Chapter 7

Algebraic Theory

7.1 Introduction

In Chapter 7, we develop the rudiments of the algebraic theory of logicoids with an eye towards developing, in Chapter 8, a model theory, paralleling the one in [12] and that developed for logicates in Part I. We first introduce the key concept of *logical grid congruence*. Based on those, we define the *Leibniz grid congruence* of a logical matrix and the *Tarski grid congruence* of a logicoid. We then study *bilogical morphisms* between logicoids, which, unlike those used for logicates, respect the logical consequence and not merely the theories of the structure. So, in that respect, they resemble more closely those introduced by Font and Jansana [12]. We then look at *quotient logicoids* and prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. This gives us the chance to look closely at *reductions* and at *reduced logicoids*. We then turn to analogs of *interpretations*, *filters* and *matrix models* and study many of their properties, including the way they interact with grid morphisms, their interplay with closure systems and their transformations via bilogical morphisms.

In more detail, Section 7.2 undertakes the study of *logical grid congruences*. Recall that, given an algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, a congruence θ on $\mathbf{A}$ is called a *grid congruence* on $\hat{\mathbf{A}}$ if $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ based on the grid $\hat{\mathbf{A}}$, θ is a *logical grid congruence* of $\mathbb{L}$ if it is a grid congruence on $\hat{\mathbf{A}}$, such that $\mathcal{C} \subseteq \mathrm{Cmp}(\theta)$. It is shown that the collection of all logical grid congruences of $\mathbb{L}$ forms a principal ideal of the complete lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\widetilde{\Omega}(\mathbb{L})$ is called the *Tarski grid congruence* of $\mathbb{L}$. An analogous situation occurs if one considers logical matrices $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$ based on an algebraic grid $\hat{\mathbf{A}}$. Here a *matrix grid congruence* is a grid congruence θ on $\hat{\mathbf{A}}$, such that $X \in \mathrm{Cmp}(\theta)$. Again, the collection of all matrix grid congruences of $\mathfrak{A}$ forms a principal ideal of the lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\Omega(\mathfrak{A})$ is called the *Leibniz grid congruence* of $\mathfrak{A}$. Two of the most useful observations related to these concepts are that $\widetilde{\Omega}$ is monotone on logicoids over the same grid and that, given a logicoid, its Tarski grid congruence is the intersection of all Leibniz grid congruences of those logical matrices formed by each of its theories.

In Section 7.3, we introduce and study *logical* and *bilogical morphisms* between logicoids. Since logicoids are based on algebraic grids, all these morphisms are algebraic grid morphisms, which were studied extensively in Section 6.5, and we rely quite heavily on that machinery. A *logical morphism* $h : \mathbb{L} \to \mathbb{L}'$ from a logicoid $\mathbb{L}$ based on $\hat{\mathbf{A}}$ to a logicoid $\mathbb{L}'$ based on $\hat{\mathbf{A}}'$ is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$. In case $h^{-1}(\mathcal{C}') = \mathcal{C}$ we say that $\mathbb{L}$ is *projectively generated from* $\mathbb{L}'$ *by* h. A logical morphism $h : \mathbb{L} \to \mathbb{L}'$ is a *bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}'$ between $\mathbb{L}$ and $\mathbb{L}'$ if it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. We provide a characterization theorem for bilogical morphisms along the lines of Proposition 1.4 of [12] and we show that, if $h : \mathbb{L} \to_b \mathbb{L}'$, then

$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}'))$. Finally, the notion of *isomorphism* between logicoids is introduced as a bijective mapping $h : A \to A'$ for which both $h : \mathbb{L} \to \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to \mathbb{L}$ are logical. It is shown that this is tantamount to requiring that $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ and $h : \mathbb{L} \to_b \mathbb{L}'$.

In Section 7.4, given an algebraic grid $\hat{\mathbf{A}}$ and a grid congruence θ on $\hat{\mathbf{A}}$, we define the *quotient closure operator* C^θ on the quotient grid $\hat{\mathbf{A}}/\theta$ of an operator C on $\hat{\mathbf{A}}$. This gives rise to the *quotient logicoid* $\mathbb{L}^\theta = \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$ of a given logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and, moreover, makes the quotient grid morphism $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ into a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta$. Quotient logicoids are important because, among other things, they allow us to prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. We prove analogs of the Homomorphism Theorem, of the Second Isomorphism Theorem and of the Correspondence Theorem. The latter, in particular, enables us to show that the Tarski grid congruence of a quotient logicoid is the quotient of the Tarski grid congruence of the parent. We also look at *reductions* of logicoids. We show the important results that the reduction of a quotient logicoid coincides with the reduction of its parent and that the reductions of two logicoids related via a bilogical morphism are isomorphic logicoids.

In Section 7.5, the goal is to develop a theory of matrix models for logicoids along the lines of the traditional theory for monotonic logics and the theory developed in Section 3.5 for logicates. We start with a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ over a base algebraic grid $\hat{\mathbf{B}}$. Lack of structurality compels us to consider structures over fixed *interpretations*. These are pairs $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where $\hat{\mathbf{A}}$ is an algebraic grid and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism. A *matrix* is a pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, where $F \subseteq A$. If F is an $\mathbb{L}$-*filter*, i.e., if $h^{-1}(F) \in \mathcal{C}^\flat$, then $\mathfrak{A}$ is called an $\mathbb{L}$-*matrix*. Moreover, $\mathfrak{A}$ is *reduced* if $\Omega_{\mathcal{A}}(F) = \Delta_{\mathbf{A}}$. On any interpretation $\mathcal{A}$, there is, induced by $\mathbb{L}$ and h, a closure operator $C_{\mathcal{A}}$. In case $\mathrm{Ker}(h)$ is a logical grid congruence of $\mathbb{L}$, the induced structure $\mathbb{L}_{\mathcal{A}} = \langle \hat{\mathbf{A}}, C_{\mathcal{A}} \rangle$ is a logicoid and, moreover, the mapping $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ becomes a bilogical morphism $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$. Further, it can be shown that the theories of $\mathbb{L}_{\mathcal{A}}$ coincide with the $\mathbb{L}$-filters of $\mathbb{L}$ on $\mathcal{A}$.

In the remainder of the section we look at ways grid morphisms interact with filters. For instance, we show that, for two interpretations connected by a grid morphism, inverse images of $\mathbb{L}$-filters are $\mathbb{L}$-filters and conversely. Considering quotient interpretations, it is shown that for an $\mathbb{L}$-filter F on $\mathcal{A}$ to be the inverse image under a quotient morphism π_θ of an $\mathbb{L}$-filter on $\mathcal{A}/\theta$ it is necessary and sufficient that θ be compatible with F. Two interpretations that are related by a grid morphism may, under certain circumstances, establish very close ties between corresponding $\mathbb{L}$-filters. The closest connection occurs when the grid morphism in question is a bilogical morphism between the filter structures. It then establishes an isomorphism between the two posets of filters under the corresponding grid orderings. If this happens between two closure structures, one in the source interpretation and another in the target, and the structure in the source interpretation consists of all $\mathbb{L}$-filters, then so does the one in the target. This yields that the $\mathbb{L}$-filters on a reduced interpretation coincide with the reductions of the $\mathbb{L}$-filters on the parent interpretation. At the end of the section, we present an analog of a standard result asserting that a base logicoid is complete with respect to all its matrix models as well as with respect to all its reduced matrix models.

7.2 Logical Congruences

Let $\mathcal{L}$ be a logical (or algebraic) language. That is, $\mathcal{L}$ is a set of connectives (or operation symbols) of finite arities. We consider algebras of type $\mathcal{L}$, or $\mathcal{L}$-algebras, $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$. If, in addition, a complete lattice order $\leq$ on $\mathcal{P}(A)$ is given, then the structure $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ is called an (*algebraic*) *grid*, in analogy with the *grids* $\hat{A} = \langle A, \leq \rangle$ on sets. Both structures were introduced and studied in Chapter 6. Continuing our review from Chapter 6, a *closure operator* on $\hat{\mathbf{A}}$ is a function

$$C : \mathcal{P}(A) \to \mathcal{P}(A),$$

satisfying, for all $X, Y \subseteq A$,

(Inflationarity) $X \leq C(X)$;

(Monotonicity) $X \leq Y$ implies $C(X) \leq C(Y)$;

(Idempotency) $C(C(X)) = C(X)$.

An *algebraic logicoid* is a pair $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$, where:

- $\hat{\mathbf{A}}$ is an algebraic grid;
- C is a closure operator on $\hat{\mathbf{A}}$.

We use $\mathcal{C}$ for the set of its *theories*,

$$\mathcal{C} = \{X \subseteq A : C(X) = X\}.$$

Recall from Proposition 102 that, with the order inherited from the grid, $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$ form a complete lattice, with meets identical with those in the grid.

Let $\theta \in \mathrm{Con}(\mathbf{A})$ be a congruence on the $\mathcal{L}$-algebra $\mathbf{A}$. We call θ *compatible with* a set $X \subseteq A$ if, for all $a, b \in A$,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in X \quad \text{imply} \quad b \in X.$$

Compatibility of θ with X is tantamount to X being a union of θ-congruence classes. We denoted by $\mathrm{Cmp}(\theta)$ the set of all $X \subseteq A$ with which θ is compatible.

Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ be an algebraic grid. A congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is called a *grid congruence* of $\hat{\mathbf{A}}$, written $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ if $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. It was shown in Proposition 120 that $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \mathrm{Con}(\hat{\mathbf{A}}), \subseteq \rangle$ is a complete lattice whose joins coincide with those of $\mathbf{Con}(\mathbf{A})$.

Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ be a grid and $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ an algebraic logicoid on $\hat{\mathbf{A}}$. We say that $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ is a **logical grid congruence of** C, or **of** $\mathbb{L}$, if θ is compatible with every theory of C, i.e.,

$$\mathcal{C} \subseteq \mathrm{Cmp}(\theta).$$

We write $\mathrm{Con}(\mathbb{L})$ for the collection of all logical congruences of $\mathbb{L}$. Moreover, $\mathbf{Con}(\mathbb{L}) = \langle \mathrm{Con}(\mathbb{L}), \subseteq \rangle$ denotes the collection of logical congruences, ordered by the subset relation between congruences. We show that this partially ordered set has a maximum element.

Proposition 129 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *be an algebraic logicoid. Then* $\mathbf{Con}(\mathbb{L})$ *is a principal ideal of* $\mathbf{Con}(\hat{\mathbf{A}})$*.*

Proof: We use elements of the proofs of Lemma 119 and Proposition 120. First note that $\Delta_{\mathbf{A}} \in \mathrm{Con}(\mathbb{L})$, whence $\mathrm{Con}(\mathbb{L}) \neq \varnothing$. Now consider $\theta := \bigvee \mathrm{Con}(\mathbb{L})$. By Proposition 120, $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Moreover, since, for all $\eta \in \mathrm{Con}(\mathbb{L})$, $\mathcal{C} \subseteq \mathrm{Cmp}(\eta)$, we get, using the equivalence in the proof of Lemma 119,

$$\mathcal{C} \subseteq \bigcap_{\eta \in \mathrm{Con}(\mathbb{L})} \mathrm{Cmp}(\eta) = \mathrm{Cmp}\left(\bigvee_{\eta \in \mathrm{Con}(\mathbb{L})} \eta \right) = \mathrm{Cmp}(\theta).$$

Hence, $\theta \in \mathrm{Con}(\mathbb{L})$ and, therefore, θ is the maximum element in $\mathrm{Con}(\mathbb{L})$. ■

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid on the grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$. The **Tarski grid congruence of** $\mathbb{L}$ (Definition 1.1 of [12]) is

$$\widetilde{\Omega}(\mathbb{L}) = \max \mathbf{Con}(\mathbb{L}),$$

that is, $\widetilde{\Omega}(\mathbb{L})$ is the largest logical grid congruence of $\mathbb{L}$. The **Tarski operator on** $\hat{\mathbf{A}}$ is the mapping

$$\widetilde{\Omega}_{\hat{\mathbf{A}}} : \mathbb{L} \longmapsto \widetilde{\Omega}(\mathbb{L}),$$

i.e., it is the mapping $\mathbb{L} \mapsto \widetilde{\Omega}(\mathbb{L})$ restricted to algebraic logicoids over the same underlying grid $\hat{\mathbf{A}}$. This notation can be extended by writing $\widetilde{\Omega}_{\hat{\mathbf{A}}}(C)$ for the Tarski congruence of the algebraic logicate $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$. It follows from the definition of $\widetilde{\Omega}(\mathbb{L})$ that

$$\mathrm{Con}(\mathbb{L}) = \{\theta \in \mathrm{Con}(\hat{\mathbf{A}}) : \theta \subseteq \widetilde{\Omega}(\mathbb{L})\}.$$

To obtain a better understanding of the Tarski congruence, we consider **logical matrices** over $\hat{\mathbf{A}}$, i.e., pairs $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$, where $X \subseteq A$ (see, e.g., Section

1.4 of [3] or Page 16 of [12]). We say that a grid congruence $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ is a **grid congruence of** $\mathfrak{A}$, written $\theta \in \mathrm{Con}(\mathfrak{A}) = \mathrm{Con}(\langle \hat{\mathbf{A}}, X \rangle)$, if θ is compatible with X. Moreover, we use $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}), \subseteq \rangle$ for the corresponding partially ordered set.

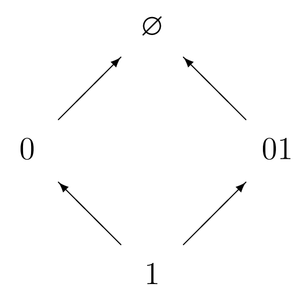
Corollary 130 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *be a grid and* $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$ *a logical matrix over* $\hat{\mathbf{A}}$. *Then* $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}), \subseteq \rangle$ *is a complete lattice and a principal ideal of* $\mathbf{Con}(\mathbf{A})$.

Proof: This is a direct consequence of Proposition 129, since, given $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$, one can construct an algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ whose only theories are X and $M := \max \langle \mathcal{P}(A), \leq \rangle$, i.e., set, for all $Y \subseteq A$,

$$C(Y) = \begin{cases} X, & \text{if } Y \leq X, \\ M, & \text{otherwise.} \end{cases}$$

Note that $M \in \mathrm{Cmp}(\theta)$, for all $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. It follows that $\mathbf{Con}(\mathfrak{A}) = \mathbf{Con}(\mathbb{L})$. ■

The construction in the proof of Corollary 130 is very similar to the traditional one, but the nontraditional grid introduces potential nonmonotonic inferences. E.g., consider the set $A = \{0, 1\}$ and the grid shown below, where we use $0, 1$ and 01 as abbreviations for the subsets $\{0\}, \{1\}$ and $\{0, 1\}$ of $A = \{0, 1\}$, respectively.



Suppose that $X = 01$. Then, the closure operator C defined in the proof is

$$C(0) = C(\varnothing) = \varnothing, \ C(1) = C(01) = 01.$$

Notice the nonmonotonic inference $C(0) = \varnothing$. Notice also that, in the closure system $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$, with the order inherited from the grid (see Chapter 6), the theory $\varnothing$ is greater than the theory 01.

Corollary 130 permits us to define the **Leibniz grid congruence of** $\mathfrak{A}$ or the **Leibniz grid congruence of** X **on** $\hat{\mathbf{A}}$, written $\Omega(\mathfrak{A}) = \Omega_{\hat{\mathbf{A}}}(X)$, as the largest grid congruence on $\hat{\mathbf{A}}$ that is compatible with X. Then, given an algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$, it is clear, by the definition of $\widetilde{\Omega}(\mathbb{L})$, that

$$\widetilde{\Omega}(\mathbb{L}) = \bigcap \{\Omega_{\hat{\mathbf{A}}}(X) : X \in \mathcal{C}\}.$$

A consequence is that the Tarski operator on a grid $\hat{\mathbf{A}}$ is monotone.

Lemma 131 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *be a grid. Then, for all logicoids* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}, C' \rangle$ *on* $\hat{\mathbf{A}}$,

$$\mathbb{L} \leq \mathbb{L}' \quad \textit{implies} \quad \widetilde{\Omega}(\mathbb{L}) \subseteq \widetilde{\Omega}(\mathbb{L}').$$

Proof: We have that

$$\begin{array}{rcll} \mathbb{L} \leq \mathbb{L}' & \text{iff} & \mathcal{C}' \subseteq \mathcal{C} & \text{(Proposition 106)} \\ & \text{implies} & \widetilde{\Omega}(\mathbb{L}) \subseteq \widetilde{\Omega}(\mathbb{L}'), & \end{array}$$

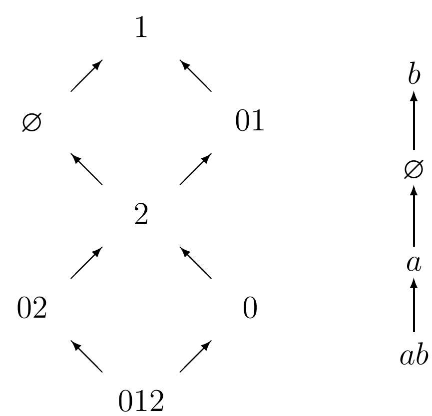
where the last inclusion follows directly from the definition of Tarski congruence. ∎

7.3 Bilogical Morphisms

Given two algebraic logicoids $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ on the grids $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$, respectively, a **logical morphism** $h : \mathbb{L} \to \mathbb{L}'$ is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$, i.e., such that, for all $X' \subseteq A'$,

$$C'(X') = X' \quad \text{implies} \quad C(h^{-1}(X')) = h^{-1}(X').$$

Consider, e.g., the sets $A = \{0, 1, 2\}$ and $A' = \{a, b\}$, which form two grids when the following two orderings are applied on their powersets.



The mapping $h : A \to A'$, with $0 \mapsto a$, $1 \mapsto b$ and $2 \mapsto a$ is a grid morphism as can be checked quickly by hand. We have

$$012 \leq 02 \leq \varnothing \leq 1 \quad \text{iff} \quad ab \leq' a \leq' \varnothing \leq' b.$$

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ be the two logicoids determined by

$$\mathcal{C} = \{\{0, 2\}, \{1\}\}, \quad \mathcal{C}' = \{\{a\}, \{b\}\}.$$

Then $h : \mathbb{L} \to \mathbb{L}'$ becomes a logical morphism.

We say that $\mathbb{L}$ is **projectively generated from $\mathbb{L}'$ by h** if

$$\mathcal{C} = h^{-1}(\mathcal{C}').$$

Finally, h is a **bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$**, or a **bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$**, written $h : \mathbb{L} \to_b \mathbb{L}'$, if it is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ and it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. For these definitions, as applied to the traditional setting, see Page 20 of [12] (also [4] for the original notions). The following proposition is an analog for logicoids of Proposition 1.4 of [12]. Note that, contrary to the case for logicates (see Chapter 3), we can recover the six equivalent conditions, due to the presence of grids and grid morphisms.

Proposition 132 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ be algebraic logicoids and $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ a grid morphism. The following conditions are equivalent.*

(i) h *is a bilogical morphism from* $\mathbb{L}$ *onto* $\mathbb{L}'$;

(ii) *For all* $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$, $C(X) = h^{-1}(C'(h(X)))$;

(iii) *For all* $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$, $h(C(X)) = C'(h(X))$ *and* $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$;

(iv) *For all* $Y \subseteq A'$, $C'(Y) = h(C(h^{-1}(Y)))$ *and* $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$;

(v) $\mathcal{C}' = h(\mathcal{C})$ *and* $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$;

(vi) $\mathcal{C} = h^{-1}(\mathcal{C}')$.

Proof:

(i)⇒(ii) Suppose that $h : \mathbb{L} \to_b \mathbb{L}'$ and $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$. Since C' is $\leq$-inflationary, $h(X) \leq' C'(h(X))$. Since h^{-1} is a complete lattice embedding, $X \leq h^{-1}(C'(h(X)))$. By $\leq$-monotonicity of C and the hypothesis, $C(X) \leq h^{-1}(C'(h(X)))$. Conversely, by $\leq$-inflationarity of C, $X \leq C(X)$. By the fact that h^{-1} is a complete lattice embedding, $h(X) \leq' h(C(X))$. By hypothesis and the $\leq'$-monotonicity of C', $C'(h(X)) \leq' h(C(X))$. Since h^{-1} is a complete lattice embedding, $h^{-1}(C'(h(X))) \leq C(X)$. We have shown that, for all $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$, $C(X) = h^{-1}(C'(h(X)))$.

(ii)⇒(iii) Since h is surjective, the first equality follows immediately from the hypothesis. The second statement also follows from the hypothesis and Lemma 117.

(iii)⇒(iv) Let $Y \subseteq A'$. Then, there exists $X = h^{-1}(Y) \in \mathrm{Cmp}(\mathrm{Ker}(h))$, such that $Y = h(X)$. Thus, by assumption,

$$C'(Y) = C'(h(X)) = h(C(X)) = h(C(h^{-1}(Y))).$$

(iv)⇒(v) Both statements follow from the hypothesis.

(v)⇒(vi) Since $\mathcal{C}' = h(\mathcal{C})$ and $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$, we get $h^{-1}(\mathcal{C}') = \mathcal{C}$.

(iv)⇒(i) By the definition of bilogical morphism.

■

Proposition 132 yields immediately the following results revealing a very tight relation between theories of two algebraic logicoids related via a bilogical morphism. We start with an analog of Proposition 1.5 of [12].

Proposition 133 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ be two algebraic logicoids and $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ be a grid morphism. Then $h : \mathbb{L} \to_b \mathbb{L}'$ if and only if the closure systems $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq\rangle$ and $\hat{\mathcal{C}} = \langle \mathcal{C}', \leq\rangle$ are isomorphic via the mapping induced by h.*

Proof: By Proposition 132,

$$h: \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}'; \\ X & \longmapsto & h(X), \end{array}$$

is a bijection. By the definition of grid morphisms, ir preserves and reflects order. Thus, $h : \hat{\mathcal{C}} \to \hat{\mathcal{C}}'$ is an isomorphism. Conversely, the hypothesis yields that $\mathcal{C} = h^{-1}(\mathcal{C}')$. Thus, by Proposition 132, $h : \mathbb{L} \to_b \mathbb{L}'$. ■

We now show that logical congruence systems correspond under the action of inverse bilogical morphisms. Let us write

$$\mathrm{Con}(\mathbb{L})^\eta := \{\theta \in \mathrm{Con}(\mathbb{L}) : \eta \subseteq \theta\}.$$

Proposition 134 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ be algebraic logicoids and $h : \mathbb{L} \to_b \mathbb{L}'$. Then*

$$\mathrm{Con}(\mathbb{L})^{\mathrm{Ker}(h)} = h^{-1}(\mathrm{Con}(\mathbb{L}')).$$

Proof: We consider the mapping

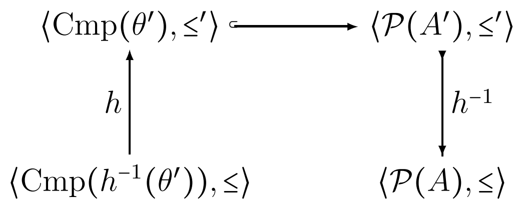
$$\mathrm{Con}(\mathbb{L})^{\mathrm{Ker}(h)} \underset{h^{-1}}{\overset{h}{\rightleftarrows}} \mathrm{Con}(\mathbb{L}')$$

Note that, if $\theta' \in \mathrm{Con}(\mathbf{A}')$, then $h^{-1}(\theta') \in \mathrm{Con}(\mathbf{A})$ and, clearly, since θ' is reflexive, $\mathrm{Ker}(h) \subseteq h^{-1}(\theta')$. Conversely, if $\theta \in \mathrm{Con}(\mathbf{A})$, such that $\mathrm{Ker}(h) \subseteq \theta$, it is not difficult to see that $h(\theta) \in \mathrm{Con}(\mathbf{A}')$. The hypothesis $\mathrm{Ker}(h) \subseteq \theta$ is used in showing the transitivity property of $h(\theta)$.

Next, we use the fact that $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ is a grid morphism to show that the mappings are well defined between grid congruences. If $X \subseteq A$ and $\theta' \in \mathrm{Con}(\mathbf{A}')$, we have

$$X \in \mathrm{Cmp}(h^{-1}(\theta')) \quad \text{iff} \quad h(X) \in \mathrm{Cmp}(\theta').$$

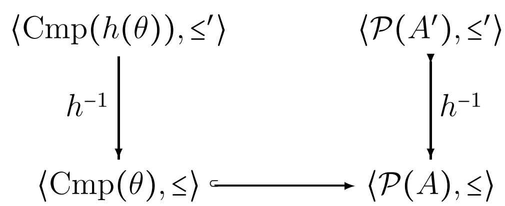
So the fact that, if $\theta' \in \mathrm{Con}(\hat{\mathbf{A}}')$, then $h^{-1}(\theta) \in \mathrm{Con}(\hat{\mathbf{A}})$ is established by looking at the following diagram.



A consequence of the preceding equivalence is that, if $X' \subseteq A'$ and $\theta \in \mathrm{Con}(\mathbf{A})$, such that $\mathrm{Ker}(h) \subseteq \theta$, then

$$X' \in \mathrm{Cmp}(h(\theta)) \quad \text{iff} \quad h^{-1}(X') \in \mathrm{Cmp}(\theta).$$

So the fact that, if $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$, with $\mathrm{Ker}(h) \subseteq \theta$, then $h(\theta) \in \mathrm{Con}(\hat{\mathbf{A}}')$ is established by looking at the following diagram.



Finally, since $h : \mathbb{L} \to_b \mathbb{L}'$ is a bilogical morphism, the preceding mappings are also well defined between logical congruences. ■

Proposition 135 *Let* $\mathbb{L} = \langle \mathbf{A}, C \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', C' \rangle$ *be algebraic logicates and* $h : \mathbb{L} \to_b \mathbb{L}'$. *Then*

$$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}')).$$

Proof: This is a consequence of Proposition 134, the fact that $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ implies that h^{-1} is order preserving and order reflecting and the definition of Tarski congruence. ■

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ be two algebraic logicoids. A bijection $h : A \to A'$ is an **isomorphism** between $\mathbb{L}$ and $\mathbb{L}'$, written $h : \mathbb{L} \cong \mathbb{L}'$, if both $h : \mathbb{L} \to \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to \mathbb{L}$ are logical morphisms.

Lemma 136 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ *be two algebraic logicoids and* $h : A \to A'$ *a bijection.* $h : \mathbb{L} \cong \mathbb{L}'$ *if and only if* $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ *and* $h : \mathbb{L} \to_b \mathbb{L}'$.

Proof: To establish the equivalence, it suffices to show that

$$(h^{-1}(\mathcal{C}') \subseteq \mathcal{C} \quad \text{and} \quad h(\mathcal{C}) \subseteq \mathcal{C}') \quad \text{iff} \quad \mathcal{C} = h^{-1}(\mathcal{C}').$$

Suppose the conjunction in the parenthesis holds and let $X \in \mathcal{C}$. Then, by hypothesis, $X' = h(X) \in \mathcal{C}'$. Thus, $X = h^{-1}(h(X)) \in h^{-1}(\mathcal{C}')$. This proves that $\mathcal{C} \subseteq h^{-1}(\mathcal{C}')$. Since the reverse inclusion is part of the hypothesis, the conclusion follows.

Suppose, conversely, that $\mathcal{C} = h^{-1}(\mathcal{C}')$. Then, clearly, $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$. So, suppose, $X \in \mathcal{C}$. By hypothesis, there exists $X' \in \mathcal{C}'$, such that $X = h^{-1}(X')$. Thus, $h(X) = h(h^{-1}(X')) = X' \in \mathcal{C}'$. This proves that $h(\mathcal{C}) \subseteq \mathcal{C}'$. ∎

7.4 Quotients

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid on a grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and suppose $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Consider the quotient grid $\hat{\mathbf{A}}/\theta = \langle \mathbf{A}/\theta, \leq^\theta \rangle$ and define a mapping

$$C^\theta := C/\theta : \mathcal{P}(A/\theta) \to \mathcal{P}(A/\theta)$$

by setting, for all $S \subseteq A/\theta$,

$$C^\theta(S) = \pi_\theta(C(\pi_\theta^{-1}(S))),$$

where $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ is the quotient grid morphism.

We show that, if θ happens to be a logical congruence, then the quotient is a legitimate logicoid on the quotient grid. For the corresponding result in the traditional framework, see Page 21 of [12]. For the one related to logicates, see Proposition 23.

Proposition 137 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid and suppose $\theta \in \mathrm{Con}(\mathbb{L})$. Then*

$$\mathbb{L}^\theta = \mathbb{L}/\theta := \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$$

is also an algebraic logicoid.

Proof: Since $\theta \in \mathrm{Con}(\mathbb{L})$, a fortiori $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Hence, by Lemma 122, $\hat{\mathbf{A}}/\theta$ is a legitimate grid and $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ becomes a grid morphism. Let $Y, Y' \subseteq \mathcal{P}(A/\theta)$.

- Since $C^\theta(Y) = \pi_\theta(C(\pi_\theta^{-1}(Y)))$, we get

  $$\pi_\theta^{-1}(Y) \leq C(\pi_\theta^{-1}(Y)) = \pi_\theta^{-1}(C^\theta(Y)).$$

  Hence, by definition, $Y \leq^\theta C^\theta(Y)$.

- Suppose, next, that $Y \leq^\theta Y'$. By definition of $\leq^\theta$, $\pi_\theta^{-1}(Y) \leq \pi_\theta^{-1}(Y')$. By monotonicity of C on $\hat{\mathbf{A}}$, $C(\pi_\theta^{-1}(Y)) \leq C(\pi_\theta^{-1}(Y'))$. By definition of $\leq^\theta$ and the fact that $\theta \in \mathrm{Con}(\mathbb{L})$, $\pi_\theta(C(\pi_\theta^{-1}(Y))) \leq^\theta \pi_\theta(C(\pi_\theta^{-1}(Y)))$. Thus, by definition of C^θ, $C^\theta(Y) \leq^\theta C^\theta(Y')$.

- Finally, we have

$$\begin{aligned} C^\theta(C^\theta(S)) &= \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))))) \quad \text{(Definition of } C^\theta) \\ &= \pi_\theta(C(C(\pi_\theta^{-1}(S)))) \quad (\theta \in \mathrm{Con}(\mathbb{L})) \\ &= \pi_\theta(C(\pi_\theta^{-1}(S))) \quad \text{(Idempotency)} \\ &= C^\theta(S). \quad \text{(Definition of } C^\theta) \end{aligned}$$

Thus, C^θ is inflationary, monotone and idempotent with respect to $\leq^\theta$, showing that $\mathbb{L}^\theta$ is an algebraic logicoid on $\hat{\mathbf{A}}^\theta$. ■

We call $\mathbb{L}^\theta = \mathbb{L}/\theta := \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$ the **quotient logicoid** of $\mathbb{L}$ by the logical grid congruence θ.

Now that we know that $\mathbb{L}$ and its quotient $\mathbb{L}^\theta$ are logicoids, we show that the quotient grid morphism $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ becomes a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}/\theta$.

Proposition 138 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid and suppose $\theta \in \mathrm{Con}(\mathbb{L})$. Then $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ is a bilogical morphism.*

Proof: By Lemma 122, $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ is a grid morphism. So it suffices to show that it projectively generates $\mathbb{L}$ from $\mathbb{L}^\theta$. Suppose, first $X \in \mathcal{C}$. Then

$$\begin{aligned} \pi_\theta^{-1}(C^\theta(\pi_\theta(X))) &= \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X))))) \quad \text{(Definition of } C^\theta) \\ &= C(X) \quad (\theta \in \mathrm{Con}(\mathbb{L})) \\ &= X. \quad (X \in \mathcal{C}) \end{aligned}$$

This shows that $\mathcal{C} \subseteq \pi_\theta^{-1}(\mathcal{C}^\theta)$. Conversely, let $S \in \mathcal{C}^\theta$. Then

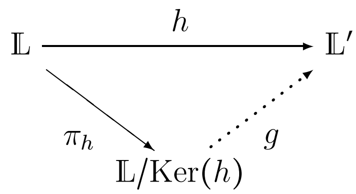
$$\begin{aligned} \pi_\theta^{-1}(S) &= \pi_\theta^{-1}(C^\theta(S)) \quad (S \in \mathcal{C}^\theta) \\ &= \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))) \quad \text{(Definition of } C^\theta) \\ &= C(\pi_\theta^{-1}(S)). \quad (\theta \in \mathrm{Con}(\mathbb{L})) \end{aligned}$$

Hence $\pi_\theta^{-1}(\mathcal{C}^\theta) \subseteq \mathcal{C}$. Thus, $\mathcal{C} = \pi_\theta^{-1}(\mathcal{C}^\theta)$, showing that π_θ projectively generates $\mathbb{L}$ from $\mathbb{L}/\theta$. So $\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta$ is a bilogical morphism. ■

$\pi_\theta : \mathbb{L} \to_b \mathbb{L}/\theta$ is called the **quotient (bilogical) morphism** or the **canonical projection (bilogical) morphism**.

Theorems 1.8, 1.9 and 1.10 of [12] adapt to the framework of abstract logics the well-known Homomorphism Theorems of universal algebra [5, 18, 1]. Theorems 25, 26 and 27 further reformulated those results to fit in the framework of logicates. We undertake here a similar adaptation for logicoids. The flavor is still the same.

Theorem 139 (Homomorphism) *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ *be two algebraic logicoids and* $h : \mathbb{L} \to_b \mathbb{L}'$. *Then* $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$ *via a unique isomorphism* g, *such that* $h = g \circ \pi_h$,



where $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ *is the bilogical projection morphism.*

Proof: By hypothesis, h is a bilogical morphism. Thus, by Proposition 132, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. It follows, by Proposition 138, that $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ is a bilogical morphism. By Theorem 125, there exists a unique $g : \hat{\mathbf{A}}/\theta \cong \hat{\mathbf{A}}'$, such that $h = g \circ \pi_h$. So it suffices to show that this is a bilogical morphism. We have

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{Ker}(h)} & = & \pi_h(\mathcal{C}) \quad (\pi_h : \mathbb{L} \to_b \mathbb{L}/\mathrm{Ker}(h)) \\ & = & g^{-1}(h(\mathcal{C})) \quad (h = g \circ \pi_h \text{ and } g : \hat{\mathbf{A}}/\theta \cong \hat{\mathbf{A}}') \\ & = & g^{-1}(\mathcal{C}'). \quad (h : \mathbb{L} \to_b \mathbb{L}') \end{array}$$

Thus, by definition, $g : \mathbb{L}/\mathrm{Ker}(h) \to \mathbb{L}'$ is a bilogical morphism. ■

Theorem 140 (Second Isomorphism) *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ *be an algebraic logicoid and* $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, *such that* $\theta \subseteq \theta'$. *Then* $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$ *and*

$$(\mathbb{L}/\theta)/(\theta'/\theta) \cong \mathbb{L}/\theta',$$

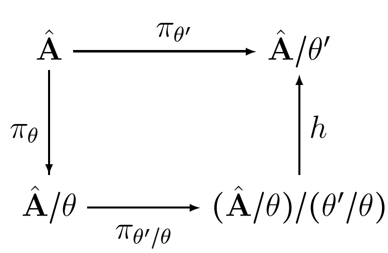
where the isomorphism is given by

$$(a/\theta)/(\theta'/\theta) \mapsto a/\theta'.$$

Proof: By Theorem 126, we know that

$$\begin{array}{rrcl} h : & (\hat{\mathbf{A}}/\theta)/(\theta'/\theta) & \longrightarrow & \hat{\mathbf{A}}/\theta'; \\ & (a/\theta)/(\theta'/\theta) & \longmapsto & a/\theta', \end{array}$$

is an isomorphism that makes the following rectangle commute,



where $\pi_\theta, \pi_{\theta'}$ and $\pi_{\theta'/\theta}$ are the natural projections. Since $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, by Proposition 138, π_θ and $\pi_{\theta'}$ are bilogical morphisms. We can also show that $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$. Let $a, b \in A$ and $S \subseteq A/\theta$, such that $\langle a/\theta, b/\theta\rangle \in \theta'/\theta$ and $a/\theta \in C^\theta(S)$. Then, by the Definition of C^θ, $\langle a, b\rangle \in \theta'$ and $a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$. Hence, since $\theta \in \mathrm{Con}(\mathbb{L})$, $\langle a, b\rangle \in \theta'$ and $a \in C(\pi_\theta^{-1}(S))$. Since $\theta' \in \mathrm{Con}(\mathbb{L})$, this yields $b \in C(\pi_\theta^{-1}(S))$, whence $b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$, i.e., $b/\theta \in C^\theta(S)$. Hence $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. Now, again by Proposition 138, the projection $\pi_{\theta'/\theta}$ is also a bilogical morphism. We check that h, which is already known to be a grid isomorphism, is also a bilogical morphism. We have

$$\begin{array}{rcll}
(\mathcal{C}^\theta)^{\theta'/\theta} & = & \pi_{\theta'/\theta}(\mathcal{C}^\theta) & (\pi_{\theta'/\theta} : \mathbb{L}^\theta \to_b (\mathbb{L}^\theta)^{\theta'/\theta}) \\
& = & \pi_{\theta'/\theta}(\pi_\theta(\mathcal{C})) & (\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta) \\
& = & h^{-1}(\pi_{\theta'}(\mathcal{C})) & (h \circ \pi_{\theta'/\theta} \circ \pi_\theta = \pi_{\theta'} \text{ and } h \text{ an iso}) \\
& = & h^{-1}(\mathcal{C}^{\theta'}). & (\pi_{\theta'} : \mathbb{L} \to_b \mathbb{L}^{\theta'})
\end{array}$$

Thus, $h : (\mathbb{L}/\theta)/(\theta'/\theta) \to \mathbb{L}/\theta'$ is indeed a bilogical morphism and, hence, an isomorphism. ■

To formulate an analog of the Correspondence Theorem (see Theorem 1.10 of [12] for abstract logics and Theorem 27 for logicates), recall that $\widetilde{\Omega}(\mathbb{L})$ denotes the Tarski congruence of $\mathbb{L}$, i.e., the largest grid congruence on $\hat{\mathbf{A}}$ that is compatible with all theories of $\mathbb{L}$.

Theorem 141 (Correspondence) *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ be an algebraic logicoid and $\theta \in \mathrm{Con}(\mathbb{L})$. Then the segment $[\theta, \widetilde{\Omega}(\mathbb{L})]$ of the lattice $\mathbf{Con}(\hat{\mathbf{A}})$ is isomorphic to the lattice $\mathbf{Con}(\mathbb{L}^\theta)$ by the mapping $\theta' \mapsto \theta'/\theta$.*

Proof: By Theorem 140, if $\theta \subseteq \theta' \in \mathrm{Con}(\mathbb{L})$, then $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. By Theorem 126, it suffices to prove that, for all $\theta \subseteq \theta' \in \mathrm{Con}(\hat{\mathbf{A}})$, if $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$, then $\theta' \in \mathrm{Con}(\mathbb{L})$. So let $a, b \in A$ and $X \in \mathcal{C}$, such that $\langle a, b\rangle \in \theta'$ and $a \in X$. As $\theta \in \mathrm{Con}(\mathbb{L})$, θ is compatible with X. Thus X is the union of θ-classes, that is $X = \pi_\theta^{-1}(\pi_\theta(X))$. Now, starting with the assumption, we get

$$\begin{array}{ll}
\langle a, b\rangle \in \theta' \text{ and } a \in X & \\
\quad \text{iff} & \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X)))) \\
\quad \text{iff} & \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in C^\theta(\pi_\theta(X)) \\
\quad \text{implies} & b/\theta \in C^\theta(\pi_\theta(X)) \\
\quad \text{iff} & b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X)))) \\
\quad \text{iff} & b \in X.
\end{array}$$

Therefore, $\theta' \in \mathrm{Con}(\mathbb{L})$ and the correspondence is established. ■

The Correspondence Theorem has a significant consequence in relation to the Tarski congruences.

Corollary 142 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ be an algebraic logicoid and $\theta \in \mathrm{Con}(\mathbb{L})$. Then*

$$\widetilde{\Omega}(\mathbb{L}^\theta) = \widetilde{\Omega}(\mathbb{L})/\theta.$$

Proof: By definition, the largest element in $\mathrm{Con}(\mathbb{L}^\theta)$ is $\widetilde{\Omega}(\mathbb{L}^\theta)$, whereas the largest element in $[\theta, \widetilde{\Omega}(\mathbb{L})]$ is clearly $\widetilde{\Omega}(\mathbb{L})$. Since, under the established correspondence of Theorem 141, these two elements correspond, we get the conclusion. ∎

It follows that, for any algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$,

$$\widetilde{\Omega}(\mathbb{L}/\widetilde{\Omega}(\mathbb{L})) = \widetilde{\Omega}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}/\widetilde{\Omega}(\mathbb{L})}.$$

This leads us to the definition of reduction (see Definition 1.12 of [12]). We say that an algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ is **reduced** when it has only one logical congruence, i.e., when $\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}}$. Given an algebraic logicoid $\mathbb{L}$, we define the **reduction** $\mathbb{L}^*$ **of** $\mathbb{L}$ by

$$\mathbb{L}^* = \mathbb{L}/\widetilde{\Omega}(\mathbb{L}).$$

If L is a class of algebraic logicoids, then we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

If $\mathbb{L}$ is an algebraic logicoid, then $\mathbb{L}^*$ is always reduced. Moreover, if $\mathbb{L}$ happens to already be reduced, then $\mathbb{L}$ and $\mathbb{L}^*$ are isomorphic and they may be identified.

We prove, next, some analogs of Propositions 1.13 and 1.14 of [12] (see, also, Propositions 29 and 30 for the case of logicates). The first asserts that the reduction of a quotient of a logicoid by a logical morphism is isomorphic to the reduction of the logicoid itself. The second proves that the reductions of two logicoids related via a bilogical morphism are isomorphic.

Proposition 143 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid and $\theta \in \mathrm{Con}(\mathbb{L})$. Then*

$$(\mathbb{L}^\theta)^* \cong \mathbb{L}^*.$$

Proof: We have

$$\begin{array}{rcl}(\mathbb{L}^\theta)^* & = & \mathbb{L}^\theta/\widetilde{\Omega}(\mathbb{L}^\theta) \quad \text{(Definition of Reduction)}\\ & = & \mathbb{L}^\theta/(\widetilde{\Omega}(\mathbb{L})/\theta) \quad \text{(Corollary 142)}\\ & \cong & \mathbb{L}/\widetilde{\Omega}(\mathbb{L}) \quad \text{(Theorem 140)}\\ & = & \mathbb{L}^*. \quad \text{(Definition of Reduction)}\end{array}$$

The conclusion follows. ∎

Proposition 144 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ be algebraic logicoids and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical morphism. Then*

$$\mathbb{L}^* \cong \mathbb{L}'^*.$$

Proof: By Theorem 139, $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$. By Proposition 135,

$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}'^*.$$

Since, by Proposition 132, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$, by Proposition 143,

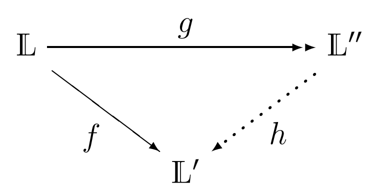
$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}^*.$$

Therefore, $\mathbb{L}'^* \cong \mathbb{L}^*$. ∎

We close the section with an analog of Proposition 1.15 of [12] (Proposition 31 for logicates), a sort of a "fill-in" theorem for arrows.

Proposition 145 *Let $\mathbb{L}$, $\mathbb{L}'$ and $\mathbb{L}''$ be algebraic logicoids, $f : \mathbb{L} \to \mathbb{L}'$ a logical morphism and $g : \mathbb{L} \to \mathbb{L}''$ a bilogical morphism, such that $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. Then, there is a unique logical morphism $h : \mathbb{L}'' \to \mathbb{L}'$, such that*

$$h \circ g = f.$$



Moreover, f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ if and only if h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$.

Proof: By Lemma 128, there exists a unique $h : \hat{\mathbf{A}}'' \to \hat{\mathbf{A}}'$, such that $h \circ g = f$. Now we have

$$\begin{array}{rcl} g^{-1}(h^{-1}(\mathcal{C}')) & = & f^{-1}(\mathcal{C}') \quad (f = h \circ g) \\ & \subseteq & \mathcal{C} \quad (f : \mathbb{L} \to \mathbb{L}') \\ & = & g^{-1}(\mathcal{C}''). \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

We now get $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}''$. Thus, $h : \mathbb{L} \to \mathbb{L}'$ is a logical morphism. That it projectively generates $\mathbb{L}''$ from $\mathbb{L}'$ if f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ follows from the fact that, in that case, the inclusion becomes an equality. Conversely, assume that h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. Then we have

$$\begin{array}{rcl} f^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) \quad (f = h \circ g) \\ & = & g^{-1}(\mathcal{C}'') \quad (\text{Assumption}) \\ & = & \mathcal{C}. \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

So f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. ∎

7.5 Interpretations, Filters and Matrices

In this section, taking after the theory of logical matrices (see, e.g., [24, 3, 12, 8]), we present a similar theory suitable for algebraic logicoids along the lines of the theory developed for logicates in Section 3.5. Here, also, because of lack of structurality, one has to fix interpretations, i.e., grid morphisms that help interpret the underlying grid of the logicoid. A model theory along similar lines was devised for π-institutions in [21].

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, with $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$, be an algebraic logicoid. This is thought of as the focal object of our study, for which models are to be devised. So it is referred to as a **base (algebraic) logicoid** and its underlying algebraic grid $\hat{\mathbf{B}}$ as the **base (algebraic) grid**. The most appropriate notion of **(grid) interpretation** is that of a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where:

- $\hat{\mathbf{A}}$ is a grid algebra of the same type as the base grid $\hat{\mathbf{B}}$;
- $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism.

We say that $F \subseteq A$ is an $\mathbb{L}$-**filter** on $\hat{\mathcal{A}}$, if

$$h^{-1}(F) \in \mathcal{C}^\flat,$$

i.e., the inverse image under h of the $\mathbb{L}$-filter is a theory of the logicoid. By $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is denoted the collection of all $\mathbb{L}$-filters on the grid interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. If F is an $\mathbb{L}$-filter on $\mathcal{A}$, the pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is called a **(grid) matrix for** $\mathbb{L}$ or a **(grid)** $\mathbb{L}$-**matrix**. The class of all grid $\mathbb{L}$-matrices is denoted $\mathrm{Mat}(\mathbb{L})$.

A grid congruence $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ is called a **(grid) matrix congruence** of $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ if $F \in \mathrm{Cmp}(\theta)$, i.e., if θ is compatible with F. The **Leibniz grid congruence of** $\mathfrak{A}$ or the **Leibniz grid congruence of** F **on** $\mathcal{A}$, denoted $\Omega(\mathfrak{A})$ or $\Omega_{\mathcal{A}}(F)$ is the largest grid matrix congruence of $\mathfrak{A}$, provided such a congruence exists. An grid $\mathbb{L}$-matrix $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is **reduced** if $\Omega_{\mathcal{A}}(F) = \Delta_{\mathbf{A}}$. The class of all reduced grid $\mathbb{L}$-matrices is denoted $\mathrm{Mat}^*(\mathbb{L})$.

Among the most important features of interpretations is that, if the kernel of their interpretation morphism is a grid congruence of the base logicoid, then they induce an algebraic logicoid on the algebra into which the interpretation takes place. Moreover, if this is the case, the mapping of the interpretation becomes a bilogical morphism from the original logicoid into the induced logicoid. This is similar to the situation encountered in the case of logicates (Proposition 32).

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. Define $C_{\mathcal{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $Y \subseteq A$,

$$C_{\mathcal{A}}(Y) = h(C^\flat(h^{-1}(Y))).$$

Proposition 146 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, *with* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, *an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. *Then* $\mathbb{L}_\mathcal{A} = \langle \hat{\mathbf{A}}, C_\mathcal{A} \rangle$ *is an algebraic logicoid. Moreover, the grid morphism* $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ *is a bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}_\mathcal{A}$.

Proof: To see that $\mathbb{L}_\mathcal{A}$ is a logicoid, we must show inflationarity, monotonicity and idempotence of $C_\mathcal{A}$ with respect to $\leq$. Let $Y \subseteq A$. Since $C^\flat$ is inflationary, $h^{-1}(Y) \leq^\flat C^\flat(h^{-1}(Y))$. Since h^{-1} is a complete lattice embedding, $Y \leq h(C^\flat(h^{-1}(Y)))$. Thus, by definition of $C_\mathcal{A}$, $Y \leq C_\mathcal{A}(Y)$ and $C_\mathcal{A}$ is inflationary.

Next, let $Y, Y' \subseteq A$, such that $Y \leq Y'$. Then, since h^{-1} is a complete lattice embedding, $h^{-1}(Y) \leq^\flat h^{-1}(Y')$. By monotonicity of $C^\flat$ with respect to $\leq^\flat$, we get $C^\flat(h^{-1}(Y)) \leq^\flat C^\flat(h^{-1}(Y'))$. Therefore, again by the embedding property of h^{-1}, $h(C^\flat(h^{-1}(Y))) \leq h(C^\flat(h^{-1}(Y')))$, which, by definition of $C_\mathcal{A}$, amounts to $C_\mathcal{A}(Y) \leq C_\mathcal{A}(Y')$, showing that $C_\mathcal{A}$ is also monotone.

Finally, suppose $Y \subseteq A$. Then

$$\begin{array}{rcll} C_\mathcal{A}(C_\mathcal{A}(Y)) & = & h(C^\flat(h^{-1}(h(C^\flat(h^{-1}(Y)))))) & \text{(Definition of } C_\mathcal{A}) \\ & = & h(C^\flat(C^\flat(h^{-1}(Y)))) & (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ & = & h(C^\flat(h^{-1}(Y))) & (\mathbb{L} \text{ a logicoid}) \\ & = & C_\mathcal{A}(Y). & \text{(Definition of } C_\mathcal{A}) \end{array}$$

Thus, $C_\mathcal{A}$ is a closure operator with respect to $\leq$ and, hence, $\mathbb{L}_\mathcal{A}$ is a logicoid.

To see that h becomes a logical morphism, we must show that $h^{-1}(\mathcal{C}_\mathcal{A}) \subseteq \mathcal{C}^\flat$. So let $Y \in \mathcal{C}_\mathcal{A}$. Then, we have

$$\begin{array}{rcll} h^{-1}(Y) & = & h^{-1}(C_\mathcal{A}(Y)) & (Y \in \mathcal{C}_\mathcal{A}) \\ & = & h^{-1}(h(C^\flat(h^{-1}(Y)))) & \text{(Definition of } C_\mathcal{A}) \\ & = & C^\flat(h^{-1}(Y)) & (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ & \in & \mathcal{C}^\flat. & \text{(Definition of } \mathcal{C}^\flat) \end{array}$$

To see that it is a bilogical morphism, let $X \in \mathcal{C}^\flat$. Then we have

$$\begin{array}{rcll} h^{-1}(C_\mathcal{A}(h(X))) & = & h^{-1}(h(C^\flat(h^{-1}(h(X))))) & \text{(Definition of } C_\mathcal{A}) \\ & = & C^\flat(X) & (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ & = & X. & (X \in \mathcal{C}^\flat) \end{array}$$

We conclude that $\mathcal{C}^\flat = h^{-1}(\mathcal{C}_\mathcal{A})$ and, therefore, $h : \mathbb{L} \to \mathbb{L}_\mathcal{A}$ is a bilogical morphism. ■

We call $\mathbb{L}_\mathcal{A} = \langle \mathcal{A}, C_\mathcal{A} \rangle$ the **logicoid induced on** $\mathcal{A}$ **by** $\mathbb{L}$.

An additional property of these algebraic logicoids is that the theories of the algebraic logicoid coincide with the $\mathbb{L}$-filters on the underlying interpretation.

Proposition 147 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, *with* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, *an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. *Then*

$$\mathcal{C}_\mathcal{A} = \mathrm{Fi}_\mathbb{L}(\mathcal{A}).$$

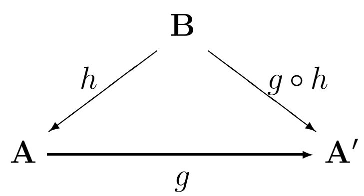
Proof: We have

$$\begin{array}{rcll}\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & = & \{Y \subseteq A : h^{-1}(Y) \in \mathcal{C}^\flat\} & \text{(Definition of an } \mathbb{L}\text{-filter)}\\ & = & \{Y \subseteq A : h(h^{-1}(Y)) \in \mathcal{C}_{\mathcal{A}}\} & \text{(Proposition 146)}\\ & = & \{Y \subseteq A : Y \in \mathcal{C}_{\mathcal{A}}\} & (h \text{ Surjective})\\ & = & \mathcal{C}_{\mathcal{A}}. & \end{array}$$

So the displayed equality in the statements holds. ■

We embark, next, on a series of results that clarify the interaction between filterhood and morphisms and, in particular, between filters and quotients. The next proposition shows that, for two interpretations, one of which results from the other by composition with a grid morphism, inverse images of filters are filters and conversely (see Proposition 34 for logicates).

Proposition 148 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$ be algebraic grids and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ and $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ grid morphisms. Setting $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h \rangle$, we have,*



for all $G \subseteq A'$, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ if and only if $g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: We have

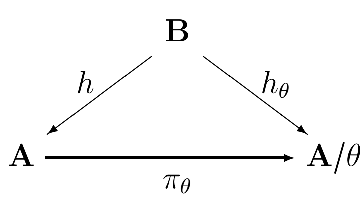
$$\begin{array}{rll}G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') & \text{iff} \quad (g \circ h)^{-1}(G) \in \mathcal{C}^\flat & \text{(Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'))\\ & \text{iff} \quad h^{-1}(g^{-1}(G)) \in \mathcal{C}^\flat & ((g \circ h)^{-1} = h^{-1} \circ g^{-1})\\ & \text{iff} \quad g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}). & \text{(Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))\end{array}$$

■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Then we set

$$\mathcal{A}^\theta = \mathcal{A}/\theta = \langle \hat{\mathbf{A}}/\theta, h_\theta \rangle,$$

where, $h_\theta : \hat{\mathbf{B}} \to \hat{\mathbf{A}}/\theta$ is defined by



$$h_\theta := \pi_\theta \circ h,$$

with $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ the quotient grid morphism.

We next show that a necessary and sufficient condition for an $\mathbb{L}$-filter F on $\mathcal{A}$ to be the inverse image under the quotient mapping of an $\mathbb{L}$-filter on the quotient $\mathcal{A}/\theta$ is that θ be compatible with F (see Proposition 35 for logicates).

Proposition 149 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Then θ is compatible with F if and only if $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$.*

Proof: Suppose, first, that θ is compatible with F. Set $G = \pi_\theta(F)$. Then, we have

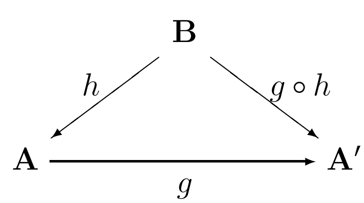
$$\begin{aligned} h_\theta^{-1}(G) &= (\pi_\theta \circ h)^{-1}(\pi_\theta(F)) \quad (h_\theta := \pi_\theta \circ h) \\ &= h^{-1}(\pi_\theta^{-1}(\pi_\theta(F))) \quad ((\pi_\theta \circ h)^{-1} = h^{-1} \circ \pi_\theta^{-1}) \\ &= h^{-1}(F). \quad \text{(Compatibility)} \end{aligned}$$

Since $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $h^{-1}(F) \in \mathcal{C}^\flat$ and, thus, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Moreover, by compatibility, $F = \pi_\theta^{-1}(\pi_\theta(F)) = \pi_\theta^{-1}(G)$.

Suppose, conversely, that $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Let $a, b \in A$, such that $\langle a, b \rangle \in \theta$ and $a \in F$. Then $a \in \pi_\theta^{-1}(G)$, whence $a/\theta \in G$. So $b/\theta = a/\theta \in G$. This gives $b \in \pi_\theta^{-1}(G) = F$. So θ is compatible with F. ■

Next, we investigate conditions under which the $\mathbb{L}$-filters on two interpretations that are related via a grid morphism are in correspondence. This result forms an analog for logicoids of Proposition 36.

Proposition 150 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation and $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ a grid morphism. Set $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h \rangle$.*



The following statements are equivalent:

(i) $g : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$, *with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, *is a bilogical morphism;*

(ii) For all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ *and* $\mathrm{Ker}(g) \in \mathrm{Con}(\langle \mathcal{A}, C \rangle)$*;*

(iii) g *induces an isomorphism between the poset* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \leq \rangle$ *and the poset* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'), \leq' \rangle$.

Proof:

(i)⇒(ii) The implication (i)⇒(ii) follows by the hypothesis and Proposition 132.

(ii)⇒(iii) By hypothesis, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Conversely, if $Y \in \mathcal{C}'$, then by Proposition 148, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By surjectivity, $g(g^{-1}(Y)) = Y$, for all $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By compatibility, $g^{-1}(g(X)) = X$, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Since g is a grid morphism, g^{-1} is both order preserving and order reflecting with respect to $\leq'$ and $\leq$. So the conclusion follows.

(iii)⇒(i) By hypothesis, $g^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. So, by definition, $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$ is a bilogical morphism.

■

If two interpretations are related via a grid morphism and the morphism happens to be a bilogical morphism between two closure structures, one on each interpretation, then it turns out that, if the structure on the source interpretation consists of the entire collection of $\mathbb{L}$-filters, then so does the structure on the target interpretation. Again, for logicates, see Proposition 37.

Proposition 151 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ an interpretation and $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ a grid morphism. Set $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h\rangle$. If $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a bilogical morphism, then*

$$\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}').$$

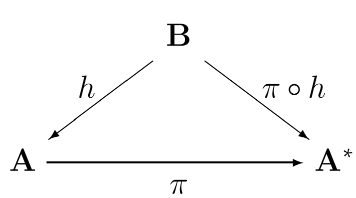
Proof: Suppose, first, that $Y \in \mathcal{C}'$. Then, since $g : \langle \mathcal{A}, C_{\mathcal{A}}\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $g^{-1}(Y) \in \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 148, $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. So $\mathcal{C}' \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Assume, conversely, that $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Then, by Proposition 148, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) = \mathcal{C}$. Thus, since $g : \langle \mathcal{A}, C\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $Y = g(g^{-1}(Y)) \in \mathcal{C}'$. So $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') \subseteq \mathcal{C}'$. We conclude that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. ■

As a corollary we get that the collection of $\mathbb{L}$-filters on a quotient structure coincides with the reductions of the $\mathbb{L}$-filters on the original structure.

Corollary 152 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ an interpretation. Then*

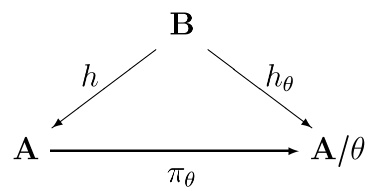
$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

Proof: One works with the diagram



where $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\widetilde{\Omega}_{\mathcal{A}}(\mathcal{C}_{\mathcal{A}})$ is the quotient grid morphism. Recalling that, by Proposition 138, it is a bilogical morphism, we may apply Proposition 151 to get the conclusion. ■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. Suppose $\mathfrak{A} = \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ and $\theta \in \mathrm{Con}(\mathfrak{A})$.



Then, using the compatibility of θ with F, we can see that $\mathfrak{A}/\theta = \langle \mathcal{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$. $\mathfrak{A}/\theta$ is called the **quotient (grid) matrix** of $\mathfrak{A}$ by θ. In particular, $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathcal{A}}(F) \in \mathrm{Mat}^*(\mathbb{L})$. $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathcal{A}}(F)$ is called the **reduction** of $\mathfrak{A}$. We let $\mathrm{Alg}^*(\mathbb{L})$ be the class of algebraic (grid interpretation) reducts of matrices in $\mathrm{Mat}^*(\mathbb{L})$.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid and consider a class M of matrices. We say that $\mathbb{L}$ is **complete with respect to** M if

$$\mathcal{C}^\flat = \{h^{-1}(F) : \langle \langle \hat{\mathbf{A}}, h \rangle, F \rangle \in \mathsf{M}\}.$$

Observe that by definition of an $\mathbb{L}$-filter, the theories of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ are captured as the $\mathbb{L}$-filters on the interpretation $\langle \hat{\mathbf{B}}, i_{\mathbf{B}} \rangle$. With this in mind, it is not difficult to see that, as is the case in the classical theory of logical matrices, $\mathbb{L}$ is complete both with respect to the class of all its grid matrices and with respect to the class of all its reduced grid matrices.

Proposition 153 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete both with respect to $\mathrm{Mat}(\mathbb{L})$ and with respect to $\mathrm{Mat}^*(\mathbb{L})$.*

Proof: By the definition of $\mathrm{Mat}(\mathbb{L})$, we have

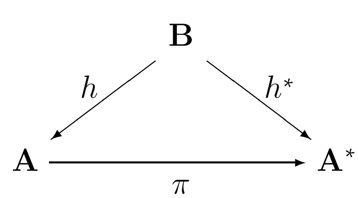
$$\{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\} \subseteq \mathcal{C}^\flat.$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then the pair $\langle \langle \hat{\mathbf{B}}, i_{\mathbf{B}} \rangle, X \rangle \in \mathrm{Mat}(\mathbb{L})$ and $i_{\mathbf{B}}^{-1}(X) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}(\mathbb{L})$.

Let $\langle \langle \hat{\mathbf{A}}^*, h^* \rangle, X^* \rangle \in \mathrm{Mat}^*(\mathbb{L})$. Then, by the definition of $\mathrm{Mat}^*(\mathbb{L})$,



$$\begin{array}{rcll}(h^*)^{-1}(X^*) & = & (\pi\circ h)^{-1}(\pi(X)) & \text{(Definition of } h^*\text{)}\\ & = & h^{-1}(\pi^{-1}(\pi(X))) & ((\pi\circ h)^{-1}=h^{-1}\circ\pi^{-1})\\ & = & h^{-1}(X) & \text{(Compatibility)}\\ & \in & \mathcal{C}^\flat. & (X\in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))\end{array}$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then, letting $\pi : \hat{\mathbf{B}} \to \hat{\mathbf{B}}^*$, where $\hat{\mathbf{B}}^* = \hat{\mathbf{B}}/\Omega_{\hat{\mathbf{B}}}(X)$, the pair $\langle\langle\hat{\mathbf{B}}^*, \pi\rangle, X^*\rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $\pi^{-1}(X^*) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle\mathcal{A}, F\rangle \in \mathrm{Mat}^*(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{L})$. ∎

Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid. Given a class K of grid interpretations, a grid interpretation $\mathcal{A} = \langle\hat{\mathbf{A}}, h\rangle$, not necessarily in the class K, and a congruence $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$, one writes $\theta \in \mathrm{Con}_{\mathsf{K}}(\mathcal{A})$ to signify that the quotient interpretation $\mathcal{A}/\theta \in \mathsf{K}$. In this case, θ is termed a **grid K-congruence**. So $\mathrm{Con}_{\mathsf{K}}(\mathcal{A})$ is the collection of all grid K-congruences on the grid interpretation $\mathcal{A}$. Using a variant of this notation, we may write

$$\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}).$$

The quotient grid $\hat{\mathbf{B}}^* := \hat{\mathbf{B}}/\widetilde{\Omega}(\mathbb{L})$ is called the **Lindenbaum-Tarski grid** of $\mathbb{L}$. The quotient logicoid $\mathbb{L}^* := \langle\hat{\mathbf{B}}^*, C^{\flat *}\rangle$ is called the **Lindenbaum-Tarski quotient** of $\mathbb{L}$.