Chapter 8

Model Theory

8.1 Introduction

This chapter discusses *logicoid models* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$. They are based on grid interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, which consist of a grid $\hat{\mathbf{A}}$ together with a grid morphism h from the base grid $\hat{\mathbf{B}}$ onto $\hat{\mathbf{A}}$. We also define and study the reduction $\mathbb{A}^*$ of such a logicoid interpretation $\mathbb{A}$. Among those models, we single out the *full models*, which are the ones whose reductions are *basic full models*, i.e., consist of all possible $\mathbb{L}$-filters on their underlying interpretations. We characterize this class of models. Moreover, we show that it consists of those models that have all possible filters corresponding to filters on the reduced interpretations. Reduced $\mathbb{L}$-models give rise to $\mathbb{L}$-*algebras*, i.e., interpretations that are reducts of reduced $\mathbb{L}$-models. Their class is shown to be the class of subdirect intersections of interpretations in $\mathrm{Alg}^*(\mathbb{L})$, which consists of all interpretation reducts of reduced $\mathbb{L}$-matrices. Our study culminates with an Isomorphism Theorem for logicoids asserting that the Tarksi operator on a fixed interpretation $\mathcal{A}$ is an isomorphism between the ordered set of full $\mathbb{L}$-models on $\mathcal{A}$ and the partially ordered set of grid $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$.

In Section 8.2, we introduce the notion of an *interpretation* of a base algebraic grid $\hat{\mathbf{B}}$ and that of a *logicoid interpretation*. The first is a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, consisting of an algebraic grid $\hat{\mathbf{A}}$ and a grid morphism $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$. The second consists of an algebraic logicoid $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is an interpretation and $\langle \hat{\mathbf{A}}, C \rangle$ is a logicoid based on $\hat{\mathbf{A}}$. A logicoid interpretation of $\hat{\mathbf{B}}$ induces a logicoid structure $\mathbb{L}^{\mathbb{A}} = \langle \hat{\mathbf{B}}, C^{\mathbb{A}} \rangle$ on $\hat{\mathbf{B}}$ in such a way that h becomes a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{A}$. Further, if two logicoid interpretations are related via a bilogical morphism, then they induce identical logicoids on the base grid $\hat{\mathbf{B}}$. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a *model* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ if $\mathbb{L} \leq \mathbb{L}^{\mathbb{A}}$ or, equivalently, if $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. The section continues with a discussion of completeness of a base logicoid with respect to a class of models. In this context, *reductions* of models and *reduced models* are discussed and analogs of classical completeness results with respect to the class of all models and with respect to the class of all reduced models are formulated. The section closes by connecting the notion of logicoid model with that of a grid matrix model, introduced and studied in Section 7.5.

In Section 8.3, we study *full models* of logicoids. Given a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, a logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a *basic full model* of $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., if its set of theories is the entire collection of $\mathbb{L}$-filters on its underlying interpretation. A *full model* of $\mathbb{L}$ is one whose reduction is a basic full model. Full models are indeed models and basic full models are indeed full models. So the terminology chosen is sound. It turns out that bilogical morphisms preserve fullness in both directions, which implies that $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if its reduction $\mathbb{A}^*$ is also a full $\mathbb{L}$-model. Additionally, $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if there exists a bilogical morphism from it onto a basic full $\mathbb{L}$-model. As a consequence we get that the class of full $\mathbb{L}$-models is the smallest class containing all basic full $\mathbb{L}$-models and closed under bilogical morphisms (in both directions). The section concludes with a result providing an additional justification of the term "full". It shows that full $\mathbb{L}$-models are those whose collection of $\mathbb{L}$-filters consists of all possible ones corresponding to $\mathbb{L}$-filters on the reduced interpretation.

Section 8.4 introduces $\mathbb{L}$-algebras (more accurately $\mathbb{L}$-interpretations) for a logicoid $\mathbb{L}$. These are interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$, such that $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))$ is the identity grid congruence on $\hat{\mathbf{A}}$. Some results relating $\mathbb{L}$-algebras with full $\mathbb{L}$-models and with their theories are provided. It is shown that, for every full $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C\rangle$, the reduction $\mathcal{A}^*$ of the interpretation $\mathcal{A}$ is an $\mathbb{L}$-algebra. Additionally, the class $\mathrm{Alg}(\mathbb{L})$ of $\mathbb{L}$-algebras is characterized as the class of interpretation reducts of reduced $\mathbb{L}$-models. Moreover, it is shown that $\mathrm{Alg}(\mathbb{L})$ is the class of all subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$ of all interpretation reducts of reduced grid matrix models of $\mathbb{L}$. This characterization yields that $\mathrm{Alg}^*(\mathbb{L})$ is contained in $\mathrm{Alg}(\mathbb{L})$ and, also, that given logicoids $\mathbb{L}$, $\mathbb{L}'$ over the same base grid $\hat{\mathbf{B}}$, such that $\mathbb{L} \leq^\flat \mathbb{L}'$, we have that $\mathrm{Alg}(\mathbb{L}')$ is contained in the class $\mathrm{Alg}(\mathbb{L})$.

In Section 8.5, the last section of the chapter, we prove an analog of the Isomorphism Theorem 13 of [12] for logicoids. The result parallels the Isomorphism Theorem for logicates (Theorem 75) and the proof is similar.

8.2 Models of Logicoids

We consider a **base algebraic grid** $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq\rangle$, where $\mathbf{B} = \langle B, \mathcal{L}^{\mathbf{B}}\rangle$ is an algebra, which, in this context, is termed the **base algebra** and $\leq$ is a complete lattice order on $\mathcal{P}(B)$. An **interpretation** is a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$, where $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$ is an algebraic grid and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism from the base algebra onto $\hat{\mathbf{A}}$. An *algebraic logicoid* $\langle \hat{\mathbf{A}}, C\rangle$ consists of an algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$ and a $\leq$-closure operator C on $\hat{\mathbf{A}}$. A **logicoid interpretation** is a pair $\mathbb{A} = \langle \mathcal{A}, C\rangle$, where:

- $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ is an interpretation;
- $\langle \hat{\mathbf{A}}, C\rangle$ is an algebraic logicoid on the grid $\hat{\mathbf{A}}$.

The logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C\rangle$ induces a function

$$C^{\mathbb{A}} : \mathcal{P}(B) \to \mathcal{P}(B),$$

defined, for all $X \subseteq B$, by

$$C^{\mathbb{A}}(X) = \bigwedge\nolimits^{\flat} h^{-1}(\mathcal{C})^X,$$

where

$$h^{-1}(\mathcal{C})^X = \{h^{-1}(Y) : Y \in \mathcal{C} \text{ and } X \leq^\flat h^{-1}(Y)\}.$$

We write

$$\mathbb{L}^{\mathbb{A}} := \langle \hat{\mathbf{B}}, C^{\mathbb{A}} \rangle.$$

This construction forms an analog of the construction in Definition 2.1 of [12]. We show that $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicoid on the base grid and that the epimorphism h is a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{L}$.

Proposition 154 *Let* $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ *be a base grid and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *a logicoid interpretation, with* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ *and* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$.

(a) $\mathbb{L}^{\mathbb{A}}$ *is an algebraic logicoid.*

(b) $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{L}$ *is a bilogical morphism.*

Proof:

(a) We must show that $C^{\mathbb{A}}$ is inflationary, monotone and idempotent with respect to $\leq^\flat$. First, let $X \subseteq B$. By definition, $X \leq^\flat \bigwedge^\flat h^{-1}(\mathcal{C})^X = C^{\mathbb{A}}(X)$. So $C^{\mathbb{A}}$ is inflationary. Next, let $X, Y \subseteq B$, such that $X \leq^\flat Y$. Then

$$C^{\mathbb{A}}(X) = \bigwedge{}^\flat h^{-1}(\mathcal{C})^X \leq^\flat \bigwedge{}^\flat h^{-1}(\mathcal{C})^Y = C^{\mathbb{A}}(Y).$$

So $C^{\mathbb{A}}$ is also monotone. Finally, let $X \subseteq B$. Then, taking into account that $\mathcal{C}$ is closed under intersections and that h^{-1} is a complete lattice embedding,

$$\begin{array}{rcl} C^{\mathbb{A}}(C^{\mathbb{A}}(X)) & = & \bigwedge^\flat h^{-1}(\mathcal{C})^{C^{\mathbb{A}}(X)} \\ & = & \bigwedge^\flat h^{-1}(\mathcal{C})^{\bigwedge^\flat h^{-1}(\mathcal{C})^X} \\ & = & \bigwedge^\flat h^{-1}(\mathcal{C})^X \\ & = & C^{\mathbb{A}}(X). \end{array}$$

Thus, $C^{\mathbb{A}}$ is inflationary, monotone and idempotent with respect to $\leq^\flat$ and, therefore, $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicoid.

(b) We show that $\mathcal{C}^{\mathbb{A}} = h^{-1}(\mathcal{C})$. Then, by Proposition 132, it will follow that $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{L}$ is a bilogical morphism. If $Y \in \mathcal{C}$, then

$$C^{\mathbb{A}}(h^{-1}(Y)) = \bigwedge{}^\flat h^{-1}(\mathcal{C})^{h^{-1}(Y)} = h^{-1}(Y).$$

Hence $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^{\mathbb{A}}$. Assume, conversely, that $X \in \mathcal{C}^{\mathbb{A}}$. Then

$$X = C^{\mathbb{A}}(X) = \bigwedge{}^\flat h^{-1}(\mathcal{C})^X \in h^{-1}(\mathcal{C}),$$

where membership follows from closure of $\mathcal{C}$ under meets and the fact that h^{-1} is a complete lattice embedding.

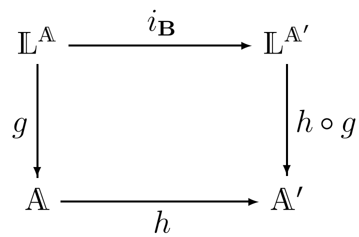
■

We call $\mathbb{L}^{\mathbb{A}}$ the **logicoid induced on B by** $\mathbb{A}$.

An analog of Proposition 2.3 of [12] ensures that logicoid interpretations related via "compatible" bilogical morphisms induce the same logicoid on the base grid.

Proposition 155 *Let* $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ *be a base grid and* $\mathbb{A} = \langle \langle \hat{\mathbf{A}}, g \rangle, C \rangle$ *and* $\mathbb{A}' = \langle \langle \hat{\mathbf{A}}', h \circ g \rangle, C' \rangle$ *two logicoid interpretations, with* $h : \mathbb{A} \to_b \mathbb{A}'$ *a bilogical morphism. Then* $\mathbb{L}^{\mathbb{A}} = \mathbb{L}^{\mathbb{A}'}$.

Proof: Using the diagram below, we have, for all $X \subseteq B$,



$$\begin{array}{rcll} C^{\mathbb{A}}(X) & = & \bigwedge^\flat g^{-1}(\mathcal{C})^X & \text{(Definition of } C^{\mathbb{A}}) \\ & = & \bigwedge^\flat g^{-1}(h^{-1}(\mathcal{C}'))^X & (h : \mathbb{A} \to_b \mathbb{A}') \\ & = & \bigwedge^\flat (h \circ g)^{-1}(\mathcal{C}')^X & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & = & C^{\mathbb{A}'}(X). & \text{(Definition of } C^{\mathbb{A}'}) \end{array}$$

So $\mathbb{L}^{\mathbb{A}} = \mathbb{L}^{\mathbb{A}'}$. ■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, with $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$, be a base logicoid, perceived as constituting the main object of investigation. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, is called a **model of** $\mathbb{L}$ or an $\mathbb{L}$**-model** if, $C^\flat \leq^\flat C^{\mathbb{A}}$, i.e., for all $X \subseteq B$,

$$C^\flat(X) \leq^\flat C^{\mathbb{A}}(X).$$

We denote by $\mathrm{Mod}(\mathbb{L})$ the class of all models of $\mathbb{L}$.

Lemma 156 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. A logicoid interpretation* $\mathbb{A} = \langle \mathcal{A}, C \rangle$*, with* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ *is a model of* $\mathbb{L}$ *if and only if*

$$h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat.$$

Proof: We have the following equivalences

$$\begin{array}{rll} C^\flat \leq^\flat C^{\mathbb{A}} & \text{iff} & \mathcal{C}^{\mathbb{A}} \subseteq \mathcal{C}^\flat \quad \text{(Proposition 106)} \\ & \text{iff} & h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat. \quad \text{(Part (b) of Proposition 154)} \end{array}$$

■

Let L be a class of models of $\mathbb{L}$. $\mathbb{L}$ is said to be **complete with respect to** L if, for all $X \subseteq B$,

$$C^\flat(X) = \textstyle\bigwedge^\flat_{\mathbb{A} \in \mathsf{L}} C^{\mathbb{A}}(X).$$

Lemma 157 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* L *be a class of models of* $\mathbb{L}$*.* $\mathbb{L}$ *is complete with respect to* L *if and only if* $\mathcal{C}^\flat$ *is the* $\leq^\flat$*-closure system generated by* $\bigcup\{h^{-1}(\mathcal{C}) : \langle \langle \hat{\mathbf{A}}, h \rangle, C \rangle \in \mathsf{L}\}$.

Proof: For the "if" direction, let $X \subseteq B$. We then have

$$\begin{array}{rcll} \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} C^{\mathbb{A}}(X) & = & \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} \bigwedge^\flat h^{-1}(\mathcal{C})^X & \text{(Definition of } C^{\mathbb{A}}) \\ & = & \bigwedge^\flat \{T \in \mathcal{C}^\flat : X \leq^\flat T\} & \text{(Hypothesis)} \\ & = & C^\flat(X). & \text{(Proposition 102)} \end{array}$$

For the "only if", assume that, for all $X \subseteq B$, $C^\flat(X) = \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} C^{\mathbb{A}}(X)$. By Lemma 156, we know that $\bigcup\{h^{-1}(\mathcal{C}) : \langle\langle\hat{\mathbf{A}}, h\rangle, C\rangle \in \mathsf{L}\} \subseteq \mathcal{C}^\flat$. Conversely, if $X \in \mathcal{C}^\flat$, then

$$X = C^\flat(X) = \bigwedge\nolimits^\flat_{\mathbb{A}\in\mathsf{L}} C^{\mathbb{A}}(X) = \bigwedge\nolimits^\flat_{\mathbb{A}\in\mathsf{L}} \bigwedge\nolimits^\flat h^{-1}(\mathcal{C})^X.$$

So X is in the $\leq^\flat$-closure system generated by $\bigcup\{h^{-1}(\mathcal{C}) : \langle\langle\hat{\mathbf{A}}, h\rangle, C\rangle \in \mathsf{L}\}$. ∎

The following result is an analog of part of Proposition 2.5 of [12] for logicoid interpretations.

Proposition 158 *Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid, $\mathbb{A} = \langle\langle\hat{\mathbf{A}}, g\rangle, C\rangle$, $\mathbb{A}' = \langle\langle\hat{\mathbf{A}}', h \circ g\rangle, C'\rangle$ be logicoid interpretations and $h : \mathbb{A} \to \mathbb{A}'$ be a bilogical morphism. Then $\mathbb{A}$ is a model of $\mathbb{L}$ if and only if $\mathbb{A}'$ is a model of $\mathbb{L}$.*

Proof: Suppose $\mathbb{A}$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcll} (h \circ g)^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & = & g^{-1}(\mathcal{C}) & (h : \mathbb{A} \to_b \mathbb{A}') \\ & \subseteq & \mathcal{C}^\flat. & (\mathbb{A} \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}'$ is a model of $\mathbb{L}$. Assume, conversely, that $\mathbb{A}'$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcll} g^{-1}(\mathcal{C}) & = & g^{-1}(h^{-1}(\mathcal{C}')) & (h : \mathbb{A} \to_b \mathbb{A}') \\ & = & (h \circ g)^{-1}(\mathcal{C}') & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & \subseteq & \mathcal{C}^\flat. & (\mathbb{A}' \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}$ is a model of $\mathbb{L}$. ∎

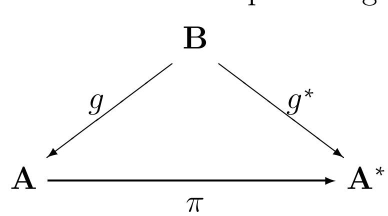
In order to formulate another part of Proposition 2.5 of [12], we need to define the Tarski reduction of a logicoid interpretation.

Let $\mathbb{L} = \langle\mathbf{B}, C^\flat\rangle$ be an base logicoid. Consider the pair $\mathbb{A} = \langle\langle\hat{\mathbf{A}}, g\rangle, C\rangle$. Recall the Tarski congruence $\widetilde{\Omega}(\mathbb{A}) \coloneqq \widetilde{\Omega}_{\hat{\mathbf{A}}}(C)$. We define the pair

$$\mathbb{A}^* = \langle\langle\hat{\mathbf{A}}^*, g^*\rangle, C^*\rangle$$

by setting:

- $\hat{\mathbf{A}}^* = \hat{\mathbf{A}}/\widetilde{\Omega}_{\hat{\mathbf{A}}}(C)$;
- $g^* = \pi \circ g$, where $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}^*$ is the quotient grid morphism;



- $C^* : \mathcal{P}(A/\widetilde{\Omega}_{\mathbf{A}}(C)) \to \mathcal{P}(A/\widetilde{\Omega}_{\mathbf{A}}(C))$, where, for all $S \subseteq A/\widetilde{\Omega}_{\mathbf{A}}(C)$,
$$C^*(S) = \pi(C(\pi^{-1}(S))).$$

Based on results already obtained, we may show that, if $\mathbb{A}$ is a model of $\mathbb{L}$, then so is $\mathbb{A}^*$.

Corollary 159 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. A pair $\mathbb{A} = \langle\langle \hat{\mathbf{A}}, g \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if $\mathbb{A}^*$ is a model of $\mathbb{L}$.*

Proof: This follows directly from Proposition 158, since, by Proposition 138, the natural projection $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism. ∎

Corollary 160 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete with respect to a class L of models if and only if it is complete with respect to the class L^*.*

Proof: Let
$$\mathsf{L} = \{\langle\langle \hat{\mathbf{A}}_i, g_i \rangle, C_i \rangle : i \in I\}.$$
Denote by $\pi_i : \hat{\mathbf{A}}_i \to \hat{\mathbf{A}}_i^*$, $i \in I$, the quotient grid morphisms. By Proposition 138, for all $i \in I$, $\pi_i : \mathbb{A}_i \to_b \mathbb{A}_i^*$ is a bilogical morphism. So we have, for all $i \in I$,
$$\bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i) = \bigcup_{i\in I} g_i^{-1}(\pi_i^{-1}(\mathcal{C}_i^*)) = \bigcup_{i\in I} (\pi_i \circ g_i)^{-1}(\mathcal{C}_i^*) = \bigcup_{i\in I} (g_i^*)^{-1}(\mathcal{C}_i^*).$$
Thus, $\mathbb{L}$ is complete with respect to L if and only if, by definition, $\mathcal{C}^\flat$ is $\le^\flat$-generated by $\bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i)$ if and only if, by the displayed equality, $\mathcal{C}^\flat$ is $\le^\flat$-generated by $\bigcup_{i\in I}(g_i^*)^{-1}(\mathcal{C}_i^*)$ if and only if, by definition, $\mathbb{L}$ is complete with respect to L^*. ∎

As far as completeness properties go, note that $\langle\langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle, C^\flat \rangle$ is a model of $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$. This yields the following results, forming together an analog of Proposition 2.6 of [12].

Proposition 161 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete with respect to any class L of models that includes $\mathbb{A} = \langle\langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle, C^\flat \rangle$ or $\mathbb{A}^*$.*

Proof: Let $\mathsf{L} = \{\langle\langle \hat{\mathbf{A}}_i, g_i \rangle, C_i \rangle : i \in I\}$. On the one hand, since $\mathsf{L} \subseteq \mathrm{Mod}(\mathbb{L})$, $\bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i) \subseteq \mathcal{C}^\flat$. On the other, since $\mathbb{A} \in \mathsf{L}$, $\mathcal{C}^\flat = i_{\hat{\mathbf{B}}}^{-1}(\mathcal{C}^\flat) \subseteq \bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i)$. Thus, $\mathbb{L}$ is complete with respect to L.

Assume, now, that $\mathbb{A}^* \in \mathsf{L}$. The first inclusion is justified in the same way. For the second, letting $\pi : \hat{\mathbf{B}} \to \hat{\mathbf{B}}^*$ be the quotient grid morphism, we have
$$\mathcal{C}^\flat = i_{\hat{\mathbf{B}}}^{-1}(\mathcal{C}^\flat) = i_{\hat{\mathbf{B}}}^{-1}(\pi^{-1}(\mathcal{C}^{\flat *})) = \pi^{-1}(\mathcal{C}^{\flat *}) \subseteq \bigcup_{i\in I} g_i^{-1}(\mathcal{C}_i).$$
Thus, $\mathbb{L}$ is again complete with respect to L. ∎

Corollary 162 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete with respect to the class of all its models and with respect to the class of all its reduced models.*

Proof: Clearly, $\mathbb{A} = \langle \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle, C^\flat \rangle \in \mathrm{Mod}(\mathbb{L})$ and $\mathbb{A}^* \in \mathrm{Mod}^*(\mathbb{L})$. Thus, by Proposition 161, $\mathbb{L}$ is complete both with respect to $\mathrm{Mod}(\mathbb{L})$ and with respect to $\mathrm{Mod}^*(\mathbb{L})$. ■

Logicoid models are closely connected with matrix models. The connection is given in the following proposition, which parallels Proposition 2.7 of [12].

Proposition 163 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\langle \langle \hat{\mathbf{A}}, h \rangle, C \rangle$ *is a model of* $\mathbb{L}$ *if and only if, for all* $Y \in \mathcal{C}$, $\langle \langle \hat{\mathbf{A}}, h \rangle, Y \rangle$ *is an* $\mathbb{L}$*-matrix.*

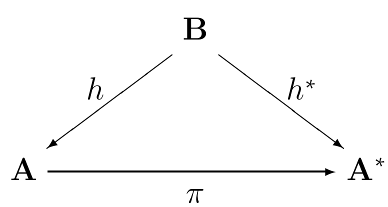
Proof: We have that $\langle \langle \hat{\mathbf{A}}, h \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if, by Lemma 156, $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$ if and only if, for all $Y \in \mathcal{C}$, $h^{-1}(Y) \in \mathcal{C}^\flat$ if and only if, by definition, for all $Y \in \mathcal{C}$, $\langle \langle \hat{\mathbf{A}}, h \rangle, Y \rangle$ is an $\mathbb{L}$-matrix. ■

Proposition 163 asserts that the weakest model of $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ on an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is the one determined by

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}).$$

8.3 Full Models

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, is called a **full model of** $\mathbb{L}$ or a **full** $\mathbb{L}$**-model** (see Definition 2.8 of [12]) if



$$\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a **basic full model of** $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, rephrasing the definition, we may say that $\mathbb{A}$ is a full model of $\mathbb{L}$ if and only if its reduction is a basic full model of $\mathbb{L}$.

$\mathrm{FMod}(\mathbb{L})$ denotes the class of all full models of $\mathbb{L}$. $\mathrm{FMod}^*(\mathbb{L})$ is the class of all reduced full models of $\mathbb{L}$. Given an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$ is the class of all full models of $\mathbb{L}$ on $\mathcal{A}$.

An analog of Part (1) of Proposition 2.9 of [12] and of Proposition 58 for logicates provides a justification for the use of the term "model" for full models.

Proposition 164 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *a full model of* $\mathbb{L}$. *Then* $\mathbb{A}$ *is a model of* $\mathbb{L}$.

Proof: Suppose $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$. By definition, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, by Proposition 163, $\mathbb{A}^*$ is a model of $\mathbb{L}$. Thus, by Corollary 159, $\mathbb{A}$ is also a model of $\mathbb{L}$. ■

The next result, an analog of Proposition 2.10 of [12] and of Proposition 59 for logicates, asserts that every basic full model is actually a full model, justifying the "full" in the definition of basic full models.

Proposition 165 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. A logicoid interpretation* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *on* $\mathcal{A}$, *such that* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, *is a full model of* $\mathbb{L}$ *and is the weakest full model of* $\mathbb{L}$ *on* $\mathcal{A}$.

Proof: By Proposition 138, the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism. By Corollary 152, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$, i.e., $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, $\mathbb{A}$ is a full model of $\mathbb{L}$. Taking into account Proposition 164, it is the weakest full model on $\mathcal{A}$, since it is the weakest model on $\mathcal{A}$, by Proposition 163. ■

Proposition 2.11 of [12], concerning closure of the class of full models under bilogical morphisms, has the following analog (see, also, Proposition 60 for logicates).

Proposition 166 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. The class* $\mathrm{FMod}(\mathbb{L})$ *is closed under bilogical morphisms, i.e., if* $h : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$, *where* $\mathcal{A} = \langle \hat{\mathbf{A}}, g \rangle$ *and* $\mathcal{A}' = \langle \hat{\mathbf{A}}', h \circ g \rangle$, *is a bilogical morphism, then*

$$\langle \mathcal{A}, C \rangle \in \mathrm{FMod}(\mathbb{L}) \quad \textit{iff} \quad \langle \mathcal{A}', C' \rangle \in \mathrm{FMod}(\mathbb{L}).$$

Proof: Suppose $h : \mathbb{A} \to \mathbb{A}'$ is a bilogical morphism. By Proposition 144, there exists an isomorphism $h^* : \mathbb{A}^* \cong \mathbb{A}'^*$. Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. Then $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Thus, by Proposition 151, $\mathcal{C}'^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'^*)$. But $\mathbb{A}'^*$ is reduced, whence $\mathbb{A}'^*$ is a full model of $\mathbb{L}$. A similar reasoning yields the converse. ■

Corollary 167 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$ *if and only if* $\mathbb{A}^* \in \mathrm{FMod}(\mathbb{L})$.

Proof: Directly from Proposition 166, since, by Proposition 138, the quotient grid morphism $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}^*$ is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$. ■

Corollary 168 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *is a full model of* $\mathbb{L}$ *if and only if there exists a bilogical morphism from* $\mathbb{A}$ *onto a model* $\langle \mathcal{A}', C' \rangle$, *with* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$.

Proof: The "only if" follows directly from the definition of full model, as the projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

Conversely, assume there is a bilogical morphism $h : \mathbb{A} \to \mathbb{A}'$, such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By Proposition 165, $\mathbb{A}'$ is a full model, whence, by Proposition 166, $\mathbb{A}$ is also a full model. ∎

Our work culminates in a characterization of the class of full models of a logicate $\mathbb{L}$ along the lines of Corollary 2.13 of [12] (Corollary 63 for logicates).

Corollary 169 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\mathrm{FMod}(\mathbb{L})$ *is the smallest class containing all* $\langle \mathcal{A}, C \rangle$*, with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, and closed under bilogical morphisms.*

Proof: Let L be the smallest class containing all $\langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, and closed under bilogical morphisms.

On the one hand, every $\langle \mathcal{A}, C \rangle$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a full model, by Proposition 165. Moreover, by Proposition 166, the class of full models is closed under bilogical morphisms. This shows that $\mathsf{L} \subseteq \mathrm{FMod}(\mathbb{L})$. The reverse inclusion is a direct consequence of Corollary 168. ∎

Corollary 169 provides one justification for the "fullness" property of full models. According to this justification, a full model is one that is obtained via a bilogical morphism by a model whose theories constitute a full set of $\mathbb{L}$-filters. A second justification is given in the following theorem. According to this, a model's "fullness" rests on the fact that its theories contain all possible $\mathbb{L}$-filters corresponding to $\mathbb{L}$-filters of the reduction of the model.

Theorem 170 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be an base logicoid. Then* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *is a full model of* $\mathbb{L}$ *if and only if*

$$\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}.$$

Proof: Suppose, first, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$. Let $X \in \mathcal{C}$. By Proposition 163, $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By definition of $\widetilde{\Omega}(\mathbb{A})$, it always holds that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. For the reverse inclusion, let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. Then, by Proposition 149, there exists $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$, such that $X = \pi^{-1}(Y)$, where $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\widetilde{\Omega}(\mathbb{A})$ is the quotient grid morphism. But, by Proposition 138, $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism and, moreover, since $\mathbb{A}$ is full, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $X \in \mathcal{C}$.

Suppose, conversely, that $\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}$. Since the natural projection $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $\mathbb{A}$ is a full model of $\mathbb{L}$. ∎

8.4 $\mathbb{L}$-Algebras

Reduced full models of $\mathbb{L}$ are those models of the form $\langle \mathcal{A}, C\rangle$, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, that are reduced. The interpretation reducts of such models are given a special name.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid. An interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ is an **$\mathbb{L}$-algebra** (see Definition 2.16 of [12]) if

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) = \Delta_{\mathbf{A}}.$$

The class of all $\mathbb{L}$-algebras is denoted by $\mathrm{Alg}(\mathbb{L})$.

The following characterization takes after Proposition 2.17 of [12] (see, also, Proposition 65 for logicates).

Proposition 171 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathbb{A} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$. Then the following statements are equivalent:*

(i) $\mathbb{A} = \langle \mathcal{A}, C\rangle$ *is a reduced full model of* $\mathbb{L}$*;*

(ii) $\mathbb{A} = \langle \mathcal{A}, C\rangle$ *is reduced and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*;*

(iii) $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$ *and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof:

(i)⇒(ii) By the definition of a reduced full model.

(ii)⇒(iii) By the definition of an $\mathbb{L}$-algebra, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$.

(iii)⇒(i) Since $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, there exists $C' : \mathcal{P}(A) \to \mathcal{P}(A)$, such that $\mathbb{A} = \langle \mathcal{A}, C'\rangle$ is a reduced full model of $\mathbb{L}$. But then $\mathcal{C}' = \mathcal{C}$ and $\mathbb{A} = \langle \mathcal{A}, C\rangle$ is a reduced full model of $\mathbb{L}$.

■

Proposition 172 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathbb{A} = \langle \mathcal{A}, C\rangle$ a full model of $\mathbb{L}$. Then $\mathcal{A}^* := \mathcal{A}/\widetilde{\Omega}(\mathbb{A})$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$.*

Proof: By Corollary 167, $\mathbb{A}^*$ is a full model of $\mathbb{L}$ and it is clearly reduced. Hence, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra. This also yields the second statement using the definition of $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. ■

A characterization of $\mathbb{L}$-algebras, an analog of Proposition 2.19 of [12] and of Proposition 67 for logicates, shows that the notion of model, without reference to fullness, suffices to characterize the class $\mathrm{Alg}(\mathbb{L})$.

Proposition 173 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid. The class $\mathrm{Alg}(\mathbb{L})$ is the class of algebraic reducts of all reduced models of $\mathbb{L}$.*

Proof: By definition, if $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, then $\mathcal{A}$ is the algebraic reduct of a reduced full model; in particular of a reduced model. Assume, conversely, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a reduced model of $\mathbb{L}$. Let $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$, be such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 163, $\mathbb{A}'$ is a model of $\mathbb{L}$ and, by Proposition 165, it is clearly full. It is also reduced, since

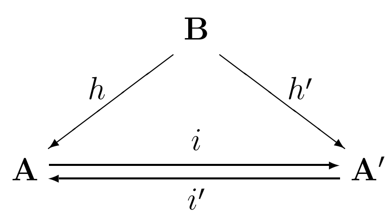
$$\widetilde{\Omega}(\mathbb{A}') \subseteq \widetilde{\Omega}(\mathbb{A}) = \Delta_{\mathbf{A}}.$$

Therefore, by definition, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ■

Closure under grid isomorphisms is guaranteed by the following proposition, an analog of Proposition 2.20 of [12] (see Proposition 68 for logicates).

Proposition 174 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. The class* $\mathrm{Alg}(\mathbb{L})$ *is closed under grid isomorphisms (commuting with the interpretations).*

Proof: Let $i : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$. We have the following diagram.



Suppose that $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for some C, $\langle \mathcal{A}, C \rangle$ is a reduced full model of $\mathbb{L}$. Consider $\mathcal{A}' = \langle \hat{\mathbf{A}}', h' \rangle = \langle \hat{\mathbf{A}}', i \circ h \rangle$. We have, $\langle \mathcal{A}', C' \rangle$, with $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, is a reduced full model of $\mathbb{L}$. Thus, $\mathcal{A}' \in \mathrm{Alg}(\mathbb{L})$. The reverse implication can be proved similarly. ■

Putting together several of the previous results, we get the following alternative characterizations of full models involving $\mathbb{L}$-algebras, an analog of Proposition 69 regarding logicates.

Proposition 175 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$*, with* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ *an interpretation. Then the following statements are equivalent.*

(i) $\mathbb{A}$ *is a full model of* $\mathbb{L}$*;*

(ii) $\mathcal{A}^*$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$*;*

(iii) There exists a bilogical morphism $g : \mathbb{A} \to \mathbb{A}'$*, with* $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$*, such that* $\mathcal{A}'$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$*.*

Proof:

(i)⇒(ii) Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. By definition, $\mathbb{A}^*$ is a basic full model of $\mathbb{L}$. Thus, by Proposition 171, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

(ii)⇒(iii) Assume $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Then (iii) is immediate by considering the quotient grid morphism $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}^*$, which is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$ and such that $\mathbb{A}^*$ fulfills the required conditions by (ii).

(iii)⇒(i) By Proposition 171, $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h \rangle$, is a reduced full model of $\mathbb{L}$. Therefore, by Corollary 168, $\mathbb{A}$ is a full model of $\mathbb{L}$.

■

An analog of the Completeness Theorem 2.22 of [12], and also of Theorem 70, asserts that the class of full models, the class of reduced full models, as well as the class of all basic full models of a logicoid can serve as a complete semantics for the logicoid.

Theorem 176 (Completeness) *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete with respect to the following classes of models:*

1. *The class* $\mathrm{FMod}(\mathbb{L})$ *of all full models of* $\mathbb{L}$*;*

2. *The class of all basic full models of* $\mathbb{L}$*;*

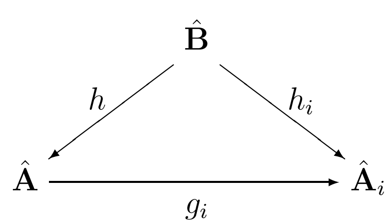
3. *The class* $\mathrm{FMod}^*(\mathbb{L})$ *of all reduced full models of* $\mathbb{L}$*.*

Proof: All three classes consist of models of $\mathbb{L}$. In addition each contains the model $\langle \langle \hat{\mathbf{B}}^*, \pi \rangle, C^{\flat *} \rangle$, where $\pi : \hat{\mathbf{B}} \to \hat{\mathbf{B}}/\widetilde{\Omega}(\mathbb{L})$ is the quotient grid morphism. Thus, by Proposition 161, $\mathbb{L}$ is complete with respect to each of these three classes. ■

We now establish an analog of the well known theorem (Theorem 2.23 of [12]) relating the classes $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$. Recall that $\mathrm{Alg}^*(\mathbb{L})$ is the class of all algebraic reducts of reduced matrix models of $\mathbb{L}$, whereas $\mathrm{Alg}(\mathbb{L})$ is the class of all algebraic reducts of reduced full models of $\mathbb{L}$. In the present setting, however, due to the presence of morphisms from the base logicoid in the interpretations involved, one has to replace subdirect products by a different operation, named subdirect intersection.

Let $\mathcal{A}_i = \langle \hat{\mathbf{A}}_i, h_i \rangle$, $i \in I$, be a collection of interpretations. We say that an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is a **subdirect intersection of** the $\mathcal{A}_i$ **relative to** $\hat{\mathbf{B}}$ if:

- There exist grid morphisms $g_i : \hat{\mathbf{A}} \to \hat{\mathbf{A}}_i$, $i \in I$, such that the following diagram commutes for all $i \in I$.

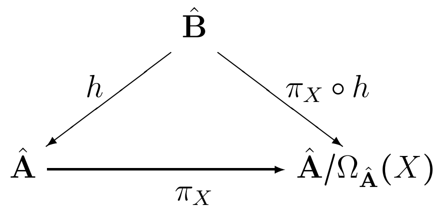


- $\bigcap_{i \in I} \mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$.

Since the role of $\hat{\mathbf{B}}$ is going to be played by the base algebraic grid (the algebraic grid reduct of the base logicoid), we usually omit the "relative to $\hat{\mathbf{B}}$" in the terminology.

Theorem 177 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^{\flat} \rangle$ be a base logicoid. The class* $\mathrm{Alg}(\mathbb{L})$ *is the class of all subdirect intersections of interpretations in* $\mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. Then there exist C, such that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a reduced full model of $\mathbb{L}$. We form the commutative triangle of grid morphisms



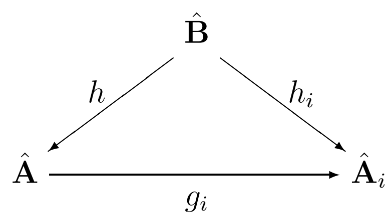
where $\pi_X : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\Omega_{\hat{\mathbf{A}}}(X)$ denotes the canonical projection. Note that

$$\bigcap_{X \in \mathcal{C}} \mathrm{Ker}(\pi_X) = \bigcap_{X \in \mathcal{C}} \Omega_{\hat{\mathbf{A}}}(X) = \widetilde{\Omega}(\mathbb{A}) = \Delta_{\mathbf{A}}.$$

Hence $\mathcal{A}$ is a subdirect intersection of

$$\mathcal{A}/\Omega_{\mathcal{A}}(X) = \langle \hat{\mathbf{A}}/\Omega_{\hat{\mathbf{A}}}(X), \pi_X \circ h \rangle \in \mathrm{Alg}^*(\mathbb{L}), \quad X \in \mathcal{C}.$$

Assume, conversely, that $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is a subdirect intersection of a collection $\mathcal{A}_i = \langle \hat{\mathbf{A}}_i, h_i \rangle \in \mathrm{Alg}^*(\mathbb{L})$, $i \in I$. Then, by hypothesis, we have commutative diagrams of grid morphisms,



such that $\bigcap_{i \in I} \mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$. Moreover, since, for all $i \in I$, $\mathcal{A}_i \in \mathrm{Alg}^*(\mathbb{L})$, there exists $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, such that $\Omega_{\hat{\mathbf{A}}_i}(X) = \Delta_{\mathbf{A}_i}$. Let $C : \mathcal{P}(A) \to \mathcal{P}(A)$, be such that

$$\mathcal{C} = \{g_i^{-1}(X_i) : i \in I\}$$

and set $\mathbb{A} = \langle \mathcal{A}, C \rangle$. Since $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, $i \in I$, we have that $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover,

$$\begin{array}{rcll}
\widetilde{\Omega}(\mathbb{A}) & = & \bigcap_{i \in I} \Omega_{\hat{\mathbf{A}}}(g_i^{-1}(X_i)) & (\text{Definition of } \widetilde{\Omega}(\mathbb{A})) \\
& = & \bigcap_{i \in I} g_i^{-1}(\Omega_{\hat{\mathbf{A}}_i}(X_i)) & (\text{Property of } \Omega) \\
& = & \bigcap_{i \in I} g_i^{-1}(\Delta_{\mathbf{A}_i}) & (\Omega_{\mathcal{A}_i}(X) = \Delta_{\mathbf{A}_i}) \\
& = & \bigcap_{i \in I} \mathrm{Ker}(g_i) & (\text{Definition of } \mathrm{Ker}(g_i)) \\
& = & \Delta_{\mathbf{A}}. & (\text{Assumption})
\end{array}$$

Thus, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ∎

Corollary 178 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be an algebraic logicoid. Then* $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$. *Moreover,* $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$ *if and only if* $\mathrm{Alg}^*(\mathbb{L})$ *is closed under subdirect intersections.*

We may also relate the classes of algebras of two logicoids over the same grid that are themselves related by the $\leq^\flat$ relation. Recall that, given an algebraic grid $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ and two logicoids $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{B}}, C'^\flat \rangle$, we write $\mathbb{L} \leq^\flat \mathbb{L}'$ if and only if, for all $X \subseteq B$,

$$C'^\flat(X) \leq^\flat C^\flat(X).$$

Recall, also, that this is equivalent to $\mathcal{C}'^\flat \subseteq \mathcal{C}^\flat$. Proposition 179 is an analog of Proposition 73 in the context of logicoids.

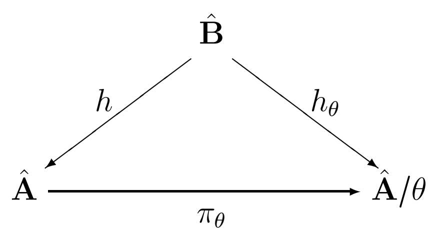
Proposition 179 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{B}}, C'^\flat \rangle$ *be algebraic logicoids over the same grid* $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$*, such that* $\mathbb{L} \leq^\flat \mathbb{L}'$. *Then* $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$ *and* $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathbb{L} \leq^\flat \mathbb{L}'$. Then, for all interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}'}(\mathcal{A}) \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$. By Theorem 177, we also have $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$. ∎

8.5 An Isomorphism Theorem

We fix a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ and an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. Let $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Recall that

$$\mathcal{A}/\theta = \langle \hat{\mathbf{A}}/\theta, h_\theta \rangle,$$



where $h_\theta = \pi_\theta \circ h$, with $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ being the quotient grid morphism. Consider the model $\langle \mathcal{A}/\theta, C \rangle$, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Define

$$\widetilde{H}_{\mathcal{A}}(\theta) \coloneqq \langle \mathcal{A}, C_\theta \rangle,$$

where $\langle \mathcal{A}, C_\theta \rangle$ is the algebraic logicoid induced by $\langle \langle \hat{\mathbf{A}}/\theta, \pi_\theta \rangle, C \rangle$ on $\mathbf{A}$. This defines a function

$$\begin{array}{rrcl} \widetilde{H}_{\mathcal{A}}(\theta) : & \mathrm{Con}(\hat{\mathbf{A}}) & \longrightarrow & \mathrm{Lgcd}(\mathcal{A}); \\ & \theta & \longmapsto & \langle \mathcal{A}, C_\theta \rangle. \end{array}$$

Note that, by Proposition 154, we have that

$$\pi_\theta : \widetilde{H}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C \rangle$$

is a bilogical morphism.

Lemma 180 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid,* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ *a fixed interpretation and* $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$.

(a) $\theta \in \mathrm{Con}(\widetilde{H}_{\mathcal{A}}(\theta))$;

(b) $\widetilde{H}_{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}/\theta, C \rangle$;

(c) $\widetilde{H}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$;

(d) $\theta \mapsto \widetilde{H}_{\mathcal{A}}(\theta)$ *is order preserving, i.e., if* $\theta \subseteq \theta'$*, then* $\widetilde{H}_{\mathcal{A}}(\theta) \leq^\flat \widetilde{H}_{\mathcal{A}}(\theta')$.

Proof:

(a) By Proposition 154, $\pi_\theta : \widetilde{H}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C \rangle$ is a bilogical morphism. By Proposition 132, $\theta \in \mathrm{Con}(\widetilde{H}_{\mathcal{A}}(\theta))$.

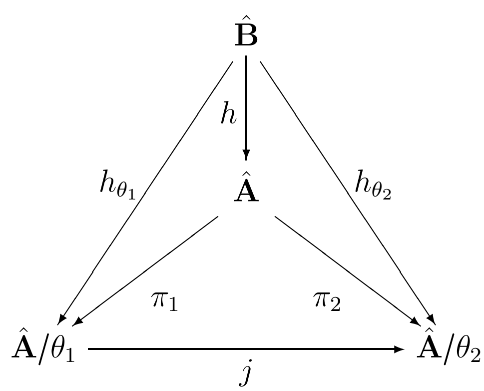
(b) We have, for all $S \subseteq A/\theta$,

$$\begin{array}{rcll} (C_\theta/\theta)(S) & = & \pi_\theta(C_\theta(\pi_\theta^{-1}(S))) & \text{(Definition of } C_\theta/\theta) \\ & = & \pi_\theta\left(\bigwedge \pi_\theta^{-1}(\mathcal{C})^{\pi_\theta^{-1}(S)}\right) & \text{(Definition of } C_\theta) \\ & = & \pi_\theta(\pi_\theta^{-1}(C(S))) & (\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta) \\ & = & C(S). & \text{(Surjectivity)} \end{array}$$

Thus, $\widetilde{H}_{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}/\theta, C \rangle$.

(c) By hypothesis, $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Thus, by Proposition 165, $\langle \mathcal{A}/\theta, C \rangle$ is a full model of $\mathbb{L}$. Thus, by Proposition 166, $\widetilde{H}_{\mathcal{A}}(\theta)$ is also a full model of $\mathbb{L}$.

(d) Let $\theta_1, \theta_2 \in \mathrm{Con}(\hat{\mathbf{A}})$, such that $\theta_1 \subseteq \theta_2$. Let $\pi_1 : \mathcal{A} \to \mathcal{A}/\theta_1$ and $\pi_2 : \mathcal{A} \to \mathcal{A}/\theta_2$ be the canonical projections. Let, also, $j : \mathcal{A}/\theta_1 \to \mathcal{A}/\theta_2$ be the map given by $a/\theta_1 \mapsto a/\theta_2$, which is well defined due to the inclusion $\theta_1 \subseteq \theta_2$. In addition, we have the following commutative diagram.



Now we get

$$\begin{array}{rcll}\mathcal{C}_{\theta_2} & = & \pi_2^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2)) & (\pi_2 : \widetilde{H}_{\mathcal{A}}(\theta_2) \to_b \langle \mathcal{A}/\theta_2, C_2\rangle) \\ & = & \pi_1^{-1}(j^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2)) & (\pi_2 = j \circ \pi_1) \\ & \subseteq & \pi_1^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_1)) & (\text{Proposition 148}) \\ & = & \mathcal{C}_{\theta_1}. & (\pi_1 : \widetilde{H}_{\mathcal{A}}(\theta_1) \to_b \langle \mathcal{A}/\theta_1, C_1\rangle)\end{array}$$

This shows that $\widetilde{H}_{\mathcal{A}}(\theta_1) \leq \widetilde{H}_{\mathcal{A}}(\theta_2)$.

■

Now we are in a position to prove a general analog of the Isomorphism Theorem of Font and Jansana (Theorem 2.30 of [12]) which is applicable even in contexts involving non-monotonicity. This result becomes applicable very widely, while its restriction to traditional $\subseteq$-closure operators is an Isomorphism Theorem very much resembling the one of Font and Jansana modulo the introduction of fixed interpretations. In this latter respect, we follow more closely the generalization of Font and Jansana's result that was presented as Theorem 13 of [21] for logical systems formalized as π-institutions. The Isomorphism Theorem 75 for logicates, proved in Chapter 4, is also a result in the same tradition.

Theorem 181 (Isomorphism) *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ a fixed interpretation. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is an order isomorphism between the ordered set $\mathbf{FMod}_{\mathbb{L}}(\mathcal{A}) = \langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A}), \leq\rangle$ of full models of $\mathbb{L}$ on $\mathcal{A}$ and the ordered set $\mathbf{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}) = \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ of $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$, ordered under inclusion. The mapping $\widetilde{H}_{\mathcal{A}}$ is its inverse.*

Proof: By Proposition 172, if $\mathbb{A} \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$, then $\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By Lemma 180, if $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$, then $\widetilde{H}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. So it suffices to show that $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse mappings and that they are both order preserving.

Let $\mathbb{A} = \langle \mathcal{A}, C\rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. By Proposition 172, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}_{\mathcal{A}}(C) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. As $\mathbb{A}$ is induced by its reduction $\mathbb{A}^* = \langle \mathcal{A}^*, C^*\rangle$, with $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, along the quotient grid morphism $\pi : \mathcal{A} \to \mathcal{A}^*$, we get, by definition, that $\mathbb{A} = \widetilde{H}_{\mathcal{A}}(\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}))$.

Suppose, conversely, that $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Consider $\mathbb{A}^\theta = \langle \mathcal{A}^\theta, C\rangle$, where $\mathcal{A}^\theta = \langle \hat{\mathbf{A}}/\theta, \pi_\theta \circ h\rangle$ and $C = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^\theta)$. Then $\widetilde{\Omega}_{\mathcal{A}^\theta}(C) = \Delta_{\mathbf{A}/\theta}$. Thus, by Proposition 148,

$$\begin{array}{rcl}\widetilde{\Omega}_{\mathcal{A}}(\widetilde{H}_{\mathcal{A}}(\theta)) & = & \widetilde{\Omega}_{\mathcal{A}}(\pi_\theta^{-1}(C)) \\ & = & \pi_\theta^{-1}(\widetilde{\Omega}_{\mathcal{A}^\theta}(C)) \\ & = & \pi_\theta^{-1}(\Delta_{\mathbf{A}/\theta}) \\ & = & \theta.\end{array}$$

Hence, $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse bijections. $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving by definition. Finally, by Lemma 180, $\widetilde{H}_{\mathcal{A}}$ is also order preserving. This shows that

$$\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A}), \leq \rangle \underset{\widetilde{H}_{\mathcal{A}}}{\overset{\widetilde{\Omega}_{\mathcal{A}}}{\rightleftarrows}} \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle$$

are inverse order isomorphisms. ■