Chapter 1

Introduction

1.1 Introduction

The purpose of this monograph is to present some attempts towards the algebraization of *non-monotonic logical systems*. By this, we mean, essentially, operators that are idempotent but, perhaps, fail to be inflationary or monotone. We deal with two approaches. In Part I, the structure of *logicate* is used, which does not assume any underlying ordering and, thus, leads by necessity to a coarser treatment, reflecting some of the challenges of the nonmonotonic framework. In Part II, the structure of *logicoid* is used, which assumes an underlying complete lattice ordering $\leq$ on the powerset of the underlying set of sentences that is "commensurate" with the logical consequence, i.e., that makes the idempotent operator inflationary and monotone. This is an attempt at reimposing some order and get the "chaotic" framework to resemble more to the classical monotonic one. The latter would result if one imposes, instead of an arbitrary $\leq$ ordering, the ordinary $\subseteq$ ordering on the powerset of the set of sentences.

We follow, throughout, the development of Font and Jansana in their pioneering monograph on "A General Algebraic Semantics for Sentential Logics" [12]. So we take the approach of "non-monotonic abstract logics", as it were, rather than that of logical matrices. But we also use logical matrices for certain tasks and for comparison purposes. After introducing in each part the basic framework and a bit of general theory (Chapters 2 and 6), we look at some algebraic rudiments; more specifically, at congruences, bilogical morphisms, quotients, interpretations, logical filters and logical matrices (Chapters 3 and 7). The model theoretic developments, paralleling those constituting the backbone of [12], are developed next (Chapters 4 and 8). Here we find the notions of logicate and logicoid models, which play the role of an abstract logic, of full models and of $\mathbb{L}$-algebras. We also see some attempts at establishing analogs of the celebrated Isomorphism Theorem 2.30 of [12] in each of the two cases of logicates and logicoids, involving their models and algebraic congruences. It should be noted, however, that, when discussing algebras of a logic in this setting, one means pairs of algebraic systems and attached interpretations. So, in that respect, the structure of "algebra" more closely resembles the models of π-institutions of [23].

Since, in the abstract treatment of algebraic logic, the crown jewel and the underlying unifying thread is the Leibniz hierarchy (see, e.g., [8, 12, 14, 10]), in Chapters 5 and 9, we return to those and present some results in accord with our nonmonotonic endeavors. We treat primarily protoalgebraicity (see [2] and, also, [8] for the classical theory), the well known correspondence theorem and the notion of Leibniz filters [12, 11, 17]. We also look, rather briefly, at weak algebraizability [9] and at truth equationality [19] to give a flavor of some of the fundamental classes of the hierarchy and how they would look like in our experimental framework.

We give, next, a more detailed outline of contents by chapter.

1.2 Chapter 2

We make an attempt at developing an abstract theory of algebraic logic incorporating features of non-monotonicity. The objects of study are *consequence operators*, which, for our purposes, are mappings on the powerset of a set which are only required to satisfy idempotency. Thus, the inflationarity and monotonicity aspects of traditional closure operators may be missing.

In the traditional abstract studies in algebraic logic [24, 3, 12, 8, 14], a central role is played by closure operators or, equivalently, closure systems. Closure operators are operators on the powerset of a given set that are required to satisfy inflationarity, monotonicity and idempotency. If one wishes to relax this framework to accommodate non-monotonicity, then, at least in a first attempt, the axioms that should be shed, are those of inflationarity and of monotonicity. We look at a few steps one can make in this direction. Namely, we introduce "consequence operators" $C : \mathcal{P}(A) \to \mathcal{P}(A)$, on an underlying set A, which are only required to satisfy idempotency. They are suppose to simulate, or stand for, "raw logics", which we call "logicates".

In the traditional theory, after introducing the basic objects of study, one compares those that are "compatible". Here, compatibility means that, as operators, they apply on the same objects. Thus, only logics over the same underlying set are compared. One defines a closure operator C to be weaker than a closure operator C', and C' to be stronger than C, if, for all $X \subseteq A$, $C(X) \subseteq C'(X)$. However, once monotonicity is out of the picture, this definition makes little sense. Instead, for logicates, one has to devise new ways of performing meaningful comparisons. In this treatment, we focus on two natural ways of doing so. One is kind of intrinsic to the framework, since it only takes into account the fixed points or theories, a fact which makes sense since our operators only satisfy idempotency. The second is an attempt to emulate more closely the comparison in the classical framework. Here, one also considers the overall structure of the logicate; not solely its theories. This comparison is more "structure preserving" at the expense of being, somehow, more "artificial", since the structure is not intrinsic but rather devised. This artificiality is mended in a way in the second part of the monograph, where we switch focus from logicates to logicoids, in which the "structure" is inserted into the formalism, thus becoming "more natural".

Another important construct in both the abstract and concrete studies in algebraic logic is that of axiomatic extensions. When monotonicity is present, a closure operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is viewed as a consequence operator of a logic. One may need to add a subset $T \subseteq A$ as a new set of axioms to axiomatically strengthen the consequence relation. This is done by defining a new operator, based on the original, by setting, for all $X \subseteq A$, $C^T(X) = C(X \cup T)$. Note that both inflationarity and monotonicity are critical here. The first ensures that, for all X, $T \subseteq C^T(X)$, that is the new axioms become genuine consequences of the new operator. The second yields that $C \leq C^T$, i.e., the new operator is indeed an *extension* of the former via the adoption of the elements in T as new axioms. The criticality of these two axioms and the fact that they are missing in the nonmonotonic framework adopted here give an indication of why the task of emulating the extension process would necessarily involve difficulties and may ultimately prove insufficient and unsatisfactory. Nevertheless, we do the best we can by devising two different operators along these lines.

The first uses a more conservative approach. It "lifts" the consequences of $X \subseteq A$ to $C(T)$ if either X or $C(X)$ are contained in T. But some emphasis must be placed on the pejorative use of "lift" here, since, in fact, $C(T)$ may be a much smaller subset of A than either X or T, due to lack of inflationarity and monotonicity. This approach has the drawback that it does not give an operator which strengthens the original operator according to the "structure preserving" comparison of operators that we alluded to in the preceding paragraph. We take this as hinting to the need of an alternative, more "aggressive", line of attack. The more liberal approach, on the other hand, allows consequences to be lifted to $C(T)$ whenever the consequences of X happen to coincide with the consequences of some $Y \subseteq T$. This construct gives rise to an operator that does strengthen the original operator C and, as it turns out, strengthens also the operator obtained by the more conservative approach.

In Section 2.2, the basic objects of study, called *logicates*, which are idempotent operators on the powerset of a set are introduced. The directed graphs that reflect the structure of logicates are called *necropoleis*,[1] since they consist of components called *pyramids*. By imposing a linear ordering on $\mathcal{P}(A)$, one may recast both as linearly ordered structures, with additional features, called linearized consequences and linearized necropoleis, respectively. However, the process of linearization introduces redundancy, which one sheds by passing to equivalence classes of those ordered structures under appropriately defined equivalence relations. We call an equivalence class of linearized necropoleis a *cemetery*.

In Section 2.3 we encounter ways we may use to compare logicates over the same underlying set. *Equipotency* is the equivalence resulting by having identical sets of theories. By comparing sets of theories by the subset relation, we may also impose a partial ordering on the set of equipotency classes. Being *weaker*, on the other hand, is a relation that also takes into account sets of theories but, in addition, it considers the consequence structure. These comparisons are also investigated from the point of view of alternative presentations of logicates, namely, using necropoleis, classes of linearized consequences and cemeteries.

In Section 2.4, we introduce and compare the two notions that attempt to replace axiomatic extensions in the nonmonotonic context. The first is

[1]Plural of **necropolis**, pronounced the same, but stressed *necropóleis* vs. *necrópolis*.

called *boosting*. It seems a natural one to adopt, based on Occam's Razor. However, it fails to produce a strengthened version of the original logicate under the comparison criterion that takes the consequence structure of the logicate into account. To atone for this failure, we fortify boosting to what we call *strong boosting*. This adjustment produces an operator that strengthens both the original and the boosted version of the original operator.

1.3 Chapter 3

In [3], Blok and Pigozzi introduced the notion of an *algebraizable logic*. Logics in their study, and in all related abstract studies in algebraic logic, can be represented by structural closure operators, i.e., functions on the powerset of an absolutely free algebra on countably many generators that satisfy the axioms of inflationarity, monotonicity, idempotency and structurality (see, e.g., Page 25 of [12]). *Logical matrices* were traditionally used in this context as models of logics. Classes of algebras that were algebraic reducts of those matrices were used as algebraic counterparts. Later, based on the study of concrete examples, it was discovered that in many less well behaved logics, the algebras obtained in this way via matrices were not the "right" ones. This led Font and Jansana in [12] to suggest a more general methodology. Instead of logical matrices, they considered *abstract logics* as models of logical systems. Following a similar process, but now taking algebraic reducts of abstract logics instead of logical matrices, they obtained a class of algebras that was seeing, via already acquired experience with a wide variety of particular logics, as being the "right" one in all known logical systems.

One of the limitations of this "traditional" theory is that it requires the logics under consideration to satisfy both inflationarity and monotonicity. Thus, one cannot accommodate potential non-monotonic operators. It is conceivable, however, that many aspects and features of the theory could be carried over to such a context. In Chapter 2, the notion of a *logicate* was introduced. It is an idempotent operator C on the powerset of a given set A. Several related representations were given and logicates over the same underlying set were compared in a couple of different ways.

If one wishes to study logicates from an algebraic point of view, the preceding framework is clearly insufficient, since it involves no algebraic structure. We remedy this by logicates over algebras. So the basic object of study here is an *algebraic logicate*, consisting of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and an idempotent operator C on the powerset of the universe A of $\mathbf{A}$. It turns out that this framework is sufficient for accommodating quite a large part of the monotonic theory, without requiring that the underlying operator be either inflationary or monotone.

Our study relies on the set $\mathcal{C}$ of *fixed points* or *theories*. No intrinsic ordering or structure on those may be assumed, in contrast with the traditional framework. Still, as in the traditional framework, we may define, based on theories, *logical congruences*. It may be shown that these form a principal ideal of the lattice of all congruences on the underlying algebra. Hence, it makes sense to define the *Tarski congruence* as the largest logical congruence of the logicate, exactly as done for abstract logics in the monotonic framework [12]. In addition, a slightly modified, but very similar, version of its well-known characterization holds (see Pages 18-19 of [12]).

An important notion in the study of abstract logics is that of a *bilogical morphism* (Page 20 of [12]). Because the framework of [12] is based on closure operators, bilogical morphisms tie very closely the consequence structures of the abstract logics they relate. As a byproduct, they also connect very strongly the theories. On the other hand, in the present context, due to lack of monotonicity, one cannot expect morphism with such tight properties. As a result, instead of focusing on consequence preservation, we take the effect on theories as primary. Thus, we adopt a notion of *bimorphism* by stipulating that it preserve theories, even though it may not have any preservation properties as regards the corresponding consequence structures. This fact becomes manifest in the formulation of an analog of a well known characterization. Proposition 1.4 of [12] consists of six equivalent statements, three referring to theories and three to the preservation of the closure structures. The analog only retains the three parts referring to theories, whereas those on consequences are not valid in general. Despite this drawback, using bilogical morphisms one can prove some analogs of the Homomorphism Theorems of Universal Algebra [5, 18, 1]. These parallel the generalized versions proven in the monotonic framework (see, e.g. [4, 12]).

Additionally, if one defines *logical matrices* for logicates by relying solely on theories and not on consequence relations, many of the results of the monotonic theory can be adapted and still shown to hold. In this setting, however, because of the absence of structurality, one has to build matrices over specific interpretations, i.e., surjective homomorphisms onto similar algebras. Some clues as to how one may proceed may be taken from the context of models of π-institutions [21]. On the other hand, if structurality is added, as is done briefly in the Addendum to this chapter, then matrices resembling the ordinary ones more closely may again be used as models and some of the flavor of the traditional treatment may be recovered.

In Section 3.2, we adapt the study of *logical congruences* on abstract logics of Font and Jansana [12] to the nonmonotonic setting. Logical congruences on a logicate form a principal ideal of the lattice of all logical congruences on the underlying algebra. The generator of this ideal is called the *Tarski congruence* of the logicate. It is important to notice that all equipotent logicates share the same Tarski congruence. Further, the Blok-Pigozzi style characterization of the Tarski congruence in the traditional setting carries over virtually unchanged. If one thinks of the Tarski operator as acting on equipotency classes, so that the set in question is partially ordered, then it is a monotone operator.

In Section 3.3 we revisit *bilogical morphisms*, but apply them to arbitrary logicates. These are surjective homomorphisms that preserve theories. A characterization is provided, as well as the important result that the Tarski operator of a logicate is preserved under the action of inverse bilogical morphisms.

In Section 3.4 we study aspects of Universal Algebra, that were applied by Brown and Suszko [4] and by Font and Jansana [12] to abstract logics, in the context of nonmonotonicity. We define *quotients* of logicates by logical congruences and show that natural projections form bilogical morphisms. Then we embark on revisiting analogs of the Homomorphism Theorems (see, e.g., [5] and Pages 22-23 of [12]) for logicates. In particular, the Correspondence Theorem, playing a role similar to the one played in the monotonic theory, leads to the definition of a *reduced logicate* and the process of *reduction*. Several properties of reductions carry over to this more general setting and are presented in detail in this section.

In Section 3.5 we introduce *interpretations* of algebraic logicates. An interpretation is essentially a surjective mapping from the underlying algebra of the logicate onto an algebra of the same type. Interpretations form the cornerstone in defining *logical filters* and *logical matrices*, as well as *reduced matrices*, which play a key role in both the traditional theory (see, e.g., [3, 12, 8]) and the more general theory presented here. One important feature of interpretations is that, if the kernel is a logical congruence of the original logicate, then one may define a logicate in the target of the interpretation in such a way that its theories coincide with the filters and the interpretations becomes a bilogical morphism between the two logicates. Several properties governing the relation between interpretations and (sets of) filters are also presented in this section. In closing the section, we define that class of *matrices* and of *reduced matrices* of a logicate and the corresponding classes of algebraic reducts and prove analogs of the well-known *completeness results* for sentential logics in the context of logicates.

In the Addendum, we briefly overview a possible alternative formulation of filters and matrices, applicable in case the consequence operator of the logicate happens to be structural. In that case filters and matrices may be defined as in the traditional monotonic theory, without recourse to fixed interpretation morphisms.

1.4 Chapter 4

In this, third chapter, on logicates, we focus specifically on the role that algebraic logicates play as models of other logicates. Logicates are models more suitable for many purposes than simple logical matrices, even though logicate models can be viewed as bundles of matrices over the same underlying interpretation.

Our framework and starting point is the study in Chapter 3 of interpretations. We are assuming a given logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ which consists of an algebra and an idempotent operator, called a *consequence operator*, on its powerset. Logicates are supposed to represent logical systems for which inflationarity and monotonicity may fail. $\mathbb{L}$ is viewed as the focus of study and it is called, accordingly, a *base logicate*. An *interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ consists of a surjective homomorphism from the algebra $\mathbf{B}$ of $\mathbb{L}$ onto a similar algebra $\mathbf{A}$. If on the target algebra, there is given a logicate structure, say $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}$ induces a logicate on the base algebra. We say that $\mathbb{A}$ is a *model* of $\mathbb{L}$ if the inverse images under h of the theories of $\mathbb{A}$ form a subset of the theories of $\mathbb{L}$, written $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. Two logicates connected by a bilogical morphism that commutes with interpretations share the property of being simultaneously models or not being models.

One of the key constructions in our framework is passing from a model to its Tarski reduction. The *Tarski operator* was used as a key ingredient in the theory of Font and Jansana (Page 19 of [12]) and, as our work is based on theirs, it continues to play a crucial role here as well. Given a logicate model, one may construct its *reduction* by moding out both the interpretation and the idempotent operator by the Tarski congruence of the logicate. A first result is that a logicate is complete with respect to both its class of logicate models and its class of reduced logicate models. Completeness here simply means that collecting all inverse images of theories of the models of the class yields the full collection of theories of the base logicate.

Connecting the theory of logicate models with the theory of matrix models of the base logicate, which was detailed in Chapter 3, we obtain the fact, well-known in classical theory (Proposition 2.7 of [12]), that, a logicate, viewed as a bundle of matrices, is a model of $\mathbb{L}$ if and only if every member of the bundle is a matrix model of $\mathbb{L}$.

The next key concept adapted here from the theory of abstract logics of [12] is that of a *full model*. A *basic (full logicate) model* is a model whose collection of theories consists of all filters on its interpretation. A *full (logicate) model* is one whose Tarski reduction is a basic full logicate model. The terminology is justified by the fact that a basic model turns out to be a full model according to these definitions. It is shown here, in a result that parallels one pertaining to abstract logics, that logicate models connected via bilogical morphisms commuting with interpretations are either both full or both fail to be full. As a consequence the property of being full is also preserved and reflected by reductions. These results yield a characterization of the class of full models as the smallest class that contains all basic full models and is closed (in both directions) under bilogical morphisms.

Full models are the first ingredient in establishing a key *Isomorphism Theorem*, along the lines of the Isomorphism Theorem (Theorem 2.30) of Font and Jansana, which is one of the main results of the abstract treatment in the theory they present in [12]. The second ingredient relates to congruences whose quotients are algebras in $\mathrm{Alg}(\mathbb{L})$. The class $\mathrm{Alg}(\mathbb{L})$ consists of the underlying interpretations of reduced full models of $\mathbb{L}$. Another class of interpretations that is related to a class of algebras traditionally studied in algebraic logic is the class $\mathrm{Alg}^*(\mathbb{L})$. It consists of all underlying interpretations of reduced matrix models of $\mathbb{L}$. The tight connection, mentioned previously, between logicate models and matrix models, yields a (sort of induced) relationship between the two classes. In the traditional setting one class turns out to be the class of subdirect products of the other (see, e.g., Theorem 2.23 of [12]). In the present setting, because of the presence of fixed interpretation morphisms, we find it convenient (and perhaps necessary) to define a related but different operation on interpretations, called a *subdirect intersection*. It is shown that the class $\mathrm{Alg}(\mathbb{L})$ consists exactly of subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$.

Our work in this part culminates with proving an analog of the Isomorphism Theorem 2.30 of [12] for the present context. We view this as one of the main results of the work. The analog proven here has some significant deviations as compared to its predecessor. First, all parts are taken to be over fixed underlying interpretations. This is compelled by the absence of structurality for logicates. If one added structurality, then something closer, perhaps, to the original could be obtained. But this seemed rather restrictive and, in addition, the framework of π-institutions [21] has provided some experience in dealing with fixed interpretations. Second, one cannot expect to establish an isomorphism theorem dealing with all full models, since full models that are equipotent, have identical Tarski congruences. So one has, by necessity, to pass to equipotency classes of full models over fixed interpretations. Taking these comments into account, we establish an order isomorphism between the set of equipotency classes of full logicate models, ordered under the superset relation between sets of theories, and the set of $\mathrm{Alg}(\mathbb{L})$-congruences under inclusion. It is also shown that the latter poset is a complete lattice. As a consequence, one obtains that the former has the same structure as well.

In Section 4.2 we recall the notion of interpretation and use it to define *logicate interpretations*. These, in turn, serve in defining models of a logicate. Logicate models have a tight relationship with matrix models. We also define reductions. We show that, if two logicate interpretations are related via a bilogical morphism, then one is a model if and only if the other is. This implies that a given one is a model if and only if its reduction is. We also formulate analogs of the well-known Completeness Theorems of Algebraic Logic both with respect to the class of all models and with respect to the class of all reduced models.

In Section 4.3, the notion of *full model* for logicates is introduced, taking after the corresponding notion for the monotonic framework (see Definition 2.8 of [12]). A model is a *basic (full) model* if the set of its theories coincides with the set of all filters on the underlying interpretation. A model is a *full* *model* if its reduction is a basic full model. Several properties, paralleling ones proved by Font and Jansana for sentential logics in [12], are adapted and proved in this setting. They culminate in two different characterizations of full models, which, as Font and Jansana explain, may be taken as justifications of the term "full". The class of full models is shown to be the smallest class that includes all basic full models and is closed under bilogical morphisms (see Corollary 2.13 of [12]). It is also the class of all models whose sets of theories consist of all preimages under canonical projections of all filters on the Tarski reduction of the model (see Theorem 2.14 of [12]).

In Section 4.4 we define the notion of $\mathbb{L}$-*algebra* for a given logicate $\mathbb{L}$. These parallel $\mathcal{S}$-algebras for a sentential logic [12]. In the present context, however, they should be referred to as $\mathbb{L}$-*interpretations*, since they are pairs consisting of an algebra together with a mapping from the base algebra of $\mathbb{L}$ onto the algebra. But the term "algebra" is retained because of the similarity of the role they play. Several results encapsulating the interaction of these algebras with full models and reduced full models are given. We also revisit the relation between the class of algebras which are reducts of reduced matrix models and the class of $\mathbb{L}$-algebras, which are reducts of reduced full models of $\mathbb{L}$. An operation, called *subdirect intersection*, paralleling that of subdirect product in the ordinary framework, is defined and comes in handy in this task. The result is an analog of Theorem 2.23 of [12].

In Section 4.5 the main goal is establishing an *Isomorphism Theorem*, along the lines of Theorem 2.30 of [12]. The Tarski operator over a fixed interpretation forms a mapping from logicates over that interpretation into congruences. Moreover, it is constant over equipotency classes of logicates. Thus, it may be viewed as an operator over equipotency classes of logicates to congruences. We introduce here an operator from $\mathrm{Alg}(\mathbb{L})$-congruences that is seeing as being the inverse of the restriction of the Tarski operator on equipotency classes of full models. Moreover, both operators are order preserving, when order is taken to be the one reflecting the superset relation between sets of theories. So they establish an isomorphism between equipotency classes of full models and $\mathbb{L}$-algebra congruences. Some consequences of this isomorphism are encountered here, among which is the fact that the equipotency classes of full models form a complete lattice. This is proven using the isomorphism theorem and a result showing that the collection of $\mathrm{Alg}(\mathbb{L})$-congruences under the subset relation form a complete lattice.

1.5 Chapter 5

One of the main achievements of the abstract theory of Algebraic Logic is the classification of logics in an algebraic hierarchy. A logic is represented by a structural closure operator on the algebra of terms (or formulas) over an algebraic type generated by countably many variables. The theory prescribes a method (or, rather, methods) one may follow to select a particular class of algebras over the same signature as the logic to associate with the logic. The higher the logic is classified in the hierarchy, the closer the ties between the logic and its associated class of algebras. Because of their clarity and comprehensiveness, but, also because they were written by pioneers, two monographs [3, 12], a survey [14], a book [8] and a textbook [10] have been used for many years as guides in being introduced to, in understanding and in delving deeper into the theory.

Since the *algebraic hierarchy* is one of the crowns (and jewels) of the traditional theory, it is only fair to, at least start to, investigate and give a first idea of how one could attempt to keep alive aspects of the theory in a rougher terrain. This is the effort we expend in the present and last chapter of Part I.

Among the major, perhaps most important, classes in the traditional hierarchy are protoalgebraic logics [2] (see, also, [8, 14, 10]). These are the logics in which, roughly speaking, indistinguishability modulo a theory implies interderivability modulo the theory. Another important characterization asserts that they are the logics on whose lattices of theories, the Leibniz operator is monotone. In Section 5.2, we use the definition from the classical framework to define *protoalgebraic logicates* and try to establish some equivalent conditions, some with and some without extra assumptions.

One of the key consequences of protoalgebraicity, which forms an important feature in their study, is the so-called *Correspondence Theorem*. This result is partly the reason why Blok and Pigozzi declared that protoalgebraic logics form the widest class of logics amenable to algebraic techniques of study, even though they are not "algebraizable", i.e., do not belong to the highest step in the hierarchy but are, rather, located near the bottom. The Correspondence Theorem establishes an isomorphism between the lattice of filters of the logic on a given algebra including a fixed filter and the lattice of filters on the quotient algebra, formed by dividing out by the Leibniz congruence of the fixed filter including the quotient of the fixed filter. We discuss this result and some of its consequences in Section 5.3. Again our focus remains to safeguard some of the result from the traditional theory, with or without provisos, in this less robust environment.

As is the case in the traditional theory [12], and as was shown provisionally in Chapter 4, full models play a key role in the investigation of the logical structure. In the context of protoalgebraic logics, full models are inextricably connected to, so-called, *Leibniz filters* [12, 17]. So, in Section 5.4, we investigate the relation between full models and Leibniz filters in the context of protoalgebraic logicates. Their study naturally segues into the study of *weakly algebraizable logicates* in Section 5.5. These are defined by analogy with the corresponding class in the monotonic framework [9] (see, also, [8, 14, 10]). Several characterizations paralleling the ones from the traditional setting are provided, but, generally, they require the additional hypothesis that the set of theories be closed under intersection.

As is well known in the ordinary setting, weak algebraizability [9] results by simultaneously insisting that a logic be protoalgebraic [2] and truth equational [19]. So in the last section, Section 5.6, we use the original definition to identify the class of *truth equational logicates*. They are characterized by the Leibniz operator on their theories being monotone and injective. Again assuming closure of the set of theories under intersection, we prove that, for logicates also, weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

1.6 Chapter 6

In our work in Part I, we already glimpsed, at least twice, how, in studying a logicate, having an underlying order on the subsets of its universe A may be beneficial. E.g., in Chapter 2, when we looked at linearized consequences, we saw that artificially linearizing allow us to study instead of an arbitrary idempotent operator, an operator that also satisfies all three properties of an ordinary closure operator. Furthermore, in Chapter 5, we saw how many of the results related to classes in the algebraic hierarchy required that the set of theories was closed under intersections or has a minimum element. These observations lead us in Part II to look at structures in which order plays a role from the get go. To take advantage of the powerful machinery of the traditional framework of monotonic logics [12], without, however, losing sight of the fact that we are dealing with, possibly, nonmonotonic systems, we introduce a complete lattice ordering on the powerset of the underlying set A.

Comparing to the development in Part I, we could say that, in Part I, we took the logical notion of consequence operator as foundational and constructed, based on it, an “ordered” consequence, which involved a type of imposed ordering, either “artificial”, e.g., a linearization, or “natural”, e.g., based on $\subseteq$, reflecting, necessarily in a rather loose way, to the extent possible the “chaotic” logical structure. On the other hand, in Part II, a reversal of roles occurs. More precisely, we presume an underlying order on the powerset $\mathcal{P}(A)$ of the set A and then build a logical structure that is, in some way, commensurate with the underlying ordering. We visualize the presumed preexisting ordering as an artificially created “molecular” shape and, since the logic is developed on that construct, it is termed a “logicoid”. This approach imitates more closely, and captures more accurately, many of the features of more traditional logical systems. On the other hand, expectations must be tempered, since the ordering is one among many that could possibly be chosen, and as such, its role is not quite natural. As noted, also, in comments in Part I,, we attempt to do what we can in a challenging setting, among rather adverse features as compared with those naturally available in the monotonic framework.

In Section 6.2, we introduce the notion of a *grid*, which consists of an underlying set A (viewed as a set of abstract sentences), together with an arbitrary complete lattice ordering on the powerset of A. The fact that this ordering is arbitrary and not the "subset" ordering is what permits accommodating nonmonotonicity and make the framework suitable for our purposes, while still maintaining many of the advantages afforded by the complete lattice structure. Naturally enough, we then introduce *grid morphisms* that connect grids. They are surjective mapping between the underlying sets that make their induced inverse powerset mappings complete lattice embeddings. Continuing, we define *closure operators* as ones that are inflationary, monotone and idempotent, but not with respect to the natural subset ordering, but, rather, with respect to the "artificial" ordering of the grid. We also define *closure systems* and, using the notion of *theory*, we show that, as in the ordinary framework, losure operators and closure systems (on the grid, as it were) are still in one to one correspondence and, thus, interchangeable.

In Section 6.3, we introduce the "*weaker than*" and "*finer than*" relations to compare closure operators of logicoids and closure systems on the underlying grids, respectively. These relationships parallel the ones in the classical (monotonic) framework, except that, instead of being with respect to the subset relation, they are based on the grid ordering.

In Section 6.4, we look at *boosting* for logicoids by a chosen set of axioms, which corresponds to taking the axiomatic extension of a sentential logic in the ordinary monotonic context. We saw the difficulties inherent in defining such an operation for logicates in Section 2.4. Here, the presence of a complete lattice ordering in the grid on which a logicoid is based, creates an environment in which some of the nice features may be recovered, albeit with respect to the $\leq$ ordering of the grid rather than the natural subset ordering that serves the same purpose in the monotonic framework.

Our main interest is in what we call *algebraic logicoids*, which are logicoids built on *algebraic grids*, that is, grids on sets having an algebraic structure. Naturally enough, treating them algebraically requires having some algebraic fundamentals available for handling them. This is precisely the purpose that Section 6.5 is supposed to fulfill. Here, we formally define *algebraic grids*, which consist of an algebra together with a complete lattice ordering on its powerset. We also define *grid morphisms* and *grid congruences*. We show that these constructs interact as expected. We then employ them to develop analogs of the fundamental Homomorphism Theorems of Universal Algebra for algebraic grids, their homomorphisms and their congruences.

1.7 Chapter 7

In Chapter 7, we develop the rudiments of the algebraic theory of logicoids with an eye towards developing, in Chapter 8, a model theory, paralleling the one in [12] and that developed for logicates in Part I. We first introduce the key concept of *logical grid congruence*. Based on those, we define the *Leibniz grid congruence* of a logical matrix and the *Tarski grid congruence* of a logicoid. We then study *bilogical morphisms* between logicoids, which, unlike those used for logicates, respect the logical consequence and not merely the theories of the structure. So, in that respect, they resemble more closely those introduced by Font and Jansana [12]. We then look at *quotient logicoids* and prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. This gives us the chance to look closely at *reductions* and at *reduced logicoids*. We then turn to analogs of *interpretations*, *filters* and *matrix models* and study many of their properties, including the way they interact with grid morphisms, their interplay with closure systems and their transformations via bilogical morphisms.

In more detail, Section 7.2 undertakes the study of *logical grid congruences*. Recall that, given an algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, a congruence θ on $\mathbf{A}$ is called a *grid congruence* on $\hat{\mathbf{A}}$ if $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ based on the grid $\hat{\mathbf{A}}$, θ is a *logical grid congruence* of $\mathbb{L}$ if it is a grid congruence on $\hat{\mathbf{A}}$, such that $\mathcal{C} \subseteq \mathrm{Cmp}(\theta)$. It is shown that the collection of all logical grid congruences of $\mathbb{L}$ forms a principal ideal of the complete lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\widetilde{\Omega}(\mathbb{L})$ is called the *Tarski grid congruence* of $\mathbb{L}$. An analogous situation occurs if one considers logical matrices $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$ based on an algebraic grid $\hat{\mathbf{A}}$. Here a *matrix grid congruence* is a grid congruence θ on $\hat{\mathbf{A}}$, such that $X \in \mathrm{Cmp}(\theta)$. Again, the collection of all matrix grid congruences of $\mathfrak{A}$ forms a principal ideal of the lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\Omega(\mathfrak{A})$ is called the *Leibniz grid congruence* of $\mathfrak{A}$. Two of the most useful observations related to these concepts are that $\widetilde{\Omega}$ is monotone on logicoids over the same grid and that, given a logicoid, its Tarski grid congruence is the intersection of all Leibniz grid congruences of those logical matrices formed by each of its theories.

In Section 7.3, we introduce and study *logical* and *bilogical morphisms* between logicoids. Since logicoids are based on algebraic grids, all these morphisms are algebraic grid morphisms, which were studied extensively in Section 6.5, and we rely quite heavily on that machinery. A *logical morphism* $h : \mathbb{L} \to \mathbb{L}'$ from a logicoid $\mathbb{L}$ based on $\hat{\mathbf{A}}$ to a logicoid $\mathbb{L}'$ based on $\hat{\mathbf{A}}'$ is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$. In case $h^{-1}(\mathcal{C}') = \mathcal{C}$ we say that $\mathbb{L}$ is *projectively generated from* $\mathbb{L}'$ *by* h. A logical morphism $h : \mathbb{L} \to \mathbb{L}'$ is a *bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}'$ between $\mathbb{L}$ and $\mathbb{L}'$ if it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. We provide a characterization theorem for bilogical morphisms along the lines of Proposition 1.4 of [12] and we show that, if $h : \mathbb{L} \to_b \mathbb{L}'$, then

$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}'))$. Finally, the notion of *isomorphism* between logicoids is introduced as a bijective mapping $h : A \to A'$ for which both $h : \mathbb{L} \to \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to \mathbb{L}$ are logical. It is shown that this is tantamount to requiring that $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ and $h : \mathbb{L} \to_b \mathbb{L}'$.

In Section 7.4, given an algebraic grid $\hat{\mathbf{A}}$ and a grid congruence θ on $\hat{\mathbf{A}}$, we define the *quotient closure operator* C^θ on the quotient grid $\hat{\mathbf{A}}/\theta$ of an operator C on $\hat{\mathbf{A}}$. This gives rise to the *quotient logicoid* $\mathbb{L}^\theta = \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$ of a given logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and, moreover, makes the quotient grid morphism $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ into a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta$. Quotient logicoids are important because, among other things, they allow us to prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. We prove analogs of the Homomorphism Theorem, of the Second Isomorphism Theorem and of the Correspondence Theorem. The latter, in particular, enables us to show that the Tarski grid congruence of a quotient logicoid is the quotient of the Tarski grid congruence of the parent. We also look at *reductions* of logicoids. We show the important results that the reduction of a quotient logicoid coincides with the reduction of its parent and that the reductions of two logicoids related via a bilogical morphism are isomorphic logicoids.

In Section 7.5, the goal is to develop a theory of matrix models for logicoids along the lines of the traditional theory for monotonic logics and the theory developed in Section 3.5 for logicates. We start with a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ over a base algebraic grid $\hat{\mathbf{B}}$. Lack of structurality compels us to consider structures over fixed *interpretations*. These are pairs $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where $\hat{\mathbf{A}}$ is an algebraic grid and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism. A *matrix* is a pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, where $F \subseteq A$. If F is an $\mathbb{L}$-*filter*, i.e., if $h^{-1}(F) \in \mathcal{C}^\flat$, then $\mathfrak{A}$ is called an $\mathbb{L}$-*matrix*. Moreover, $\mathfrak{A}$ is *reduced* if $\Omega_{\mathcal{A}}(F) = \Delta_{\mathbf{A}}$. On any interpretation $\mathcal{A}$, there is, induced by $\mathbb{L}$ and h, a closure operator $C_{\mathcal{A}}$. In case $\mathrm{Ker}(h)$ is a logical grid congruence of $\mathbb{L}$, the induced structure $\mathbb{L}_{\mathcal{A}} = \langle \hat{\mathbf{A}}, C_{\mathcal{A}} \rangle$ is a logicoid and, moreover, the mapping $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ becomes a bilogical morphism $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$. Further, it can be shown that the theories of $\mathbb{L}_{\mathcal{A}}$ coincide with the $\mathbb{L}$-filters of $\mathbb{L}$ on $\mathcal{A}$.

In the remainder of the section we look at ways grid morphisms interact with filters. For instance, we show that, for two interpretations connected by a grid morphism, inverse images of $\mathbb{L}$-filters are $\mathbb{L}$-filters and conversely. Considering quotient interpretations, it is shown that for an $\mathbb{L}$-filter F on $\mathcal{A}$ to be the inverse image under a quotient morphism π_θ of an $\mathbb{L}$-filter on $\mathcal{A}/\theta$ it is necessary and sufficient that θ be compatible with F. Two interpretations that are related by a grid morphism may, under certain circumstances, establish very close ties between corresponding $\mathbb{L}$-filters. The closest connection occurs when the grid morphism in question is a bilogical morphism between the filter structures. It then establishes an isomorphism between the two posets of filters under the corresponding grid orderings. If this happens between two closure structures, one in the source interpretation and another in the target, and the structure in the source interpretation consists of all $\mathbb{L}$-filters, then so does the one in the target. This yields that the $\mathbb{L}$-filters on a reduced interpretation coincide with the reductions of the $\mathbb{L}$-filters on the parent interpretation. At the end of the section, we present an analog of a standard result asserting that a base logicoid is complete with respect to all its matrix models as well as with respect to all its reduced matrix models.

1.8 Chapter 8

This chapter discusses *logicoid models* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$. They are based on grid interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, which consist of a grid $\hat{\mathbf{A}}$ together with a grid morphism h from the base grid $\hat{\mathbf{B}}$ onto $\hat{\mathbf{A}}$. We also define and study the reduction $\mathbb{A}^*$ of such a logicoid interpretation $\mathbb{A}$. Among those models, we single out the *full models*, which are the ones whose reductions are *basic full models*, i.e., consist of all possible $\mathbb{L}$-filters on their underlying interpretations. We characterize this class of models. Moreover, we show that it consists of those models that have all possible filters corresponding to filters on the reduced interpretations. Reduced $\mathbb{L}$-models give rise to $\mathbb{L}$-*algebras*, i.e., interpretations that are reducts of reduced $\mathbb{L}$-models. Their class is shown to be the class of subdirect intersections of interpretations in $\mathrm{Alg}^*(\mathbb{L})$, which consists of all interpretation reducts of reduced $\mathbb{L}$-matrices. Our study culminates with an Isomorphism Theorem for logicoids asserting that the Tarksi operator on a fixed interpretation $\mathcal{A}$ is an isomorphism between the ordered set of full $\mathbb{L}$-models on $\mathcal{A}$ and the partially ordered set of grid $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$.

In Section 8.2, we introduce the notion of an *interpretation* of a base algebraic grid $\hat{\mathbf{B}}$ and that of a *logicoid interpretation*. The first is a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, consisting of an algebraic grid $\hat{\mathbf{A}}$ and a grid morphism $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$. The second consists of an algebraic logicoid $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is an interpretation and $\langle \hat{\mathbf{A}}, C \rangle$ is a logicoid based on $\hat{\mathbf{A}}$. A logicoid interpretation of $\hat{\mathbf{B}}$ induces a logicoid structure $\mathbb{L}^{\mathbb{A}} = \langle \hat{\mathbf{B}}, C^{\mathbb{A}} \rangle$ on $\hat{\mathbf{B}}$ in such a way that h becomes a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{A}$. Further, if two logicoid interpretations are related via a bilogical morphism, then they induce identical logicoids on the base grid $\hat{\mathbf{B}}$. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a *model* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ if $\mathbb{L} \leq \mathbb{L}^{\mathbb{A}}$ or, equivalently, if $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. The section continues with a discussion of completeness of a base logicoid with respect to a class of models. In this context, *reductions* of models and *reduced models* are discussed and analogs of classical completeness results with respect to the class of all models and with respect to the class of all reduced models are formulated. The section closes by connecting the notion of logicoid model with that of a grid matrix model, introduced and studied in Section 7.5.

In Section 8.3, we study *full models* of logicoids. Given a base logicoid

$\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, a logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a *basic full model* of $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., if its set of theories is the entire collection of $\mathbb{L}$-filters on its underlying interpretation. A *full model* of $\mathbb{L}$ is one whose reduction is a basic full model. Full models are indeed models and basic full models are indeed full models. So the terminology chosen is sound. It turns out that bilogical morphisms preserve fullness in both directions, which implies that $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if its reduction $\mathbb{A}^*$ is also a full $\mathbb{L}$-model. Additionally, $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if there exists a bilogical morphism from it onto a basic full $\mathbb{L}$-model. As a consequence we get that the class of full $\mathbb{L}$-models is the smallest class containing all basic full $\mathbb{L}$-models and closed under bilogical morphisms (in both directions). The section concludes with a result providing an additional justification of the term "full". It shows that full $\mathbb{L}$-models are those whose collection of $\mathbb{L}$-filters consists of all possible ones corresponding to $\mathbb{L}$-filters on the reduced interpretation.

Section 8.4 introduces $\mathbb{L}$-algebras (more accurately $\mathbb{L}$-interpretations) for a logicoid $\mathbb{L}$. These are interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, such that $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))$ is the identity grid congruence on $\hat{\mathbf{A}}$. Some results relating $\mathbb{L}$-algebras with full $\mathbb{L}$-models and with their theories are provided. It is shown that, for every full $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, the reduction $\mathcal{A}^*$ of the interpretation $\mathcal{A}$ is an $\mathbb{L}$-algebra. Additionally, the class $\mathrm{Alg}(\mathbb{L})$ of $\mathbb{L}$-algebras is characterized as the class of interpretation reducts of reduced $\mathbb{L}$-models. Moreover, it is shown that $\mathrm{Alg}(\mathbb{L})$ is the class of all subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$ of all interpretation reducts of reduced grid matrix models of $\mathbb{L}$. This characterization yields that $\mathrm{Alg}^*(\mathbb{L})$ is contained in $\mathrm{Alg}(\mathbb{L})$ and, also, that given logicoids $\mathbb{L}$, $\mathbb{L}'$ over the same base grid $\hat{\mathbf{B}}$, such that $\mathbb{L} \leq^\flat \mathbb{L}'$, we have that $\mathrm{Alg}(\mathbb{L}')$ is contained in the class $\mathrm{Alg}(\mathbb{L})$.

In Section 8.5, the last section of the chapter, we prove an analog of the Isomorphism Theorem 13 of [12] for logicoids. The result parallels the Isomorphism Theorem for logicates (Theorem 75) and the proof is similar.

1.9 Chapter 9

Chapter 8 is intended to only provide a relatively superficial flavor of a semantically defined algebraic hierarchy of classes of logicoids based on properties of their Leibniz operator, paralleling the classical one for monotonic logics (see, e.g., [8, 14, 10]). The best part of it (Sections 9.2-9.4) is dedicated to the study of *protoalgebraicity*, Section 9.5 looks briefly at *weak algebraizability* and Section 9.6 looks even more briefly at *truth equationality*. We give several characterizations of *protoalgebraicity*, which is the property of having a monotone Leibniz operator, and we study the *Correspondence Theorem* and several of its consequences. This segues nicely into the introduction of *Leibniz filters* and some of their properties. *Weak algebraizability* is the property of having both a monotone and an order reflecting Leibniz operator, whereas *truth equationality* is the property of having a completely order reflecting Leibniz operator. As in the traditional monotonic case, it turns out that weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

In Section 9.2, we introduce *protoalgebraic logicoids*. A logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ is *protoalgebraic* if, for all theories X and all $a, b \in B$, $\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ implies that $\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X)$, where $\Lambda_{\mathbb{L}}(X)$ is the relation holding if, for every theory X', with $X \leq^\flat X'$, $a \in X'$ iff $b \in X'$. We show that protoalgebraicity is equivalent to the monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on the theories of $\mathbb{L}$. Moreover, $\mathbb{L}$ is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. An additional characterization asserts that $\mathcal{L}$ is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is submeetive on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, meaning that, for all $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $\Omega_{\mathcal{A}}(\bigwedge_{i \in I} X_i) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i)$.

In Section 9.3, the central task is proving a Correspondence Theorem, an analog of Theorem 6.19 of [10], for protoalgebraic logicoids. After doing this, we explore some of its consequences. Among these are some additional characterizations of protoalgebraicity using the Tarski operator. We show, e.g., that $\mathbb{L}$ is protoalgebraic if and only if, for every $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$, where $\min \mathcal{C}$ is the least theory of $\mathbb{A}$ (or the set of theorems of $\mathbb{A}$). Another consequence is that, if the logicoid $\mathbb{L}$ happens to be protoalgebraic, then the classes of interpretations $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$ coincide. In addition, if $\mathbb{L}$ is protoalgebraic, then any two logicoid models $\mathbb{A}$ and $\mathbb{A}'$ over the same underlying interpretation that share the same minimum theories must be identical. The last result of the section is a theorem characterizing the full models of a protoalgebraic logicoid, while, at the same time providing yet another characterization of protoalgebraicity. It asserts that $\mathbb{L}$ is protoalgebraic if and only if its full models are of the form $\langle \mathcal{A}, C^F \rangle$, where $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, for some interpretation $\mathcal{A}$ and some filter F in $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Here, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ denotes the collection of all $\mathbb{L}$-filters on $\mathcal{A}$ dominating F in the $\leq$ ordering of the subsets of A in the underlying algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ of $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$.

Section 9.4 considers a question that arises naturally from the characterization of full models of protoalgebraic logicoids. More precisely, it attempts to characterize those $\mathbb{L}$-filters F on an interpretation $\mathcal{A}$ for which $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ is a full $\mathbb{L}$-model. To do this, we form the subset of such filters $\mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A})$. These filters are termed *Leibniz filters*. If $\mathbb{L}$ is protoalgebraic, $\Omega_{\mathcal{A}}$ turns out to be an order isomorphism from $\mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A})$, ordered by $\leq$ onto $\mathrm{Con}_{\mathrm{Alg}^*}(\mathcal{A})$, ordered by $\subseteq$. Further, we introduce an equivalence $\sim_\Omega$ between $\mathbb{L}$-filters on an interpretation $\mathcal{A}$ that "identifies" two filters if they have the same Leibniz grid congruence. The $\sim_\Omega$-class of an $\mathbb{L}$-filter F is denoted by $[F]_\Omega$. If $\mathbb{L}$ is protoalgebraic, then F is the minimum element in the $\leq$ ordering in $[F]_\Omega$. This affords the characterization of Leibniz filters as those $\mathbb{L}$-filters on an interpretation that are minimum in their $\sim_\Omega$-equivalence classes. Equivalently, they are the $\mathbb{L}$-filters F, whose quotients $F/\Omega_{\mathcal{A}}(F)$ are minimum $\mathbb{L}$-filters in the $\leq^{\Omega_{\mathcal{A}}(F)}$ ordering on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Section 9.5 deals with a second question that may be seen to arise from the characterization of full models of a protoalgebraic logicoid. Namely, identify those situations for which the collection of Leibniz filters is the entire collection of filters on an interpretation. We call a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *weakly algebraizable* if the Leibniz operator is order preserving and order reflecting on $\mathcal{C}^\flat$. This is equivalent to the Leibniz operator $\Omega_{\mathcal{A}}$ being order preserving and order reflecting on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. Furthermore, it turns out that $\mathbb{L}$ is weakly algebraizable if and only if $\mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. Thus, weak algebraizability settles the initial problem of discovering a property under which the collection of the Leibniz $\mathbb{L}$-filters on any interpretation coinciding with the entire collection of $\mathbb{L}$-filters. This characterization, combined with the results of Section 9.4, provides several additional characterizations of weak algebraizability. E.g., we get that $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A}$ and all $\mathbb{L}$-filters F on $\mathcal{A}$, $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$ and that $\mathbb{L}$ is weakly algebraizable if and only if $\Omega_{\mathcal{A}}$ is a lattice isomorphism from $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$.

In Section 9.6, we briefly introduce the property of *truth equationality*. We say that a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ is *truth equational* if $\Omega_{\hat{\mathbf{B}}}$ is completely order reflecting on $\mathcal{C}^\flat$, that is, if, for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$,

$$\bigcap_{i\in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies} \quad \bigwedge_{i\in I}^{\flat} X_i \leq^\flat X.$$

We show that this is equivalent to the complete order reflectivity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for all interpretations $\mathcal{A}$ of $\mathbb{L}$. Finally, we prove that weak algebraizability is characterized as the conjunction of protoalgebraicity and truth equationality.