

# Chapter 6

## General Theory

## 6.1 Introduction

In our work in Part I, we already glimpsed, at least twice, how, in studying a logicate, having an underlying order on the subsets of its universe  $A$  may be beneficial. E.g., in Chapter 2, when we looked at linearized consequences, we saw that artificially linearizing allow us to study instead of an arbitrary idempotent operator, an operator that also satisfies all three properties of an ordinary closure operator. Furthermore, in Chapter 5, we saw how many of the results related to classes in the algebraic hierarchy required that the set of theories was closed under intersections or has a minimum element. These observations lead us in Part II to look at structures in which order plays a role from the get go. To take advantage of the powerful machinery of the traditional framework of monotonic logics [12], without, however, losing sight of the fact that we are dealing with, possibly, nonmonotonic systems, we introduce a complete lattice ordering on the powerset of the underlying set  $A$ .

Comparing to the development in Part I, we could say that, in Part I, we took the logical notion of consequence operator as foundational and constructed, based on it, an “ordered” consequence, which involved a type of imposed ordering, either “artificial”, e.g., a linearization, or “natural”, e.g., based on  $\subseteq$ , reflecting, necessarily in a rather loose way, to the extent possible the “chaotic” logical structure. On the other hand, in Part II, a reversal of roles occurs. More precisely, we presume an underlying order on the powerset  $\mathcal{P}(A)$  of the set  $A$  and then build a logical structure that is, in some way, commensurate with the underlying ordering. We visualize the presumed preexisting ordering as an artificially created “molecular” shape and, since the logic is developed on that construct, it is termed a “logicoid”. This approach imitates more closely, and captures more accurately, many of the features of more traditional logical systems. On the other hand, expectations must be tempered, since the ordering is one among many that could possibly be chosen, and as such, its role is not quite natural. As noted, also, in comments in Part I., we attempt to do what we can in a challenging setting, among rather adverse features as compared with those naturally available in the monotonic framework.

In Section 6.2, we introduce the notion of a *grid*, which consists of an underlying set  $A$  (viewed as a set of abstract sentences), together with an arbitrary complete lattice ordering on the powerset of  $A$ . The fact that this ordering is arbitrary and not the “subset” ordering is what permits accommodating nonmonotonicity and make the framework suitable for our purposes, while still maintaining many of the advantages afforded by the complete lattice structure. Naturally enough, we then introduce *grid morphisms* that connect grids. They are surjective mapping between the underlying sets that make their induced inverse powerset mappings complete lattice embeddings. Continuing, we define *closure operators* as ones that are inflationary, mono-

tone and idempotent, but not with respect to the natural subset ordering, but, rather, with respect to the “artificial” ordering of the grid. We also define *closure systems* and, using the notion of *theory*, we show that, as in the ordinary framework, closure operators and closure systems (on the grid, as it were) are still in one to one correspondence and, thus, interchangeable.

In Section 6.3, we introduce the “weaker than” and “finer than” relations to compare closure operators of logicoids and closure systems on the underlying grids, respectively. These relationships parallel the ones in the classical (monotonic) framework, except that, instead of being with respect to the subset relation, they are based on the grid ordering.

In Section 6.4, we look at *boosting* for logicoids by a chosen set of axioms, which corresponds to taking the axiomatic extension of a sentential logic in the ordinary monotonic context. We saw the difficulties inherent in defining such an operation for logicates in Section 2.4. Here, the presence of a complete lattice ordering in the grid on which a logicoid is based, creates an environment in which some of the nice features may be recovered, albeit with respect to the  $\leq$  ordering of the grid rather than the natural subset ordering that serves the same purpose in the monotonic framework.

Our main interest is in what we call *algebraic logicoids*, which are logicoids built on *algebraic grids*, that is, grids on sets having an algebraic structure. Naturally enough, treating them algebraically requires having some algebraic fundamentals available for handling them. This is precisely the purpose that Section 6.5 is supposed to fulfill. Here, we formally define *algebraic grids*, which consist of an algebra together with a complete lattice ordering on its powerset. We also define *grid morphisms* and *grid congruences*. We show that these constructs interact as expected. We then employ them to develop analogs of the fundamental Homomorphism Theorems of Universal Algebra for algebraic grids, their homomorphisms and their congruences.

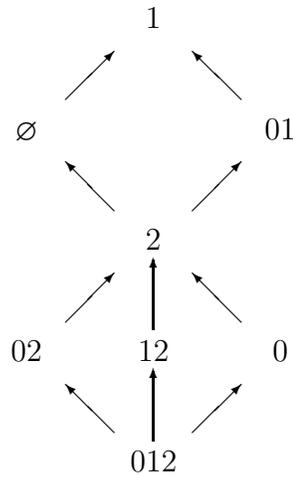
## 6.2 Logicoids

In this section, we introduce the basic notion of *logicoid* which forms the underlying object of study throughout.

Let  $A$  be a set. Let  $\mathcal{P}(A)$  denote the powerset of  $A$ . A **grid** is a pair  $\hat{A} = \langle A, \leq \rangle$ , where  $\leq$  is a complete lattice ordering on  $\mathcal{P}(A)$ .

**Example 99** Consider the set  $A = \{0, 1, 2\}$ . Let  $\leq$  be the ordering on its

powerset  $\mathcal{P}(A)$  shown in the diagram.



Then  $\hat{A} = \langle A, \leq \rangle$  is a grid.

**Example 100** Consider the set  $A' = \{a, b\}$ . Let  $\leq'$  be the ordering on its powerset shown in the diagram.

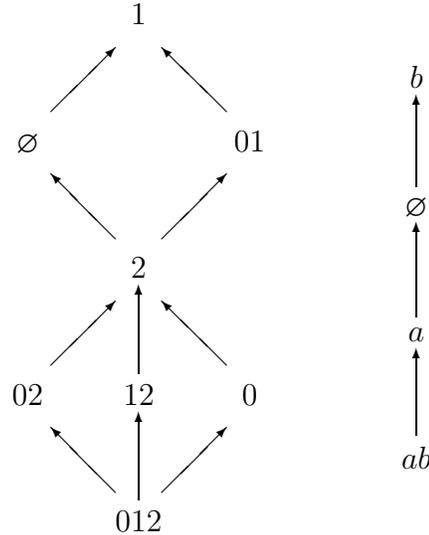


Then  $\hat{A}' = \langle A', \leq' \rangle$  is a grid.

A **grid morphism**  $h : \hat{A} \rightarrow \hat{A}'$  is a surjective mapping  $h : A \rightarrow A'$ , such that  $h^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \rightarrow \langle \mathcal{P}(A), \leq \rangle$  is a complete lattice embedding.

**Example 101** Consider, again, the grids  $\hat{A}$  and  $\hat{A}'$  of the preceding two

examples.



The mapping  $h : A \rightarrow A'$ , with  $0 \mapsto a$ ,  $1 \mapsto b$  and  $2 \mapsto a$  is a grid morphism as can be checked quickly by hand. We have

$$012 \leq 02 \leq \emptyset \leq 1 \quad \text{iff} \quad ab \leq' a \leq' \emptyset \leq' b.$$

A **closure operator on  $\hat{A}$**  is a mapping  $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  such that, for all  $X, Y \subseteq A$ ,

- (Inflationarity)  $X \leq C(X)$ ;
- (Monotonicity)  $X \leq Y$  implies  $C(X) \leq C(Y)$ ;
- (Idempotency)  $C(C(X)) = C(X)$ .

A **logicoid** is a pair  $\mathbb{L} = \langle \hat{A}, C \rangle$ , where:

- $\hat{A} = \langle A, \leq \rangle$  is a grid;
- $C$  a closure operator on  $\hat{A}$ .

We denote by  $\text{Lgcd}(\hat{A})$  the collection of all logicoids on  $\hat{A}$ .

A **closure system on a grid  $\hat{A} = \langle A, \leq \rangle$**  is a pair  $\hat{\mathcal{X}} = \langle \mathcal{X}, \leq \rangle$ , where  $\mathcal{X} \subseteq \mathcal{P}(A)$  is closed under arbitrary grid meets and  $\leq$  is the grid order. We denote the collection of all closure systems on the grid  $\hat{A}$  by  $\text{Clos}(\hat{A})$ .

Let  $\hat{A} = \langle A, \leq \rangle$  be a grid and  $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  a closure operator on  $\hat{A}$ . We consider the set of its **fixed points** or **theories**

$$\mathcal{C} = \{X \subseteq A : C(X) = X\}.$$

We define the partially ordered set

$$\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle,$$

where  $\leq$  is the grid order.

On the other hand, let  $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$  be a closure system on a grid  $\hat{A}$ . Then define an operator  $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  by setting, for all  $X \subseteq A$ ,

$$C(X) = \bigwedge \{Y \in \mathcal{C} : X \leq Y\}.$$

One can prove immediately the following properties paralleling properties that hold in the monotonic setting.

**Proposition 102** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid.*

- (a) *If  $C$  is a closure operator on  $\hat{A}$ , then  $\hat{\mathcal{C}}$  is a closure system on  $\hat{A}$ ;*
- (b) *If  $\hat{\mathcal{C}}$  is a closure system on  $\hat{A}$ , then  $C$  is a closure operator on  $\hat{A}$ ;*
- (c) *The two mappings are inverses of one another, whence closure operators on  $\hat{A}$  are in one-to-one correspondence with closure systems on  $\hat{A}$ .*

**Proof:**

- (a) Let  $C$  be a closure operator and  $\{X_i : i \in I\} \subseteq \mathcal{C}$ . By Inflationarity,  $\bigwedge_{i \in I} X_i \leq C(\bigwedge_{i \in I} X_i)$ . On the other hand, for all  $i \in I$ , by Monotonicity and the fact that  $X_i \in \mathcal{C}$ ,  $C(\bigwedge_{i \in I} X_i) \leq C(X_i) = X_i$ , whence  $C(\bigwedge_{i \in I} X_i) \leq \bigwedge_{i \in I} X_i$ . Hence,  $\bigwedge_{i \in I} X_i \in \mathcal{C}$  and  $\mathcal{C}$  is closed under arbitrary meets, i.e.,  $\hat{\mathcal{C}}$  is a closure system on  $\hat{A}$ .

- (b) Suppose  $\hat{\mathcal{C}}$  is a closure system on  $\hat{A}$ . Then, we have the following:

- For all  $X \subseteq A$ ,  $X \leq \bigwedge \{Y \in \mathcal{C} : X \leq Y\} = C(X)$ .
- For all  $X, Y \subseteq A$ , such that  $X \leq Y$ ,

$$C(X) = \bigwedge \{Z \in \mathcal{C} : X \leq Z\} \leq \bigwedge \{Z \in \mathcal{C} : Y \leq Z\} = C(Y).$$

- For all  $X \subseteq A$ ,

$$\begin{aligned} C(C(X)) &= \bigwedge \{Z \in \mathcal{C} : C(X) \leq Z\} \\ &= \bigwedge \{Z \in \mathcal{C} : \bigwedge \{Y \in \mathcal{C} : X \leq Y\} \leq Z\} \\ &= \bigwedge \{Z \in \mathcal{C} : X \leq Z\} \\ &= C(X). \end{aligned}$$

Thus,  $C$  is a closure operator on  $\hat{A}$ .

- (c) Suppose, first, that  $C$  is a closure operator. Consider the closure operator  $C'$  associated with the closure system  $\hat{\mathcal{C}}$ . Then, for all  $X \subseteq A$ ,

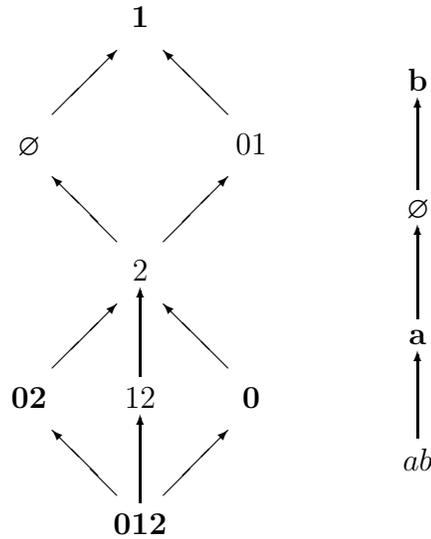
$$\begin{aligned} C'(X) &= \bigwedge \{Y \in \mathcal{C} : X \leq Y\} \quad (\text{Definition of } C') \\ &= \bigwedge \{Y \in \mathcal{C} : C(X) \leq Y\} \quad (\text{Monotonicity of } C) \\ &= C(X). \end{aligned}$$

Moreover, if  $\hat{\mathcal{C}}$  is a closure system, let  $\hat{\mathcal{C}}'$  be the closure system associated with its closure operator  $C$ . Then, we have, for all  $X \subseteq A$ ,

$$\begin{aligned} X \in \mathcal{C}' &\text{ iff } C(X) = X && \text{(Definition of } \mathcal{C}') \\ &\text{ iff } \bigwedge \{Y \in \mathcal{C} : X \leq Y\} = X && \text{(Definition of } C) \\ &\text{ iff } \bigwedge \{Y \in \mathcal{C} : C(X) \leq Y\} = X && \text{(Monotonicity of } C) \\ &\text{ iff } C(X) = X \\ &\text{ iff } X \in \mathcal{C}. \end{aligned}$$

This shows that closure operators and closure systems on a grid  $\hat{A}$  are in one-to-one correspondence. ■

**Example 103** Consider the grids  $\hat{A}$  and  $\hat{A}'$  seen previously.



Take  $\mathcal{C} = \{\{0, 1, 2\}, \{0, 2\}, \{0\}, \{1\}\}$  and  $\mathcal{C}' = \{\{a\}, \{b\}\}$ . Denote by  $C$  and  $C'$  the corresponding closure operators on  $\hat{A}$  and  $\hat{A}'$ , respectively. Then  $\mathbb{L} = \langle \hat{A}, C \rangle$  and  $\mathbb{L}' = \langle \hat{A}', C' \rangle$  are logicoïds on  $\hat{A}$  and  $\hat{A}'$ , respectively.

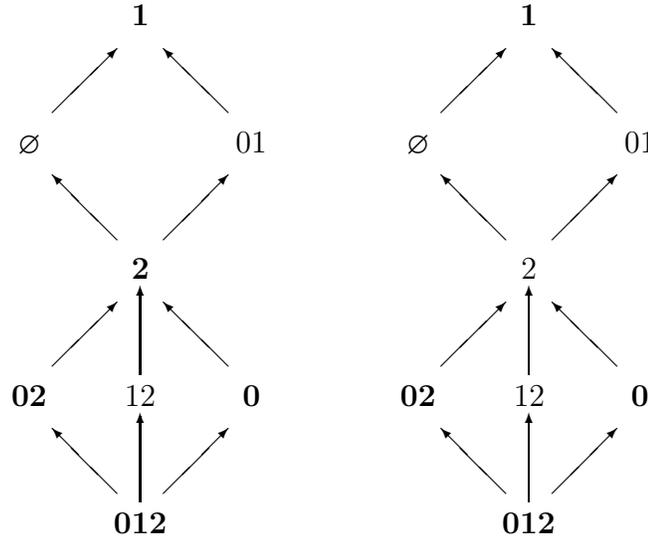
### 6.3 Comparing Logicoïds

Let  $\hat{A} = \langle A, \leq \rangle$  be a grid. Given two closure operators  $C$  and  $C'$  on  $\hat{A}$ , we say  $C$  is **weaker than**  $C'$  or  $C'$  is **stronger than**  $C$ , written  $C \leq C'$ , if, for all  $X \subseteq A$ ,

$$C(X) \leq C'(X).$$

We extend the same terminology for the two logicoïds  $\mathbb{L} = \langle \hat{A}, C \rangle$  and  $\mathbb{L}' = \langle \hat{A}', C' \rangle$ , that is, we set  $\mathbb{L} \leq \mathbb{L}'$  if and only if  $C \leq C'$ .

**Example 104** Consider the grid  $\hat{A}$  seen previously.



Take  $\mathcal{C} = \{\{0, 1, 2\}, \{0, 2\}, \{0\}, \{2\}, \{1\}\}$  and  $\mathcal{C}' = \{\{0, 1, 2\}, \{0, 2\}, \{0\}, \{1\}\}$ . Denote by  $C$  and  $C'$  the corresponding closure operators on  $\hat{A}$ . Then  $\mathbb{L} = \langle \hat{A}, C \rangle$  and  $\mathbb{L}' = \langle \hat{A}, C' \rangle$  are such that  $\mathbb{L} \leq \mathbb{L}'$ .

**Proposition 105** Let  $\hat{A} = \langle A, \leq \rangle$  be a grid. Then  $\mathbf{Lgcd}(\hat{A}) = \langle \mathbf{Lgcd}(\hat{A}), \leq \rangle$  is a complete lattice.

**Proof:** Let  $\mathbb{L}_i = \langle \hat{A}, C_i \rangle$ ,  $i \in I$ , be a collection of logicoïds on  $\hat{A}$ . Define  $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  by setting, for all  $X \subseteq A$ ,

$$C(X) = \bigwedge_{i \in I} C_i(X).$$

It can be shown that  $C$  is a closure operator on  $\hat{A}$ .

- For all  $X \subseteq A$ ,  $X \leq C_i(X)$ , for all  $i \in I$ , whence,  $X \leq \bigwedge_{i \in I} C_i(X) = C(X)$ .
- For all  $X, Y \subseteq A$ , such that  $X \leq Y$ ,  $C_i(X) \leq C_i(Y)$ , for all  $i \in I$ , whence  $C(X) = \bigwedge_{i \in I} C_i(X) \leq \bigwedge_{i \in I} C_i(Y) = C(Y)$ .
- Finally, let  $X \subseteq A$ . We have, for all  $i \in I$ ,

$$C(C(X)) = \bigwedge_{i \in I} C_i \left( \bigwedge_{i \in I} C_i(X) \right) \leq C_i(C_i(X)) = C_i(X).$$

Thus,  $C(C(X)) \leq \bigwedge_{i \in I} C_i(X) = C(X)$ . Since the reverse inclusion holds by Inflationarity, we get that  $C$  is also idempotent.

Moreover,  $C = \bigwedge_{i \in I} C_i$ , where the meet is with respect to the  $\leq$  ordering on logicoïds on  $\hat{A}$ . Clearly, by definition, for all  $X$ ,  $C(X) = \bigwedge_{i \in I} C_i(X) \leq C_i(X)$ . Thus,  $C$  is a  $\leq$ -lower bound of the  $C_i$ ,  $i \in I$ . Further, if  $C'$  is also a lower

bound, one has, for all  $X \subseteq A$ ,  $C'(X) \leq C_i(X)$ , whence,  $C'(X) \leq \bigwedge_{i \in I} C_i(X) = C(X)$ , i.e.,  $C$  is the meet of the  $C_i$  in  $\leq$ . Therefore,  $\mathbf{Lgcd}(\hat{A})$  is a complete lattice. ■

A similar comparison applies to closure systems. Given two closure systems  $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$  and  $\hat{\mathcal{C}}' = \langle \mathcal{C}', \leq \rangle$  on  $\hat{A}$ , we say that  $\mathcal{C}$  is **finer than**  $\mathcal{C}'$  or that  $\mathcal{C}'$  is **coarser than**  $\mathcal{C}$  if

$$\mathcal{C}' \subseteq \mathcal{C}.$$

**Proposition 106** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid and  $\mathbb{L} = \langle \hat{A}, \mathcal{C} \rangle$ ,  $\mathbb{L}' = \langle \hat{A}, \mathcal{C}' \rangle$  be logicoids on  $\hat{A}$ . Then*

$$\mathbb{L} \leq \mathbb{L}' \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

**Proof:** Suppose, first, that  $\mathcal{C} \subseteq \mathcal{C}'$ . Let  $X \in \mathcal{C}'$ . Then  $C(X) \leq C'(X) = X$ , whence, since, by Inflationarity,  $X \leq C(X)$ ,  $C(X) = X$  and  $X \in \mathcal{C}$ . Thus,  $\mathcal{C}' \subseteq \mathcal{C}$ . Assume, conversely, that  $\mathcal{C}' \subseteq \mathcal{C}$  and let  $X \subseteq A$ . Then

$$C(X) = \bigwedge \{Y \in \mathcal{C} : X \leq Y\} \leq \bigwedge \{Y \in \mathcal{C}' : X \leq Y\} = C'(X).$$

Hence  $\mathcal{C} \leq \mathcal{C}'$ . ■

**Proposition 107** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid. Then the partially ordered sets  $\mathbf{Lgcd}(\hat{A}) = \langle \mathbf{Lgcd}(\hat{A}), \leq \rangle$  and  $\mathbf{Clos}(\hat{A}) = \langle \mathbf{Clos}(\hat{A}), \supseteq \rangle$  are isomorphic.*

**Proof:** The correspondence established in Proposition 102 is, by Proposition 106, order preserving and order reflecting, whence it establishes the required isomorphism. ■

**Corollary 108** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid. Then  $\mathbf{Clos}(\hat{A}) = \langle \mathbf{Clos}(\hat{A}), \subseteq \rangle$  is a complete lattice with meet being set intersection.*

**Proof:** By Proposition 107, using Proposition 105. ■

## 6.4 Boosting

In this section we build operations that are, in the nonmonotonic context, “parallel” to axiomatic extensions in the traditional monotonic framework. We could call this operation a “nonmonotonic axiomatic extension”, but, in analogy with the operation introduced in the context of logicates, we call it *boosting* instead.

Let  $\hat{A} = \langle A, \leq \rangle$  be a grid,  $\mathbb{L} = \langle \hat{A}, \mathcal{C} \rangle$  a logicoid on  $\hat{A}$  and  $T \subseteq A$ . The **boosting of  $\mathcal{C}$  by  $T$**  is the operator  $C^T : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  that is defined, for all  $X \subseteq A$ , by

$$C^T(X) = C(T \vee X).$$

We show that this recipe gives a bona fide consequence operator.

**Proposition 109** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid,  $\mathbb{L} = \langle \hat{A}, C \rangle$  a logicoid and  $T \subseteq A$ . Then  $C^T : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  is a closure operator on  $\hat{A}$ .*

**Proof:** Let  $X \subseteq A$ . Then

$$X \leq X \vee T \leq C(X \vee T) = C^T(X).$$

Let  $X, Y \subseteq A$ , such that  $X \leq Y$ . Then  $X \vee T \leq Y \vee T$ . By Monotonicity,  $C(X \vee T) \leq C(Y \vee T)$  and, therefore,  $C^T(X) \leq C^T(Y)$ . Finally, let  $X \subseteq A$ . Then

$$\begin{aligned} C^T(C^T(X)) &= C(C^T(X) \vee T) \\ &= C(C(X \vee T) \vee T) \\ &= C(C(X \vee T)) \\ &= C(X \vee T) \\ &= C^T(X). \end{aligned}$$

So  $C^T$  satisfies Inflationarity, Monotonicity and Idempotency and is, therefore, a closure operator on  $\hat{A}$ . ■

We call the logicoid  $\mathbb{L}^T = \langle \hat{A}, C^T \rangle$  the **boosting of  $\mathbb{L}$  by  $T$** .

We also describe the way the two closure systems are related. Consider a closure system  $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$  on  $\hat{A}$  and let  $T \subseteq A$ . Define the **boosted closure system  $\hat{\mathcal{C}}^T = \langle \mathcal{C}^T, \leq \rangle$  of  $\hat{\mathcal{C}}$  by  $T$  on  $\hat{A}$** , by setting

$$\mathcal{C}^T = \{X \in \mathcal{C} : T \leq X\}$$

and taking  $\leq$  be the restriction of  $\leq$  on  $\mathcal{C}^T$ .

**Proposition 110** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid,  $\mathbb{L} = \langle \hat{A}, C \rangle$  a logicoid and  $T \subseteq A$ . If  $\hat{\mathcal{C}}$  is the closure system corresponding to  $\mathbb{L}$ , then  $\hat{\mathcal{C}}^T$  is the closure system corresponding to  $\mathbb{L}^T$ .*

**Proof:** To prove the claim, we must show that, for all  $X \subseteq A$ ,

$$C^T(X) = X \quad \text{iff} \quad X \in \hat{\mathcal{C}}^T.$$

Suppose  $C^T(X) = X$ . By definition,  $C(X \vee T) = X$ . On the one hand,  $C(X) \leq C(X \vee T) = X$ , whence  $X \in \mathcal{C}$ . On the other,  $T \leq X \vee T \leq C(X \vee T) = C^T(X) = X$ . Thus,  $X \in \hat{\mathcal{C}}^T$ .

Assume, conversely, that  $X \in \hat{\mathcal{C}}^T$ . Then  $C(X) = X$  and  $T \leq X$ . It follows that  $X = C(X)$  and  $T \leq C(X)$ . Hence,  $X \vee T \leq C(X)$ , which yields  $C(X \vee T) \leq C(X)$ . Thus, we get

$$C^T(X) = C(X \vee T) \leq C(X) = X,$$

whence  $C^T(X) = X$ . ■

Finally, we obtain the following characterization of boosting.

**Proposition 111** *Let  $\hat{A} = \langle A, \leq \rangle$  be a grid,  $\mathbb{L} = \langle \hat{A}, C \rangle$  a logicoid and  $T \subseteq A$ . Then  $\mathbb{L}^T$  is the weakest logicoid in  $\mathbf{Lgd}(\hat{A})$ , satisfying*

- $C \leq C^T$ ;
- $T \leq C^T(X)$ , for all  $X \subseteq A$ .

**Proof:** That  $C^T$  satisfies the two statement is shown as follows:

- $C(X) \leq C(X \vee T) = C^T(X)$ ;
- $T \leq X \vee T \leq C(X \vee T) = C^T(X)$ .

Suppose that  $C'$  is a closure operator on  $\hat{A}$ , such that  $C \leq C'$  and  $T \leq C'(X)$ , for all  $X \subseteq A$ . Note that these give, for all  $X \subseteq A$ ,  $X \vee T \leq C'(X)$ . Then, we get

$$\begin{aligned} C^T(X) &= C(X \vee T) \\ &\leq C(C'(X)) \\ &\leq C'(C'(X)) \\ &= C'(X). \end{aligned}$$

This shows that  $C^T \leq C'$  and, hence,  $C^T$  is the weakest closure operator satisfying the two conditions. ■

## 6.5 Algebraic Grids and Logicoids

In the sequel, we will focus on logicoids built not simply on an underlying set  $A$ , but on an algebra  $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$  of an arbitrary, but fixed, type  $\mathcal{L}$ . We call those *algebraic logicoids* and start their study per se in Chapter 7. However, since there is substantial interaction between the algebraic structure  $\mathbf{A}$  and the complete lattice ordering  $\leq$  on the powerset  $\mathcal{P}(A)$ , we need to delve a little into universal algebra. This we do in this section as preparation for what is to follow.

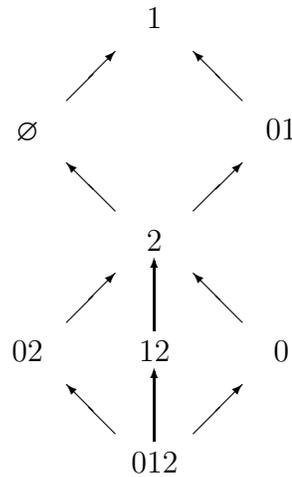
Fix an algebraic type  $\mathcal{L}$ . An (**algebraic**) **grid** is a pair  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ , where:

- $\mathbf{A}$  is an  $\mathcal{L}$ -algebra;
- $\leq$  is a complete lattice ordering on  $\mathcal{P}(A)$ .

**Example 112** *Let  $\mathcal{L} = \{*\}$ , where  $*$  is a binary operation symbol. Consider the  $\mathcal{L}$ -algebra  $\mathbf{A} = \langle A, *^{\mathbf{A}} \rangle$ , where  $A = \{0, 1, 2\}$  and  $*^{\mathbf{A}}$  is given by the following table.*

$*^{\mathbf{A}}$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	0

Let  $\leq$  be the ordering on its powerset  $\mathcal{P}(A)$  shown in the diagram.



Then  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  is an algebraic grid.

**Example 113** Let  $\mathcal{L} = \{*\}$ , where  $*$  is a binary operation symbol. Consider the  $\mathcal{L}$ -algebra  $\mathbf{A}' = \langle A', *^{\mathbf{A}'} \rangle$ , where  $A' = \{a, b\}$  and  $*^{\mathbf{A}'}$  is given by the following table.

$*^{\mathbf{A}'}$	$a$	$b$
$a$	$a$	$b$
$b$	$b$	$a$

Let  $\leq'$  be the ordering on its powerset shown in the diagram.

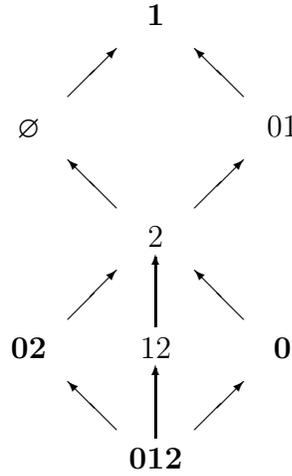


Then  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  is an algebraic grid.

An **algebraic logicoid** is a pair  $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ , where:

- $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  is an algebraic grid;
- $C$  is a closure operator on  $\hat{\mathbf{A}}$ .

**Example 114** Consider the algebraic grid  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  constructed above.



Let

$$\mathcal{C} = \{\{0, 1, 2\}, \{0, 2\}, \{0\}, \{1\}\}.$$

Then  $\mathbb{L} = \langle \hat{\mathbf{A}}, \mathcal{C} \rangle$  is an algebraic logicoid.

**Example 115** Consider the algebraic grid  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  constructed above.



Let  $\mathcal{C}' = \{\{a\}, \{b\}\}$ . Then  $\mathbb{L}' = \langle \hat{\mathbf{A}}', \mathcal{C}' \rangle$  is an algebraic logicoid.

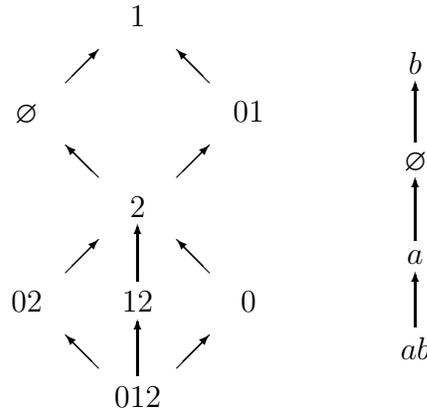
Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  and  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  be two algebraic grids. An (**algebraic**) **grid morphism**  $h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'$  is a surjective (algebra) homomorphism  $h : \mathbf{A} \rightarrow \mathbf{A}'$ , such that

$$h^{-1} : \langle \mathcal{P}(\mathbf{A}'), \leq' \rangle \rightarrow \langle \mathcal{P}(\mathbf{A}), \leq \rangle$$

is a complete lattice embedding.

**Example 116** Consider the algebraic grids  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  and  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$

constructed above.



The mapping  $h : A \rightarrow A'$ , determined by  $0 \mapsto a$ ,  $1 \mapsto b$  and  $2 \mapsto a$  is a homomorphism  $h : \mathbf{A} \rightarrow \mathbf{A}'$ . Moreover, as can be checked by hand, it is a grid morphism.

To define *grid congruences*, we look at some preliminary concepts. Let  $A$  be a set and  $\theta$  be an equivalence relation on  $A$ . We say that  $\theta$  is **compatible with** a subset  $X \subseteq A$ , if, for all  $a, b \in A$ ,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in X \quad \text{imply} \quad b \in X.$$

We denote by  $\text{Cmp}(\theta)$  the set of all subsets of  $A$  that are compatible with  $\theta$ . These are characterized in various ways.

**Lemma 117** *Let  $A$  be a set,  $\theta$  an equivalence relation on  $A$  and  $X \subseteq A$ . The following statements are equivalent.*

- (i)  $X \in \text{Cmp}(\theta)$ ;
- (ii)  $X$  is a union of  $\theta$ -equivalence classes;
- (iii)  $X = \pi_\theta^{-1}(Y)$ , for some  $Y \subseteq A/\theta$ .

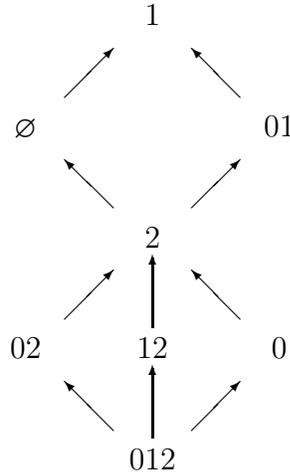
**Proof:**

- (i) $\Rightarrow$ (ii) Let  $X \subseteq A$  be compatible with  $\theta$ . Suppose  $x \in X$  and  $a \in A$ , such that  $\langle x, a \rangle \in \theta$ . Since  $x \in X$ , by compatibility,  $a \in X$ . Thus,  $X$  is a union of  $\theta$ -equivalence classes.
- (ii) $\Rightarrow$ (iii) Let  $X$  be a union of  $\theta$ -equivalence classes. Set  $Y = \pi_\theta(X) \in A/\theta$ . Then  $X = \pi_\theta^{-1}(\pi_\theta(X)) = \pi_\theta^{-1}(Y)$ , where the hypothesis is used to ensure that the first equality holds.
- (iii) $\Rightarrow$ (i) Let  $X = \pi_\theta^{-1}(Y)$ , for some  $Y \subseteq A/\theta$ . Consider  $a, b \in A$ , such that  $\langle a, b \rangle \in \theta$  and  $a \in X$ . Then  $\pi_\theta(a) \in Y$ . Hence,  $\pi_\theta(b) = \pi_\theta(a) \in Y$ . So  $b \in \pi_\theta^{-1}(Y) = X$ . This shows that  $\theta$  is compatible with  $X$ .

■

Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  be an algebraic grid. A **grid congruence**  $\theta$  is a congruence on the algebra  $\mathbf{A}$ , such that  $\langle \text{Cmp}(\theta), \leq \rangle$  is a complete sublattice of the complete lattice  $\langle \mathcal{P}(A), \leq \rangle$ .

**Example 118** Consider again the algebraic grid  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ .



Since  $\star^{\mathbf{A}}$  is given by

$\star^{\mathbf{A}}$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	0

the equivalence  $\theta$  whose classes are  $\{0, 2\}$  and  $\{1\}$  is a grid congruence on  $\hat{\mathbf{A}}$ .

We denote by  $\text{Con}(\hat{\mathbf{A}})$  the set of all grid congruences on the grid  $\hat{\mathbf{A}}$ . We show that  $\text{Con}(\hat{\mathbf{A}})$ , ordered by  $\subseteq$ , forms a complete lattice  $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \text{Con}(\hat{\mathbf{A}}), \subseteq \rangle$ , whose join coincides with the join in  $\mathbf{Con}(\mathbf{A})$ .

**Lemma 119** Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  be an algebraic grid. The poset  $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \text{Con}(\hat{\mathbf{A}}), \subseteq \rangle$  is closed under arbitrary joins in  $\mathbf{Con}(\mathbf{A})$ .

**Proof:** Let  $\{\theta_i : i \in I\} \subseteq \text{Con}(\hat{\mathbf{A}})$ . Consider the join  $\bigvee_{i \in I} \theta_i$  in  $\mathbf{A}$ . It suffices to show that  $\langle \text{Cmp}(\bigvee_{i \in I} \theta_i), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ . Let  $X \subseteq A$ . We show

$$X \in \text{Cmp}\left(\bigvee_{i \in I} \theta_i\right) \quad \text{iff} \quad X \in \bigcap_{i \in I} \text{Cmp}(\theta_i).$$

For the left-to-right implication, let  $a, b \in A$ , such that  $\langle a, b \rangle \in \theta_i$  and  $a \in X$ . Then  $\langle a, b \rangle \in \bigvee_{i \in I} \theta_i$  and  $a \in X$ . By hypothesis,  $b \in X$ . Hence,  $X$  is compatible with  $\theta_i$ , for all  $i \in I$ , i.e.,  $X \in \bigcap_{i \in I} \text{Cmp}(\theta_i)$ .

For the right-to-left implication, suppose  $\langle a, b \rangle \in \bigvee_{i \in I} \theta_i$  and  $a \in X$ . Then, there exist  $k \in \omega$  and  $i_0, i_1, \dots, i_{k-1} \in I$ , such that

$$a = c_0 \theta_{i_0} c_1 \theta_{i_1} \cdots \theta_{i_{k-2}} c_{k-1} \theta_{i_{k-1}} c_k = b,$$

for some  $c_0, c_1, \dots, c_k \in A$ . Since, by hypothesis,  $X \in \bigcap_{i \in I} \text{Cmp}(\theta_i)$ , we get

$$a = c_0, c_1, c_2, \dots, c_{k-1}, c_k = b \in X.$$

This shows that  $X \in \text{Cmp}(\bigvee_{i \in I} \theta_i)$ .

Since  $\langle \text{Cmp}(\theta_i), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ , for all  $i \in I$ , we have that  $\langle \bigcap_{i \in I} \text{Cmp}(\theta_i), \leq \rangle$  is also a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ , which shows that  $\langle \text{Cmp}(\bigvee_{i \in I} \theta_i), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ . ■

**Proposition 120** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  be an algebraic grid. The poset  $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \text{Con}(\hat{\mathbf{A}}), \subseteq \rangle$  forms a complete lattice, with joins coinciding with those in  $\mathbf{Con}(\mathbf{A})$ .*

**Proof:** Apply Lemma 119. ■

We may now establish a correspondence between grid morphisms and grid congruences, taking after the one between homomorphisms and congruences in Universal Algebra. For the kernel of a homomorphism  $h : \mathbf{A} \rightarrow \mathbf{A}'$  we use the notation

$$\text{Ker}(h) = \{ \langle a, b \rangle \in A^2 : h(a) = h(b) \}.$$

**Lemma 121** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ ,  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  be algebraic grids and  $h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'$  be a grid morphism. Then  $\text{Ker}(h)$  is a grid congruence on  $\hat{\mathbf{A}}$ .*

**Proof:** First, since  $h : \mathbf{A} \rightarrow \mathbf{A}'$  is a homomorphism, its kernel is a congruence on  $\mathbf{A}$ . In addition, by hypothesis,  $h^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \rightarrow \langle \mathcal{P}(A), \leq \rangle$  is a complete lattice embedding. By Lemma 117, the image of  $h^{-1}$  in  $\mathcal{P}(A)$  is exactly  $\text{Cmp}(\text{Ker}(h))$ . Thus,  $\langle \text{Cmp}(\text{Ker}(h)), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ . ■

Let, now  $\hat{\mathbf{A}}$  be a grid and  $\theta \in \text{Con}(\hat{\mathbf{A}})$ . Construct the pair

$$\hat{\mathbf{A}}^\theta = \hat{\mathbf{A}}/\theta := \langle \mathbf{A}/\theta, \leq^\theta \rangle$$

by setting, for all  $Y, Y' \in \mathcal{P}(A/\theta)$ ,

$$Y \leq^\theta Y' \quad \text{iff} \quad \pi_\theta^{-1}(Y) \leq \pi_\theta^{-1}(Y').$$

**Lemma 122** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  be an algebraic grid and  $\theta \in \text{Con}(\hat{\mathbf{A}})$ .*

(a)  $\hat{\mathbf{A}}/\theta = \langle \mathbf{A}/\theta, \leq^\theta \rangle$  is an algebraic grid.

(b)  $\pi_\theta : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}/\theta$  is a grid morphism.

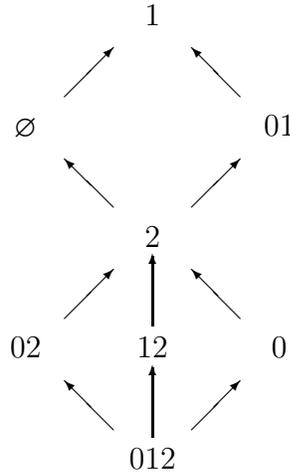
**Proof:**

(a) Since  $\theta$  is a congruence,  $\mathbf{A}/\theta$  is an algebra. So to prove that the pair forms a grid, it suffices to show that  $\leq^\theta$  is a complete lattice ordering on  $\mathcal{P}(A/\theta)$ . But,  $\theta$  is also a grid congruence, which means that  $\langle \text{Cmp}(\theta), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ . By Lemma 117,  $\langle \pi_\theta^{-1}(A/\theta), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ . Hence, by definition of  $\leq^\theta$ ,  $\leq^\theta$  is a complete lattice ordering on  $\mathcal{P}(A/\theta)$ .

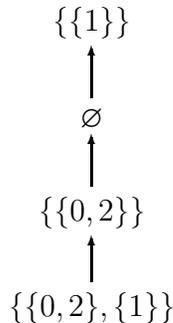
(b) We know that  $\pi_\theta$  is a homomorphism. By the proof of Part (a),  $\pi_\theta^{-1}$  is also a complete lattice embedding of  $\langle \mathcal{P}(A/\theta), \leq^\theta \rangle$  into  $\langle \mathcal{P}(A), \leq \rangle$ . Therefore,  $\pi_\theta : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}/\theta$  is a grid morphism. ■

We call  $\hat{\mathbf{A}}^\theta = \hat{\mathbf{A}}/\theta := \langle \mathbf{A}/\theta, \leq^\theta \rangle$  the **quotient (algebraic) grid** of  $\hat{\mathbf{A}}$  by  $\theta$  and  $\pi_\theta$  the **quotient (algebraic) grid morphism** or the **canonical projection grid morphism**.

**Example 123** Consider, once more, the algebraic grid  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ .



In a previous example, we looked at the grid congruence  $\theta = \{\{0, 2\}, \{1\}\}$ . The quotient grid  $\hat{\mathbf{A}}/\theta$  consists of the algebra  $\mathbf{A}/\theta$  and the complete lattice ordering  $\leq^\theta$  on  $\mathcal{P}(A/\theta)$ , given in the diagram.



Finally, we define isomorphisms. Consider two algebraic grids  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  and  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$ . A bijection  $h : A \rightarrow A'$  is an **isomorphism** between  $\hat{\mathbf{A}}$  and  $\hat{\mathbf{A}}'$  if both  $h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'$  and  $h^{-1} : \hat{\mathbf{A}}' \rightarrow \hat{\mathbf{A}}$  are grid homomorphisms. We then write  $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ .

**Lemma 124** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  and  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  be algebraic grids and  $h : A \rightarrow A'$  a bijection.  $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$  if and only if  $h : \mathbf{A} \cong \mathbf{A}'$  and  $h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'$ .*

**Proof:** The left to right implication is trivial. For the right to left, it suffices to show that, for all  $X, Y \subseteq A$ ,

$$X \leq Y \quad \text{iff} \quad h(X) \leq' h(Y).$$

Since  $h$  is a bijection, for the subsets  $X' = h(X)$  and  $Y' = h(Y)$  of  $A'$ , we have  $X = h^{-1}(X')$  and  $Y = h^{-1}(Y')$ . Thus, we get

$$\begin{aligned} X \leq Y & \quad \text{iff} \quad h^{-1}(X') \leq h^{-1}(Y') \quad (X = h^{-1}(X') \text{ and } Y = h^{-1}(Y')) \\ & \quad \text{iff} \quad X' \leq' Y' \quad (h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}') \\ & \quad \text{iff} \quad h(X) \leq h(Y). \quad (X' = h(X) \text{ and } Y' = h(Y)) \end{aligned}$$

Thus,  $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ . ■

Once we have this machinery available, we can work out versions of the Homomorphism Theorems of Universal Algebra for grid morphisms.

**Theorem 125 (Homomorphism)** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  and  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  be algebraic grids and  $h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'$  be a grid morphism. Then, there exists a unique grid isomorphism  $g : \hat{\mathbf{A}}/\text{Ker}(h) \cong \hat{\mathbf{A}}'$ , such that  $h = g \circ \pi_h$ ,*

$$\begin{array}{ccc} \hat{\mathbf{A}} & \xrightarrow{h} & \hat{\mathbf{A}}' \\ & \searrow \pi_h & \nearrow g \\ & \hat{\mathbf{A}}/\text{Ker}(h) & \end{array}$$

where  $\pi_h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}/\text{Ker}(h)$  is the quotient grid morphism.

**Proof:** By the Homomorphism Theorem of Universal Algebra, we know that there exists a unique  $g : \hat{\mathbf{A}}/\text{Ker}(h) \cong \hat{\mathbf{A}}'$ . Thus, by Lemma 124, we must show that  $g^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \rightarrow \langle \mathcal{P}(A/\text{Ker}(h)), \leq^{\text{Ker}(h)} \rangle$  is a complete lattice embedding. For this, it suffices to show that, for all  $Y, Y' \in \mathcal{P}(A')$ ,

$$g^{-1}(Y) \leq^{\text{Ker}(h)} g^{-1}(Y') \quad \text{iff} \quad Y \leq' Y'.$$

Indeed, we have, for all  $Y, Y' \in \mathcal{P}(A')$ ,

$$\begin{aligned} g^{-1}(Y) \leq^{\text{Ker}(h)} g^{-1}(Y') & \quad \text{iff} \quad \pi_h^{-1}(g^{-1}(Y)) \leq \pi_h^{-1}(g^{-1}(Y')) \quad (\text{Lemma 122}) \\ & \quad \text{iff} \quad h^{-1}(Y) \leq h^{-1}(Y') \quad (h = g \circ \pi_h) \\ & \quad \text{iff} \quad Y \leq' Y'. \quad (h : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}') \end{aligned}$$

We conclude that  $g : \hat{\mathbf{A}}/\text{Ker}(h) \cong \hat{\mathbf{A}}'$ . ■

We continue with a version of the Second Isomorphism Theorem.

**Theorem 126 (Second Isomorphism Theorem)** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  be an algebraic lattice and  $\theta, \theta' \in \text{Con}(\hat{\mathbf{A}})$ , such that  $\theta \subseteq \theta'$ . Then  $\theta'/\theta \in \text{Con}(\hat{\mathbf{A}}/\theta)$  and*

$$(\hat{\mathbf{A}}/\theta)/(\theta'/\theta) \cong \hat{\mathbf{A}}/\theta',$$

where the isomorphism is given by

$$(a/\theta)/(\theta'/\theta) \mapsto a/\theta'.$$

**Proof:** We consider the commutative diagram of natural quotient homomorphisms.

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\pi_\theta} & \mathbf{A}/\theta \\ & \searrow \pi'_\theta & \swarrow \pi \\ & \mathbf{A}/\theta' & \end{array}$$

By Lemma 122,  $\pi_\theta : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}/\theta$  and  $\pi_{\theta'} : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}/\theta'$  are grid morphisms. We can show that  $\pi : \hat{\mathbf{A}}/\theta \rightarrow \hat{\mathbf{A}}/\theta'$  is also a grid morphism. E.g., we have, for all  $\{Y'_i : i \in I\} \subseteq \mathcal{P}(A/\theta')$ ,

$$\begin{aligned} \pi^{-1}(\bigwedge_{i \in I}^{\theta'} Y'_i) &= \pi_\theta(\pi_{\theta'}^{-1}(\bigwedge_{i \in I}^{\theta'} Y'_i)) \\ &= \pi_\theta(\bigwedge_{i \in I} \pi_{\theta'}^{-1}(Y'_i)) \\ &= \bigwedge_{i \in I}^{\theta} (\pi_\theta(\pi_{\theta'}^{-1}(Y'_i))) \\ &= \bigwedge_{i \in I}^{\theta} (\pi^{-1}(Y'_i)). \end{aligned}$$

We know that  $\text{Ker}(\pi) = \theta'/\theta$ , which, by Theorem 26, is a grid congruence on  $\hat{\mathbf{A}}/\theta$ . So we consider the diagram

$$\begin{array}{ccc} \mathbf{A}/\theta & \xrightarrow{\pi} & \mathbf{A}/\theta' \\ & \searrow \pi_{\theta'/\theta} & \swarrow h \\ & (\mathbf{A}/\theta)/(\theta'/\theta) & \end{array}$$

where  $h$  is the unique grid isomorphism given by Theorem 26. Thus,  $h : (\hat{\mathbf{A}}/\theta)/(\theta'/\theta) \cong \hat{\mathbf{A}}/\theta'$ . ■

We finally prove an analog of the Correspondence Theorem.

**Theorem 127 (Correspondence Theorem)** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$  be an algebraic lattice and  $\theta \in \text{Con}(\hat{\mathbf{A}})$ . Then the segment  $[\theta, \nabla^{\mathbf{A}}]$  of the poset  $\text{Con}(\hat{\mathbf{A}})$  is isomorphic to the poset  $\text{Con}(\hat{\mathbf{A}}/\theta)$  by the mapping  $\theta' \mapsto \theta'/\theta$ .*

**Proof:** By Theorem 26, if  $\theta \subseteq \theta'$  are grid congruences on  $\hat{\mathbf{A}}$ , then  $\theta'/\theta \in \text{Con}(\hat{\mathbf{A}}/\theta)$ . By Universal Algebra, it suffices to prove that, for all  $\theta \subseteq \theta' \in$

$\text{Con}(\mathbf{A})$ , if  $\theta'/\theta \in \text{Con}(\hat{\mathbf{A}}^\theta)$ , then  $\theta' \in \text{Con}(\hat{\mathbf{A}})$ .

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\pi_\theta} & \mathbf{A}/\theta \\ \pi_{\theta'} \downarrow & & \downarrow \pi \\ \mathbf{A}/\theta' & \xrightarrow{\cong} & (\mathbf{A}/\theta)/(\theta'/\theta) \end{array}$$

Since, by hypothesis,  $\theta \in \text{Con}(\hat{\mathbf{A}})$  and  $\theta'/\theta \in \text{Con}(\hat{\mathbf{A}}^\theta)$ , we have  $\pi_\theta$  and  $\pi$  are grid morphisms. Identifying elements corresponding under the bottom isomorphism, we have, for all  $X \subseteq A$ ,

$$\begin{aligned} X \in \text{Cmp}(\text{Ker}(\pi \circ \pi_\theta)) & \text{ iff } X = \pi_\theta^{-1}(\pi^{-1}(Y)), \text{ for } Y \subseteq \mathcal{P}((A/\theta)/(\theta'/\theta)), \\ & \text{ iff } X = \pi_{\theta'}^{-1}(Y), \text{ for } Y \subseteq \mathcal{P}(A/\theta'), \\ & \text{ iff } X \in \text{Cmp}(\theta'). \end{aligned}$$

Thus, since  $\langle \text{Cmp}(\text{Ker}(\pi \circ \pi_\theta)), \leq \rangle$  is a complete sublattice of  $\langle \mathcal{P}(A), \leq \rangle$ , then so is  $\langle \text{Cmp}(\theta'), \leq \rangle$ , showing that  $\theta' \in \text{Con}(\hat{\mathbf{A}})$ .  $\blacksquare$

We close the section with a “fill-in” lemma for arrows, which will play a role later in establishing for logicoïds an analog of Proposition 1.15 of [12] (see, also, Proposition 31 for logicates).

**Lemma 128** *Let  $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ ,  $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$  and  $\hat{\mathbf{A}}'' = \langle \mathbf{A}'', \leq'' \rangle$  be algebraic grids,  $f : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'$  and  $g : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}''$  be grid morphisms, such that  $\text{Ker}(g) \subseteq \text{Ker}(f)$ . Then, there exists a unique grid morphism  $h : \hat{\mathbf{A}}'' \rightarrow \hat{\mathbf{A}}'$ , such that*

$$h \circ g = f.$$

$$\begin{array}{ccc} \hat{\mathbf{A}} & \xrightarrow{g} & \hat{\mathbf{A}}'' \\ & \searrow f & \nearrow h \\ & \hat{\mathbf{A}}' & \end{array}$$

**Proof:** We know from Universal Algebra that there exists a unique  $h : \mathbf{A}'' \rightarrow \mathbf{A}'$ , such that  $h \circ g = f$ . Moreover, by the definition of grid morphism, the following mappings are complete lattice embeddings.

$$\begin{aligned} g^{-1} : \langle \mathcal{P}(A''), \leq'' \rangle & \rightarrow \langle \mathcal{P}(A), \leq \rangle, \\ f^{-1} : \langle \mathcal{P}(A'), \leq' \rangle & \rightarrow \langle \mathcal{P}(A), \leq \rangle. \end{aligned}$$

These suffice to show that

$$h^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \rightarrow \langle \mathcal{P}(A''), \leq'' \rangle$$

is also a complete lattice embedding. Indeed, for all  $X', Y' \subseteq A'$ , we have

$$\begin{aligned} X' \leq' Y' & \text{ iff } f^{-1}(X') \leq f^{-1}(Y') \quad (f : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}') \\ & \text{ iff } g^{-1}(h^{-1}(X')) \leq g^{-1}(h^{-1}(Y')) \quad (h \circ g = f) \\ & \text{ iff } h^{-1}(X') \leq'' h^{-1}(Y'). \quad (g : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}'') \end{aligned}$$

Therefore,  $h : \hat{\mathbf{A}}'' \rightarrow \hat{\mathbf{A}}'$  is a grid morphism.  $\blacksquare$