Chapter 24

Special Topics

24.1 Rule Based π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.

An **F-rule** is a pair $\langle P, \rho \rangle$, where $P \cup \{\rho\} : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ is a finite set of natural transformations in $N^\flat$. If $P = \varnothing$, then $\langle \varnothing, \rho \rangle$ is called an **F-axiom** and it is ordinarily identified with ρ.

Let $R = \langle P, \rho \rangle$ be an **F**-rule, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We say ϕ R-**follows from** Φ, written $\Phi \to_\Sigma^R \phi$, if there exists $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$P_\Sigma(\vec{\chi}) \subseteq \Phi \quad \text{and} \quad \rho_\Sigma(\vec{\chi}) = \phi.$$

Consider, now, a set $\mathcal{R}$ of **F**-rules. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, we say ϕ is $\mathcal{R}$-**provable from** Φ, written $\phi \in C_\Sigma^{\mathcal{R}}(\Phi)$ or $\Phi \vdash_\Sigma^{\mathcal{R}} \phi$, if there exists a sequence

$$\phi_0, \phi_1, \phi_2, \dots, \phi_{n-1}, \phi_n$$

in $\mathrm{SEN}^\flat(\Sigma)$, such that $\phi_n = \phi$ and, for all $i \le n$,

- $\phi_i \in \Phi$ or
- ϕ_i R-follows from $\{\phi_0, \phi_1, \dots, \phi_{i-1}\}$, for some $R \in \mathcal{R}$.

A sequence $\phi_0, \phi_1, \dots, \phi_n$ witnessing $\Phi \vdash_\Sigma^{\mathcal{R}} \phi$ is called an $\mathcal{R}$-**proof** of ϕ from Φ.

We show that $C^{\mathcal{R}}$, as defined here, is indeed a closure system on the base algebraic system **F**.

Proposition 1815 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{R}$ *a collection of* **F***-rules. Then* $C^{\mathcal{R}} = \{C_\Sigma^{\mathcal{R}}\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ *is a closure system on* **F**.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

(i) If $\phi \in \Phi$, then ϕ is an $\mathcal{R}$-proof of ϕ from Φ. So $\phi \in C_\Sigma^{\mathcal{R}}(\Phi)$ and $C^{\mathcal{R}}$ is inflationary.

(ii) If $\Phi \subseteq \Psi$ and $\phi \in C_\Sigma^{\mathcal{R}}(\Phi)$, then, there exists an $\mathcal{R}$-proof of ϕ from Φ. The same sequence is then an $\mathcal{R}$-proof of ϕ from Ψ. So $\phi \in C_\Sigma^{\mathcal{R}}(\Psi)$ and $C^{\mathcal{R}}$ is monotone.

(iii) Suppose $\phi \in C_\Sigma^{\mathcal{R}}(C_\Sigma^{\mathcal{R}}(\Phi))$. Then, there exists an $\mathcal{R}$-proof of ϕ from $C_\Sigma^{\mathcal{R}}(\Phi)$, say

$$\phi_0, \phi_1, \dots, \phi_{n-1}, \phi_n = \phi.$$

Then, for each $\phi_i \in C_\Sigma^{\mathcal{R}}(\Phi)$, there exists an $\mathcal{R}$-proof of ϕ_i from Φ. For each such ϕ_i, we insert its $\mathcal{R}$-proof from Φ in its place in the sequence. The new sequence is an $\mathcal{R}$-proof of ϕ from Φ. Thus, we get that $\phi \in C_\Sigma^{\mathcal{R}}(\Phi)$ and $C^{\mathcal{R}}$ is also idempotent.

(iv) Finally, it remains to show structurality. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathcal{R}}_\Sigma(\Phi)$. Let $\phi_0, \phi_1, \dots, \phi_{n-1}, \phi_n = \phi$ be an $\mathcal{R}$-proof of ϕ from Φ. We consider the sequence

$$\mathrm{SEN}^\flat(f)(\phi_0), \mathrm{SEN}^\flat(f)(\phi_1), \dots, \mathrm{SEN}^\flat(f)(\phi_{n-1}), \mathrm{SEN}^\flat(f)(\phi_n).$$

Then $\mathrm{SEN}^\flat(f)(\phi_n) = \mathrm{SEN}^\flat(f)(\phi)$ and, moreover, for all $i \le n$, if $\phi_i \in \Phi$, then $\mathrm{SEN}^\flat(f)(\phi_i) \in \mathrm{SEN}^\flat(f)(\Phi)$, and, if ϕ_i R-follows from $\{\phi_0, \phi_1, \dots, \phi_{i-1}\}$, for some $R \in \mathcal{R}$, then $\mathrm{SEN}^\flat(f)(\phi_i)$ R-follows from $\{\mathrm{SEN}^\flat(f)(\phi_0), \mathrm{SEN}^\flat(f)(\phi_1), \dots, \mathrm{SEN}^\flat(f)(\phi_{i-1})\}$ because of the naturality of R. So, the displayed sequence is an $\mathcal{R}$-proof of $\mathrm{SEN}^\flat(f)(\phi)$ from $\mathrm{SEN}^\flat(f)(\Phi)$ and $C^{\mathcal{R}}$ is also structural.

We conclude that $C^{\mathcal{R}}$ is a closure system on $\mathbf{F}$. ∎

We denote by $\mathcal{I}^{\mathcal{R}} = \langle \mathbf{F}, C^{\mathcal{R}} \rangle$ the π-institution corresponding to $C^{\mathcal{R}}$.

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, we say that $\mathcal{I}$ is **rule based** if there exists a collection $\mathcal{R}$ of $\mathbf{F}$-rules, such that $C = C^{\mathcal{R}}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{R}$ a collection of $\mathbf{F}$-rules, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that T is **closed under** $\mathcal{R}$ or is $\mathcal{R}$**-closed** if, for all $R = \langle P, \rho \rangle \in \mathcal{R}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

$$P^{\mathcal{A}}_\Sigma(\vec{\chi}) \subseteq T_\Sigma \quad \text{implies} \quad \rho^{\mathcal{A}}_\Sigma(\vec{\chi}) \in T_\Sigma.$$

This terminology allows the following elegant characterization of $\mathcal{I}^{\mathcal{R}}$-filter families of $\mathcal{A}$.

Proposition 1816 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{R}$ *a collection of* $\mathbf{F}$*-rules,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{SenFam}(\mathcal{A})$*. Then* $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathcal{R}}}(\mathcal{A})$ *if and only if* T *is* $\mathcal{R}$*-closed.*

Proof: Assume, first, that $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathcal{R}}}(\mathcal{A})$, $R = \langle P, \rho \rangle \in \mathcal{R}$ and, using surjectivity of $\langle F, \alpha \rangle$, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$P^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}.$$

Then we get $\alpha_\Sigma(P_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}$. Since, by the definition of $C^{\mathcal{I}^{\mathcal{R}}}$, $\rho_\Sigma(\vec{\chi}) \in C^{\mathcal{I}^{\mathcal{R}}}_\Sigma(P_\Sigma(\vec{\chi}))$ and, by hypothesis, $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathcal{R}}}(\mathcal{A})$, we get $\alpha_\Sigma(\rho_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$ or, equivalently, $\rho^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$. Thus, T is $\mathcal{R}$-closed.

Suppose, conversely, that $T \in \mathrm{SenFam}(\mathcal{A})$ is $\mathcal{R}$-closed. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathcal{I}^{\mathcal{R}}}_\Sigma(\Phi)$ and consider $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, such that

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}.$$

Since $\phi \in C_\Sigma^{\mathcal{I}^\mathcal{R}}(\Phi)$, there exists an $\mathcal{R}$-proof of ϕ from Φ, say

$$\phi_0, \phi_1, \dots, \phi_{n-1}, \phi_n = \phi.$$

We prove by induction on $i \le n$ that, every member of the sequence

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_0)), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_1)), \dots, \\ \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_{n-1})), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_n))$$

belongs to $T_{F(\Sigma')}$. The case $i = n$, will yield the desired conclusion.

First, if $\phi_i \in \Phi$, then $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) \in \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}$, where the latter inclusion holds by hypothesis.

Suppose, on the other hand, that ϕ_i R-follows from $\{\phi_0, \phi_1, \dots, \phi_{i-1}\}$, for some $R = \langle P, \rho\rangle \in \mathcal{R}$. Thus, there exists $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$P_\Sigma(\vec{\chi}) \subseteq \{\phi_0, \phi_1, \dots, \phi_{i-1}\} \quad \text{and} \quad \rho_\Sigma(\vec{\chi}) = \phi_i.$$

But then

$$\begin{array}{rcl} P^\mathcal{A}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\chi}))) & = & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(P_\Sigma(\vec{\chi}))) \\ & \subseteq & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\{\phi_0, \dots, \phi_{i-1}\})) \\ & \subseteq & T_{F(\Sigma')}, \end{array}$$

where the last inclusion follows by the induction hypothesis, and, hence, since T is $\mathcal{R}$-closed, we get that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\rho_\Sigma(\vec{\chi}))) = \rho^\mathcal{A}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\chi}))) \in T_{F(\Sigma')}$. This concludes the induction step and shows that, for all $i \le n$, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) \in T_{F(\Sigma')}$. ■

In addition, we can characterize $\mathcal{I}^\mathcal{R}$-filter families generated by a given sentence family as follows.

Proposition 1817 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{R}$ *a collection of* $\mathbf{F}$*-rules,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, an* $\mathbf{F}$*-algebraic system and* $X \in \mathrm{SenFam}(\mathcal{A})$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$*,*

$$C_\Sigma^{\mathcal{I}^\mathcal{R}, \mathcal{A}}(X) = \{\phi \in \mathrm{SEN}(\Sigma) : X_\Sigma \vdash_\Sigma^\mathcal{R} \phi\}.$$

Proof: Define $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$T_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : X_\Sigma \vdash_\Sigma^\mathcal{R} \phi\}.$$

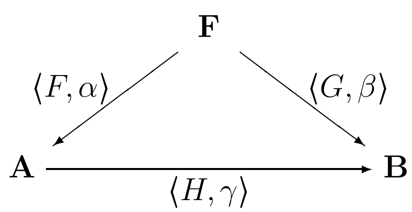
It is not difficult to see that $X \le T$ and T is $\mathcal{R}$-closed. Thus, by Proposition 1816, $C^{\mathcal{I}^\mathcal{R}, \mathcal{A}}(X) \le T$. On the other hand, if $T' \in \mathrm{SenFam}(\mathcal{I})$ contains X and is $\mathcal{R}$-closed, then $T \le T'$. Therefore, we conclude that $T \le C^{\mathcal{I}^\mathcal{R}, \mathcal{A}}(X)$. Equality now follows. ■

24.2 Operators on Classes of Matrix Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Recall that an $\mathbf{F}$-algebraic system is a pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, where $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an $N^\flat$-algebraic system and $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$ is a surjective $N^\flat$-algebraic system morphism. Recall, also, that an $\mathbf{F}$-matrix family is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathbf{A})$ is a sentence family on $\mathbf{A}$.

We define now some class operators on classes of $\mathbf{F}$-matrix families, i.e., operators that, given, as input a class of $\mathbf{F}$-matrix families, produce a new class of $\mathbf{F}$-matrix families.

Given $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, and $\mathbf{F}$-matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$, we say that $\mathfrak{B}$ is a **morphic image** of $\mathfrak{A}$ and write $\mathfrak{B} \in \mathbb{M}(\mathfrak{A})$, if there exists a surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ (that is, such that $\langle G, \beta \rangle = \langle H, \gamma \rangle \circ \langle F, \alpha \rangle$)



such that

$$\gamma^{-1}(T') = T.$$

In this case, we call $\mathfrak{A}$ an **inverse morphic image** or a **morphic preimage** of $\mathfrak{B}$ and write $\mathfrak{A} \in \mathbb{M}^{-1}(\mathfrak{B})$.

Given a class M of $\mathbf{F}$-matrix families, we write $\mathfrak{B} \in \mathbb{M}(\mathsf{M})$ if there exists $\mathfrak{A} \in \mathsf{M}$, such that $\mathfrak{B} \in \mathbb{M}(\mathfrak{A})$.

Similarly, we write $\mathfrak{A} \in \mathbb{M}^{-1}(\mathsf{M})$ if there exists $\mathfrak{B} \in \mathsf{M}$, such that $\mathfrak{A} \in \mathbb{M}^{-1}(\mathfrak{B})$.

It is not difficult to show that both $\mathbb{M}$ and $\mathbb{M}^{-1}$ are closure operators on the collection of all $\mathbf{F}$-matrix families.

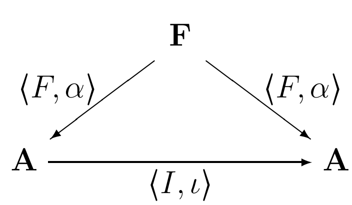
Lemma 1818 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{M}, \mathbb{M}^{-1} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$$

are closure operators on $\mathrm{MatFam}(\mathbf{F})$.

Proof: We prove the statement for $\mathbb{M}$ in detail. The proof for $\mathbb{M}^{-1}$ is similar.

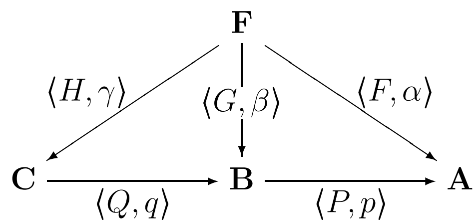
Suppose, first, that M is a class of $\mathbf{F}$-matrix families and $\mathfrak{A} \in \mathsf{M}$. Then, the diagram



where $\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}$ is the identity morphism, shows that $\mathfrak{A} \in \mathbb{M}(\mathsf{M})$. Therefore, $\mathbb{M}$ is inflationary.

Monotonicity is obvious, since, if M, N are classes of $\mathbf{F}$-matrix families, such that $\mathsf{M} \subseteq \mathsf{N}$, and $\mathfrak{A} \in \mathbb{M}(\mathsf{M})$, then, by definition, $\mathfrak{A} \in \mathbb{M}(\mathfrak{B})$, with $\mathcal{B} \in \mathsf{M}$. But then, since $\mathsf{M} \subseteq \mathsf{N}$, $\mathfrak{A} \in \mathbb{M}(\mathfrak{B})$, with $\mathcal{B} \in \mathsf{N}$ and, again, by definition, $\mathfrak{A} \in \mathbb{M}(\mathsf{N})$. Thus, we have $\mathbb{M}(\mathsf{M}) \subseteq \mathbb{M}(\mathsf{N})$.

Finally, assume that M is a class of $\mathbf{F}$-matrix families and $\mathfrak{A} \in \mathbb{M}(\mathbb{M}(\mathsf{M}))$. Then, there exists $\mathfrak{B} \in \mathbb{M}(\mathsf{M})$, such that $\mathfrak{A} \in \mathbb{M}(\mathfrak{B})$. Furthermore, there exists $\mathfrak{C} \in \mathsf{M}$, such that $\mathfrak{B} \in \mathbb{M}(\mathfrak{C})$. But these two statements combined reveal the existence of the following diagram, in which the two small triangles commute.



As a result, the big triangle also commutes and this ensures that $\mathfrak{A} \in \mathbb{M}(\mathfrak{C})$, which yields $\mathfrak{A} \in \mathbb{M}(\mathsf{M})$. ■

Next, we introduce another class operator on classes of $\mathbf{F}$-matrix families.

Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathfrak{A}^i = \langle \mathcal{A}, T^i\rangle$, $i \in I$, a collection of $\mathbf{F}$-matrix families, all over $\mathcal{A}$. Define the **intersection** of the $\mathfrak{A}^i$, $i \in I$, as the $\mathbf{F}$-matrix family, with the same underlying $\mathbf{F}$-algebraic system $\mathcal{A}$ and with filter family the intersection of the T^i's; more formally

$$\bigcap_{i\in I} \mathfrak{A}^i = \langle \mathcal{A}, \bigcap_{i\in I} T^i\rangle.$$

Given a class M of $\mathbf{F}$-matrix families and an $\mathbf{F}$-matrix family $\mathfrak{B}$, we write $\mathfrak{B} \in \mathbb{\Pi}(\mathsf{M})$ if $\mathfrak{B}$ is the intersection of members of M, i.e., $\mathfrak{B} = \bigcap_{i\in I} \mathfrak{A}^i$, with $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$.

Again, it is not difficult to show that $\mathbb{\Pi}$ is a closure operator on the collection of $\mathbf{F}$-matrix families.

Lemma 1819 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then*

$$\mathbb{\Pi} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}(\mathbf{F})$.

Proof: To show inflationarity, notice that, trivially, for all $\mathfrak{A} \in \mathsf{M}$, $\mathfrak{A} = \bigcap\{\mathfrak{A}\}$, whence $\mathfrak{A} \in \mathbb{\Pi}(\mathsf{M})$.

Monotonicity is straightforward, since, if $\mathsf{M} \subseteq \mathsf{N}$ and $\mathfrak{A} \in \mathbb{\Pi}(\mathsf{M})$, then $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}^i$, with $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$, and, hence, $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}^i$, with $\mathfrak{A}^i \in \mathsf{N}$, for all $i \in I$. So $\mathfrak{A} \in \mathbb{\Pi}(\mathsf{N})$.

Finally, for transitivity, if $\mathfrak{A} \in \mathbb{II}(\mathbb{II}(\mathsf{M}))$, then $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}^i$, where $\mathfrak{A}^i \in \mathbb{II}(\mathsf{M})$, for all $i \in I$. Thus, for all $i \in I$, $\mathfrak{A}^i = \bigcap_{j\in J_i} \mathfrak{A}^{ij}$, where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $j \in J_i$. Therefore, we get

$$\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}^i = \bigcap_{i\in I} \bigcap_{j\in J_i} \mathfrak{A}^{ij},$$

where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I$, $j \in J_i$, and, hence, $\mathfrak{A} \in \mathbb{II}(\mathsf{M})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and M a class of $\mathbf{F}$-matrix families. Recall the closure system $C^{\mathsf{M}} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ on $\mathbf{F}$ generated by M. It is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by $\phi \in C^{\mathsf{M}}_\Sigma(\Phi)$ if and only if, for all $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathsf{M}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

$\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$ denotes the corresponding π-institution generated by M.

Now, given a π-institution $\mathcal{I}$, one can consider its matrix family models, i.e., those $\mathbf{F}$-matrix families $\mathfrak{A}$, such that

$$\mathcal{I} \leq \mathcal{I}^{\mathfrak{A}}.$$

Doing this for the specific π-institution $\mathcal{I}^{\mathsf{M}}$, generated by the class M of $\mathbf{F}$-matrix families, we consider the class $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$ of $\mathcal{I}^{\mathsf{M}}$-matrix families. Clearly, since, for every $\mathfrak{A} \in \mathsf{M}$, $C^{\mathsf{M}} \leq C^{\mathfrak{A}}$,

$$\mathsf{M} \subseteq \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}).$$

In the spirit of many classical problems in universal algebraic logic, the following question naturally arises:

> Characterize $\mathrm{MathFam}(\mathcal{I}^{\mathsf{M}})$, i.e., find a list of operators on classes of $\mathbf{F}$-matrix families so that, when applied to M consecutively, they generate the class $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$.

Our goal here is to show that the list of operators that are needed consists of $\mathbb{M}\mathbb{II}\mathbb{M}^{-1}$, i.e., that, given any class M of $\mathbf{F}$-matrix families, we have

$$\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\mathbb{II}\mathbb{M}^{-1}(\mathsf{M}).$$

We start by showing that applying each of the three operators to classes of matrix family models of a π-institution $\mathcal{I}$ always results in classes of the same character.

Proposition 1820 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

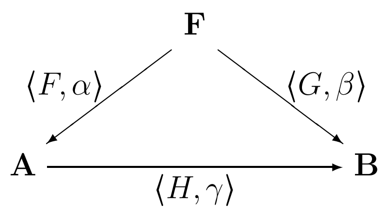
(a) $\mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I})$;

(b) $\mathbb{P}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I})$;

(c) $\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I})$.

Proof:

(a) Let $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}(\mathcal{I})$ and consider a surjective morphism $\langle H, \gamma\rangle : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{B} = \langle \mathcal{B}, T'\rangle$, as in the diagram.



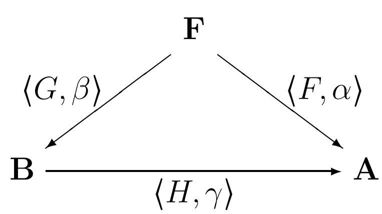
We now have

$$\beta^{-1}(T') = \alpha^{-1}(\gamma^{-1}(T')) = \alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}),$$

where the last membership follows by the hypothesis and Lemma 51. Thus, again by Lemma 51, we get that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\mathfrak{B} \in \mathrm{MatFam}(\mathcal{I})$.

(b) Suppose, next, that $\mathfrak{A}^i = \langle \mathcal{A}, T^i\rangle$, $i \in I$, are $\mathcal{I}$-matrix families. Then $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $i \in I$. Since the collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ forms a closure system on $\mathcal{A}$, it follows that $\bigcap_{i\in I} T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, we get that $\bigcap_{i\in I} \mathfrak{A}^i \in \mathrm{MatFam}(\mathcal{I})$. So $\mathrm{MatFam}(\mathcal{I})$ is closed under $\mathbb{P}$.

(c) Let $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}(\mathcal{I})$ and consider a surjective morphism $\langle H, \gamma\rangle : \mathfrak{B} \to \mathfrak{A}$, where $\mathfrak{B} = \langle \mathcal{B}, T'\rangle$, as in the diagram.



We now have

$$\beta^{-1}(T') = \beta^{-1}(\gamma^{-1}(T)) = \alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}),$$

where the last membership follows by the hypothesis and Lemma 51. Thus, again by Lemma 51, we get that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\mathfrak{B} \in \mathrm{MatFam}(\mathcal{I})$.

■

Proposition 1820 gives

Corollary 1821 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathbb{M}\Pi\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I}).$$

Proof: We have, using Proposition 1820,

$$\begin{array}{rcl}\mathbb{M}\Pi\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) & \subseteq & \mathbb{M}\Pi(\mathrm{MatFam}(\mathcal{I})) \\ & \subseteq & \mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}).\end{array}$$

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that a Lindenbaum $\mathcal{I}$-matrix family is an $\mathcal{I}$-matrix family of the form $\langle \mathcal{F}, T \rangle$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and $T \in \mathrm{ThFam}(\mathcal{I})$. We show, next, that the class of all $\mathcal{I}$-matrix families can be obtained by applying the $\mathbb{M}$ operator on the class of all Lindenbaum matrix families, i.e., $\mathrm{MatFam}(\mathcal{I}) = \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}))$.

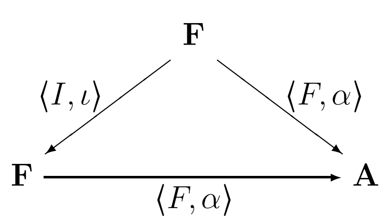
Lemma 1822 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathrm{MatFam}(\mathcal{I}) = \mathbb{M}(\mathrm{LMatFam}(\mathcal{I})).$$

Proof: First, observe that, since $\mathrm{LMatFam}(\mathcal{I}) \subseteq \mathrm{MatFam}(\mathcal{I})$, we have, by Proposition 1820,

$$\mathbb{M}(\mathrm{LMatFam}(\mathcal{I})) \subseteq \mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I}).$$

Suppose, conversely, that $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}(\mathcal{I})$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$. Then, we have, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Hence, $\langle \mathcal{F}, \alpha^{-1}(T) \rangle \in \mathrm{LMatFam}(\mathcal{I})$. Now, it suffices to consider the surjective morphism $\langle F, \alpha \rangle : \langle \mathcal{F}, \alpha^{-1}(T) \rangle \to \mathfrak{A}$



to conclude that $\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}))$. Therefore, we obtain $\mathrm{MatFam}(\mathcal{I}) \subseteq \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}))$. ■

Now, to complete our task, we turn again to the specific π-institution $\mathcal{I}^{\mathsf{M}}$, generated by a given class M of $\mathbf{F}$-matrix families. We show that all its Lindenbaum matrix families, i.e., all matrix families of the form $\langle \mathcal{F}, T \rangle$, where $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{M}})$, can be obtained by applying the operator $\Pi\mathbb{M}^{-1}$ on the class M.

Lemma 1823 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}).$$

Proof: Let $\mathfrak{F} = \langle \mathcal{F}, T \rangle \in \mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}})$, i.e., $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{M}})$. Thus, there exist $\mathfrak{A}^i = \langle \mathcal{A}^i, T^i \rangle \in \mathsf{M}$, with $\mathcal{A} = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, such that

$$T = \bigcap_{i \in I} (\alpha^i)^{-1}(T^i).$$

Consider the collection $\mathfrak{F}^i = \langle \mathcal{F}, (\alpha^i)^{-1}(T^i) \rangle$, $i \in I$. Taking into account the surjective morphisms $\langle F^i, \alpha^i \rangle : \mathfrak{F}^i \to \mathfrak{A}^i$, $i \in I$, and the fact that $\mathfrak{A}^i \in \mathsf{M}$, we conclude that $\mathfrak{F}^i \in \mathbb{M}^{-1}(\mathsf{M})$, for all $i \in I$. Finally, observing that $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}^i$, we get that $\mathfrak{F} \in \mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$. Therefore, $\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$. ∎

Now we are ready to provide the promised characterization of $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$ in terms of M and the class operators $\mathbb{M}$, $\mathbb{I}\mathbb{I}$ and $\mathbb{M}^{-1}$, introduced in this section.

Theorem 1824 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}).$$

Proof: First, since $\mathsf{M} \subseteq \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$, we have, using Corollary 1821,

$$\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}) \subseteq \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})) \subseteq \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}).$$

Conversely, let $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$. Then, by Lemmas 1822 and 1823,

$$\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}})) \subseteq \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}).$$

Therefore, $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$. ∎

As a consequence of this characterization, we can also show that the operator $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}$ is a closure operator on classes of $\mathbf{F}$-matrix families and, moreover, given any such class M, applying the operator to the class results in the smallest class of $\mathbf{F}$-matrix systems that contains M and is closed under the operations $\mathbb{M}$, $\mathbb{I}\mathbb{I}$ and $\mathbb{M}^{-1}$.

Theorem 1825 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of* $\mathbf{F}$*-matrix families.*

(a) $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$ *is a closure operator;*

(b) $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$ *is the smallest class of* $\mathbf{F}$*-matrix families containing* M *and closed under the operators* $\mathbb{M}$*,* $\mathbb{I}\mathbb{I}$ *and* $\mathbb{M}^{-1}$*.*

Proof:

(a) Inflationarity and monotonicity follow from the corresponding properties of the three operators, which were established in Lemmas 1818 and 1819. For idempotency, we have

$$\begin{array}{rcl} \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})) & = & \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})) \\ & & \text{(by Theorem 1824)} \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) \\ & & \text{(by Corollary 1821)} \\ & = & \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}). \\ & & \text{(again by Theorem 1824)} \end{array}$$

(b) By Part (a), $\mathsf{M} \subseteq \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$. Moreover, if $\mathbb{O} \in \{\mathbb{M}, \mathbb{I}\mathbb{I}, \mathbb{M}^{-1}\}$, then

$$\begin{array}{rcll} \mathbb{O}(\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})) & = & \mathbb{O}(\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})) & \text{(by Theorem 1824)} \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) & \text{(by Corollary 1821)} \\ & = & \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}). & \text{(by Theorem 1824)} \end{array}$$

Hence, $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$ is closed under all three operators. If N is a class of $\mathbf{F}$-matrix families such that $\mathsf{M} \subseteq \mathsf{N}$ and N closed under the three operators, then, clearly, $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}) \subseteq \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{N}) = \mathsf{N}$. Therefore, $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M})$ is the smallest class satisfying these properties.

■

24.3 Classes of Reduced Matrix Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathrm{LMatFam}^*(\mathcal{I})$ is the class of all reduced Lindenbaum $\mathcal{I}$-matrix families, i.e., all $\mathbf{F}$-matrix families of the form $\langle \mathcal{F}^{\Omega(T)}, T/\Omega(T) \rangle$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and $T \in \mathrm{ThFam}(\mathcal{I})$, and that $\mathcal{I}$ is complete with respect to $\mathrm{LMatFam}^*(\mathcal{I})$.

Recall, also, that $\mathrm{MatFam}^*(\mathcal{I})$ is the collection of all reduced $\mathcal{I}$-matrix families, i.e., $\mathbf{F}$-matrix families of the form $\langle \mathcal{A}, T \rangle$, where $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Moreover, $\mathcal{I}$ is also complete with respect to $\mathrm{MatFam}^*(\mathcal{I})$.

Our first goal is to show that the class $\mathrm{MatFam}^*(\mathcal{I})$ is, in fact, the class generated by applying the morphic image operator $\mathbb{M}$, introduced in the previous section, on the class $\mathrm{LMatFam}^*(\mathcal{I})$.

We prove, first, that the operator

$$\mathbb{M} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})),$$

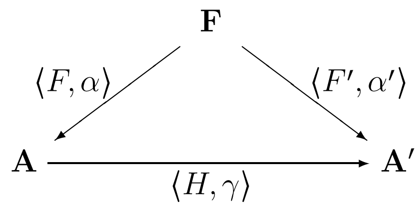
i.e., the operator $\mathbb{M}$, introduced in Section 24.2, restricted to reduced $\mathbf{F}$-matrix families, is also a closure operator.

Proposition 1826 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{M} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}^*(\mathbf{F})$.

Proof: Since we know, by Lemma 1818, that $\mathbb{M}$ is inflationary, monotone and idempotent, it suffices to show that it is well-defined, i.e., that, when applied to collections of reduced $\mathbf{F}$-matrix families, it produces collections of the same kind. In turn, it suffices to show that, given a reduced $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, an $\mathbf{F}$-matrix family $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, and a strict surjective morphism $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$, then $\mathfrak{A}'$ is also reduced.



Taking into account the surjectivity of $\langle F', \alpha' \rangle$, we reason as follows. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$\begin{array}{rl} \multicolumn{2}{l}{\langle \alpha'_\Sigma(\phi), \alpha'_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}'}_{F'(\Sigma)}(T')} \\ \text{iff} & \langle \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)), \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi)) \rangle \in \Omega^{\mathcal{A}'}_{G(F(\Sigma))}(T') \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \gamma^{-1}_{F(\Sigma)}(\Omega^{\mathcal{A}'}_{G(F(\Sigma))}(T')) \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(\gamma^{-1}(T')) \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Delta^{\mathcal{A}}_{F(\Sigma)}(T) \\ \text{iff} & \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi) \\ \text{implies} & \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi)) \\ \text{iff} & \alpha'_\Sigma(\phi) = \alpha'_\Sigma(\psi). \end{array}$$

Therefore $\Omega^{\mathcal{A}'}(T') = \Delta^{\mathcal{A}'}$ and, hence $\mathfrak{A}'$ is also reduced. ■

Next, we show that, given π-institution $\mathcal{I}$, the class $\mathrm{MatFam}^*(\mathcal{I})$ is obtained by applying the operator $\mathbb{M}$ on the class $\mathrm{LMatFam}^*(\mathcal{I})$.

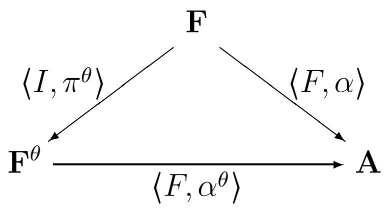
Proposition 1827 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$\mathrm{MatFam}^*(\mathcal{I}) = \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I})).$$

Proof: The inclusion $\mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$ is obtained by observing that $\mathrm{LMatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}^*(\mathcal{I})$ and applying $\mathbb{M}$:

$$\begin{array}{rcll} \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I})) & \subseteq & \mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) & \text{(Lemma 1818)} \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}). & \text{(Propositions 1820 and 1826)} \end{array}$$

Suppose, conversely, that $\mathfrak{A} = \langle \mathcal{A}, T\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, is a reduced $\mathcal{I}$-matrix family. Let $\theta = \mathrm{Ker}(\langle F, \alpha\rangle)$ and consider the commutative diagram



where, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha^\theta_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\phi).$$

It now suffices to show that $\mathfrak{F} := \langle \mathcal{F}^\theta, \alpha^{-1}(T)/\theta\rangle \in \mathrm{LMatFam}^*(\mathcal{I})$. First, note that since $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}(\mathcal{I})$, then

$$\mathfrak{F} \in \mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I}),$$

by Proposition 1820. So it suffices to show that $\Omega^{\mathcal{F}^\theta}(\alpha^{-1}(T)/\theta) = \Delta^{\mathcal{F}^\theta}$. We have

$$\begin{array}{rcl}\Omega^{\mathcal{F}^\theta}(\alpha^{-1}(T)/\theta) & = & \Omega^{\mathcal{F}^\theta}((\alpha^\theta)^{-1}(T))\\ & = & (\alpha^\theta)^{-1}(\Omega^{\mathcal{A}}(T))\\ & = & (\alpha^\theta)^{-1}(\Delta^{\mathcal{A}})\\ & = & \mathrm{Ker}(\langle F, \alpha^\theta\rangle) = \Delta^{\mathcal{F}^\theta}.\end{array}$$

Now we conclude that $\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I}))$. ■

Consider, again, a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and a collection M of reduce $\mathbf{F}$-matrix families. We pose now a problem similar to that posed in Section 24.2, but for classes of reduced matrix families.

> Characterize the class $\mathrm{MatFam}^*(\mathcal{I})$, i.e., find a list of operators on classes of reduced $\mathbf{F}$-matrix families so that, when applied to M consecutively, they generate the class $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$.

Unlike the operator $\mathbb{M}$ that, when applied to reduced matrix families yields reduced matrix families, the other two operators that we considered in Section 24.2, namely $\mathbb{\Pi}$ and $\mathbb{M}^{-1}$, do not share this property. So to "localize" them to reduced matrix families, we must take the output classes of $\mathbf{F}$-matrix families that they produce and "reduce" them so that the output produced becomes a collection of reduced $\mathbf{F}$-matrix families. According to this scheme, we consider the following operators, induced by the operators $\mathbb{\Pi}$ and $\mathbb{M}^{-1}$ on class of matrix families, introduced in Section 24.2.

- $\mathbb{\Pi}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$ is given, by setting, for all $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,

$$\mathbb{\Pi}^*(\mathsf{M}) = \{\mathfrak{A}^* : \mathfrak{A} \in \mathbb{\Pi}(\mathsf{M})\};$$

- $\mathbb{M}^{-1*} : \mathcal{P}(\text{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\text{MatFam}^*(\mathbf{F}))$ is given, by setting, for all $\mathsf{M} \subseteq \text{MatFam}^*(\mathbf{F})$,

$$\mathbb{M}^{-1*}(\mathsf{M}) = \{\mathfrak{A}^* : \mathfrak{A} \in \mathbb{M}^{-1}(\mathsf{M})\}.$$

It is not very difficult to prove that both $\mathbb{P}\mathbb{I}^*$ and $\mathbb{M}^{-1*}$ are closure operators on the class of reduced $\mathbf{F}$-matrix families.

Proposition 1828 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{P}\mathbb{I}^* : \mathcal{P}(\text{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\text{MatFam}^*(\mathbf{F}))$$

is a closure operator on $\text{MatFam}^*(\mathbf{F})$.

Proof: Let $\mathsf{M} \subseteq \text{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathsf{M}$. Then, by Proposition 1819, $\mathfrak{A} \in \mathbb{P}\mathbb{I}(\mathsf{M})$ and, as $\mathfrak{A}$ is reduced, we get $\mathfrak{A} \in \mathbb{P}\mathbb{I}^*(\mathsf{M})$. Thus, $\mathbb{P}\mathbb{I}^*$ is inflationary.

Suppose, next, that $\mathsf{M} \subseteq \mathsf{N} \subseteq \text{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathbb{P}\mathbb{I}^*(\mathsf{M})$. Then $\mathfrak{A} = (\bigcap_{i\in I} \mathfrak{A}^i)^*$, with $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$. But then, since $\mathsf{M} \subseteq \mathsf{N}$, $\mathfrak{A} = (\bigcap_{i\in I} \mathfrak{A}^i)^*$, with $\mathfrak{A}^i \in \mathsf{N}$, for all $i \in I$, and, hence, $\mathfrak{A} \in \mathbb{P}\mathbb{I}^*(\mathsf{N})$. Therefore $\mathbb{P}\mathbb{I}^*$ is also monotone.

Suppose, finally, that $\mathsf{M} \subseteq \text{MatFam}^*(\mathbf{F})$ and that $\mathfrak{A} \in \mathbb{P}\mathbb{I}^*(\mathbb{P}\mathbb{I}^*(\mathsf{M}))$. Then $\mathfrak{A} = (\bigcap_{i\in I} \mathfrak{A}^i)^*$, where $\mathfrak{A}^i \in \mathbb{P}\mathbb{I}^*(\mathsf{M})$. Hence, for all $i \in I$, $\mathfrak{A}^i = (\bigcap_{j\in J_i} \mathfrak{A}^{ij})^*$, where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I$ and all $j \in J_i$. Now note the following:

- For every $i \in I$, for $\bigcap_{j\in J_i} \mathfrak{A}^{ij}$ to be defined, we must have $\mathfrak{A}^{ij} = \langle \mathcal{A}^i, T^{ij} \rangle$, for all $j \in J_i$.
- For $\bigcap_{i\in I} \mathcal{A}^i = \bigcap_{i\in I}(\bigcap_{j\in J_i} \mathfrak{A}^{ij})^*$ to be defined, we must have, for all $i \in I$, $\mathcal{A}^i = \mathcal{A}$, for some $\mathbf{F}$-algebraic system $\mathcal{A}$, and, moreover, for all $i \in I$, $\Omega^{\mathcal{A}}(\bigcap_{j\in J_i} T^{ij}) = \theta$, for some $\theta \in \text{ConSys}(\mathcal{A})$.

Under these restrictions, it is easy to show that

$$\langle I, \pi \rangle : \langle \mathcal{A}, \bigcap_{i\in I} \bigcap_{j\in J_i} T^{ij} \rangle \rangle \to \mathcal{A}^\theta / \Omega^{\mathcal{A}^\theta}(\bigcap_{i\in I}((\bigcap_{j\in J_i} T^{ij})/\theta))$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \text{SEN}(\Sigma)$, by

$$\pi_\Sigma(\phi) = (\phi/\theta_\Sigma)/\Omega_\Sigma^{\mathcal{A}^\theta}(\bigcap_{i\in I}((\bigcap_{j\in J_i} T^{ij})/\theta)),$$

is a strict surjective matrix morphism, with kernel

$$\text{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(\bigcap_{i\in I} \bigcap_{j\in J_i} T^{ij}).$$

Therefore, we get an isomorphism

$$\mathfrak{A}/\Omega^{\mathcal{A}}(\bigcap_{i\in I} \bigcap_{j\in J_i} T^{ij}) \cong (\mathfrak{A}^\theta)(\Omega^{\mathcal{A}^\theta}(\bigcap_{i\in I}((\bigcap_{j\in J_i} T^{ij})/\theta))).$$

We conclude that $\mathfrak{A} \in \mathbb{P}\mathbb{I}^*(\mathsf{M})$ and, therefore, $\mathbb{P}\mathbb{I}^*$ is also idempotent. ■

To show that $\mathbb{M}^{-1*}$ is a closure operator, we employ a lemma to the effect that, given a class M of reduced $\mathbf{F}$-matrix families, $\mathbb{M}^{-1*}(\mathsf{M}) \subseteq \mathbb{M}^{-1}(\mathsf{M})$.

Lemma 1829 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. For every* $\mathbf{F}$*-matrix family* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$*, every reduced* $\mathbf{F}$*-matrix family* $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ *and strict surjective morphism* $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$*, there exists a strict surjective morphism* $\langle H, \gamma^* \rangle : \mathfrak{A}^* \to \mathfrak{A}'$*, such that the following triangle commutes,*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle H, \gamma \rangle} & \mathfrak{A}' \\ & {\scriptstyle \langle I, \pi \rangle} \searrow \quad \nearrow {\scriptstyle \langle H, \gamma^* \rangle} & \\ & \mathfrak{A}^* & \end{array}$$

where $\langle I, \pi \rangle : \mathfrak{A} \to \mathfrak{A}^*$ *is the quotient morphism.*

Proof: We define $\gamma^* : \mathrm{SEN}^* \to \mathrm{SEN}' \circ H$ by setting, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\gamma^*_\Sigma(\phi^*) = \gamma_\Sigma(\phi).$$

This makes sense, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi^* = \psi^*$, we have $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T) = \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))$, whence $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$ and, hence, $\langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \Delta^{\mathcal{A}'}_{H(\Sigma)}$, i.e., $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$.

Moreover, $\gamma : \mathrm{SEN}^* \to \mathrm{SEN} \circ H$ is a natural transformation, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc} \mathrm{SEN}^*(\Sigma) & \xrightarrow{\gamma^*_\Sigma} & \mathrm{SEN}'(H(\Sigma)) \\ {\scriptstyle \mathrm{SEN}^*(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{SEN}'(H(f))} \\ \mathrm{SEN}^*(\Sigma') & \xrightarrow[\gamma^*_{\Sigma'}]{} & \mathrm{SEN}'(H(\Sigma')) \end{array}$$

$$\begin{array}{rcl} \mathrm{SEN}'(H(f))(\gamma^*_\Sigma(\phi^*)) & = & \mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)) \\ & = & \gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\ & = & \gamma^*_{\Sigma'}(\mathrm{SEN}(f)(\phi)^*) \\ & = & \gamma^*_{\Sigma'}(\mathrm{SEN}^*(g)(\phi^*)). \end{array}$$

Further, the triangle commutes, by the definition of $\langle H, \gamma^* \rangle$ and, finally, $\langle H, \gamma^* \rangle : \mathfrak{A}^* \to \mathfrak{A}'$ is strict since $\pi^{-1}((\gamma^*)^{-1}(T')) = \gamma^{-1}(T') = T$ and, therefore, $(\gamma^*)^{-1}(T') = \pi(T) = T^*$. ■

Now, we show $\mathbb{M}^{-1*}$ is a closure operator.

Proposition 1830 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{M}^{-1*} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}^*(\mathbf{F})$*.*

Proof: Suppose, first, that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathsf{M}$. Then, we have, by Proposition 1818, $\mathfrak{A} \in \mathbb{M}^{-1}(\mathsf{M})$ and, since $\mathfrak{A}$ is reduced, we get $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{M})$. So $\mathbb{M}^{-1*}$ is inflationary.

Suppose, next, that $\mathsf{M} \subseteq \mathsf{N} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{M})$. Then $\mathfrak{A} = \mathfrak{B}^*$, with $\mathfrak{B} \in \mathbb{M}^{-1}(\mathsf{M})$. Thus, by Proposition 1818, we get $\mathfrak{A} = \mathfrak{B}^*$, with $\mathfrak{B} \in \mathbb{M}^{-1}(\mathsf{N})$. We conclude that $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{N})$ and, therefore, $\mathbb{M}^{-1*}$ is also monotone.

Finally, suppose that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and that $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathbb{M}^{-1*}(\mathsf{M}))$. Then, using Lemma 1829, we get

$$\mathfrak{A} \in \mathbb{M}^{-1*}(\mathbb{M}^{-1*}(\mathsf{M})) \subseteq \mathbb{M}^{-1}(\mathbb{M}^{-1*}(\mathsf{M})) \subseteq \mathbb{M}^{-1}(\mathbb{M}^{-1}(\mathsf{M})) \subseteq \mathbb{M}^{-1}(\mathsf{M}),$$

and, since $\mathfrak{A}$ is reduced, we get $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{M})$. Therefore $\mathbb{M}^{-1*}$ is also idempotent. ∎

We need one more operator on reduced classes of $\mathbf{F}$-matrix families.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. We define

$$\overleftarrow{\mathbb{\Pi}}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$

by setting, for all $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,

$$\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}) = (\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}))^*.$$

Note that this operator dominates both $\mathbb{\Pi}^*$ and $\mathbb{M}^{-1*}$.

Proposition 1831 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then, for all* $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,

$$\mathbb{\Pi}^*(\mathsf{M}) \subseteq \overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}) \quad \textit{and} \quad \mathbb{M}^{-1*}(\mathsf{M}) \subseteq \overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}).$$

Proof: The proofs of both statements are parallel. We have

$$\begin{array}{rclcrcl}
\mathbb{\Pi}^*(\mathsf{M}) & = & (\mathbb{\Pi}(\mathsf{M}))^* & \qquad & \mathbb{M}^{-1}(\mathsf{M}) & = & (\mathbb{M}^{-1}(\mathsf{M}))^* \\
& \subseteq & (\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}))^* & & & \subseteq & (\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}))^* \\
& = & \overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}) & & & = & \overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})
\end{array}$$

where the inclusions follow from Lemmas 1818 and 1819, respectively. ∎

Our next goal is to show that the list of operators that are needed to obtain the class of all reduced $\mathcal{I}^{\mathsf{M}}$-matrix families from a class M of reduced $\mathbf{F}$-matrix families generating a closure operator C^{M} (of a π-institution $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$) consists of $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*$, i.e., that, given any class M of reduced $\mathbf{F}$-matrix families, we have

$$\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}).$$

We start by showing that applying each of these operators to classes of reduced matrix family models of a π-institution $\mathcal{I}$ always results in classes of the same character. This forms an analog of Proposition 1820.

Proposition 1832 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$*;*

(b) $\overleftarrow{\mathbb{I}\!\Pi}^*(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$*.*

Proof:

(a) We have

$$\begin{array}{rcl}\mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) & \subseteq & \mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \cap \mathbb{M}(\mathrm{MatFam}^*(\mathbf{F})) \\ & & \text{(Proposition 1818)} \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}) \cap \mathrm{MatFam}^*(\mathbf{F}) \\ & & \text{(Propositions 1820 and 1826)} \\ & = & \mathrm{MatFam}^*(\mathcal{I}). \quad \text{(Definition)}\end{array}$$

(b) Similarly,

$$\begin{array}{rcl}\overleftarrow{\mathbb{I}\!\Pi}^*(\mathrm{MatFam}^*(\mathcal{I})) & = & (\mathbb{I}\!\Pi\mathbb{M}^{-1}(\mathrm{MatFam}^*(\mathcal{I})))^* \\ & \subseteq & (\mathbb{I}\!\Pi\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})))^* \\ & & \text{(Lemmas 1818 and 1819)} \\ & \subseteq & (\mathrm{MatFam}(\mathcal{I}))^* \\ & & \text{(Proposition 1820)} \\ & = & \mathrm{MatFam}^*(\mathcal{I}).\end{array}$$

■

Proposition 1832, together with Proposition 1820, give the following

Corollary 1833 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathbb{M}\overleftarrow{\mathbb{I}\!\Pi}^*(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I}).$$

Proof: We have, using Propositions 1820 and 1832,

$$\begin{array}{rcl}\mathbb{M}\overleftarrow{\mathbb{I}\!\Pi}^*(\mathrm{MatFam}^*(\mathcal{I})) & \subseteq & \mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}).\end{array}$$

■

In order to establish our final result, we must show that, given a class M of reduced $\mathbf{F}$-matrix families, all reduced Lindenbaum matrix families of the π-institution $\mathcal{I}^{\mathsf{M}}$, i.e., all matrix families of the form $\langle \mathcal{F}/\Omega(T), T/\Omega(T) \rangle$, where $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{M}})$, can be obtained by applying the operator $\overleftarrow{\mathbb{I}\!\Pi}^*$ on the class M.

Lemma 1834 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of reduced* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{LMatFam}^*(\mathcal{I}^{\mathsf{M}}) \subseteq \overleftarrow{\mathbb{P}}^*(\mathsf{M}).$$

Proof: we have

$$\begin{array}{lcll}\mathrm{LMatFam}^*(\mathcal{I}^{\mathsf{M}}) & = & (\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}}))^* & \text{(Definition)}\\ & \subseteq & (\mathbb{P}\mathbb{M}^{-1}(M))^* & \text{(Lemma 1823)}\\ & = & \overleftarrow{\mathbb{P}}^*(\mathsf{M}). & \text{(Definition)}\end{array}$$

■

Now we provide the promised characterization of $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$ in terms of the class M of reduced $\mathbf{F}$-matrix families and the class operators $\mathbb{M}$ and $\overleftarrow{\mathbb{P}}^*$.

Theorem 1835 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of reduced* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathsf{M}).$$

Proof: First, since $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$, we have, using Corollary 1833,

$$\mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathsf{M}) \subseteq \mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})) \subseteq \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}).$$

Conversely, let $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$. Then, by Proposition 1827 and Lemma 1834,

$$\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I}^{\mathsf{M}})) \subseteq \mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathsf{M}).$$

Therefore, $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathsf{M})$, and equality follows. ■

As a consequence of this characterization, we can also show that the operator $\mathbb{M}\overleftarrow{\mathbb{P}}^*$ is a closure operator on classes of reduced $\mathbf{F}$-matrix families and, moreover, given any such class M, applying the operator to the class results in the smallest class of reduced $\mathbf{F}$-matrix systems that contains M and is closed under the operations $\mathbb{M}$, $\overleftarrow{\mathbb{P}}^*$.

Theorem 1836 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of reduced* $\mathbf{F}$*-matrix families.*

(a) $\mathbb{M}\overleftarrow{\mathbb{P}}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$ *is a closure operator;*

(b) $\mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathsf{M})$ *is the smallest class of* $\mathbf{F}$*-matrix families containing* M *and closed under the operators* $\mathbb{M}$ *and* $\overleftarrow{\mathbb{P}}^*$*.*

Proof:

(a) Inflationarity and monotonicity follow from the corresponding properties of the operators $\mathbb{M}$ and $\mathbb{\Pi}$, which were established in Lemmas 1818 and 1819. For idempotency, we have

$$\begin{array}{rcl}\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})) & = & \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})) \\ & & \text{(by Theorem 1835)} \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) \\ & & \text{(by Corollary 1833)} \\ & = & \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}). \\ & & \text{(again by Theorem 1835)}\end{array}$$

(b) By Part (a), $\mathsf{M} \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$. Moreover, if $\mathbb{O} \in \{\mathbb{M}, \overleftarrow{\mathbb{\Pi}}^*\}$, then

$$\begin{array}{rcl}\mathbb{O}(\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})) & = & \mathbb{O}(\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})) \quad \text{(by Theorem 1835)} \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) \quad \text{(by Corollary 1833)} \\ & = & \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}). \quad \text{(by Theorem 1835)}\end{array}$$

Hence, $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$ is closed under both operators. If N is a class of reduced $\mathbf{F}$-matrix families such that $\mathsf{M} \subseteq \mathsf{N}$ and N closed under both operators, then, clearly, $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}) \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{N}) = \mathsf{N}$. Therefore, $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$ is the smallest class satisfying these properties.

■

24.4 Protoclasses of Matrix Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. A class of $\mathbf{F}$-matrix families M is called a **protoclass** if it is the class of all reduced $\mathcal{I}$-matrix families for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ based on $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$, $i \in I$, a collection of $\mathbf{F}$-algebraic systems and $\mathfrak{A}^i = \langle \mathcal{A}^i, T^i\rangle$ a collection of $\mathbf{F}$-matrix families. We say that an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, is a **subdirect intersection** of the collection $\mathfrak{A}^i$, $i \in I$, if there exist surjective morphisms

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathfrak{A}^i, \quad i \in I,$$

such that:

- $T = \bigcap_{i\in I}(\gamma^i)^{-1}(T^i)$;
- $\bigcap_{i\in I}\mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}$.

Let M be a class of $\mathbf{F}$-matrix families. Given an $\mathbf{F}$-matrix family $\mathfrak{A}$, we write $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{M})$ to denote the fact that $\mathfrak{A}$ is a subdirect intersection of members of M.

It is not difficult to see that $\overset{\triangleleft}{\mathbb{I}\Pi}$ is a closure operator on classes of $\mathbf{F}$-matrix families.

Lemma 1837 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\overset{\triangleleft}{\mathbb{I}\Pi} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}(\mathbf{F})$*.*

Proof: Assume, first, that $\mathsf{M} \subseteq \mathrm{MatFam}(\mathbf{F})$ and $\mathfrak{A} \in \mathsf{M}$. Then $\langle I, \iota \rangle : \mathfrak{A} \to \mathfrak{A}$ is a subdirect intersection morphism and, therefore, since $\mathfrak{A} \in \mathsf{M}$, we get $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{M})$. Therefore $\overset{\triangleleft}{\mathbb{I}\Pi}$ is inflationary.

Suppose, next, that $\mathsf{M} \subseteq \mathsf{N} \subseteq \mathrm{MatFam}(\mathbf{F})$. Let $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$, $i \in I$, be a collection of subdirect intersection morphisms, with $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$. Since, then, $\mathfrak{A}'^i \in \mathsf{N}$, for all $i \in I$, the same collection of morphisms witnesses that $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{N})$. Therefore, $\overset{\triangleleft}{\mathbb{I}\Pi}$ is also monotone.

Finally, assume that $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{M}))$, where $\mathsf{M} \subseteq \mathrm{MatFam}(\mathbf{F})$. Thus, there exists a collection of subdirect intersection morphisms

$$\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^i, \quad i \in I,$$

where $\mathfrak{A}^i \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{M})$, for all $i \in I$. It now follows that, for each $i \in I$, there exists a collection of subdirect intersection morphisms

$$\langle H^{ij}, \gamma^{ij} \rangle : \mathfrak{A}^i \to \mathfrak{A}^{ij}, \quad j \in J_i,$$

where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I$ and all $j \in J_i$. We look at the collection

$$\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^{ij}, \quad i \in I, j \in J_i,$$

with $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I, j \in J_i$. We have

- For filter family intersections,

$$\begin{array}{rcl}\bigcap_{i \in I} \bigcap_{j \in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(T^{ij})) & = & \bigcap_{i \in I} (\gamma^i)^{-1}(\bigcap_{j \in J_i} (\gamma^{ij})^{-1}(T^{ij})) \\ & = & \bigcap_{i \in I} (\gamma^i)^{-1}(T^i) \\ & = & T.\end{array}$$

- Similarly, for kernels,

$$\begin{array}{l}\bigcap_{i \in I} \bigcap_{j \in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle) \\ \quad = \bigcap_{i \in I} \bigcap_{j \in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(\Delta^{\mathcal{A}^{ij}})) \\ \quad = \bigcap_{i \in I} (\gamma^i)^{-1}(\bigcap_{j \in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle)) \\ \quad = \bigcap_{i \in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\ \quad = \bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) \\ \quad = \Delta^{\mathcal{A}}.\end{array}$$

Therefore, $\langle H^{ij},\gamma^{ij}\rangle \circ \langle H^i,\gamma^i\rangle$, $i \in I$, $j \in J_i$, is also a collection of subdirect intersection morphisms, and, hence $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}}(\mathsf{M})$. We conclude that $\overset{\triangleleft}{\mathbb{I}}$ is also idempotent. ■

In general, given a class M of reduced $\mathbf{F}$-matrix families, its closures under both operators $\mathbb{I}$ and $\mathbb{M}^{-1*}$ are included in its closure under $\overset{\triangleleft}{\mathbb{I}}$.

Proposition 1838 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* M *a class of reduced* $\mathbf{F}$*-matrix families. Then*

$$\mathbb{I}(\mathsf{M}) \subseteq \overset{\triangleleft}{\mathbb{I}}(\mathsf{M}) \quad \textit{and} \quad \mathbb{M}^{-1*}(\mathsf{M}) \subseteq \overset{\triangleleft}{\mathbb{I}}(\mathsf{M}).$$

Proof: Assume, first, that $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathbb{I}(\mathsf{M})$. Thus, there exists a collection $\mathfrak{A}^i = \langle \mathcal{A}, T^i\rangle \in \mathsf{M}$, such that

$$T = \bigcap_{i\in I} T^i.$$

Consider the family of surjective morphisms

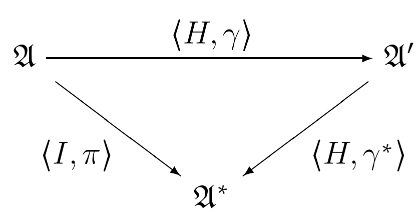
$$\langle I,\iota\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{A}, T^i\rangle, \quad i \in I.$$

We have

- $T = \bigcap_{i\in I} T^i = \bigcap_{i\in I} \iota^{-1}(T^i)$, by hypothesis;
- $\bigcap_{i\in I} \mathrm{Ker}(\langle I,\iota\rangle) = \bigcap_{i\in I} \Delta^{\mathcal{A}} = \Delta^{\mathcal{A}}$.

Therefore, since $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$, $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}}(\mathsf{M})$.

Assume, next, that $\mathfrak{A}^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \mathbb{M}^{-1*}(\mathsf{M})$, where $\langle H,\gamma\rangle : \mathfrak{A} \to \mathfrak{A}'$ is a strict surjective morphism, with $\mathfrak{A}' = \langle \mathcal{A}', T'\rangle \in \mathsf{M}$. Since $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$, there exists a factorization



Moreover, we have

- $\pi^{-1}(\gamma^{*-1}(T')) = \gamma^{-1}(T') = T$, whence $\gamma^{*-1}(T') = T/\Omega^{\mathcal{A}}(T)$;
- $\mathrm{Ker}(\langle H,\gamma^*\rangle) = \Delta^{\mathcal{A}^*}$ holds, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\langle H,\gamma^*\rangle) & \text{iff} & \gamma^*_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(T)) = \gamma^*_\Sigma(\psi/\Omega^{\mathcal{A}}_\Sigma(T)) \\ & \text{iff} & \gamma_\Sigma(\phi) = \gamma_\Sigma(\psi) \\ & \text{iff} & \langle\phi,\psi\rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')) \\ & \text{iff} & \langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')) \\ & \text{iff} & \langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T).\end{array}$$

Therefore, $\mathfrak{A}^* \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{M})$. ■

Another useful feature of the operator $\overset{\triangleleft}{\mathbb{\Pi}}$ is that among model classes of matrix families, it characterizes those that are protoclasses.

Theorem 1839 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{M} = \mathrm{MatFam}^*(\mathcal{I})$ *a class of reduced* $\mathbf{F}$*-matrix families. Then* M *is a protoclass if and only if* $\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{M}) \subseteq \mathsf{M}$.

Proof: Suppose, first, that $\mathsf{M} = \mathrm{MatFam}^*(\mathcal{I})$, with $\mathcal{I}$ protoalgebraic and let $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$, $i \in I$, be a collection of subdirect intersection morphisms. Then, clearly,

$$\begin{array}{lll} \mathfrak{A} & \in & \mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}) \quad (\text{Definition of } \overset{\triangleleft}{\mathbb{\Pi}}) \\ & \subseteq & \mathbb{\Pi}\mathbb{M}^{-1}(\mathrm{MatFam}^*(\mathcal{I})) \quad (\text{Lemmas 1818 and 1819}) \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}). \quad (\text{Proposition 1820}) \end{array}$$

It suffices now to show that $\mathcal{A}$ is reduced. We have

$$\begin{array}{lll} \Omega^{\mathcal{A}}(T) & = & \Omega^{\mathcal{A}}(\bigcap_{i \in I} (\gamma^i)^{-1}(T'^i)) \quad (\text{Subdirect Intersection}) \\ & = & \bigcap_{i \in I} \Omega^{\mathcal{A}}((\gamma^i)^{-1}(T'^i)) \quad (\mathcal{I} \text{ protoalgebraic}) \\ & = & \bigcap_{i \in I} (\gamma^i)^{-1}(\Omega^{\mathcal{A}'^i}(T'^i)) \quad (\langle H^i, \gamma^i \rangle \text{ surjective}) \\ & = & \bigcap_{i \in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}'^i}) \quad (\mathfrak{A}'^i \text{ reduced}) \\ & = & \bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) \\ & = & \Delta^{\mathcal{A}}. \quad (\text{Subdirect Intersection}) \end{array}$$

Since $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I})$ and $\mathfrak{A}$ is reduced, we conclude that $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I})$. So $\overset{\triangleleft}{\mathbb{\Pi}}(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$.

Suppose, conversely, that $\overset{\triangleleft}{\mathbb{\Pi}}(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$ and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, with $T \leq T'$. We set

$$\mathfrak{F} \coloneqq \langle \mathcal{F}/(\Omega(T) \cap \Omega(T')), (T \cap T')/(\Omega(T) \cap \Omega(T')) \rangle$$

and consider the surjective natural projection morphisms

$$\begin{array}{ll} \langle I, \pi \rangle : & \mathfrak{F} \to \langle \mathcal{F}/\Omega(T), T/\Omega(T) \rangle \\ \langle I, \pi' \rangle : & \mathfrak{F} \to \langle \mathcal{F}/\Omega(T'), T'/\Omega(T') \rangle. \end{array}$$

We observe that

- As far as filter families, we have

$$\begin{array}{l} (T \cap T')/(\Omega(T) \cap \Omega(T')) \\ = T/(\Omega(T) \cap \Omega(T')) \cap T'/(\Omega(T) \cap \Omega(T')) \\ = \pi^{-1}(T/\Omega(T')) \cap \pi'^{-1}(T'/\Omega(T')); \end{array}$$

- As far as kernels, we get

$$\begin{aligned}&\mathrm{Ker}(\langle I,\pi\rangle)\cap\mathrm{Ker}(\langle I,\pi'\rangle)\\ &=\Omega(T)/(\Omega(T)\cap\Omega(T'))\cap\Omega(T')/(\Omega(T)\cap\Omega(T'))\\ &=(\Omega(T)\cap\Omega(T'))/(\Omega(T)\cap\Omega(T'))=\Delta^{\mathfrak{F}}.\end{aligned}$$

Therefore,

$$\mathfrak{F}\in\overset{\triangleleft}{\mathbb{P}}(\mathrm{MatFam}^*(\mathcal{I}))\subseteq\mathrm{MatFam}^*(\mathcal{I}).$$

Hence $\Omega(T)=\Omega(T\cap T')=\Omega(T)\cap\Omega(T')$, which implies that $\Omega(T)\leq\Omega(T')$. Thus, Ω is monotone on theory families and, hence, $\mathcal{I}$ is protoalgebraic. ■

Finally, we work to obtain expressions for the protoclass $\mathrm{MatFam}^*(\mathcal{I})$ based on a reduced class M of generating $\mathbf{F}$-matrix families for $\mathcal{I}$.

We show, first, that if M is a class of reduced models of a protoalgebraic π-institution, then its closure under $\overset{\triangleleft}{\mathbb{P}}$ is included in its closure under $\overleftarrow{\mathbb{P}}^*$.

Proposition 1840 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a protoalgebraic π-institution based on $\mathbf{F}$ and $\mathsf{M}\subseteq\mathrm{MatFam}^*(\mathcal{I})$ a class of reduced $\mathcal{I}$-matrix families. Then*

$$\overset{\triangleleft}{\mathbb{P}}(\mathsf{M})\subseteq\overleftarrow{\mathbb{P}}^*(\mathsf{M}).$$

Proof: Let $\mathfrak{A}=\langle\mathcal{A},T\rangle\in\overset{\triangleleft}{\mathbb{P}}(\mathsf{M})$. Then, there exists a collection of subdirect intersection morphisms

$$\langle H^i,\gamma^i\rangle:\langle\mathcal{A},T\rangle\to\langle\mathcal{A}'^i,T'^i\rangle,\quad i\in I,$$

where $\mathfrak{A}'^i=\langle\mathcal{A}'^i,T'^i\rangle\in\mathsf{M}$, for all $i\in I$. By using the same morphisms,

$$\langle H^i,\gamma^i\rangle:\langle\mathcal{A},(\gamma^i)^{-1}(T'^i)\rangle\to\mathfrak{A}'^i,\quad i\in I,$$

which have now become strict and surjective, we get that, for all $i\in I$, $\langle\mathcal{A},(\gamma^i)^{-1}(T'^i)\rangle\in\mathbb{M}^{-1}(\mathsf{M})$. Moreover, since, by the definition of a subdirect intersection, $\mathfrak{A}=\langle\mathcal{A},T\rangle=\langle\mathcal{A},\bigcap_{i\in I}(\gamma^i)^{-1}(T'^i)\rangle$, we get that $\mathfrak{A}\in\mathbb{P}\mathbb{M}^{-1}(\mathsf{M})$. Now, by Theorem 1839, $\mathfrak{A}$ is reduced, whence $\mathfrak{A}\in(\mathbb{P}\mathbb{M}^{-1}(\mathsf{M}))^*=\overleftarrow{\mathbb{P}}^*(\mathsf{M})$. ■

Next, it is shown that, if M is a class of reduced models of a protoalgebraic π-institution, then its closure under $\overleftarrow{\mathbb{P}}^*$ is included in its closure under the operator $\overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}$.

Proposition 1841 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a protoalgebraic π-institution based on $\mathbf{F}$ and $\mathsf{M}\subseteq\mathrm{MatFam}^*(\mathcal{I})$ a class of reduced $\mathcal{I}$-matrix families. Then*

$$\overleftarrow{\mathbb{P}}^*(\mathsf{M})\subseteq\overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}(\mathsf{M}).$$

Proof: Suppose that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and let $\mathfrak{A}^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \overleftarrow{\mathbb{P}}^*(\mathsf{M})$, where $\mathfrak{A} = \langle \mathcal{A}, T\rangle = \langle \mathcal{A}, \bigcap_{i\in I} T^i\rangle$ is such that there exist strict surjective morphisms

$$\langle H^i, \gamma^i\rangle : \langle \mathcal{A}, T^i\rangle \to \langle \mathcal{A}'^i, T'^i\rangle, \quad i \in I,$$

with $\mathfrak{A}'^i = \langle \mathcal{A}'^i, T'^i\rangle \in \mathsf{M}$, for all $i \in I$. The key now is to look at the collection of the projection morphisms

$$\langle I, \pi^i\rangle : \mathfrak{A}^* \to \langle \mathcal{A}/\Omega^{\mathcal{A}}(T^i), T^i/\Omega^{\mathcal{A}}(T^i)\rangle, \quad i \in I,$$

where, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\pi^i_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(\bigcap_{i\in I} T^i)) = \phi/\Omega^{\mathcal{A}}_\Sigma(T^i).$$

Since $\langle H^i, \gamma^i\rangle$ is strict and surjective, we have that $\langle \mathcal{A}, T^i\rangle \in \mathbb{M}^{-1}(\mathsf{M})$, for all $i \in I$. Thus, $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T^i), T^i/\Omega^{\mathcal{A}}(T^i)\rangle \in \mathbb{M}^{-1*}(\mathsf{M})$. Therefore, to complete the proof, it suffices to show that the collection $\langle I, \pi^i\rangle$, $i \in I$, constitutes a collection of subdirect intersection morphisms. This is not difficult to verify. We have

- $\bigcap_{i\in I}(\pi^i)^{-1}(T^i/\Omega^{\mathcal{A}}(T^i)) = \bigcap_{i\in I} T^i/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) = (\bigcap_{i\in I} T^i)/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)$;
- For kernels,

$$\begin{array}{rcll}\bigcap_{i\in I}\mathrm{Ker}(\langle I, \pi^i\rangle) & = & \bigcap_{i\in I}\Omega^{\mathcal{A}}(T^i)/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) & \\ & = & \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) & (\mathcal{I}\text{ protoalgebraic}) \\ & = & \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)}. & \end{array}$$

Now we have $\mathfrak{A}^* \in \overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1}(\mathsf{M})$. ■

We are now able to obtain, under protoalgebraicity, some equivalent expressions for the operator $\overleftarrow{\mathbb{P}}^*$, which, based on Theorem 1835, will allow us to provide characterizations for the class $\mathrm{MatFam}^*(\mathcal{I})$, in case $\mathcal{I}$ is protoalgebraic.

Theorem 1842 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ based on $\mathbf{F}$. Then*

$$\overset{\triangleleft}{\mathbb{P}}(\mathsf{M}) = \overleftarrow{\mathbb{P}}^*(\mathsf{M}) = \overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}(\mathsf{M}).$$

Proof: Suppose $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$. Then, we have

$$\begin{array}{rcll}\overset{\triangleleft}{\mathbb{P}}(\mathsf{M}) & \subseteq & \overleftarrow{\mathbb{P}}^*(\mathsf{M}) & (\text{Proposition 1840}) \\ & \subseteq & \overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}(\mathsf{M}) & (\text{Proposition 1841}) \\ & \subseteq & \overset{\triangleleft}{\mathbb{P}}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{M})) & (\text{Proposition 1838}) \\ & = & \overset{\triangleleft}{\mathbb{P}}(\mathsf{M}). & (\text{Lemma 1837})\end{array}$$

The conclusion follows. ∎

Finally, we get the following characterization of $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$ in terms of closure operators on M, under the hypothesis that M is a subclass of a protoclass of $\mathbf{F}$-matrix families.

Theorem 1843 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, *for a protoalgebraic* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *based on* $\mathbf{F}$. *Then*

$$\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\overset{\triangleleft}{\mathbb{II}}(\mathsf{M}).$$

Proof: Suppose $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$. Then,

$$\begin{array}{rcll} \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) & = & \mathbb{M}\overset{\leftarrow}{\mathbb{II}}^*(\mathsf{M}) & \text{(Theorem 1835)} \\ & = & \mathbb{M}\overset{\triangleleft}{\mathbb{II}}(\mathsf{M}). & \text{(Theorem 1842)} \end{array}$$

∎

24.5 Irreducibility

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, an $\mathbf{F}$-algebraic system and $\mathfrak{A} = \langle \mathcal{A}, X \rangle \in \mathrm{MatFam}(\mathcal{I})$.

An $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ is **completely meet irreducible in** $\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ if, for all $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$,

$$T = \bigcap_{i \in I} T^i \quad \text{implies} \quad T = T^i, \text{ for some } i \in I.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, a collection of $\mathbf{F}$-algebraic systems and $\mathfrak{A}^i = \langle \mathcal{A}^i, T^i \rangle$ a collection of $\mathbf{F}$-matrix families. Recall that an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, is a **subdirect intersection** of the collection $\mathfrak{A}^i$, $i \in I$, if there exist surjective morphisms

$$\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^i, \quad i \in I,$$

such that $T = \bigcap_{i \in I} (\gamma^i)^{-1}(T^i)$ and $\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}$. This subdirect intersection is called a **special subdirect intersection** if $H^i : \mathbf{Sign} \to \mathbf{Sign}^i$ is an isomorphism, for all $i \in I$.

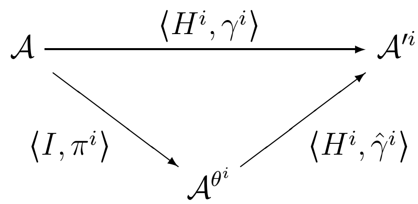
It turns out that $\mathbf{F}$-matrix families are representable as subdirect intersections if and only they are representable as special subdirect intersections.

Proposition 1844 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ *an* $\mathbf{F}$*-matrix family. Then* $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$*,* $i \in I$*, is a collection of subdirect intersection morphisms if and only if*

$$\langle I, \pi^i \rangle : \mathfrak{A} \to \langle \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle), (\gamma^i)^{-1}(T'^i)/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \rangle, \quad i \in I,$$

is a collection of special subdirect intersection morphisms.

Proof: Suppose, first, that $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathcal{A}'^i$, $i \in I$, is a sibdirect intersection representation of $\mathfrak{A}$. For convenience, denote $\theta^i = \mathrm{Ker}(\langle H^i, \gamma^i \rangle)$, $i \in I$. Note that there exist algebraic system morphisms $\langle H^i, \hat{\gamma}^i \rangle : \mathcal{A}^{\theta^i} \to \mathcal{A}'^i$, such that



$$\langle H^i, \gamma^i \rangle = \langle H^i, \hat{\gamma}^i \rangle \circ \langle I, \pi^i \rangle,$$

where $\langle I, \pi^i \rangle : \mathcal{A} \to \mathcal{A}^{\theta^i}$, $i \in I$, are the quotient morphisms. Moreover, these morphisms are well-defined $\mathbf{F}$-matrix family morphisms, since, for all $i \in I$, we have, on the one hand, $T \le (\gamma^i)^{-1}(T'^i) = (\pi^i)^{-1}((\gamma^i)^{-1}(T'^i)/\theta^i)$, and, on the other, $(\pi^i)^{-1}((\gamma^i)^{-1}(T'^i)/\theta^i) = (\gamma^i)^{-1}(T'^i) = (\pi^i)^{-1}((\hat{\gamma}^i)^{-1}(T'^i))$ and, hence, by the surjectivity of $\langle I, \pi^i \rangle$, $(\gamma^i)^{-1}(T'^i)/\theta^i = (\hat{\gamma}^i)^{-1}(T'^i)$. Now we compute:

- For the filter families:

$$\begin{array}{l} \bigcap_{i \in I} (\pi^i)^{-1}((\gamma^i)^{-1}(T'^i)/\theta^i) \\ \qquad = \bigcap_{i \in I} (\pi^i)^{-1}(\pi^i((\gamma^i)^{-1}(T'^i))) \\ \qquad = \bigcap_{i \in I} (\gamma^i)^{-1}(T'^i) \\ \qquad \quad (\theta^i \text{ compatible with } (\gamma^i)^{-1}(T'^i)) \\ \qquad = T. \quad \text{(by hypothesis)} \end{array}$$

- For the kernels

$$\begin{array}{rcl} \bigcap_{i \in I} \mathrm{Ker}(\langle I, \pi^i \rangle) & = & \bigcap_{i \in I} \theta^i \\ & = & \Delta^{\mathcal{A}}. \quad \text{(by hypothesis)} \end{array}$$

Therefore,

$$\langle I, \pi^i \rangle : \mathfrak{A} \to \langle \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle), (\gamma^i)^{-1}(T'^i)/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \rangle, \quad i \in I,$$

is a collection of special subdirect intersection morphisms. ■

Special subdirect intersections of reduced matrix families have a characterization similar to the one applicable for subdirect products of reduced matriced.

Proposition 1845 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle$ *be an* $\mathbf{F}$*-algebraic system and* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ *an* $\mathbf{F}$*-matrix family.* $\mathfrak{A}$ *is a special subdirect intersection of the system* $\{\mathfrak{A}'^i = \langle \mathcal{A}'^i, T'^i \rangle : i \in I\}$ *of reduced* $\mathbf{F}$*-matrix families if and only if, there exists a corresponding system of sentence families* $\{T^i : i \in I\} \subseteq \mathrm{SenFam}(\mathcal{A})$*, such that:*

(i) $\bigcap_{i\in I} T^i = T$*;*

(ii) $\mathfrak{A}/T^i \cong \mathfrak{A}'^i$*, for all* $i \in I$*.*

Proof: Suppose, first, that $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$, $i \in I$, is a collection of special subdirect intersection morphisms. Define $T^i = (\gamma^i)^{-1}(T'^i)$, $i \in I$. Then, we have

- $\bigcap_{i\in I} T^i = \bigcap_{i\in I} (\gamma^i)^{-1}(T'^i) = T$;
- Noting that

$$\begin{array}{rcll}\Omega^{\mathcal{A}}(T^i) & = & \Omega^{\mathcal{A}}((\gamma^i)^{-1}(T'^i)) & \text{(definition of } T^i)\\ & = & (\gamma^i)^{-1}(\Omega^{\mathcal{A}'^i}(T'^i)) & \text{(Proposition 24)}\\ & = & (\gamma^i)^{-1}(\Delta^{\mathcal{A}'^i}) & (\mathfrak{A}'^i \text{ reduced})\\ & = & \mathrm{Ker}(\langle H^i, \gamma^i \rangle), & \text{(set theory)}\end{array}$$

  we obtain

$$\begin{array}{rcl}\mathfrak{A}/T^i & = & \langle \mathcal{A}/\Omega^{\mathcal{A}}(T^i), T^i/\Omega^{\mathcal{A}}(T^i) \rangle\\ & = & \langle \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle), (\gamma^i)^{-1}(T'^i)/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \rangle\\ & \cong & \mathfrak{A}'^i,\end{array}$$

  where the last isomorphism is established by the morphism $\langle H^i, \hat{\gamma}^i \rangle : \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \to \mathcal{A}'^i$, given in Proposition 1844.

Thus, (i) and (ii) of the statement hold.

Assume, conversely, that there exists a system $\{T^i : i \in I\} \subseteq \mathrm{SenFam}(\mathcal{A})$ satisfying (i) and (ii). Consider $\langle I, \pi^i \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T^i)$, $i \in I$. This forms a well-defined system of $\mathbf{F}$-matrix family morphisms

$$\langle I, \pi^i \rangle : \mathfrak{A} \to \mathfrak{A}/T^i, \quad i \in I.$$

Since, by hypothesis, $\mathfrak{A}/T^i \cong \mathfrak{A}'^i$, for all $i \in I$, it suffices to show that the above system of morphisms constitutes a subdirect intersection. We indeed have

- $\bigcap_{i\in I} (\pi^i)^{-1}(T^i/\Omega^{\mathcal{A}}(T^i)) = \bigcap_{i\in I} T^i = T$;
- $\bigcap_{i\in I} \mathrm{Ker}(\langle I, \pi^i \rangle) = \bigcap_{i\in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$.

So $\{\langle I,\pi^i\rangle : \mathfrak{A} \to \mathfrak{A}/T^i : i \in I\}$ is a system of special subdirect intersection morphisms. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, M a class of reduced $\mathbf{F}$-matrix families and $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$.

The $\mathbf{F}$-matrix family $\mathfrak{A} \in \mathsf{M}$ is called **subdirectly irreducible relative to** M if, for every subdirect intersection

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathfrak{A}'^i, \quad i \in I,$$

with $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$, there exists $i \in I$, such that

(i) $T = (\gamma^i)^{-1}(T'^i)$ and

(ii) $\mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}$.

We write $\mathsf{M}^\S$ for the class of all relatively subdirectly irreducible members of M.

If $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is a π-institution based on $\mathbf{F}$ and $\mathsf{M} = \mathrm{MatFam}^*(\mathcal{I})$ is the class of all reduced $\mathcal{I}$-matrix families, then a subdirectly irreducible $\mathfrak{A}$ relative to M is also called **subdirectly irreducible relative to** $\mathcal{I}$.

It turns out that relative subdirect irreducibility and complete meet irreducibility have a close relationship. To detail the relationship, we need an additional operator on classes of $\mathbf{F}$-matrix families.

Proposition 1846 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, \mathbb{N}^\flat\rangle$ *be an algebraic system and* M *a class of reduced* $\mathbf{F}$*-matrix families closed under reduced inverse morphic images, i.e., such that* $\mathbb{M}^{-1*}(\mathsf{M}) \subseteq \mathsf{M}$*. Then an* $\mathbf{F}$*-matrix family* $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}$ *is subdirectly irreducible relative to* M *if and only if* T *is completely meet irreducible in* $\mathcal{X} = \{X \in \mathrm{SenFam}(\mathcal{A}) : \mathfrak{A}/X \in \mathsf{M}\}$.

Proof: Suppose, first, that $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}^\S$ and let $\{X^i : i \in I\} \subseteq \mathcal{X}$, such that $T = \bigcap_{i\in I} X^i$. Then, by Proposition 1844,

$$\langle I, \pi^i\rangle : \mathfrak{A} \to \mathfrak{A}/X^i, \quad i \in I,$$

constitutes a special subdirect intersection. Moreover, since $X^i \in \mathcal{X}$, for all $i \in I$, we have that $\mathfrak{A}/X^i \in \mathsf{M}$, for all $i \in I$. By hypothesis, there exists an $i \in I$, such that $T = (\pi^i)^{-1}(X^i/\Omega^{\mathcal{A}}(X^i)) = X^i$. We conclude that T is completely meet irreducible in $\mathcal{X}$.

Assume, conversely, that T is completely meet irreducible in $\mathcal{X}$ and let

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathfrak{A}'^i, \quad i \in I,$$

be a system of subdirect intersection morphisms, with $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$. By Proposition 1844,

$$\langle I, \pi^i\rangle : \mathfrak{A} \to \mathfrak{A}/(\gamma^i)^{-1}(T'^i), \quad i \in I,$$

is a collection of special subdirect intersection morphisms. Moreover, $\langle H^i, \hat{\gamma}^i\rangle : \mathfrak{A}/(\gamma^i)^{-1}(T'^i) \to \mathfrak{A}'^i$, $i \in I$, are strict surjective morphisms and $\mathfrak{A}/(\gamma^i)^{-1}(T'^i)$ is reduced. Thus, since $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$ and $\mathbb{M}^{-1*}(\mathsf{M}) \subseteq \mathsf{M}$, we get that $\mathfrak{A}/(\gamma^i)^{-1}(T'^i) \in \mathsf{M}$, for all $i \in I$. This shows that $(\gamma^i)^{-1}(T'^i) \in \mathcal{X}$, for all $i \in I$. But, by the subdirect intersection property, $T = \bigcap_{i\in I}(\gamma^i)^{-1}(T'^i)$, whence, by hypothesis, there exists $i \in I$, such that $T = (\gamma^i)^{-1}(T'^i)$. Moreover, $\mathrm{Ker}(\langle H^i, \gamma^i\rangle) = (\gamma^i)^{-1}(\Delta^{\mathcal{A}'^i}) = (\gamma^i)^{-1}(\Omega^{\mathcal{A}'^i}(T'^i)) = \Omega^{\mathcal{A}}((\gamma^i)^{-1}(T'^i)) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Therefore, $\mathfrak{A}$ is subdirectly irreducible relative to M. ■