Chapter 3

Algebraic Theory

3.1 Introduction

In [3], Blok and Pigozzi introduced the notion of an *algebraizable logic*. Logics in their study, and in all related abstract studies in algebraic logic, can be represented by structural closure operators, i.e., functions on the powerset of an absolutely free algebra on countably many generators that satisfy the axioms of inflationarity, monotonicity, idempotency and structurality (see, e.g., Page 25 of [12]). *Logical matrices* were traditionally used in this context as models of logics. Classes of algebras that were algebraic reducts of those matrices were used as algebraic counterparts. Later, based on the study of concrete examples, it was discovered that in many less well behaved logics, the algebras obtained in this way via matrices were not the "right" ones. This led Font and Jansana in [12] to suggest a more general methodology. Instead of logical matrices, they considered *abstract logics* as models of logical systems. Following a similar process, but now taking algebraic reducts of abstract logics instead of logical matrices, they obtained a class of algebras that was seeing, via already acquired experience with a wide variety of particular logics, as being the "right" one in all known logical systems.

One of the limitations of this "traditional" theory is that it requires the logics under consideration to satisfy both inflationarity and monotonicity. Thus, one cannot accommodate potential non-monotonic operators. It is conceivable, however, that many aspects and features of the theory could be carried over to such a context. This work is a first attempt at such a treatment. In Chapter 2, the notion of a *logicate* was introduced. It is an idempotent operator C on the powerset of a given set A. Several related representations were given and logicates over the same underlying set were compared in a couple of different ways.

If one wishes to study logicates from an algebraic point of view, the preceding framework is clearly insufficient, since it involves no algebraic structure. We remedy this by logicates over algebras. So the basic object of study here is an *algebraic logicate*, consisting of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and an idempotent operator C on the powerset of the universe A of $\mathbf{A}$. It turns out that this framework is sufficient for accommodating quite a large part of the monotonic theory, without requiring that the underlying operator be either inflationary or monotone.

Our study relies on the set $\mathcal{C}$ of *fixed points* or *theories*. No intrinsic ordering or structure on those may be assumed, in contrast with the traditional framework. Still, as in the traditional framework, we may define, based on theories, *logical congruences*. It may be shown that these form a principal ideal of the lattice of all congruences on the underlying algebra. Hence, it makes sense to define the *Tarski congruence* as the largest logical congruence of the logicate, exactly as done for abstract logics in the monotonic framework [12]. In addition, a slightly modified, but very similar, version of its well-known characterization holds (see Pages 18-19 of [12]).

An important notion in the study of abstract logics is that of a *bilogical morphism* (Page 20 of [12]). Because the framework of [12] is based on closure operators, bilogical morphisms tie very closely the consequence structures of the abstract logics they relate. As a byproduct, they also connect very strongly the theories. On the other hand, in the present context, due to lack of monotonicity, one cannot expect morphism with such tight properties. As a result, instead of focusing on consequence preservation, we take the effect on theories as primary. Thus, we adopt a notion of *bimorphism* by stipulating that it preserve theories, even though it may not have any preservation properties as regards the corresponding consequence structures. This fact becomes manifest in the formulation of an analog of a well known characterization. Proposition 1.4 of [12] consists of six equivalent statements, three referring to theories and three to the preservation of the closure structures. The analog only retains the three parts referring to theories, whereas those on consequences are not valid in general. Despite this drawback, using bilogical morphisms one can prove some analogs of the Homomorphism Theorems of Universal Algebra [5, 18, 1]. These parallel the generalized versions proven in the monotonic framework (see, e.g. [4, 12]).

Additionally, if one defines *logical matrices* for logicates by relying solely on theories and not on consequence relations, many of the results of the monotonic theory can be adapted and still shown to hold. In this setting, however, because of the absence of structurality, one has to build matrices over specific interpretations, i.e., surjective homomorphisms onto similar algebras. Some clues as to how one may proceed may be taken from the context of models of π-institutions [21]. On the other hand, if structurality is added, as is done briefly in the Addendum to this chapter, then matrices resembling the ordinary ones more closely may again be used as models and some of the flavor of the traditional treatment may be recovered.

In Section 3.2, we adapt the study of *logical congruences* on abstract logics of Font and Jansana [12] to the nonmonotonic setting. Logical congruences on a logicate form a principal ideal of the lattice of all logical congruences on the underlying algebra. The generator of this ideal is called the *Tarski congruence* of the logicate. It is important to notice that all equipotent logicates share the same Tarski congruence. Further, the Blok-Pigozzi style characterization of the Tarski congruence in the traditional setting carries over virtually unchanged. If one thinks of the Tarski operator as acting on equipotency classes, so that the set in question is partially ordered, then it is a monotone operator.

In Section 3.3 we revisit *bilogical morphisms*, but apply them to arbitrary logicates. These are surjective homomorphisms that preserve theories. A characterization is provided, as well as the important result that the Tarski operator of a logicate is preserved under the action of inverse bilogical morphisms.

In Section 3.4 we study aspects of Universal Algebra, that were applied by

Brown and Suszko [4] and by Font and Jansana [12] to abstract logics, in the context of nonmonotonicity. We define *quotients* of logicates by logical congruences and show that natural projections form bilogical morphisms. Then we embark on revisiting analogs of the Homomorphism Theorems (see, e.g., [5] and Pages 22-23 of [12]) for logicates. In particular, the Correspondence Theorem, playing a role similar to the one played in the monotonic theory, leads to the definition of a *reduced logicate* and the process of *reduction*. Several properties of reductions carry over to this more general setting and are presented in detail in this section.

In Section 3.5 we introduce *interpretations* of algebraic logicates. An interpretation is essentially a surjective mapping from the underlying algebra of the logicate onto an algebra of the same type. Interpretations form the cornerstone in defining *logical filters* and *logical matrices*, as well as *reduced matrices*, which play a key role in both the traditional theory (see, e.g., [3, 12, 8]) and the more general theory presented here. One important feature of interpretations is that, if the kernel is a logical congruence of the original logicate, then one may define a logicate in the target of the interpretation in such a way that its theories coincide with the filters and the interpretations becomes a bilogical morphism between the two logicates. Several properties governing the relation between interpretations and (sets of) filters are also presented in this section. In closing the section, we define that class of *matrices* and of *reduced matrices* of a logicate and the corresponding classes of algebraic reducts and prove analogs of the well-known *completeness results* for sentential logics in the context of logicates.

In the Addendum, we briefly overview a possible alternative formulation of filters and matrices, applicable in case the consequence operator of the logicate happens to be structural. In that case filters and matrices may be defined as in the traditional monotonic theory, without recourse to fixed interpretation morphisms.

3.2 Logical Congruences

Let $\mathcal{L}$ be a logical (or algebraic) language. That is, $\mathcal{L}$ is a set of connectives (or operation symbols) of finite arities. We consider algebras of type $\mathcal{L}$, or $\mathcal{L}$-algebras, $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$. Recall from Chapter 2, that a *logicate* or a *consequence operator* C on A is an idempotent mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Following tradition, we may write $\mathbb{L} = \langle \mathbf{A}, C \rangle$ for the structure consisting of the algebra $\mathbf{A}$ and the consequence operator C on A. We call such a structure an **algebraic logicate**. Moreover, we use $\mathcal{C}$ for the set of its theories,

$$\mathcal{C} = \{X \subseteq A : C(X) = X\}.$$

Recalling from Chapter 2 the *equipotency* equivalence relation $\triangleq$, which relates two logicates if their sets of theories coincide, we may also write

$$\mathbb{L}/\triangleq \,:= \langle \mathbf{A}, \mathcal{C} \rangle$$

to denote the equipotency class of the algebraic logicate $\mathbb{L}$. Note that this notation is justified by the fact that the exact consequence structure of a representative does not matter, since the equipotency class is determined solely by its theories.

Let $\theta \in \mathrm{Con}(\mathbf{A})$ be a congruence on the $\mathcal{L}$-algebra $\mathbf{A}$. We say that θ is **compatible with** a set $X \subseteq A$ if, for all $a, b \in A$,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in X \quad \text{imply} \quad b \in X.$$

Compatibility of θ with X is tantamount to X being a union of θ-congruence classes.

Suppose, next, that we have given an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C \rangle$. We say that θ is a **logical congruence of** C, or **of** $\mathbb{L}$, if θ is compatible with every theory of C, i.e., with every $X \in \mathcal{C}$. We write $\mathrm{Con}(\mathbb{L})$ for the collection of all logical congruences of $\mathbb{L}$. Moreover, $\mathbf{Con}(\mathbb{L}) = \langle \mathrm{Con}(\mathbb{L}), \subseteq \rangle$ denotes the collection of logical congruences, ordered by the subset relation between congruences. It can be shown that this partially ordered set is a complete lattice and, in fact, a principal ideal of the complete lattice of all congruences on the algebra $\mathbf{A}$ (see Page 18 of [12]).

Proposition 15 *Let $\mathbb{L} = \langle \mathbf{A}, C \rangle$ be an algebraic logicate. Then $\mathbf{Con}(\mathbb{L})$ is a complete lattice and a principal ideal of $\mathbf{Con}(\mathbf{A})$.*

Proof: We first show that $\mathrm{Con}(\mathbb{L})$ is a downset in $\mathbf{Con}(\mathbf{A})$. Let $\theta, \theta' \in \mathrm{Con}(\mathbf{A})$, such that $\theta \subseteq \theta' \in \mathrm{Con}(\mathbb{L})$. For all $X \in \mathcal{C}$ and all $a, b \in A$, we have

$$\begin{array}{rll} \langle a, b \rangle \in \theta \text{ and } a \in X & \text{imply} & \langle a, b \rangle \in \theta' \text{ and } a \in X \\ & \text{imply} & b \in X. \end{array}$$

Thus, $\theta \in \mathrm{Con}(\mathbb{L})$ and $\mathrm{Con}(\mathbb{L})$ is a downset in $\mathbf{Con}(\mathbf{A})$.

Next, we show that $\mathrm{Con}(\mathbb{L})$ is closed under joins. Because of the way joins are determined in $\mathbf{Con}(A)$, to do this it suffices to show that, for all $X \in \mathcal{C}$, if $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, then $\theta \circ \theta'$ is compatible with X. So, let $a, b \in A$, such that $\langle a, b \rangle \in \theta \circ \theta'$ and $a \in X$. Thus, there exists $c \in A$, such that $\langle a, c \rangle \in \theta$, $\langle c, b \rangle \in \theta'$ and $a \in X$. By compatibility of θ and θ' with X, we get, first, $b \in X$ and, then, $c \in X$. Hence, $\mathrm{Con}(\mathbb{L})$ is closed under joins. Thus, it forms an ideal of $\mathbf{Con}(\mathbf{A})$.

Finally, to see that it is principal, we note that the union of every chain in $\mathbf{Con}(\mathbb{L})$ is an upper bound of the chain and lies in $\mathrm{Con}(\mathbb{L})$. Hence, by Zorn's Lemma, $\mathrm{Con}(\mathbb{L})$ has a maximal element. However, as $\mathrm{Con}(\mathbb{L})$ is closed under joins, a maximal element must be a maximum element. Therefore $\mathbf{Con}(\mathbb{L})$ is a principal ideal of $\mathbf{Con}(\mathbf{A})$. ∎

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate. The **Tarski congruence of** $\mathbb{L}$ (Definition 1.1 of [12]) is

$$\widetilde{\Omega}(\mathbb{L}) = \max \mathbf{Con}(\mathbb{L}),$$

that is, $\widetilde{\Omega}(\mathbb{L})$ is the greatest logical congruence of $\mathbb{L}$. The **Tarski operator on A** is the mapping

$$\widetilde{\Omega}_{\mathbf{A}} : \mathbb{L} \longmapsto \widetilde{\Omega}(\mathbb{L}),$$

i.e., it is the mapping $\mathbb{L} \mapsto \widetilde{\Omega}(\mathbb{L})$ restricted to algebraic logicates over the same underlying algebra $\mathbf{A}$. This notation can be extended by writing $\widetilde{\Omega}_{\mathbf{A}}(C)$ for the Tarski congruence of the algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$. It follows from the definition of $\widetilde{\Omega}_{\mathbf{A}}(C)$ that

$$\mathrm{Con}(\mathbb{L}) = \{\theta \in \mathrm{Con}(\mathbf{A}) : \theta \subseteq \widetilde{\Omega}(\mathbb{L})\}.$$

To characterize the Tarski congruence, one uses the same way employed in the traditional theory. Namely, we consider **logical matrices** over $\mathbf{A}$, i.e., pairs $\mathfrak{A} = \langle \mathbf{A}, X\rangle$, where $X \subseteq A$ (see, e.g., Section 1.4 of [3] or Page 16 of [12]). We say that a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is a **congruence of** $\mathfrak{A}$, written $\theta \in \mathrm{Con}(\mathfrak{A}) = \mathrm{Con}(\langle \mathbf{A}, X\rangle)$, if θ is compatible with X.

Corollary 16 *Let* $\mathfrak{A} = \langle \mathbf{A}, X\rangle$ *be a logical matrix. Then* $\mathbf{Con}(\mathfrak{A})$ *is a complete lattice and a principal ideal of* $\mathbf{Con}(\mathbf{A})$.

Proof: This is a direct consequence of Proposition 15, since, given $\mathfrak{A} = \langle \mathbf{A}, X\rangle$, one can construct an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$ whose only theory is X, i.e., one can set, for all $Y \subseteq A$,

$$C(Y) = X.$$

Then, clearly, $\mathbf{Con}(\mathfrak{A}) = \mathbf{Con}(\mathbb{L})$. ∎

Corollary 16 permits us to define the **Leibniz congruence of** $\mathfrak{A}$ or the **Leibniz congruence of** X **on A**, written $\Omega(\mathfrak{A}) = \Omega_{\mathbf{A}}(X)$, as the largest congruence on $\mathbf{A}$ that is compatible with X. Then, given an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$, it is clear, by the definition of $\widetilde{\Omega}(\mathbb{L})$, that

$$\widetilde{\Omega}(\mathbb{L}) = \bigcap\{\Omega_{\mathbf{A}}(X) : X \in \mathcal{C}\}.$$

In particular, note that, since the value of $\widetilde{\Omega}(\mathbb{L})$ only depends on the set $\mathcal{C}$ of theories and not on the details of the consequence structure, it makes sense to also consider $\widetilde{\Omega}$ as an operator on equipotency classes,

$$\widetilde{\Omega}_{\mathbf{A}} : \mathcal{C} \longmapsto \mathrm{Con}(\mathbf{A}).$$

The following characterization of the Leibniz congruence of a logical matrix was given by Blok and Pigozzi in their ground breaking monograph (see Theorem 1.5 of [3]).

Theorem 17 (Blok-Pigozzi) *Let $\mathfrak{A} = \langle \mathbf{A}, X \rangle$ be a logical matrix and $a, b \in A$. Then $\langle a, b \rangle \in \Omega_{\mathbf{A}}(X)$ if and only if, for every formula $\varphi(x, \overline{z})$ in $\mathrm{Fm}_{\mathcal{L}}(V)$ and every tuple $\overline{c}$ of elements in A,*

$$\varphi^{\mathbf{A}}(a, \overline{c}) \in X \quad \textit{iff} \quad \varphi^{\mathbf{A}}(b, \overline{c}) \in X.$$

From this theorem and the discussion preceding it, formalizing the relation between the Tarski congruence of an algebraic logicate and the Leibniz congruences of the associated class of logical matrices, one obtains easily a characterization of the Tarski congruence (see Page 29 of [12]).

Corollary 18 *Let $\mathbb{L} = \langle \mathbf{A}, C \rangle$ be an algebraic logicate and $a, b \in A$. Then $\langle a, b \rangle \in \Omega_{\mathbf{A}}(C)$ if and only if, for every $X \in \mathcal{C}$, every formula $\varphi(x, \overline{z})$ in $\mathrm{Fm}_{\mathcal{L}}(V)$ and every tuple $\overline{c}$ of elements in A,*

$$\varphi^{\mathbf{A}}(a, \overline{c}) \in X \quad \textit{iff} \quad \varphi^{\mathbf{A}}(b, \overline{c}) \in X.$$

Proof: One has $\langle a, b \rangle \in \Omega_{\mathbf{A}}(C)$ iff, for all $X \in \mathcal{C}$, $\langle a, b \rangle \in \Omega_{\mathbf{A}}(X)$ iff, by Theorem 17, the asserted condition in the statement holds. ■

A consequence of Corollary 18 is that the Tarski operator on an algebra $\mathbf{A}$ is monotone. But one should exercise caution in formalizing this result to avoid pitfalls. Namely, when referring to logicates, we understand their equipotency classes. Logicates are preordered by $\trianglelefteq$ (see Chapter 2), given by

$$C \trianglelefteq C' \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

This induces a partial order (denoted by the same symbol) on the equipotency classes, which we may write

$$C/{\triangleq} \trianglelefteq C'/{\triangleq} \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

Accordingly, the Tarski operator is seen as applying on equipotency classes of logicates.

Corollary 19 *Let $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ be an algebra. Then, for all logicates C and C' on $\mathbf{A}$,*

$$C \trianglelefteq C' \quad \textit{implies} \quad \widetilde{\Omega}_{\mathbf{A}}(C) \subseteq \widetilde{\Omega}_{\mathbf{A}}(C').$$

Proof: We have that

$$\begin{array}{rll} C \trianglelefteq C' & \text{iff} & \mathcal{C}' \subseteq \mathcal{C} \quad \text{(Definition of } \trianglelefteq) \\ & \text{implies} & \widetilde{\Omega}_{\mathbf{A}}(C) \subseteq \widetilde{\Omega}_{\mathbf{A}}(C'). \quad \text{(Corollary 18)} \end{array}$$

Thus, the Tarski operator on $\mathbf{A}$ is monotone. ■

3.3 Bilogical Morphisms

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates. A **logical morphism** from $\mathbb{L}$ to $\mathbb{L}'$, written $h : \mathbb{L} \to \mathbb{L}'$, is a homomorphism $h : \mathbf{A} \to \mathbf{A}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$, that is, for all $X' \subseteq A'$,

$$C'(X') = X' \quad \text{implies} \quad C(h^{-1}(X')) = h^{-1}(X').$$

We say that $\mathbb{L}$ is **projectively generated from** $\mathbb{L}'$ **by** h if

$$\mathcal{C} = h^{-1}(\mathcal{C}').$$

Finally, h is a **bilogical morphism from** $\mathbb{L}$ **onto** $\mathbb{L}'$, or a **bilogical morphism between** $\mathbb{L}$ **and** $\mathbb{L}'$, written $h : \mathbb{L} \to_b \mathbb{L}'$, if it is an epimorphism $h : \mathbf{A} \twoheadrightarrow \mathbf{A}'$ and it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. For these definitions, as applied to the traditional setting, see Page 20 of [12] (also [4] for the original notions). The following proposition is the analog in the nonmonotonic setting of Proposition 1.4 of [12]. Note that instead of six equivalent conditions, it provides only three, and this is due to the fact that the consequence operators are not required to be inflationary or monotone.

Proposition 20 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be algebraic logicates and $h : \mathbf{A} \to \mathbf{A}'$ an epimorphism. The following conditions are equivalent.*

(i) h is a bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$;

(ii) $\mathcal{C}' = h(\mathcal{C})$ and $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$;

(iii) $\mathcal{C} = h^{-1}(\mathcal{C}')$.

Proof: First observe that, since, by hypothesis, h is an epimorphism, Conditions (i) and (iii) are equivalent by the definition of bilogical morphism. Thus, it suffices to show (i)⇒(ii) and (ii)⇒(iii).

(i)⇒(ii) Let $a, b \in A$ and $X \in \mathcal{C}$, such that $\langle a, b\rangle \in \mathrm{Ker}(h)$ and $a \in X$. By hypothesis, $\langle a, b\rangle \in \mathrm{Ker}(h)$ and $a \in h^{-1}(Y)$, for some $Y \in \mathcal{C}'$. Hence, $h(a) = h(b)$ and $h(a) \in Y$. This yields $h(b) \in Y$ and, hence, $b \in h^{-1}(Y) = X$. So $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$.

Let $Y \in \mathcal{C}'$. Then

$$\begin{array}{rcll} Y & = & h(h^{-1}(Y)) & (h \text{ surjective}) \\ & = & h(X), & (\text{for } X = h^{-1}(Y)) \end{array}$$

where, by hypothesis, $X = h^{-1}(Y)$ is a theory of C. Conversely, if X is a theory of C,

$$\begin{array}{rcll} h(X) & = & h(h^{-1}(Y)) & (\text{for some } Y \in \mathcal{C}') \\ & = & Y. & (h \text{ surjective}) \end{array}$$

Thus, Condition (ii) holds.

(ii)⇒(iii) Note that, since $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$, we have, for every $X \in \mathcal{C}$,

$$X = h^{-1}(h(X)).$$

Suppose $Y \in \mathcal{C}'$. By hypothesis, there exists $X \in \mathcal{C}$, such that $Y = h(X)$. Hence, by the remark above,

$$h^{-1}(Y) = h^{-1}(h(X)) = X.$$

Suppose, conversely, $X \in \mathcal{C}$. Then $X = h^{-1}(h(X)) = h^{-1}(Y)$, where $Y = h(X)$ is a C'-theory by the hypothesis. Thus, Condition (iii) holds.

■

Proposition 20 yields immediately the following results revealing a very tight relation between theories of two algebraic logicates related via a bilogical morphism. We start with an analog of Proposition 1.5 of [12]. Note, however, that, as C and C' may not be closure operators, the collections $\mathcal{C}$ and $\mathcal{C}'$ may not be complete lattices. So we may only establish a poset isomorphism between them.

Proposition 21 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates and $h : \mathbf{A} \to \mathbf{A}'$ be an epimorphism. Then $h : \mathbb{L} \to_b \mathbb{L}'$ if and only if the posets $\langle \mathcal{C}, \subseteq\rangle$ and $\langle \mathcal{C}', \subseteq\rangle$ are isomorphic via the mapping induced by h.*

Proof: The conclusion follows directly from Proposition 20. ■

We conclude the section by showing that Tarski congruence systems are preserved under the action of inverse bilogical morphisms.

Proposition 22 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be algebraic logicates and $h : \mathbb{L} \to_b \mathbb{L}'$. Then*

$$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}')).$$

Proof: We have, for all $a, b \in A$,

$$\begin{array}{ll}
\langle a, b\rangle \in h^{-1}(\widetilde{\Omega}(\mathbb{L}')) & \\
\quad \text{iff} & \langle h(a), h(b)\rangle \in \widetilde{\Omega}(\mathbb{L}') \\
\quad \text{iff,} & \text{for all } X' \in \mathcal{C}', \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c}' \text{ in } A', \\
& \varphi^{\mathbf{A}'}(h(a), \overline{c}') \in X' \quad \text{iff} \quad \varphi^{\mathbf{A}'}(h(b), \overline{c}') \in X' \\
\quad \text{iff,} & \text{for all } X \in \mathcal{C}, \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \text{ in } A, \\
& \varphi^{\mathbf{A}'}(h(a), h(\overline{c})) \in h(X) \quad \text{iff} \quad \varphi^{\mathbf{A}'}(h(b), h(\overline{c})) \in h(X) \\
\quad \text{iff,} & \text{for all } X \in \mathcal{C}, \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \text{ in } A, \\
& h(\varphi^{\mathbf{A}}(a, \overline{c})) \in h(X) \quad \text{iff} \quad h(\varphi^{\mathbf{A}'}(b, \overline{c})) \in h(X) \\
\quad \text{iff,} & \text{for all } X \in \mathcal{C}, \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \text{ in } A, \\
& \varphi^{\mathbf{A}}(a, \overline{c}) \in X \quad \text{iff} \quad \varphi^{\mathbf{A}'}(b, \overline{c}) \in X \\
\quad \text{iff} & \langle a, b\rangle \in \widetilde{\Omega}(\mathbb{L}).
\end{array}$$

We conclude that $\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}'))$. ■

3.4 Quotients

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates. $\mathbb{L}$ and $\mathbb{L}'$ are **isomorphic**, written $\mathbb{L} \cong \mathbb{L}'$, if there exists a bijective logical morphism $h : \mathbb{L} \to \mathbb{L}'$, whose inverse is also a logical morphism. Equivalently, h is a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$, which happens to be an algebra isomorphism.

We need to make a comment here to prevent a possible misunderstanding. The term "isomorphic" is used because the definition is very similar (almost a repetition) of the one used for isomorphisms in the monotonic setting (see Page 21 of [12]). However, in the present context, because of lack of monotonicity, many properties are lacking and the term may be misconstrued as implying that an isomorphism is structure preserving, whereas it only preserves theories in both directions. So, perhaps, instead of the term "isomorphism", a better term would be "isopotency". In any case, we keep the term "isomorphism" for now, pretending that theories are the only objects that matter, since they are, after all, of utmost importance.

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate and suppose $\theta \in \mathrm{Con}(\mathbf{A})$. Consider the quotient algebra $\mathbf{A}/\theta = \langle A/\theta, \mathcal{L}^{\mathbf{A}/\theta}\rangle$ and define a mapping

$$C^\theta := C/\theta : \mathcal{P}(A/\theta) \to \mathcal{P}(A/\theta)$$

by setting, for all $S \subseteq A/\theta$,

$$C^\theta(S) = \pi_\theta(C(\pi_\theta^{-1}(S))),$$

where $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ is the canonical projection homomorphism onto the quotient.

We show that, if θ happens to be a logical congruence, then the quotient is a legitimate logicate on the quotient algebra. For the corresponding result in the traditional framework, see Page 21 of [12].

Proposition 23 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate and suppose $\theta \in \mathrm{Con}(\mathbb{L})$. Then*

$$\mathbb{L}^\theta = \mathbb{L}/\theta := \langle \mathbf{A}/\theta, C^\theta\rangle$$

is also an algebraic logicate.

Proof: Let $S \subseteq A/\theta$. Then

$$\begin{array}{rcll} C^\theta(C^\theta(S)) & = & \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))))) & \text{(Definition of } C^\theta) \\ & = & \pi_\theta(C(C(\pi_\theta^{-1}(S)))) & (\theta \in \mathrm{Con}(\mathbb{L})) \\ & = & \pi_\theta(C(\pi_\theta^{-1}(S))) & \text{(Idempotency)} \\ & = & C^\theta(S). & \text{(Definition of } C^\theta) \end{array}$$

Thus, C^θ is idempotent and $\mathbb{L}^\theta$ is an algebraic logicate, as claimed. ■

$\mathbb{L}^\theta = \mathbb{L}/\theta$ is called the **quotient logicate of $\mathbb{L}$ by θ**.

Now that we know that, in case θ is a logical congruence, $\mathbb{L}$ and its quotient $\mathbb{L}^\theta$ are logicates, we show that the canonical projection $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ becomes a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}/\theta$.

Proposition 24 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate and suppose $\theta \in$ Con$(\mathbb{L})$. Then $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ is a bilogical morphism.*

Proof: Given that π_θ is an epimorphism, it suffices to show that it projectively generates $\mathbb{L}$ from $\mathbb{L}/\theta$. Suppose, first $X \in \mathcal{C}$. Then

$$\begin{array}{rcll} \pi_\theta^{-1}(C^\theta(\pi_\theta(X))) & = & \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X))))) & \text{(Definition of } C^\theta) \\ & = & C(X) & (\theta \in \mathrm{Con}(\mathbb{L})) \\ & = & X. & (X \in \mathcal{C}) \end{array}$$

This shows that $\mathcal{C} \subseteq \pi_\theta^{-1}(\mathcal{C}^\theta)$. Conversely, let $S \in \mathcal{C}^\theta$. Then

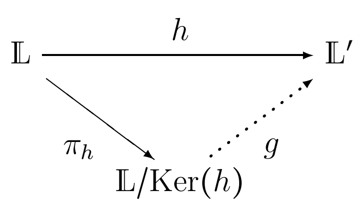
$$\begin{array}{rcll} \pi_\theta^{-1}(S) & = & \pi_\theta^{-1}(C^\theta(S)) & (S \in \mathcal{C}^\theta) \\ & = & \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))) & \text{(Definition of } C^\theta) \\ & = & C(\pi_\theta^{-1}(S)). & (\theta \in \mathrm{Con}(\mathbb{L})) \end{array}$$

Hence $\pi_\theta^{-1}(\mathcal{C}^\theta) \subseteq \mathcal{C}$. Thus, $\mathcal{C} = \pi_\theta^{-1}(\mathcal{C}^\theta)$, showing that π_θ projectively generates $\mathbb{L}$ from $\mathbb{L}/\theta$. So $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ is a bilogical morphism. ∎

We call $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ the **quotient morphism**, or the **natural projection morphism**, from $\mathbb{L}$ to its quotient $\mathbb{L}^\theta$ by θ.

Theorems 1.8, 1.9 and 1.10 of [12] adapt to the framework of abstract logics the well-known Homomorphism Theorems of universal algebra [5, 18, 1]. We undertake here their adaptation to the general nonmonotonic setting. The fundamentals, of course, remain the same.

Theorem 25 (Homomorphism Theorem) *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates and $h : \mathbb{L} \to_b \mathbb{L}'$. Then $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$ via a unique isomorphism g, such that $h = g \circ \pi_h$,*



where $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ is the bilogical projection morphism.

Proof: By hypothesis, h is a bilogical morphism. Thus, by Proposition 20, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. It follows, by Proposition 24, that $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ is a bilogical morphism. By Universal Algebra, there exists a unique $g : \mathbf{A}/\theta \cong \mathbf{A}'$, such that $h = g \circ \pi_h$. So it suffices to show that this is a bilogical morphism. We have

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{Ker}(h)} & = & \pi_h(\mathcal{C}) \quad (\pi_h : \mathbb{L} \to_b \mathbb{L}/\mathrm{Ker}(h)) \\ & = & g^{-1}(h(\mathcal{C})) \quad (h = g \circ \pi_h \text{ and } g : \mathbf{A}/\theta \cong \mathbf{A}') \\ & = & g^{-1}(\mathcal{C}'). \quad (h : \mathbb{L} \to_b \mathbb{L}') \end{array}$$

Thus, by definition, $g : \mathbb{L}/\mathrm{Ker}(h) \to \mathbb{L}'$ is a bilogical morphism. ∎

Theorem 26 (Second Isomorphism Theorem) *Let* $\mathbb{L} = \langle \mathbf{A}, C \rangle$ *be an algebraic logicate and* $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$*, such that* $\theta \subseteq \theta'$*. Then* $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$ *and*

$$(\mathbb{L}/\theta)/(\theta'/\theta) \cong \mathbb{L}/\theta',$$

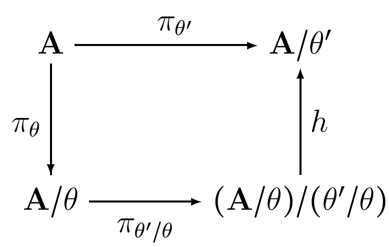
where the isomorphism is given by

$$(a/\theta)/(\theta'/\theta) \mapsto a/\theta'.$$

Proof: By Universal Algebra, we know that

$$\begin{array}{rrcl} h : & (\mathbf{A}/\theta)/(\theta'/\theta) & \longrightarrow & \mathbf{A}/\theta'; \\ & (a/\theta)/(\theta'/\theta) & \longmapsto & a/\theta', \end{array}$$

is an isomorphism that makes the following rectangle commute,



where $\pi_\theta, \pi_{\theta'}$ and $\pi_{\theta'/\theta}$ are the natural projections. Since $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, by Proposition 24, π_θ and $\pi_{\theta'}$ are bilogical morphisms. We can also show that $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$. Let $a, b \in A$ and $S \subseteq A/\theta$, such that $\langle a/\theta, b/\theta \rangle \in \theta'/\theta$ and $a/\theta \in C^\theta(S)$. Then, by the definition of C^θ, $\langle a, b \rangle \in \theta'$ and $a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$. Hence, since $\theta \in \mathrm{Con}(\mathbb{L})$, $\langle a, b \rangle \in \theta'$ and $a \in C(\pi_\theta^{-1}(S))$. Since $\theta' \in \mathrm{Con}(\mathbb{L})$, this yields $b \in C(\pi_\theta^{-1}(S))$, whence $b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$, i.e., $b/\theta \in C^\theta(S)$. Hence $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. Now, again by Proposition 24, the projection $\pi_{\theta'/\theta}$ is also a bilogical morphism. Now we check that h, which is already known to be an algebraic isomorphism, is also a bilogical morphism. We have

$$\begin{array}{rcl} (\mathcal{C}^\theta)^{\theta'/\theta} & = & \pi_{\theta'/\theta}(\mathcal{C}^\theta) \quad (\pi_{\theta'/\theta} : \mathbb{L}^\theta \to_b (\mathbb{L}^\theta)^{\theta'/\theta}) \\ & = & \pi_{\theta'/\theta}(\pi_\theta(\mathcal{C})) \quad (\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta) \\ & = & h^{-1}(\pi_{\theta'}(\mathcal{C})) \quad (h \circ \pi_{\theta'/\theta} \circ \pi_\theta = \pi_{\theta'} \text{ and } h \text{ an iso}) \\ & = & h^{-1}(\mathcal{C}^{\theta'}). \quad (\pi_{\theta'} : \mathbb{L} \to_b \mathbb{L}^{\theta'}) \end{array}$$

Thus, $h : (\mathbb{L}/\theta)/(\theta'/\theta) \to \mathbb{L}/\theta'$ is indeed a bilogical morphism and, hence, an isomorphism. ∎

Theorem 27 (Correspondence Theorem) *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then the segment* $[\theta, \widetilde{\Omega}(\mathbb{L})]$ *of the lattice* $\mathbf{Con}(\mathbb{L})$ *is isomorphic to the lattice* $\mathbf{Con}(\mathbb{L}^\theta)$ *by the mapping* $\theta' \mapsto \theta'/\theta$.

Proof: By Theorem 26, if $\theta \subseteq \theta' \subseteq \widetilde{\Omega}(\mathbb{L})$, then $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. By Universal Algebra, it suffices to prove that, for all $\theta \subseteq \theta' \in \mathrm{Con}(\mathbf{A})$, if $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$, then $\theta' \in \mathrm{Con}(\mathbb{L})$. So let $a, b \in A$ and $X \in \mathcal{C}$, such that $\langle a, b\rangle \in \theta'$ and $a \in X$. As $\theta \in \mathrm{Con}(\mathbb{L})$, θ is compatible with X. Thus X is the union of θ-classes, that is $X = \pi_\theta^{-1}(\pi_\theta(X))$. Now, starting with the assumption, we get

$$\begin{aligned}
&\langle a, b\rangle \in \theta' \text{ and } a \in X\\
&\qquad \text{iff} \quad \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X))))\\
&\qquad \text{iff} \quad \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in C^\theta(\pi_\theta(X))\\
&\qquad \text{implies} \quad b/\theta \in C^\theta(\pi_\theta(X))\\
&\qquad \text{iff} \quad b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X))))\\
&\qquad \text{iff} \quad b \in X.
\end{aligned}$$

Therefore, $\theta' \in \mathrm{Con}(\mathbb{L})$ and the correspondence is established. ■

The Correspondence Theorem has a significant consequence in relation to the Tarski congruences.

Corollary 28 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then*

$$\widetilde{\Omega}(\mathbb{L}^\theta) = \widetilde{\Omega}(\mathbb{L})/\theta.$$

Proof: By definition, the largest element in $\mathrm{Con}(\mathbb{L}^\theta)$ is $\widetilde{\Omega}(\mathbb{L}^\theta)$, whereas the largest element in $[\theta, \widetilde{\Omega}(\mathbb{L})]$ is clearly $\widetilde{\Omega}(\mathbb{L})$. Since, under the established correspondence of Theorem 27, these two elements correspond, we get the conclusion. ■

It follows that, for any algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$,

$$\widetilde{\Omega}(\mathbb{L}/\widetilde{\Omega}(\mathbb{L})) = \widetilde{\Omega}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}/\widetilde{\Omega}(\mathbb{L})}.$$

This leads us to the core definition of *reduction* (see Definition 1.12 of [12]). We say that an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$ is **reduced** when it has only one logical congruence, i.e., when $\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}}$. Given an arbitrary algebraic logicate $\mathbb{L}$, we define the **reduction** $\mathbb{L}^*$ **of** $\mathbb{L}$ by

$$\mathbb{L}^* = \mathbb{L}/\widetilde{\Omega}(\mathbb{L}).$$

If L is a class of algebraic logicates, then we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

If $\mathbb{L}$ is an algebraic logicate, then $\mathbb{L}^*$ is always reduced. Moreover, if $\mathbb{L}$ happens to already be reduced, then $\mathbb{L}$ and $\mathbb{L}^*$ are isomorphic and they may be identified.

We prove, next, some analogs of Propositions 1.13 and 1.14 of [12]. The first asserts that the reduction of a quotient of a logicate by a logical congruence is isomorphic to the reduction of the logicate itself. The second proves that the reductions of two logicates related via a bilogical morphism are isomorphic.

Proposition 29 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then*

$$(\mathbb{L}^\theta)^* \cong \mathbb{L}^*.$$

Proof: We have

$$\begin{array}{rcl}(\mathbb{L}^\theta)^* & = & \mathbb{L}^\theta/\widetilde{\Omega}(\mathbb{L}^\theta) \quad \text{(Definition of Reduction)}\\ & = & \mathbb{L}^\theta/(\widetilde{\Omega}(\mathbb{L})/\theta) \quad \text{(Corollary 28)}\\ & \cong & \mathbb{L}/\widetilde{\Omega}(\mathbb{L}) \quad \text{(Theorem 26)}\\ & = & \mathbb{L}^*. \quad \text{(Definition of Reduction)}\end{array}$$

The conclusion follows. ■

Proposition 30 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ *be algebraic logicates and* $h : \mathbb{L} \to_b \mathbb{L}'$. *Then*

$$\mathbb{L}^* \cong \mathbb{L}'^*.$$

Proof: By Theorem 25, $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$. By Proposition 22,

$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}'^*.$$

Since, by Proposition 20, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$, by Proposition 29,

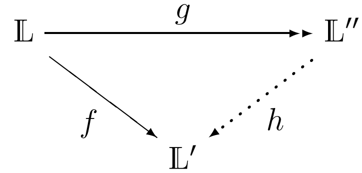
$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}^*.$$

Therefore, $\mathbb{L}'^* \cong \mathbb{L}^*$. ■

We close the section with an analog of Proposition 1.15 of [12], a sort of "fill-in" theorem for arrows.

Proposition 31 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$, $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ *and* $\mathbb{L}'' = \langle \mathbf{A}'', C''\rangle$ *be algebraic logicates,* $f : \mathbb{L} \to \mathbb{L}'$ *a logical morphism and* $g : \mathbb{L} \to_b \mathbb{L}''$ *a bilogical morphism, such that* $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. *Then, there is a unique logical morphism* $h : \mathbb{L}'' \to \mathbb{L}'$, *such that*

$$h \circ g = f.$$



Moreover, f *projectively generates* $\mathbb{L}$ *from* $\mathbb{L}'$ *if and only if* h *projectively generates* $\mathbb{L}''$ *from* $\mathbb{L}'$.

Proof: Let $a'' \in A''$. By surjectivity of g, there exists $a \in A$, such that $g(a) = a''$. We define

$$h(a'') = f(a).$$

Immediately observe that, since $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$, this assignment is well defined, that is, independent of the choice of $a \in A$. Now we have

$$\begin{array}{rcl} g^{-1}(h^{-1}(\mathcal{C}')) & = & f^{-1}(\mathcal{C}') \quad (f = h \circ g) \\ & \subseteq & \mathcal{C} \quad (f : \mathbb{L} \to \mathbb{L}') \\ & = & g^{-1}(\mathcal{C}''). \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

We now get $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}''$. Thus, $h : \mathbb{L} \to \mathbb{L}'$ is a logical morphism. That it projectively generates $\mathbb{L}''$ from $\mathbb{L}'$ if f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ follows from the fact that, in that case, the inclusion becomes an equality. Conversely, assume that h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. Then we have

$$\begin{array}{rcl} f^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) \quad (f = h \circ g) \\ & = & g^{-1}(\mathcal{C}'') \quad \text{(Assumption)} \\ & = & \mathcal{C}. \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

So f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. ■

3.5 Interpretations, Filters and Matrices

In this section, taking after the theory of logical matrices (see, e.g., [24, 3, 12, 8]), we present a similar theory suitable for algebraic logicates. Note, however, that, due to lack of structurality, one has to fix interpretations, i.e., homomorphisms onto which the underlying algebra of the logicate is interpreted. A model theory along similar lines was devised for π-institutions in [21] (again based on the work of Font and Jansana on abstract logics [12]). On the other hand, when structurality is added to the mix, as will be done in a short overview in the Addendum, then considering other models, resembling more closely the ordinary matrices in Algebraic Logic, also makes sense.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. This is thought of as the focal object of our study, for which models are to be devised. So it is referred to as a **base logicate**. This explains the use of $\mathbf{B}$ for its underlying algebra and $C^\flat$ for its consequence operator. The most appropriate notion of **interpretation** is that of a pair $\mathcal{A} = \langle \mathbf{A}, h \rangle$, where:

- $\mathbf{A}$ is an algebra of the same type as the base algebra $\mathbf{B}$;
- $h : \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism.

We say that $F \subseteq A$ is an $\mathbb{L}$-**filter** (or a **filter of** $\mathbb{L}$) **on** $\mathcal{A}$, if

$$h^{-1}(F) \in \mathcal{C}^\flat,$$

i.e., the inverse image under h of the $\mathbb{L}$-filter is a theory of the logicate. If this is the case, the pair $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ is called a **matrix for** $\mathbb{L}$ or an **$\mathbb{L}$-matrix**. The class of all $\mathbb{L}$-matrices is denoted $\mathrm{Mat}(\mathbb{L})$. An $\mathbb{L}$-matrix $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ is **reduced** if $\Omega_{\mathbf{A}}(F) = \Delta_{\mathbf{A}}$. The class of all reduced $\mathbb{L}$-matrices is denoted $\mathrm{Mat}^*(\mathbb{L})$. By $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is denoted the collection of all $\mathbb{L}$-filters on the interpretation $\mathcal{A} = \langle \mathbf{A}, h\rangle$.

Among the most important features of interpretations is that, if the kernel of their interpretation morphisms is a congruence of the base logicate, then they induce an algebraic logicate on the algebra into which the interpretation takes place. Moreover, if this is the case, the mapping of the interpretation becomes a bilogical morphism from the base logicate into the induced logicate.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation. Define $C_{\mathcal{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $Y \subseteq A$,

$$C_{\mathcal{A}}(Y) = h(C^\flat(h^{-1}(Y))).$$

Proposition 32 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation, such that $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. Then $\mathbb{L}_{\mathcal{A}} = \langle \mathbf{A}, C_{\mathcal{A}}\rangle$ is an algebraic logicate. Moreover, $h : \mathbf{B} \to \mathbf{A}$ is a bilogical morphism $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$.*

Proof: To see that $\mathbb{L}_{\mathcal{A}}$ is a logicate, we must show idempotence of $C_{\mathcal{A}}$. Let $Y \subseteq A$. Then, we have

$$\begin{aligned} C_{\mathcal{A}}(C_{\mathcal{A}}(Y)) &= h(C^\flat(h^{-1}(h(C^\flat(h^{-1}(Y)))))) \quad \text{(Definition of } C_{\mathcal{A}})\\ &= h(C^\flat(C^\flat(h^{-1}(Y)))) \quad (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L}))\\ &= h(C^\flat(h^{-1}(Y))) \quad (\mathbb{L} \text{ a logicate})\\ &= C_{\mathcal{A}}(Y). \quad \text{(Definition of } C_{\mathcal{A}}) \end{aligned}$$

Thus, $\mathbb{L}_{\mathcal{A}}$ is a logicate. To see that h becomes a logical morphism, we must show that $h^{-1}(\mathcal{C}_{\mathcal{A}}) \subseteq \mathcal{C}^\flat$. So let $Y \in \mathcal{C}_{\mathcal{A}}$. Then, we have

$$\begin{aligned} h^{-1}(Y) &= h^{-1}(C_{\mathcal{A}}(Y)) \quad (Y \in \mathcal{C}_{\mathcal{A}})\\ &= h^{-1}(h(C^\flat(h^{-1}(Y)))) \quad \text{(Definition of } C_{\mathcal{A}})\\ &= C^\flat(h^{-1}(Y)) \quad (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L}))\\ &\in \mathcal{C}^\flat. \quad \text{(Definition of } \mathcal{C}^\flat) \end{aligned}$$

To see that it is a bilogical morphism, let $X \in \mathcal{C}^\flat$. Then we have

$$\begin{aligned} h^{-1}(C_{\mathcal{A}}(h(X))) &= h^{-1}(h(C^\flat(h^{-1}(h(X))))) \quad \text{(Definition of } C_{\mathcal{A}})\\ &= C^\flat(X) \quad (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L}))\\ &= X. \quad (X \in \mathcal{C}^\flat) \end{aligned}$$

We conclude that $\mathcal{C}^\flat = h^{-1}(\mathcal{C}_{\mathcal{A}})$ and, as, by hypothesis, h is surjective, $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$ is a bilogical morphism. ■

An additional property of these algebraic logicates is that the theories of the algebraic logicate coincide with the $\mathcal{S}$-filters on the underlying interpretation.

Proposition 33 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate and* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. *Then*

$$\mathcal{C}_{\mathcal{A}} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}).$$

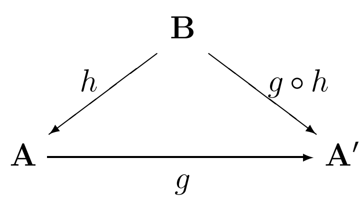
Proof: We have

$$\begin{array}{rcl} \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & = & \{Y \subseteq A : h^{-1}(Y) \in \mathcal{C}^\flat\} \quad \text{(Definition of an } \mathbb{L}\text{-filter)} \\ & = & \{Y \subseteq A : h(h^{-1}(Y)) \in \mathcal{C}_{\mathcal{A}}\} \quad \text{(Proposition 32)} \\ & = & \{Y \subseteq A : Y \in \mathcal{C}_{\mathcal{A}}\} \quad (h \text{ Surjective}) \\ & = & \mathcal{C}_{\mathcal{A}}. \end{array}$$

So the displayed equality in the statements holds. ∎

Next, we show that filters of interpretations that are related via surjective homomorphisms interact in a nice way.

Proposition 34 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate,* $\mathbf{A}$ *and* $\mathbf{A}'$ *be algebras and* $h : \mathbf{B} \to \mathbf{A}$ *and* $g : \mathbf{A} \to \mathbf{A}'$ *surjective homomorphisms and set* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$.



For all $G \subseteq A'$, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ *if and only if* $g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: We have

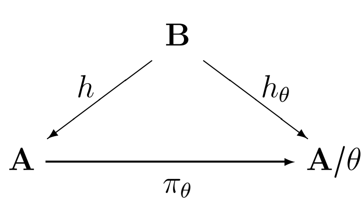
$$\begin{array}{rl} G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') & \text{iff} \quad (g \circ h)^{-1}(G) \in \mathcal{C}^\flat \quad (\text{Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')) \\ & \text{iff} \quad h^{-1}(g^{-1}(G)) \in \mathcal{C}^\flat \quad ((g \circ h)^{-1} = h^{-1} \circ g^{-1}) \\ & \text{iff} \quad g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}). \quad (\text{Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) \end{array}$$

∎

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $\theta \in \mathrm{Con}(\mathbf{A})$. Then we set

$$\mathcal{A}^\theta = \mathcal{A}/\theta = \langle \mathbf{A}/\theta, h_\theta \rangle,$$

where, $h_\theta : \mathbf{B} \to \mathbf{A}/\theta$ is defined by



$$h_\theta := \pi_\theta \circ h,$$

with $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ the natural projection homomorphism.

Proposition 35 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\theta \in \mathrm{Con}(\mathbf{A})$. Then θ is compatible with F, i.e., $\theta \subseteq \Omega_{\mathbf{A}}(F)$, if and only if $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$.*

Proof: Suppose, first, that θ is compatible with F. Set $G = \pi_\theta(F)$. Then, we have

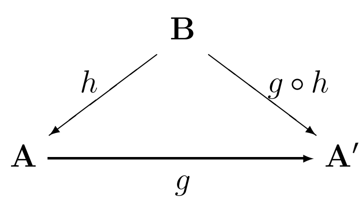
$$\begin{array}{rcll} h_\theta^{-1}(G) & = & (\pi_\theta \circ h)^{-1}(\pi_\theta(F)) & (h_\theta := \pi_\theta \circ h) \\ & = & h^{-1}(\pi_\theta^{-1}(\pi_\theta(F))) & ((\pi_\theta \circ h)^{-1} = h^{-1} \circ \pi_\theta^{-1}) \\ & = & h^{-1}(F). & (\theta \text{ compatible with } F) \end{array}$$

Since $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $h_\theta^{-1}(G) = h^{-1}(F) \in \mathcal{C}^\flat$ and, thus, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Moreover, by compatibility, $F = \pi_\theta^{-1}(\pi_\theta(F)) = \pi_\theta^{-1}(G)$.

Suppose, conversely, that $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Let $a, b \in A$, such that $\langle a, b \rangle \in \theta$ and $a \in F$. Then $a \in \pi_\theta^{-1}(G)$, whence $a/\theta \in G$. So $b/\theta = a/\theta \in G$. This gives $b \in \pi_\theta^{-1}(G) = F$. So θ is compatible with F. ■

Continuing from Proposition 34, we characterize the property of a connecting homomorphism between two interpretations being a bilogical morphism. In this, we are helped by Proposition 20, which contained a general characterization of bilogical morphisms of logicates. This is an "analog" in the present context of Proposition 1.21 of [12].

Proposition 36 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation and $g : \mathbf{A} \to \mathbf{A}'$ an epimorphism. Set $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$.*



The following statements are equivalent:

(i) $g : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, is a bilogical morphism;

(ii) For all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ and $\mathrm{Ker}(g) \in \mathrm{Con}(\langle \mathcal{A}, C \rangle)$;

(iii) g induces an isomorphism between the poset $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \subseteq \rangle$ and the poset $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'), \subseteq \rangle$.

Proof:

(i)⇒(ii) The implication (i)⇒(ii) follows by the hypothesis and Proposition 20.

(ii)⇒(iii) By hypothesis, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Conversely, if $Y \in \mathcal{C}'$, then by Proposition 34, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By surjectivity, $g(g^{-1}(Y)) = Y$, for all $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By compatibility, $g^{-1}(g(X)) = X$, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Since it is clear that both g and g' are order preserving, the conclusion follows.

(iii)⇒(i) By hypothesis, $g^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. So, by definition, $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$ is a bilogical morphism.

■

It turns out that a bilogical morphism between two models $\langle\langle \mathbf{A}, h\rangle, C\rangle$ and $\langle\langle \mathbf{A}', g \circ h\rangle, C'\rangle$ of a base logicate $\mathbb{L}$, where $g : \mathbf{A} \to \mathbf{A}'$ is an epimorphism, forces $\mathcal{C}'$ to be the full collection of $\mathbb{L}$-filters on $\mathcal{A}'$, provided that $\mathcal{C}$ is a full collection of $\mathbb{L}$-filters on $\mathcal{A}$.

Proposition 37 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation and $g : \mathbf{A} \to \mathbf{A}'$ an epimorphism. Set $\mathcal{A}' = \langle \mathbf{A}', g \circ h\rangle$. If $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a bilogical morphism, then*

$$\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}').$$

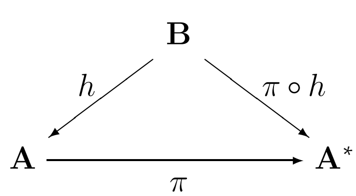
Proof: Suppose, first, that $Y \in \mathcal{C}'$. Then, since $g : \langle \mathcal{A}, C\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $g^{-1}(Y) \in \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 34, $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. So $\mathcal{C}' \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Assume, conversely, that $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Then, by Proposition 34, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) = \mathcal{C}$. Thus, since $g : \langle \mathcal{A}, C\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $Y = g(g^{-1}(Y)) \in \mathcal{C}'$. So $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') \subseteq \mathcal{C}'$. We conclude that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. ■

Corollaries 38 and 39 include consequences that constitute analogs in the context of logicates of the contents of Proposition 1.22 of [12].

Corollary 38 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation. Then*

$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

Proof: One works with the diagram



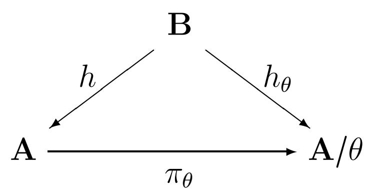
where $\pi : \mathbf{A} \to \mathbf{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))$ is the natural projection homomorphism. Recalling that, by Proposition 24, it is a bilogical morphism, we may apply Proposition 37 to get the conclusion. ■

Corollary 39 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a formula logicate,* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation and* $g : \mathbf{A} \to \mathbf{A}'$ *an isomorphism. Set* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$. *If* $g : \langle \mathcal{A}, C \rangle \cong \langle \mathcal{A}', C' \rangle$ *is an isomorphism, then*

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) \quad \textit{iff} \quad \mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}').$$

Proof: By Proposition 37. ■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Suppose $\mathfrak{A} = \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ and $\theta \in \mathrm{Con}(\mathfrak{A})$.



Then, using the compatibility of θ with F, we can see that $\mathfrak{A}/\theta = \langle \mathcal{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$. We call $\mathfrak{A}/\theta = \langle \mathcal{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$ the **quotient matrix of $\mathfrak{A}$ by** θ. In particular, $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathcal{A}}(F) \in \mathrm{Mat}^*(\mathbb{L})$. $\mathfrak{A}^*$ is the **reduction** of $\mathfrak{A}$. We let $\mathrm{Alg}^*(\mathbb{L})$ be the class of algebraic reducts of matrices in $\mathrm{Mat}^*(\mathbb{L})$.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and consider a class M of matrices. We say that $\mathbb{L}$ is **complete with respect to** M if

$$\mathcal{C}^\flat = \{h^{-1}(F) : \langle \langle \mathbf{A}, h \rangle, F \rangle \in \mathsf{M}\}.$$

In this case we say M is a **complete semantics for** $\mathbb{L}$. Observe that by definition of an $\mathbb{L}$-filter, the C-theories of a base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ are captured as the $\mathbb{L}$-filters on the interpretation $\langle \mathbf{B}, i_{\mathbf{B}} \rangle$. With this in mind, it is not difficult to see that, as is the case in the classical theory of logical matrices, $\mathbb{L}$ is complete both with respect to the class of all its matrices and with respect to the class of all its reduced matrices.

Proposition 40 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete both with respect to* $\mathrm{Mat}(\mathbb{L})$ *and with respect to* $\mathrm{Mat}^*(\mathbb{L})$.

Proof: By the definition of $\mathrm{Mat}(\mathbb{L})$, we have

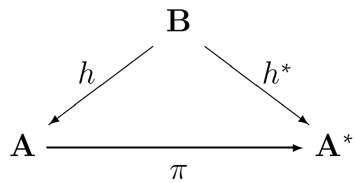
$$\{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\} \subseteq \mathcal{C}^\flat.$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then the pair $\langle \langle \mathbf{B}, i_{\mathbf{B}} \rangle, X \rangle \in \mathrm{Mat}(\mathbb{L})$ and $i_{\mathbf{B}}^{-1}(X) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}(\mathbb{L})$.

Let $\langle\langle \mathbf{A}^*, h^*\rangle, X^*\rangle \in \mathrm{Mat}^*(\mathbb{L})$. Then, by the definition of $\mathrm{Mat}^*(\mathbb{L})$,



$$\begin{array}{rcll} (h^*)^{-1}(X^*) & = & (\pi \circ h)^{-1}(\pi(X)) & \text{(Definition of } h^*) \\ & = & h^{-1}(\pi^{-1}(\pi(X))) & ((\pi\circ h)^{-1} = h^{-1}\circ\pi^{-1}) \\ & = & h^{-1}(X) & \text{(Compatibility)} \\ & \in & \mathcal{C}^\flat. & (X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) \end{array}$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then, letting $\pi : \mathbf{B} \to \mathbf{B}^*$, where $\mathbf{B}^* = \mathbf{B}/\Omega_{\mathbf{B}}(X)$, the pair $\langle\langle \mathbf{B}^*, \pi\rangle, X^*\rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $\pi^{-1}(X^*) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle \mathcal{A}, F\rangle \in \mathrm{Mat}^*(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{L})$. ■

Given a class K of $\mathcal{L}$-algebras, an $\mathcal{L}$-algebra $\mathbf{A}$, not necessarily in the class, and a congruence $\theta \in \mathrm{Con}(\mathbf{A})$, one writes $\theta \in \mathrm{Con}_{\mathsf{K}}(\mathbf{A})$ to signify that the quotient algebra $\mathbf{A}/\theta \in \mathsf{K}$. In this case, θ is termed a **K-congruence**. So $\mathrm{Con}_{\mathsf{K}}(\mathbf{A})$ is the collection of all K-congruences on the algebra $\mathbf{A}$. Using a variant of this notation, we may write

$$\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}).$$

Given a base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ and a subset $X \subseteq B$, we have, for all $b, b' \in B$,

$$\begin{array}{lll} \langle b, b'\rangle \in \Omega_{\mathbf{B}}(X) & \text{iff} & \text{for all } \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \in B \\ & & \varphi^{\mathbf{B}}(b, \overline{c}) \in X \quad \text{iff} \quad \varphi^{\mathbf{B}}(b', \overline{c}) \in X. \end{array}$$

Similarly,

$$\begin{array}{lll} \langle b, b'\rangle \in \widetilde{\Omega}(\mathbb{L}) & \text{iff} & \text{for all } X \in \mathcal{C}^\flat,\ \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \in B \\ & & \varphi^{\mathbf{B}}(b, \overline{c}) \in X \quad \text{iff} \quad \varphi^{\mathbf{B}}(b', \overline{c}) \in X. \end{array}$$

The quotient algebra $\mathbf{B}^* := \mathbf{B}/\widetilde{\Omega}(\mathbb{L})$ is called the **Lindenbaum-Tarski algebra** of $\mathbb{L}$. The quotient logicate $\mathbb{L}^* := \langle \mathbf{B}^*, C^{\flat *}\rangle$ is called the **Lindenbaum-Tarski quotient** of $\mathbb{L}$. The variety generated by $\mathbf{B}^*$ is denoted by $\mathsf{K}_{\mathbb{L}}$.

Addendum: Structural Logicates

In this Addendum, we briefly overview an alternative formulation of filters and matrices applicable in case the consequence operator of the logicate, in addition to idempotency, satisfies also structurality. In that case, filters and matrices may be defined as in the traditional monotonic theory, without recourse to fixed interpretation morphisms.

A base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ is said to be **structural** if the consequence operator C, in addition to being idempotent, satisfies, for all $e : \mathbf{B} \to \mathbf{B}$,

(**Structurality**) $e^{-1}(\mathcal{C}^\flat) \subseteq \mathcal{C}^\flat$,

that is, $\mathcal{C}^\flat$ is closed under inverse endomorphisms.

For a given structural logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ it makes sense to consider another notion of *interpretation* that simulates more closely the classical treatment. Namely, we consider algebras $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ of the same type as the structural logicate (without specific morphisms attached). Let us call this a **structural interpretation**. We say that $F \subseteq A$ is an (**structural**) $\mathbb{L}$-**filter** on $\mathbf{A}$ if, for all $h : \mathbf{B} \to \mathbf{A}$,

$$h^{-1}(F) \in \mathcal{C}^\flat,$$

i.e., the inverse image of an $\mathbb{L}$-filter under all homomorphisms from the formula algebra into $\mathbf{A}$ is a theory of $\mathbb{L}$. If this is the case, the pair $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is called a (**structural**) **matrix for** $\mathbb{L}$ or a (**structural**) $\mathbb{L}$-**matrix**. The class of all structural $\mathbb{L}$-matrices is denoted $\mathrm{Mat}(\mathbb{L})$. An $\mathbb{L}$-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is **reduced** if $\Omega_{\mathbf{A}}(F) = \Delta_{\mathbf{A}}$. The class of all reduced $\mathbb{L}$-matrices is denoted $\mathrm{Mat}^*(\mathbb{L})$. By $\mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$ is denoted the collection of all $\mathbb{L}$-filters on the algebra $\mathbf{A}$.

Many of the results established previously continue to hold, in an appropriately recalibrated form, for structural filters and matrices. We present here some samples, pointing to corresponding results in Section 3.5 of which they form analogs. E.g., Proposition 41 is an analog of Proposition 34 for structural filters.

Proposition 41 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a structural logicate,* $\mathbf{A}$ *and* $\mathbf{A}'$ *be algebras,* $h : \mathbf{A} \to \mathbf{A}'$ *a homomorphism and* $G \subseteq A'$.

(a) If $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$, *then* $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$.

(b) If h *is surjective and* $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$, *then* $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$.

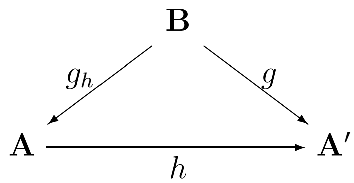
Proof:

(a) Suppose that $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$. Let $g : \mathbf{B} \to \mathbf{A}$. Then we have

$$g^{-1}(h^{-1}(G)) = (h \circ g)^{-1}(G) \in \mathcal{C}^\flat.$$

Thus, $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$.

(b) Suppose, now, that $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. Let $g : \mathbf{B} \to \mathbf{A}'$.



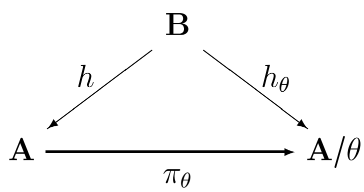
Then, there exists $g_h : \mathbf{B} \to \mathbf{A}$, such that $h \circ g_h = g$. So we have

$$g^{-1}(G) = (h \circ g_h)^{-1}(G) = g_h^{-1}(h^{-1}(G)) \in \mathcal{C}^\flat.$$

Thus, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$.

■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate, $\mathbf{A}$ an algebra and $\theta \in \mathrm{Con}(\mathbf{A})$. we write $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ for the canonical projection homomorphism and, given $h : \mathbf{B} \to \mathbf{A}$, we denote by h_θ the composition $h_\theta = \pi_\theta \circ h$.



Proposition 42 forms an analog of Proposition 35 for structural filters.

Proposition 42 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate, $\mathbf{A}$ an algebra, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{A})$. Then θ is compatible with F, i.e., $\theta \subseteq \Omega_{\mathbf{A}}(F)$, if and only if $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}/\theta)$.*

Proof: Suppose, first, that θ is compatible with F. Set $G = \pi_\theta(F)$. Then $\pi_\theta^{-1}(\pi_\theta(F)) = F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. By Proposition 41, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}/\theta)$. Since $F = \pi_\theta^{-1}(G)$, this proves necessity.

Suppose, conversely, that $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}/\theta)$. Let $a, b \in A$, such that $\langle a, b \rangle \in \theta$ and $a \in F$. Then $\pi_\theta(a) = \pi_\theta(b)$ and $a \in \pi_\theta^{-1}(G)$. Thus, $\pi_\theta(b) = \pi_\theta(a) \in G$, whence $b \in \pi_\theta^{-1}(G) = F$. So θ is compatible with F. ■

As far as an analog of Proposition 36, we get the following

Proposition 43 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate, $\mathbf{A}$, $\mathbf{A}'$ be algebras and $h : \mathbf{A} \to \mathbf{A}'$ an epimorphism. The following statements are equivalent:*

(i) $h : \langle \mathbf{A}, C \rangle \to \langle \mathbf{A}', C' \rangle$ is a bilogical morphism, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$ and $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$;

(ii) $h : \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}) \to \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$ is a bijection;

(ii) For all $F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$*,* $h(F) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$ *and* $\mathrm{Ker}(h) \in \mathrm{Con}(\langle \mathcal{A}, C\rangle)$*.*

Proof:

(i)⇒(ii) By Proposition 20.

(ii)⇒(iii) The first statement follows by hypothesis. For the second, suppose $\langle a, b\rangle \in \mathrm{Ker}(h)$ and $a \in F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. Then $h(a) = h(b)$ and $h(a) \in h(F)$. Thus, $h(b) \in h(F)$. Hence, $b \in h^{-1}(h(F)) = F$. This establishes that $\mathrm{Ker}(h) \in \mathrm{Con}(\langle \mathbf{A}, C\rangle)$.

(iii)⇒(i) By Proposition 20, we must show that

$$\mathcal{C}' = h(\mathcal{C}).$$

The right-to-left inclusion holds by hypothesis. Suppose, next, that $G \in \mathcal{C}'$. Let $F = h^{-1}(G)$. Then, by Proposition 41, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. By surjectivity, $G = h(h^{-1}(G)) = h(F)$. This proves the left-to-right inclusion. Now Proposition 20 gives that $h : \langle \mathbf{A}, C\rangle \to \langle \mathbf{A}', C'\rangle$ is a bilogical morphism.

■

The next proposition is an analog of Proposition 37 for structural logicates.

Proposition 44 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a structural logicate,* $\mathbf{A}$*,* $\mathbf{A}'$ *be algebras and* $h : \mathbf{A} \to \mathbf{A}'$ *surjective. If* $h : \langle \mathbf{A}, C\rangle \to \langle \mathbf{A}', C'\rangle$ *is a bilogical morphism and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$*, then*

$$\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}').$$

Proof: By Proposition 41, $\mathcal{C}' \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$. Assume, conversely, that $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$. Then, by Proposition 41, $h^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}) = \mathcal{C}$. Thus, by hypothesis, $Y = h(h^{-1}(Y)) \in \mathcal{C}'$. Hence, $\mathrm{Fi}_{\mathbb{L}}(\mathbf{A}') \subseteq \mathcal{C}'$. ■

Corollary 45 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a structural logicate and* $\mathbf{A}$ *an algebra. Then*

$$\mathrm{Fi}_{\mathbb{L}}(\mathbf{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}^*).$$

Proof: By Proposition 44. ■

Corollary 46 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a structural logicate,* $\mathbf{A}$*,* $\mathbf{A}'$ *algebras and and* $h : \langle \mathbf{A}, C\rangle \to \langle \mathbf{A}', C'\rangle$ *an isomorphism. Then*

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}) \quad \textit{iff} \quad \mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}').$$

Proof: By Proposition 44. ■

Many of the definitions and results concerning classes of algebras also have counterparts for structural logicates.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate. Suppose $\mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ be a structural matrix and $\theta \in \mathrm{Con}(\mathfrak{A})$. Then $\mathfrak{A}/\theta = \langle \mathbf{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$. $\mathfrak{A}/\theta$ is called the **quotient structural matrix of $\mathfrak{A}$ by θ**. In particular, $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathbf{A}}(F) \in \mathrm{Mat}^*(\mathbb{L})$. $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathbf{A}}(F)$ is called the **reduction** of $\mathfrak{A}$. We let $\mathrm{Alg}^*(\mathbb{L})$ be the class of algebraic reducts of matrices in $\mathrm{Mat}^*(\mathbb{L})$. So we have $\Omega_{\mathbf{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathbf{A})$. In the structural case, $\mathcal{C}^\flat$ coincides with the set of $\mathbb{L}$-filters on $\mathbf{B}$. This fact allows us, here also, to obtain a completeness analog of Proposition 40, which more closely resembles the one from the traditional setting.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate and consider a class M of structural matrices. We say that $\mathbb{L}$ is **complete with respect to M** if

$$\mathcal{C}^\flat = \{h^{-1}(F) : \langle \mathbf{A}, F \rangle \in \mathsf{M} \text{ and } h : \mathbf{B} \to \mathbf{A}\}.$$

Proposition 47 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate. $\mathbb{L}$ is complete both with respect to $\mathrm{Mat}(\mathbb{L})$ and with respect to $\mathrm{Mat}^*(\mathbb{L})$.*

Proof: By the definition of $\mathrm{Mat}(\mathbb{L})$, if $\langle \mathbf{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ and $h : \mathbf{B} \to \mathbf{A}$, then $h^{-1}(F) \in \mathcal{C}^\flat$. Conversely, if $T \in \mathcal{C}^\flat$, then, by structurality, $\langle \mathbf{B}, T \rangle \in \mathrm{Mat}(\mathbb{L})$ and $i_{\mathbf{B}}^{-1}(T) = T$. Hence, $\mathbb{L}$ is complete with respect to $\mathrm{Mat}(\mathbb{L})$.

By the definition of $\mathrm{Mat}^*(\mathbb{L})$, if $\mathfrak{A}^* = \langle \mathbf{A}^*, F^* \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $h : \mathbf{B} \to \mathbf{A}^*$, then there exists $h^* : \mathbf{B} \to \mathbf{A}$, such that

$$\pi \circ h^* = h,$$

where $\pi : \mathbf{A} \to \mathbf{A}^*$ is the canonical projection. Thus

$$\begin{array}{rcl} h^{-1}(F^*) & = & (\pi \circ h^*)^{-1}(F^*) \\ & = & (h^*)^{-1}(\pi^{-1}(F^*)) \\ & = & (h^*)^{-1}(F) \\ & \in & \mathcal{C}^\flat. \end{array}$$

Assume, conversely, that $T \in \mathcal{C}^\flat$. Then the pair $\langle \mathbf{B}^*, T^* \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $\pi^{-1}(T^*) = T$. This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{L})$. ■

We close this section with another result borrowed from the classical theory pertaining to the class $\mathrm{Alg}^*(\mathbb{L})$ of algebras of a structural logicate whose underlying algebra is a free algebra with countably many generators.

Proposition 48 *Let $\mathbb{L} = \langle \mathbf{Fm}_{\mathcal{L}}(V), C^\flat \rangle$ be a structural logicate. The class $\mathsf{K}_{\mathbb{L}}$ is the variety generated by the class $\mathrm{Alg}^*(\mathbb{L})$.*

Proof: Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. We have

$$\begin{array}{lll} \mathsf{K}_{\mathbb{L}} \vDash \varphi \approx \psi & \text{iff} & \langle \varphi, \psi \rangle \in \widetilde{\Omega}(\mathbb{L}) \\ & \text{iff} & \text{for all } \chi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V) \text{ and } T \in \mathcal{C}^{\flat}, \\ & & \chi(\varphi, \overline{z}) \in T \quad \text{iff} \quad \chi(\psi, \overline{z}) \in T \\ & \text{iff} & \text{for all } \chi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V),\ \langle \mathbf{A}, F \rangle \in \mathrm{Mat}^*(\mathbb{L}),\ \overline{a}, \overline{c} \text{ in } A, \\ & & \chi^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}), \overline{c}) \in F \quad \text{iff} \quad \chi^{\mathbf{A}}(\psi^{\mathbf{A}}(\overline{a}), \overline{c}) \in F \\ & \text{iff} & \text{for all } \langle \mathbf{A}, F \rangle \in \mathrm{Mat}^*(\mathbb{L}),\ \overline{a} \in A, \\ & & \langle \varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a}) \rangle \in \Omega_{\mathbf{A}}(F) \\ & \text{iff} & \text{for all } \langle \mathbf{A}, F \rangle \in \mathrm{Mat}^*(\mathbb{L}),\ \overline{a} \in A, \\ & & \varphi^{\mathbf{A}}(\overline{a}) = \psi^{\mathbf{A}}(\overline{a}) \\ & \text{iff} & \mathrm{Mat}^*(\mathbb{L}) \vDash \varphi \approx \psi. \end{array}$$

■