

Chapter 9

Aspects of the Hierarchy

9.1 Introduction

Chapter 8 is intended to only provide a relatively superficial flavor of a semantically defined algebraic hierarchy of classes of logicoids based on properties of their Leibniz operator, paralleling the classical one for monotonic logics (see, e.g., [8, 14, 10]). The best part of it (Sections 9.2-9.4) is dedicated to the study of *protoalgebraicity*, Section 9.5 looks briefly at *weak algebraizability* and Section 9.6 looks even more briefly at *truth equationality*. We give several characterizations of *protoalgebraicity*, which is the property of having a monotone Leibniz operator, and we study the *Correspondence Theorem* and several of its consequences. This segues nicely into the introduction of *Leibniz filters* and some of their properties. *Weak algebraizability* is the property of having both a monotone and an order reflecting Leibniz operator, whereas *truth equationality* is the property of having a completely order reflecting Leibniz operator. As in the traditional monotonic case, it turns out that weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

In Section 9.2, we introduce *protoalgebraic logicoids*. A logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ is *protoalgebraic* if, for all theories X and all $a, b \in B$, $\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ implies that $\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X)$, where $\Lambda_{\mathbb{L}}(X)$ is the relation holding if, for every theory X' , with $X \leq^b X'$, $a \in X'$ iff $b \in X'$. We show that protoalgebraicity is equivalent to the monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on the theories of \mathbb{L} . Moreover, \mathbb{L} is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is monotone on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation \mathcal{A} of \mathbb{L} . An additional characterization asserts that \mathcal{L} is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is submeetive on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$, meaning that, for all $\{X_i : i \in I\} \subseteq \text{Fi}_{\mathbb{L}}(\mathcal{A})$, $\Omega_{\mathcal{A}}(\bigwedge_{i \in I} X_i) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i)$.

In Section 9.3, the central task is proving a Correspondence Theorem, an analog of Theorem 6.19 of [10], for protoalgebraic logicoids. After doing this, we explore some of its consequences. Among these are some additional characterizations of protoalgebraicity using the Tarski operator. We show, e.g., that \mathbb{L} is protoalgebraic if and only if, for every \mathbb{L} -model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min C)$, where $\min C$ is the least theory of \mathbb{A} (or the set of theorems of \mathbb{A}). Another consequence is that, if the logicoid \mathbb{L} happens to be protoalgebraic, then the classes of interpretations $\text{Alg}^*(\mathbb{L})$ and $\text{Alg}(\mathbb{L})$ coincide. In addition, if \mathbb{L} is protoalgebraic, then any two logicoid models \mathbb{A} and \mathbb{A}' over the same underlying interpretation that share the same minimum theories must be identical. The last result of the section is a theorem characterizing the full models of a protoalgebraic logicoid, while, at the same time providing yet another characterization of protoalgebraicity. It asserts that \mathbb{L} is protoalgebraic if and only if its full models are of the form $\langle \mathcal{A}, C^F \rangle$, where $C^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$, for some interpretation \mathcal{A} and some filter F in $\text{Fi}_{\mathbb{L}}(\mathcal{A})$. Here, $\text{Fi}_{\mathbb{L}}(\mathcal{A})^F$ denotes the collection of all \mathbb{L} -filters on \mathcal{A} dominating F in the \leq ordering of the subsets of A in the underlying algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ of $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$.

Section 9.4 considers a question that arises naturally from the characterization of full models of protoalgebraic logicoïds. More precisely, it attempts to characterize those \mathbb{L} -filters F on an interpretation \mathcal{A} for which $\text{Fi}_{\mathbb{L}}(\mathcal{A})^F$ is a full \mathbb{L} -model. To do this, we form the subset of such filters $\text{Fi}_{\mathbb{L}}^*(\mathcal{A})$. These filters are termed *Leibniz filters*. If \mathbb{L} is protoalgebraic, $\Omega_{\mathcal{A}}$ turns out to be an order isomorphism from $\text{Fi}_{\mathbb{L}}^*(\mathcal{A})$, ordered by \leq onto $\text{Con}_{\text{Alg}^*}(\mathcal{A})$, ordered by \subseteq . Further, we introduce an equivalence \sim_{Ω} between \mathbb{L} -filters on an interpretation \mathcal{A} that "identifies" two filters if they have the same Leibniz grid congruence. The \sim_{Ω} -class of an \mathbb{L} -filter F is denoted by $[F]_{\Omega}$. If \mathbb{L} is protoalgebraic, then F is the minimum element in the \leq ordering in $[F]_{\Omega}$. This affords the characterization of Leibniz filters as those \mathbb{L} -filters on an interpretation that are minimum in their \sim_{Ω} -equivalence classes. Equivalently, they are the \mathbb{L} -filters F , whose quotients $F/\Omega_{\mathcal{A}}(F)$ are minimum \mathbb{L} -filters in the $\leq^{\Omega_{\mathcal{A}}(F)}$ ordering on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Section 9.5 deals with a second question that may be seen to arise from the characterization of full models of a protoalgebraic logicoïd. Namely, identify those situations for which the collection of Leibniz filters is the entire collection of filters on an interpretation. We call a logicoïd $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ *weakly algebraizable* if the Leibniz operator is order preserving and order reflecting on C^b . This is equivalent to the Leibniz operator $\Omega_{\mathcal{A}}$ being order preserving and order reflecting on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation \mathcal{A} of \mathbb{L} . Furthermore, it turns out that \mathbb{L} is weakly algebraizable if and only if $\text{Fi}_{\mathbb{L}}^*(\mathcal{A}) = \text{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation \mathcal{A} of \mathbb{L} . Thus, weak algebraizability settles the initial problem of discovering a property under which the collection of the Leibniz \mathbb{L} -filters on any interpretation coinciding with the entire collection of \mathbb{L} -filters. This characterization, combined with the results of Section 9.4, provides several additional characterizations of weak algebraizability. E.g., we get that \mathbb{L} is weakly algebraizable if and only if, for every interpretation \mathcal{A} and all \mathbb{L} -filters F on \mathcal{A} , $F/\Omega_{\mathcal{A}}(F)$ is the least \mathbb{L} -filter on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$ and that \mathbb{L} is weakly algebraizable if and only if $\Omega_{\mathcal{A}}$ is a lattice isomorphism from $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\text{Con}_{\text{Alg}^*}(\mathbb{L})(\mathcal{A})$.

In Section 9.6, we briefly introduce the property of *truth equationality*. We say that a logicoïd $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ is *truth equational* if $\Omega_{\hat{\mathbf{B}}}$ is completely order reflecting on C^b , that is, if, for all $\{X_i : i \in I\} \cup \{X\} \subseteq C^b$,

$$\bigcap_{i \in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies} \quad \bigwedge_{i \in I}^b X_i \leq^b X.$$

We show that this is equivalent to the complete order reflectivity of $\Omega_{\mathcal{A}}$ on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$, for all interpretations \mathcal{A} of \mathbb{L} . Finally, we prove that weak algebraizability is characterized as the conjunction of protoalgebraicity and truth equationality.

9.2 Protoalgebraic Logicoïds

Recall that an *algebraic grid* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ consists of an algebra \mathbf{A} and a complete lattice ordering \leq on $\mathcal{P}(A)$. Recall, further, that an *algebraic logicoïd* $\langle \hat{\mathbf{A}}, C \rangle$ consists of an algebraic grid $\hat{\mathbf{A}}$ and a \leq -closure operator $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$. That is, an operator that is inflationary, monotone and idempotent with respect to \leq . We use \mathcal{C} to denote the collection of all *theories* of C , i.e., subsets $X \subseteq A$, such that $C(X) = X$.

In the abstract study of logicoïds, a particular fixed logicoïd $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$, with $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^b \rangle$, is at the focus of investigations and it is called the *base logicoïd*. Both matrix (Chapter 7) and logicoïd (Chapter 8) models of the base logicoïd are based on interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, which are grid morphisms from the base grid $\hat{\mathbf{B}}$ onto a grid $\hat{\mathbf{A}}$ over a similar algebra. A *grid morphism* $h : \hat{\mathbf{B}} \rightarrow \hat{\mathbf{A}}$ is a surjective algebra homomorphism $h : \mathbf{B} \rightarrow \mathbf{A}$, such that $h^{-1} : \langle \mathcal{P}(A), \leq \rangle \rightarrow \langle \mathcal{P}(B), \leq^b \rangle$ is a complete lattice embedding.

Let $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^b \rangle$ be an algebraic grid. A *grid congruence* θ on $\hat{\mathbf{B}}$ is a congruence on \mathbf{B} , such that $\langle \text{Cmp}(\theta), \leq^b \rangle$ is a complete sublattice of $\langle \mathcal{P}(B), \leq^b \rangle$. Here $\text{Cmp}(\theta)$ denotes the set of all subsets of B with which θ is compatible. Given an $X \subseteq B$, the *Leibniz congruence* $\Omega_{\hat{\mathbf{B}}}(X)$ of the logical matrix $\mathfrak{A} = \langle \hat{\mathbf{B}}, X \rangle$ is the largest grid congruence on $\hat{\mathbf{B}}$ that is compatible with X . Given a \leq -closure operator C on $\hat{\mathbf{B}}$, the *Tarski congruence* $\tilde{\Omega}_{\hat{\mathbf{B}}}(C)$ of the logicoïd $\mathbb{L} = \langle \hat{\mathbf{B}}, C \rangle$ is the largest grid congruence on $\hat{\mathbf{B}}$ that is compatible with \mathcal{C} .

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be an algebraic logicoïd. We say that \mathbb{L} is **protoalgebraic** (see [2] and, also, [8, 12, 10]) if, for all $a, b \in B$ and all $X \in \mathcal{C}^b$,

$$\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X) \text{ implies, for all } X \leq^b X' \in \mathcal{C}^b, \\ a \in X' \text{ iff } b \in X'.$$

For the corresponding definition for logicates, see Section 5.2. We make an observation and then introduce some notation that will help abbreviate the definition.

Observe that protoalgebraicity depends only on the collection of theories of a logicoïd. This is commensurate with the monotonic theory, where protoalgebraicity depends only on the theory lattice of a sentential logic. Note, also, that, as in the monotonic framework, in the framework of logicoïds, the theory lattice fully determines the logicoïd itself, due to the presence of the underlying grid.

Given a logicoïd $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ and $X \subseteq B$, we define a relation $\Lambda_{\mathbb{L}}(X)$ on B with the goal of capturing the defining property of protoalgebraicity. We set, for all $X \subseteq B$ and all $a, b \in B$,

$$\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X) \text{ iff, for all } X \leq^b X' \in \mathcal{C}^b, \\ a \in X' \text{ iff } b \in X'.$$

With this definition available, we may rephrase the definition of protoalgebraicity. Clearly, \mathbb{L} is **protoalgebraic** if and only if, for all $X \in \mathcal{C}^b$,

$$\Omega_{\hat{\mathbf{B}}}(X) \subseteq \Lambda_{\mathbb{L}}(X).$$

It is well known that, in the traditional framework, protoalgebraicity is tantamount to the monotonicity of the Leibniz operator (see [3]) on the lattice of theories of the logic (see, e.g., [8, 12, 10]). An analogous result was proven for protoalgebraic logicates in Proposition 80. The following proposition revisits this characterization in the context of logicooids.

Proposition 182 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be an algebraic logicooid. \mathbb{L} is protoalgebraic if and only if $\Omega_{\hat{\mathbf{B}}}$ is monotone on \mathcal{C}^b .*

Proof: Suppose \mathbb{L} is protoalgebraic and let $X, X' \in \mathcal{C}^b$, such that $X \leq^b X'$. Let $a, b \in B$, such that $\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ and $a \in X'$. By protoalgebraicity, $\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X)$ and $a \in X'$. Since $X \leq^b X'$, $b \in X'$. This shows that $\Omega_{\hat{\mathbf{B}}}(X)$ is compatible with X' . By the maximality property of $\Omega_{\hat{\mathbf{B}}}(X')$ with respect to compatibility with X' , we conclude that $\Omega_{\hat{\mathbf{B}}}(X) \subseteq \Omega_{\hat{\mathbf{B}}}(X')$. Thus, $\Omega_{\hat{\mathbf{B}}}$ is monotone on \mathcal{C}^b .

Suppose, conversely, that $\Omega_{\hat{\mathbf{B}}}$ is monotone on \mathcal{C}^b . Let $a, b \in B$, $X \in \mathcal{C}^b$, such that $\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ and $X' \in \mathcal{C}^b$, with $X \leq^b X'$. Then $\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X')$. So, by the compatibility of $\Omega_{\hat{\mathbf{B}}}(X')$ with X' , $a \in X'$ iff $b \in X'$. Thus, \mathbb{L} is protoalgebraic. ■

Protoalgebraicity of logicooids offers an opportunity of formulating another transfer theorem, analogous to Proposition 81. It extends monotonicity of the Leibniz operator to the monotonicity of the Leibniz operator on the \mathbb{L} -filters of any interpretation.

Proposition 183 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicooid. \mathbb{L} is protoalgebraic if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\Omega_{\mathcal{A}}$ is monotone on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$.*

Proof: Suppose that \mathbb{L} is protoalgebraic. Let $Y_1, Y_2 \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $Y_1 \leq Y_2$, and $a_1, a_2 \in A$. Note that, since $h : \hat{\mathbf{B}} \rightarrow \hat{\mathbf{A}}$ is a grid morphism, there exist $b_1, b_2 \in B$, such that $a_1 = h(b_1)$ and $a_2 = h(b_2)$. Now we have

$$\begin{aligned} \langle a_1, a_2 \rangle \in \Omega_{\mathcal{A}}(Y_1) & \quad \text{iff} \quad \langle h(b_1), h(b_2) \rangle \in \Omega_{\mathcal{A}}(Y_1) \\ & \quad \text{iff} \quad \langle b_1, b_2 \rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_1)) \\ & \quad \text{iff} \quad \langle b_1, b_2 \rangle \in \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y_1)) \\ & \quad \text{implies} \quad \langle b_1, b_2 \rangle \in \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y_2)) \\ & \quad \text{iff} \quad \langle b_1, b_2 \rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_2)) \\ & \quad \text{iff} \quad \langle h(b_1), h(b_2) \rangle \in \Omega_{\mathcal{A}}(Y_2) \\ & \quad \text{iff} \quad \langle a_1, a_2 \rangle \in \Omega_{\mathcal{A}}(Y_2). \end{aligned}$$

Note that we have used the property that the Leibniz operator commutes with inverse grid morphisms. We have now shown that $\Omega_{\mathcal{A}}$ is monotone on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, if the condition in the statement holds, then the monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on \mathcal{C}^b follows by taking $\mathcal{A} = \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle$. Then the conclusion follows from Proposition 182 and the observation that $\mathcal{C}^b = \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$. ■

One may also devise a slightly different characterization involving meets. Let us show, first, that, for every interpretation, the collection of \mathbb{L} -filters on the interpretation is closed under arbitrary meets.

Lemma 184 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicoid. Then, for any interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, the set $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$ of \mathbb{L} -filters on \mathcal{A} is closed under arbitrary meets.*

Proof: Let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $\{X_i : i \in I\} \subseteq \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$ be a collection of \mathbb{L} -filters on \mathcal{A} . Then, since h^{-1} is a complete lattice embedding,

$$h^{-1} \left(\bigwedge_{i \in I} X_i \right) = \bigwedge_{i \in I}^b h^{-1}(X_i) \in \mathcal{C}^b,$$

where membership follows by the definition of \mathbb{L} -filter and Proposition 102. This shows that $\bigwedge_{i \in I} X_i \in \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$. ■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. We say that the Leibniz operator is **submeective** on $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, if, for all $\{X_i : i \in I\} \subseteq \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\Omega_{\mathcal{A}} \left(\bigwedge_{i \in I} X_i \right) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i).$$

We show that protoalgebraicity is equivalent to the Leibniz operator on the filters of any interpretation being submeective. This property is an adaptation of the “difficult half” of the well-known property of “commuting with intersections”. Except that, in the present setting, since in $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, \subseteq has been replaced by (an arbitrary complete lattice ordering) \leq , it may be that the “easy half” may not hold.

Proposition 185 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicoid. \mathbb{L} is protoalgebraic if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\Omega_{\mathcal{A}}$ is submeective on $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., for all $\{X_i : i \in I\} \subseteq \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$,*

$$\Omega_{\mathcal{A}} \left(\bigwedge_{i \in I} X_i \right) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i).$$

Proof: By Proposition 183, protoalgebraicity is equivalent to the monotonicity of $\Omega_{\mathcal{A}}$ on $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$.

Suppose, first, that $\Omega_{\mathcal{A}}$ is monotone. Let $\{X_i : i \in I\} \subseteq \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$. By Lemma 184, $\bigwedge_{i \in I} X_i \in \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$. By monotonicity, $\Omega_{\mathcal{A}}(\bigwedge_{i \in I} X_i) \subseteq \Omega_{\mathcal{A}}(X_i)$, for all $i \in I$. Thus, $\Omega_{\mathcal{A}}(\bigwedge_{i \in I} X_i) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i)$.

Suppose, next, that, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\Omega_{\mathcal{A}}$ is submeective on $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$. Let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $X, X' \in \mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \leq X'$. Then $X \wedge X' = X$ and we have

$$\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X \wedge X') \subseteq \Omega_{\mathcal{A}}(X) \cap \Omega_{\mathcal{A}}(X') \subseteq \Omega_{\mathcal{A}}(X').$$

So $\Omega_{\mathcal{A}}$ is monotone on $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$ and, hence, \mathbb{L} is protoalgebraic. ■

9.3 Correspondence Theorem

Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, \leq^b \rangle$, an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, with $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, and a filter $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, we write

$$\text{Fi}_{\mathbb{L}}(\mathcal{A})^F := \{X \in \text{Fi}_{\mathbb{L}}(\mathcal{A}) : F \leq X\}.$$

The well-known Correspondence Theorem for protoalgebraic logics (see, e.g., Theorem 6.19 of [10] and Proposition 85 for a logicate version) dealing with the structure of theories and filters may be adapted to the present context. Here, it establishes an isomorphism between complete lattices of filters on an interpretation and on the quotient of an interpretation by a Leibniz congruence.

Proposition 186 (Correspondence) *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a base logicoid. If \mathbb{L} is protoalgebraic, then, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and every $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, letting $\pi : \hat{\mathbf{A}} \rightarrow \hat{\mathbf{A}}/\Omega_{\mathcal{A}}(F)$ be the quotient grid morphism,*

$$\begin{aligned} \pi : \text{Fi}_{\mathbb{L}}(\mathcal{A})^F &\longrightarrow \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}; \\ X &\longmapsto \pi(X), \end{aligned}$$

establishes an isomorphism between the complete lattice $\langle \text{Fi}_{\mathbb{L}}(\mathcal{A})^F, \leq \rangle$ and the complete lattice $\langle \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \leq^{\Omega_{\mathcal{A}}(F)} \rangle$.

Proof: Let $X \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $F \leq X$. By protoalgebraicity, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(X)$. Hence $\Omega_{\mathcal{A}}(F)$ is compatible with X . It follows that, for $\pi : \mathcal{A} \rightarrow \mathcal{A}/\Omega_{\mathcal{A}}(F)$, $X = \pi^{-1}(\pi(X))$. Thus, by Proposition 148, we obtain $\pi(X) \in \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Clearly, since $F \leq X$ and π^{-1} is a complete lattice embedding, $\pi(F) \leq^{\Omega_{\mathcal{A}}(F)} \pi(X)$. On the other hand, if $Y \in \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, then, again by Proposition 148, $\pi^{-1}(Y) \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover, $\pi(F) \leq^{\Omega_{\mathcal{A}}(F)} Y$ and π^{-1} a complete lattice embedding imply $F = \pi^{-1}(\pi(F)) \leq \pi^{-1}(Y)$. Thus, π establishes an isomorphism between the ordered set $\langle \text{Fi}_{\mathbb{L}}(\mathcal{A})^F, \leq \rangle$ and the ordered set $\langle \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \leq^{\Omega_{\mathcal{A}}(F)} \rangle$, as claimed. \blacksquare

We now provide some additional characterizations of protoalgebraicity in terms of the Tarski operator. Given an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, with $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, a logicoid $\mathbb{A} = \langle \hat{\mathbf{A}}, C \rangle$ and a theory F of C , we shall write $\mathbb{A}^F = \langle \mathcal{A}, C^F \rangle$ for the logicoid with

$$C^F = \{X \in C : F \leq X\}.$$

Proposition 187 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a base logicate. The following statements are equivalent.*

- (i) \mathbb{L} is protoalgebraic;
- (ii) For any \mathbb{L} -model $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle$, $\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$;
- (iii) For any \mathbb{L} -model $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle$, with $Y \in \mathcal{C}$, $\tilde{\Omega}(\mathbb{A}^Y) = \Omega_{\mathcal{A}}(Y)$;
- (iv) For any $X \in \mathcal{C}^b$, $\tilde{\Omega}(\mathbb{L}^X) = \Omega_{\hat{\mathbb{B}}}(X)$.

Proof:

- (i) \Rightarrow (ii) Suppose $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle \in \text{Mod}(\mathbb{L})$, where $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. Then, by Proposition 163, $\mathcal{C} \subseteq \text{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 183, $\Omega_{\mathcal{A}}$ is order preserving on \mathcal{C} . By definition of the Leibniz congruence and its monotonicity, $\Omega_{\mathcal{A}}(\min \mathcal{C})$ is a grid congruence on $\hat{\mathbf{A}}$ compatible with all $X \in \mathcal{C}$. Hence, by the definition of the Tarski congruence, $\Omega_{\mathcal{A}}(\min \mathcal{C}) \subseteq \tilde{\Omega}(\mathbb{A})$. On the other hand, $\tilde{\Omega}(\mathbb{A})$ is a grid congruence on $\hat{\mathbf{A}}$, compatible with all $X \in \mathcal{C}$ and, thus, in particular with $\min \mathcal{C}$. Hence, by the definition of the Leibniz congruence, $\tilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(\min \mathcal{C})$. So we get $\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$.
- (ii) \Rightarrow (iii) Trivial.
- (iii) \Rightarrow (iv) Trivial.
- (iv) \Rightarrow (i) Let $X, X' \in \mathcal{C}^b$, such that $X \subseteq X'$. Then $X' \in \mathcal{C}^{b^X}$. Thus, we get

$$\begin{aligned} \Omega_{\hat{\mathbb{B}}}(X) &= \tilde{\Omega}(\mathbb{L}^X) \quad (\text{Hypothesis (iv)}) \\ &\subseteq \bigcap_{Y \in \mathcal{C}^{b^X}} \Omega_{\hat{\mathbb{B}}}(Y) \quad (\text{Tarski Congruence}) \\ &\subseteq \Omega_{\hat{\mathbb{B}}}(X'). \quad (X' \in \mathcal{C}^{b^X}) \end{aligned}$$

So $\Omega_{\hat{\mathbb{B}}}$ is monotone on \mathcal{C}^b , showing that \mathbb{L} is protoalgebraic. ■

Let $\mathbb{L} = \langle \hat{\mathbb{B}}, \mathcal{C}^b \rangle$ be a protoalgebraic logicoid and consider an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. By Proposition 187, for $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle$, with $\mathcal{C} = \text{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \text{Fi}_{\mathbb{L}}(\mathcal{A})).$$

The following proposition is an analog of Proposition 3.2 of [12] in the present setting (see, also Proposition 87 for logicates).

Proposition 188 *Let $\mathbb{L} = \langle \hat{\mathbb{B}}, \mathcal{C}^b \rangle$ be a protoalgebraic logicoid. Then*

$$\text{Alg}(\mathbb{L}) = \text{Alg}^*(\mathbb{L}).$$

Proof: By Corollary 178, we have $\text{Alg}^*(\mathbb{L}) \subseteq \text{Alg}(\mathbb{L})$, without any preconditions. Suppose, conversely, that $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle \in \text{Alg}(\mathbb{L})$. Then, for $\mathcal{C} = \text{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $\tilde{\Omega}(\langle \mathcal{A}, \mathcal{C} \rangle) = \Delta_{\mathbf{A}}$. By hypothesis and Proposition 187,

$$\Omega_{\mathcal{A}}(\min \mathcal{C}) = \tilde{\Omega}(\langle \mathcal{A}, \mathcal{C} \rangle) = \Delta_{\mathbf{A}}.$$

This shows that $\mathcal{A} \in \text{Alg}^*(\mathbb{L})$. ■

Lemma 3.3 of [12] asserts that two protoalgebraic logics over the same signature that share the same sets of theorems are identical. The following analog requires the two protoalgebraic logicoids compared to have identical minimum theories and, in that case, asserts that the logicoids in question coincide.

Lemma 189 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a protoalgebraic logicoid and $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$ be two full models of \mathbb{L} over the same interpretation. Then*

$$\min C = \min C' \quad \text{implies} \quad C = C'.$$

Proof: By hypothesis and Proposition 187,

$$\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min C) = \Omega_{\mathcal{A}}(\min C') = \tilde{\Omega}(\mathbb{A}').$$

Thus, by the Isomorphism Theorem 181, $C = C'$. ■

Protoalgebraicity in the monotonic theory was characterized in terms of full models in Theorem 3.4 of [12]. A similar characterization is possible in the case of logicoids. This forms a logicoid analog of Theorem 89 applicable to logicates.

Theorem 190 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a base logicoid. \mathbb{L} is protoalgebraic if and only if all full models of \mathbb{L} are of the form $\langle \mathcal{A}, C^F \rangle$, with $C^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$, for some interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and some $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$.*

Proof: We work, first, to prove the “only if”. Let $\mathbb{A} = \langle \mathcal{A}, C \rangle$ be a full model of \mathbb{L} , with $F = \min C$. By protoalgebraicity and Proposition 187, $\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(F)$. Hence, the quotient grid morphism $\pi : \mathcal{A} \rightarrow \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \langle \mathcal{A}, C \rangle \rightarrow \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), C^{\Omega_{\mathcal{A}}(F)} \rangle.$$

Since, by hypothesis, \mathbb{A} is a full model of \mathbb{L} , $C^{\Omega_{\mathcal{A}}(F)} = \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Consider $X \in \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Then $F \leq X$ and, by protoalgebraicity, $\Omega_{\mathcal{A}}(F)$ is compatible with X . Thus, $X = \pi^{-1}(\pi(X))$. By Proposition 148, $\pi(X) \in \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Hence, since $X = \pi^{-1}(\pi(X))$, $X \in C$. This proves that $C = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$.

We turn, next, to the “if”. Suppose that all models of \mathbb{L} have the indicated form. Let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $X, X' \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \leq X'$. Consider $\Omega_{\mathcal{A}}(X)$. By Corollary 178, $\text{Alg}^*(\mathbb{L}) \subseteq \text{Alg}(\mathbb{L})$. Hence $\Omega_{\mathcal{A}}(X) \in \text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A})$. By the Isomorphism Theorem 181, there exists a full model $\mathbb{A} = \langle \mathcal{A}, C \rangle$ of \mathbb{L} , such that $\tilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(X)$. Moreover, by hypothesis,

there exists $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\mathcal{C} = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Since \mathbb{A} is full, the quotient grid morphism $\pi : \mathcal{A} \rightarrow \mathcal{A}/\Omega_{\mathcal{A}}(X)$ is a biological morphism

$$\pi : \mathbb{A} \rightarrow \langle \mathcal{A}/\Omega_{\mathcal{A}}(X), \mathcal{C}^{\Omega_{\mathcal{A}}(X)} \rangle,$$

where $\mathcal{C}^{\Omega_{\mathcal{A}}(X)} = \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(X))$. Moreover, as $X = \pi^{-1}(\pi(X))$, we get $X \in \mathcal{C}$. Hence, $F \leq X \leq X'$, whence, $X' \in \mathcal{C}$. Now we get

$$\Omega_{\mathcal{A}}(X) = \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X'),$$

i.e., $\Omega_{\mathcal{A}}$ is monotone on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 183, \mathbb{L} is protoalgebraic. ■

9.4 Leibniz Filters

Leibniz filters were introduced by Font and Jansana in [12] (see Page 63), extensively studied in [11] and [17], and used further in applications of the theory in [13] and, in the case of logicates, in Section 5.4. Here we define an analog in the framework of logicoïds.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a protoalgebraic logicoïd and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. We define

$$\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A}) = \{F \in \text{Fi}_{\mathbb{L}}(\mathcal{A}) : \text{if } \mathcal{C} = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F, \text{ then } \langle \mathcal{A}, \mathcal{C} \rangle \in \text{FMod}_{\mathbb{L}}(\mathcal{A})\}.$$

The elements in $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$ are called **Leibniz filters of \mathbb{L} on \mathcal{A}** .

Proposition 191 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a protoalgebraic logicoïd and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. Then $\Omega_{\mathcal{A}}$ is a poset isomorphism*

$$\Omega_{\mathcal{A}} : \langle \text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A}), \leq \rangle \cong \langle \text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle = \langle \text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A}), \subseteq \rangle.$$

Proof: Consider the mapping

$$F \mapsto \langle \mathcal{A}, C^F \rangle,$$

where $C^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$. By the definition of $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$, this is a mapping from $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$ to $\text{FMod}_{\mathbb{L}}(\mathcal{A})$. It is injective and it is \leq -order preserving and order reflecting. By protoalgebraicity and Theorem 190, it is also surjective. So it is an order isomorphism from $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$ to $\text{FMod}_{\mathbb{L}}(\mathcal{A})$. By the Isomorphism Theorem 181, $\text{FMod}_{\mathbb{L}}(\mathcal{A})$ is isomorphic to $\text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A})$ via the Tarski operator. Thus, the composition

$$F \mapsto \widetilde{\Omega}_{\mathcal{A}}(\text{Fi}_{\mathbb{L}}(\mathcal{A})^F)$$

establishes an order isomorphism between $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$ and $\text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 187, $\widetilde{\Omega}(\text{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$. By protoalgebraicity and Proposition 188, $\text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A}) = \text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A})$. Therefore, the Leibniz operator is an order isomorphism from $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$ to $\text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A})$. ■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a protoalgebraic logicoid. The \mathbb{L} -filters in $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$ on a given interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ may be characterized without reference to full models. To show this, we consider a binary relation \sim_{Ω} on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ defined as the kernel of the Leibniz operator on \mathcal{A} , i.e., for all $X, X' \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$X \sim_{\Omega} X' \quad \text{iff} \quad \Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X').$$

We write $[X]_{\Omega}$ for the \sim_{Ω} -equivalence class of an \mathbb{L} -filter X . By Lemma 184, $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ is closed under arbitrary meets. A fortiori, every full model of \mathbb{L} on \mathcal{A} has a minimum filter. Thus, if \mathbb{L} is protoalgebraic, by Proposition 191, at most one \mathbb{L} -filter in each \sim_{Ω} -equivalence class is in $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$. As in Proposition 3.6 of [12], it is possible to characterize this filter. However, since in the present setting \sqsubseteq -monotonicity is lost, it is not necessarily the case that this filter is the intersection of all filters in the class, as in Page 64 of [12] (see paragraph before Proposition 3.6). Instead, we need to use a slightly different argument taking into account the definition of a Leibniz filter.

Lemma 192 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a protoalgebraic logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. If $F \in \text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$, then $F = \min [F]_{\Omega}$.*

Proof: Suppose $F \in \text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$. By definition of Leibniz filters, $\langle \mathcal{A}, \text{Fi}_{\mathbb{L}}(\mathcal{A})^F \rangle \in \text{FMod}(\mathbb{L})$. By protoalgebraicity, Proposition 187 and the definition of a full model, the quotient grid morphism $\pi : \mathcal{A} \rightarrow \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a biological morphism

$$\pi : \langle \mathcal{A}, \text{Fi}_{\mathbb{L}}(\mathcal{A})^F \rangle \rightarrow_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F)) \rangle.$$

Now consider $X \in [F]_{\Omega}$. Then

$$\begin{aligned} (\pi \circ h)^{-1}(X/\Omega_{\mathcal{A}}(F)) &= h^{-1}(\pi^{-1}(X/\Omega_{\mathcal{A}}(F))) \quad ((\pi \circ h)^{-1} = h^{-1} \circ \pi^{-1}) \\ &= h^{-1}(X) \quad (\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(F)) \\ &\in C^b. \quad (X \in \text{Fi}_{\mathbb{L}}(\mathcal{A})) \end{aligned}$$

Hence, by definition of a filter, $X/\Omega_{\mathcal{A}}(F) \in \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Then, since π is a biological morphism,

$$X = \pi^{-1}(X/\Omega_{\mathcal{A}}(F)) \in \text{Fi}_{\mathbb{L}}(\mathcal{A})^F.$$

This shows that $F \leq X$. Since X was an arbitrary element in $[F]_{\Omega}$, we conclude that $F = \min [F]_{\Omega}$. ■

Now we may characterize the class of Leibniz filters of \mathbb{L} on \mathcal{A} . This is an analog for logicoids of Proposition 3.6 of [12] and of Proposition 91.

Proposition 193 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a protoalgebraic logicate, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation and $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$. The following statements are equivalent:*

- (i) $F \in \text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A})$, i.e., $\langle \mathcal{A}, C^F \rangle$, with $C^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of \mathbb{L} ;
- (ii) F is the minimum element in its \sim_{Ω} -equivalence class;
- (iii) $F/\Omega_{\mathcal{A}}(F)$ is the least \mathbb{L} -filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Proof:

(i) \Rightarrow (ii) This is the content of Lemma 192.

(ii) \Rightarrow (iii) Suppose $F = \min[F]$ and let $G \in \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Our goal is to show that $F/\Omega_{\mathcal{A}}(F) \leq^{\Omega_{\mathcal{A}}(F)} G$. Let $\pi : \mathcal{A} \rightarrow \mathcal{A}/\Omega_{\mathcal{A}}(F)$ be the quotient grid morphism and set $F' = \pi^{-1}(G) \wedge F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, where membership is due to Proposition 148, the hypothesis and Lemma 184. Then

$$\begin{aligned} F' &= \pi^{-1}(G) \wedge \pi^{-1}(\pi(F)) \quad (\text{Compatibility of } \Omega_{\mathcal{A}}(F) \text{ with } F) \\ &= \pi^{-1}(G \wedge^{\Omega_{\mathcal{A}}(F)} \pi(F)). \quad (\pi \text{ grid morphism}) \end{aligned}$$

Hence, F' is a union of $\Omega_{\mathcal{A}}(F)$ -classes, i.e., $\Omega_{\mathcal{A}}(F)$ is compatible with F' . By the maximality property of the Leibniz congruence, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(F')$. As, by definition, $F' \leq F$, by protoalgebraicity, $\Omega_{\mathcal{A}}(F') \subseteq \Omega_{\mathcal{A}}(F)$. Consequently, $\Omega_{\mathcal{A}}(F') = \Omega_{\mathcal{A}}(F)$, i.e., $F \sim_{\Omega} F'$. By hypothesis, $F \leq F'$ and, since, by definition, $F' \leq F$, $F = F'$. Thus, $F \leq \pi^{-1}(G)$. This yields

$$F/\Omega_{\mathcal{A}}(F) = \pi(F) \leq^{\Omega_{\mathcal{A}}(F)} \pi(\pi^{-1}(G)) = G.$$

Therefore, $F/\Omega_{\mathcal{A}}(F)$ is the least \mathbb{L} -filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

(iii) \Rightarrow (i) Assume $F/\Omega_{\mathcal{A}}(F) = \min \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. By protoalgebraicity and the correspondence established in Proposition 186, the quotient grid morphism $\pi : \mathcal{A} \rightarrow \mathcal{A}/\Omega_{\mathcal{A}}(F)$ gives an order isomorphism between $\text{Fi}_{\mathbb{L}}(\mathcal{A})^F$ and $\text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{F/\Omega_{\mathcal{A}}(F)} = \text{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Also by protoalgebraicity and Proposition 187,

$$\widetilde{\Omega}_{\mathcal{A}}(\text{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F).$$

Hence $\langle \mathcal{A}, C^F \rangle$, with $C^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of \mathbb{L} . ■

9.5 Weak Algebraizability

In [3], Blok and Pigozzi introduced the notion of algebraizable logic. As they explain, the notion was a natural abstraction from many well-known examples, the most prototypical, perhaps, being that of classical propositional

logic, of intuitionistic logic and the various implicative logics of Rasiowa [20]. Making an exact notion of algebraizability precise had, besides unification and clarification, the advantage of being able to show, for the first time, that logics that were known not to be amenable to algebraizability techniques, were somehow intrinsically non-algebraizable, since they did not fall under the scope of Blok and Pigozzi's definition. Blok and Pigozzi worked with finitary sentential logics, but their results were soon generalized further to cover many additional systems. One of the earliest generalizations was by Herrmann [15, 16] to cover infinitary logics. Algebraizability was shown to be equivalent to the conjunction of equivalentiality [6, 7] and of truth equationality [19]. Equivalentiality is a stronger property than protoalgebraicity, since it requires that the Leibniz operator be both monotone and commute with substitutions. If equivalentiality is weakened to protoalgebraicity, that is, if one requires that the logic be protoalgebraic and truth equational, then weak algebraizability [9] is obtained. All these properties and their characterizations and interconnections are studied, e.g., in the surveys [8, 12, 14]. In Section 5.5, we studied weak algebraizability in the context of logicates. We study, next, weak algebraizability in the context of logicoids.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be an algebraic logicoid. We say that \mathbb{L} is **weakly algebraizable** [9] (see, also, [12, 8]) if the Leibniz operator is monotone (order preserving) and order reflecting on \mathcal{C}^b .

Note that the fact that in the traditional framework, in which the Leibniz operator commutes with intersections, monotonicity and injectivity are together equivalent to the property of the Leibniz operator being order preserving and order reflecting. So, as far as the traditional framework is concerned, postulating order reflectivity instead of injectivity, together with monotonicity, neither adds nor subtracts to the power of weak algebraizability. Recall, however, that, in the present setting, the Leibniz operator is only submeetive (and does not necessarily commute with meets). So order reflectivity postulated on top of monotonicity here adds more power than simply imposing injectivity in addition to monotonicity.

Proposition 194 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicoid. \mathbb{L} is weakly algebraizable if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ is order preserving and order reflecting.*

Proof: First, by Proposition 183, monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on \mathcal{C}^b is equivalent to monotonicity of $\Omega_{\mathcal{A}}$ on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation \mathcal{A} . So it suffices to see that order reflectivity of $\Omega_{\hat{\mathbf{B}}}$ on \mathcal{C}^b is equivalent to order reflectivity of $\Omega_{\mathcal{A}}$ on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation \mathcal{A} .

Assume, first, that $\Omega_{\hat{\mathbf{B}}}$ is order reflective on \mathcal{C}^b . Consider an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and let $X, X' \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) \subseteq \Omega_{\mathcal{A}}(X')$. Applying h^{-1} , we get $h^{-1}(\Omega_{\mathcal{A}}(X)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(X'))$. By commutativity of the Leibniz

operator with inverse grid morphisms, $\Omega_{\hat{\mathbf{B}}}(h^{-1}(X)) \subseteq \Omega_{\hat{\mathbf{B}}}(h^{-1}(X'))$. By hypothesis, $h^{-1}(X) \leq^b h^{-1}(X')$. Since h is a grid morphism, $X \leq X'$. Thus $\Omega_{\mathcal{A}}$ is order reflecting on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, suppose $\Omega_{\mathcal{A}}$ on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ is order reflecting, for every interpretation \mathcal{A} . Then, by considering $\mathcal{A} = \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle$, we get that $\Omega_{\hat{\mathbf{B}}}$ is order reflecting on \mathcal{C}^b . ■

The work of the preceding section on characterizing Leibniz filters of \mathbb{L} on an interpretation \mathcal{A} comes in handy in case one wants to provide a characterization of weakly algebraizable logicoïds inside the class of protoalgebraic logicoïds.

Proposition 195 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a protoalgebraic logicoïd. \mathbb{L} is weakly algebraizable if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A}) = \text{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., for all $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, $\mathbb{A} = \langle \mathcal{A}, \mathcal{C}^F \rangle$, with $\mathcal{C}^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of \mathbb{L} .*

Proof: Suppose, first, that \mathbb{L} is weakly algebraizable. Then, for every interpretation \mathcal{A} , $\Omega_{\mathcal{A}}$ is order preserving and order reflecting and, hence, a fortiori, $\Omega_{\mathcal{A}}$ is injective. Thus, for all \mathcal{A} and all $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, $[F]_{\Omega} = \{F\}$. Hence, by Proposition 193, $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A}) = \text{Fi}_{\mathbb{L}}(\mathcal{A})$.

Assume, conversely, that $\text{Fi}_{\mathbb{L}}^{\star}(\mathcal{A}) = \text{Fi}_{\mathbb{L}}(\mathcal{A})$ and let $F, G \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(G)$. Thus, $\Omega_{\mathcal{A}}(F)$ is compatible with G . This implies that $G/\Omega_{\mathcal{A}}(F)$ is an \mathbb{L} -filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$. By Proposition 193, $F/\Omega_{\mathcal{A}}(F) \leq^{\Omega_{\mathcal{A}}(F)} G/\Omega_{\mathcal{A}}(F)$. Since π is a grid morphism, we get that $F \leq G$. This proves that $\Omega_{\mathcal{A}}$ is also order reflecting and, hence, \mathbb{L} is weakly algebraizable. ■

This leads to several additional characterizations of weak algebraizability.

Theorem 196 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a protoalgebraic logicoïd. The following statements are equivalent:*

- (i) \mathbb{L} is weakly algebraizable;
- (ii) For every interpretation \mathcal{A} , $\Omega_{\mathcal{A}}$ is monotone and injective on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$;
- (iii) \mathbb{L} is protoalgebraic and, for every interpretation \mathcal{A} and every filter $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, $F/\Omega_{\mathcal{A}}(F)$ is the least filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$;
- (iv) For every interpretation \mathcal{A} , the mapping $F \mapsto \langle \mathcal{A}, \mathcal{C}^F \rangle$, with $\mathcal{C}^F = \text{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a bijection between $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\text{FMod}_{\mathbb{L}}(\mathcal{A})$;
- (v) For every interpretation \mathcal{A} , $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A})$;
- (vi) For every interpretation \mathcal{A} , $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A})$.

Proof:

(i) \Leftrightarrow (ii) By Proposition 194.

(ii) \Leftrightarrow (iii) By Propositions 193 and 195.

(iii) \Rightarrow (iv) Consider the mapping $F \mapsto \langle \mathcal{A}, C^F \rangle$. It is injective. By Proposition 193 and the hypothesis, it is well defined. By Theorem 190, it is also surjective. Thus, it is a bijection. Since it is clearly order preserving and order reflecting, we get that it is a lattice isomorphism.

(iv) \Rightarrow (v) Consider again the mapping $F \mapsto \langle \mathcal{A}, C^F \rangle$. Since, by hypothesis, it is onto, by Theorem 190, \mathbb{L} is protoalgebraic. Further, the composition of this mapping with the mapping $\tilde{\Omega}_{\mathcal{A}}$ from the Isomorphism Theorem 181 gives an isomorphism from $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 187, the mapping is identical to $F \mapsto \tilde{\Omega}(\langle \mathcal{A}, C^F \rangle) = \Omega_{\mathcal{A}}(F)$.

(v) \Rightarrow (vi) In general, $\text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A}) \subseteq \text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A})$. Also in general, $\Omega_{\mathcal{A}}(F) \in \text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A})$. By hypothesis, each $\text{Alg}(\mathbb{L})$ -congruence is of the form $\Omega_{\mathcal{A}}(F)$, for some interpretation \mathcal{A} and some $F \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus,

$$\text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A}) = \text{Con}_{\text{Alg}(\mathbb{L})}(\mathcal{A}).$$

This yields (vi).

(vi) \Rightarrow (i) Trivial. ■

For a weakly algebraizable logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$, we call a class \mathbf{K} of interpretations an **equivalent algebraic semantics** for \mathbb{L} if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$,

$$\langle \text{Fi}_{\mathbb{L}}(\mathcal{A}), \leq \rangle \cong \langle \text{Con}_{\mathbf{K}}(\mathcal{A}), \subseteq \rangle.$$

Proposition 197 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^b \rangle$ be a weakly algebraizable logicoid. Then $\text{Alg}^*(\mathbb{L})$ is an equivalent algebraic semantics for \mathbb{L} .*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation. Define

$$\begin{aligned} \Omega_{\mathcal{A}} : \quad \text{Fi}_{\mathbb{L}}(\mathcal{A}) &\longrightarrow \text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A}); \\ X &\longmapsto \Omega_{\mathcal{A}}(X). \end{aligned}$$

This mapping is well defined since $\langle \mathcal{A}/\Omega_{\mathcal{A}}(X), X/\Omega_{\mathcal{A}}(X) \rangle \in \text{Mat}^*(\mathbb{L})$ and, hence, $\mathcal{A}/\Omega_{\mathcal{A}}(X) \in \text{Alg}^*(\mathbb{L})$. By weak algebraizability and Proposition 194, it is both order preserving and order reflecting and, hence, a fortiori, injective. So it suffices to show that it is surjective.

Let $\theta \in \text{Con}_{\text{Alg}^*(\mathbb{L})}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \text{Alg}^*(\mathbb{L})$, that is, there exists $X \in \text{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) = \theta$. Hence, $\Omega_{\mathcal{A}}$ is surjective. ■

Corollary 198 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a weakly algebraizable logicoid. Then $\text{Alg}(\mathbb{L})$ is an equivalent algebraic semantics for \mathbb{L} .*

Proof: By hypothesis and Proposition 197, $\text{Alg}^*(\mathbb{L})$ is an equivalent algebraic semantics for \mathbb{L} . Also by hypothesis and Proposition 188, $\text{Alg}^*(\mathbb{L}) = \text{Alg}(\mathbb{L})$. Therefore, $\text{Alg}(\mathbb{L})$ is an equivalent algebraic semantics for \mathbb{L} . ■

9.6 Truth Equationality

Recall that weak algebraizability [9] is the combination of protoalgebraicity [2] and truth equationality [19]. Having studied both protoalgebraicity and weak algebraizability in the context of logicoids, we, now, look briefly at truth equationality. We introduce a definition adapted from [19], we show that it transfers and then prove the main result that weak algebraizability is indeed the conjunction of protoalgebraicity and truth equationality.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be an algebraic logicoid. \mathbb{L} is **truth equational** if the Leibniz operator $\Omega_{\hat{\mathbf{B}}}$ is **completely order reflecting on \mathcal{C}^b** , i.e., if for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^b$,

$$\bigcap_{i \in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies} \quad \bigwedge_{i \in I} X_i \leq^b X.$$

Lemma 199 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicoid. \mathbb{L} is truth equational if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, with $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, the Leibniz operator on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ is completely order reflecting, i.e., for all $\{Y_i : i \in I\} \cup \{Y\} \subseteq \text{Fi}_{\mathbb{L}}(\mathcal{A})$,*

$$\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) \quad \text{implies} \quad \bigwedge_{i \in I} Y_i \leq Y.$$

Proof: The right to left implication is again obtained by applying the hypothesis to the interpretation $\langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle$. For the left to right implication, let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $\{Y_i : i \in I\} \cup \{Y\} \subseteq \text{Fi}_{\mathbb{L}}(\mathcal{A})$. Then we have

$$\begin{aligned} \bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) & \quad \text{iff} \quad h^{-1}(\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\ & \quad \text{iff} \quad \bigcap_{i \in I} h^{-1}(\Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\ & \quad \text{iff} \quad \bigcap_{i \in I} \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y_i)) \subseteq \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y)) \\ & \quad \text{implies} \quad \bigwedge_{i \in I} h^{-1}(Y_i) \leq^b h^{-1}(Y) \\ & \quad \quad \quad \text{(Truth Equationality)} \\ & \quad \text{iff} \quad h^{-1}(\bigwedge_{i \in I} Y_i) \subseteq h^{-1}(Y) \\ & \quad \quad \quad (h \text{ a grid morphism}) \\ & \quad \text{iff} \quad \bigwedge_{i \in I} Y_i \subseteq Y. \end{aligned}$$

Hence, the Leibniz operator on $\text{Fi}_{\mathbb{L}}(\mathcal{A})$ is completely order reflecting. ■

Finally, we prove the equivalence of weak algebraizability with protoalgebraicity and truth equationality for logicoids.

Theorem 200 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, \mathcal{C}^b \rangle$ be a base logicoid. Then \mathbb{L} is weakly algebraizable if and only if it is protoalgebraic and truth equational.*

Proof: Suppose \mathbb{L} is weakly algebraizable. Since, by definition $\Omega_{\hat{\mathbf{B}}}$ is monotone, \mathbb{L} is certainly protoalgebraic. To show that it is also truth equational, consider $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^b$, such that $\bigcap_{i \in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X)$. Then

$$\begin{aligned} \Omega_{\hat{\mathbf{B}}}(\bigwedge_{i \in I}^b X_i) &\subseteq \bigcap_{i \in I} \Omega_{\hat{\mathbf{B}}}(X_i) \quad (\text{Corollary 185}) \\ &\subseteq \Omega_{\hat{\mathbf{B}}}(X). \quad (\text{Hypothesis}) \end{aligned}$$

As $\Omega_{\hat{\mathbf{B}}}$ is order reflecting, $\bigwedge_{i \in I}^b X_i \leq^b X$. This shows that $\Omega_{\hat{\mathbf{B}}}$ is completely order reflecting and, therefore, \mathbb{L} is also truth equational.

Suppose, conversely, that \mathbb{L} is protoalgebraic and truth equational. By protoalgebraicity, $\Omega_{\hat{\mathbf{B}}}$ is monotone. So it suffices to show that it is order reflecting. Let $X, Y \in \mathcal{C}^b$, such that $\Omega_{\hat{\mathbf{B}}}(X) \subseteq \Omega_{\hat{\mathbf{B}}}(Y)$. As $\Omega_{\hat{\mathbf{B}}}$ is, by hypothesis, completely order reflective, we get $X \leq^b Y$. Hence $\Omega_{\hat{\mathbf{B}}}$ is also order reflective, showing that \mathbb{L} is weakly algebraizable. ■

