Chapter 2

Basic Theory

2.1 Introduction

We make an attempt at developing an abstract theory of algebraic logic incorporating features of non-monotonicity. The objects of study are *consequence operators*, which, for our purposes, are mappings on the powerset of a set which are only required to satisfy idempotency. Thus, the inflationarity and monotonicity aspects of traditional closure operators may be missing.

In the traditional abstract studies in algebraic logic [24, 3, 12, 8, 14], a central role is played by closure operators or, equivalently, closure systems. Closure operators are operators on the powerset of a given set that are required to satisfy inflationarity, monotonicity and idempotency. If one wishes to relax this framework to accommodate non-monotonicity, then, at least in a first attempt, the axioms that should be shed, are those of inflationarity and of monotonicity. We look at a few steps one can make in this direction. Namely, we introduce "consequence operators" $C : \mathcal{P}(A) \to \mathcal{P}(A)$, on an underlying set A, which are only required to satisfy idempotency. They are suppose to simulate, or stand for, "raw logics", which we call "logicates".

In the traditional theory, after introducing the basic objects of study, one compares those that are "compatible". Here, compatibility means that, as operators, they apply on the same objects. Thus, only logics over the same underlying set are compared. One defines a closure operator C to be weaker than a closure operator C', and C' to be stronger than C, if, for all $X \subseteq A$, $C(X) \subseteq C'(X)$. However, once monotonicity is out of the picture, this definition makes little sense. Instead, for logicates, one has to devise new ways of performing meaningful comparisons. In this treatment, we focus on two natural ways of doing so. One is kind of intrinsic to the framework, since it only takes into account the fixed points or theories, a fact which makes sense since our operators only satisfy idempotency. The second is an attempt to emulate more closely the comparison in the classical framework. Here, one also considers the overall structure of the logicate; not solely its theories. This comparison is more "structure preserving" at the expense of being, somehow, more "artificial", since the structure is not intrinsic but rather devised. This artificiality is mended in a way in the second part of the monograph, where we switch focus from logicates to logicoids, in which the "structure" is inserted into the formalism, thus becoming "more natural".

Another important construct in both the abstract and concrete studies in algebraic logic is that of axiomatic extensions. When monotonicity is present, a closure operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is viewed as a consequence operator of a logic. One may need to add a subset $T \subseteq A$ as a new set of axioms to axiomatically strengthen the consequence relation. This is done by defining a new operator, based on the original, by setting, for all $X \subseteq A$, $C^T(X) = C(X \cup T)$. Note that both inflationarity and monotonicity are critical here. The first ensures that, for all X, $T \subseteq C^T(X)$, that is the new axioms become genuine consequences of the new operator. The second yields that $C \leq C^T$, i.e., the new operator is indeed an *extension* of the former via the adoption of the elements in T as new axioms. The criticality of these two axioms and the fact that they are missing in the nonmonotonic framework adopted here give an indication of why the task of emulating the extension process would necessarily involve difficulties and may ultimately prove insufficient and unsatisfactory. Nevertheless, we do the best we can by devising two different operators along these lines.

The first uses a more conservative approach. It "lifts" the consequences of $X \subseteq A$ to $C(T)$ if either X or $C(X)$ are contained in T. But some emphasis must be placed on the pejorative use of "lift" here, since, in fact, $C(T)$ may be a much smaller subset of A than either X or T, due to lack of inflationarity and monotonicity. This approach has the drawback that it does not give an operator which strengthens the original operator according to the "structure preserving" comparison of operators that we alluded to in the preceding paragraph. We take this as hinting to the need of an alternative, more "aggressive", line of attack. The more liberal approach, on the other hand, allows consequences to be lifted to $C(T)$ whenever the consequences of X happen to coincide with the consequences of some $Y \subseteq T$. This construct gives rise to an operator that does strengthen the original operator C and, as it turns out, strengthens also the operator obtained by the more conservative approach.

In Section 2.2, the basic objects of study, called *logicates*, which are idempotent operators on the powerset of a set are introduced. The directed graphs that reflect the structure of logicates are called *necropoleis*,[1] since they consist of components called *pyramids*. By imposing a linear ordering on $\mathcal{P}(A)$, one may recast both as linearly ordered structures, with additional features, called linearized consequences and linearized necropoleis, respectively. However, the process of linearization introduces redundancy, which one sheds by passing to equivalence classes of those ordered structures under appropriately defined equivalence relations. We call an equivalence class of linearized necropoleis a *cemetery*.

In Section 2.3 we encounter ways we may use to compare logicates over the same underlying set. *Equipotency* is the equivalence resulting by having identical sets of theories. By comparing sets of theories by the subset relation, we may also impose a partial ordering on the set of equipotency classes. Being *weaker*, on the other hand, is a relation that also takes into account sets of theories but, in addition, it considers the consequence structure. These comparisons are also investigated from the point of view of alternative presentations of logicates, namely, using necropoleis, classes of linearized consequences and cemeteries.

In Section 2.4, we introduce and compare the two notions that attempt to replace axiomatic extensions in the nonmonotonic context. The first is

[1] Plural of **necropolis**, pronounced the same, but stressed *necropóleis* vs. *necrópolis*.

called *boosting*. It seems a natural one to adopt, based on Occam's Razor. However, it fails to produce a strengthened version of the original logicate under the comparison criterion that takes the consequence structure of the logicate into account. To atone for this failure, we fortify boosting to what we call *strong boosting*. This adjustment produces an operator that strengthens both the original and the boosted version of the original operator.

2.2 Logicates

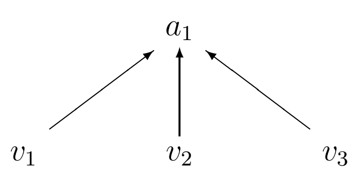
In this section, we introduce the basic notion of *logicate* which forms the underlying object of study throughout. We spend some time giving different representations that may help looking at these objects from different points of view, developing some intuition about them and, also, perhaps, in visualizing their behavior.

Let A be a set. Let $\mathcal{P}(A)$ denote the powerset of A. An **idempotent mapping on** $\mathcal{P}(A)$ is a mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for all $X \subseteq A$,

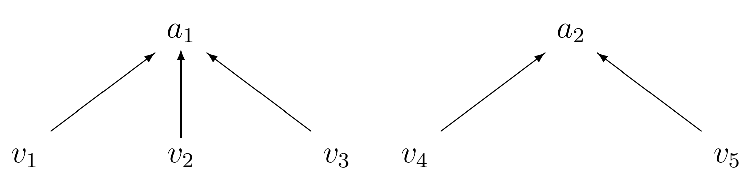
(**Idempotency**) $C(C(X)) = C(X)$.

A **consequence operator on** A or a **logicate on** A is an idempotent mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$.

A **pyramid** is a simple kind of directed graph. It has a distinguished vertex, called the **apex**, and all its other vertices, called **base vertices**, are connected to the apex.



A **necropolis** is a collection of (disjoint) pyramids.



A **consequence system on** A is a necropolis on A, i.e., a necropolis $G = \langle \mathcal{P}(A), E \rangle$, with vertex set $\mathcal{P}(A)$.

Proposition 1 assures that, for all intends and purposes, consequence operators and consequence systems are interchangeable. So which one to use depends entirely on the point of view taken and on the convenience for the specific application.

Proposition 1 *Let A be a set. Consequence operators on A are in one to one correspondence with consequence systems on A.*

Proof: Consider a consequence operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Create the graph of the function C, i.e., the graph having set of vertices $\mathcal{P}(A)$ and edges $X \to Y$ iff $C(X) = Y$ Then throw away the loops. Because of idempotency, this gives rise to a consequence system on A.

Conversely, consider a consequence system $\langle \mathcal{P}(A), E \rangle$ on A. Define $C : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $X \subseteq A$,

$$C(X) = \begin{cases} Y, & \text{if } (X, Y) \in E, \\ X, & \text{if } X \text{ has outdegree } 0. \end{cases}$$

It should be fairly clear that, since $\langle \mathcal{P}(A), E \rangle$ is a consequence system, C is well defined and idempotent. Thus, C is a consequence operator.

Starting from a consequence operator on A, constructing its graph and, then, obtaining the operator associated with it, results to the original consequence operator. And, similarly, stating from a consequence system, creating the function depicted by it and, then, obtaining its graph, results to the original consequence system. Thus, the established correspondence is a one-to-one correspondence. ■

A **linearized consequence** $\langle \mathcal{P}(A), \leq, L \rangle$ **on** A consists of a linear ordering $\leq$ on $\mathcal{P}(A)$ and a function $L : \mathcal{P}(A) \to \mathcal{P}(A)$, which satisfies, for all $X, Y \subseteq A$:

(**Idempotency**) $L(L(X)) = X$;

(**≤-Inflationarity**) $X \leq L(X)$;

(**≤-Monotonicity**) $X \leq Y$ implies $L(X) \leq L(Y)$.

Two linearized consequences $\langle \mathcal{P}(A), \leq, L \rangle$ and $\langle \mathcal{P}(A), \leq', L' \rangle$ on A are **equivalent**, written

$$\langle \mathcal{P}(A), \leq, L \rangle \sim \langle \mathcal{P}(A), \leq', L' \rangle,$$

if, for all $X \subseteq A$,

$$L(X) = L'(X).$$

It is immediate from the definition that the equivalence $\sim$ is a bona fide equivalence relation on linearized consequences on a given set A. Consequently, it partitions the collection of linearized consequences into equivalence classes. The collection of equivalence classes are in one-to-one correspondence with consequence operators (and, hence, by Proposition 1, with consequence systems) on A.

Proposition 2 *Consequence operators on a set A are in one-to-one correspondence with equivalence classes of linearized consequences on A.*

Proof: Let $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a consequence operator on A. Create its consequence system. Arbitrarily order its pyramids, say $P_0, P_1, \ldots$. For each pyramid P_i, create a linear ordering $o(P_i)$ of is vertices by arbitrarily ordering the base vertices and then placing the apex as the largest vertex. Finally, create a linear ordering $\leq$ of $\mathcal{P}(A)$ by juxtaposing the linear orderings $o(P_i)$ of the pyramids according to the originally adopted ordering of the set of pyramids,

$$o(P_0), o(P_1), \ldots.$$

Now consider the structure $\langle \mathcal{P}(A), \leq, C\rangle$. Since, by hypothesis, C is a consequence operator, $C(C(X)) = C(X)$, for all $X \subseteq A$. So idempotency holds. Since the apex of a pyramid follows all vertices of its base, $X \leq C(X)$, whence $\leq$-Inflationarity also holds. Finally, consider $X, Y \subseteq A$, such that $X \leq Y$.

- Suppose X, Y are vertices in the same pyramid. Then $C(X) = C(Y)$.
- Suppose X, Y are vertices in different pyramids. Then the pyramid of X precedes in the linear ordering the pyramid of Y, whence $C(X) < C(Y)$.

In either case $C(X) \leq C(Y)$ and $\leq$-Monotonicity also holds. Thus, $\langle \mathcal{P}(A), \leq, C\rangle$ is a linearized consequence on A. We let its equivalence class $L(C)$ be the class associated with the consequence operator C. By the definition of equivalence, any two linearized consequences constructed in this way from the same consequence operator are equivalent, whence $C \mapsto L(C)$ is well defined.

Suppose, conversely, that an equivalence class of linearized consequences is given. Consider a representative $\langle \mathcal{P}(A), \leq, L\rangle$. To this class we associate the consequence operator $L : \mathcal{P}(A) \to \mathcal{P}(A)$. By definition, if one changes the representative, the consequence operator remains invariant. It follows that this association is also well defined.

The two mappings just described are inverses of one another. So they establish a one-to-one correspondence between consequence operators on A and equivalence classes of linearized consequences on A. ■

A **linearized necropolis** $\langle \mathcal{P}(A), \leq, c\rangle$ **on** A consists of a linear ordering $\leq$ on $\mathcal{P}(A)$, having a maximum element, together with a 2-coloring

$$c : \mathcal{P}(A) \to \{w, p\}$$

of the vertices, with colors, say, white and purple, such that the $\leq$-maximum element is colored purple.

Two linearized necropoleis $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c'\rangle$ on $\mathcal{P}(A)$ are **equivalent**, written

$$\langle \mathcal{P}(A), \leq, c\rangle \bowtie \langle \mathcal{P}(A), \leq', c'\rangle,$$

if the colorings are identical, that is, $c = c'$, and each purple vertex is preceded in each ordering (before the appearance of another purple vertex) by the same (unordered) set of white vertices, i.e., for any intervals in $\leq$ and $\leq'$, respectively,

$$p_0, w_0, w_1, \ldots, p_1$$
$$p'_0, w'_0, w'_1, \ldots, p'_1,$$

where the vertices denoted by w's are colored white and the vertices denoted by p's are colored purple, if $p_1 = p'_1$, then $\{w_0, w_1, \ldots\} = \{w'_0, w'_1, \ldots\}$.

A **cemetery on** A is an equivalence class (with respect to $\bowtie$) of linearized necropoleis on A.

Proposition 3 *Let A be a set. Consequence operators are in one-to-one correspondence with cemeteries on A.*

Proof: By Proposition 2, it suffices to show that equivalence classes of linearized consequences are in one-to-one correspondence with cemeteries.

Consider, first, an equivalence class of linearized consequences, with representative $\langle \mathcal{P}(A), \leq, L\rangle$. Construct the linearized necropolis $\langle \mathcal{P}(A), \leq, c\rangle$ by defining the coloring c by setting, for all $X \subseteq A$,

$$c(X) = \begin{cases} p, & \text{if } L(X) = X, \\ w, & \text{otherwise.} \end{cases}$$

Suppose two linearized consequences $\langle \mathcal{P}(A), \leq, L\rangle$ and $\langle \mathcal{P}(A), \leq', L'\rangle$ are equivalent. Then, by definition, $L = L'$. Thus, directly by definition, $c = c'$. So to establish that the two linearized necropoleis $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c\rangle$ are equivalent, it suffices to show that the two orderings are related in the required way. Consider two intervals in $\leq$ and $\leq'$, respectively,

$$p_0, w_0, w_1, \ldots, p_1$$
$$p'_0, w'_0, w'_1, \ldots, p'_1,$$

where the vertices denoted by w's are colored white, the vertices denoted by p's are colored purple and $p_1 = p'_1$. Assume, without loss of generality, that $p'_1 = p_1 <' w_0 <' p'_2$, where p'_2 is the first purple vertex following w_0 in $<'$. The axioms of linearized consequences, then, give $L(w_0) = p_1$ and $L(w_0) = L'(w_0) = p'_2 \neq p'_1 = p_1$, a contradiction. Thus, the two linearized necropoleis $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c'\rangle$ are equivalent.

Suppose, conversely, that a cemetery is given. Let $\langle \mathcal{P}(A), \leq, c\rangle$ be a representative linearized necropolis. Define the triple $\langle \mathcal{P}(A), \leq, L\rangle$ by setting, for all $X \subseteq A$,

$$L(X) = \min\{Y : X \leq Y \text{ and } c(Y) = p\}.$$

We show that $\langle \mathcal{P}(A), \leq, L\rangle$ is a linearized consequence. Indeed, for all $X, Y \subseteq A$, we have:

- $X \leq \min\{Y : X \leq Y \text{ and } c(Y) = p\} = L(X)$;

- Further,

$$\begin{array}{lll} X \leq Y & \text{implies} & \min\{Z : X \leq Z \text{ and } c(Z) = p\} \\ & & \quad \leq \min\{Z : Y \leq Z \text{ and } c(Z) = p\} \\ & \text{implies} & L(X) \leq L(Y); \end{array}$$

- Finally,

$$\begin{array}{lll} L(L(X)) & = & \min\{Y : L(X) \leq Y \text{ and } c(Y) = p\} \\ & = & \min\{Y : \min\{Z : X \leq Z \text{ and } c(Z) = p\} \leq Y \\ & & \qquad \text{and } c(Y) = p\} \\ & = & \min\{Z : X \leq Z \text{ and } c(Z) = p\} \\ & = & L(X). \end{array}$$

Suppose $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c\rangle$ are equivalent linearized necropoleis. We show that $\langle \mathcal{P}(A), \leq, L\rangle$ and $\langle \mathcal{P}(A), \leq', L'\rangle$ are equivalent linearized consequences. Suppose, to the contrary, that, for some $X \subseteq A$, $L(X) \neq L'(X)$. Then X is in the interval preceding the purple element $L(X)$ in $\leq$, but X is not in the interval preceding the purple element $L(X)$ in $\leq'$. This contradicts the equivalence of the two linearized necropoleis $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c\rangle$.

The two associations are inverses of one another. So equivalence classes of linearized consequences correspond to cemeteries. As a result, consequence operators on A are also in one-to-one correspondence with cemeteries on A, as claimed. ∎

2.3 Comparing Logicates

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a logicate. We set

$$\mathcal{C} = \{X \subseteq A : C(X) = X\},$$

i.e., $\mathcal{C}$ is the set of its **fixed points** or **theories**. Let us denote the collection of all logicates on the same set A by $\mathrm{Lgct}(A)$.

A critical role in our considerations is played by the theories of a logicate. In fact, on several occasions we may need to construct a logicate for which the only feature that matters is the set of its theories and is otherwise arbitrary. To prepare for this eventuality, we define the notion of *equipotency*.

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ be logicates on A. We say that C and C' are **equipotent**, written $C \triangleq C'$, if $\mathcal{C} = \mathcal{C}'$,

$$C \triangleq C' \quad \text{iff} \quad \mathcal{C} = \mathcal{C}'.$$

To refer to the $\triangleq$-equivalence class of a logicate C, we may write either $\mathcal{C}$ or $C/\!\triangleq$. The first notation, which adds overloading, requires a typographical correspondence that must be respected to avoid confusion.

In accordance, two consequence systems $\langle \mathcal{P}(A), E\rangle$ and $\langle \mathcal{P}(A), E'\rangle$ are **equipotent**, written

$$\langle \mathcal{P}(A), E\rangle \triangleq \langle \mathcal{P}(A), E'\rangle,$$

if the set of apexes of their pyramids are identical.

Proposition 4 *Suppose A is a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ and $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ are two logicates. C and C' are equipotent if and only if the corresponding consequence systems are equipotent.*

Proof: This is a direct consequence of the correspondence established in Proposition 1. In fact, we have

$$\begin{array}{lll} C \triangleq C' & \text{iff} & \text{for all } X,\ C(X) = X \text{ iff } C'(X) = X \\ & \text{iff} & \text{for all } X,\ X \text{ has outdegree 0 in } E \text{ iff} \\ & & \qquad\qquad X \text{ has outdegree 0 in } E' \\ & \text{iff} & \langle \mathcal{P}(A), E\rangle \text{ and } \langle \mathcal{P}(A), E'\rangle \text{ have same apexes} \\ & \text{iff} & \langle \mathcal{P}(A), E\rangle \triangleq \langle \mathcal{P}(A), E'\rangle. \end{array}$$

■

Logicates may be preordered using the inverse inclusion relation between their theories, i.e.,

$$C \trianglelefteq C' \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

This preorder induces a partial ordering between $\triangleq$-equivalence classes, which is expressed by

$$C/{\triangleq} \trianglelefteq C'/{\triangleq} \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

The unpleasant features of this construct are, first, that it compares equivalence classes and, second, that it hides important details of the consequence relation. On the other hand, we get the advantage of having a nice ordered set on the equivalence classes (almost a complete lattice, but without top). This is an important feature that we shall build upon, attempting to simulate, to the extent possible, the traditional algebraic theory pertaining to monotonic logics.

Proposition 5 *Let A be a set. The quotient $\mathrm{Lgct}(A)/{\triangleq}$, ordered by $\trianglelefteq$, is isomorphic to the ordered set $\langle \mathcal{P}(\mathcal{P}(A))\backslash\varnothing, \supseteq\rangle$.*

Proof: To see this, it suffices to show that the mapping

$$\begin{array}{rrcl} \mathrm{Idp}: & \mathrm{Lgct}(A)/{\triangleq} & \longrightarrow & \mathcal{P}(\mathcal{P}(A))\backslash\varnothing; \\ & C/{\triangleq} & \longmapsto & \mathcal{C}, \end{array}$$

is an order isomorphism from $\langle \mathrm{Lgct}(A)/{\triangleq}, \trianglelefteq\rangle$ onto $\langle \mathcal{P}(\mathcal{P}(A))\backslash\varnothing, \supseteq\rangle$. Note that Idp is well defined and one-to-one, since, by definition of $\triangleq$, for all $C, C' \in \mathrm{Lgct}(A)$,

$$C/{\triangleq} = C'/{\triangleq} \quad \text{iff} \quad \mathcal{C} = \mathcal{C}'.$$

It is also onto. Given $\varnothing \neq \mathcal{X} \subseteq \mathcal{P}(A)$, we let $X_0 \in \mathcal{X}$ be a fixed element of $\mathcal{X}$ and define $C : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $Y \subseteq A$,

$$C(X) = \begin{cases} X, & \text{if } X \in \mathcal{X}, \\ X_0, & \text{otherwise.} \end{cases}$$

Then, clearly, $\mathcal{C} = \mathcal{X}$. So Idp is a bijection. Moreover, by the definition of $\trianglelefteq$, for all $C, C' \in \mathrm{Lct}(A)$,

$$C/{\triangleq} \trianglelefteq C'/{\triangleq} \quad \text{iff} \quad \mathcal{C} \supseteq \mathcal{C}'.$$

Thus, Idp is both order preserving and order reflecting. ■

We next introduce a different ordering that takes also into account the consequence structure. We write $C \leq C'$ and say that C is **weaker** than C' and that C' is **stronger** than C if the following conditions hold, for all $X, Y \subseteq A$:

- $C'(X) = X$ implies $C(X) = X$, i.e., all C'-theories are also C-theories;
- $C(X) = Y$ implies $C'(X) = C'(Y)$, i.e., any consequence in C must be between elements having the same image under C'.

Note that the first condition may be equivalently formulated by saying that, for all $X \subseteq A$,

$$C'(X) = C(C'(X)).$$

Note, also, that the second condition may be equivalently formulated by saying that, for all $X \subseteq A$,

$$C'(X) = C'(C(X)).$$

Consequently, we may summarize by saying that $C \leq C'$ if and only if, for all $X \subseteq A$,

$$C'(X) = C(C'(X)) = C'(C(X)).$$

The $\leq$ relation has the advantage that it is a genuine partial ordering on $\mathrm{Lgct}(A)$. So there is no need to take a quotient. On the other hand, it is not difficult to see that its order structure is less well behaved than $\trianglelefteq$.

Proposition 6 *The structure* $\mathbf{Lgct}(A) = \langle \mathrm{Lgct}(A), \leq \rangle$ *is a partially ordered set.*

Proof: We have to prove reflexivity, antisymmetry and transitivity. To this end, let $C, C', C'' \in \mathrm{Lct}(A)$.

- By idempotency, for all X, $C(X) = C(C(X))$, whence $C \leq C$ and $\leq$ is reflexive.

- Suppose $C \leq C'$ and $C' \leq C$, Then, we get, for all X,

$$C(X) = C(C'(X)) = C'(X).$$

  Thus $C = C'$ and $\leq$ is anti-symmetric.

- Let $C \leq C'$ and $C' \leq C''$. Then, by definition, for all X,

$$\begin{array}{l} C'(X) = C'(C(X)) = C(C'(X)), \\ C''(X) = C''(C'(X)) = C'(C''(X)). \end{array}$$

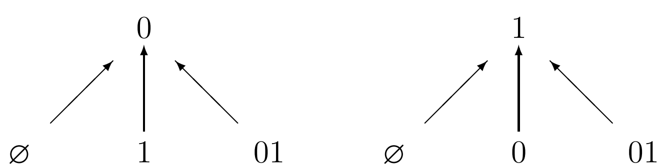
  Therefore, we get

$$\begin{array}{lcl} C(C''(X)) & = & C(C'(C''(X))) = C'(C''(X)) = C''(X); \\ C''(C(X)) & = & C''(C'(C(X))) = C''(C'(X)) = C''(X). \end{array}$$

  Hence, by definition, $C \leq C''$ and $\leq$ is also transitive.

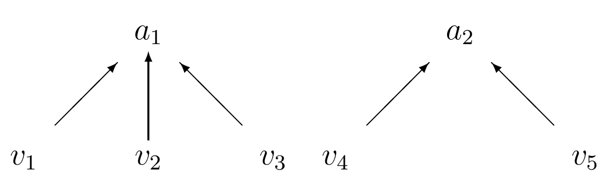
Thus, $\mathbf{Lgct}(A) = \langle \mathrm{Lgct}(A), \leq \rangle$ is a partially ordered set. ∎

To see that the order is not lattice-like consider, e.g., The following two logicates on $\{0, 1\}$, given in their necropolis representation.

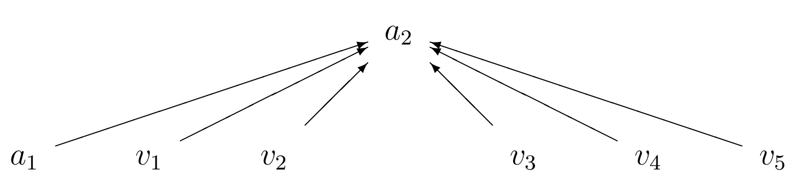


The following two logicates are lower bounds. There is, however, no greatest lower bound.

Consider, next two or more disjoint pyramids with vertices in $\mathcal{P}(A)$.



A **merge** results by taking one of the apexes as a new apex and all other vertices as base vertices of a new pyramid. E.g., a merge of the two pyramids above is



Suppose G' is a necropolis obtained from a necropolis G by some merges. Then we say that G is **finer** than G' and write $G' \leq_n G$. The same terminology carries to consequence systems. That is, given two consequence systems $G = \langle \mathcal{P}(A), E\rangle$ and $G' = \langle \mathcal{P}(A), E'\rangle$, we say that G is **finer** than G' and write $G' \leq_n G$, if the necropolis G' is the result of some merges applied on the necropolis G.

Proposition 7 *Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ consequence operators on A, with corresponding consequence systems $G = \langle \mathcal{P}(A), E\rangle$, $G' = \langle \mathcal{P}(A), E'\rangle$. The $C \leq C'$ if and only if $G' \leq_n G$.*

Proof: Suppose, first, that $C \leq C'$. Consider a pyramid in G'. If X is its apex, then $C'(X) = X$. Thus, by the definition of $\leq$, we get that $C(X) = X$. This shows that X is the apex of a pyramid in G. Let Y be a base vertex of the same pyramid in G. Then $C(Y) = X$. Therefore, $C'(Y) = C'(X) = X$. So Y is a base vertex of the pyramid with apex X in G'. Suppose Z is a base vertex of the pyramid in G' that is not a base vertex of the pyramid in G with apex X, and let Z' be another vertex of the same pyramid in G as Z. Then $C(Z) = C(Z') = Z''$, which imply $C'(Z) = C'(Z') = C'(Z'') = X$. Hence all these vertices are base vertices of the pyramid in G', with apex X. This shows that G' results from merges of pyramids in G. Therefore $G' \leq_n G$.

Suppose, conversely, that G is finer that G'. We show that $C \leq C'$. Suppose that for some $X \subseteq A$, $C'(X) = X$. Then, in G', X is the apex of a pyramid. Since that pyramid is the result of a merge of one or more pyramids in G, X must be an apex in G. Thus, $C(X) = X$. Assume, next, that, for some $X, Y \subseteq A$, $C(X) = Y$. Thus, in G, X and Y are elements of the same pyramid. Therefore, in G' also, X and Y must belong to the same pyramid (into which the one in G has been merged). It follows that $C'(X) = C'(Y)$. By the definition of $\leq$, we conclude that $C \leq C'$. ∎

A different characterization may be obtained by looking at equivalence classes of linearized consequences. Let $\langle \mathcal{P}(A), \leq, L\rangle$ and $\langle \mathcal{P}(A), \leq', L'\rangle$ be two linearized consequences. We say that $\langle \mathcal{P}(A), \leq, L\rangle$ is **weaker** than $\langle \mathcal{P}(A), \leq', L'\rangle$ and that $\langle \mathcal{P}(A), \leq', L'\rangle$ is **stronger** than $\langle \mathcal{P}(A), \leq, L\rangle$, written

$$\langle \mathcal{P}(A), \leq, L\rangle \leq_\ell \langle \mathcal{P}(A), \leq', L'\rangle,$$

if the following conditions hold:

- $\leq = \leq'$;
- $L(X) \leq L'(X)$, for all $X \subseteq A$.

This ordering induces an ordering on the collection of equivalence classes of linearized congruences. Namely, we say that an equivalence class of linearized consequences is **weaker** than another class if there exist a representative

$\langle \mathcal{P}(A), \leq, L\rangle$ of the first class and a representative $\langle \mathcal{P}(A), \leq', L'\rangle$ of the second class, such that

$$\langle \mathcal{P}(A), \leq, L\rangle \leq_\ell \langle \mathcal{P}(A), \leq', L'\rangle.$$

Further, this ordering on equivalence classes reflects the ordering on consequence operators.

Proposition 8 *Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ be consequence operators on A. Then $C \leq C'$ if and only if the equivalence class of linearized consequences associated with C is weaker that the class associated with C'.*

Proof: Suppose, first, that $C \leq C'$. One has to choose a representative of the linearized consequence corresponding to C carefully so as to be able to accommodate the strengthening to C' using the same linear ordering. Using Propositions 1 and 7, we order the pyramids of the consequence system so that pyramids of C merged by C' are placed in adjacent positions in the ordering and, moreover, so that the pyramid whose apex is used as the apex in the merge is placed last. Define $\leq$ to be the ordering of $\mathcal{P}(A)$ constructed in this way. Then it is not difficult to see that both C and C' satisfy Idempotency, $\leq$-Inflationarity and $\leq$-Monotonicity. Further, by construction, $C(X) \leq C'(X)$, for all $X \subseteq A$. Thus, the class of linearized consequences represented by $\langle \mathcal{P}(A), \leq, C\rangle$ is indeed weaker than the class represented by $\langle \mathcal{P}(A), \leq, C'\rangle$.

Suppose, conversely, that C is represented by the linearized consequence $\langle \mathcal{P}(A), \leq, C\rangle$, C' is represented by the linearized congruence $\langle \mathcal{P}(A), \leq, C'\rangle$ and that $C(X) \leq C'(X)$, for all $X \subseteq A$. If $C'(X) = X$, then $C(X) \leq C'(X) = X$ and, since the reverse inequality holds by $\leq$-Inflationarity, we get $C(X) = X$. Finally, if $C(X) = Y$, then

$$C'(Y) = C'(C(X)) \leq C'(C'(X)) = C'(X)$$

and, since $X \leq C(X) = Y$, $C'(X) \leq C'(Y)$. Hence, $C'(X) = C'(Y)$. So the two properties demanded by the definition of $\leq$ for consequence operators are satisfied, showing that $C \leq C'$. ■

Finally, we turn to cemeteries to establish similar comparison criteria. Let $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c'\rangle$ be linearized necropoleis on $\mathcal{P}(A)$. We say that $\langle \mathcal{P}(A), \leq, c\rangle$ is **finer** than $\langle \mathcal{P}(A), \leq', c'\rangle$, written

$$\langle \mathcal{P}(A), \leq', c'\rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c\rangle,$$

if the following hold:

- $\leq = \leq'$;
- $c'(X) = p$ implies $c(X) = p$, i.e. the set of purple nodes under c' is a subset of those under c.

In a way similar to linearized consequences, we say that a cemetery $\mathcal{T}$ on A is **finer** than a cemetery $\mathcal{T}'$ on A if there exist a linearized necropolis $\langle \mathcal{P}(A), \leq, c\rangle$ representing $\mathcal{T}$ and a linearized necropolis $\langle \mathcal{P}(A), \leq', c'\rangle$ representing $\mathcal{T}'$, such that

$$\langle \mathcal{P}(A), \leq', c'\rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c\rangle.$$

Proposition 9 *Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ be consequence operators on A. Then $C \leq C'$ if and only if the cemetery associated with C is finer that the cemetery associated with C'.*

Proof: In the proof of Proposition 3, we showed that classes of linearized consequences and cemeteries are in one-to-one correspondence. More precisely, a class of linearized consequences represented by $\langle \mathcal{P}(A), \leq, L\rangle$ corresponds to the cemetery represented by $\langle \mathcal{P}(A), \leq, c\rangle$, where $c(X) = p$ if and only if $L(X) = X$, for all $X \subseteq A$. And, conversely, a cemetery represented by $\langle \mathcal{P}(A), \leq, c\rangle$ corresponds to the class represented by the linearized consequence $\langle \mathcal{P}(A), \leq, L\rangle$, where

$$L(X) = \min\{Y : X \leq Y \text{ and } c(Y) = p\}.$$

In view of Proposition 8, it suffices to show that a class of linearized consequences is weaker than another class if and only if the corresponding cemetery of the second is finer than the cemetery corresponding to the first.

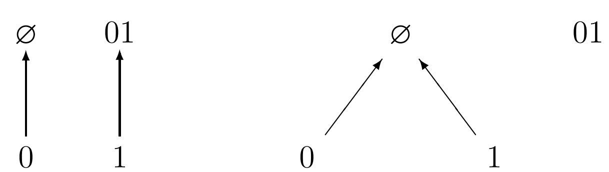
Assume, first, that $\langle \mathcal{P}(A), \leq, L\rangle \leq_\ell \langle \mathcal{P}(A), \leq, L'\rangle$. Then, by definition, $L(X) \leq L'(X)$, for all $X \subseteq A$. It follows immediately that $L'(X) = X$ implies $L(X) = X$. Thus, $c'(X) = p$ implies $c(X) = p$. Thus, by definition, $\langle \mathcal{P}(A), \leq, c'\rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c\rangle$.

Assume, conversely, that $\langle \mathcal{P}(A), \leq, c'\rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c\rangle$. Then, for the corresponding linearized consequences, we have, for all $X \subseteq A$,

$$\begin{aligned} L(X) &= \min\{Y : X \leq Y \text{ and } c(Y) = p\} \\ &\leq \min\{Y : X \leq Y \text{ and } c'(Y) = p\} \\ &= L'(X). \end{aligned}$$

Thus, $\langle \mathcal{P}(A), \leq, L\rangle \leq_\ell \langle \mathcal{P}(A), \leq, L'\rangle$. ■

In concluding this section, let us also observe that equipotent logicates are, in general, incomparable with respect to strengthening. E.g., if $A = \{0, 1\}$, the following two consequence operators are equipotent, but neither of the two is weaker than the other.



2.4 Boosting and Strong Boosting

In this section we attempt to build operations that are, in the nonmonotonic context, "parallel" to axiomatic extensions in the traditional monotonic framework. We start with an operation called *boosting*. It is devised by Occam's razor, i.e., it is seemingly the simplest possible recipe that makes sense with what we have. However, it is shown that, even though boosting gives rise to a logicate and, in fact, a logicate that is related to the given one, it does not result in a strengthening of the original according to the $\leq$ ordering. So we are compelled to fortify boosting giving rise to what we call *strong boosting*. We show that the strong boosting construction results in a logicate which is not only a strengthening of the original but also a strengthening of its boosting.

Suppose $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is a logicate and $T \subseteq A$. The **boosting of** C **by** T is the operator $C_T : \mathcal{P}(A) \to \mathcal{P}(A)$ that is defined, for all $X \subseteq A$, by

$$C_T(X) = \begin{cases} C(T), & \text{if } X \subseteq T \text{ or } C(X) \subseteq T, \\ C(X), & \text{otherwise.} \end{cases}$$

We show that this recipe gives a bona fide consequence operator.

Proposition 10 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. Then $C_T : \mathcal{P}(A) \to \mathcal{P}(A)$ is also a logicate.*

Proof: Suppose, first, that $X \nsubseteq T$ and $C(X) \nsubseteq T$. Then, by definition

$$C_T(C_T(X)) = C_T(C(X)) = C(C(X)) = C(X) = C_T(X).$$

Suppose, next, that $X \subseteq T$ or $X \subseteq C(T)$. Then

$$C_T(C_T(X)) = C_T(C(T)) = C(T) = C_T(X).$$

Thus, in either case C_T is idempotent and, therefore, it is a consequence operator. ■

We also describe the way the two consequence systems are related. Consider a consequence system $G = \langle \mathcal{P}(A), E \rangle$ and let $T \subseteq A$. Define the graph $G_T = \langle \mathcal{P}(A), E_T \rangle$ as follows. If, in G, a base vertex X of some pyramid is such that $X \subseteq T$, then, make X a base vertex of the pyramid with apex $C(T)$. If, in G, the apex X of some pyramid is such that $X \subseteq T$, then merge that pyramid with the pyramid of $C(T)$, keeping $C(T)$ as the apex of the merge.

Proposition 11 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. If G is the consequence system corresponding to C, then G_T is the consequence system corresponding to C_T.*

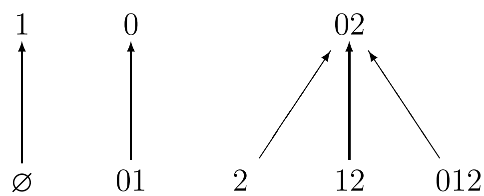
Proof: Suppose, first, that $X \subseteq T$ is a base vertex of some pyramid. Then, by definition, $C_T(X) = C(T)$. Moreover, since X is a base vertex, $C(X) \neq X$. On the other hand

$$C(C_T(T)) = C(C(T)) = C(T) = C_T(T).$$

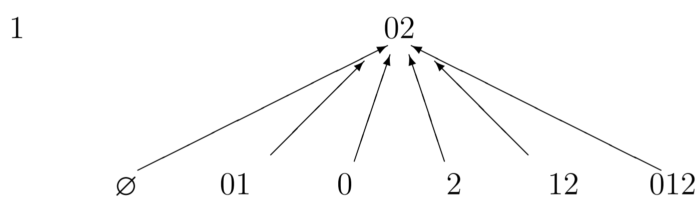
Therefore, $X \neq C_T(T)$. Hence, in G_T, X is attached as a base vertex of the pyramid with apex $C(T)$.

Suppose, finally, that $X \subseteq T$, with $X \neq C(T)$, is the apex of some pyramid. Then, for all Y in the same pyramid, $C(Y) = X \subseteq T$. Thus, $C_T(Y) = C(T) \neq X$. This shows that in G_T the entire pyramid gets merged under the pyramid with vertex $C(T)$. ∎

Note that boosting, in general, is equipotent to a logicate resulting by applying merges. E.g., consider the consequence operator C on $A = \{0, 1, 2\}$, whose consequence system is pictured below.



Consider $T = \{0, 2\}$. Then the consequence operator C_T has the consequence system G_T, shown below. It is equipotent to the system resulting from G by an application of a single merge.



So, as equipotents may be incomparable with respect to strengthening, it is not, in general, the case that C_T is a strengthening of C. This leads to the operation of strong boosting, which does lead to strengthening since it only involves merges.

Suppose $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is a consequence operator and $T \subseteq A$. The **strong boosting of** C **by** T is the consequence operator $C^T : \mathcal{P}(A) \to \mathcal{P}(A)$ that is defined, for all $X \subseteq A$, by

$$C^T(X) = \begin{cases} C(T), & \text{if } C(X) = C(Y), \text{ for some } Y \subseteq T, \\ C(X), & \text{otherwise.} \end{cases}$$

For now, let us point out that, if, for some $X \subseteq A$, $C_T(X)$ is determined by the first branch in the piecewise definition of C_T, then the same holds in the piecewise definition of C^T. Suppose that $X \subseteq T$. Then certainly $C(X) = C(Y)$, for some $Y \subseteq T$. Moreover, if $C(X) \subseteq T$, then, similarly, $C(X) = C(Y)$, for some $Y \subseteq T$ (namely, for $Y = C(X)$).

Proposition 12 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. Then $C^T : \mathcal{P}(A) \to \mathcal{P}(A)$ is also a logicate.*

Proof: Let $X \subseteq A$. Suppose, first, that $C(X) = C(Y)$, for some $Y \subseteq T$. Then we have

$$C^T(C^T(X)) = C^T(C(T)) = C(T) = C^T(T).$$

On the other hand, suppose that, for all $Y \subseteq T$, $C(X) \neq C(Y)$. Note that, in this case, one also has that $C(C(X)) = C(X) \neq C(Y)$, for all $Y \subseteq T$. Taking this into account, we get

$$C^T(C^T(X)) = C^T(C(X)) = C(C(X)) = C(X) = C^T(X).$$

Therefore, C^T is a consequence operator. ■

We describe, next, the way corresponding consequence systems are related. Consider a consequence system $G = \langle \mathcal{P}(A), E \rangle$ and let $T \subseteq A$. Define the necropolis $G^T = \langle \mathcal{P}(A), E^T \rangle$ as follows. If, in G, a vertex X of some pyramid is such that $X \subseteq T$, then merge that pyramid with the pyramid of $C(T)$, keeping $C(T)$ as the apex of the merge.

Proposition 13 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. If G is the consequence system corresponding to C, then G^T is the consequence system corresponding to C^T.*

Proof: Suppose $X \subseteq T$ is a vertex (base or apex) of some pyramid in G. Then, for all Y in the same pyramid, $C(Y) = C(X)$ and, hence, $C^T(Y) = C(T)$. Thus, the entire pyramid is merged in G^T under $C(T)$ (this operation is trivial if $C(X) = C(T)$). For Y any vertex in any pyramid in which $X \nsubseteq T$, for all X, we get, by definition, $C^T(X) = C(X)$. Thus, all these pyramids are maintained in G^T as they were in G. ■

It turns out that the strong boosting C^T is a strengthening both of C and of the boosting C_T.

Proposition 14 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. Then $C \leq C^T$ and $C_T \leq C^T$.*

Proof: That $C \leq C^T$ is a direct consequence of Proposition 13, which asserts that G^T is a merge of G, i.e., that $G^T \leq_n G$, and of Proposition 7, which, when applied to $G^T \leq_n G$, gives that $C \leq C^T$.

For the second strengthening relation, we must prove that, for all $X \subseteq A$, $C^T(X) = X$ implies $C_T(X) = X$ and $C^T(X) = C^T(C_T(X))$.

Let us, first, fix $X \subseteq A$, such that $C^T(X) = X$. If $X \subseteq T$ or $C(X) \subseteq T$,

$$\begin{aligned} X &= C^T(X) \quad \text{(Hypothesis)} \\ &= C(T) \quad (C(X) = C(C(X)) \text{ and } X \subseteq T \text{ or } C(X) \subseteq T) \\ &= C_T(X). \quad (X \subseteq T \text{ or } C(X) \subseteq T) \end{aligned}$$

Otherwise,

$$\begin{array}{rcl} C_T(X) & = & C(X) \quad (X \nsubseteq T \text{ and } C(X) \nsubseteq T) \\ & = & C(C^T(X)) \quad (C^T(X) = X) \\ & = & C^T(X) \quad (C \leq C^T) \\ & = & X. \quad (C^T(X) = X) \end{array}$$

In either case, the first condition holds.

Let us, now, fix $X \subseteq A$. One has to look at three possible cases.

- Suppose $X \subseteq T$ or $C(X) \subseteq T$. In either case, there exists $Y \subseteq T$, such that $C(X) = C(Y)$. Thus, we get

$$C^T(X) = C(T) = C^T(C(T)) = C^T(C_T(X)).$$

- Suppose, next, that $X \nsubseteq T$, $C(X) \nsubseteq T$ and $C(X) = C(Y)$, for some $Y \subseteq T$. Then

$$C^T(C_T(X)) = C^T(C(X)) = C(T) = C^T(X).$$

- If none of the above hold, that is, if $X \nsubseteq T$, $C(X) \nsubseteq T$ and $C(X) \neq C(Y)$, for all $Y \subseteq T$, then

$$C^T(C_T(X)) = C^T(C(X)) = C(X) = C^T(X).$$

We conclude that $C_T \leq C^T$. ∎