

Chapter 22

The Strong Version of a π -Institution

22.1 The Strong Version of a π -Institution

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . We define the following classes of \mathcal{I} -matrix families.

$$\begin{aligned} \mathbf{M}^{\mathcal{I}^*} &= \{ \langle \mathcal{A}, T \rangle : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \}; \\ \mathbf{M}^{\mathcal{I}, \text{Su}} &= \{ \langle \mathcal{A}, T \rangle : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A}) \}; \\ \mathbf{M}^{\mathcal{I}, m} &= \{ \langle \mathcal{A}, T \rangle : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \}. \end{aligned}$$

We show that all three classes of \mathcal{I} -matrix families generate the same closure system on \mathbf{F} .

Proposition 1699 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Then $\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}^{\mathcal{I}, m}}$.*

Proof: By Lemma 1605, we have that, for all $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$, $\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Thus, $\mathbf{M}^{\mathcal{I}, m} \subseteq \mathbf{M}^{\mathcal{I}^*}$. This implies that $\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} \leq \mathcal{I}^{\mathbf{M}^{\mathcal{I}, m}}$. To show the converse, assume that $\langle \mathcal{A}, T \rangle \in \mathbf{M}^{\mathcal{I}^*}$ and consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\Omega^{\mathcal{A}}(T)$. By Corollary 1591, $\pi(T^*)$ is the least \mathcal{I} -filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T)$. By hypothesis $T = T^*$, whence $\pi(T) = \pi(T^*)$ and, hence, since $\langle I, \pi \rangle : \langle \mathcal{A}, T \rangle \rightarrow \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T) \rangle$ is a strict surjective morphism, we get that

$$\mathcal{I}^{\langle \mathcal{A}, T \rangle} = \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T) \rangle} = \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*) \rangle}$$

and $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*) \rangle \in \mathbf{M}^{\mathcal{I}, m}$. Putting things together, we finally obtain

$$\begin{aligned} \mathcal{I}^{\mathbf{M}^{\mathcal{I}, m}} &\leq \bigcap \{ \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*) \rangle} : T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \} \\ &= \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, T \rangle} : T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \} \\ &= \mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}}. \end{aligned}$$

Therefore, $\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}^{\mathcal{I}, m}}$. ■

Proposition 1699 enables us to show that $\mathbf{M}^{\mathcal{I}^*}$ and $\mathbf{M}^{\mathcal{I}, \text{Su}}$ also generate the same closure system on \mathbf{F} .

Corollary 1700 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Then $\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}^{\mathcal{I}, \text{Su}}}$.*

Proof: By Lemma 1620, $\mathbf{M}^{\mathcal{I}, \text{Su}} \subseteq \mathbf{M}^{\mathcal{I}^*}$. Also by Lemma 1620, $\mathbf{M}^{\mathcal{I}, m} \subseteq \mathbf{M}^{\mathcal{I}, \text{Su}}$. So we get $\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} \leq \mathcal{I}^{\mathbf{M}^{\mathcal{I}, \text{Su}}} \leq \mathcal{I}^{\mathbf{M}^{\mathcal{I}, m}}$. Therefore, by Proposition 1699, $\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}^{\mathcal{I}, \text{Su}}}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Taking into account Proposition 1699 and Corollary 1700, we define the **strong version of \mathcal{I}** , denoted by $\mathcal{I}^+ = \langle \mathbf{F}, C^+ \rangle$, by

$$\mathcal{I}^+ := \mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}^{\mathcal{I}, \text{Su}}} = \mathcal{I}^{\mathbf{M}^{\mathcal{I}, m}}.$$

There are even more ways to characterize the π -institution \mathcal{I}^+ . Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Given a class \mathbf{K} of \mathbf{F} -algebraic systems, we define

$$\begin{aligned} \mathbf{M}_{\mathbf{K}}^{\mathcal{I},m} &= \{ \langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathbf{K}, T = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \}; \\ \mathbf{M}_{\mathbf{K}}^{\mathcal{I}^*} &= \{ \langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathbf{K}, T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \}; \\ \mathbf{M}_{\mathbf{K}}^{\mathcal{I},\text{Su}} &= \{ \langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathbf{K}, T \in \text{FiFam}^{\mathcal{I},\text{Su}}(\mathcal{A}) \}. \end{aligned}$$

Proposition 1701 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π -institution based on \mathbf{F} and $\mathbf{K} = \text{AlgSys}^*(\mathcal{I})$ or $\mathbf{K} = \text{AlgSys}(\mathcal{I})$. Then*

$$\mathcal{I}^+ = \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},m}} = \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},\text{Su}}}.$$

Proof: By definition and Lemma 1620, we have

$$\mathbf{M}_{\mathbf{K}}^{\mathcal{I},m} \subseteq \mathbf{M}_{\mathbf{K}}^{\mathcal{I},\text{Su}} \subseteq \mathbf{M}_{\mathbf{K}}^{\mathcal{I}^*} \subseteq \mathbf{M}^{\mathcal{I}^*}.$$

Therefore, we get

$$\mathcal{I}^+ \leq \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I}^*}} \leq \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},\text{Su}}} \leq \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},m}}.$$

For the converse, suppose $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$ and $T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. By Proposition 1609, $T/\Omega^{\mathcal{A}}(T)$ is the least \mathcal{I} -filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \text{AlgSys}^*(\mathcal{I}) \subseteq \text{AlgSys}(\mathcal{I})$. Therefore, we get

$$\begin{aligned} \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},m}} &\leq \bigcap \{ \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T) \rangle} : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \} \\ &= \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, T \rangle} : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \} \\ &= \mathcal{I}^+. \end{aligned}$$

We conclude that $\mathcal{I}^+ = \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},m}} = \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I}^*}} = \mathcal{I}^{\mathbf{M}_{\mathbf{K}}^{\mathcal{I},\text{Su}}}$. ■

The following proposition lists some of the properties of the strong version \mathcal{I}^+ of a π -institution \mathcal{I} .

Proposition 1702 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} .*

- (a) $\mathcal{I} \leq \mathcal{I}^+$;
- (b) $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{A} ;
- (c) $\text{FiFam}^{\mathcal{I},\text{Su}}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{A} ;
- (d) If \mathcal{I} is family reflective, then $\mathcal{I}^+ = \mathcal{I}$.

Proof:

- (a) Since $\mathbf{M}^{\mathcal{I},m} \subseteq \text{MatFam}(\mathcal{I})$, we get $\mathcal{I} = \mathcal{I}^{\text{MatFam}(\mathcal{I})} \leq \mathcal{I}^{\mathbf{M}^{\mathcal{I},m}} = \mathcal{I}^+$.

- (b) Since, by Part (a), $\mathcal{I} \leq \mathcal{I}^+$, we get that $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$.
- (c) By definition of \mathcal{I}^+ , we have, for all $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$, all $T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$ and all $T' \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$, $C^+ \leq C^{\langle \mathcal{A}, T \rangle}$ and $C^+ \leq C^{\langle \mathcal{A}, T' \rangle}$. Moreover, by Lemma 1620, every Suszko filter family is a Leibniz filter family. We conclude that $\text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.
- (d) By the hypothesis and Proposition 1610, $\text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{A} . Therefore, $\mathcal{I}^+ = \mathcal{I}^{\text{M}^{\mathcal{I}^*}} = \mathcal{I}^{\text{MatFam}(\mathcal{I})} = \mathcal{I}$. ■

It turns out that the strong version \mathcal{I}^+ is mostly interesting when \mathcal{I} itself has theorems. In the absence of theorems \mathcal{I}^+ has only trivial theory families.

Proposition 1703 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . If \mathcal{I} does not have theorems, then \mathcal{I} is almost inconsistent.*

Proof: Assume that \mathcal{I} does not have theorems. Then, for every \mathbf{F} -algebraic system \mathcal{A} , $\emptyset \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, by definition $\mathcal{I}^+ = \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, \emptyset \rangle} : \mathcal{A} \in \text{AlgSys}(\mathbf{F}) \}$. This implies that, for all $\Sigma \in |\mathbf{Sign}^b|$ and all $\phi \in \text{SEN}^b(\Sigma)$, we have, vacuously, for all $\psi \in \text{SEN}^b(\Sigma)$, $\psi \in C_{\Sigma}^+(\phi)$. Therefore, the only Σ -theory families of \mathcal{I}^+ are \emptyset and $\text{SEN}^b(\Sigma)$. So \mathcal{I}^+ is almost inconsistent. ■

The least \mathcal{I} -filter family on every algebraic system \mathcal{A} coincides with the least \mathcal{I}^+ -filter family. As a consequence \mathcal{I} and \mathcal{I}^+ share the same theorems.

Lemma 1704 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} ,*

$$\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

In particular, $\text{ThFam}(\mathcal{I}^+) = \text{ThFam}(\mathcal{I})$.

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system. By Proposition 1702, $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, we have $\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \subseteq \bigcap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. On the other hand, by Lemma 1605, $\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, whence, by Proposition 1702, $\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, $\bigcap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Equality now follows. ■

Lemma 1704 implies the idempotency of the strong version operator on π -institutions.

Corollary 1705 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . Then $(\mathcal{I}^+)^+ = \mathcal{I}^+$.*

Proof: We have

$$\begin{aligned} (\mathcal{I}^+)^+ &= \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, T \rangle} : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T = \bigcap \text{FiFam}^{\mathcal{I}^+}(\mathcal{I}) \} \\ &= \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, T \rangle} : \mathcal{A} \in \text{AlgSys}(\mathbf{F}), T = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{I}) \} \\ &= \mathcal{I}^+. \end{aligned}$$

The first and last equalities follow by the definition of $^+$, and the main equality is due to Lemma 1704. \blacksquare

The next proposition provides sufficient conditions for recognizing that a given π -institution is the strong version of another π -institution based on the same algebraic system.

Proposition 1706 *Let $\mathbf{F} = \langle \text{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ π -institutions based on \mathbf{F} , such that*

1. \mathcal{I}' is family reflective;
2. $\text{AlgSys}(\mathcal{I}') = \text{AlgSys}(\mathcal{I})$;
3. For all $\mathcal{A} \in \text{AlgSys}(\mathcal{I}')$, $\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \text{FiFam}^{\mathcal{I}'}(\mathcal{A})$.

Then $\mathcal{I}' = \mathcal{I}^+$.

Proof: We have

$$\begin{aligned} \mathcal{I}' &= \mathcal{I}'^+ \quad (\text{by 1 and Proposition 1702}) \\ &= \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, T \rangle} : \mathcal{A} \in \text{AlgSys}(\mathcal{I}'), T = \bigcap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}) \} \\ &\quad (\text{by Proposition 1701}) \\ &= \bigcap \{ \mathcal{I}^{\langle \mathcal{A}, T \rangle} : \mathcal{A} \in \text{AlgSys}(\mathcal{I}), T = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \} \\ &\quad (\text{by 2 and 3}) \\ &= \mathcal{I}^+. \quad (\text{by Proposition 1701}) \end{aligned}$$

This proves the claim. \blacksquare

We now show that Suszko and Leibniz \mathcal{I} -filter families form subclasses, respectively, of the classes of Suszko and Leibniz \mathcal{I}^+ -filter families on every \mathbf{F} -algebraic system.

Proposition 1707 *Let $\mathbf{F} = \langle \text{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} ,*

$$\text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A}) \quad \text{and} \quad \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

Proof: By Proposition 1702, $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FFam}^{\mathcal{I}}(\mathcal{A})$. Thus, for all $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, $\llbracket T \rrbracket^{\mathcal{I}^+} \subseteq \llbracket T \rrbracket^{\mathcal{I}^*}$ and $\llbracket T \rrbracket^{\mathcal{I}^+, \text{Su}} \subseteq \llbracket T \rrbracket^{\mathcal{I}, \text{Su}}$.

Suppose that $T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. Then, by Proposition 1702, $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and, moreover, $T = \bigcap \llbracket T \rrbracket^{\mathcal{I}, \text{Su}} \leq \bigcap \llbracket T \rrbracket^{\mathcal{I}^+, \text{Su}}$. Thus, since $T \in \llbracket T \rrbracket^{\mathcal{I}^+, \text{Su}}$, we get that $T = \bigcap \llbracket T \rrbracket^{\mathcal{I}^+, \text{Su}} \in \text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A})$.

The second inclusion may be shown similarly. \blacksquare

But the Leibniz counterpart of an \mathcal{I}^+ -filter family is identical whether it be considered with respect to \mathcal{I} or with respect to \mathcal{I}^+ .

Lemma 1708 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} , and all $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, $T^{\mathcal{I}^*} = T^{\mathcal{I}^{+*}}$.*

Proof: By Proposition 1702, $[[T]]^{\mathcal{I}^{+*}} \subseteq [[T]]^{\mathcal{I}^*}$. Therefore, $T^{\mathcal{I}^*} \leq T^{\mathcal{I}^{+*}}$. On the other hand,

$$\begin{aligned} T^{\mathcal{I}^*} &\in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \quad (\text{by Proposition 1607}) \\ &\subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad (\text{by Proposition 1707}) \end{aligned}$$

and, since $T^{\mathcal{I}^*} \in [[T]]^{\mathcal{I}^*}$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\mathcal{I}^*})$. Thus, $T^{\mathcal{I}^*} \subseteq [[T]]^{\mathcal{I}^{+*}}$, which gives $T^{\mathcal{I}^{+*}} \leq T^{\mathcal{I}^*}$. We conclude that $T^{\mathcal{I}^*} = T^{\mathcal{I}^{+*}}$. ■

And this implies that the Leibniz \mathcal{I} -filter families and the Leibniz \mathcal{I}^+ -filter families coincide on every \mathbf{F} -algebraic system.

Corollary 1709 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} ,*

$$\text{FiFam}^{\mathcal{I}^{+*}}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}).$$

Proof: The right-to-left inclusion was shown in Proposition 1707. For the reverse, assume that $T \in \text{FiFam}^{\mathcal{I}^{+*}}(\mathcal{A})$. Then, by Proposition 1702, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by Lemma 1708, $T = T^{\mathcal{I}^{+*}} = T^{\mathcal{I}^*}$. Therefore, $T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. ■

22.2 Leibniz and Suszko \mathcal{I}^+ -Filter Families

There is a relation between the \mathcal{I}^+ -filter families on algebraic systems and the Leibniz and Suszko \mathcal{I} -filter families on the same algebraic systems. The following proposition shows how these relations interplay with family c-reflectivity.

Proposition 1710 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} .*

- (a) *If, for all \mathbf{F} -algebraic systems \mathcal{A} , $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$, then \mathcal{I}^+ is family c-reflective.*
- (b) *If \mathcal{I}^+ is family c-reflective, then $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, for all \mathbf{F} -algebraic systems \mathcal{A} .*

Proof:

- (a) Suppose, for all \mathbf{F} -algebraic systems \mathcal{A} , $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. Let \mathcal{A} be an \mathbf{F} -algebraic system. By Proposition 1707, $\text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A})$. Hence, by hypothesis, $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A})$. Thus, $\text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By Theorem 1627, \mathcal{I}^+ is family c-reflective.
- (b) Suppose \mathcal{I}^+ is family c-reflective and let \mathcal{A} be an \mathbf{F} -algebraic system. By Theorem 1627, $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A})$. Since, by Lemma 1620 and Corollary 1709, $\text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, we get that $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. The reverse inclusion holds by Proposition 1702. ■

A necessary and sufficient condition for the \mathcal{I}^+ -filter families to coincide with the Leibniz \mathcal{I} -filter families is the universal reflectivity of the Leibniz operator on \mathcal{I}^+ -filter families.

Proposition 1711 *Let $\mathbf{F} = \langle \text{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} ,*

$$\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$$

if and only if, for every \mathbf{F} -algebraic system \mathcal{A} , $\Omega^{\mathcal{A}}$ is order reflecting on $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

Proof: By Corollary 1709, for every \mathbf{F} -algebraic system \mathcal{A} , $\text{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. By Proposition 1610, $\Omega^{\mathcal{A}}$ is reflective on $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for all \mathcal{A} , if and only if $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^+*}(\mathcal{A})$, for all \mathcal{A} . Thus, we get that $\Omega^{\mathcal{A}}$ is reflective on $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for all \mathcal{A} , if and only if $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, for all \mathcal{A} . ■

Under the stipulation that the strong version of \mathcal{I} be protoalgebraic, the identification of \mathcal{I}^+ -filter families with the Leibniz \mathcal{I} -families have several characterizations.

Proposition 1712 *Let $\mathbf{F} = \langle \text{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , such that \mathcal{I}^+ is protoalgebraic. The following conditions are equivalent:*

- (i) $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{A} ;
- (ii) $\text{ThFam}(\mathcal{I}^+) = \text{ThFam}^*(\mathcal{I})$;
- (iii) \mathcal{I}^+ is weakly family algebraizable;
- (iv) \mathcal{I}^+ is family c-reflective;

Proof:

- (i) \Rightarrow (ii) Trivial.
- (ii) \Rightarrow (iii) Suppose that $\text{ThFam}(\mathcal{I}^+) = \text{ThFam}^*(\mathcal{I})$. By Proposition 1565, Ω is injective on $\text{ThFam}^*(\mathcal{I})$. By definition it is onto $\text{FiFam}^{\mathcal{I}^*}(\mathcal{F})$. Thus, by hypothesis and Corollary 1709, $\Omega : \text{FiFam}(\mathcal{I}^+) \rightarrow \text{ConSys}^{\mathcal{I}^*}(\mathcal{F})$ is a bijection. By hypothesis it is monotone and, by Proposition 1565, it is order reflecting. Therefore, it is an order isomorphism. By Theorem 296, \mathcal{I}^+ is weakly family algebraizable.
- (iii) \Rightarrow (iv) Every weakly family algebraizable π -institution is a fortiori family c-reflective.
- (iv) \Rightarrow (i) By hypothesis, \mathcal{I}^+ is protoalgebraic, whence, by Proposition 1638 and Corollary 1709,

$$\text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}).$$

By hypothesis and Theorem 1627, $\text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, we get that $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. ■

We close the section by looking at various consequences of the condition imposed on a π -institution \mathcal{I} that $\Omega^{\mathcal{A}}$ be an order isomorphism from the Leibniz \mathcal{I} -filter families of \mathcal{A} onto the \mathcal{I}^* -congruence systems on \mathcal{A} , for every \mathcal{I} -algebraic system \mathcal{A} . First, we show that this condition ensures that \mathcal{I} -algebraic systems, \mathcal{I}^* -algebraic systems, \mathcal{I}^+ -algebraic systems and $(\mathcal{I}^+)^*$ -algebraic systems all coincide.

Lemma 1713 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathcal{I} , such that, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{AlgSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order isomorphism. Then

$$\text{AlgSys}(\mathcal{I}^+) = \text{AlgSys}^*(\mathcal{I}^+) = \text{AlgSys}^*(\mathcal{I}) = \text{AlgSys}(\mathcal{I}).$$

Proof: We show, first, that $\text{AlgSys}^*(\mathcal{I}^+) = \text{AlgSys}^*(\mathcal{I})$. The left-to-right inclusion holds because, by Proposition 1702, $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{I} . Assume, conversely, that $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$. Then $\Delta^{\mathcal{A}} \in \text{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. By hypothesis, then, there exists $T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. By Proposition 1702 again, $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Hence, $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I}^+)$.

Now we have

$$\begin{aligned} \text{AlgSys}(\mathcal{I}) &= \text{AlgSys}^*(\mathcal{I}) && \text{(by Lemma 1660)} \\ &= \text{AlgSys}^*(\mathcal{I}^+) && \text{(shown above)} \\ &\subseteq \text{AlgSys}(\mathcal{I}^+) && \text{(by Proposition 65)} \\ &\subseteq \text{AlgSys}(\mathcal{I}). && \text{(by Proposition 1702).} \end{aligned}$$

We conclude that all four classes of algebraic system coincide. \blacksquare

Next we show that, under the same hypothesis the Leibniz congruence systems of a filter family and its Leibniz counterpart coincide and that the Suszko congruence system of a filter family coincides with the Leibniz congruence system of its Suszko counterpart.

Proposition 1714 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathcal{I} , such that, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{AlgSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order isomorphism. Then, for every \mathbf{F} -algebraic system and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*) \quad \text{and} \quad \tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathcal{I}, \text{Su}}).$$

Proof: By Proposition 1659, for all $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism.

Let $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$ and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\Omega^{\mathcal{A}}(T) \in \text{ConSys}^{\mathcal{I}^*}(\mathcal{A})$, there exists $T' \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Hence, $\llbracket T \rrbracket^* = \llbracket T' \rrbracket^*$, which gives $T^* = T'^* = T'$. Thus, we get $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T^*)$.

By hypothesis and Lemma 1660, $\text{AlgSys}^*(\mathcal{I}) = \text{AlgSys}(\mathcal{I})$. Since we have $\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T) \in \text{ConSys}^{\mathcal{I}}(\mathcal{A})$, there exists $T'' \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T'') = \tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)$. Thus, we get $\llbracket T \rrbracket^{\text{Su}} = \llbracket T'' \rrbracket^*$ and, therefore, $T^{\mathcal{I}, \text{Su}} = T''^* = T''$. This gives $\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T^{\mathcal{I}, \text{Su}})$. \blacksquare

Under the same hypothesis, it turns out that the coincidence of the class of Leibniz filter families with Suszko filter families on every algebraic system characterizes protoalgebraicity.

Corollary 1715 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathcal{I} , such that, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{AlgSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order isomorphism. \mathcal{I} is protoalgebraic if and only if, for every \mathbf{F} -algebraic system \mathcal{A} , $\text{FiFam}^{\mathcal{I}^}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$.*

Proof: If \mathcal{I} is protoalgebraic, then, by Proposition 1638, Leibniz and Suszko classes coincide and, therefore, $\text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$, for all $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$.

Suppose, conversely, that, for all \mathbf{F} -algebraic systems \mathcal{A} , $\text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. Let $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$ and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1620, $T^{\mathcal{I}, \text{Su}} \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. By the hypothesis and Lemma 1623, $T^{\mathcal{I}, \text{Su}}$ is the largest Leibniz \mathcal{I} -filter family included in T . Since, by Lemma

1620, $T^{\mathcal{I}, \text{Su}} \leq T^* \leq T$ and, by Proposition 1607, T^* is a Leibniz \mathcal{I} -filter family, we get $T^{\mathcal{I}, \text{Su}} = T^*$. Therefore, using Proposition 1607, we get

$$\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathcal{I}, \text{Su}}) = \Omega^{\mathcal{A}}(T^*) = \Omega^{\mathcal{A}}(T).$$

Thus, on every \mathbf{F} -algebraic system \mathcal{A} , the Suszko and the Leibniz operators coincide and, therefore, by Lemma 1555, \mathcal{I} is protoalgebraic. ■

We already have the tools to show that the property that $\Omega^{\mathcal{A}}$ be an isomorphism between Leibniz filter families and reduced algebraic systems is bequeathed by a π -institution \mathcal{I} to its strong version \mathcal{I}^+ .

Lemma 1716 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathcal{I} , such that, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{AlgSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order isomorphism. Then, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^{+}}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^{+*}}(\mathcal{A})$ is also an order isomorphism.*

Proof: By Corollary 1709, we have $\text{FiFam}^{\mathcal{I}^{+*}}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. By Lemma 1713, $\text{AlgSys}^*(\mathcal{I}) = \text{AlgSys}^*(\mathcal{I}^+)$. Now, taking into account the hypothesis, we get the conclusion. ■

In a proposition analogous to Proposition 1712, we provide under our working hypothesis, of the Leibniz operator being an order isomorphism, a characterization of the property of \mathcal{I}^+ being weakly family algebraizable.

Proposition 1717 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathcal{I} , such that, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{AlgSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order isomorphism. The following conditions are equivalent:

- (i) $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{A} ;
- (ii) $\text{ThFam}(\mathcal{I}^+) = \text{ThFam}^*(\mathcal{I})$;
- (iii) \mathcal{I}^+ is weakly family algebraizable;
- (iv) \mathcal{I}^+ is family c -reflective;
- (v) Ω is injective on the collection of reduced \mathcal{I}^+ -filter families.

Proof:

(i) \Rightarrow (ii) Trivial.

- (ii) \Rightarrow (iii) By hypothesis and Lemma 1713, $\Omega : \text{ThFam}(\mathcal{I}^+) \rightarrow \text{ConSys}^{\mathcal{I}^*}(\mathcal{F})$ is an order isomorphism. Thus Ω is both monotone and family c-reflective, whence \mathcal{I}^+ is weakly family algebraizable.
- (iii) \Rightarrow (iv) Weak family algebraizability implies family c-reflectivity.
- (iv) \Rightarrow (v) If \mathcal{I}^+ is family c-reflective, then it is a fortiori injective. Therefore, by Theorem 214, $\Omega^{\mathcal{A}}$ is injective on the \mathcal{I} -filter families of every \mathbf{F} -algebraic system \mathcal{A} .
- (v) \Rightarrow (i) Suppose (v) holds and let $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$. By Proposition 1702, we have $\text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. So it suffices to prove the reverse inclusion. To this end, suppose $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

$\text{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$, the last inclusion, since, by Proposition 1562, $T^* \in \llbracket T \rrbracket^{\mathcal{I}^*}$. Hence, by Corollary 56,

$$\pi(T), \pi(T^*) \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$$

and, by compatibility, $\pi^{-1}(\pi(T)) = T$ and $\pi^{-1}(\pi(T^*)) = T^*$. By Corollary 1591, $\pi(T^*) = \pi(T)^*$. Now we get

$$\begin{aligned} \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} &= \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)) \quad (\text{by Lemma 1594}) \\ &= \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)^*) \quad (\text{by Proposition 1714}) \\ &= \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T^*)). \end{aligned}$$

This, both $\pi(T)$ and $\pi(T^*)$ are reduced \mathcal{I}^+ -filter families and, therefore, by the injectivity hypothesis, $\pi(T) = \pi(T^*)$. Now we conclude that $T = \pi^{-1}(\pi(T)) = \pi^{-1}(\pi(T^*)) = T^*$. This proves that, for all \mathcal{A} , $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Equality now follows. \blacksquare

22.3 Full \mathcal{I}^+ -Structures

We now explore the relation between full \mathcal{I} -structures and full \mathcal{I}^+ -structures.

Proposition 1718 *Let $\mathbf{F} = \langle \text{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} and \mathcal{A} an \mathbf{F} -algebraic system. $\langle \mathcal{A}, \mathcal{D} \rangle \in \text{FStr}^{\mathcal{I}^+}(\mathcal{A})$ if and only if, there exists $\mathcal{T} \subseteq \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle \mathcal{A}, \mathcal{T} \rangle \in \text{FStr}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{D} = \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, i.e.,*

$$\text{FStr}(\mathcal{I}^+) = \{ \langle \mathcal{A}, \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rangle : \langle \mathcal{A}, \mathcal{T} \rangle \in \text{FStr}(\mathcal{I}) \}.$$

Proof:

(\Rightarrow) Suppose that $\langle \mathcal{A}, \mathcal{D} \rangle \in \text{FStr}(\mathcal{I}^+)$. Set

$$\mathcal{T} = \{T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)\}.$$

If $T \in \mathcal{D}$, then $\tilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$ and $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}}(T)$. Thus, $T \in \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. On the other hand, let $T \in \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Then $\tilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$ and, since $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\langle \mathcal{A}, \mathcal{D} \rangle \in \text{FStr}(\mathcal{I}^+)$, we must have, by Theorem 1432, $T \in \mathcal{D}$. We conclude that $\mathcal{D} = \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Thus, it only remains to show that $\langle \mathcal{A}, \mathcal{T} \rangle \in \text{FStr}(\mathcal{I})$.

To this end, let $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)$. Then, we get $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \bigcap_{T' \in \mathcal{D}} \Omega^{\mathcal{A}}(T') = \tilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$. Thus, by definition, $T \in \mathcal{T}$. We conclude, using Theorem 1432, that $\langle \mathcal{A}, \mathcal{T} \rangle \in \text{FStr}(\mathcal{I})$.

(\Leftarrow) Suppose, now, that $\langle \mathcal{A}, \mathcal{T} \rangle \in \text{FStr}(\mathcal{I})$ and $\mathcal{D} = \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Since, by Proposition 1600, the least element of a full \mathcal{I} -structure is a Leibniz \mathcal{I} -filter family, we get that $\bigcap \mathcal{T} \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. To see that $\langle \mathcal{A}, \mathcal{D} \rangle$ is a dull \mathcal{I}^+ -structure, let $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, such that $\tilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$. Then, we infer

$$\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \tilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T).$$

Since $\langle \mathcal{A}, \mathcal{T} \rangle \in \text{FStr}(\mathcal{I})$, then, by Theorem 1432, $T \in \mathcal{T}$. Since, in addition, by hypothesis, $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, we get $T \in \mathcal{D}$. Thus, again by Theorem 1432, $\langle \mathcal{A}, \mathcal{D} \rangle \in \text{FStr}(\mathcal{I}^+)$. ■

Next, we show that the association

$$\langle \mathcal{A}, \mathcal{T} \rangle \mapsto \langle \mathcal{A}, \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rangle$$

of full \mathcal{I}^+ -structures to full \mathcal{I} -structures, given in Proposition 1718, is one-to-one, provided that \mathcal{I} - and \mathcal{I}^+ -algebraic systems coincide.

Proposition 1719 *Let $\mathbf{F} = \langle \text{Sign}^{\flat}, \text{SEN}^{\flat}, N^{\flat} \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , such that $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}(\mathcal{I}^+)$, and \mathcal{A} an \mathbf{F} -algebraic system. For all $\langle \mathcal{A}, \mathcal{T} \rangle, \langle \mathcal{A}, \mathcal{T}' \rangle \in \text{FStr}(\mathcal{I})$,*

$$\mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathcal{T}' \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \text{implies} \quad \mathcal{T} = \mathcal{T}'.$$

Proof: We start with some preparatory remarks. Suppose \mathcal{A} is an \mathbf{F} -algebraic system. Since, by hypothesis, $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}(\mathcal{I}^+)$, we get that $\text{ConSys}^{\mathcal{I}}(\mathcal{A}) = \text{ConSys}^{\mathcal{I}^+}(\mathcal{A})$. Now, using Theorem 1445 (or, alternatively, Corollary 1602), we have that $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A}) \cong \text{FStr}^{\mathcal{I}^+}(\mathcal{A})$, through

$$\mathcal{T} \mapsto \bar{\mathcal{T}} = \{T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)\}.$$

This is obtained, by applying Theorem 1445 to get an isomorphism

$$\begin{aligned} \gamma : \text{FiFam}^{\mathcal{I}}(\mathcal{A}) &\rightarrow \text{ConSys}^{\mathcal{I}}(\mathcal{A}); \\ \mathcal{T} &\xrightarrow{\gamma} \tilde{\Omega}^{\mathcal{A}}(\mathcal{T}), \end{aligned}$$

then, applying Theorem 1445 to get an isomorphism

$$\begin{aligned} \delta : \text{ConSys}^{\mathcal{I}^+}(\mathcal{A}) &\rightarrow \text{FStr}^{\mathcal{I}^+}(\mathcal{A}); \\ \theta &\xrightarrow{\delta} \{T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \theta \leq \Omega^{\mathcal{A}}(T)\} \end{aligned}$$

and, finally, composing these two, taking into account the hypothesis.

Now let $\mathcal{T}, \mathcal{T}' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle \mathcal{A}, \mathcal{T} \rangle, \langle \mathcal{A}, \mathcal{T}' \rangle \in \text{FStr}^{\mathcal{I}}(\mathcal{A})$, and suppose that $\mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathcal{T}' \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

Claim 1: $\overline{\mathcal{T}} = \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\overline{\mathcal{T}'} = \mathcal{T}' \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

We show the first equality. The second one is shown in exactly the same way. First, if $T \in \overline{\mathcal{T}}$, then $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)$. Since $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full \mathcal{I} -structure, by Theorem 1432, $T \in \mathcal{T}$. Thus, $T \in \mathcal{Y} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. If, on the other hand, $T \in \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, then $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $T \in \mathcal{T}$. Thus, $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)$. Therefore, $T \in \overline{\mathcal{T}}$.

Claim 2: $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}})$ and $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}') = \tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}'})$.

Again, it suffices to show the first equality, since the second is proven in exactly the same way. By Claim 1 and Proposition 1718, $\langle \mathcal{A}, \overline{\mathcal{T}} \rangle \in \text{FStr}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, by Theorem 1432, $\overline{\mathcal{T}} = \{T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}) \leq \Omega^{\mathcal{A}}(T)\}$. Thus, we get $\delta(\tilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \delta(\gamma(\mathcal{T})) = \overline{\mathcal{T}} = \delta(\tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}))$. Since δ is an isomorphism, we get that $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}})$.

To finish the proof, we get $\tilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}) = \tilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}'}) = \tilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$. Therefore, by Theorem 1445, $\mathcal{T} = \mathcal{T}'$. ■

Now we can formulate an order isomorphism between full \mathcal{I} - and full \mathcal{I}^+ -structures, subject to the condition that \mathcal{I} - and \mathcal{I}^+ -algebraic systems coincide.

Corollary 1720 *Let $\mathbf{F} = \langle \text{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , such that $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}(\mathcal{I}^+)$, and \mathcal{A} an \mathbf{F} -algebraic system.*

$$\begin{aligned} h : \text{FStr}^{\mathcal{I}}(\mathcal{A}) &\rightarrow \text{FStr}^{\mathcal{I}^+}(\mathcal{A}); \\ \langle \mathcal{A}, \mathcal{T} \rangle &\xrightarrow{h} \langle \mathcal{A}, \mathcal{T} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rangle \end{aligned}$$

is an order isomorphism.

Proof: By Propositions 1718 and 1719. ■

We turn next to relationships between full classes of filter families with respect to a π -institution \mathcal{I} and its strong version \mathcal{I}^+ . Recall that, given any

$\mathcal{A} \in \text{AlgSys}(\mathbf{F})$, we have $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. So we get immediately the following inclusions, for all $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

$$\begin{aligned} \llbracket T \rrbracket^{\mathcal{I}^+*} &= \{T' \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &\subseteq \{T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &= \llbracket T \rrbracket^{\mathcal{I}*}. \end{aligned}$$

Moreover, taking into account

$$\tilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \tilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \tilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T) = \tilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T),$$

we infer

$$\begin{aligned} \llbracket T \rrbracket^{\mathcal{I}^+,\text{Su}} &= \{T' \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &\subseteq \{T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &= \llbracket T \rrbracket^{\mathcal{I},\text{Su}}. \end{aligned}$$

These relationships may be strengthened to apply to all extensions to a π -institution rather than only its strong version. More precisely, we obtain

Lemma 1721 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ be π -institutions based on \mathbf{F} , such that $\mathcal{I} \leq \mathcal{I}'$, \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}'}(\mathcal{A})$. Then*

$$\llbracket T \rrbracket^{\mathcal{I}'*} = \llbracket T \rrbracket^{\mathcal{I}*} \cap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}) \quad \text{and} \quad \llbracket T \rrbracket^{\mathcal{I}',\text{Su}} \subseteq \llbracket T \rrbracket^{\mathcal{I},\text{Su}} \cap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}).$$

Proof: We have, mimicking the process preceding the statement, applied to the extension \mathcal{I}' rather than specifically \mathcal{I}^+ :

$$\begin{aligned} \llbracket T \rrbracket^{\mathcal{I}'*} &= \{T' \in \text{FiFam}^{\mathcal{I}'}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &= \{T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \cap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}) \\ &= \llbracket T \rrbracket^{\mathcal{I}*} \cap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}). \end{aligned}$$

Moreover, taking into account

$$\tilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \tilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \tilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}'}(\mathcal{A})^T) = \tilde{\Omega}^{\mathcal{I}',\mathcal{A}}(T),$$

we infer

$$\begin{aligned} \llbracket T \rrbracket^{\mathcal{I}',\text{Su}} &= \{T' \in \text{FiFam}^{\mathcal{I}'}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{I}',\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ &\subseteq \{T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) : \tilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \cap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}) \\ &= \llbracket T \rrbracket^{\mathcal{I},\text{Su}} \cap \text{FiFam}^{\mathcal{I}'}(\mathcal{A}). \end{aligned}$$

Thus, we have the equality and the inclusion claimed. ■

Since \mathcal{I}^+ is an extension of \mathcal{I} , then we immediately deduce

Corollary 1722 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , \mathcal{A} an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Then*

$$[[T]]^{\mathcal{I}^+*} = [[T]]^{\mathcal{I}^+} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \text{and} \quad [[T]]^{\mathcal{I}^+, \text{Su}} \subseteq [[T]]^{\mathcal{I}, \text{Su}} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

Proof: By Lemma 1721, since $\mathcal{I} \leq \mathcal{I}^+$. ■

Finally, we strengthen the preceding relation between Suszko classes to an equality, in the special case, where T happens to be a Suszko \mathcal{I} -filter family of \mathcal{I} (recalling that $\text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{I}) \subseteq \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$).

Lemma 1723 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , \mathcal{A} an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. Then $[[T]]^{\mathcal{I}^+, \text{Su}} = [[T]]^{\mathcal{I}, \text{Su}} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.*

Proof: Let $T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. Then, by Lemma 1620, $[[T]]^{\mathcal{I}, \text{Su}} = \text{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since $T = \bigcap [[T]]^{\mathcal{I}, \text{Su}}$, $[[T]]^{\mathcal{I}^+, \text{Su}} \subseteq [[T]]^{\mathcal{I}, \text{Su}}$ and $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, we get $T = \bigcap [[T]]^{\mathcal{I}^+, \text{Su}}$. Hence $T \in \text{FiFam}^{\mathcal{I}^+, \text{Su}}(\mathcal{A})$. Again, using Lemma 1620, we get $[[T]]^{\mathcal{I}^+, \text{Su}} = \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T$. Therefore, we conclude that

$$\begin{aligned} [[T]]^{\mathcal{I}^+, \text{Su}} &= \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T \\ &= \text{FiFam}^{\mathcal{I}}(\mathcal{A})^T \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \\ &= [[T]]^{\mathcal{I}, \text{Su}} \cap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}). \end{aligned}$$
■

22.4 Leibniz Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} is **Leibniz truth equational** if there exists $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b , such that, for every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathbf{SEN}^b(\Sigma)$,

$$\phi \in T_\Sigma^* \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T).$$

It follows directly by the definition that, if \mathcal{I} is Leibniz truth equational, then, for all $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$ and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \quad \text{iff} \quad T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)).$$

Moreover, we can easily see that family truth equationality implies Leibniz truth equationality.

Lemma 1724 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . If \mathcal{I} is family truth equational, then \mathcal{I} is Leibniz truth equational.*

Proof: Suppose that \mathcal{I} is family truth equational, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . Thus, by Theorem 849, for every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Let \mathcal{A} be an \mathbf{F} -algebraic system, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathbf{SEN}(\Sigma)$. We have

$$\begin{aligned} \phi \in T_\Sigma & \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T) \quad (\mathcal{I} \text{ truth equational}) \\ & \quad \text{implies} \quad \tau^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T^*) \quad (T^* \in \llbracket T \rrbracket^*) \\ & \quad \text{iff} \quad \phi \in T_\Sigma^*. \quad (\mathcal{I} \text{ truth equational}) \end{aligned}$$

Thus, we get $T \leq T^*$. On the other hand, by Lemma 1605, $T^* \leq T$, whence $T = T^*$. This gives $T^* = T$ and, hence $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$, showing that \mathcal{I} is Leibniz truth equational. \blacksquare

If \mathcal{I} is Leibniz truth equational, then the collection of all its Leibniz filters on every algebraic system forms a closure family.

Proposition 1725 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} , $\text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$ is closed under signature-wise intersections and, hence, forms a closure family on \mathcal{A} .*

Proof: Suppose \mathcal{I} is Leibniz truth-equational, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . Let \mathcal{A} be an \mathbf{F} -algebraic system and $\{T^i : i \in I\} \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$ be a collection of Leibniz \mathcal{I} -filter families. Then

$$\begin{aligned} \bigcap_{i \in I} T^i & = \bigcap_{i \in I} (T^i)^* \quad (T^i \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})) \\ & = \bigcap_{i \in I} \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^i)) \quad (\mathcal{I} \text{ Leibniz truth equational}) \\ & \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(\bigcap_{i \in I} T^i)) \quad (\bigcap_{i \in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i \in I} T^i)) \\ & = (\bigcap_{i \in I} T^i)^*. \quad (\mathcal{I} \text{ Leibniz truth equational}) \end{aligned}$$

Since, by Lemma 1605, $(\bigcap_{i \in I} T^i)^* \leq \bigcap_{i \in I} T^i$, we get that $(\bigcap_{i \in I} T^i)^* = \bigcap_{i \in I} T^i$ and, therefore, $\bigcap_{i \in I} T^i \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. \blacksquare

The next proposition shows that to check that a given π -institution \mathcal{I} is Leibniz truth equational, it is sufficient to work with \mathcal{I}^* -algebraic systems only. That is, if the defining property holds for all Leibniz filters of \mathcal{I}^* -algebraic systems, then it extends to Leibniz filters over arbitrary \mathbf{F} -algebraic systems.

Proposition 1726 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} is Leibniz truth equational if and only if, there exists $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b , such that, for all $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$ and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$.*

Proof: The implication left-to-right follows from the definition of Leibniz truth equationality. Suppose, conversely, that there exists $\tau^b : (\text{SEN}^b)^\omega \rightarrow (\text{SEN}^b)^2$ in N^b , such that, for all $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$ and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Let \mathcal{A} be an arbitrary \mathbf{F} -algebraic system, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, and consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\Omega^{\mathcal{A}}(T)$. Then, by Corollary 1591, $\pi(T^*) = \pi(T)^*$ and, by Proposition 1567, $\pi(T)^*$ is the least \mathcal{I} -filter family on $\mathcal{A}/\Omega^{\mathcal{A}}(T)$. Since $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \text{AlgSys}^*(\mathcal{I})$, we get, by hypothesis,

$$\pi(T)^* = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}).$$

Hence, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \text{SEN}(\Sigma)$,

$$\begin{aligned} \phi \in T_\Sigma^* & \text{ iff } \phi/\Omega_\Sigma^{\mathcal{A}}(T) \in \pi_\Sigma(T_\Sigma^*) \quad (\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)) \\ & \text{ iff } \phi/\Omega_\Sigma^{\mathcal{A}}(T) \in \pi(T)_\Sigma^* \\ & \text{ iff } \phi/\Omega_\Sigma^{\mathcal{A}}(T) \in \tau_\Sigma^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & \text{ iff } \phi \in \tau_\Sigma^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)). \end{aligned}$$

Thus, \mathcal{I} is Leibniz truth equational. ■

A fortiori, it suffices to show that the condition in the statement of Proposition 1726 holds for all \mathcal{I} -algebraic systems, since this class encompasses all \mathcal{I}^* -algebraic systems.

Corollary 1727 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} is Leibniz truth equational if and only if, there exists $\tau^b : (\text{SEN}^b)^\omega \rightarrow (\text{SEN}^b)^2$ in N^b , such that, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$ and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$.*

Proof: The conclusion follows from Proposition 1726, taking into account the fact that $\text{AlgSys}^*(\mathcal{I}) \subseteq \text{AlgSys}(\mathcal{I})$. ■

Next, we provide another characterization of Leibniz truth equationality by showing that it is equivalent to $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})$ being the least \mathcal{I} -filter family on every \mathcal{I} - (or \mathcal{I}^* -) algebraic system.

Proposition 1728 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} and $\tau^b : (\text{SEN}^b)^\omega \rightarrow (\text{SEN}^b)^2$ in N^b . The following conditions are equivalent.*

- (i) \mathcal{I} is Leibniz truth equational, with witnessing transformations τ^b ;
- (ii) For all $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$;
- (iii) For all $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof:

- (i) \Rightarrow (iii) Suppose \mathcal{I} is Leibniz truth equational, with witnessing transformations τ^b . Let $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$ and $T^m = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 1605, $T^m \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Since $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^m))$, we get, by hypothesis, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq T^m$. On the other hand, since $T^m = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, for all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^m \leq T^*$, whence, by hypothesis, $T^m \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Since, this holds for all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, taking into account that $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$,

$$T^m \leq \tau^{\mathcal{A}}(\tilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}).$$

Therefore, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = T^m$.

- (iii) \Rightarrow (ii) Trivial, since $\text{AlgSys}^*(\mathcal{I}) \subseteq \text{AlgSys}(\mathcal{I})$.

- (ii) \Rightarrow (i) Suppose, for all $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let \mathcal{A} be an \mathbf{F} -algebraic system, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

Then, $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \text{AlgSys}^*(\mathcal{I})$ and, by Corollary 1591, $\pi(T^*) = \pi(T)^*$ and, by Proposition 1567, $\pi(T)^* = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Thus, by hypothesis, $\pi(T^*) = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)})$. Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \text{SEN}(\Sigma)$,

$$\begin{aligned} \phi \in T_{\Sigma}^* & \text{ iff } \phi/\Omega_{\Sigma}^{\mathcal{A}}(T) \in \pi_{\Sigma}(T_{\Sigma}^*) \\ & \text{ iff } \phi/\Omega_{\Sigma}^{\mathcal{A}}(T) \in \tau_{\Sigma}^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & \text{ iff } \phi \in \tau_{\Sigma}^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)). \end{aligned}$$

Hence, τ^b witnesses the Leibniz truth equationality of \mathcal{I} . ■

If \mathcal{I} -algebraic systems and \mathcal{I}^+ -algebraic systems coincide, then truth equationality of \mathcal{I}^+ guarantees the Leibniz truth equationality of \mathcal{I} .

Proposition 1729 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} and $\tau^b : (\text{SEN}^b)^{\omega} \rightarrow (\text{SEN}^b)^2$ in N^b . If \mathcal{I}^+ is family truth equational, with witnessing transformations τ^b and $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}(\mathcal{I}^+)$, then \mathcal{I} is Leibniz truth equational, with witnessing transformations τ^b .*

Proof: We use Proposition 1728. Suppose \mathcal{I}^+ is family truth equational via τ^b and $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}(\mathcal{I}^+)$. Let $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$. Since, by hypothesis $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, we get, by hypothesis, Lemma 1724 and Proposition 1728, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By Lemma 1704, $\bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Hence, we get $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Proposition 1728, \mathcal{I} is Leibniz truth equational via τ^b . ■

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} , $\tau^b : (\text{SEN}^b)^\omega \rightarrow (\text{SEN}^b)^2$ in N^b and \mathbf{K} a class of \mathbf{F} -algebraic systems. We define, as before, on \mathbf{F} the closure system $C^{\mathbf{K}, \tau} = \{C_\Sigma^{\mathbf{K}, \tau}\}_{\Sigma \in |\mathbf{Sign}^b|}$, where, for all $\Sigma \in |\mathbf{Sign}^b|$, $C_\Sigma^{\mathbf{K}, \tau} : \mathcal{P}(\text{SEN}^b(\Sigma)) \rightarrow \mathcal{P}(\text{SEN}^b(\Sigma))$ is given, for all $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$, by

$$\phi \in C_\Sigma^{\mathbf{K}, \tau}(\Phi) \quad \text{iff} \quad \tau_\Sigma^b[\phi] \leq C^{\mathbf{K}}(\tau_\Sigma^b[\Phi]).$$

Then we say that \mathbf{K} is a τ^b -**algebraic semantics for \mathcal{I}** if $C = C^{\mathbf{K}, \tau}$.

We show that, if a π -institution \mathcal{I} is Leibniz truth equational, with witnessing transformations τ^b , then any of the four classes $\text{AlgSys}^*(\mathcal{I}^+)$, $\text{AlgSys}(\mathcal{I}^+)$, $\text{AlgSys}^*(\mathcal{I})$ or $\text{AlgSys}(\mathcal{I})$ serves as a τ^b -algebraic semantics for \mathcal{I}^+ .

Theorem 1730 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution based on \mathbf{F} , with witnessing transformations $\tau^b : (\text{SEN}^b)^\omega \rightarrow (\text{SEN}^b)^2$ in N^b . Set $\mathbf{K} = \text{AlgSys}^*(\mathcal{I}^+)$ or $\text{AlgSys}(\mathcal{I}^+)$ or $\text{AlgSys}^*(\mathcal{I})$ or $\text{AlgSys}(\mathcal{I})$. Then \mathbf{K} is a τ^b -algebraic semantics for \mathcal{I}^+ .*

Proof: Let, first, $K = \text{AlgSys}^*(\mathcal{I})$ or $\text{AlgSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^b|$ and $\Phi \cup \{\phi\} \subseteq \text{SEN}^b(\Sigma)$. Then, we have $\phi \in C_\Sigma^+(\Phi)$ if and only if, by Proposition 1701, $\phi \in C_\Sigma^{M_{\mathbf{K}}^{\mathcal{I}, m}}(\Phi)$ if and only if, for all $\mathcal{A} \in \mathbf{K}$,

$$\alpha_\Sigma(\Phi) \subseteq C_{F(\Sigma)}^{\mathcal{I}, \mathcal{A}}(\emptyset) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in C_{F(\Sigma)}^{\mathcal{I}, \mathcal{A}}(\emptyset).$$

if and only if, by hypothesis and Proposition 1728,

$$\alpha_\Sigma(\Phi) \subseteq \tau_{F(\Sigma)}^{\mathcal{A}}(\Delta^{\mathcal{A}}) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in \tau_{F(\Sigma)}^{\mathcal{A}}(\Delta^{\mathcal{A}}),$$

if and only if

$$\tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}} \quad \text{implies} \quad \tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}},$$

if and only if

$$\alpha(\tau_\Sigma^b[\Phi]) \leq \Delta^{\mathcal{A}} \quad \text{implies} \quad \alpha(\tau_\Sigma^b[\phi]) \leq \Delta^{\mathcal{A}},$$

if and only if $\tau_\Sigma^b[\phi] \leq C^{\mathbf{K}}(\tau_\Sigma^b[\Phi])$ if and only if $\phi \in C_\Sigma^{\mathbf{K}, \tau}(\Phi)$. Thus, \mathbf{K} is a τ^b -algebraic semantics of \mathcal{I}^+ .

Finally, note that, by hypothesis and Lemma 1708, \mathcal{I}^+ is Leibniz truth equational via τ^b , as well. Moreover, by Corollary 1705, $(\mathcal{I}^+)^+ = \mathcal{I}^+$. Applying, therefore, what was shown above to \mathcal{I}^+ , we get the result for $\mathbf{K} = \text{AlgSys}^*(\mathcal{I}^+)$ or $\text{AlgSys}(\mathcal{I}^+)$. \blacksquare

Theorem 1730 implies that for $\text{AlgSys}(\mathcal{I})$ to be a τ^b -algebraic semantics of a Leibniz truth equational π -institution \mathcal{I} , where τ^b is a set of witnessing transformations, \mathcal{I} and \mathcal{I}^+ must be identical.

Corollary 1731 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution based on \mathbf{F} , with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . $\text{AlgSys}(\mathcal{I})$ is a τ^b -algebraic semantics for \mathcal{I} if and only if $\mathcal{I} = \mathcal{I}^+$.*

Proof: By Theorem 1730, $C^+ = C^{\text{AlgSys}(\mathcal{I}), \tau}$. Therefore, we get that $\text{AlgSys}(\mathcal{I})$ is a τ^b -algebraic semantics of \mathcal{I} if and only if, by definition $C = C^{\text{AlgSys}(\mathcal{I}), \tau}$ if and only if $C = C^+$. ■

Moreover, we can show that Leibniz truth equationality of \mathcal{I} implies the family truth equationality of \mathcal{I}^+ .

Corollary 1732 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . If \mathcal{I} is Leibniz truth equational, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$, then \mathcal{I}^+ is family truth equational via τ^b .*

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By hypothesis and Theorem 1730, \mathcal{I}^+ has a τ^b -algebraic semantics. Therefore, by Corollary 825, $T = \tau^{\mathcal{A}}(\tilde{\Omega}^{\mathcal{I}^+, \mathcal{A}}(T)) \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Conversely, by hypothesis and the fact that, by Proposition 1702, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, using Lemma 1605, $\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)) = T^* \leq T$. We now conclude that $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Thus, \mathcal{I}^+ is family truth equational, with witnessing transformations τ^b . ■

As another consequence, we get that, under Leibniz truth equationality, \mathcal{I}^+ filter families coincide with Leibniz \mathcal{I} -filter families on any algebraic system.

Corollary 1733 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . If \mathcal{I} is Leibniz truth equational, then, for every \mathbf{F} -algebraic system \mathcal{A} , $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$.*

Proof: Suppose \mathcal{I} is Leibniz truth equational. Then, by Corollary 1732, \mathcal{I}^+ is family truth equational. Thus, by Proposition 1710, $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, for every \mathbf{F} -algebraic system \mathcal{A} . ■

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . Let, also, \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by definition $T^{\mathcal{I}, \text{Su}} = \bigcap \llbracket T \rrbracket^{\mathcal{I}, \text{Su}}$ and, by Proposition 1621, $\langle \mathcal{A}, \llbracket T \rrbracket^{\mathcal{I}, \text{Su}} \rangle \in \text{FStr}(\mathcal{I})$. Thus, by Proposition 1621, $T^{\mathcal{I}, \text{Su}} \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Now it follows, by hypothesis, that

$$T^{\mathcal{I}, \text{Su}} = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^{\mathcal{I}, \text{Su}})).$$

There is also an additional characterization of the Suszko filter family, using the Suszko operator.

Proposition 1734 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . For every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T^{\mathcal{I}, \text{Su}} = \tau^{\mathcal{A}}(\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)).$$

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T).$$

Then $\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T) \in \text{AlgSys}(\mathcal{I})$. Moreover, by Lemma 1594, $\pi(T^{\mathcal{I}, \text{Su}}) = \pi(T)^{\mathcal{I}, \text{Su}}$ and, by Proposition 1624, $\pi(T)^{\mathcal{I}, \text{Su}} = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T))$. Thus, by Proposition 1728,

$$\pi(T^{\mathcal{I}, \text{Su}}) = \tau^{\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)}(\Delta^{\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)}).$$

Now we get

$$\begin{aligned} T^{\mathcal{I}, \text{Su}} &= \pi^{-1}(\pi(T^{\mathcal{I}, \text{Su}})) \\ &= \pi^{-1}(\tau^{\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)}(\Delta^{\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)})) \\ &= \tau^{\mathcal{A}}(\pi^{-1}(\Delta^{\mathcal{A}/\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)})) \\ &= \tau^{\mathcal{A}}(\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)). \end{aligned}$$

This proves the statement. ■

Proposition 1734 enables us to characterize the Suszko filter counterpart $T^{\mathcal{I}, \text{Su}}$ of a given filter family T as the intersection of all Leibniz filter family companions of filter families in the upset of T .

Corollary 1735 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . For every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T^{\mathcal{I}, \text{Su}} = \bigcap \{T'^* : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{aligned} T^{\mathcal{I}, \text{Su}} &= \tau^{\mathcal{A}}(\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)) \quad (\text{by Proposition 1734}) \\ &= \tau^{\mathcal{A}}(\bigcap \{\Omega^{\mathcal{A}}(T') : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\ &\quad (\text{definition of } \tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)) \\ &= \bigcap \{\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T')) : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ &= \bigcap \{T'^* : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})\}. \\ &\quad (\text{Leibniz truth equationality}) \end{aligned}$$

This proves the corollary. ■

We now get immediately

Corollary 1736 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . For every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{I}) \quad \text{iff} \quad T \leq T'^*, \text{ for all } T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})^T.$$

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have $T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$ if and only if, by definition, $T = T^{\mathcal{I}, \text{Su}}$ if and only if, by Corollary 1735, $T = \bigcap \{T'^* : T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})^T\}$, if and only if, taking into account that $T^* \leq T$, $T \leq T'^*$, for all $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. ■

We close the section with a characterization of weak family algebraizability of the strong version of \mathcal{I} among those π -institutions that are Leibniz truth equational.

Proposition 1737 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . \mathcal{I}^+ is weakly family algebraizable if and only if, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^+}(\mathcal{A})$ is an order isomorphism.*

Proof: If \mathcal{I}^+ is weakly family algebraizable, then it is, a fortiori, protoalgebraic. Therefore, by Proposition 1658, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^+}(\mathcal{A})$ is an order isomorphism.

Assume, conversely, that the condition in the statement holds. Then, for every \mathbf{F} -algebraic system \mathcal{A} ,

$$\begin{aligned} \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) &= \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \quad (\text{by Corollary 1709}) \\ &= \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}). \quad (\text{by Corollary 1733}) \end{aligned}$$

Thus, for all $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^+}(\mathcal{A})$ is an order isomorphism. Hence, by Theorem 296, \mathcal{I}^+ is weakly family algebraizable. ■

Proposition 1737 gives a sufficient condition for the weak family algebraizability of \mathcal{I}^+ that involves only \mathcal{I} -algebraic systems.

Corollary 1738 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π -institution, with witnessing transformations $\tau^b : (\mathbf{SEN}^b)^\omega \rightarrow (\mathbf{SEN}^b)^2$ in N^b . If, for every $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism, then \mathcal{I}^+ is weakly family algebraizable.*

Proof: By hypothesis and Lemma 1716, for every $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^+}(\mathcal{A})$ is an order isomorphism. Hence, by Proposition 1737, \mathcal{I}^+ is weakly family algebraizable. ■

22.5 Leibniz Definability

Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} is **Leibniz definable** if, there exists $\mu^b : (\mathbf{SEN}^b)^\omega \rightarrow \mathbf{SEN}^b$ in N^b , such that, for every \mathbf{F} -algebraic system \mathcal{A} , and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = \mu^{\mathcal{A}}(T),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathbf{SEN}(\Sigma)$,

$$\phi \in T_\Sigma^* \quad \text{iff} \quad \mu_\Sigma^{\mathcal{A}}[\phi] \leq T.$$

We show that it suffices to consider only \mathcal{I}^* -algebraic systems to establish Leibniz definability.

Proposition 1739 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π -institution based on \mathbf{F} . \mathcal{I} is Leibniz definable if and only if, there exists $\mu^b : (\mathbf{SEN}^b)^\omega \rightarrow \mathbf{SEN}^b$ in N^b , such that, for all $\mathcal{A} \in \text{AlgSys}^*(\mathcal{I})$ and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \mu^{\mathcal{A}}(T)$.*

Proof: The “only if” is trivial. For the “if”, suppose the stated condition holds and let \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \rightarrow \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

Then $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \text{AlgSys}^*(\mathcal{I})$ and, moreover, $\text{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$, since $T^* \in \llbracket T \rrbracket^*$. Now we have

$$\begin{aligned} T^* &= \pi^{-1}(\pi(T^*)) \quad (\text{Ker}(\langle I, \pi \rangle) \text{ compatible with } T^*) \\ &= \pi^{-1}(\pi(T)^*) \quad (\text{by Lemma 1594}) \\ &= \pi^{-1}(\mu^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T))) \quad (\text{by hypothesis}) \\ &= \mu^{\mathcal{A}}(\pi^{-1}(\pi(T))) \quad (\text{algebra and surjectivity of } \langle I, \pi \rangle) \\ &= \mu^{\mathcal{A}}(T). \quad (\text{Ker}(\langle I, \pi \rangle) \text{ compatible with } T) \end{aligned}$$

Therefore, \mathcal{I} is Leibniz definable via μ^b . ■

Leibniz definability ensures that the mapping sending a filter family to its Leibniz counterpart is monotone and this, in turn, implies that T^* is the largest Leibniz filter family below T .

Lemma 1740 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π -institution based on \mathbf{F} , with witnessing transformations $\mu^b : (\mathbf{SEN}^b)^\omega \rightarrow \mathbf{SEN}^b$ in N^b . For every \mathbf{F} -algebraic system \mathcal{A} and all $T, T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \leq T' \quad \text{implies} \quad T^* \leq T'^*.$$

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system and $T, T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then $T^* = \mu^{\mathcal{A}}(T) \leq \mu^{\mathcal{A}}(T') = T'^*$. ■

Corollary 1741 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π -institution based on \mathbf{F} , with witnessing transformations $\mu^b : (\mathbf{SEN}^b)^\omega \rightarrow \mathbf{SEN}^b$ in N^b . For every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, T^* is the largest Leibniz filter family below T .*

Proof: Let \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Suppose $T' \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, such that $T' \leq T$. Then we have $T' = T'^* \leq T^*$, where the last inclusion is due to Lemma 1740. ■

Under Leibniz definability, the condition that $\Omega^{\mathcal{A}}$ be an order isomorphism from Leibniz filter families of \mathcal{A} onto \mathcal{I}^* -congruence systems on \mathcal{A} , for every \mathcal{I} -algebraic system yields protoalgebraicity.

Proposition 1742 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π -institution based on \mathbf{F} , with witnessing transformations $\mu^b : (\mathbf{SEN}^b)^\omega \rightarrow \mathbf{SEN}^b$ in N^b . If, for every $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism, then \mathcal{I} is protoalgebraic.*

Proof: Suppose the stated condition holds and let \mathcal{A} be an \mathbf{F} -algebraic system and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{aligned}
\tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T) &= \bigcap \{ \Omega^{\mathcal{A}}(T') : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \} \\
&\quad (\text{definition of } \tilde{\Omega}^{\mathcal{I}, \mathcal{A}}(T)) \\
&= \bigcap \{ \Omega^{\mathcal{A}}(T'^*) : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \} \\
&\quad (\text{by Proposition 1714}) \\
&= \Omega^{\mathcal{A}}(\bigcap \{ T'^* : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \}) \\
&\quad (\text{by the hypothesis}) \\
&= \Omega^{\mathcal{A}}(T^*) \quad (\text{by Lemma 1740}) \\
&= \Omega^{\mathcal{A}}(T). \quad (\text{by Proposition 1714})
\end{aligned}$$

Hence, the Leibniz and Suszko operators on every \mathbf{F} -algebraic system coincide, whence, by Lemma 1555, \mathcal{I} is protoalgebraic. ■

We show, next, that, under Leibniz definability, the collection of Leibniz \mathcal{I} -filter families on every \mathbf{F} -algebraic system is closed under morphic images and preimages and under intersections.

Proposition 1743 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π -institution based on \mathbf{F} , with witnessing transformations $\mu^b : (\mathbf{SEN}^b)^\omega \rightarrow \mathbf{SEN}^b$ in N^b .*

- (a) $\mathbb{M}(M^{\mathcal{I}^*}) \subseteq M^{\mathcal{I}^*}$ and $\mathbb{M}^{-1}(M^{\mathcal{I}^*}) \subseteq M^{\mathcal{I}^*}$;
- (b) $\mathbb{III}(M^{\mathcal{I}^*}) \subseteq M^{\mathcal{I}^*}$.

Proof:

- (a) Let \mathcal{A}, \mathcal{B} be \mathbf{F} -algebraic systems, $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \rightarrow \langle \mathcal{B}, T' \rangle$ a strict surjective morphism. We then have

$$\begin{aligned}
T = T^* & \text{ iff } T = \mu^{\mathcal{A}}(T) \\
& \text{ iff } \gamma^{-1}(T') = \mu^{\mathcal{A}}(\gamma^{-1}(T')) \\
& \text{ iff } \gamma^{-1}(T') = \gamma^{-1}(\mu^{\mathcal{B}}(T')) \\
& \text{ iff } T' = \mu^{\mathcal{B}}(T') \\
& \text{ iff } T' = T'^*.
\end{aligned}$$

Thus, $\langle \mathcal{A}, T \rangle \in \mathbf{M}^{\mathcal{I}^*}$ if and only if $\langle \mathcal{B}, T' \rangle \in \mathbf{M}^{\mathcal{I}^*}$.

- (b) Let \mathcal{A} be an \mathbf{F} -algebraic system and $\{T^i : i \in I\} \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Then we have

$$\begin{aligned}
\bigcap_{i \in I} T^i & = \bigcap_{i \in I} (T^i)^* \\
& = \bigcap_{i \in I} \mu^{\mathcal{A}}(T^i) \\
& = \mu^{\mathcal{A}}(\bigcap_{i \in I} T^i) \\
& = (\bigcap_{i \in I} T^i)^*.
\end{aligned}$$

Therefore $\bigcap_{i \in I} T^i \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Thus, if $\langle \mathcal{A}, T^i \rangle \in \mathbf{M}^{\mathcal{I}^*}$, for all $i \in I$, then $\langle \mathcal{A}, \bigcap_{i \in I} T^i \rangle \in \mathbf{M}^{\mathcal{I}^*}$. ■

Proposition 1743, in conjunction with the characterization Theorem 1824 of the $\mathcal{I}^{\mathbf{M}}$ -matrix families for a class \mathbf{M} of \mathbf{F} -matrix families, allow us to prove that, under Leibniz definability, \mathcal{I}^+ -filter families and Leibniz \mathcal{I} -filter families on any \mathbf{F} -algebraic system coincide.

Theorem 1744 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π -institution based on \mathbf{F} , with witnessing transformations $\mu^b : (\text{SEN}^b)^\omega \rightarrow \text{SEN}^b$ in N^b . For every \mathbf{F} -algebraic system \mathcal{A} ,*

$$\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}).$$

Proof: We have

$$\begin{aligned}
\text{MatFam}(\mathcal{I}^+) & = \text{MatFam}(\mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}}) \quad (\mathcal{I}^+ = \mathcal{I}^{\mathbf{M}^{\mathcal{I}^*}}, \text{ by definition}) \\
& = \mathbf{MIIM}^{-1}(\mathbf{M}^{\mathcal{I}^*}) \quad (\text{by Theorem 1824}) \\
& \subseteq \mathbf{M}^{\mathcal{I}^*}. \quad (\text{by Proposition 1743})
\end{aligned}$$

This shows that $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. But, by Proposition 1702, the reverse inclusion always holds. Therefore, for every \mathbf{F} -algebraic system \mathcal{A} , $\text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. ■

We give several conditions involving the strong version of \mathcal{I} that turn out to characterize both the protoalgebraicity of \mathcal{I} and the protoalgebraicity of \mathcal{I}^+ , under the proviso that \mathcal{I} be Leibniz definable.

Corollary 1745 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π -institution based on \mathbf{F} . The following conditions are equivalent:*

- (i) \mathcal{I}^+ is protoalgebraic;
- (ii) \mathcal{I} is protoalgebraic;
- (iii) For every $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism;
- (iv) For every $\mathcal{A} \in \text{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^{+*}}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^{+*}}(\mathcal{A})$ is an order isomorphism;
- (v) \mathcal{I}^+ is weakly family algebraizable.

Proof:

(i) \Rightarrow (ii) Suppose \mathcal{I}^+ is protoalgebraic. Let \mathcal{A} be an \mathbf{F} -algebraic system and $T, T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. By Lemma 1740, $T^* \leq T'^*$. Hence, by Proposition 1702 and the hypothesis, $\Omega^{\mathcal{A}}(T^*) \leq \Omega^{\mathcal{A}}(T'^*)$. By hypothesis, Proposition 1658 and Proposition 1714, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Thus, the Leibniz operator is monotone on the \mathcal{I} -filter families of every \mathbf{F} -algebraic system and, therefore, \mathcal{I} is protoalgebraic.

(ii) \Rightarrow (iii) By Proposition 1658.

(iii) \Rightarrow (iv) By Lemma 1716.

(iv) \Rightarrow (v) We have, for every \mathbf{F} -algebraic system \mathcal{A} ,

$$\begin{aligned} \text{FiFam}^{\mathcal{I}^{+*}}(\mathcal{A}) &= \text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \quad (\text{by Corollary 1709}) \\ &= \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}). \quad (\text{by Theorem 1744}) \end{aligned}$$

Therefore, by hypothesis, $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rightarrow \text{ConSys}^{\mathcal{I}^{+*}}(\mathcal{A})$ is an order isomorphism. By Theorem 296, \mathcal{I}^+ is weakly family algebraizable.

(v) \Rightarrow (i) If \mathcal{I}^+ is weakly family algebraizable, then it is, a fortiori, protoalgebraic. ■

Finally, we give some consequences of imposing both Leibniz definability and Leibniz truth equationality. The combination is strong enough to guarantee that Leibniz filter families and Suszko filter families coincide.

Proposition 1746 *Let $\mathbf{F} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable and Leibniz truth equational π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} and all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T^* = T^{\mathcal{I}, \text{Su}}.$$

Proof: Let $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$ and $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$\begin{aligned} T^{\mathcal{I}, \text{Su}} &= \bigcap \{T'^* : T \leq T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad (\text{by Corollary 1735}) \\ &= T^*. \quad (\text{by Lemma 1740}) \end{aligned}$$

This proves the statement. ■

Corollary 1747 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \text{SEN}^{\flat}, N^{\flat} \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable and Leibniz truth equational π -institution based on \mathbf{F} . For every \mathbf{F} -algebraic system \mathcal{A} ,*

$$\text{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A}).$$

Proof: Let $\mathcal{A} \in \text{AlgSys}(\mathbf{F})$. By Lemma 1620, $\text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A}) \subseteq \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. On the other hand, if $T \in \text{FiFam}^{\mathcal{I}^*}(\mathcal{A})$, then, by Proposition 1746, $T = T^* = T^{\mathcal{I}, \text{Su}}$. Thus, $T \in \text{FiFam}^{\mathcal{I}, \text{Su}}(\mathcal{A})$. ■

