Chapter 5

Aspects of the Hierarchy

5.1 Introduction

One of the main achievements of the abstract theory of Algebraic Logic is the classification of logics in an algebraic hierarchy. A logic is represented by a structural closure operator on the algebra of terms (or formulas) over an algebraic type generated by countably many variables. The theory prescribes a method (or, rather, methods) one may follow to select a particular class of algebras over the same signature as the logic to associate with the logic. The higher the logic is classified in the hierarchy, the closer the ties between the logic and its associated class of algebras. Because of their clarity and comprehensiveness, but, also because they were written by pioneers, two monographs [3, 12], a survey [14], a book [8] and a textbook [10] have been used for many years as guides in being introduced to, in understanding and in delving deeper into the theory.

Since the *algebraic hierarchy* is one of the crowns (and jewels) of the traditional theory, it is only fair to, at least start to, investigate and give a first idea of how one could attempt to keep alive aspects of the theory in a rougher terrain. This is the effort we expend in the present and last chapter of Part I.

Among the major, perhaps most important, classes in the traditional hierarchy are protoalgebraic logics [2] (see, also, [8, 14, 10]). These are the logics in which, roughly speaking, indistinguishability modulo a theory implies interderivability modulo the theory. Another important characterization asserts that they are the logics on whose lattices of theories, the Leibniz operator is monotone. In Section 5.2, we use the definition from the classical framework to define *protoalgebraic logicates* and try to establish some equivalent conditions, some with and some without extra assumptions.

One of the key consequences of protoalgebraicity, which forms an important feature in their study, is the so-called *Correspondence Theorem*. This result is partly the reason why Blok and Pigozzi declared that protoalgebraic logics form the widest class of logics amenable to algebraic techniques of study, even though they are not "algebraizable", i.e., do not belong to the highest step in the hierarchy but are, rather, located near the bottom. The Correspondence Theorem establishes an isomorphism between the lattice of filters of the logic on a given algebra including a fixed filter and the lattice of filters on the quotient algebra, formed by dividing out by the Leibniz congruence of the fixed filter including the quotient of the fixed filter. We discuss this result and some of its consequences in Section 5.3. Again our focus remains to safeguard some of the result from the traditional theory, with or without provisos, in this less robust environment.

As is the case in the traditional theory [12], and as was shown provisionally in Chapter 4, full models play a key role in the investigation of the logical structure. In the context of protoalgebraic logics, full models are inextricably connected to, so-called, *Leibniz filters* [12, 17]. So, in Section

5.4, we investigate the relation between full models and Leibniz filters in the context of protoalgebraic logicates. Their study naturally segues into the study of *weakly algebraizable logicates* in Section 5.5. These are defined by analogy with the corresponding class in the monotonic framework [9] (see, also, [8, 14, 10]). Several characterizations paralleling the ones from the traditional setting are provided, but, generally, they require the additional hypothesis that the set of theories be closed under intersection.

As is well known in the ordinary setting, weak algebraizability [9] results by simultaneously insisting that a logic be protoalgebraic [2] and truth equational [19]. So in the last section, Section 5.6, we use the original definition to identify the class of *truth equational logicates*. They are characterized by the Leibniz operator on their theories being monotone and injective. Again assuming closure of the set of theories under intersection, we prove that, for logicates also, weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

5.2 Protoalgebraic Logicates

Recall that an algebraic logicate $\langle \mathbf{A}, C\rangle$ consists of an algebra $\mathbf{A}$ and an idempotent operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Recall, also, that we use $\mathcal{C}$ to denote the collection of all fixed points or theories of C.

In the abstract study of logicates, a particular fixed logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ is at the focus of investigations and it is called the *base logicate*. Both *matrix* (Chapter 3) and *logicate* (Chapter 4) *models* of the base logicate are based on *interpretations* $\mathcal{A} = \langle \mathbf{A}, h\rangle$, which are epimorphisms from the base algebra $\mathbf{B}$ onto a similar algebra $\mathbf{A}$.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be an algebraic logicate. We say that $\mathbb{L}$ is **protoalgebraic** (see [2] and, also, [8, 12, 10]) if, for all $a, b \in B$ and all $X \in \mathcal{C}^\flat$,

$$\langle a, b\rangle \in \Omega_{\mathbf{B}}(X) \quad \text{implies,} \quad \begin{array}{l}\text{for all } X \subseteq X' \in \mathcal{C}^\flat,\\ a \in X' \text{ iff } b \in X'.\end{array}$$

We make an observation and then introduce some notation that will help abbreviate the definition.

Observe that protoalgebraicity depends only on the collection of theories of a logicate. This is commensurate with the monotonic theory, where protoalgebraicity depends only on the theory lattice of the sentential logic, even though, in contrast with the present framework, in the monotonic framework the theory lattice fully determines the logic itself. Thus, if two logicates are equipotent (see Chapter 2), then they are either both protoalgebraic or none of the two is. This implies that protoalgebraicity may be viewed, without harm, as a property of equipotency classes instead of individual logicates. We may implicitly use this fact whenever convenient.

Given a logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ and $X \subseteq B$, we define a **logical indistinguishability relation** $\Lambda_\mathbb{L}(X)$ on B with the goal of capturing the defining property of protoalgebraicity. We set, for all $X \subseteq B$ and all $a, b \in B$,

$$\langle a, b \rangle \in \Lambda_\mathbb{L}(X) \quad \text{iff,} \quad \begin{array}{l} \text{for all } X \subseteq X' \in \mathcal{C}^\flat, \\ a \in X' \text{ iff } b \in X'. \end{array}$$

With this definition available, we may rephrase the definition of protoalgebraicity. Clearly, $\mathbb{L}$ is **protoalgebraic** if and only if, for all $X \in \mathcal{C}^\flat$,

$$\Omega_\mathbf{B}(X) \subseteq \Lambda_\mathbb{L}(X).$$

It is well known that, in the traditional framework, protoalgebraicity is tantamount to the monotonicity of the Leibniz operator (see [3]) on the lattice of theories of the logic (see, e.g., [8, 12, 10]). The following proposition revisits this characterization in the context of logicates.

Proposition 80 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. $\mathbb{L}$ is protoalgebraic if and only if $\Omega_\mathbf{B}$ is monotone on $\mathcal{C}^\flat$.*

Proof: Suppose $\mathbb{L}$ is protoalgebraic and let $X, X' \in \mathcal{C}^\flat$, such that $X \subseteq X'$. Let $a, b \in B$, such that $\langle a, b \rangle \in \Omega_\mathbf{B}(X)$ and $a \in X'$. By protoalgebraicity, $\langle a, b \rangle \in \Lambda_\mathbb{L}(X)$ and $a \in X'$. Since $X \subseteq X'$, $b \in X'$. This shows that $\Omega_\mathbf{B}(X)$ is compatible with X'. By the maximality property of $\Omega_\mathbf{B}(X')$ with respect to compatibility with X', we conclude that $\Omega_\mathbf{B}(X) \subseteq \Omega_\mathbf{B}(X')$. Thus, $\Omega_\mathbf{B}$ is monotone on $\mathcal{C}^\flat$.

Suppose, conversely, that $\Omega_\mathbf{B}$ is monotone on $\mathcal{C}^\flat$. Let $a, b \in B$, $X \in \mathcal{C}^\flat$, such that $\langle a, b \rangle \in \Omega_\mathbf{B}(X)$ and $X' \in \mathcal{C}^\flat$, with $X \subseteq X'$. Then $\langle a, b \rangle \in \Omega_\mathbf{B}(X')$. So, by the compatibility of $\Omega_\mathbf{B}(X')$ with X', $a \in X'$ iff $b \in X'$. Thus, $\mathbb{L}$ is protoalgebraic. ■

The characterization extends to the monotonicity of the Leibniz operator on the $\mathbb{L}$-filters of any interpretation. This is one of many results of this type in the abstract theory of Algebraic Logic. They are collectively known as "transfer theorems" and assert that a certain property that holds on the theories of a logic, also holds, more generally, on the filters of the logic on any interpretation structure.

Proposition 81 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is protoalgebraic if and only if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_\mathcal{A}$ is monotone on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$.*

Proof: Suppose that $\mathbb{L}$ is protoalgebraic. Let $Y_1, Y_2 \in \mathrm{Fi}_\mathbb{L}(\mathcal{A})$, such that $Y_1 \subseteq Y_2$, and $a_1, a_2 \in A$. Note that, since $h : \mathbf{B} \to \mathbf{A}$ is surjective, there exist

$b_1, b_2 \in B$, such that $a_1 = h(b_1)$ and $a_2 = h(b_2)$. Now we have

$$\begin{array}{rcl} \langle a_1, a_2\rangle \in \Omega_{\mathcal{A}}(Y_1) & \text{iff} & \langle h(b_1), h(b_2)\rangle \in \Omega_{\mathcal{A}}(Y_1) \\ & \text{iff} & \langle b_1, b_2\rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_1)) \\ & \text{iff} & \langle b_1, b_2\rangle \in \Omega_{\mathbf{B}}(h^{-1}(Y_1)) \\ & \text{implies} & \langle b_1, b_2\rangle \in \Omega_{\mathbf{B}}(h^{-1}(Y_2)) \\ & \text{iff} & \langle b_1, b_2\rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_2)) \\ & \text{iff} & \langle h(b_1), h(b_2)\rangle \in \Omega_{\mathcal{A}}(Y_2) \\ & \text{iff} & \langle a_1, a_2\rangle \in \Omega_{\mathcal{A}}(Y_2). \end{array}$$

Note that we have used the well-known property that the Leibniz operator commutes with inverse surjective homomorphisms. Even though this fact was not proven here, the proof closely parallels the one included in showing that the Tarski operator satisfies the same property in Proposition 22. We have now shown that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, if the condition in the statement holds, then the monotonicity of $\Omega_{\mathbf{B}}$ on $\mathcal{C}^\flat$ follows by taking $\mathcal{A} = \langle \mathbf{B}, i_{\mathbf{B}}\rangle$. Then the conclusion follows from Proposition 80 and the observation that $\mathcal{C}^\flat = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. ∎

One may also devise a slightly different characterization in the special case in which the set of theories is closed under intersections. We show, first, that, in this case, the collection of all $\mathbb{L}$-filters on any interpretation is also closed under intersections.

Lemma 82 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, such that its set $\mathcal{C}^\flat$ of theories is closed under arbitrary (respectively, nonempty, binary) intersections. Then, for any interpretation $\mathcal{A} = \langle \mathbf{A}, h\rangle$, the set $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ of $\mathbb{L}$-filters on $\mathcal{A}$ is also closed under arbitrary (respectively, nonempty, binary) intersections.*

Proof: All three statements are proven similarly. E.g., for the first one, consider an interpretation $\mathcal{A} = \langle \mathbf{A}, h\rangle$ and let $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ be a collection of $\mathbb{L}$-filters on $\mathcal{A}$. Then

$$h^{-1}\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} h^{-1}(X_i) \in \mathcal{C}^\flat,$$

where membership follows by the definition of $\mathbb{L}$-filter and the hypothesis. This shows that $\bigcap_{i\in I} X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. ∎

Now we show that, if the set of theories of the base logicate is closed under intersections, then protoalgebraicity is equivalent to the property that the Leibniz operator on the filters of any interpretation commutes with arbitrary intersections.

Proposition 83 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate.*

(i) *If the set* $\mathcal{C}^\flat$ *of theories is closed under arbitrary intersections and* $\mathbb{L}$ *is protoalgebraic, then, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$*,* $\Omega_{\mathcal{A}}$ *commutes with arbitrary intersections on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, i.e., for all* $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*,*

$$\Omega_{\mathcal{A}}\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i).$$

(ii) *If the set* $\mathcal{C}^\flat$ *of theories is closed under binary intersection and, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$*,* $\Omega_{\mathcal{A}}$ *commutes with binary intersections on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, then* $\mathbb{L}$ *is protoalgebraic.*

Proof: By Proposition 81, protoalgebraicity is equivalent to the monotonicity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$.

(i) Suppose, first, that $\mathcal{C}^\flat$ is closed under arbitrary intersection and $\Omega_{\mathcal{A}}$ is monotone. Let $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Lemma 82, $\bigcap_{i\in I} X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By monotonicity, $\Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i) \subseteq \Omega_{\mathcal{A}}(X_i)$, for all $i \in I$. Thus, $\Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i)$. On the other hand, $\bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i)$ is compatible with $\bigcap_{i\in I} X_i$. Thus, by the maximality property of $\Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i)$ with respect to compatibility with $\bigcap_{i\in I} X_i$, we obtain $\bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i) \subseteq \Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i)$.

(ii) Suppose, next, that $\mathcal{C}^\flat$ is closed under binary intersection and, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_{\mathcal{A}}$ commutes with binary intersections on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \subseteq X'$. Then $X \cap X' = X$ and we have

$$\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X \cap X') = \Omega_{\mathcal{A}}(X) \cap \Omega_{\mathcal{A}}(X').$$

Therefore, $\Omega_{\mathcal{A}}(X) \subseteq \Omega_{\mathcal{A}}(X')$, showing that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and, hence, $\mathbb{L}$ is protoalgebraic.

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Corollary 84 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate, such that its set* $\mathcal{C}^\flat$ *of theories is closed under arbitrary intersection.* $\mathbb{L}$ *is protoalgebraic if and only if, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$*,* $\Omega_{\mathcal{A}}$ *commutes with arbitrary intersections on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*.*

5.3 Correspondence Theorem

Given a logicate $\mathbb{L}$, an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ and a filter $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we write

$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F := \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : F \subseteq X\}.$$

Parts of the well-known Correspondence Theorem for protoalgebraic logics (see, e.g., Theorem 6.19 of [10]) may also be retained in the present context, since they are concerned solely with the structure of theories and filters. Note, however, that what is an isomorphism between complete lattices in the monotonic framework, has to be replaced here by, merely, an isomorphism between ordered sets.

Proposition 85 (Correspondence) *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. If $\mathbb{L}$ is protoalgebraic, then, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ and every $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, letting $\pi : \mathbf{A} \to \mathbf{A}/\Omega_{\mathcal{A}}(F)$ be the natural projection,*

$$\begin{array}{rrcl}\pi : & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F & \longrightarrow & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}; \\ & X & \longmapsto & \pi(X),\end{array}$$

establishes an isomorphism between the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \subseteq \rangle$ and the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \subseteq \rangle$.

Proof: Let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $F \subseteq X$. By protoalgebraicity, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(X)$. Hence $\Omega_{\mathcal{A}}(F)$ is compatible with X. It follows that, for $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$, $X = \pi^{-1}(\pi(X))$. Thus, by Proposition 34, we obtain $\pi(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Clearly, since $F \subseteq X$, $\pi(F) \subseteq \pi(X)$. On the other hand, if $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, then, again by Proposition 34, $\pi^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover, $\pi(F) \subseteq Y$ implies $F = \pi^{-1}(\pi(F)) \subseteq \pi^{-1}(Y)$. Thus, π establishes an isomorphism between the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \subseteq \rangle$ and the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \subseteq \rangle$, as claimed. ■

Note that, for any algebra $\mathbf{A}$, we have $\Omega_{\mathbf{A}}(\varnothing) = \nabla_{\mathbf{A}}$. Thus, by the definition of protoalgebraicity, the only protoalgebraic logicates $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ for which $\varnothing$ is a theory are the ones with $\mathcal{C}^\flat = \{\varnothing\}$ or $\mathcal{C}^\flat = \{\varnothing, B\}$.

We now provide some additional characterizations of protoalgebraicity in terms of the Tarski operator. Given an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, a logicate $\mathbb{A} = \langle \mathcal{A}, C \rangle$ and a theory F of C, we shall write $\mathbb{A}^F = \langle \mathcal{A}, C^F \rangle$ for a logicate with

$$\mathcal{C}^F = \{X \in \mathcal{C} : F \subseteq X\},$$

assuming that only the theories matter and that the exact consequence structure is irrelevant, i.e., thinking of $\mathbb{A}^F$ as a representative of its equipotency class.

Proposition 86 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. The following statements are equivalent.*

(i) *$\mathbb{L}$ is protoalgebraic;*

(ii) *For any $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, if $\mathcal{C}$ has a minimum element, then $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$;*

(iii) *For any* $\mathbb{L}$*-model* $\mathbb{A} = \langle \mathcal{A}, C \rangle$, *with* $Y \in \mathcal{C}$, $\widetilde{\Omega}(\mathbb{A}^Y) = \Omega_{\mathcal{A}}(Y)$;

(iv) *For any* $X \in \mathcal{C}^\flat$, $\widetilde{\Omega}(\mathbb{L}^X) = \Omega_{\mathbf{B}}(X)$.

Proof:

(i)⇒(ii) Suppose $\mathbb{A} \in \mathrm{Mod}(\mathbb{L})$. Then, by Proposition 57, $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 81, $\Omega_{\mathcal{A}}$ is order preserving on $\mathcal{C}$. So we get

$$\widetilde{\Omega}(\mathbb{A}) = \bigcap_{X \in \mathcal{C}} \Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(\min \mathcal{C}).$$

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Trivial.

(iv)⇒(i) Let $X, X' \in \mathcal{C}^\flat$, such that $X \subseteq X'$. Then $X' \in \mathcal{C}^{\flat X}$. Thus, we get

$$\begin{array}{rcll} \Omega_{\mathbf{B}}(X) & = & \widetilde{\Omega}(\mathbb{L}^X) & \text{(Hypothesis (iv))} \\ & = & \bigcap_{Y \in \mathcal{C}^{\flat X}} \Omega_{\mathbf{B}}(Y) & \text{(Tarski Congruence)} \\ & \subseteq & \Omega_{\mathbf{B}}(X'). & (X' \in \mathcal{C}^{\flat X}) \end{array}$$

So $\Omega_{\mathbf{B}}$ is monotone on $\mathcal{C}^\flat$, showing that $\mathbb{L}$ is protoalgebraic.

■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate and consider an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$. If $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ has a minimum element, which, e.g., is the case when $\mathcal{C}^\flat$ is closed under intersections, then, by Proposition 86, for $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})).$$

The following proposition is an analog of Proposition 3.2 of [12] in the present setting.

Proposition 87 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a protoalgebraic logicate, such that, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *has a minimum element. Then*

$$\mathrm{Alg}(\mathbb{L}) = \mathrm{Alg}^*(\mathbb{L}).$$

Proof: By Corollary 72, we have $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$, without any preconditions. Suppose, conversely, that $\mathcal{A} = \langle \mathbf{A}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $\widetilde{\Omega}(\langle \mathcal{A}, C \rangle) = \Delta_{\mathbf{A}}$. By hypothesis and Proposition 86,

$$\Omega_{\mathcal{A}}(\min \mathcal{C}) = \widetilde{\Omega}(\langle \mathcal{A}, C \rangle) = \Delta_{\mathbf{A}}.$$

This shows that $\mathbf{A} \in \mathrm{Alg}^*(\mathbb{L})$. ■

Lemma 3.3 of [12] asserts that two full models of a protoalgebraic logic that share the same sets of theorems are identical. The following analog requires the two logicate models compared to share the same minimum theories and, in that case, asserts that the logicates in question must be equipotent.

Lemma 88 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a protoalgebraic logicate and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$ *two full models of* $\mathbb{L}$ *over the same interpretation that have minimum theories* X_0, X_0', *respectively. If* $X_0 = X_0'$, *then* $\mathcal{C} = \mathcal{C}'$.

Proof: By hypothesis and Proposition 86,

$$\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(X_0) = \Omega_{\mathcal{A}}(X_0') = \widetilde{\Omega}(\mathbb{A}').$$

Thus, by the Isomorphism Theorem 75, $\mathbb{A}$ and $\mathbb{A}'$ are in the same equipotency class, i.e., $\mathcal{C} = \mathcal{C}'$. ■

From the previous few results, it has become apparent that full models that have minimum theories play a somewhat important role. So one may use, if needed, special notation, such as $\mathrm{FMod}^m(\mathbb{L})$ for the class of all full models of $\mathbb{L}$ with a minimum theory and $\mathrm{FMod}^m_{\mathbb{L}}(\mathcal{A})$ for the class of all full models of $\mathbb{L}$ on $\mathcal{A}$ with a minimum theory.

Protoalgebraicity in the monotonic theory was characterized in terms of full models in Theorem 3.4 of [12]. A similar characterization is possible here, provided that all full models of a logicate have minimum theories.

Theorem 89 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a protoalgebraic logicate all of whose full models have a minimum theory. Then* $\mathbb{L}$ *is protoalgebraic if and only if all full models of* $\mathbb{L}$ *are of the form* $\langle \mathcal{A}, C^F \rangle$, *with* $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, *for some interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *and some* $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: We work, first, to prove the "only if". Let $\mathbb{A} = \langle \mathcal{A}, C \rangle$ be a full model of $\mathbb{L}$, with $F = \min \mathcal{C}$. By protoalgebraicity and Proposition 86, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(F)$. Hence, the natural projection $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \langle \mathcal{A}, C \rangle \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), C^{\Omega_{\mathcal{A}}(F)} \rangle.$$

By hypothesis, $\mathbb{A}$ is a full model of $\mathbb{L}$. It follows that $\mathcal{C}^{\Omega_{\mathcal{A}}(F)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Consider $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Then $F \subseteq X$ and, by protoalgebraicity, $\Omega_{\mathcal{A}}(F)$ is compatible with X. Thus, $X = \pi^{-1}(\pi(X))$. By Proposition 34, $\pi(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Hence, since $X = \pi^{-1}(\pi(X))$, $X \in \mathcal{C}$. This proves that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$.

We turn, next, to the "if". Suppose that all models of $\mathbb{L}$ have the indicated form. Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \subseteq X'$. Consider $\Omega_{\mathcal{A}}(X)$. By Corollary 72, $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$. Hence $\Omega_{\mathcal{A}}(X) \in \mathrm{Alg}(\mathbb{L})$. By the Isomorphism Theorem 75, there exists a full model $\mathbb{A} = \langle \mathcal{A}, C \rangle$ of $\mathbb{L}$, such that $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(X)$. Moreover, by hypothesis, there exists $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Since $\mathbb{A}$ is full, the natural projection $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(X)$ is a bilogical morphism

$$\pi : \mathbb{A} \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(X), \mathcal{C}^{\Omega_{\mathcal{A}}(X)} \rangle,$$

where $\mathcal{C}^{\Omega_{\mathcal{A}}(X)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(X))$. Moreover, as $X = \pi^{-1}(\pi(X))$, we get $X \in \mathcal{C}$. Hence, $F \subseteq X \subseteq X'$, whence, $X' \in \mathcal{C}$. Now we get

$$\Omega_{\mathcal{A}}(X) = \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X'),$$

i.e., $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 81, $\mathbb{L}$ is protoalgebraic. ■

5.4 Leibniz Filters

Leibniz filters were introduced by Font and Jansana in [12] (see Page 63), extensively studied in [11] and [17], and used further in applications of the theory in [13]. Here we define an analog in the framework of logicates.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation. We define

$$\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}) = \{F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \text{if } \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \text{ then } \langle \mathcal{A}, C \rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})\}.$$

The elements in $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ are called **Leibniz filters of $\mathbb{L}$ on $\mathcal{A}$**.

Proposition 90 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation over which all full models have a minimum theory. Then $\Omega_{\mathcal{A}}$ is a lattice isomorphism*

$$\Omega_{\mathcal{A}} : \langle \mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}), \subseteq \rangle \cong \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle = \langle \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}), \subseteq \rangle.$$

Proof: Consider the mapping

$$F \longmapsto \langle \mathcal{A}, C^F \rangle,$$

where $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. By the definition of $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$, this is a mapping from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$. It is injective and it is order preserving and order reflecting. By protoalgebraicity and Theorem 89, it is also surjective. So it is an order isomorphism from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$. By the Isomorphism Theorem 75, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$ is isomorphic to $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$ via the Tarski operator. Thus, the composition

$$F \longmapsto \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F\right)$$

establishes an order isomorphism between $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 86, $\widetilde{\Omega}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$. By protoalgebraicity and Proposition 87, $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}) = \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. Therefore, the Leibniz operator is an order isomorphism from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. ■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate. In case $\mathcal{C}^\flat$ is closed under intersections, the $\mathbb{L}$-filters in $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ on a given interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$

may be characterized without reference to full models. To show this, we consider a binary relation $\sim_\Omega$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ defined as the kernel of the Leibniz operator on $\mathcal{A}$, i.e., for all $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$X \sim_\Omega X' \quad \text{iff} \quad \Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X').$$

Under the assumption that $\mathcal{C}^\flat$ is closed under intersection, we get, by Lemma 82, that $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is also closed under intersection. A fortiori, every full model of $\mathbb{L}$ on $\mathcal{A}$ has a minimum filter. Thus, if $\mathbb{L}$ is protoalgebraic, by Proposition 90, at most one $\mathbb{L}$-filter in each $\sim_\Omega$-equivalence class is in $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$. As in Proposition 3.6 of [12], it is possible in this setting as well to characterize this filter.

Suppose $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ is protoalgebraic, with $\mathcal{C}^\flat$ closed under intersection, and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation. Then each $\sim_\Omega$-class in $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ has a minimum element. In fact, if $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $[X]_\Omega$ denotes its $\sim_\Omega$-class, then, by hypothesis, $\bigcap [X]_\Omega \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and

$$\begin{aligned}
\Omega_{\mathcal{A}}(\bigcap [X]_\Omega) &= \Omega_{\mathcal{A}}(\bigcap \{Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \Omega_{\mathcal{A}}(Y) = \Omega_{\mathcal{A}}(X)\} \\
&\qquad \text{(Definition of } [X]_\Omega) \\
&= \bigcap \{\Omega_{\mathcal{A}}(Y) : Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \Omega_{\mathcal{A}}(Y) = \Omega_{\mathcal{A}}(X)\} \\
&\qquad \text{(Proposition 83)} \\
&= \Omega_{\mathcal{A}}(X).
\end{aligned}$$

Hence, $\bigcap [X]_\Omega \in [X]_\Omega$. The identification of $\bigcap [X]_\Omega$ as the least member in the $\sim_\Omega$-equivalence class of X is the key in helping us characterize the class of Leibniz filters of $\mathbb{L}$ on $\mathcal{A}$. This is the promised analog of Proposition 3.6 of [12] for logicates. The proof remains virtually the same.

Proposition 91 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate, with $\mathcal{C}^\flat$ closed under intersection, $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation and $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. The following statements are equivalent:*

(i) $F \in \mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$, i.e., $\langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of $\mathbb{L}$;

(ii) F is the minimum element in its $\sim_\Omega$-equivalence class;

(iii) $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Proof:

(ii)⇒(iii) Suppose $F = \min [F]_\Omega$ and let $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Our goal is to show that $F/\Omega_{\mathcal{A}}(F) \subseteq G$. Let $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ be the natural projection and set $F' = \pi^{-1}(G) \cap F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, where membership is due to Proposition 34, the hypothesis and Lemma 82. Then

$$\begin{aligned}
F' &= \pi^{-1}(G) \cap \pi^{-1}(\pi(F)) \quad \text{(Compatibility of } \Omega_{\mathcal{A}}(F) \text{ with } F) \\
&= \pi^{-1}(G \cap \pi(F)). \quad \text{(Set Theoretical)}
\end{aligned}$$

Hence, F' is a union of $\Omega_{\mathcal{A}}(F)$-classes, i.e., $\Omega_{\mathcal{A}}(F)$ is compatible with F'. By the maximality property of the Leibniz congruence, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(F')$. As, by definition, $F' \subseteq F$, by protoalgebraicity, $\Omega_{\mathcal{A}}(F') \subseteq \Omega_{\mathcal{A}}(F)$. Consequently, $\Omega_{\mathcal{A}}(F') = \Omega_{\mathcal{A}}(F)$, i.e., $F \sim_{\Omega} F'$. By hypothesis, $F \subseteq F'$ and, since, by definition, $F' \subseteq F$, $F = F'$. Thus, $F \subseteq \pi^{-1}(G)$. This yields

$$F/\Omega_{\mathcal{A}}(F) = \pi(F) \subseteq \pi(\pi^{-1}(G)) = G.$$

Therefore, $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

(iii)⇒(i) Assume $F/\Omega_{\mathcal{A}}(F) = \min \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. By protoalgebraicity and the correspondence established in Proposition 85, the natural projection $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ gives an order isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ and $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{F/\Omega_{\mathcal{A}}(F)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Also by protoalgebraicity and Proposition 86,

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F).$$

Hence $\langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of $\mathbb{L}$.

(i)⇒(ii) Suppose $F \in \mathrm{Fi}^{\bigstar}_{\mathbb{L}}(\mathcal{A})$ and $G = \min [F]_{\Omega}$. By (ii)⇒(iii)⇒(i), $\mathbb{A}^G = \langle \mathcal{A}, C^G \rangle$, with $\mathcal{C}^G = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^G$, is a full model of $\mathbb{L}$. By hypothesis, $\mathbb{A}^F = \langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is also a full model of $\mathbb{L}$. But

$$\begin{array}{rcll} \widetilde{\Omega}(\mathbb{A}^F) & = & \Omega_{\mathcal{A}}(F) & \text{(Proposition 86)} \\ & = & \Omega_{\mathcal{A}}(G) & (G \sim_{\Omega} F) \\ & = & \widetilde{\Omega}(\mathbb{A}^G). & \text{(Proposition 86)} \end{array}$$

Thus, by the Isomorphism Theorem 75, $\mathcal{C}^F = \mathcal{C}^G$, showing that $F = G$. Therefore, F is the minimum element in the class $[F]_{\Omega}$.

■

5.5 Weak Algebraizability

In [3], Blok and Pigozzi introduced the notion of *algebraizable logic*. As they explain, the notion was a natural abstraction from many well-known examples, the most prototypical ones, perhaps, being that of classical propositional logic, of intuitionistic logic and the various implicative logics of Rasiowa [20]. Making an exact notion of algebraizability precise had, besides unification and clarification, the advantage of being able to show, for the first time, that logics that were known not to be amenable to algebraizability techniques, were somehow intrinsically non-algebraizable, since they did not fall under the scope of Blok and Pigozzi's definition. Blok and Pigozzi worked with finitary sentential logics, but their results were soon generalized further to cover many additional systems. One of the earliest generalizations was by

Herrmann [15, 16] to cover infinitary logics. Algebraizability was shown to be equivalent to the conjunction of equivalentiality [6, 7] and of truth equationality [19]. Equivalentiality is a stronger property than protoalgebraicity, since it requires that the Leibniz operator be both monotone and commute with substitutions. If equivalentiality is weakened to protoalgebraicity, that is, if one requires that the logic be protoalgebraic and truth equational, then weak algebraizability [9] is obtained. All these properties and their characterizations and interconnections are studied in surveys on abstract Algebraic Logic, e.g., [8, 12, 14]. *Weak algebraizability* is the property studied here in the context of logicates.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. We say that $\mathbb{L}$ is **weakly algebraizable** [9] (see, also, [12, 8]) if the Leibniz operator is monotone and injective on $\mathcal{C}^\flat$.

Proposition 92 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is weakly algebraizable if and only if, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$*, the Leibniz operator* $\Omega_{\mathcal{A}}$ *on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *is injective and monotone.*

Proof: First, by Proposition 81, monotonicity of $\Omega_{\mathbf{B}}$ on $\mathcal{C}^\flat$ is equivalent to monotonicity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$. So it suffices to see that injectivity of $\Omega_{\mathbf{B}}$ on $\mathcal{C}^\flat$ is equivalent to injectivity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$.

Assume, first, that $\Omega_{\mathbf{B}}$ is injective on $\mathcal{C}^\flat$. Consider an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ and let $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X')$. Applying h^{-1}, we get $h^{-1}(\Omega_{\mathcal{A}}(X)) = h^{-1}(\Omega_{\mathcal{A}}(X'))$. By commutativity of the Leibniz operator with inverse surjective homomorphisms, $\Omega_{\mathbf{B}}(h^{-1}(X)) = \Omega_{\mathbf{B}}(h^{-1}(X'))$. By hypothesis, $h^{-1}(X) = h^{-1}(X')$. By the surjectivity of h, $X = X'$. Thus $\Omega_{\mathcal{A}}$ is injective on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, if $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is injective, for every interpretation $\mathcal{A}$, then, by considering $\mathcal{A} = \langle \mathbf{B}, i_{\mathbf{B}} \rangle$, we get that $\Omega_{\mathbf{B}}$ is injective on $\mathcal{C}^\flat$. ∎

The work of the preceding section on characterizing Leibniz filters of $\mathbb{L}$ on an interpretation $\mathcal{A}$ comes in handy in case one wants to provide a characterization of weakly algebraizable logicates inside the class of protoalgebraic logicates (at least in some better behaved cases).

Proposition 93 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a protoalgebraic logicate, with* $\mathcal{C}^\flat$ *closed under intersection. Then* $\mathbb{L}$ *is weakly algebraizable if and only if, for every interpretation* $\mathcal{A}$*,* $\mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, i.e., for all* $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*,* $\mathbb{A} = \langle \mathcal{A}, C^F \rangle$*, with* $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$*, is a full model of* $\mathbb{L}$*.*

Proof: We have

$$\begin{array}{rll} \mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & \text{iff,} & \text{for all } F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), [F]_\Omega = \{F\} \\ & \text{iff,} & \text{for all } F, G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \\ & & \Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(G) \text{ implies } F = G \\ & \text{iff} & \Omega_{\mathcal{A}} \text{ is injective on } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) \\ & \text{iff} & \mathbb{L} \text{ is weakly algebraizable,} \end{array}$$

the last equivalence since, by hypothesis, $\mathbb{L}$ is protoalgebraic. ■

This leads to several additional characterizations of weak algebraizability in the special case in which the set of theories of the logicate is closed under intersection.

Theorem 94 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate, with $\mathcal{C}^\flat$ closed under intersection. The following statements are equivalent:*

(i) $\mathbb{L}$ *is weakly algebraizable;*

(ii) For every interpretation $\mathcal{A}$, $\Omega_\mathcal{A}$ is monotone and injective on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$*;*

(iii) $\mathbb{L}$ *is protoalgebraic and, for every interpretation $\mathcal{A}$ and every filter $F \in \mathrm{Fi}_\mathbb{L}(\mathcal{A})$, $F/\Omega_\mathcal{A}(F)$ is the least filter on $\mathcal{A}/\Omega_\mathcal{A}(F)$;*

(iv) For every interpretation $\mathcal{A}$, the mapping $F \mapsto \langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_\mathbb{L}(\mathcal{A})^F$, is a bijection between $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$ and $\mathrm{FMod}_\mathbb{L}(\mathcal{A})/\triangleq$;

(v) For every interpretation $\mathcal{A}$, $\Omega_\mathcal{A}$ is a lattice isomorphism between $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$;

(vi) For every interpretation $\mathcal{A}$, $\Omega_\mathcal{A}$ is a lattice isomorphism between $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$.

Proof:

(i)⇔(ii) By Proposition 92.

(ii)⇔(iii) By Propositions 91 and 93.

(iii)⇒(iv) Consider $F \mapsto \langle \mathcal{A}, C^F \rangle$, viewed as a map into equipotency classes. It is injective. By Proposition 91 and the hypothesis, it is well defined. By Theorem 89, it is also surjective. Thus, it is a bijection. Since it is clearly order preserving and order reflecting, we get that it is a lattice isomorphism.

(iv)⇒(v) Consider $F \mapsto \langle \mathcal{A}, C^F \rangle$, again viewed as a map into equipotency classes. Since, by hypothesis, it is onto, by Theorem 89, $\mathbb{L}$ is protoalgebraic. Further, the composition of this mapping with the mapping $\widetilde{\Omega}_\mathcal{A}$ from the Isomorphism Theorem 75 gives an isomorphism from $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$ onto $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 86, the mapping is identical to $F \mapsto \widetilde{\Omega}(\langle \mathcal{A}, C^F \rangle) = \Omega_\mathcal{A}(F)$.

(v)⇒(vi) In general, $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}) \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Also in general, $\Omega_\mathcal{A}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. By hypothesis, each $\mathrm{Alg}(\mathbb{L})$-congruence is of the form $\Omega_\mathcal{A}(F)$, for some interpretation $\mathcal{A}$ and some $F \in \mathrm{Fi}_\mathbb{L}(\mathcal{A})$. Thus,

$$\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}) = \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}).$$

This yields (vi).

(vi)⇒(i) Trivial.

■

For a weakly algebraizable logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$, we call a class K of $\mathcal{L}$-algebras an **equivalent algebraic semantics** for $\mathbb{L}$ if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, there exists an isomorphism

$$\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \subseteq \rangle \cong \langle \mathrm{Con}_{\mathsf{K}}(\mathcal{A}), \subseteq \rangle.$$

Proposition 95 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a weakly algebraizable logicate. Then* $\mathrm{Alg}^*(\mathbb{L})$ *is an equivalent algebraic semantics for* $\mathbb{L}$*.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation. Define

$$\begin{array}{rccl} \Omega_{\mathcal{A}}: & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & \longrightarrow & \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}); \\ & X & \longmapsto & \Omega_{\mathcal{A}}(X). \end{array}$$

This mapping is well defined since $\langle \mathcal{A}/\Omega_{\mathcal{A}}(X), X/\Omega_{\mathcal{A}}(X) \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and, hence, $\mathbf{A}/\Omega_{\mathcal{A}}(X) \in \mathrm{Alg}^*(\mathbb{L})$. By weak algebraizability and Proposition 92, it is injective and monotone. So it suffices to show that it is surjective.

Let $\theta \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{Alg}^*(\mathbb{L})$, that is, there exists $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) = \theta$. Hence, $\Omega_{\mathcal{A}}$ is surjective. ■

Corollary 96 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be an algebraizable logicate, such that, for every interpretation* $\mathcal{A}$*,* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *has a minimum element. Then* $\mathrm{Alg}(\mathbb{L})$ *is an equivalent algebraic semantics for* $\mathbb{L}$*.*

Proof: By hypothesis and Proposition 95, $\mathrm{Alg}^*(\mathbb{L})$ is an equivalent algebraic semantics for $\mathbb{L}$. Also by hypothesis and Proposition 87, $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$. Therefore, $\mathrm{Alg}(\mathbb{L})$ is an equivalent algebraic semantics for $\mathbb{L}$. ■

5.6 Truth Equationality

Recall that weak algebraizability [9] is the combination of protoalgebraicity [2] and truth equationality [19]. Since we briefly studied both protoalgebraicity and weak algebraizability in the context of logicates, it is only fair that we, at least briefly, look also at truth equationality as a property on its own. We introduce a definition adapted from [19], we show that it transfers and then prove the main result that weak algebraizability is indeed the conjunction of protoalgebraicity and truth equationality.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. $\mathbb{L}$ is **truth equational** if the Leibniz operator $\Omega_{\mathbf{B}}$ is **completely order reflecting on** $\mathcal{C}^\flat$, i.e., if for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$, such that $\bigcap_{i \in I} X_i \in \mathcal{C}^\flat$,

$$\bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \subseteq \Omega_{\mathbf{B}}(X) \quad \text{implies} \quad \bigcap_{i \in I} X_i \subseteq X.$$

Lemma 97 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is truth equational if and only if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, the Leibniz operator on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is completely order reflecting, i.e., for all $\{Y_i : i \in I\} \cup \{Y\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\bigcap_{i \in I} Y_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,*

$$\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) \quad \textit{implies} \quad \bigcap_{i \in I} Y_i \subseteq Y.$$

Proof: The right to left implication is again obtained by applying the hypothesis to the interpretation $\langle \mathbf{B}, i_{\mathbf{B}} \rangle$. For the left to right implication, let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $\{Y_i : i \in I\} \cup \{Y\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, with $\bigcap_{i \in I} Y_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Then we have

$$\begin{array}{rcl}
\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) & \text{iff} & h^{-1}(\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\
& \text{iff} & \bigcap_{i \in I} h^{-1}(\Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\
& \text{iff} & \bigcap_{i \in I} \Omega_{\mathbf{B}}(h^{-1}(Y_i)) \subseteq \Omega_{\mathbf{B}}(h^{-1}(Y)) \\
& \text{implies} & \bigcap_{i \in I} h^{-1}(Y_i) \subseteq h^{-1}(Y) \\
& \text{iff} & h^{-1}(\bigcap_{i \in I} Y_i) \subseteq h^{-1}(Y) \\
& \text{iff} & \bigcap_{i \in I} Y_i \subseteq Y.
\end{array}$$

Hence, the Leibniz operator on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is completely order reflecting. ■

And, finally the equivalence of weak algebraizability with protoalgebraicity and truth equationality for logicates whose theory sets are closed under intersection.

Theorem 98 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, with $\mathcal{C}^\flat$ closed under intersection. Then $\mathbb{L}$ is weakly algebraizable if and only if it is protoalgebraic and truth equational.*

Proof: Suppose $\mathbb{L}$ is weakly algebraizable. Since, by definition $\Omega_{\mathbf{B}}$ is monotone, $\mathbb{L}$ is certainly protoalgebraic. To show that it is also truth-equational, consider $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$, with $\bigcap_{i \in I} X_i \in \mathcal{C}^\flat$, such that $\bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \subseteq \Omega_{\mathbf{B}}(X)$. Then

$$\begin{array}{rcl}
\Omega_{\mathbf{B}}(\bigcap_{i \in I} X_i \cap X) & = & \bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \cap \Omega_{\mathbf{B}}(X) \quad \text{(Corollary 84)} \\
& = & \bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \quad (\bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \subseteq \Omega_{\mathbf{B}}(X)) \\
& = & \Omega_{\mathbf{B}}(\bigcap_{i \in I} X_i). \quad \text{(Corollary 84)}
\end{array}$$

As $\Omega_{\mathbf{B}}$ is injective, $\bigcap_{i \in I} X_i \cap X = \bigcap_{i \in I} X_i$. So $\bigcap_{i \in I} X_i \subseteq X$. This shows that $\Omega_{\mathbf{B}}$ is completely order reflecting and, therefore, $\mathbb{L}$ is also truth equational.

Suppose, conversely, that $\mathbb{L}$ is protoalgebraic and truth equational. By protoalgebraicity, $\Omega_{\mathbf{B}}$ is monotone. So it suffices to show that it is injective. Let $X, Y \in \mathcal{C}^\flat$, such that $\Omega_{\mathbf{B}}(X) = \Omega_{\mathbf{B}}(Y)$. Then, as $\Omega_{\mathbf{B}}$ is, by hypothesis, completely order reflective, we get $X \subseteq Y$ and $Y \subseteq X$, which yields $X = Y$. Hence $\Omega_{\mathbf{B}}$ is also injective, showing that $\mathbb{L}$ is weakly algebraizable. ■