Chapter 4

Model Theory

4.1 Introduction

In this, third chapter, on logicates, we focus specifically on the role that algebraic logicates play as models of other logicates. Logicates are models more suitable for many purposes than simple logical matrices, even though logicate models can be viewed as bundles of matrices over the same underlying interpretation.

Our framework and starting point is the study in Chapter 3 of interpretations. We are assuming a given logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ which consists of an algebra and an idempotent operator, called a *consequence operator*, on its powerset. Logicates are supposed to represent logical systems for which inflationarity and monotonicity may fail. $\mathbb{L}$ is viewed as the focus of study and it is called, accordingly, a *base logicate*. An *interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ consists of a surjective homomorphism from the algebra $\mathbf{B}$ of $\mathbb{L}$ onto a similar algebra $\mathbf{A}$. If on the target algebra, there is given a logicate structure, say $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}$ induces a logicate on the base algebra. We say that $\mathbb{A}$ is a *model* of $\mathbb{L}$ if the inverse images under h of the theories of $\mathbb{A}$ form a subset of the theories of $\mathbb{L}$, written $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. Two logicates connected by a bilogical morphism that commutes with interpretations share the property of being simultaneously models or not being models.

One of the key constructions in our framework is passing from a model to its Tarski reduction. The *Tarski operator* was used as a key ingredient in the theory of Font and Jansana (Page 19 of [12]) and, as our work is based on theirs, it continues to play a crucial role here as well. Given a logicate model, one may construct its *reduction* by moding out both the interpretation and the idempotent operator by the Tarski congruence of the logicate. A first result is that a logicate is complete with respect to both its class of logicate models and its class of reduced logicate models. Completeness here simply means that collecting all inverse images of theories of the models of the class yields the full collection of theories of the base logicate.

Connecting the theory of logicate models with the theory of matrix models of the base logicate, which was detailed in Chapter 3, we obtain the fact, well-known in classical theory (Proposition 2.7 of [12]), that, a logicate, viewed as a bundle of matrices, is a model of $\mathbb{L}$ if and only if every member of the bundle is a matrix model of $\mathbb{L}$.

The next key concept adapted here from the theory of abstract logics of [12] is that of a *full model*. A *basic (full logicate) model* is a model whose collection of theories consists of all filters on its interpretation. A *full (logicate) model* is one whose Tarski reduction is a basic full logicate model. The terminology is justified by the fact that a basic model turns out to be a full model according to these definitions. It is shown here, in a result that parallels one pertaining to abstract logics, that logicate models connected via bilogical morphisms commuting with interpretations are either both full or both fail to be full. As a consequence the property of being full is also preserved and reflected by reductions. These results yield a characterization of the class of full models as the smallest class that contains all basic full models and is closed (in both directions) under bilogical morphisms.

Full models are the first ingredient in establishing a key *Isomorphism Theorem*, along the lines of the Isomorphism Theorem (Theorem 2.30) of Font and Jansana, which is one of the main results of the abstract treatment in the theory they present in [12]. The second ingredient relates to congruences whose quotients are algebras in $\mathrm{Alg}(\mathbb{L})$. The class $\mathrm{Alg}(\mathbb{L})$ consists of the underlying interpretations of reduced full models of $\mathbb{L}$. Another class of interpretations that is related to a class of algebras traditionally studied in algebraic logic is the class $\mathrm{Alg}^*(\mathbb{L})$. It consists of all underlying interpretations of reduced matrix models of $\mathbb{L}$. The tight connection, mentioned previously, between logicate models and matrix models, yields a (sort of induced) relationship between the two classes. In the traditional setting one class turns out to be the class of subdirect products of the other (see, e.g., Theorem 2.23 of [12]). In the present setting, because of the presence of fixed interpretation morphisms, we find it convenient (and perhaps necessary) to define a related but different operation on interpretations, called a *subdirect intersection*. It is shown that the class $\mathrm{Alg}(\mathbb{L})$ consists exactly of subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$.

Our work in this part culminates with proving an analog of the Isomorphism Theorem 2.30 of [12] for the present context. We view this as one of the main results of the work. The analog proven here has some significant deviations as compared to its predecessor. First, all parts are taken to be over fixed underlying interpretations. This is compelled by the absence of structurality for logicates. If one added structurality, then something closer, perhaps, to the original could be obtained. But this seemed rather restrictive and, in addition, the framework of π-institutions [21] has provided some experience in dealing with fixed interpretations. Second, one cannot expect to establish an isomorphism theorem dealing with all full models, since full models that are equipotent, have identical Tarski congruences. So one has, by necessity, to pass to equipotency classes of full models over fixed interpretations. Taking these comments into account, we establish an order isomorphism between the set of equipotency classes of full logicate models, ordered under the superset relation between sets of theories, and the set of $\mathrm{Alg}(\mathbb{L})$-congruences under inclusion. It is also shown that the latter poset is a complete lattice. As a consequence, one obtains that the former has the same structure as well.

In Section 4.2 we recall the notion of interpretation and use it to define *logicate interpretations*. These, in turn, serve in defining models of a logicate. Logicate models have a tight relationship with matrix models. We also define reductions. We show that, if two logicate interpretations are related via a bilogical morphism, then one is a model if and only if the other is. This implies that a given one is a model if and only if its reduction is. We also formulate analogs of the well-known Completeness Theorems of Algebraic

Logic both with respect to the class of all models and with respect to the class of all reduced models.

In Section 4.3, the notion of *full model* for logicates is introduced, taking after the corresponding notion for the monotonic framework (see Definition 2.8 of [12]). A model is a *basic (full) model* if the set of its theories coincides with the set of all filters on the underlying interpretation. A model is a *full model* if its reduction is a basic full model. Several properties, paralleling ones proved by Font and Jansana for sentential logics in [12], are adapted and proved in this setting. They culminate in two different characterizations of full models, which, as Font and Jansana explain, may be taken as justifications of the term "full". The class of full models is shown to be the smallest class that includes all basic full models and is closed under bilogical morphisms (see Corollary 2.13 of [12]). It is also the class of all models whose sets of theories consist of all preimages under canonical projections of all filters on the Tarski reduction of the model (see Theorem 2.14 of [12]).

In Section 4.4 we define the notion of $\mathbb{L}$-*algebra* for a given logicate $\mathbb{L}$. These parallel $\mathcal{S}$-algebras for a sentential logic [12]. In the present context, however, they should be referred to as $\mathbb{L}$-*interpretations*, since they are pairs consisting of an algebra together with a mapping from the base algebra of $\mathbb{L}$ onto the algebra. But the term "algebra" is retained because of the similarity of the role they play. Several results encapsulating the interaction of these algebras with full models and reduced full models are given. We also revisit the relation between the class of algebras which are reducts of reduced matrix models and the class of $\mathbb{L}$-algebras, which are reducts of reduced full models of $\mathbb{L}$. An operation, called *subdirect intersection*, paralleling that of subdirect product in the ordinary framework, is defined and comes in handy in this task. The result is an analog of Theorem 2.23 of [12].

In Section 4.5 the main goal is establishing an *Isomorphism Theorem*, along the lines of Theorem 2.30 of [12]. The Tarski operator over a fixed interpretation forms a mapping from logicates over that interpretation into congruences. Moreover, it is constant over equipotency classes of logicates. Thus, it may be viewed as an operator over equipotency classes of logicates to congruences. We introduce here an operator from $\mathrm{Alg}(\mathbb{L})$-congruences that is seeing as being the inverse of the restriction of the Tarski operator on equipotency classes of full models. Moreover, both operators are order preserving, when order is taken to be the one reflecting the superset relation between sets of theories. So they establish an isomorphism between equipotency classes of full models and $\mathbb{L}$-algebra congruences. Some consequences of this isomorphism are encountered here, among which is the fact that the equipotency classes of full models form a complete lattice. This is proven using the isomorphism theorem and a result showing that the collection of $\mathrm{Alg}(\mathbb{L})$-congruences under the subset relation form a complete lattice.

4.2 Models of Logicates

Let $\mathbf{B} = \langle B, \mathcal{L}^{\mathbf{B}} \rangle$ be an algebra, which, in this context, is termed **base algebra**. An **interpretation** is a pair $\mathcal{A} = \langle \mathbf{A}, h \rangle$, where

- $\mathbf{A}$ is an $\mathcal{L}$-algebra;
- $h : \mathbf{B} \twoheadrightarrow \mathbf{A}$ is an epimorphism from the base algebra onto $\mathbf{A}$.

A **logicate interpretation**, is a pair $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where:

- $\mathcal{A} = \langle \mathbf{A}, h \rangle$ is an interpretation;
- $\langle \mathbf{A}, C \rangle$ is an algebraic logicate on the algebra $\mathbf{A}$.

The logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ induces a function

$$C^{\mathbb{A}} : \mathcal{P}(B) \to \mathcal{P}(B),$$

where, for all $X \subseteq B$,

$$C^{\mathbb{A}}(X) = h^{-1}(C(h(X))).$$

We write

$$\mathbb{L}^{\mathbb{A}} := \langle \mathbf{B}, C^{\mathbb{A}} \rangle.$$

This construction forms an analog of the construction in Definition 2.1 of [12]. We show that $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicate on the base algebra and that the epimorphism h is a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{A}$.

Proposition 49 *Let* $\mathbf{B}$ *be a base algebra and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *a logicate interpretation, with* $\mathcal{A} = \langle \mathbf{A}, h \rangle$.

(a) $\mathbb{L}^{\mathbb{A}}$ *is an algebraic logicate.*

(b) $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{A}$ *is a bilogical morphism.*

Proof:

(a) We must show that $C^{\mathbb{A}}$ is idempotent. Let $X \subseteq B$. Then

$$\begin{aligned} C^{\mathbb{A}}(C^{\mathbb{A}}(X)) &= h^{-1}(C(h(h^{-1}(C(h(X)))))) \quad \text{(Definition of } C^{\mathbb{A}}) \\ &= h^{-1}(C(C(h(X)))) \quad (h \text{ Surjective}) \\ &= h^{-1}(C(h(X))) \quad (C \text{ Idempotent}) \\ &= C^{\mathbb{A}}(X). \quad \text{(Definition of } C^{\mathbb{A}}) \end{aligned}$$

Thus, $C^{\mathbb{A}}$ is idempotent and, therefore, $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicate.

(b) By the surjectivity of h and the definition of $C^{\mathbb{A}}$, we have, for all $X \subseteq B$,

$$h(C^{\mathbb{A}}(X)) = h(h^{-1}(C(h(X)))) = C(h(X)).$$

Suppose $X \in \mathcal{C}^{\mathbb{A}}$. Then

$$h(X) = h(C^{\mathbb{A}}(X)) = C(h(X)).$$

Hence $h(X) \in \mathcal{C}$. Suppose, conversely, that $Y \in \mathcal{C}$. Consider $X \subseteq B$, such that $h(X) = Y$. Then

$$h(C^{\mathbb{A}}(X)) = C(h(X)) = Y.$$

Thus Y is the image of a $C^{\mathbb{A}}(X) \in \mathcal{C}^{\mathbb{A}}$. Finally, if $b, b' \in B$, such that $\langle b, b' \rangle \in \mathrm{Ker}(h)$ and $b \in C^{\mathbb{A}}(X)$, then $h(b) = h(b')$ and, by definition of $C^{\mathbb{A}}$, $b \in h^{-1}(C(h(X)))$, whence $h(b') = h(b) \in C(h(X))$, showing that $b' \in h^{-1}(C(h(X))) = C^{\mathbb{A}}(X)$. Hence $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L}^{\mathbb{A}})$. By Proposition 20, we get that $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{L}$ is a bilogical morphism.

■

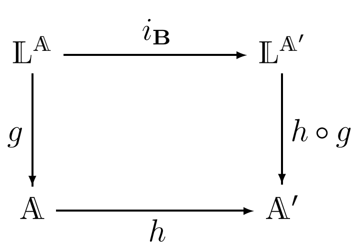
We call $\mathbb{L}^{\mathbb{A}}$ the **logicate induced on B** by $\mathbb{A}$.

An analog of Proposition 2.3 of [12] ensures that logicate interpretations related via "compatible" bilogical morphisms induce isomorphic logicates on the base algebra. Recall, however, the discussion on "isomorphisms" from Section 3.4 intended to thwart any misunderstandings concerning the term and its rather counterintuitive meaning. One may be better off as thinking of "isomorphisms" as "isopotent" bijections, meaning that they preserve and reflect theories without necessarily preserving consequences.

Proposition 50 *Let* $\mathbf{B}$ *be a base algebra and* $\mathbb{A} = \langle\langle \mathbf{A}, g\rangle, C\rangle$ *and* $\mathbb{A}' = \langle\langle \mathbf{A}', h \circ g\rangle, C'\rangle$ *two logicate interpretations, with* $h : \mathbb{A} \to \mathbb{A}'$ *a bilogical morphism. The identity* $i_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ *is an isomorphism*

$$i_{\mathbf{B}} : \mathbb{L}^{\mathbb{A}} \cong \mathbb{L}^{\mathbb{A}'}.$$

Proof: Using the diagram below, we have



$$\begin{aligned}
\mathcal{C}^{\mathbb{A}} &= g^{-1}(\mathcal{C}) \quad (g : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{A}) \\
&= g^{-1}(h^{-1}(\mathcal{C}')) \quad (h : \mathbb{A} \to_b \mathbb{A}') \\
&= (h \circ g)^{-1}(\mathcal{C}') \quad ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\
&= \mathcal{C}^{\mathbb{A}'}. \quad (h \circ g : \mathbb{L}^{\mathbb{A}'} \to_b \mathbb{A}')
\end{aligned}$$

So $i_{\mathbf{B}} : \mathbb{L}^{\mathbb{A}} \cong \mathbb{L}^{\mathbb{A}'}$ is an isomorphism. ■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, perceived as constituting the main object of investigation. A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, h \rangle$, is called a **model of** $\mathbb{L}$ or an $\mathbb{L}$**-model** if

$$h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat.$$

We denote by $\mathrm{Mod}(\mathbb{L})$ the class of all models of $\mathbb{L}$.

Recalling from Chapter 2 the ordering $\trianglelefteq$ on logicates, we can relate the notion of model with the construction of the logicate induced by a logicate interpretation.

Proposition 51 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, h \rangle$, a logicate interpretation. Then $\mathbb{A}$ is an $\mathbb{L}$-model if and only if $\mathbb{L} \trianglelefteq \mathbb{L}^{\mathbb{A}}$.*

Proof: Note that $\mathbb{A}$ being an $\mathbb{L}$-model means that $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$, whereas $\mathbb{L} \trianglelefteq \mathbb{L}^{\mathbb{A}}$ means that $\mathcal{C}^{\mathbb{A}} \subseteq \mathcal{C}^\flat$. Thus, to prove the result, it suffices to show that $\mathcal{C}^{\mathbb{A}} = h^{-1}(\mathcal{C})$.

Suppose, first, that $X \in \mathcal{C}^{\mathbb{A}}$. Consider $C(h(X)) \in \mathcal{C}$. We have

$$\begin{array}{rcl} h^{-1}(C(h(X))) & = & C^{\mathbb{A}}(X) \quad (\text{Definition of } C^{\mathbb{A}}) \\ & = & X. \quad (X \in \mathcal{C}^{\mathbb{A}}) \end{array}$$

This shows that $\mathcal{C}^{\mathbb{A}} \subseteq h^{-1}(\mathcal{C})$.

Assume, conversely, that $X \in h^{-1}(\mathcal{C})$. Then, $X = h^{-1}(Y)$, for some $Y \in \mathcal{C}$. Thus, we get

$$\begin{array}{rcl} C^{\mathbb{A}}(X) & = & h^{-1}(C(h(X))) \quad (\text{Definition of } C^{\mathbb{A}}) \\ & = & h^{-1}(C(h(h^{-1}(Y)))) \quad (X = h^{-1}(Y)) \\ & = & h^{-1}(C(Y)) \quad (h \text{ surjective}) \\ & = & h^{-1}(Y) \quad (Y \in \mathcal{C}) \\ & = & X. \quad (X = h^{-1}(Y)) \end{array}$$

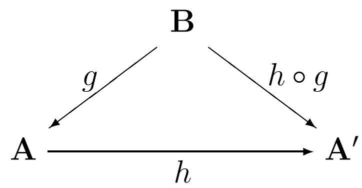
Hence, $X \in \mathcal{C}^{\mathbb{A}}$ and $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^{\mathbb{A}}$. ■

Let L be a class of models of $\mathbb{L}$. $\mathbb{L}$ is said to be **complete with respect to** L if

$$\mathcal{C}^\flat = \bigcup \{ h^{-1}(\mathcal{C}) : \langle \langle \mathbf{A}, h \rangle, C \rangle \in \mathsf{L} \}.$$

The following result is an analog of Part (1) of Proposition 2.5 of [12] for logicate interpretations.

Proposition 52 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate,* $\mathbb{A} = \langle \langle \mathbf{A}, g \rangle, C \rangle$*,* $\mathbb{A}' = \langle \langle \mathbf{A}', h \circ g \rangle, C' \rangle$ *be logicate interpretations, where* $h : \mathbb{A} \to \mathbb{A}'$ *be a bilogical morphism.*



Then $\mathbb{A}$ *is a model of* $\mathbb{L}$ *if and only if* $\mathbb{A}'$ *is a model of* $\mathbb{L}$*.*

Proof: Suppose $\mathbb{A}$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcll} (h \circ g)^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & = & g^{-1}(\mathcal{C}) & (h : \mathbb{A} \to_b \mathbb{A}') \\ & \subseteq & \mathcal{C}^\flat. & (\mathbb{A} \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}'$ is a model of $\mathbb{L}$. Assume, conversely, that $\mathbb{A}'$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcll} g^{-1}(\mathcal{C}) & = & g^{-1}(h^{-1}(\mathcal{C}')) & (h : \mathbb{A} \to_b \mathbb{A}') \\ & = & (h \circ g)^{-1}(\mathcal{C}') & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & \subseteq & \mathcal{C}^\flat. & (\mathbb{A}' \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}$ is a model of $\mathbb{L}$. ■

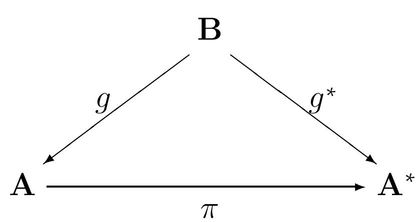
In order to formulate another Part (2) of Proposition 2.5 of [12], we need to define the Tarski reduction of a logicate interpretation.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an base logicate. Consider the pair $\mathbb{A} = \langle \langle \mathbf{A}, g \rangle, C \rangle$. Recall the Tarski congruence $\widetilde{\Omega}(\mathbb{A}) := \widetilde{\Omega}_{\mathbf{A}}(C)$. We define the pair

$$\mathbb{A}^* = \langle \langle \mathbf{A}^*, g^* \rangle, C^* \rangle$$

by setting:

- $\mathbf{A}^* = \mathbf{A}/\widetilde{\Omega}_{\mathbf{A}}(C)$;
- $g^* = \pi \circ g$, where $\pi : \mathbf{A} \to \mathbf{A}^*$ is the natural projection;



- $C^* : \mathcal{P}(A/\widetilde{\Omega}_{\mathbf{A}}(C)) \to \mathcal{P}(A/\widetilde{\Omega}_{\mathbf{A}}(C))$, where, for all $S \subseteq A/\widetilde{\Omega}_{\mathbf{A}}(C)$,

$$C^*(S) = \pi(C(\pi^{-1}(S))).$$

Based on results already obtained, we may show that, if $\mathbb{A}$ is a model of $\mathbb{L}$, then so is $\mathbb{A}^*$.

Corollary 53 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. A pair $\mathbb{A} = \langle \langle \mathbf{A}, g \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if $\mathbb{A}^*$ is a model of $\mathbb{L}$.*

Proof: This follows directly from Proposition 52, since, by Proposition 24, the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism. ■

Corollary 54 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is complete with respect to a class L of models if and only if it is complete with respect to the class L^*.*

Proof: Let

$$\mathsf{L} = \{\langle \langle \mathbf{A}_i, g_i \rangle, C_i \rangle : i \in I\}.$$

Denote by $\pi_i : \mathbf{A}_i \to \mathbf{A}_i^*$, $i \in I$, the natural projections. By Proposition 24, for all $i \in I$, $\pi_i : \mathbb{A}_i \to \mathbb{A}_i^*$ is a bilogical morphism. So we have, for all $i \in I$,

$$\begin{array}{rcll} \bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i) & = & \bigcup_{i\in I} g_i^{-1}(\pi_i^{-1}(\mathcal{C}_i^*)) & (\pi_i : \mathbb{A}_i \to_b \mathbb{A}_i^*) \\ & = & \bigcup_{i\in I} (\pi_i \circ g_i)^{-1}(\mathcal{C}_i^*) & ((\pi_i \circ g_i)^{-1} = g_i^{-1} \circ \pi_i^{-1}) \\ & = & \bigcup_{i\in I} (g_i^*)^{-1}(\mathcal{C}_i^*). & (g_i^* = \pi_i \circ g_i) \end{array}$$

Thus, $\mathbb{L}$ is complete with respect to L if and only if, by definition, $\mathcal{C}^\flat = \bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i)$ if and only if, by the displayed equality, $\mathcal{C}^\flat = \bigcup_{i\in I} (g_i^*)^{-1}(\mathcal{C}_i^*)$ if and only if, by definition, $\mathbb{L}$ is complete with respect to L^*. ■

As far as completeness properties go, note that $\langle \langle \mathbf{B}, i_\mathbf{B} \rangle, C^\flat \rangle$ is a model of $\mathbb{L}$. This yields the following results, forming, together, an analog of Proposition 2.6 of [12].

Proposition 55 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is complete with respect to any class L of models that includes $\mathbb{A} = \langle \langle \mathbf{B}, i_\mathbf{B} \rangle, C^\flat \rangle$ or $\mathbb{A}^*$.*

Proof: Let $\mathsf{L} = \{\langle \langle \mathbf{A}_i, g_i \rangle, C_i \rangle : i \in I\}$. On the one hand, since $\mathsf{L} \subseteq \mathrm{Mod}(\mathbb{L})$, $\bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i) \subseteq \mathcal{C}^\flat$. On the other, since $\mathbb{A} \in \mathsf{L}$, $\mathcal{C}^\flat = i_\mathbf{B}^{-1}(\mathcal{C}^\flat) \subseteq \bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i)$. Thus, $\mathbb{L}$ is complete with respect to L.

Assume, now, that $\mathbb{A}^* \in \mathsf{L}$. The first inclusion is justified in the same way. For the second, letting $\pi : \mathbf{B} \to \mathbf{B}^*$ be the natural projection, we have

$$\begin{array}{rcll} \mathcal{C}^\flat & = & i_\mathbf{B}^{-1}(\mathcal{C}^\flat) & (i_\mathbf{B} \text{ identity}) \\ & = & i_\mathbf{B}^{-1}(\pi^{-1}(\mathcal{C}^{\flat *})) & (\pi : \mathbb{A} \to_b \mathbb{A}^*) \\ & = & (\pi \circ i_\mathbf{B})^{-1}(\mathcal{C}^{\flat *}) & ((\pi \circ i_\mathbf{B})^{-1} = i_\mathbf{B}^{-1} \circ \pi^{-1}) \\ & \subseteq & \bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i). & (\mathbb{A}^* \in \mathsf{L}) \end{array}$$

Thus, $\mathbb{L}$ is again complete with respect to L. ■

Corollary 56 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is complete with respect to the class of all its models and with respect to the class of all its reduced models.*

Proof: Clearly, $\mathbb{A} = \langle \langle \mathbf{B}, i_{\mathbf{B}} \rangle, C^\flat \rangle \in \mathrm{Mod}(\mathbb{L})$ and $\mathbb{A}^* \in \mathrm{Mod}^*(\mathbb{L})$. Thus, by Proposition 55, $\mathbb{L}$ is complete both with respect to $\mathrm{Mod}(\mathbb{L})$ and with respect to $\mathrm{Mod}^*(\mathbb{L})$. ∎

Logicate models are closely connected with matrix models. The connection is given in the following proposition, which parallels Proposition 2.7 of [12].

Proposition 57 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Then $\langle \langle \mathbf{A}, h \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if, for all $Y \in \mathcal{C}$, $\langle \langle \mathbf{A}, h \rangle, Y \rangle$ is an $\mathbb{L}$-matrix.*

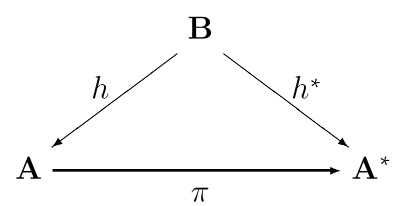
Proof: We have that $\langle \langle \mathbf{A}, h \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if, by definition, $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$ if and only if, for all $Y \in \mathcal{C}$, $h^{-1}(Y) \in \mathcal{C}^\flat$ if and only if, by definition, for all $Y \in \mathcal{C}$, $\langle \langle \mathbf{A}, h \rangle, Y \rangle$ is an $\mathbb{L}$-matrix. ∎

Proposition 57 asserts that the weakest equipotency class of models of $\mathbb{L}$ on $\mathcal{A}$ with respect to the $\trianglelefteq$ relation on equipotency classes of logicates is the one determined by

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}).$$

4.3 Full Models

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, h \rangle$, is called a **full model of** $\mathbb{L}$ or a **full** $\mathbb{L}$**-model** (see Definition 2.8 of [12]) if



$$\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a **basic (full) model of** $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, rephrasing the definition, we may say that $\mathbb{A}$ is a full model of $\mathbb{L}$ if and only if its reduction is a basic full model of $\mathbb{L}$.

$\mathrm{FMod}(\mathbb{L})$ denotes the class of all full models of $\mathbb{L}$. $\mathrm{FMod}^*(\mathbb{L})$ is the class of all reduced full models of $\mathbb{L}$. Given an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$ is the class of all full models of $\mathbb{L}$ on $\mathcal{A}$.

An analog of Part (1) of Proposition 2.9 of [12] provides a justification for the use of the term "model" for full models.

Proposition 58 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and $\mathbb{A} = \langle \mathcal{A}, C \rangle$ a full model of $\mathbb{L}$. Then $\mathbb{A}$ is a model of $\mathbb{L}$.*

Proof: Suppose $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$. By definition, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, by Proposition 57, $\mathbb{A}^*$ is a model of $\mathbb{L}$. Thus, by Corollary 53, $\mathbb{A}$ is also a model of $\mathbb{L}$. ■

The next result, an analog of Proposition 2.10 of [12], asserts that every basic full model is actually a full model, justifying the "full" in the definition of basic (full) models.

Proposition 59 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ on $\mathcal{A}$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a full model of $\mathbb{L}$ and is among the $\trianglelefteq$-weakest full models of $\mathbb{L}$ on $\mathcal{A}$.*

Proof: By Proposition 24, the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism. By Corollary 38, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$, i.e., $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, $\mathbb{A}$ is a full model of $\mathbb{L}$. It is among the $\trianglelefteq$-weakest full models since it is among the $\trianglelefteq$-weakest models, by Proposition 57. ■

Proposition 2.11 of [12], concerning closure of the class of full models under bilogical morphisms, has the following analog.

Proposition 60 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. The class $\mathrm{FMod}(\mathbb{L})$ is closed under bilogical morphisms, i.e., if $h : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$, where $\mathcal{A} = \langle \mathbf{A}, g \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', h \circ g \rangle$, is a bilogical morphism, then*

$$\langle \mathcal{A}, C \rangle \in \mathrm{FMod}(\mathbb{L}) \quad \textit{iff} \quad \langle \mathcal{A}', C' \rangle \in \mathrm{FMod}(\mathbb{L}).$$

Proof: Suppose $h : \mathbb{A} \to \mathbb{A}'$ is a bilogical morphism. By Proposition 30, there exists an isomorphism $h^* : \mathbb{A}^* \cong \mathbb{A}'^*$. Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. Then $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Thus, by Corollary 39, $\mathcal{C}'^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'^*)$. But $\mathbb{A}'^*$ is reduced, whence $\mathbb{A}'^*$ is a full model of $\mathbb{L}$. A similar reasoning yields the converse. ■

Corollary 61 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Then $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$ if and only if $\mathbb{A}^* \in \mathrm{FMod}(\mathbb{L})$.*

Proof: Directly from Proposition 60, since the natural projection $\pi : \mathbf{A} \to \mathbf{A}^*$ is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$. ■

Corollary 62 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Then $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$ if and only if there exists a bilogical morphism from $\mathbb{A}$ onto a model $\langle \mathcal{A}', C' \rangle$, with $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$.*

Proof: The "only if" follows directly from the definition of full model, as the projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

Conversely, assume there is a bilogical morphism $h : \mathbb{A} \to_b \mathbb{A}'$, such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By Proposition 59, $\mathbb{A}'$ is a full model, whence, by Proposition 60, $\mathbb{A}$ is also a full model. ∎

Our work culminates in a characterization of the class of full models of a logicate $\mathbb{L}$ along the lines of Corollary 2.13 of [12].

Corollary 63 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. Then* $\mathrm{FMod}(\mathbb{L})$ *is the smallest class containing all* $\langle \mathcal{A}, C \rangle$*, with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, and closed under bilogical morphisms.*

Proof: Let L be the smallest class containing all $\langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, and closed under bilogical morphisms.

On the one hand, every $\langle \mathcal{A}, C \rangle$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a full model, by Proposition 59. Moreover, by Proposition 11, the class of full models is closed under bilogical morphisms. This shows that $\mathsf{L} \subseteq \mathrm{FMod}(\mathbb{L})$. The reverse inclusion is a direct consequence of Corollary 62. ∎

Corollary 63 provides one justification for the "fullness" property of full models. According to this justification, a full model is one that is obtained via a bilogical morphism by a model whose theories constitute a full set of $\mathbb{L}$-filters. A second justification is given in the following theorem. According to this, a model's "fullness" rests on the fact that its theories contain all possible $\mathbb{L}$-filters corresponding to $\mathbb{L}$-filters of the Tarski-reduction of the model.

Theorem 64 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be an algebraic logicate. Then* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *is a full model of* $\mathbb{L}$ *if and only if*

$$\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}.$$

Proof: Suppose, first, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$. Let $X \in \mathcal{C}$. By Proposition 57, $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By definition of $\widetilde{\Omega}(\mathbb{A})$, it always holds that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. For the reverse inclusion, let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. Then, by Proposition 35, there exists $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$, such that $X = \pi^{-1}(Y)$, where $\pi : \mathbf{A} \to \mathbf{A}/\widetilde{\Omega}(\mathbb{A})$ is the natural projection. But, by Proposition 24, $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism and, moreover, since $\mathbb{A}$ is full, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $X \in \mathcal{C}$.

Suppose, conversely, that $\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}$. Since the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $\mathbb{A}$ is a full model of $\mathbb{L}$. ∎

4.4 $\mathbb{L}$-Algebras

Reduced full models of $\mathbb{L}$ are those models of the form $\langle\mathcal{A}, C\rangle$, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, that are reduced. The interpretation reducts of such models are given a special name.

Let $\mathbb{L} = \langle\mathbf{B}, C^\flat\rangle$ be a base logicate. An interpretation $\mathcal{A} = \langle\mathbf{A}, h\rangle$ is an $\mathbb{L}$-**algebra** (see Definition 2.16 of [12]) if

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) = \Delta_{\mathbf{A}}.$$

The class of all $\mathbb{L}$-algebras is denoted by $\mathrm{Alg}(\mathbb{L})$. Perhaps, a more suitable term and notation here would have been $\mathbb{L}$-*interpretation* and $\mathrm{Int}(\mathbb{L})$, but we keep the ones used in the traditional theory, even though the entities differ, since they play analogous roles as the $\mathbb{L}$-algebras in the traditional theory.

The following characterization takes after Proposition 2.17 of [12].

Proposition 65 *Let $\mathbb{L} = \langle\mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathbb{A} = \langle\mathcal{A}, C\rangle$, with $\mathcal{A} = \langle\mathbf{A}, h\rangle$. Then the following statements are equivalent:*

(i) $\mathbb{A} = \langle\mathcal{A}, C\rangle$ *is a reduced full model of* $\mathbb{L}$*;*

(ii) $\mathbb{A} = \langle\mathcal{A}, C\rangle$ *is reduced and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*;*

(iii) $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$ *and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof:

(i)⇒(ii) By the definition of a reduced full model.

(ii)⇒(iii) By the definition of an $\mathbb{L}$-algebra, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$.

(iii)⇒(i) Since $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, there exists $C' : \mathcal{P}(A) \to \mathcal{P}(A)$, such that $\mathbb{A} = \langle\mathcal{A}, C'\rangle$ is a reduced full model of $\mathbb{L}$. But then $\mathcal{C}' = \mathcal{C}$ and $\mathbb{A} = \langle\mathcal{A}, C\rangle$ is a reduced full model of $\mathbb{L}$.

■

Proposition 66 *Let $\mathbb{L} = \langle\mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathbb{A} = \langle\mathcal{A}, C\rangle$ a full model of $\mathbb{L}$. Then $\mathcal{A}^* \coloneqq \mathcal{A}/\widetilde{\Omega}(\mathbb{A})$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$.*

Proof: By Corollary 61, $\mathbb{A}^*$ is a full model of $\mathbb{L}$ and it is clearly reduced. Hence, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra. This also yields the second statement using the definition of $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. ■

A characterization of $\mathbb{L}$-algebras, an analog of Proposition 2.19 of [12], shows that the notion of model, without reference to fullness, suffices to characterize the class $\mathrm{Alg}(\mathbb{L})$.

Proposition 67 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. The class* $\mathrm{Alg}(\mathbb{L})$ *is the class of algebraic reducts of all reduced models of* $\mathbb{L}$.

Proof: By definition, if $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, then $\mathcal{A}$ is the algebraic reduct of a reduced full model; in particular of a reduced model. Assume, conversely, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a reduced model of $\mathbb{L}$. Let $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$, be such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 57, $\mathbb{A}'$ is a model of $\mathbb{L}$ and, by Proposition 59, it is clearly full. It is also reduced, since

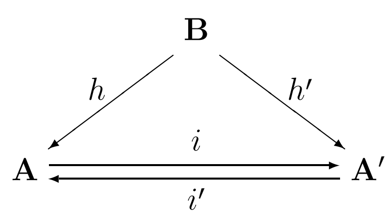
$$\widetilde{\Omega}(\mathbb{A}') \subseteq \widetilde{\Omega}(\mathbb{A}) = \Delta_{\mathbf{A}}.$$

Therefore, by definition, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ■

Closure under isomorphisms is guaranteed by the following proposition, an analog of Proposition 2.20 of [12].

Proposition 68 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. The class* $\mathrm{Alg}(\mathbb{L})$ *is closed under isomorphisms (commuting with the interpretations).*

Proof: Let $i : \mathbf{A} \cong \mathbf{A}'$. We have the following diagram.



Suppose that $\mathcal{A} = \langle \mathbf{A}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for some C, $\langle \mathcal{A}, C \rangle$ is a reduced full model of $\mathbb{L}$. Consider $\mathcal{A}' = \langle \mathbf{A}', h' \rangle = \langle \mathbf{A}', i \circ h \rangle$. We have, $\langle \mathcal{A}', C' \rangle$, with $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, is a reduced full model of $\mathbb{L}$. Thus, $\mathcal{A}' \in \mathrm{Alg}(\mathbb{L})$. The reverse implication can be proved similarly. ■

Putting together several of the previous results, we get the following alternative characterizations of full models involving $\mathbb{L}$-algebras.

Proposition 69 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$, *with* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation. Then the following statements are equivalent.*

(i) $\mathbb{A}$ *is a full model of* $\mathbb{L}$;

(ii) $\mathcal{A}^*$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$;

(iii) There exists a bilogical morphism $g : \mathbb{A} \to \mathbb{A}'$, *with* $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$, *such that* $\mathcal{A}'$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$.

Proof:

(i)⇒(ii) Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. By definition, $\mathbb{A}^*$ is a basic full model of $\mathbb{L}$. Thus, by Proposition 65, $\mathcal{A}^* = \langle \mathbf{A}^*, h^* \rangle$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

(ii)⇒(iii) Assume $\mathcal{A}^* = \langle \mathbf{A}^*, h^* \rangle$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Then (iii) is immediate by considering the natural projection $\pi : \mathbf{A} \to \mathbf{A}^*$, which is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$ and such that $\mathbb{A}^*$ fulfills the required conditions by (ii).

(iii)⇒(i) By Proposition 65, $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$, is a reduced full model of $\mathbb{L}$. Therefore, by Corollary 62, $\mathbb{A}$ is a full model of $\mathbb{L}$.

■

An analog of the Completeness Theorem 2.22 of [12] asserts that the class of full models, the class of reduced full models, as well as the class of all basic full models of a logicate can serve as a complete semantics for the logicate.

Theorem 70 (Completeness) *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete with respect to the following classes of models:*

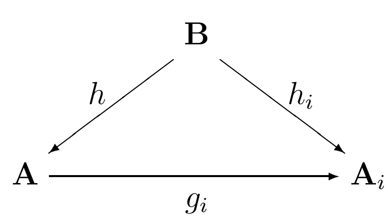
1. *The class* $\mathrm{FMod}(\mathbb{L})$ *of all full models of* $\mathbb{L}$*;*
2. *The class of all basic full models of* $\mathbb{L}$*;*
3. *The class* $\mathrm{FMod}^*(\mathbb{L})$ *of all reduced full models of* $\mathbb{L}$*.*

Proof: All three classes consist of models of $\mathbb{L}$. In addition each contains the model $\langle \langle \mathbf{B}^*, \pi \rangle, C^{\flat *} \rangle$, where $\pi : \mathbf{B} \to \mathbf{B}^*$ is the canonical projection. Thus, by Proposition 55, $\mathbb{L}$ is complete with respect to each of these three classes. ■

We now establish an analog of the well known theorem (Theorem 2.23 of [12]) relating the classes $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$. Recall that $\mathrm{Alg}^*(\mathbb{L})$ is the class of all algebraic (interpretation) reducts of reduced matrix models of $\mathbb{L}$, whereas $\mathrm{Alg}(\mathbb{L})$ is the class of all algebraic (interpretation) reducts of reduced full models of $\mathbb{L}$. In the present setting, however, due to the presence of morphisms from the base logicate in the interpretations involved, one has to replace subdirect products by a different operation, named *subdirect intersection*.

Let $\mathcal{A}_i = \langle \mathbf{A}_i, h_i \rangle$, $i \in I$, be a collection of interpretations. We say that an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ is a **subdirect intersection of** the $\mathcal{A}_i$ **relative to B** if:

- There exist surjective homomorphisms $g_i : \mathbf{A} \to \mathbf{A}_i$, $i \in I$, such that the following diagram commutes for all $i \in I$.

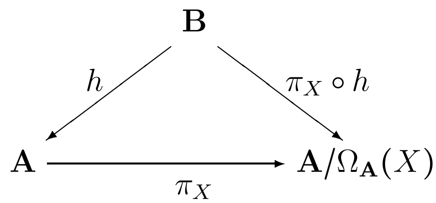


- $\bigcap_{i\in I}\mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$.

Since the role of $\mathbf{B}$ is going to be played by the base algebra, we usually omit the "relative to $\mathbf{B}$" in the terminology.

Theorem 71 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate. The class* $\mathrm{Alg}(\mathbb{L})$ *is the class of all subdirect intersections of interpretations in* $\mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathcal{A} = \langle \mathbf{A}, h\rangle \in \mathrm{Alg}(\mathbb{L})$. Then there exists C, such that $\mathbb{A} = \langle \mathcal{A}, C\rangle$ is a reduced full model of $\mathbb{L}$. For every $X \in \mathcal{C}$, we form the commutative triangle of epimorphisms



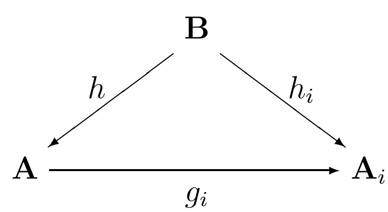
where $\pi_X : \mathbf{A} \to \mathbf{A}/\Omega_{\mathbf{A}}(X)$ denotes the canonical projection. Note that

$$\begin{array}{rcl}\bigcap_{X\in\mathcal{C}}\mathrm{Ker}(\pi_X) & = & \bigcap_{X\in\mathcal{C}}\Omega_{\mathbf{A}}(X) \quad (\pi_X : \mathbf{A} \to \mathbf{A}/\Omega_{\mathbf{A}}(X))\\ & = & \widetilde{\Omega}(\mathbb{A}) \quad (\text{Definition of } \widetilde{\Omega}(\mathbb{A}))\\ & = & \Delta_{\mathbf{A}}. \quad (\mathbb{A} = \langle \mathcal{A}, C\rangle \text{ reduced})\end{array}$$

Hence $\mathbf{A}$ is a subdirect intersection of

$$\mathcal{A}/\Omega_{\mathcal{A}}(X) = \langle \mathbf{A}/\Omega_{\mathcal{A}}(X), \pi_X \circ h\rangle \in \mathrm{Alg}^*(\mathbb{L}), \quad X \in \mathcal{C}.$$

Assume, conversely, that $\mathcal{A} = \langle \mathbf{A}, h\rangle$ is a subdirect intersection of a collection $\mathcal{A}_i = \langle \mathbf{A}_i, h_i\rangle \in \mathrm{Alg}^*(\mathbb{L})$, $i \in I$. Then, by hypothesis, we have commutative diagrams of epimorphisms,



such that $\bigcap_{i\in I}\mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$. Moreover, since, for all $i \in I$, $\mathcal{A}_i \in \mathrm{Alg}^*(\mathbb{L})$, there exists $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, such that $\Omega_{\mathcal{A}_i}(X) = \Delta_{\mathbf{A}_i}$. Let $C : \mathcal{P}(A) \to \mathcal{P}(A)$, be such that

$$\mathcal{C} = \{g_i^{-1}(X_i) : i \in I\}$$

and set $\mathbb{A} = \langle \mathcal{A}, C\rangle$. Since $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, $i \in I$, we have, by Proposition 34, that $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover,

$$\begin{array}{rcl}\widetilde{\Omega}(\mathbb{A}) & = & \bigcap_{i\in I}\Omega_{\mathcal{A}}(g_i^{-1}(X_i)) \quad (\text{Definition of } \widetilde{\Omega}(\mathbb{A}))\\ & = & \bigcap_{i\in I} g_i^{-1}(\Omega_{\mathcal{A}_i}(X_i)) \quad (\text{Property of } \Omega)\\ & = & \bigcap_{i\in I} g_i^{-1}(\Delta_{\mathbf{A}_i}) \quad (\Omega_{\mathcal{A}_i}(X) = \Delta_{\mathbf{A}_i})\\ & = & \bigcap_{i\in I}\mathrm{Ker}(g_i) \quad (\text{Definition of } \mathrm{Ker}(g_i))\\ & = & \Delta_{\mathbf{A}}. \quad (\text{Assumption})\end{array}$$

Thus, by Proposition 67, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ■

Corollary 72 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. Then* $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$. *Moreover,* $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$ *if and only if* $\mathrm{Alg}^*(\mathbb{L})$ *is closed under subdirect intersections.*

Recall from Chapter 2 the preorder $\trianglelefteq$ on logicates over the same underlying set A. Here, of course, we understand it to be between logicates over the same underlying algebra. We write $\langle \mathbf{A}, C \rangle \trianglelefteq \langle \mathbf{A}, C' \rangle$ to signify that $\mathcal{C}' \subseteq \mathcal{C}$. This becomes a partial order on equipotency classes of logicates (equipotency is the relation of having identical sets of theories).

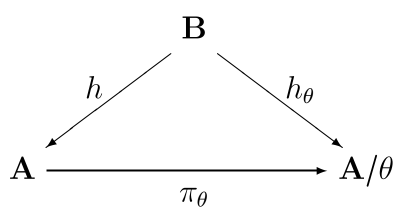
Proposition 73 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *and* $\mathbb{L}' = \langle \mathbf{B}, C'^\flat \rangle$ *be algebraic logicates over the same algebra, such that* $\mathbb{L} \trianglelefteq \mathbb{L}'$. *Then* $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$ *and* $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathbb{L} \trianglelefteq \mathbb{L}'$. Then, for all interpretations $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}'}(\mathcal{A}) \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$. By Theorem 71, we also have $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$. ■

4.5 The Lattice of Full Models

We fix a base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ and an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$. Let $\theta \in \mathrm{Con}(\mathbf{A})$. Recall that

$$\mathcal{A}/\theta = \langle \mathbf{A}/\theta, h_\theta \rangle,$$



where $h_\theta = \pi_\theta \circ h$, with $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ being the canonical projection. Consider the equipotency class of algebraic logicates (here meant over the same fixed interpretation $\mathcal{A}$) represented by $\langle \mathcal{A}/\theta, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Define

$$\widetilde{h}_{\mathcal{A}}(\theta) \coloneqq \langle \mathcal{A}, C_\theta \rangle,$$

where $\langle \mathcal{A}, C_\theta \rangle$ is the algebraic logicate induced by $\langle \langle \mathbf{A}/\theta, \pi_\theta \rangle, C \rangle$ on $\mathbf{A}$ (viewing $\mathbf{A}$ as the base algebra momentarily). The induced logicate $\langle \mathcal{A}, C_\theta \rangle$ depends on the choice of representative $\langle \mathcal{A}/\theta, C \rangle$ from the equipotency class on $\mathcal{A}/\theta$. However, any two representatives result in equipotent logicates on $\mathcal{A}$. So, it makes sense to define the function

$$\begin{array}{rrcl} \widetilde{H}_{\mathcal{A}}(\theta): & \mathrm{Con}(\mathbf{A}) & \longrightarrow & \mathrm{Log}(\mathcal{A})/\triangleq; \\ & \theta & \longmapsto & \langle \mathcal{A}, C_\theta \rangle/\triangleq. \end{array}$$

Note that, restricting to representatives, we have, by Proposition 49, that

$$\pi_\theta : \widetilde{h}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C\rangle$$

is a bilogical morphism.

Lemma 74 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be an algebraic logicate,* $\mathcal{A} = \langle \mathbf{A}, h\rangle$ *a fixed interpretation and* $\theta \in \mathrm{Con}(\mathbf{A})$.

(a) $\theta \in \mathrm{Con}(\widetilde{h}_{\mathcal{A}}(\theta))$;

(b) $\widetilde{h}_{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}/\theta, C\rangle$;

(c) $\widetilde{h}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$;

(d) $\theta \mapsto \widetilde{H}_{\mathcal{A}}(\theta)$ *is order preserving, i.e., if* $\theta \subseteq \theta'$, *then* $\widetilde{H}_{\mathcal{A}}(\theta) \trianglelefteq \widetilde{H}_{\mathcal{A}}(\theta')$.

Proof:

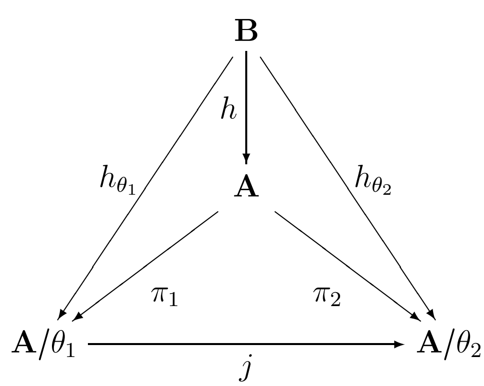
(a) By Proposition 49, $\pi_\theta : \widetilde{h}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C\rangle$ is a bilogical morphism. By Proposition 20, $\theta \in \mathrm{Con}(\widetilde{h}_{\mathcal{A}}(\theta))$.

(b) We have

$$\pi_\theta(C_\theta(X)) = \pi_\theta(\pi_\theta^{-1}(C(\pi_\theta(X)))) = C(\pi_\theta(X)).$$

(c) By hypothesis, $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Thus, by Proposition 59, $\langle \mathcal{A}/\theta, C\rangle$ is a full model of $\mathbb{L}$. Thus, by Proposition 60, $\widetilde{h}_{\mathcal{A}}(\theta)$ is also a full model of $\mathbb{L}$.

(d) Let $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$, such that $\theta_1 \subseteq \theta_2$. Let $\pi_1 : \mathcal{A} \to \mathcal{A}/\theta_1$ and $\pi_2 : \mathcal{A} \to \mathcal{A}/\theta_2$ be the canonical projections. Let, also, $j : \mathcal{A}/\theta_1 \to \mathcal{A}/\theta_2$ be the map given by $a/\theta_1 \mapsto a/\theta_2$, which is well defined due to the inclusion $\theta_1 \subseteq \theta_2$. In addition, we have the following commutative diagram.



Now we get

$$\begin{array}{rcl}\mathcal{C}_{\theta_2} & = & \pi_2^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2)) \quad (\pi_2 : \widetilde{h}_{\mathcal{A}}(\theta_2) \to_b \langle \mathcal{A}/\theta_2, C_2\rangle) \\ & = & \pi_1^{-1}(j^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2))) \quad (\pi_2 = j \circ \pi_1) \\ & \subseteq & \pi_1^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_1)) \quad (\text{Proposition 34}) \\ & = & \mathcal{C}_{\theta_1}. \quad (\pi_1 : \widetilde{h}_{\mathcal{A}}(\theta_1) \to_b \langle \mathcal{A}/\theta_1, C_1\rangle)\end{array}$$

This shows that $\widetilde{H}_{\mathcal{A}}(\theta_1) \le \widetilde{H}_{\mathcal{A}}(\theta_2)$.

■

Now we are in a position to prove a general analog of the Isomorphism Theorem of Font and Jansana (Theorem 2.30 of [12]) which is applicable even in contexts involving non-monotonicity. Of course, one has to pay the price that, instead of individual full models, equipotency classes of full models are considered. This has the undesirable effect that consequence is taken into account only as a determinator of the set of theories. On the other hand, this is to be expected and it creates an advantage. It is to be expected, since the Tarski operator is constant on equipotency classes and, hence, only restricting to equipotency classes is there a possibility of it becoming injective. The advantage created is that the result becomes applicable more widely, while its restriction to closure operators (i.e., inflationary and monotone consequence operators) is an Isomorphism Theorem very much resembling the one of Font and Jansana modulo the introduction of fixed interpretations. In this latter respect, we follow more closely the generalization of Font and Jansana's result that was presented as Theorem 13 of [21] for logical systems formalized as π-institutions.

Theorem 75 (Isomorphism) *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ a fixed interpretation. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is an order isomorphism between the ordered set $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ of equipotency classes of full models of $\mathbb{L}$ and the ordered set $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ of $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$, ordered under inclusion. Moreover, the mapping $\widetilde{H}_{\mathcal{A}}$ is its inverse.*

Proof: By Proposition 66, if $\mathbb{A} \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$, then $\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By Lemma 74, if $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$, then $\widetilde{H}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$. So it suffices to show that $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse mappings and that they are both order preserving.

Let $\mathbb{A} = \langle \mathcal{A}, C\rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. By Proposition 66, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}_{\mathcal{A}}(C) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. As $\mathbb{A}$ is induced by its reduction $\mathbb{A}^* = \langle \mathcal{A}^*, C^*\rangle$, with $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, along the natural projection $\pi : \mathcal{A} \to \mathcal{A}^*$, we get, by definition, that $\mathbb{A}/\triangleq = \widetilde{H}_{\mathcal{A}}(\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}))$.

Suppose, conversely, that $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Consider $\mathbb{A}^\theta = \langle \mathcal{A}^\theta, C\rangle$, where $\mathcal{A}^\theta = \langle \mathbf{A}/\theta, \pi_\theta \circ h\rangle$ and $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^\theta)$. Then, since $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$, $\widetilde{\Omega}_{\mathcal{A}^\theta}(\mathcal{C}) =$

$\Delta_{\mathbf{A}/\theta}$. Thus, by Proposition 34,

$$\begin{array}{rcll}\widetilde{\Omega}_{\mathcal{A}}(\widetilde{H}_{\mathcal{A}}(\theta)) & = & \widetilde{\Omega}_{\mathcal{A}}(\mathcal{C}_\theta) & (\text{Definition of } \widetilde{H}_{\mathcal{A}}(\theta)) \\ & = & \widetilde{\Omega}_{\mathcal{A}}(\pi_\theta^{-1}(\mathcal{C})) & (\text{Definition of } \mathcal{C}_\theta) \\ & = & \pi_\theta^{-1}(\widetilde{\Omega}_{\mathcal{A}^\theta}(\mathcal{C})) & (\text{Proposition 22}) \\ & = & \pi_\theta^{-1}(\Delta_{\mathbf{A}/\theta}) & (\widetilde{\Omega}_{\mathcal{A}^\theta}(\mathcal{C}) = \Delta_{\mathbf{A}/\theta}) \\ & = & \theta. & (\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta \text{ natural projection})\end{array}$$

Hence, $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse bijections. $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving by definition. Finally, by Lemma 74, $\widetilde{H}_{\mathcal{A}}$ is also order preserving. This shows that

$$\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle \underset{\widetilde{H}_{\mathcal{A}}}{\overset{\widetilde{\Omega}_{\mathcal{A}}}{\rightleftarrows}} \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$$

are inverse order isomorphisms. ■

The structure of $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ is not very hard to obtain. This is done in the following analog of Theorem 2.31 of [12].

Theorem 76 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation. Then $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ is a complete lattice with meet coinciding with intersection.*

Proof: Let $\varnothing \neq \{\theta_i : i \in I\} \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Set $\theta = \bigcap_{i\in I}\theta_i$. We must show that $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Note that, for all $a \in A$,

$$a/\theta = a/\left(\bigcap_{i\in I}\theta_i\right) = \bigcap_{i\in I} a/\theta_i.$$

Let $h_i : \mathcal{A}/\theta \to \mathcal{A}/\theta_i$ be the mapping $a/\theta \mapsto a/\theta_i$. By hypothesis, $\mathbb{A}_i = \langle \mathcal{A}/\theta_i, C_i\rangle$, with $\mathcal{C}_i = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_i)$, is reduced. We must show that $\mathbb{A} = \langle \mathcal{A}/\theta, C\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$, is also reduced. By Proposition 34, $h_i^{-1}(\mathcal{C}_i) \subseteq \mathcal{C}$, for all $i \in I$. Hence, for all $i \in I$,

$$\begin{array}{rcl}\widetilde{\Omega}(\mathbb{A}) & \subseteq & \widetilde{\Omega}_{\mathcal{A}/\theta}(h_i^{-1}(\mathcal{C}_i)) \\ & = & h_i^{-1}(\widetilde{\Omega}_{\mathcal{A}/\theta_i}(\mathcal{C}_i)) \\ & = & h_i^{-1}(\Delta_{\mathbf{A}/\theta_i}).\end{array}$$

Thus, $\langle a, b\rangle \in \widetilde{\Omega}(\mathbb{A})$ implies $h_i(a) = h_i(b)$, for all $i \in I$, whence, $a/\theta = b/\theta$. So $\mathbb{A}$ is reduced and $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$.

We still need to investigate what happens when we take the collection of congruences to be empty. We then get $\bigcap\varnothing = \nabla_{\mathbf{A}}$. Thus, $\mathcal{A}/\theta = \langle \mathbf{1}, 1\rangle$, where $\mathbf{1}$ is the trivial one-element algebra and $1 : \mathbf{B} \to \mathbf{1}$ denotes the only available homomorphism. In this case, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ can be any of the sets $\{\varnothing\}$, $\{A\}$ or $\{\varnothing, A\}$ depending on the theories of $\mathbb{L}$. In all three cases, $\langle \mathcal{A}, C\rangle$, with

$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is reduced. Hence, it is a full model of $\mathbb{L}$ on $\mathcal{A}$ and, therefore, $\nabla^{\mathbf{A}} \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. ∎

Based on the Isomorphism Theorem 75, Theorem 76 helps us determine the structure of the partially ordered set $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$.

Corollary 77 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation. The ordered set $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ is a complete lattice and the Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is a lattice isomorphism from $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ onto $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$.*

Proof: This is an immediate consequence of the Theorem 75. ∎

Note that, via the isomorphism established in Theorem 75, we can say that, given a collection of equipotency classes $\{\mathbb{A}_i/\triangleq : i \in I\}$ of full models on $\mathcal{A}$, the meet is obtained by first taking the $\mathrm{Alg}(\mathbb{L})$-congruence $\theta = \bigcap_{i\in I} \widetilde{\Omega}(\mathbb{A}_i)$ and, then constructing the class represented by the full model induced by $\langle \mathcal{A}/\theta, C\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$, along the canonical projection $\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta$.

Proposition 78 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, $\mathbb{A} = \langle \mathcal{A}, C\rangle$ and $\mathbb{A}' = \langle \mathcal{A}', C'\rangle$ be two full models of $\mathbb{L}$, with $h : \mathbb{A} \to \mathbb{A}'$ a bilogical morphism between them. Then*

$$\mathbb{B} \stackrel{h}{\longmapsto} \mathbb{B}'$$

establishes an isomorphism between the lattice of equipotency classes of all full models of $\mathbb{L}$ on $\mathcal{A}$ $\trianglelefteq$-extending $\mathbb{A}$ and the lattice of equipotency classes of all full models of $\mathbb{L}$ on $\mathcal{A}'$ $\trianglelefteq$-extending $\mathbb{A}'$. Moreover, the principal filters of $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}')$ determined by $\widetilde{\Omega}(\mathbb{A})$ and $\widetilde{\Omega}(\mathbb{A}')$, respectively, are isomorphic.

Proof: By Proposition 21, h establishes an isomorphism between the lattice of equivalence classes of logicates on $\mathcal{A}$ extending $\mathbb{A}$ and the lattice of equivalence classes of logicates on $\mathcal{A}'$ extending $\mathbb{A}'$. The mapping establishes a bilogical morphism between representatives of each of the corresponding classes. Hence, by Proposition 60, one of those is full if and only if the other is. ∎

Corollary 79 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation and $g : \mathbf{A} \to \mathbf{A}'$ and epimorphism, such that $g : \langle \mathcal{A}, C\rangle \to \langle\langle \mathbf{A}, g \circ h\rangle, C'\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\langle \mathbf{A}, g\circ h\rangle)$, is a bilogical morphism. Then g induces an isomorphism between the complete lattices $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ and $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A}')/\triangleq, \trianglelefteq\rangle$, where $\mathcal{A}' = \langle \mathbf{A}, g \circ h\rangle$. Further, the complete lattices $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ and $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}'), \subseteq\rangle$ are isomorphic.*

Proof: By Proposition 59, for any interpretation $\mathcal{A}$, the equipotency class of $\langle \mathcal{A}, C\rangle$ is the least element in $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$. Thus, the result follows by combining Proposition 36 and Proposition 78. ∎