin{array}{rcl}\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) & = & \tau^{\mathcal{A}}(\bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle))\\ & = & \bigcap_{i\in I}\tau^{\mathcal{A}}(\mathrm{Ker}(\langle H^i,\gamma^i\rangle))\\ & = & \bigcap_{i\in I}\tau^{\mathcal{A}}((\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\tau^{\mathcal{A}^i}(\Delta^{\mathcal{A}^i}))\\ & \in & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}),\end{array}$$

where membership follows from the fact that $\tau^{\mathcal{A}^i}(\Delta^{\mathcal{A}^i}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, for all $i \in I$, by Corollary 55 and by closure of $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ under intersections. We conclude, using again Proposition 1683, that $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. ■

Recall the classes of **F**-algebraic systems

$$\begin{array}{rcl}\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) & = & \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) = \Delta^{\mathcal{A}})\};\\ \mathrm{AlgSys}^{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) & = & \{\mathcal{A}/\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.\end{array}$$

Proposition 1698 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, with a* $\tau^\flat$*-algebraic semantics* K*, such that* $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$*. Then*

$$\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \mathrm{AlgSys}^{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{P}}(\mathsf{K}).$$

Proof: First, by Proposition 1595, $\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \mathrm{AlgSys}^{\Psi^{\mathsf{K},\tau}}(\mathcal{I})$. So it suffices to show that $\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$.

Suppose, first, that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. By Corollary 1684, $\mathsf{K} \subseteq \mathsf{K}(\mathcal{I},\tau)$. By Lemma 1697, $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \overset{\triangleleft}{\mathbb{P}}(\mathsf{K}(\mathcal{I},\tau)) \subseteq \mathsf{K}(\mathcal{I},\tau)$, whence $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. Thus, by

Proposition 1683, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, $\Delta^{\mathcal{A}} \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})}(\mathcal{A})$. We now get

$$\Psi^{\mathsf{K},\tau,\mathcal{A}}(\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})) = \Theta^{\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})]) \leq \Theta^{\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}),\mathcal{A}}(\Delta^{\mathcal{A}}) = \Delta^{\mathcal{A}}.$$

We conclude that $\mathcal{A} \in \mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I})$.

Suppose, conversely, that $\mathcal{A} \in \mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) = \Delta^{\mathcal{A}}$, that is, $\Theta^{\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) = \Delta^{\mathcal{A}}$. This shows that $\Delta^{\mathcal{A}}$ is an $\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$-congruence system on $\mathcal{A}$. Hence $\mathcal{A} \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$. Then we have

$$\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K} \subseteq \mathsf{K}(\mathcal{I}, \tau).$$

Assume, now, that $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat$. By Proposition 1685, $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$ and, by Proposition 65, $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Thus, in the case of truth equationality $\mathcal{I}$ has a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K} \subseteq \mathsf{K}(\mathcal{I}, \tau)$.

- If $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I})$, then, by Proposition 1674 and Theorem 1441, we would have $\Psi^{\mathsf{K},\tau} = \widetilde{\Omega}^{\mathcal{I}}$;
- At the other extreme, if $\mathsf{K} = \mathsf{K}(\mathcal{I}, \tau)$, then, we get, by Proposition 1698 and Lemma 1697, a semi-coherent family of congruential monotone compatibility $\mathcal{I}$-operators $\Psi^{\mathsf{K},\tau}$, such that, similarly, $\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \mathsf{K}(\mathcal{I}, \tau)$.