Algebraic Logic for Non-Monotonicity

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Chapter 1

Introduction

1.1 Introduction

The purpose of this monograph is to present some attempts towards the algebraization of *non-monotonic logical systems*. By this, we mean, essentially, operators that are idempotent but, perhaps, fail to be inflationary or monotone. We deal with two approaches. In Part I, the structure of *logicate* is used, which does not assume any underlying ordering and, thus, leads by necessity to a coarser treatment, reflecting some of the challenges of the nonmonotonic framework. In Part II, the structure of *logicoid* is used, which assumes an underlying complete lattice ordering $\leq$ on the powerset of the underlying set of sentences that is "commensurate" with the logical consequence, i.e., that makes the idempotent operator inflationary and monotone. This is an attempt at reimposing some order and get the "chaotic" framework to resemble more to the classical monotonic one. The latter would result if one imposes, instead of an arbitrary $\leq$ ordering, the ordinary $\subseteq$ ordering on the powerset of the set of sentences.

We follow, throughout, the development of Font and Jansana in their pioneering monograph on "A General Algebraic Semantics for Sentential Logics" [12]. So we take the approach of "non-monotonic abstract logics", as it were, rather than that of logical matrices. But we also use logical matrices for certain tasks and for comparison purposes. After introducing in each part the basic framework and a bit of general theory (Chapters 2 and 6), we look at some algebraic rudiments; more specifically, at congruences, bilogical morphisms, quotients, interpretations, logical filters and logical matrices (Chapters 3 and 7). The model theoretic developments, paralleling those constituting the backbone of [12], are developed next (Chapters 4 and 8). Here we find the notions of logicate and logicoid models, which play the role of an abstract logic, of full models and of $\mathbb{L}$-algebras. We also see some attempts at establishing analogs of the celebrated Isomorphism Theorem 2.30 of [12] in each of the two cases of logicates and logicoids, involving their models and algebraic congruences. It should be noted, however, that, when discussing algebras of a logic in this setting, one means pairs of algebraic systems and attached interpretations. So, in that respect, the structure of "algebra" more closely resembles the models of π-institutions of [23].

Since, in the abstract treatment of algebraic logic, the crown jewel and the underlying unifying thread is the Leibniz hierarchy (see, e.g., [8, 12, 14, 10]), in Chapters 5 and 9, we return to those and present some results in accord with our nonmonotonic endeavors. We treat primarily protoalgebraicity (see [2] and, also, [8] for the classical theory), the well known correspondence theorem and the notion of Leibniz filters [12, 11, 17]. We also look, rather briefly, at weak algebraizability [9] and at truth equationality [19] to give a flavor of some of the fundamental classes of the hierarchy and how they would look like in our experimental framework.

We give, next, a more detailed outline of contents by chapter.

1.2 Chapter 2

We make an attempt at developing an abstract theory of algebraic logic incorporating features of non-monotonicity. The objects of study are *consequence operators*, which, for our purposes, are mappings on the powerset of a set which are only required to satisfy idempotency. Thus, the inflationarity and monotonicity aspects of traditional closure operators may be missing.

In the traditional abstract studies in algebraic logic [24, 3, 12, 8, 14], a central role is played by closure operators or, equivalently, closure systems. Closure operators are operators on the powerset of a given set that are required to satisfy inflationarity, monotonicity and idempotency. If one wishes to relax this framework to accommodate non-monotonicity, then, at least in a first attempt, the axioms that should be shed, are those of inflationarity and of monotonicity. We look at a few steps one can make in this direction. Namely, we introduce "consequence operators" $C : \mathcal{P}(A) \to \mathcal{P}(A)$, on an underlying set A, which are only required to satisfy idempotency. They are suppose to simulate, or stand for, "raw logics", which we call "logicates".

In the traditional theory, after introducing the basic objects of study, one compares those that are "compatible". Here, compatibility means that, as operators, they apply on the same objects. Thus, only logics over the same underlying set are compared. One defines a closure operator C to be weaker than a closure operator C', and C' to be stronger than C, if, for all $X \subseteq A$, $C(X) \subseteq C'(X)$. However, once monotonicity is out of the picture, this definition makes little sense. Instead, for logicates, one has to devise new ways of performing meaningful comparisons. In this treatment, we focus on two natural ways of doing so. One is kind of intrinsic to the framework, since it only takes into account the fixed points or theories, a fact which makes sense since our operators only satisfy idempotency. The second is an attempt to emulate more closely the comparison in the classical framework. Here, one also considers the overall structure of the logicate; not solely its theories. This comparison is more "structure preserving" at the expense of being, somehow, more "artificial", since the structure is not intrinsic but rather devised. This artificiality is mended in a way in the second part of the monograph, where we switch focus from logicates to logicoids, in which the "structure" is inserted into the formalism, thus becoming "more natural".

Another important construct in both the abstract and concrete studies in algebraic logic is that of axiomatic extensions. When monotonicity is present, a closure operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is viewed as a consequence operator of a logic. One may need to add a subset $T \subseteq A$ as a new set of axioms to axiomatically strengthen the consequence relation. This is done by defining a new operator, based on the original, by setting, for all $X \subseteq A$, $C^T(X) = C(X \cup T)$. Note that both inflationarity and monotonicity are critical here. The first ensures that, for all X, $T \subseteq C^T(X)$, that is the new axioms become genuine consequences of the new operator. The second yields that $C \leq C^T$, i.e., the new operator is indeed an *extension* of the former via the adoption of the elements in T as new axioms. The criticality of these two axioms and the fact that they are missing in the nonmonotonic framework adopted here give an indication of why the task of emulating the extension process would necessarily involve difficulties and may ultimately prove insufficient and unsatisfactory. Nevertheless, we do the best we can by devising two different operators along these lines.

The first uses a more conservative approach. It "lifts" the consequences of $X \subseteq A$ to $C(T)$ if either X or $C(X)$ are contained in T. But some emphasis must be placed on the pejorative use of "lift" here, since, in fact, $C(T)$ may be a much smaller subset of A than either X or T, due to lack of inflationarity and monotonicity. This approach has the drawback that it does not give an operator which strengthens the original operator according to the "structure preserving" comparison of operators that we alluded to in the preceding paragraph. We take this as hinting to the need of an alternative, more "aggressive", line of attack. The more liberal approach, on the other hand, allows consequences to be lifted to $C(T)$ whenever the consequences of X happen to coincide with the consequences of some $Y \subseteq T$. This construct gives rise to an operator that does strengthen the original operator C and, as it turns out, strengthens also the operator obtained by the more conservative approach.

In Section 2.2, the basic objects of study, called *logicates*, which are idempotent operators on the powerset of a set are introduced. The directed graphs that reflect the structure of logicates are called *necropoleis*,[1] since they consist of components called *pyramids*. By imposing a linear ordering on $\mathcal{P}(A)$, one may recast both as linearly ordered structures, with additional features, called linearized consequences and linearized necropoleis, respectively. However, the process of linearization introduces redundancy, which one sheds by passing to equivalence classes of those ordered structures under appropriately defined equivalence relations. We call an equivalence class of linearized necropoleis a *cemetery*.

In Section 2.3 we encounter ways we may use to compare logicates over the same underlying set. *Equipotency* is the equivalence resulting by having identical sets of theories. By comparing sets of theories by the subset relation, we may also impose a partial ordering on the set of equipotency classes. Being *weaker*, on the other hand, is a relation that also takes into account sets of theories but, in addition, it considers the consequence structure. These comparisons are also investigated from the point of view of alternative presentations of logicates, namely, using necropoleis, classes of linearized consequences and cemeteries.

In Section 2.4, we introduce and compare the two notions that attempt to replace axiomatic extensions in the nonmonotonic context. The first is

[1] Plural of **necropolis**, pronounced the same, but stressed *necropóleis* vs. *necrópolis*.

called *boosting*. It seems a natural one to adopt, based on Occam's Razor. However, it fails to produce a strengthened version of the original logicate under the comparison criterion that takes the consequence structure of the logicate into account. To atone for this failure, we fortify boosting to what we call *strong boosting*. This adjustment produces an operator that strengthens both the original and the boosted version of the original operator.

1.3 Chapter 3

In [3], Blok and Pigozzi introduced the notion of an *algebraizable logic*. Logics in their study, and in all related abstract studies in algebraic logic, can be represented by structural closure operators, i.e., functions on the powerset of an absolutely free algebra on countably many generators that satisfy the axioms of inflationarity, monotonicity, idempotency and structurality (see, e.g., Page 25 of [12]). *Logical matrices* were traditionally used in this context as models of logics. Classes of algebras that were algebraic reducts of those matrices were used as algebraic counterparts. Later, based on the study of concrete examples, it was discovered that in many less well behaved logics, the algebras obtained in this way via matrices were not the "right" ones. This led Font and Jansana in [12] to suggest a more general methodology. Instead of logical matrices, they considered *abstract logics* as models of logical systems. Following a similar process, but now taking algebraic reducts of abstract logics instead of logical matrices, they obtained a class of algebras that was seeing, via already acquired experience with a wide variety of particular logics, as being the "right" one in all known logical systems.

One of the limitations of this "traditional" theory is that it requires the logics under consideration to satisfy both inflationarity and monotonicity. Thus, one cannot accommodate potential non-monotonic operators. It is conceivable, however, that many aspects and features of the theory could be carried over to such a context. In Chapter 2, the notion of a *logicate* was introduced. It is an idempotent operator C on the powerset of a given set A. Several related representations were given and logicates over the same underlying set were compared in a couple of different ways.

If one wishes to study logicates from an algebraic point of view, the preceding framework is clearly insufficient, since it involves no algebraic structure. We remedy this by logicates over algebras. So the basic object of study here is an *algebraic logicate*, consisting of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and an idempotent operator C on the powerset of the universe A of $\mathbf{A}$. It turns out that this framework is sufficient for accommodating quite a large part of the monotonic theory, without requiring that the underlying operator be either inflationary or monotone.

Our study relies on the set $\mathcal{C}$ of *fixed points* or *theories*. No intrinsic ordering or structure on those may be assumed, in contrast with the traditional framework. Still, as in the traditional framework, we may define, based on theories, *logical congruences*. It may be shown that these form a principal ideal of the lattice of all congruences on the underlying algebra. Hence, it makes sense to define the *Tarski congruence* as the largest logical congruence of the logicate, exactly as done for abstract logics in the monotonic framework [12]. In addition, a slightly modified, but very similar, version of its well-known characterization holds (see Pages 18-19 of [12]).

An important notion in the study of abstract logics is that of a *bilogical morphism* (Page 20 of [12]). Because the framework of [12] is based on closure operators, bilogical morphisms tie very closely the consequence structures of the abstract logics they relate. As a byproduct, they also connect very strongly the theories. On the other hand, in the present context, due to lack of monotonicity, one cannot expect morphism with such tight properties. As a result, instead of focusing on consequence preservation, we take the effect on theories as primary. Thus, we adopt a notion of *bimorphism* by stipulating that it preserve theories, even though it may not have any preservation properties as regards the corresponding consequence structures. This fact becomes manifest in the formulation of an analog of a well known characterization. Proposition 1.4 of [12] consists of six equivalent statements, three referring to theories and three to the preservation of the closure structures. The analog only retains the three parts referring to theories, whereas those on consequences are not valid in general. Despite this drawback, using bilogical morphisms one can prove some analogs of the Homomorphism Theorems of Universal Algebra [5, 18, 1]. These parallel the generalized versions proven in the monotonic framework (see, e.g. [4, 12]).

Additionally, if one defines *logical matrices* for logicates by relying solely on theories and not on consequence relations, many of the results of the monotonic theory can be adapted and still shown to hold. In this setting, however, because of the absence of structurality, one has to build matrices over specific interpretations, i.e., surjective homomorphisms onto similar algebras. Some clues as to how one may proceed may be taken from the context of models of π-institutions [21]. On the other hand, if structurality is added, as is done briefly in the Addendum to this chapter, then matrices resembling the ordinary ones more closely may again be used as models and some of the flavor of the traditional treatment may be recovered.

In Section 3.2, we adapt the study of *logical congruences* on abstract logics of Font and Jansana [12] to the nonmonotonic setting. Logical congruences on a logicate form a principal ideal of the lattice of all logical congruences on the underlying algebra. The generator of this ideal is called the *Tarski congruence* of the logicate. It is important to notice that all equipotent logicates share the same Tarski congruence. Further, the Blok-Pigozzi style characterization of the Tarski congruence in the traditional setting carries over virtually unchanged. If one thinks of the Tarski operator as acting on equipotency classes, so that the set in question is partially ordered, then it is a monotone operator.

In Section 3.3 we revisit *bilogical morphisms*, but apply them to arbitrary logicates. These are surjective homomorphisms that preserve theories. A characterization is provided, as well as the important result that the Tarski operator of a logicate is preserved under the action of inverse bilogical morphisms.

In Section 3.4 we study aspects of Universal Algebra, that were applied by Brown and Suszko [4] and by Font and Jansana [12] to abstract logics, in the context of nonmonotonicity. We define *quotients* of logicates by logical congruences and show that natural projections form bilogical morphisms. Then we embark on revisiting analogs of the Homomorphism Theorems (see, e.g., [5] and Pages 22-23 of [12]) for logicates. In particular, the Correspondence Theorem, playing a role similar to the one played in the monotonic theory, leads to the definition of a *reduced logicate* and the process of *reduction*. Several properties of reductions carry over to this more general setting and are presented in detail in this section.

In Section 3.5 we introduce *interpretations* of algebraic logicates. An interpretation is essentially a surjective mapping from the underlying algebra of the logicate onto an algebra of the same type. Interpretations form the cornerstone in defining *logical filters* and *logical matrices*, as well as *reduced matrices*, which play a key role in both the traditional theory (see, e.g., [3, 12, 8]) and the more general theory presented here. One important feature of interpretations is that, if the kernel is a logical congruence of the original logicate, then one may define a logicate in the target of the interpretation in such a way that its theories coincide with the filters and the interpretations becomes a bilogical morphism between the two logicates. Several properties governing the relation between interpretations and (sets of) filters are also presented in this section. In closing the section, we define that class of *matrices* and of *reduced matrices* of a logicate and the corresponding classes of algebraic reducts and prove analogs of the well-known *completeness results* for sentential logics in the context of logicates.

In the Addendum, we briefly overview a possible alternative formulation of filters and matrices, applicable in case the consequence operator of the logicate happens to be structural. In that case filters and matrices may be defined as in the traditional monotonic theory, without recourse to fixed interpretation morphisms.

1.4 Chapter 4

In this, third chapter, on logicates, we focus specifically on the role that algebraic logicates play as models of other logicates. Logicates are models more suitable for many purposes than simple logical matrices, even though logicate models can be viewed as bundles of matrices over the same underlying interpretation.

Our framework and starting point is the study in Chapter 3 of interpretations. We are assuming a given logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ which consists of an algebra and an idempotent operator, called a *consequence operator*, on its powerset. Logicates are supposed to represent logical systems for which inflationarity and monotonicity may fail. $\mathbb{L}$ is viewed as the focus of study and it is called, accordingly, a *base logicate*. An *interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ consists of a surjective homomorphism from the algebra $\mathbf{B}$ of $\mathbb{L}$ onto a similar algebra $\mathbf{A}$. If on the target algebra, there is given a logicate structure, say $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}$ induces a logicate on the base algebra. We say that $\mathbb{A}$ is a *model* of $\mathbb{L}$ if the inverse images under h of the theories of $\mathbb{A}$ form a subset of the theories of $\mathbb{L}$, written $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. Two logicates connected by a bilogical morphism that commutes with interpretations share the property of being simultaneously models or not being models.

One of the key constructions in our framework is passing from a model to its Tarski reduction. The *Tarski operator* was used as a key ingredient in the theory of Font and Jansana (Page 19 of [12]) and, as our work is based on theirs, it continues to play a crucial role here as well. Given a logicate model, one may construct its *reduction* by moding out both the interpretation and the idempotent operator by the Tarski congruence of the logicate. A first result is that a logicate is complete with respect to both its class of logicate models and its class of reduced logicate models. Completeness here simply means that collecting all inverse images of theories of the models of the class yields the full collection of theories of the base logicate.

Connecting the theory of logicate models with the theory of matrix models of the base logicate, which was detailed in Chapter 3, we obtain the fact, well-known in classical theory (Proposition 2.7 of [12]), that, a logicate, viewed as a bundle of matrices, is a model of $\mathbb{L}$ if and only if every member of the bundle is a matrix model of $\mathbb{L}$.

The next key concept adapted here from the theory of abstract logics of [12] is that of a *full model*. A *basic (full logicate) model* is a model whose collection of theories consists of all filters on its interpretation. A *full (logicate) model* is one whose Tarski reduction is a basic full logicate model. The terminology is justified by the fact that a basic model turns out to be a full model according to these definitions. It is shown here, in a result that parallels one pertaining to abstract logics, that logicate models connected via bilogical morphisms commuting with interpretations are either both full or both fail to be full. As a consequence the property of being full is also preserved and reflected by reductions. These results yield a characterization of the class of full models as the smallest class that contains all basic full models and is closed (in both directions) under bilogical morphisms.

Full models are the first ingredient in establishing a key *Isomorphism Theorem*, along the lines of the Isomorphism Theorem (Theorem 2.30) of Font and Jansana, which is one of the main results of the abstract treatment in the theory they present in [12]. The second ingredient relates to congruences whose quotients are algebras in $\mathrm{Alg}(\mathbb{L})$. The class $\mathrm{Alg}(\mathbb{L})$ consists of the underlying interpretations of reduced full models of $\mathbb{L}$. Another class of interpretations that is related to a class of algebras traditionally studied in algebraic logic is the class $\mathrm{Alg}^*(\mathbb{L})$. It consists of all underlying interpretations of reduced matrix models of $\mathbb{L}$. The tight connection, mentioned previously, between logicate models and matrix models, yields a (sort of induced) relationship between the two classes. In the traditional setting one class turns out to be the class of subdirect products of the other (see, e.g., Theorem 2.23 of [12]). In the present setting, because of the presence of fixed interpretation morphisms, we find it convenient (and perhaps necessary) to define a related but different operation on interpretations, called a *subdirect intersection*. It is shown that the class $\mathrm{Alg}(\mathbb{L})$ consists exactly of subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$.

Our work in this part culminates with proving an analog of the Isomorphism Theorem 2.30 of [12] for the present context. We view this as one of the main results of the work. The analog proven here has some significant deviations as compared to its predecessor. First, all parts are taken to be over fixed underlying interpretations. This is compelled by the absence of structurality for logicates. If one added structurality, then something closer, perhaps, to the original could be obtained. But this seemed rather restrictive and, in addition, the framework of π-institutions [21] has provided some experience in dealing with fixed interpretations. Second, one cannot expect to establish an isomorphism theorem dealing with all full models, since full models that are equipotent, have identical Tarski congruences. So one has, by necessity, to pass to equipotency classes of full models over fixed interpretations. Taking these comments into account, we establish an order isomorphism between the set of equipotency classes of full logicate models, ordered under the superset relation between sets of theories, and the set of $\mathrm{Alg}(\mathbb{L})$-congruences under inclusion. It is also shown that the latter poset is a complete lattice. As a consequence, one obtains that the former has the same structure as well.

In Section 4.2 we recall the notion of interpretation and use it to define *logicate interpretations*. These, in turn, serve in defining models of a logicate. Logicate models have a tight relationship with matrix models. We also define reductions. We show that, if two logicate interpretations are related via a bilogical morphism, then one is a model if and only if the other is. This implies that a given one is a model if and only if its reduction is. We also formulate analogs of the well-known Completeness Theorems of Algebraic Logic both with respect to the class of all models and with respect to the class of all reduced models.

In Section 4.3, the notion of *full model* for logicates is introduced, taking after the corresponding notion for the monotonic framework (see Definition 2.8 of [12]). A model is a *basic (full) model* if the set of its theories coincides with the set of all filters on the underlying interpretation. A model is a *full* *model* if its reduction is a basic full model. Several properties, paralleling ones proved by Font and Jansana for sentential logics in [12], are adapted and proved in this setting. They culminate in two different characterizations of full models, which, as Font and Jansana explain, may be taken as justifications of the term "full". The class of full models is shown to be the smallest class that includes all basic full models and is closed under bilogical morphisms (see Corollary 2.13 of [12]). It is also the class of all models whose sets of theories consist of all preimages under canonical projections of all filters on the Tarski reduction of the model (see Theorem 2.14 of [12]).

In Section 4.4 we define the notion of $\mathbb{L}$-*algebra* for a given logicate $\mathbb{L}$. These parallel $\mathcal{S}$-algebras for a sentential logic [12]. In the present context, however, they should be referred to as $\mathbb{L}$-*interpretations*, since they are pairs consisting of an algebra together with a mapping from the base algebra of $\mathbb{L}$ onto the algebra. But the term "algebra" is retained because of the similarity of the role they play. Several results encapsulating the interaction of these algebras with full models and reduced full models are given. We also revisit the relation between the class of algebras which are reducts of reduced matrix models and the class of $\mathbb{L}$-algebras, which are reducts of reduced full models of $\mathbb{L}$. An operation, called *subdirect intersection*, paralleling that of subdirect product in the ordinary framework, is defined and comes in handy in this task. The result is an analog of Theorem 2.23 of [12].

In Section 4.5 the main goal is establishing an *Isomorphism Theorem*, along the lines of Theorem 2.30 of [12]. The Tarski operator over a fixed interpretation forms a mapping from logicates over that interpretation into congruences. Moreover, it is constant over equipotency classes of logicates. Thus, it may be viewed as an operator over equipotency classes of logicates to congruences. We introduce here an operator from $\mathrm{Alg}(\mathbb{L})$-congruences that is seeing as being the inverse of the restriction of the Tarski operator on equipotency classes of full models. Moreover, both operators are order preserving, when order is taken to be the one reflecting the superset relation between sets of theories. So they establish an isomorphism between equipotency classes of full models and $\mathbb{L}$-algebra congruences. Some consequences of this isomorphism are encountered here, among which is the fact that the equipotency classes of full models form a complete lattice. This is proven using the isomorphism theorem and a result showing that the collection of $\mathrm{Alg}(\mathbb{L})$-congruences under the subset relation form a complete lattice.

1.5 Chapter 5

One of the main achievements of the abstract theory of Algebraic Logic is the classification of logics in an algebraic hierarchy. A logic is represented by a structural closure operator on the algebra of terms (or formulas) over an algebraic type generated by countably many variables. The theory prescribes a method (or, rather, methods) one may follow to select a particular class of algebras over the same signature as the logic to associate with the logic. The higher the logic is classified in the hierarchy, the closer the ties between the logic and its associated class of algebras. Because of their clarity and comprehensiveness, but, also because they were written by pioneers, two monographs [3, 12], a survey [14], a book [8] and a textbook [10] have been used for many years as guides in being introduced to, in understanding and in delving deeper into the theory.

Since the *algebraic hierarchy* is one of the crowns (and jewels) of the traditional theory, it is only fair to, at least start to, investigate and give a first idea of how one could attempt to keep alive aspects of the theory in a rougher terrain. This is the effort we expend in the present and last chapter of Part I.

Among the major, perhaps most important, classes in the traditional hierarchy are protoalgebraic logics [2] (see, also, [8, 14, 10]). These are the logics in which, roughly speaking, indistinguishability modulo a theory implies interderivability modulo the theory. Another important characterization asserts that they are the logics on whose lattices of theories, the Leibniz operator is monotone. In Section 5.2, we use the definition from the classical framework to define *protoalgebraic logicates* and try to establish some equivalent conditions, some with and some without extra assumptions.

One of the key consequences of protoalgebraicity, which forms an important feature in their study, is the so-called *Correspondence Theorem*. This result is partly the reason why Blok and Pigozzi declared that protoalgebraic logics form the widest class of logics amenable to algebraic techniques of study, even though they are not "algebraizable", i.e., do not belong to the highest step in the hierarchy but are, rather, located near the bottom. The Correspondence Theorem establishes an isomorphism between the lattice of filters of the logic on a given algebra including a fixed filter and the lattice of filters on the quotient algebra, formed by dividing out by the Leibniz congruence of the fixed filter including the quotient of the fixed filter. We discuss this result and some of its consequences in Section 5.3. Again our focus remains to safeguard some of the result from the traditional theory, with or without provisos, in this less robust environment.

As is the case in the traditional theory [12], and as was shown provisionally in Chapter 4, full models play a key role in the investigation of the logical structure. In the context of protoalgebraic logics, full models are inextricably connected to, so-called, *Leibniz filters* [12, 17]. So, in Section 5.4, we investigate the relation between full models and Leibniz filters in the context of protoalgebraic logicates. Their study naturally segues into the study of *weakly algebraizable logicates* in Section 5.5. These are defined by analogy with the corresponding class in the monotonic framework [9] (see, also, [8, 14, 10]). Several characterizations paralleling the ones from the traditional setting are provided, but, generally, they require the additional hypothesis that the set of theories be closed under intersection.

As is well known in the ordinary setting, weak algebraizability [9] results by simultaneously insisting that a logic be protoalgebraic [2] and truth equational [19]. So in the last section, Section 5.6, we use the original definition to identify the class of *truth equational logicates*. They are characterized by the Leibniz operator on their theories being monotone and injective. Again assuming closure of the set of theories under intersection, we prove that, for logicates also, weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

1.6 Chapter 6

In our work in Part I, we already glimpsed, at least twice, how, in studying a logicate, having an underlying order on the subsets of its universe A may be beneficial. E.g., in Chapter 2, when we looked at linearized consequences, we saw that artificially linearizing allow us to study instead of an arbitrary idempotent operator, an operator that also satisfies all three properties of an ordinary closure operator. Furthermore, in Chapter 5, we saw how many of the results related to classes in the algebraic hierarchy required that the set of theories was closed under intersections or has a minimum element. These observations lead us in Part II to look at structures in which order plays a role from the get go. To take advantage of the powerful machinery of the traditional framework of monotonic logics [12], without, however, losing sight of the fact that we are dealing with, possibly, nonmonotonic systems, we introduce a complete lattice ordering on the powerset of the underlying set A.

Comparing to the development in Part I, we could say that, in Part I, we took the logical notion of consequence operator as foundational and constructed, based on it, an "ordered" consequence, which involved a type of imposed ordering, either "artificial", e.g., a linearization, or "natural", e.g., based on $\subseteq$, reflecting, necessarily in a rather loose way, to the extent possible the "chaotic" logical structure. On the other hand, in Part II, a reversal of roles occurs. More precisely, we presume an underlying order on the powerset $\mathcal{P}(A)$ of the set A and then build a logical structure that is, in some way, commensurate with the underlying ordering. We visualize the presumed preexisting ordering as an artificially created "molecular" shape and, since the logic is developed on that construct, it is termed a "logicoid". This approach imitates more closely, and captures more accurately, many of the features of more traditional logical systems. On the other hand, expectations must be tempered, since the ordering is one among many that could possibly be chosen, and as such, its role is not quite natural. As noted, also, in comments in Part I,, we attempt to do what we can in a challenging setting, among rather adverse features as compared with those naturally available in the monotonic framework.

In Section 6.2, we introduce the notion of a *grid*, which consists of an underlying set A (viewed as a set of abstract sentences), together with an arbitrary complete lattice ordering on the powerset of A. The fact that this ordering is arbitrary and not the "subset" ordering is what permits accommodating nonmonotonicity and make the framework suitable for our purposes, while still maintaining many of the advantages afforded by the complete lattice structure. Naturally enough, we then introduce *grid morphisms* that connect grids. They are surjective mapping between the underlying sets that make their induced inverse powerset mappings complete lattice embeddings. Continuing, we define *closure operators* as ones that are inflationary, monotone and idempotent, but not with respect to the natural subset ordering, but, rather, with respect to the "artificial" ordering of the grid. We also define *closure systems* and, using the notion of *theory*, we show that, as in the ordinary framework, losure operators and closure systems (on the grid, as it were) are still in one to one correspondence and, thus, interchangeable.

In Section 6.3, we introduce the "*weaker than*" and "*finer than*" relations to compare closure operators of logicoids and closure systems on the underlying grids, respectively. These relationships parallel the ones in the classical (monotonic) framework, except that, instead of being with respect to the subset relation, they are based on the grid ordering.

In Section 6.4, we look at *boosting* for logicoids by a chosen set of axioms, which corresponds to taking the axiomatic extension of a sentential logic in the ordinary monotonic context. We saw the difficulties inherent in defining such an operation for logicates in Section 2.4. Here, the presence of a complete lattice ordering in the grid on which a logicoid is based, creates an environment in which some of the nice features may be recovered, albeit with respect to the $\leq$ ordering of the grid rather than the natural subset ordering that serves the same purpose in the monotonic framework.

Our main interest is in what we call *algebraic logicoids*, which are logicoids built on *algebraic grids*, that is, grids on sets having an algebraic structure. Naturally enough, treating them algebraically requires having some algebraic fundamentals available for handling them. This is precisely the purpose that Section 6.5 is supposed to fulfill. Here, we formally define *algebraic grids*, which consist of an algebra together with a complete lattice ordering on its powerset. We also define *grid morphisms* and *grid congruences*. We show that these constructs interact as expected. We then employ them to develop analogs of the fundamental Homomorphism Theorems of Universal Algebra for algebraic grids, their homomorphisms and their congruences.

1.7 Chapter 7

In Chapter 7, we develop the rudiments of the algebraic theory of logicoids with an eye towards developing, in Chapter 8, a model theory, paralleling the one in [12] and that developed for logicates in Part I. We first introduce the key concept of *logical grid congruence*. Based on those, we define the *Leibniz grid congruence* of a logical matrix and the *Tarski grid congruence* of a logicoid. We then study *bilogical morphisms* between logicoids, which, unlike those used for logicates, respect the logical consequence and not merely the theories of the structure. So, in that respect, they resemble more closely those introduced by Font and Jansana [12]. We then look at *quotient logicoids* and prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. This gives us the chance to look closely at *reductions* and at *reduced logicoids*. We then turn to analogs of *interpretations*, *filters* and *matrix models* and study many of their properties, including the way they interact with grid morphisms, their interplay with closure systems and their transformations via bilogical morphisms.

In more detail, Section 7.2 undertakes the study of *logical grid congruences*. Recall that, given an algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, a congruence θ on $\mathbf{A}$ is called a *grid congruence* on $\hat{\mathbf{A}}$ if $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ based on the grid $\hat{\mathbf{A}}$, θ is a *logical grid congruence* of $\mathbb{L}$ if it is a grid congruence on $\hat{\mathbf{A}}$, such that $\mathcal{C} \subseteq \mathrm{Cmp}(\theta)$. It is shown that the collection of all logical grid congruences of $\mathbb{L}$ forms a principal ideal of the complete lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\widetilde{\Omega}(\mathbb{L})$ is called the *Tarski grid congruence* of $\mathbb{L}$. An analogous situation occurs if one considers logical matrices $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$ based on an algebraic grid $\hat{\mathbf{A}}$. Here a *matrix grid congruence* is a grid congruence θ on $\hat{\mathbf{A}}$, such that $X \in \mathrm{Cmp}(\theta)$. Again, the collection of all matrix grid congruences of $\mathfrak{A}$ forms a principal ideal of the lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\Omega(\mathfrak{A})$ is called the *Leibniz grid congruence* of $\mathfrak{A}$. Two of the most useful observations related to these concepts are that $\widetilde{\Omega}$ is monotone on logicoids over the same grid and that, given a logicoid, its Tarski grid congruence is the intersection of all Leibniz grid congruences of those logical matrices formed by each of its theories.

In Section 7.3, we introduce and study *logical* and *bilogical morphisms* between logicoids. Since logicoids are based on algebraic grids, all these morphisms are algebraic grid morphisms, which were studied extensively in Section 6.5, and we rely quite heavily on that machinery. A *logical morphism* $h : \mathbb{L} \to \mathbb{L}'$ from a logicoid $\mathbb{L}$ based on $\hat{\mathbf{A}}$ to a logicoid $\mathbb{L}'$ based on $\hat{\mathbf{A}}'$ is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$. In case $h^{-1}(\mathcal{C}') = \mathcal{C}$ we say that $\mathbb{L}$ is *projectively generated from* $\mathbb{L}'$ *by* h. A logical morphism $h : \mathbb{L} \to \mathbb{L}'$ is a *bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}'$ between $\mathbb{L}$ and $\mathbb{L}'$ if it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. We provide a characterization theorem for bilogical morphisms along the lines of Proposition 1.4 of [12] and we show that, if $h : \mathbb{L} \to_b \mathbb{L}'$, then

$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}'))$. Finally, the notion of *isomorphism* between logicoids is introduced as a bijective mapping $h : A \to A'$ for which both $h : \mathbb{L} \to \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to \mathbb{L}$ are logical. It is shown that this is tantamount to requiring that $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ and $h : \mathbb{L} \to_b \mathbb{L}'$.

In Section 7.4, given an algebraic grid $\hat{\mathbf{A}}$ and a grid congruence θ on $\hat{\mathbf{A}}$, we define the *quotient closure operator* C^θ on the quotient grid $\hat{\mathbf{A}}/\theta$ of an operator C on $\hat{\mathbf{A}}$. This gives rise to the *quotient logicoid* $\mathbb{L}^\theta = \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$ of a given logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and, moreover, makes the quotient grid morphism $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ into a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta$. Quotient logicoids are important because, among other things, they allow us to prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. We prove analogs of the Homomorphism Theorem, of the Second Isomorphism Theorem and of the Correspondence Theorem. The latter, in particular, enables us to show that the Tarski grid congruence of a quotient logicoid is the quotient of the Tarski grid congruence of the parent. We also look at *reductions* of logicoids. We show the important results that the reduction of a quotient logicoid coincides with the reduction of its parent and that the reductions of two logicoids related via a bilogical morphism are isomorphic logicoids.

In Section 7.5, the goal is to develop a theory of matrix models for logicoids along the lines of the traditional theory for monotonic logics and the theory developed in Section 3.5 for logicates. We start with a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ over a base algebraic grid $\hat{\mathbf{B}}$. Lack of structurality compels us to consider structures over fixed *interpretations*. These are pairs $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where $\hat{\mathbf{A}}$ is an algebraic grid and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism. A *matrix* is a pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, where $F \subseteq A$. If F is an $\mathbb{L}$-*filter*, i.e., if $h^{-1}(F) \in \mathcal{C}^\flat$, then $\mathfrak{A}$ is called an $\mathbb{L}$-*matrix*. Moreover, $\mathfrak{A}$ is *reduced* if $\Omega_{\mathcal{A}}(F) = \Delta_{\mathbf{A}}$. On any interpretation $\mathcal{A}$, there is, induced by $\mathbb{L}$ and h, a closure operator $C_{\mathcal{A}}$. In case $\mathrm{Ker}(h)$ is a logical grid congruence of $\mathbb{L}$, the induced structure $\mathbb{L}_{\mathcal{A}} = \langle \hat{\mathbf{A}}, C_{\mathcal{A}} \rangle$ is a logicoid and, moreover, the mapping $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ becomes a bilogical morphism $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$. Further, it can be shown that the theories of $\mathbb{L}_{\mathcal{A}}$ coincide with the $\mathbb{L}$-filters of $\mathbb{L}$ on $\mathcal{A}$.

In the remainder of the section we look at ways grid morphisms interact with filters. For instance, we show that, for two interpretations connected by a grid morphism, inverse images of $\mathbb{L}$-filters are $\mathbb{L}$-filters and conversely. Considering quotient interpretations, it is shown that for an $\mathbb{L}$-filter F on $\mathcal{A}$ to be the inverse image under a quotient morphism π_θ of an $\mathbb{L}$-filter on $\mathcal{A}/\theta$ it is necessary and sufficient that θ be compatible with F. Two interpretations that are related by a grid morphism may, under certain circumstances, establish very close ties between corresponding $\mathbb{L}$-filters. The closest connection occurs when the grid morphism in question is a bilogical morphism between the filter structures. It then establishes an isomorphism between the two posets of filters under the corresponding grid orderings. If this happens between two closure structures, one in the source interpretation and another in the target, and the structure in the source interpretation consists of all $\mathbb{L}$-filters, then so does the one in the target. This yields that the $\mathbb{L}$-filters on a reduced interpretation coincide with the reductions of the $\mathbb{L}$-filters on the parent interpretation. At the end of the section, we present an analog of a standard result asserting that a base logicoid is complete with respect to all its matrix models as well as with respect to all its reduced matrix models.

1.8 Chapter 8

This chapter discusses *logicoid models* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$. They are based on grid interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, which consist of a grid $\hat{\mathbf{A}}$ together with a grid morphism h from the base grid $\hat{\mathbf{B}}$ onto $\hat{\mathbf{A}}$. We also define and study the reduction $\mathbb{A}^*$ of such a logicoid interpretation $\mathbb{A}$. Among those models, we single out the *full models*, which are the ones whose reductions are *basic full models*, i.e., consist of all possible $\mathbb{L}$-filters on their underlying interpretations. We characterize this class of models. Moreover, we show that it consists of those models that have all possible filters corresponding to filters on the reduced interpretations. Reduced $\mathbb{L}$-models give rise to $\mathbb{L}$-*algebras*, i.e., interpretations that are reducts of reduced $\mathbb{L}$-models. Their class is shown to be the class of subdirect intersections of interpretations in $\mathrm{Alg}^*(\mathbb{L})$, which consists of all interpretation reducts of reduced $\mathbb{L}$-matrices. Our study culminates with an Isomorphism Theorem for logicoids asserting that the Tarksi operator on a fixed interpretation $\mathcal{A}$ is an isomorphism between the ordered set of full $\mathbb{L}$-models on $\mathcal{A}$ and the partially ordered set of grid $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$.

In Section 8.2, we introduce the notion of an *interpretation* of a base algebraic grid $\hat{\mathbf{B}}$ and that of a *logicoid interpretation*. The first is a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, consisting of an algebraic grid $\hat{\mathbf{A}}$ and a grid morphism $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$. The second consists of an algebraic logicoid $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is an interpretation and $\langle \hat{\mathbf{A}}, C \rangle$ is a logicoid based on $\hat{\mathbf{A}}$. A logicoid interpretation of $\hat{\mathbf{B}}$ induces a logicoid structure $\mathbb{L}^{\mathbb{A}} = \langle \hat{\mathbf{B}}, C^{\mathbb{A}} \rangle$ on $\hat{\mathbf{B}}$ in such a way that h becomes a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{A}$. Further, if two logicoid interpretations are related via a bilogical morphism, then they induce identical logicoids on the base grid $\hat{\mathbf{B}}$. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a *model* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ if $\mathbb{L} \leq \mathbb{L}^{\mathbb{A}}$ or, equivalently, if $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. The section continues with a discussion of completeness of a base logicoid with respect to a class of models. In this context, *reductions* of models and *reduced models* are discussed and analogs of classical completeness results with respect to the class of all models and with respect to the class of all reduced models are formulated. The section closes by connecting the notion of logicoid model with that of a grid matrix model, introduced and studied in Section 7.5.

In Section 8.3, we study *full models* of logicoids. Given a base logicoid

$\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, a logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a *basic full model* of $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., if its set of theories is the entire collection of $\mathbb{L}$-filters on its underlying interpretation. A *full model* of $\mathbb{L}$ is one whose reduction is a basic full model. Full models are indeed models and basic full models are indeed full models. So the terminology chosen is sound. It turns out that bilogical morphisms preserve fullness in both directions, which implies that $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if its reduction $\mathbb{A}^*$ is also a full $\mathbb{L}$-model. Additionally, $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if there exists a bilogical morphism from it onto a basic full $\mathbb{L}$-model. As a consequence we get that the class of full $\mathbb{L}$-models is the smallest class containing all basic full $\mathbb{L}$-models and closed under bilogical morphisms (in both directions). The section concludes with a result providing an additional justification of the term "full". It shows that full $\mathbb{L}$-models are those whose collection of $\mathbb{L}$-filters consists of all possible ones corresponding to $\mathbb{L}$-filters on the reduced interpretation.

Section 8.4 introduces $\mathbb{L}$-algebras (more accurately $\mathbb{L}$-interpretations) for a logicoid $\mathbb{L}$. These are interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, such that $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))$ is the identity grid congruence on $\hat{\mathbf{A}}$. Some results relating $\mathbb{L}$-algebras with full $\mathbb{L}$-models and with their theories are provided. It is shown that, for every full $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, the reduction $\mathcal{A}^*$ of the interpretation $\mathcal{A}$ is an $\mathbb{L}$-algebra. Additionally, the class $\mathrm{Alg}(\mathbb{L})$ of $\mathbb{L}$-algebras is characterized as the class of interpretation reducts of reduced $\mathbb{L}$-models. Moreover, it is shown that $\mathrm{Alg}(\mathbb{L})$ is the class of all subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$ of all interpretation reducts of reduced grid matrix models of $\mathbb{L}$. This characterization yields that $\mathrm{Alg}^*(\mathbb{L})$ is contained in $\mathrm{Alg}(\mathbb{L})$ and, also, that given logicoids $\mathbb{L}$, $\mathbb{L}'$ over the same base grid $\hat{\mathbf{B}}$, such that $\mathbb{L} \leq^\flat \mathbb{L}'$, we have that $\mathrm{Alg}(\mathbb{L}')$ is contained in the class $\mathrm{Alg}(\mathbb{L})$.

In Section 8.5, the last section of the chapter, we prove an analog of the Isomorphism Theorem 13 of [12] for logicoids. The result parallels the Isomorphism Theorem for logicates (Theorem 75) and the proof is similar.

1.9 Chapter 9

Chapter 8 is intended to only provide a relatively superficial flavor of a semantically defined algebraic hierarchy of classes of logicoids based on properties of their Leibniz operator, paralleling the classical one for monotonic logics (see, e.g., [8, 14, 10]). The best part of it (Sections 9.2-9.4) is dedicated to the study of *protoalgebraicity*, Section 9.5 looks briefly at *weak algebraizability* and Section 9.6 looks even more briefly at *truth equationality*. We give several characterizations of *protoalgebraicity*, which is the property of having a monotone Leibniz operator, and we study the *Correspondence Theorem* and several of its consequences. This segues nicely into the introduction of *Leibniz filters* and some of their properties. *Weak algebraizability* is the property of having both a monotone and an order reflecting Leibniz operator, whereas *truth equationality* is the property of having a completely order reflecting Leibniz operator. As in the traditional monotonic case, it turns out that weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

In Section 9.2, we introduce *protoalgebraic logicoids*. A logicoid $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ is *protoalgebraic* if, for all theories X and all $a, b \in B$, $\langle a, b\rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ implies that $\langle a, b\rangle \in \Lambda_{\mathbb{L}}(X)$, where $\Lambda_{\mathbb{L}}(X)$ is the relation holding if, for every theory X', with $X \leq^\flat X'$, $a \in X'$ iff $b \in X'$. We show that protoalgebraicity is equivalent to the monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on the theories of $\mathbb{L}$. Moreover, $\mathbb{L}$ is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. An additional characterization asserts that $\mathcal{L}$ is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is submeetive on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, meaning that, for all $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $\Omega_{\mathcal{A}}(\bigwedge_{i\in I} X_i) \subseteq \bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i)$.

In Section 9.3, the central task is proving a Correspondence Theorem, an analog of Theorem 6.19 of [10], for protoalgebraic logicoids. After doing this, we explore some of its consequences. Among these are some additional characterizations of protoalgebraicity using the Tarski operator. We show, e.g., that $\mathbb{L}$ is protoalgebraic if and only if, for every $\mathbb{L}$-model $\mathbb{A} = \langle\mathcal{A}, C\rangle$, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min\mathcal{C})$, where $\min\mathcal{C}$ is the least theory of $\mathbb{A}$ (or the set of theorems of $\mathbb{A}$). Another consequence is that, if the logicoid $\mathbb{L}$ happens to be protoalgebraic, then the classes of interpretations $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$ coincide. In addition, if $\mathbb{L}$ is protoalgebraic, then any two logicoid models $\mathbb{A}$ and $\mathbb{A}'$ over the same underlying interpretation that share the same minimum theories must be identical. The last result of the section is a theorem characterizing the full models of a protoalgebraic logicoid, while, at the same time providing yet another characterization of protoalgebraicity. It asserts that $\mathbb{L}$ is protoalgebraic if and only if its full models are of the form $\langle\mathcal{A}, C^F\rangle$, where $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, for some interpretation $\mathcal{A}$ and some filter F in $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Here, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ denotes the collection of all $\mathbb{L}$-filters on $\mathcal{A}$ dominating F in the $\leq$ ordering of the subsets of A in the underlying algebraic grid $\hat{\mathbf{A}} = \langle\mathbf{A}, \leq\rangle$ of $\mathcal{A} = \langle\hat{\mathbf{A}}, h\rangle$.

Section 9.4 considers a question that arises naturally from the characterization of full models of protoalgebraic logicoids. More precisely, it attempts to characterize those $\mathbb{L}$-filters F on an interpretation $\mathcal{A}$ for which $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ is a full $\mathbb{L}$-model. To do this, we form the subset of such filters $\mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A})$. These filters are termed *Leibniz filters*. If $\mathbb{L}$ is protoalgebraic, $\Omega_{\mathcal{A}}$ turns out to be an order isomorphism from $\mathrm{Fi}^\star_{\mathbb{L}}(\mathcal{A})$, ordered by $\leq$ onto $\mathrm{Con}_{\mathrm{Alg}^*}(\mathcal{A})$, ordered by $\subseteq$. Further, we introduce an equivalence $\sim_\Omega$ between $\mathbb{L}$-filters on an interpretation $\mathcal{A}$ that "identifies" two filters if they have the same Leibniz grid congruence. The $\sim_\Omega$-class of an $\mathbb{L}$-filter F is denoted by $[F]_\Omega$. If $\mathbb{L}$ is protoalgebraic, then F is the minimum element in the $\leq$ ordering in $[F]_\Omega$. This affords the characterization of Leibniz filters as those $\mathbb{L}$-filters on an interpretation that are minimum in their $\sim_\Omega$-equivalence classes. Equivalently, they are the $\mathbb{L}$-filters F, whose quotients $F/\Omega_{\mathcal{A}}(F)$ are minimum $\mathbb{L}$-filters in the $\leq^{\Omega_{\mathcal{A}}(F)}$ ordering on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Section 9.5 deals with a second question that may be seen to arise from the characterization of full models of a protoalgebraic logicoid. Namely, identify those situations for which the collection of Leibniz filters is the entire collection of filters on an interpretation. We call a logicoid $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ *weakly algebraizable* if the Leibniz operator is order preserving and order reflecting on $\mathcal{C}^\flat$. This is equivalent to the Leibniz operator $\Omega_{\mathcal{A}}$ being order preserving and order reflecting on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. Furthermore, it turns out that $\mathbb{L}$ is weakly algebraizable if and only if $\mathrm{Fi}^{\star}_{\mathbb{L}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. Thus, weak algebraizability settles the initial problem of discovering a property under which the collection of the Leibniz $\mathbb{L}$-filters on any interpretation coinciding with the entire collection of $\mathbb{L}$-filters. This characterization, combined with the results of Section 9.4, provides several additional characterizations of weak algebraizability. E.g., we get that $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A}$ and all $\mathbb{L}$-filters F on $\mathcal{A}$, $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$ and that $\mathbb{L}$ is weakly algebraizable if and only if $\Omega_{\mathcal{A}}$ is a lattice isomorphism from $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$.

In Section 9.6, we briefly introduce the property of *truth equationality*. We say that a logicoid $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ is *truth equational* if $\Omega_{\hat{\mathbf{B}}}$ is completely order reflecting on $\mathcal{C}^\flat$, that is, if, for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$,

$$\bigcap_{i\in I}\Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies} \quad \bigwedge_{i\in I}^{\flat} X_i \leq^\flat X.$$

We show that this is equivalent to the complete order reflectivity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for all interpretations $\mathcal{A}$ of $\mathbb{L}$. Finally, we prove that weak algebraizability is characterized as the conjunction of protoalgebraicity and truth equationality.

Part I

Logicates

Chapter 2

Basic Theory

2.1 Introduction

We make an attempt at developing an abstract theory of algebraic logic incorporating features of non-monotonicity. The objects of study are *consequence operators*, which, for our purposes, are mappings on the powerset of a set which are only required to satisfy idempotency. Thus, the inflationarity and monotonicity aspects of traditional closure operators may be missing.

In the traditional abstract studies in algebraic logic [24, 3, 12, 8, 14], a central role is played by closure operators or, equivalently, closure systems. Closure operators are operators on the powerset of a given set that are required to satisfy inflationarity, monotonicity and idempotency. If one wishes to relax this framework to accommodate non-monotonicity, then, at least in a first attempt, the axioms that should be shed, are those of inflationarity and of monotonicity. We look at a few steps one can make in this direction. Namely, we introduce "consequence operators" $C : \mathcal{P}(A) \to \mathcal{P}(A)$, on an underlying set A, which are only required to satisfy idempotency. They are suppose to simulate, or stand for, "raw logics", which we call "logicates".

In the traditional theory, after introducing the basic objects of study, one compares those that are "compatible". Here, compatibility means that, as operators, they apply on the same objects. Thus, only logics over the same underlying set are compared. One defines a closure operator C to be weaker than a closure operator C', and C' to be stronger than C, if, for all $X \subseteq A$, $C(X) \subseteq C'(X)$. However, once monotonicity is out of the picture, this definition makes little sense. Instead, for logicates, one has to devise new ways of performing meaningful comparisons. In this treatment, we focus on two natural ways of doing so. One is kind of intrinsic to the framework, since it only takes into account the fixed points or theories, a fact which makes sense since our operators only satisfy idempotency. The second is an attempt to emulate more closely the comparison in the classical framework. Here, one also considers the overall structure of the logicate; not solely its theories. This comparison is more "structure preserving" at the expense of being, somehow, more "artificial", since the structure is not intrinsic but rather devised. This artificiality is mended in a way in the second part of the monograph, where we switch focus from logicates to logicoids, in which the "structure" is inserted into the formalism, thus becoming "more natural".

Another important construct in both the abstract and concrete studies in algebraic logic is that of axiomatic extensions. When monotonicity is present, a closure operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is viewed as a consequence operator of a logic. One may need to add a subset $T \subseteq A$ as a new set of axioms to axiomatically strengthen the consequence relation. This is done by defining a new operator, based on the original, by setting, for all $X \subseteq A$, $C^T(X) = C(X \cup T)$. Note that both inflationarity and monotonicity are critical here. The first ensures that, for all X, $T \subseteq C^T(X)$, that is the new axioms become genuine consequences of the new operator. The second yields that $C \leq C^T$, i.e., the new operator is indeed an *extension* of the former via the adoption of the elements in T as new axioms. The criticality of these two axioms and the fact that they are missing in the nonmonotonic framework adopted here give an indication of why the task of emulating the extension process would necessarily involve difficulties and may ultimately prove insufficient and unsatisfactory. Nevertheless, we do the best we can by devising two different operators along these lines.

The first uses a more conservative approach. It "lifts" the consequences of $X \subseteq A$ to $C(T)$ if either X or $C(X)$ are contained in T. But some emphasis must be placed on the pejorative use of "lift" here, since, in fact, $C(T)$ may be a much smaller subset of A than either X or T, due to lack of inflationarity and monotonicity. This approach has the drawback that it does not give an operator which strengthens the original operator according to the "structure preserving" comparison of operators that we alluded to in the preceding paragraph. We take this as hinting to the need of an alternative, more "aggressive", line of attack. The more liberal approach, on the other hand, allows consequences to be lifted to $C(T)$ whenever the consequences of X happen to coincide with the consequences of some $Y \subseteq T$. This construct gives rise to an operator that does strengthen the original operator C and, as it turns out, strengthens also the operator obtained by the more conservative approach.

In Section 2.2, the basic objects of study, called *logicates*, which are idempotent operators on the powerset of a set are introduced. The directed graphs that reflect the structure of logicates are called *necropoleis*,[1] since they consist of components called *pyramids*. By imposing a linear ordering on $\mathcal{P}(A)$, one may recast both as linearly ordered structures, with additional features, called linearized consequences and linearized necropoleis, respectively. However, the process of linearization introduces redundancy, which one sheds by passing to equivalence classes of those ordered structures under appropriately defined equivalence relations. We call an equivalence class of linearized necropoleis a *cemetery*.

In Section 2.3 we encounter ways we may use to compare logicates over the same underlying set. *Equipotency* is the equivalence resulting by having identical sets of theories. By comparing sets of theories by the subset relation, we may also impose a partial ordering on the set of equipotency classes. Being *weaker*, on the other hand, is a relation that also takes into account sets of theories but, in addition, it considers the consequence structure. These comparisons are also investigated from the point of view of alternative presentations of logicates, namely, using necropoleis, classes of linearized consequences and cemeteries.

In Section 2.4, we introduce and compare the two notions that attempt to replace axiomatic extensions in the nonmonotonic context. The first is

[1] Plural of **necropolis**, pronounced the same, but stressed *necropóleis* vs. *necrópolis*.

called *boosting*. It seems a natural one to adopt, based on Occam's Razor. However, it fails to produce a strengthened version of the original logicate under the comparison criterion that takes the consequence structure of the logicate into account. To atone for this failure, we fortify boosting to what we call *strong boosting*. This adjustment produces an operator that strengthens both the original and the boosted version of the original operator.

2.2 Logicates

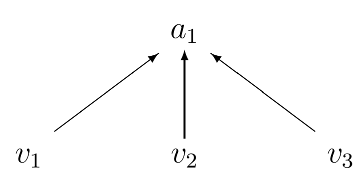
In this section, we introduce the basic notion of *logicate* which forms the underlying object of study throughout. We spend some time giving different representations that may help looking at these objects from different points of view, developing some intuition about them and, also, perhaps, in visualizing their behavior.

Let A be a set. Let $\mathcal{P}(A)$ denote the powerset of A. An **idempotent mapping on** $\mathcal{P}(A)$ is a mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for all $X \subseteq A$,

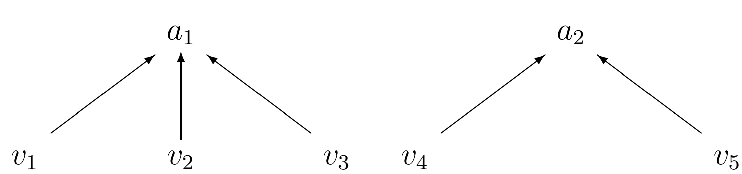
(**Idempotency**) $C(C(X)) = C(X)$.

A **consequence operator on** A or a **logicate on** A is an idempotent mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$.

A **pyramid** is a simple kind of directed graph. It has a distinguished vertex, called the **apex**, and all its other vertices, called **base vertices**, are connected to the apex.



A **necropolis** is a collection of (disjoint) pyramids.



A **consequence system on** A is a necropolis on A, i.e., a necropolis $G = \langle \mathcal{P}(A), E \rangle$, with vertex set $\mathcal{P}(A)$.

Proposition 1 assures that, for all intends and purposes, consequence operators and consequence systems are interchangeable. So which one to use depends entirely on the point of view taken and on the convenience for the specific application.

Proposition 1 *Let A be a set. Consequence operators on A are in one to one correspondence with consequence systems on A.*

Proof: Consider a consequence operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Create the graph of the function C, i.e., the graph having set of vertices $\mathcal{P}(A)$ and edges $X \to Y$ iff $C(X) = Y$ Then throw away the loops. Because of idempotency, this gives rise to a consequence system on A.

Conversely, consider a consequence system $\langle \mathcal{P}(A), E \rangle$ on A. Define $C : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $X \subseteq A$,

$$C(X) = \begin{cases} Y, & \text{if } (X, Y) \in E, \\ X, & \text{if } X \text{ has outdegree } 0. \end{cases}$$

It should be fairly clear that, since $\langle \mathcal{P}(A), E \rangle$ is a consequence system, C is well defined and idempotent. Thus, C is a consequence operator.

Starting from a consequence operator on A, constructing its graph and, then, obtaining the operator associated with it, results to the original consequence operator. And, similarly, stating from a consequence system, creating the function depicted by it and, then, obtaining its graph, results to the original consequence system. Thus, the established correspondence is a one-to-one correspondence. ■

A **linearized consequence** $\langle \mathcal{P}(A), \leq, L \rangle$ **on** A consists of a linear ordering $\leq$ on $\mathcal{P}(A)$ and a function $L : \mathcal{P}(A) \to \mathcal{P}(A)$, which satisfies, for all $X, Y \subseteq A$:

(**Idempotency**) $L(L(X)) = X$;

($\leq$-**Inflationarity**) $X \leq L(X)$;

($\leq$-**Monotonicity**) $X \leq Y$ implies $L(X) \leq L(Y)$.

Two linearized consequences $\langle \mathcal{P}(A), \leq, L \rangle$ and $\langle \mathcal{P}(A), \leq', L' \rangle$ on A are **equivalent**, written

$$\langle \mathcal{P}(A), \leq, L \rangle \sim \langle \mathcal{P}(A), \leq', L' \rangle,$$

if, for all $X \subseteq A$,

$$L(X) = L'(X).$$

It is immediate from the definition that the equivalence $\sim$ is a bona fide equivalence relation on linearized consequences on a given set A. Consequently, it partitions the collection of linearized consequences into equivalence classes. The collection of equivalence classes are in one-to-one correspondence with consequence operators (and, hence, by Proposition 1, with consequence systems) on A.

Proposition 2 *Consequence operators on a set A are in one-to-one correspondence with equivalence classes of linearized consequences on A.*

Proof: Let $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a consequence operator on A. Create its consequence system. Arbitrarily order its pyramids, say $P_0, P_1, \ldots$. For each pyramid P_i, create a linear ordering $o(P_i)$ of is vertices by arbitrarily ordering the base vertices and then placing the apex as the largest vertex. Finally, create a linear ordering $\leq$ of $\mathcal{P}(A)$ by juxtaposing the linear orderings $o(P_i)$ of the pyramids according to the originally adopted ordering of the set of pyramids,

$$o(P_0), o(P_1), \ldots.$$

Now consider the structure $\langle \mathcal{P}(A), \leq, C \rangle$. Since, by hypothesis, C is a consequence operator, $C(C(X)) = C(X)$, for all $X \subseteq A$. So idempotency holds. Since the apex of a pyramid follows all vertices of its base, $X \leq C(X)$, whence $\leq$-Inflationarity also holds. Finally, consider $X, Y \subseteq A$, such that $X \leq Y$.

- Suppose X, Y are vertices in the same pyramid. Then $C(X) = C(Y)$.
- Suppose X, Y are vertices in different pyramids. Then the pyramid of X precedes in the linear ordering the pyramid of Y, whence $C(X) < C(Y)$.

In either case $C(X) \leq C(Y)$ and $\leq$-Monotonicity also holds. Thus, $\langle \mathcal{P}(A), \leq, C \rangle$ is a linearized consequence on A. We let its equivalence class $L(C)$ be the class associated with the consequence operator C. By the definition of equivalence, any two linearized consequences constructed in this way from the same consequence operator are equivalent, whence $C \mapsto L(C)$ is well defined.

Suppose, conversely, that an equivalence class of linearized consequences is given. Consider a representative $\langle \mathcal{P}(A), \leq, L \rangle$. To this class we associate the consequence operator $L : \mathcal{P}(A) \to \mathcal{P}(A)$. By definition, if one changes the representative, the consequence operator remains invariant. It follows that this association is also well defined.

The two mappings just described are inverses of one another. So they establish a one-to-one correspondence between consequence operators on A and equivalence classes of linearized consequences on A. ■

A **linearized necropolis** $\langle \mathcal{P}(A), \leq, c \rangle$ **on** A consists of a linear ordering $\leq$ on $\mathcal{P}(A)$, having a maximum element, together with a 2-coloring

$$c : \mathcal{P}(A) \to \{w, p\}$$

of the vertices, with colors, say, white and purple, such that the $\leq$-maximum element is colored purple.

Two linearized necropoleis $\langle \mathcal{P}(A), \leq, c \rangle$ and $\langle \mathcal{P}(A), \leq', c' \rangle$ on $\mathcal{P}(A)$ are **equivalent**, written

$$\langle \mathcal{P}(A), \leq, c \rangle \bowtie \langle \mathcal{P}(A), \leq', c' \rangle,$$

if the colorings are identical, that is, $c = c'$, and each purple vertex is preceded in each ordering (before the appearance of another purple vertex) by the same (unordered) set of white vertices, i.e., for any intervals in $\leq$ and $\leq'$, respectively,

$$p_0, w_0, w_1, \dots, p_1$$
$$p'_0, w'_0, w'_1, \dots, p'_1,$$

where the vertices denoted by w's are colored white and the vertices denoted by p's are colored purple, if $p_1 = p'_1$, then $\{w_0, w_1, \dots\} = \{w'_0, w'_1, \dots\}$.

A **cemetery on** A is an equivalence class (with respect to $\bowtie$) of linearized necropoleis on A.

Proposition 3 *Let A be a set. Consequence operators are in one-to-one correspondence with cemeteries on A.*

Proof: By Proposition 2, it suffices to show that equivalence classes of linearized consequences are in one-to-one correspondence with cemeteries.

Consider, first, an equivalence class of linearized consequences, with representative $\langle \mathcal{P}(A), \leq, L\rangle$. Construct the linearized necropolis $\langle \mathcal{P}(A), \leq, c\rangle$ by defining the coloring c by setting, for all $X \subseteq A$,

$$c(X) = \begin{cases} p, & \text{if } L(X) = X, \\ w, & \text{otherwise.} \end{cases}$$

Suppose two linearized consequences $\langle \mathcal{P}(A), \leq, L\rangle$ and $\langle \mathcal{P}(A), \leq', L'\rangle$ are equivalent. Then, by definition, $L = L'$. Thus, directly by definition, $c = c'$. So to establish that the two linearized necropoleis $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c\rangle$ are equivalent, it suffices to show that the two orderings are related in the required way. Consider two intervals in $\leq$ and $\leq'$, respectively,

$$p_0, w_0, w_1, \dots, p_1$$
$$p'_0, w'_0, w'_1, \dots, p'_1,$$

where the vertices denoted by w's are colored white, the vertices denoted by p's are colored purple and $p_1 = p'_1$. Assume, without loss of generality, that $p'_1 = p_1 <' w_0 <' p'_2$, where p'_2 is the first purple vertex following w_0 in $<'$. The axioms of linearized consequences, then, give $L(w_0) = p_1$ and $L(w_0) = L'(w_0) = p'_2 \neq p'_1 = p_1$, a contradiction. Thus, the two linearized necropoleis $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c'\rangle$ are equivalent.

Suppose, conversely, that a cemetery is given. Let $\langle \mathcal{P}(A), \leq, c\rangle$ be a representative linearized necropolis. Define the triple $\langle \mathcal{P}(A), \leq, L\rangle$ by setting, for all $X \subseteq A$,

$$L(X) = \min\{Y : X \leq Y \text{ and } c(Y) = p\}.$$

We show that $\langle \mathcal{P}(A), \leq, L\rangle$ is a linearized consequence. Indeed, for all $X, Y \subseteq A$, we have:

- $X \leq \min\{Y : X \leq Y \text{ and } c(Y) = p\} = L(X)$;

- Further,

$$\begin{aligned} X \le Y \quad \text{implies} \quad & \min\{Z : X \le Z \text{ and } c(Z) = p\} \\ & \le \min\{Z : Y \le Z \text{ and } c(Z) = p\} \\ \text{implies} \quad & L(X) \le L(Y); \end{aligned}$$

- Finally,

$$\begin{aligned} L(L(X)) &= \min\{Y : L(X) \le Y \text{ and } c(Y) = p\} \\ &= \min\{Y : \min\{Z : X \le Z \text{ and } c(Z) = p\} \le Y \\ & \qquad \text{and } c(Y) = p\} \\ &= \min\{Z : X \le Z \text{ and } c(Z) = p\} \\ &= L(X). \end{aligned}$$

Suppose $\langle \mathcal{P}(A), \le, c\rangle$ and $\langle \mathcal{P}(A), \le', c\rangle$ are equivalent linearized necropoleis. We show that $\langle \mathcal{P}(A), \le, L\rangle$ and $\langle \mathcal{P}(A), \le', L'\rangle$ are equivalent linearized consequences. Suppose, to the contrary, that, for some $X \subseteq A$, $L(X) \ne L'(X)$. Then X is in the interval preceding the purple element $L(X)$ in $\le$, but X is not in the interval preceding the purple element $L(X)$ in $\le'$. This contradicts the equivalence of the two linearized necropoleis $\langle \mathcal{P}(A), \le, c\rangle$ and $\langle \mathcal{P}(A), \le', c\rangle$.

The two associations are inverses of one another. So equivalence classes of linearized consequences correspond to cemeteries. As a result, consequence operators on A are also in one-to-one correspondence with cemeteries on A, as claimed. ∎

2.3 Comparing Logicates

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a logicate. We set

$$\mathcal{C} = \{X \subseteq A : C(X) = X\},$$

i.e., $\mathcal{C}$ is the set of its **fixed points** or **theories**. Let us denote the collection of all logicates on the same set A by $\mathrm{Lgct}(A)$.

A critical role in our considerations is played by the theories of a logicate. In fact, on several occasions we may need to construct a logicate for which the only feature that matters is the set of its theories and is otherwise arbitrary. To prepare for this eventuality, we define the notion of *equipotency*.

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ be logicates on A. We say that C and C' are **equipotent**, written $C \triangleq C'$, if $\mathcal{C} = \mathcal{C}'$,

$$C \triangleq C' \quad \text{iff} \quad \mathcal{C} = \mathcal{C}'.$$

To refer to the $\triangleq$-equivalence class of a logicate C, we may write either $\mathcal{C}$ or $C/{\triangleq}$. The first notation, which adds overloading, requires a typographical correspondence that must be respected to avoid confusion.

In accordance, two consequence systems $\langle \mathcal{P}(A), E\rangle$ and $\langle \mathcal{P}(A), E'\rangle$ are **equipotent**, written

$$\langle \mathcal{P}(A), E\rangle \triangleq \langle \mathcal{P}(A), E'\rangle,$$

if the set of apexes of their pyramids are identical.

Proposition 4 *Suppose A is a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ and $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ are two logicates. C and C' are equipotent if and only if the corresponding consequence systems are equipotent.*

Proof: This is a direct consequence of the correspondence established in Proposition 1. In fact, we have

$$\begin{array}{lll} C \triangleq C' & \text{iff} & \text{for all } X,\ C(X) = X \text{ iff } C'(X) = X \\ & \text{iff} & \text{for all } X,\ X \text{ has outdegree } 0 \text{ in } E \text{ iff} \\ & & \qquad\qquad X \text{ has outdegree } 0 \text{ in } E' \\ & \text{iff} & \langle \mathcal{P}(A), E\rangle \text{ and } \langle \mathcal{P}(A), E'\rangle \text{ have same apexes} \\ & \text{iff} & \langle \mathcal{P}(A), E\rangle \triangleq \langle \mathcal{P}(A), E'\rangle. \end{array}$$

■

Logicates may be preordered using the inverse inclusion relation between their theories, i.e.,

$$C \trianglelefteq C' \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

This preorder induces a partial ordering between $\triangleq$-equivalence classes, which is expressed by

$$C/{\triangleq} \trianglelefteq C'/{\triangleq} \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

The unpleasant features of this construct are, first, that it compares equivalence classes and, second, that it hides important details of the consequence relation. On the other hand, we get the advantage of having a nice ordered set on the equivalence classes (almost a complete lattice, but without top). This is an important feature that we shall build upon, attempting to simulate, to the extent possible, the traditional algebraic theory pertaining to monotonic logics.

Proposition 5 *Let A be a set. The quotient $\mathrm{Lgct}(A)/{\triangleq}$, ordered by $\trianglelefteq$, is isomorphic to the ordered set $\langle \mathcal{P}(\mathcal{P}(A))\backslash\varnothing, \supseteq\rangle$.*

Proof: To see this, it suffices to show that the mapping

$$\begin{array}{rrcl} \mathrm{Idp}: & \mathrm{Lgct}(A)/{\triangleq} & \longrightarrow & \mathcal{P}(\mathcal{P}(A))\backslash\varnothing; \\ & C/{\triangleq} & \longmapsto & \mathcal{C}, \end{array}$$

is an order isomorphism from $\langle \mathrm{Lgct}(A)/{\triangleq}, \trianglelefteq\rangle$ onto $\langle \mathcal{P}(\mathcal{P}(A))\backslash\varnothing, \supseteq\rangle$. Note that Idp is well defined and one-to-one, since, by definition of $\triangleq$, for all $C, C' \in \mathrm{Lgct}(A)$,

$$C/{\triangleq} = C'/{\triangleq} \quad \text{iff} \quad \mathcal{C} = \mathcal{C}'.$$

It is also onto. Given $\varnothing \neq \mathcal{X} \subseteq \mathcal{P}(A)$, we let $X_0 \in \mathcal{X}$ be a fixed element of $\mathcal{X}$ and define $C : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $Y \subseteq A$,

$$C(X) = \begin{cases} X, & \text{if } X \in \mathcal{X}, \\ X_0, & \text{otherwise.} \end{cases}$$

Then, clearly, $\mathcal{C} = \mathcal{X}$. So Idp is a bijection. Moreover, by the definition of $\trianglelefteq$, for all $C, C' \in \text{Lct}(A)$,

$$C/{\triangleq} \trianglelefteq C'/{\triangleq} \quad \text{iff} \quad \mathcal{C} \supseteq \mathcal{C}'.$$

Thus, Idp is both order preserving and order reflecting. ■

We next introduce a different ordering that takes also into account the consequence structure. We write $C \leq C'$ and say that C is **weaker** than C' and that C' is **stronger** than C if the following conditions hold, for all $X, Y \subseteq A$:

- $C'(X) = X$ implies $C(X) = X$, i.e., all C'-theories are also C-theories;
- $C(X) = Y$ implies $C'(X) = C'(Y)$, i.e., any consequence in C must be between elements having the same image under C'.

Note that the first condition may be equivalently formulated by saying that, for all $X \subseteq A$,

$$C'(X) = C(C'(X)).$$

Note, also, that the second condition may be equivalently formulated by saying that, for all $X \subseteq A$,

$$C'(X) = C'(C(X)).$$

Consequently, we may summarize by saying that $C \leq C'$ if and only if, for all $X \subseteq A$,

$$C'(X) = C(C'(X)) = C'(C(X)).$$

The $\leq$ relation has the advantage that it is a genuine partial ordering on $\text{Lgct}(A)$. So there is no need to take a quotient. On the other hand, it is not difficult to see that its order structure is less well behaved than $\trianglelefteq$.

Proposition 6 *The structure* $\mathbf{Lgct}(A) = \langle \text{Lgct}(A), \leq \rangle$ *is a partially ordered set.*

Proof: We have to prove reflexivity, antisymmetry and transitivity. To this end, let $C, C', C'' \in \text{Lct}(A)$.

- By idempotency, for all X, $C(X) = C(C(X))$, whence $C \leq C$ and $\leq$ is reflexive.

- Suppose $C \leq C'$ and $C' \leq C$, Then, we get, for all X,

$$C(X) = C(C'(X)) = C'(X).$$

  Thus $C = C'$ and $\leq$ is anti-symmetric.

- Let $C \leq C'$ and $C' \leq C''$. Then, by definition, for all X,

$$\begin{array}{l} C'(X) = C'(C(X)) = C(C'(X)), \\ C''(X) = C''(C'(X)) = C'(C''(X)). \end{array}$$

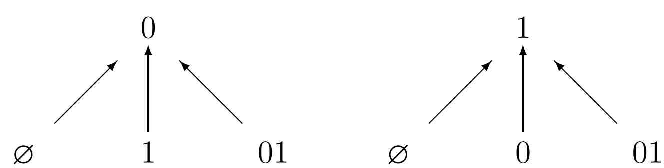
  Therefore, we get

$$\begin{array}{lll} C(C''(X)) & = & C(C'(C''(X))) = C'(C''(X)) = C''(X); \\ C''(C(X)) & = & C''(C'(C(X))) = C''(C'(X)) = C''(X). \end{array}$$

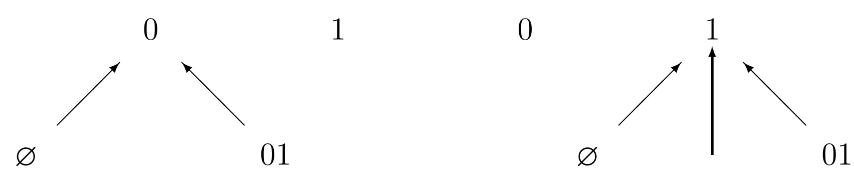
  Hence, by definition, $C \leq C''$ and $\leq$ is also transitive.

Thus, $\mathbf{Lgct}(A) = \langle \mathrm{Lgct}(A), \leq \rangle$ is a partially ordered set. ∎

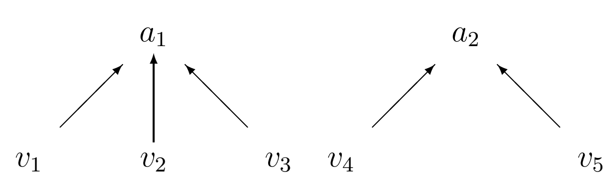
To see that the order is not lattice-like consider, e.g., The following two logicates on $\{0,1\}$, given in their necropolis representation.



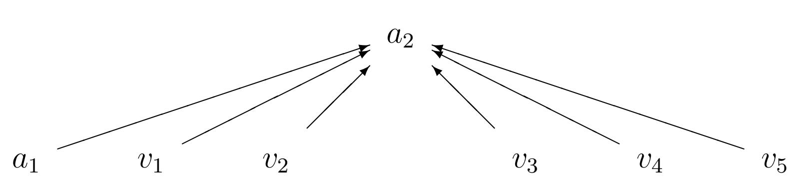
The following two logicates are lower bounds. There is, however, no greatest lower bound.



Consider, next two or more disjoint pyramids with vertices in $\mathcal{P}(A)$.



A **merge** results by taking one of the apexes as a new apex and all other vertices as base vertices of a new pyramid. E.g., a merge of the two pyramids above is



Suppose G' is a necropolis obtained from a necropolis G by some merges. Then we say that G is **finer** than G' and write $G' \leq_n G$. The same terminology carries to consequence systems. That is, given two consequence systems $G = \langle \mathcal{P}(A), E\rangle$ and $G' = \langle \mathcal{P}(A), E'\rangle$, we say that G is **finer** than G' and write $G' \leq_n G$, if the necropolis G' is the result of some merges applied on the necropolis G.

Proposition 7 *Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ consequence operators on A, with corresponding consequence systems $G = \langle \mathcal{P}(A), E\rangle$, $G' = \langle \mathcal{P}(A), E'\rangle$. The $C \leq C'$ if and only if $G' \leq_n G$.*

Proof: Suppose, first, that $C \leq C'$. Consider a pyramid in G'. If X is its apex, then $C'(X) = X$. Thus, by the definition of $\leq$, we get that $C(X) = X$. This shows that X is the apex of a pyramid in G. Let Y be a base vertex of the same pyramid in G. Then $C(Y) = X$. Therefore, $C'(Y) = C'(X) = X$. So Y is a base vertex of the pyramid with apex X in G'. Suppose Z is a base vertex of the pyramid in G' that is not a base vertex of the pyramid in G with apex X, and let Z' be another vertex of the same pyramid in G as Z. Then $C(Z) = C(Z') = Z''$, which imply $C'(Z) = C'(Z') = C'(Z'') = X$. Hence all these vertices are base vertices of the pyramid in G', with apex X. This shows that G' results from merges of pyramids in G. Therefore $G' \leq_n G$.

Suppose, conversely, that G is finer that G'. We show that $C \leq C'$. Suppose that for some $X \subseteq A$, $C'(X) = X$. Then, in G', X is the apex of a pyramid. Since that pyramid is the result of a merge of one or more pyramids in G, X must be an apex in G. Thus, $C(X) = X$. Assume, next, that, for some $X, Y \subseteq A$, $C(X) = Y$. Thus, in G, X and Y are elements of the same pyramid. Therefore, in G' also, X and Y must belong to the same pyramid (into which the one in G has been merged). It follows that $C'(X) = C'(Y)$. By the definition of $\leq$, we conclude that $C \leq C'$. ∎

A different characterization may be obtained by looking at equivalence classes of linearized consequences. Let $\langle \mathcal{P}(A), \leq, L\rangle$ and $\langle \mathcal{P}(A), \leq', L'\rangle$ be two linearized consequences. We say that $\langle \mathcal{P}(A), \leq, L\rangle$ is **weaker** than $\langle \mathcal{P}(A), \leq', L'\rangle$ and that $\langle \mathcal{P}(A), \leq', L'\rangle$ is **stronger** than $\langle \mathcal{P}(A), \leq, L\rangle$, written

$$\langle \mathcal{P}(A), \leq, L\rangle \leq_\ell \langle \mathcal{P}(A), \leq', L'\rangle,$$

if the following conditions hold:

- $\leq = \leq'$;
- $L(X) \leq L'(X)$, for all $X \subseteq A$.

This ordering induces an ordering on the collection of equivalence classes of linearized congruences. Namely, we say that an equivalence class of linearized consequences is **weaker** than another class if there exist a representative

$\langle \mathcal{P}(A), \leq, L\rangle$ of the first class and a representative $\langle \mathcal{P}(A), \leq', L'\rangle$ of the second class, such that

$$\langle \mathcal{P}(A), \leq, L\rangle \leq_\ell \langle \mathcal{P}(A), \leq', L'\rangle.$$

Further, this ordering on equivalence classes reflects the ordering on consequence operators.

Proposition 8 *Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ be consequence operators on A. Then $C \leq C'$ if and only if the equivalence class of linearized consequences associated with C is weaker that the class associated with C'.*

Proof: Suppose, first, that $C \leq C'$. One has to choose a representative of the linearized consequence corresponding to C carefully so as to be able to accommodate the strengthening to C' using the same linear ordering. Using Propositions 1 and 7, we order the pyramids of the consequence system so that pyramids of C merged by C' are placed in adjacent positions in the ordering and, moreover, so that the pyramid whose apex is used as the apex in the merge is placed last. Define $\leq$ to be the ordering of $\mathcal{P}(A)$ constructed in this way. Then it is not difficult to see that both C and C' satisfy Idempotency, $\leq$-Inflationarity and $\leq$-Monotonicity. Further, by construction, $C(X) \leq C'(X)$, for all $X \subseteq A$. Thus, the class of linearized consequences represented by $\langle \mathcal{P}(A), \leq, C\rangle$ is indeed weaker than the class represented by $\langle \mathcal{P}(A), \leq, C'\rangle$.

Suppose, conversely, that C is represented by the linearized consequence $\langle \mathcal{P}(A), \leq, C\rangle$, C' is represented by the linearized congruence $\langle \mathcal{P}(A), \leq, C'\rangle$ and that $C(X) \leq C'(X)$, for all $X \subseteq A$. If $C'(X) = X$, then $C(X) \leq C'(X) = X$ and, since the reverse inequality holds by $\leq$-Inflationarity, we get $C(X) = X$. Finally, if $C(X) = Y$, then

$$C'(Y) = C'(C(X)) \leq C'(C'(X)) = C'(X)$$

and, since $X \leq C(X) = Y$, $C'(X) \leq C'(Y)$. Hence, $C'(X) = C'(Y)$. So the two properties demanded by the definition of $\leq$ for consequence operators are satisfied, showing that $C \leq C'$. ■

Finally, we turn to cemeteries to establish similar comparison criteria. Let $\langle \mathcal{P}(A), \leq, c\rangle$ and $\langle \mathcal{P}(A), \leq', c'\rangle$ be linearized necropoleis on $\mathcal{P}(A)$. We say that $\langle \mathcal{P}(A), \leq, c\rangle$ is **finer** than $\langle \mathcal{P}(A), \leq', c'\rangle$, written

$$\langle \mathcal{P}(A), \leq', c'\rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c\rangle,$$

if the following hold:

- $\leq = \leq'$;
- $c'(X) = p$ implies $c(X) = p$, i.e. the set of purple nodes under c' is a subset of those under c.

In a way similar to linearized consequences, we say that a cemetery $\mathcal{T}$ on A is **finer** than a cemetery $\mathcal{T}'$ on A if there exist a linearized necropolis $\langle \mathcal{P}(A), \leq, c \rangle$ representing $\mathcal{T}$ and a linearized necropolis $\langle \mathcal{P}(A), \leq', c' \rangle$ representing $\mathcal{T}'$, such that

$$\langle \mathcal{P}(A), \leq', c' \rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c \rangle.$$

Proposition 9 *Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$, $C' : \mathcal{P}(A) \to \mathcal{P}(A)$ be consequence operators on A. Then $C \leq C'$ if and only if the cemetery associated with C is finer that the cemetery associated with C'.*

Proof: In the proof of Proposition 3, we showed that classes of linearized consequences and cemeteries are in one-to-one correspondence. More precisely, a class of linearized consequences represented by $\langle \mathcal{P}(A), \leq, L \rangle$ corresponds to the cemetery represented by $\langle \mathcal{P}(A), \leq, c \rangle$, where $c(X) = p$ if and only if $L(X) = X$, for all $X \subseteq A$. And, conversely, a cemetery represented by $\langle \mathcal{P}(A), \leq, c \rangle$ corresponds to the class represented by the linearized consequence $\langle \mathcal{P}(A), \leq, L \rangle$, where

$$L(X) = \min \{Y : X \leq Y \text{ and } c(Y) = p\}.$$

In view of Proposition 8, it suffices to show that a class of linearized consequences is weaker than another class if and only if the corresponding cemetery of the second is finer than the cemetery corresponding to the first.

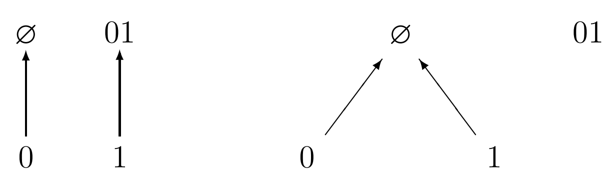
Assume, first, that $\langle \mathcal{P}(A), \leq, L \rangle \leq_\ell \langle \mathcal{P}(A), \leq, L' \rangle$. Then, by definition, $L(X) \leq L'(X)$, for all $X \subseteq A$. It follows immediately that $L'(X) = X$ implies $L(X) = X$. Thus, $c'(X) = p$ implies $c(X) = p$. Thus, by definition, $\langle \mathcal{P}(A), \leq, c' \rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c \rangle$.

Assume, conversely, that $\langle \mathcal{P}(A), \leq, c' \rangle \leq_{\ell n} \langle \mathcal{P}(A), \leq, c \rangle$. Then, for the corresponding linearized consequences, we have, for all $X \subseteq A$,

$$\begin{array}{rcl} L(X) & = & \min \{Y : X \leq Y \text{ and } c(Y) = p\} \\ & \leq & \min \{Y : X \leq Y \text{ and } c'(Y) = p\} \\ & = & L'(X). \end{array}$$

Thus, $\langle \mathcal{P}(A), \leq, L \rangle \leq_\ell \langle \mathcal{P}(A), \leq, L' \rangle$. ■

In concluding this section, let us also observe that equipotent logicates are, in general, incomparable with respect to strengthening. E.g., if $A = \{0, 1\}$, the following two consequence operators are equipotent, but neither of the two is weaker than the other.



2.4 Boosting and Strong Boosting

In this section we attempt to build operations that are, in the nonmonotonic context, "parallel" to axiomatic extensions in the traditional monotonic framework. We start with an operation called *boosting*. It is devised by Occam's razor, i.e., it is seemingly the simplest possible recipe that makes sense with what we have. However, it is shown that, even though boosting gives rise to a logicate and, in fact, a logicate that is related to the given one, it does not result in a strengthening of the original according to the $\leq$ ordering. So we are compelled to fortify boosting giving rise to what we call *strong boosting*. We show that the strong boosting construction results in a logicate which is not only a strengthening of the original but also a strengthening of its boosting.

Suppose $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is a logicate and $T \subseteq A$. The **boosting of** C **by** T is the operator $C_T : \mathcal{P}(A) \to \mathcal{P}(A)$ that is defined, for all $X \subseteq A$, by

$$C_T(X) = \begin{cases} C(T), & \text{if } X \subseteq T \text{ or } C(X) \subseteq T, \\ C(X), & \text{otherwise.} \end{cases}$$

We show that this recipe gives a bona fide consequence operator.

Proposition 10 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. Then $C_T : \mathcal{P}(A) \to \mathcal{P}(A)$ is also a logicate.*

Proof: Suppose, first, that $X \nsubseteq T$ and $C(X) \nsubseteq T$. Then, by definition

$$C_T(C_T(X)) = C_T(C(X)) = C(C(X)) = C(X) = C_T(X).$$

Suppose, next, that $X \subseteq T$ or $X \subseteq C(T)$. Then

$$C_T(C_T(X)) = C_T(C(T)) = C(T) = C_T(X).$$

Thus, in either case C_T is idempotent and, therefore, it is a consequence operator. ∎

We also describe the way the two consequence systems are related. Consider a consequence system $G = \langle \mathcal{P}(A), E \rangle$ and let $T \subseteq A$. Define the graph $G_T = \langle \mathcal{P}(A), E_T \rangle$ as follows. If, in G, a base vertex X of some pyramid is such that $X \subseteq T$, then, make X a base vertex of the pyramid with apex $C(T)$. If, in G, the apex X of some pyramid is such that $X \subseteq T$, then merge that pyramid with the pyramid of $C(T)$, keeping $C(T)$ as the apex of the merge.

Proposition 11 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. If G is the consequence system corresponding to C, then G_T is the consequence system corresponding to C_T.*

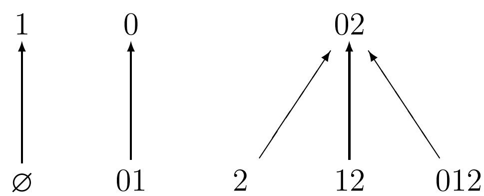
Proof: Suppose, first, that $X \subseteq T$ is a base vertex of some pyramid. Then, by definition, $C_T(X) = C(T)$. Moreover, since X is a base vertex, $C(X) \neq X$. On the other hand

$$C(C_T(T)) = C(C(T)) = C(T) = C_T(T).$$

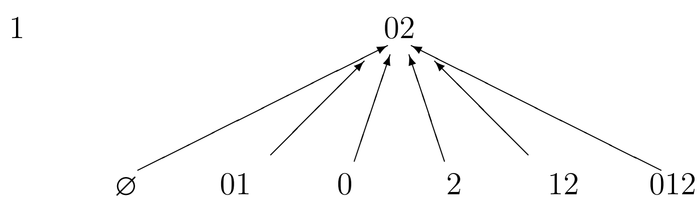
Therefore, $X \neq C_T(T)$. Hence, in G_T, X is attached as a base vertex of the pyramid with apex $C(T)$.

Suppose, finally, that $X \subseteq T$, with $X \neq C(T)$, is the apex of some pyramid. Then, for all Y in the same pyramid, $C(Y) = X \subseteq T$. Thus, $C_T(Y) = C(T) \neq X$. This shows that in G_T the entire pyramid gets merged under the pyramid with vertex $C(T)$. ■

Note that boosting, in general, is equipotent to a logicate resulting by applying merges. E.g., consider the consequence operator C on $A = \{0,1,2\}$, whose consequence system is pictured below.



Consider $T = \{0,2\}$. Then the consequence operator C_T has the consequence system G_T, shown below. It is equipotent to the system resulting from G by an application of a single merge.



So, as equipotents may be incomparable with respect to strengthening, it is not, in general, the case that C_T is a strengthening of C. This leads to the operation of strong boosting, which does lead to strengthening since it only involves merges.

Suppose $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is a consequence operator and $T \subseteq A$. The **strong boosting of C by T** is the consequence operator $C^T : \mathcal{P}(A) \to \mathcal{P}(A)$ that is defined, for all $X \subseteq A$, by

$$C^T(X) = \begin{cases} C(T), & \text{if } C(X) = C(Y), \text{ for some } Y \subseteq T, \\ C(X), & \text{otherwise.} \end{cases}$$

For now, let us point out that, if, for some $X \subseteq A$, $C_T(X)$ is determined by the first branch in the piecewise definition of C_T, then the same holds in the piecewise definition of C^T. Suppose that $X \subseteq T$. Then certainly $C(X) = C(Y)$, for some $Y \subseteq T$. Moreover, if $C(X) \subseteq T$, then, similarly, $C(X) = C(Y)$, for some $Y \subseteq T$ (namely, for $Y = C(X)$).

Proposition 12 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. Then $C^T : \mathcal{P}(A) \to \mathcal{P}(A)$ is also a logicate.*

Proof: Let $X \subseteq A$. Suppose, first, that $C(X) = C(Y)$, for some $Y \subseteq T$. Then we have

$$C^T(C^T(X)) = C^T(C(T)) = C(T) = C^T(T).$$

On the other hand, suppose that, for all $Y \subseteq T$, $C(X) \neq C(Y)$. Note that, in this case, one also has that $C(C(X)) = C(X) \neq C(Y)$, for all $Y \subseteq T$. Taking this into account, we get

$$C^T(C^T(X)) = C^T(C(X)) = C(C(X)) = C(X) = C^T(X).$$

Therefore, C^T is a consequence operator. ■

We describe, next, the way corresponding consequence systems are related. Consider a consequence system $G = \langle \mathcal{P}(A), E \rangle$ and let $T \subseteq A$. Define the necropolis $G^T = \langle \mathcal{P}(A), E^T \rangle$ as follows. If, in G, a vertex X of some pyramid is such that $X \subseteq T$, then merge that pyramid with the pyramid of $C(T)$, keeping $C(T)$ as the apex of the merge.

Proposition 13 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. If G is the consequence system corresponding to C, then G^T is the consequence system corresponding to C^T.*

Proof: Suppose $X \subseteq T$ is a vertex (base or apex) of some pyramid in G. Then, for all Y in the same pyramid, $C(Y) = C(X)$ and, hence, $C^T(Y) = C(T)$. Thus, the entire pyramid is merged in G^T under $C(T)$ (this operation is trivial if $C(X) = C(T)$). For Y any vertex in any pyramid in which $X \nsubseteq T$, for all X, we get, by definition, $C^T(X) = C(X)$. Thus, all these pyramids are maintained in G^T as they were in G. ■

It turns out that the strong boosting C^T is a strengthening both of C and of the boosting C_T.

Proposition 14 *Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a logicate and $T \subseteq A$. Then $C \leq C^T$ and $C_T \leq C^T$.*

Proof: That $C \leq C^T$ is a direct consequence of Proposition 13, which asserts that G^T is a merge of G, i.e., that $G^T \leq_n G$, and of Proposition 7, which, when applied to $G^T \leq_n G$, gives that $C \leq C^T$.

For the second strengthening relation, we must prove that, for all $X \subseteq A$, $C^T(X) = X$ implies $C_T(X) = X$ and $C^T(X) = C^T(C_T(X))$.

Let us, first, fix $X \subseteq A$, such that $C^T(X) = X$. If $X \subseteq T$ or $C(X) \subseteq T$,

$$\begin{array}{lll} X & = & C^T(X) \quad \text{(Hypothesis)} \\ & = & C(T) \quad (C(X) = C(C(X)) \text{ and } X \subseteq T \text{ or } C(X) \subseteq T) \\ & = & C_T(X). \quad (X \subseteq T \text{ or } C(X) \subseteq T) \end{array}$$

Otherwise,

$$\begin{array}{rcl} C_T(X) & = & C(X) \quad (X \nsubseteq T \text{ and } C(X) \nsubseteq T) \\ & = & C(C^T(X)) \quad (C^T(X) = X) \\ & = & C^T(X) \quad (C \leq C^T) \\ & = & X. \quad (C^T(X) = X) \end{array}$$

In either case, the first condition holds.

Let us, now, fix $X \subseteq A$. One has to look at three possible cases.

- Suppose $X \subseteq T$ or $C(X) \subseteq T$. In either case, there exists $Y \subseteq T$, such that $C(X) = C(Y)$. Thus, we get

$$C^T(X) = C(T) = C^T(C(T)) = C^T(C_T(X)).$$

- Suppose, next, that $X \nsubseteq T$, $C(X) \nsubseteq T$ and $C(X) = C(Y)$, for some $Y \subseteq T$. Then

$$C^T(C_T(X)) = C^T(C(X)) = C(T) = C^T(X).$$

- If none of the above hold, that is, if $X \nsubseteq T$, $C(X) \nsubseteq T$ and $C(X) \neq C(Y)$, for all $Y \subseteq T$, then

$$C^T(C_T(X)) = C^T(C(X)) = C(X) = C^T(X).$$

We conclude that $C_T \leq C^T$. ∎

Chapter 3

Algebraic Theory

3.1 Introduction

In [3], Blok and Pigozzi introduced the notion of an *algebraizable logic*. Logics in their study, and in all related abstract studies in algebraic logic, can be represented by structural closure operators, i.e., functions on the powerset of an absolutely free algebra on countably many generators that satisfy the axioms of inflationarity, monotonicity, idempotency and structurality (see, e.g., Page 25 of [12]). *Logical matrices* were traditionally used in this context as models of logics. Classes of algebras that were algebraic reducts of those matrices were used as algebraic counterparts. Later, based on the study of concrete examples, it was discovered that in many less well behaved logics, the algebras obtained in this way via matrices were not the "right" ones. This led Font and Jansana in [12] to suggest a more general methodology. Instead of logical matrices, they considered *abstract logics* as models of logical systems. Following a similar process, but now taking algebraic reducts of abstract logics instead of logical matrices, they obtained a class of algebras that was seeing, via already acquired experience with a wide variety of particular logics, as being the "right" one in all known logical systems.

One of the limitations of this "traditional" theory is that it requires the logics under consideration to satisfy both inflationarity and monotonicity. Thus, one cannot accommodate potential non-monotonic operators. It is conceivable, however, that many aspects and features of the theory could be carried over to such a context. This work is a first attempt at such a treatment. In Chapter 2, the notion of a *logicate* was introduced. It is an idempotent operator C on the powerset of a given set A. Several related representations were given and logicates over the same underlying set were compared in a couple of different ways.

If one wishes to study logicates from an algebraic point of view, the preceding framework is clearly insufficient, since it involves no algebraic structure. We remedy this by logicates over algebras. So the basic object of study here is an *algebraic logicate*, consisting of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and an idempotent operator C on the powerset of the universe A of $\mathbf{A}$. It turns out that this framework is sufficient for accommodating quite a large part of the monotonic theory, without requiring that the underlying operator be either inflationary or monotone.

Our study relies on the set $\mathcal{C}$ of *fixed points* or *theories*. No intrinsic ordering or structure on those may be assumed, in contrast with the traditional framework. Still, as in the traditional framework, we may define, based on theories, *logical congruences*. It may be shown that these form a principal ideal of the lattice of all congruences on the underlying algebra. Hence, it makes sense to define the *Tarski congruence* as the largest logical congruence of the logicate, exactly as done for abstract logics in the monotonic framework [12]. In addition, a slightly modified, but very similar, version of its well-known characterization holds (see Pages 18-19 of [12]).

An important notion in the study of abstract logics is that of a *bilogical morphism* (Page 20 of [12]). Because the framework of [12] is based on closure operators, bilogical morphisms tie very closely the consequence structures of the abstract logics they relate. As a byproduct, they also connect very strongly the theories. On the other hand, in the present context, due to lack of monotonicity, one cannot expect morphism with such tight properties. As a result, instead of focusing on consequence preservation, we take the effect on theories as primary. Thus, we adopt a notion of *bimorphism* by stipulating that it preserve theories, even though it may not have any preservation properties as regards the corresponding consequence structures. This fact becomes manifest in the formulation of an analog of a well known characterization. Proposition 1.4 of [12] consists of six equivalent statements, three referring to theories and three to the preservation of the closure structures. The analog only retains the three parts referring to theories, whereas those on consequences are not valid in general. Despite this drawback, using bilogical morphisms one can prove some analogs of the Homomorphism Theorems of Universal Algebra [5, 18, 1]. These parallel the generalized versions proven in the monotonic framework (see, e.g. [4, 12]).

Additionally, if one defines *logical matrices* for logicates by relying solely on theories and not on consequence relations, many of the results of the monotonic theory can be adapted and still shown to hold. In this setting, however, because of the absence of structurality, one has to build matrices over specific interpretations, i.e., surjective homomorphisms onto similar algebras. Some clues as to how one may proceed may be taken from the context of models of π-institutions [21]. On the other hand, if structurality is added, as is done briefly in the Addendum to this chapter, then matrices resembling the ordinary ones more closely may again be used as models and some of the flavor of the traditional treatment may be recovered.

In Section 3.2, we adapt the study of *logical congruences* on abstract logics of Font and Jansana [12] to the nonmonotonic setting. Logical congruences on a logicate form a principal ideal of the lattice of all logical congruences on the underlying algebra. The generator of this ideal is called the *Tarski congruence* of the logicate. It is important to notice that all equipotent logicates share the same Tarski congruence. Further, the Blok-Pigozzi style characterization of the Tarski congruence in the traditional setting carries over virtually unchanged. If one thinks of the Tarski operator as acting on equipotency classes, so that the set in question is partially ordered, then it is a monotone operator.

In Section 3.3 we revisit *bilogical morphisms*, but apply them to arbitrary logicates. These are surjective homomorphisms that preserve theories. A characterization is provided, as well as the important result that the Tarski operator of a logicate is preserved under the action of inverse bilogical morphisms.

In Section 3.4 we study aspects of Universal Algebra, that were applied by

Brown and Suszko [4] and by Font and Jansana [12] to abstract logics, in the context of nonmonotonicity. We define *quotients* of logicates by logical congruences and show that natural projections form bilogical morphisms. Then we embark on revisiting analogs of the Homomorphism Theorems (see, e.g., [5] and Pages 22-23 of [12]) for logicates. In particular, the Correspondence Theorem, playing a role similar to the one played in the monotonic theory, leads to the definition of a *reduced logicate* and the process of *reduction*. Several properties of reductions carry over to this more general setting and are presented in detail in this section.

In Section 3.5 we introduce *interpretations* of algebraic logicates. An interpretation is essentially a surjective mapping from the underlying algebra of the logicate onto an algebra of the same type. Interpretations form the cornerstone in defining *logical filters* and *logical matrices*, as well as *reduced matrices*, which play a key role in both the traditional theory (see, e.g., [3, 12, 8]) and the more general theory presented here. One important feature of interpretations is that, if the kernel is a logical congruence of the original logicate, then one may define a logicate in the target of the interpretation in such a way that its theories coincide with the filters and the interpretations becomes a bilogical morphism between the two logicates. Several properties governing the relation between interpretations and (sets of) filters are also presented in this section. In closing the section, we define that class of *matrices* and of *reduced matrices* of a logicate and the corresponding classes of algebraic reducts and prove analogs of the well-known *completeness results* for sentential logics in the context of logicates.

In the Addendum, we briefly overview a possible alternative formulation of filters and matrices, applicable in case the consequence operator of the logicate happens to be structural. In that case filters and matrices may be defined as in the traditional monotonic theory, without recourse to fixed interpretation morphisms.

3.2 Logical Congruences

Let $\mathcal{L}$ be a logical (or algebraic) language. That is, $\mathcal{L}$ is a set of connectives (or operation symbols) of finite arities. We consider algebras of type $\mathcal{L}$, or $\mathcal{L}$-algebras, $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$. Recall from Chapter 2, that a *logicate* or a *consequence operator* C on A is an idempotent mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Following tradition, we may write $\mathbb{L} = \langle \mathbf{A}, C \rangle$ for the structure consisting of the algebra $\mathbf{A}$ and the consequence operator C on A. We call such a structure an **algebraic logicate**. Moreover, we use $\mathcal{C}$ for the set of its theories,

$$\mathcal{C} = \{X \subseteq A : C(X) = X\}.$$

Recalling from Chapter 2 the *equipotency* equivalence relation $\triangleq$, which relates two logicates if their sets of theories coincide, we may also write

$$\mathbb{L}/\triangleq := \langle \mathbf{A}, \mathcal{C} \rangle$$

to denote the equipotency class of the algebraic logicate $\mathbb{L}$. Note that this notation is justified by the fact that the exact consequence structure of a representative does not matter, since the equipotency class is determined solely by its theories.

Let $\theta \in \mathrm{Con}(\mathbf{A})$ be a congruence on the $\mathcal{L}$-algebra $\mathbf{A}$. We say that θ is **compatible with** a set $X \subseteq A$ if, for all $a, b \in A$,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in X \quad \text{imply} \quad b \in X.$$

Compatibility of θ with X is tantamount to X being a union of θ-congruence classes.

Suppose, next, that we have given an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C \rangle$. We say that θ is a **logical congruence of** C, or **of** $\mathbb{L}$, if θ is compatible with every theory of C, i.e., with every $X \in \mathcal{C}$. We write $\mathrm{Con}(\mathbb{L})$ for the collection of all logical congruences of $\mathbb{L}$. Moreover, $\mathbf{Con}(\mathbb{L}) = \langle \mathrm{Con}(\mathbb{L}), \subseteq \rangle$ denotes the collection of logical congruences, ordered by the subset relation between congruences. It can be shown that this partially ordered set is a complete lattice and, in fact, a principal ideal of the complete lattice of all congruences on the algebra $\mathbf{A}$ (see Page 18 of [12]).

Proposition 15 *Let* $\mathbb{L} = \langle \mathbf{A}, C \rangle$ *be an algebraic logicate. Then* $\mathbf{Con}(\mathbb{L})$ *is a complete lattice and a principal ideal of* $\mathbf{Con}(\mathbf{A})$.

Proof: We first show that $\mathrm{Con}(\mathbb{L})$ is a downset in $\mathbf{Con}(\mathbf{A})$. Let $\theta, \theta' \in \mathrm{Con}(\mathbf{A})$, such that $\theta \subseteq \theta' \in \mathrm{Con}(\mathbb{L})$. For all $X \in \mathcal{C}$ and all $a, b \in A$, we have

$$\begin{array}{rll} \langle a, b \rangle \in \theta \text{ and } a \in X & \text{imply} & \langle a, b \rangle \in \theta' \text{ and } a \in X \\ & \text{imply} & b \in X. \end{array}$$

Thus, $\theta \in \mathrm{Con}(\mathbb{L})$ and $\mathrm{Con}(\mathbb{L})$ is a downset in $\mathbf{Con}(\mathbf{A})$.

Next, we show that $\mathrm{Con}(\mathbb{L})$ is closed under joins. Because of the way joins are determined in $\mathbf{Con}(A)$, to do this it suffices to show that, for all $X \in \mathcal{C}$, if $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, then $\theta \circ \theta'$ is compatible with X. So, let $a, b \in A$, such that $\langle a, b \rangle \in \theta \circ \theta'$ and $a \in X$. Thus, there exists $c \in A$, such that $\langle a, c \rangle \in \theta$, $\langle c, b \rangle \in \theta'$ and $a \in X$. By compatibility of θ and θ' with X, we get, first, $b \in X$ and, then, $c \in X$. Hence, $\mathrm{Con}(\mathbb{L})$ is closed under joins. Thus, it forms an ideal of $\mathbf{Con}(\mathbf{A})$.

Finally, to see that it is principal, we note that the union of every chain in $\mathbf{Con}(\mathbb{L})$ is an upper bound of the chain and lies in $\mathrm{Con}(\mathbb{L})$. Hence, by Zorn's Lemma, $\mathrm{Con}(\mathbb{L})$ has a maximal element. However, as $\mathrm{Con}(\mathbb{L})$ is closed under joins, a maximal element must be a maximum element. Therefore $\mathbf{Con}(\mathbb{L})$ is a principal ideal of $\mathbf{Con}(\mathbf{A})$. ■

Let $\mathbb{L} = \langle \mathbf{A}, C \rangle$ be an algebraic logicate. The **Tarski congruence of** $\mathbb{L}$ (Definition 1.1 of [12]) is

$$\widetilde{\Omega}(\mathbb{L}) = \max \mathbf{Con}(\mathbb{L}),$$

that is, $\widetilde{\Omega}(\mathbb{L})$ is the greatest logical congruence of $\mathbb{L}$. The **Tarski operator on A** is the mapping

$$\widetilde{\Omega}_{\mathbf{A}} : \mathbb{L} \longmapsto \widetilde{\Omega}(\mathbb{L}),$$

i.e., it is the mapping $\mathbb{L} \mapsto \widetilde{\Omega}(\mathbb{L})$ restricted to algebraic logicates over the same underlying algebra $\mathbf{A}$. This notation can be extended by writing $\widetilde{\Omega}_{\mathbf{A}}(C)$ for the Tarski congruence of the algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C \rangle$. It follows from the definition of $\widetilde{\Omega}_{\mathbf{A}}(C)$ that

$$\mathrm{Con}(\mathbb{L}) = \{\theta \in \mathrm{Con}(\mathbf{A}) : \theta \subseteq \widetilde{\Omega}(\mathbb{L})\}.$$

To characterize the Tarski congruence, one uses the same way employed in the traditional theory. Namely, we consider **logical matrices** over **A**, i.e., pairs $\mathfrak{A} = \langle \mathbf{A}, X \rangle$, where $X \subseteq A$ (see, e.g., Section 1.4 of [3] or Page 16 of [12]). We say that a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is a **congruence of** $\mathfrak{A}$, written $\theta \in \mathrm{Con}(\mathfrak{A}) = \mathrm{Con}(\langle \mathbf{A}, X \rangle)$, if θ is compatible with X.

Corollary 16 *Let $\mathfrak{A} = \langle \mathbf{A}, X \rangle$ be a logical matrix. Then $\mathbf{Con}(\mathfrak{A})$ is a complete lattice and a principal ideal of $\mathbf{Con}(\mathbf{A})$.*

Proof: This is a direct consequence of Proposition 15, since, given $\mathfrak{A} = \langle \mathbf{A}, X \rangle$, one can construct an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C \rangle$ whose only theory is X, i.e., one can set, for all $Y \subseteq A$,

$$C(Y) = X.$$

Then, clearly, $\mathbf{Con}(\mathfrak{A}) = \mathbf{Con}(\mathbb{L})$. ■

Corollary 16 permits us to define the **Leibniz congruence of** $\mathfrak{A}$ or the **Leibniz congruence of** X **on A**, written $\Omega(\mathfrak{A}) = \Omega_{\mathbf{A}}(X)$, as the largest congruence on $\mathbf{A}$ that is compatible with X. Then, given an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C \rangle$, it is clear, by the definition of $\widetilde{\Omega}(\mathbb{L})$, that

$$\widetilde{\Omega}(\mathbb{L}) = \bigcap \{\Omega_{\mathbf{A}}(X) : X \in \mathcal{C}\}.$$

In particular, note that, since the value of $\widetilde{\Omega}(\mathbb{L})$ only depends on the set $\mathcal{C}$ of theories and not on the details of the consequence structure, it makes sense to also consider $\widetilde{\Omega}$ as an operator on equipotency classes,

$$\widetilde{\Omega}_{\mathbf{A}} : \mathcal{C} \longmapsto \mathrm{Con}(\mathbf{A}).$$

The following characterization of the Leibniz congruence of a logical matrix was given by Blok and Pigozzi in their ground breaking monograph (see Theorem 1.5 of [3]).

Theorem 17 (Blok-Pigozzi) *Let $\mathfrak{A} = \langle \mathbf{A}, X\rangle$ be a logical matrix and $a, b \in A$. Then $\langle a, b\rangle \in \Omega_{\mathbf{A}}(X)$ if and only if, for every formula $\varphi(x, \overline{z})$ in $\mathrm{Fm}_{\mathcal{L}}(V)$ and every tuple $\overline{c}$ of elements in A,*

$$\varphi^{\mathbf{A}}(a, \overline{c}) \in X \quad \textit{iff} \quad \varphi^{\mathbf{A}}(b, \overline{c}) \in X.$$

From this theorem and the discussion preceding it, formalizing the relation between the Tarski congruence of an algebraic logicate and the Leibniz congruences of the associated class of logical matrices, one obtains easily a characterization of the Tarski congruence (see Page 29 of [12]).

Corollary 18 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate and $a, b \in A$. Then $\langle a, b\rangle \in \Omega_{\mathbf{A}}(C)$ if and only if, for every $X \in \mathcal{C}$, every formula $\varphi(x, \overline{z})$ in $\mathrm{Fm}_{\mathcal{L}}(V)$ and every tuple $\overline{c}$ of elements in A,*

$$\varphi^{\mathbf{A}}(a, \overline{c}) \in X \quad \textit{iff} \quad \varphi^{\mathbf{A}}(b, \overline{c}) \in X.$$

Proof: One has $\langle a, b\rangle \in \Omega_{\mathbf{A}}(C)$ iff, for all $X \in \mathcal{C}$, $\langle a, b\rangle \in \Omega_{\mathbf{A}}(X)$ iff, by Theorem 17, the asserted condition in the statement holds. ■

A consequence of Corollary 18 is that the Tarski operator on an algebra **A** is monotone. But one should exercise caution in formalizing this result to avoid pitfalls. Namely, when referring to logicates, we understand their equipotency classes. Logicates are preordered by $\trianglelefteq$ (see Chapter 2), given by

$$C \trianglelefteq C' \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

This induces a partial order (denoted by the same symbol) on the equipotency classes, which we may write

$$C/{\triangleq} \trianglelefteq C'/{\triangleq} \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

Accordingly, the Tarski operator is seen as applying on equipotency classes of logicates.

Corollary 19 *Let $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ be an algebra. Then, for all logicates C and C' on $\mathbf{A}$,*

$$C \trianglelefteq C' \quad \textit{implies} \quad \widetilde{\Omega}_{\mathbf{A}}(C) \subseteq \widetilde{\Omega}_{\mathbf{A}}(C').$$

Proof: We have that

$$\begin{array}{rll} C \trianglelefteq C' & \text{iff} & \mathcal{C}' \subseteq \mathcal{C} \quad \text{(Definition of } \trianglelefteq) \\ & \text{implies} & \widetilde{\Omega}_{\mathbf{A}}(C) \subseteq \widetilde{\Omega}_{\mathbf{A}}(C'). \quad \text{(Corollary 18)} \end{array}$$

Thus, the Tarski operator on **A** is monotone. ■

3.3 Bilogical Morphisms

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates. A **logical morphism** from $\mathbb{L}$ to $\mathbb{L}'$, written $h : \mathbb{L} \to \mathbb{L}'$, is a homomorphism $h : \mathbf{A} \to \mathbf{A}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$, that is, for all $X' \subseteq A'$,

$$C'(X') = X' \quad \text{implies} \quad C(h^{-1}(X')) = h^{-1}(X').$$

We say that $\mathbb{L}$ is **projectively generated from $\mathbb{L}'$ by** h if

$$\mathcal{C} = h^{-1}(\mathcal{C}').$$

Finally, h is a **bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$**, or a **bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$**, written $h : \mathbb{L} \to_b \mathbb{L}'$, if it is an epimorphism $h : \mathbf{A} \twoheadrightarrow \mathbf{A}'$ and it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. For these definitions, as applied to the traditional setting, see Page 20 of [12] (also [4] for the original notions). The following proposition is the analog in the nonmonotonic setting of Proposition 1.4 of [12]. Note that instead of six equivalent conditions, it provides only three, and this is due to the fact that the consequence operators are not required to be inflationary or monotone.

Proposition 20 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be algebraic logicates and $h : \mathbf{A} \to \mathbf{A}'$ an epimorphism. The following conditions are equivalent.*

(i) *h is a bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$;*

(ii) $\mathcal{C}' = h(\mathcal{C})$ *and* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$;

(iii) $\mathcal{C} = h^{-1}(\mathcal{C}')$.

Proof: First observe that, since, by hypothesis, h is an epimorphism, Conditions (i) and (iii) are equivalent by the definition of bilogical morphism. Thus, it suffices to show (i)⇒(ii) and (ii)⇒(iii).

(i)⇒(ii) Let $a, b \in A$ and $X \in \mathcal{C}$, such that $\langle a, b\rangle \in \mathrm{Ker}(h)$ and $a \in X$. By hypothesis, $\langle a, b\rangle \in \mathrm{Ker}(h)$ and $a \in h^{-1}(Y)$, for some $Y \in \mathcal{C}'$. Hence, $h(a) = h(b)$ and $h(a) \in Y$. This yields $h(b) \in Y$ and, hence, $b \in h^{-1}(Y) = X$. So $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$.

Let $Y \in \mathcal{C}'$. Then

$$\begin{aligned} Y &= h(h^{-1}(Y)) \quad (h \text{ surjective}) \\ &= h(X), \quad (\text{for } X = h^{-1}(Y)) \end{aligned}$$

where, by hypothesis, $X = h^{-1}(Y)$ is a theory of C. Conversely, if X is a theory of C,

$$\begin{aligned} h(X) &= h(h^{-1}(Y)) \quad (\text{for some } Y \in \mathcal{C}') \\ &= Y. \quad (h \text{ surjective}) \end{aligned}$$

Thus, Condition (ii) holds.

(ii)⇒(iii) Note that, since $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$, we have, for every $X \in \mathcal{C}$,

$$X = h^{-1}(h(X)).$$

Suppose $Y \in \mathcal{C}'$. By hypothesis, there exists $X \in \mathcal{C}$, such that $Y = h(X)$. Hence, by the remark above,

$$h^{-1}(Y) = h^{-1}(h(X)) = X.$$

Suppose, conversely, $X \in \mathcal{C}$. Then $X = h^{-1}(h(X)) = h^{-1}(Y)$, where $Y = h(X)$ is a C'-theory by the hypothesis. Thus, Condition (iii) holds.

■

Proposition 20 yields immediately the following results revealing a very tight relation between theories of two algebraic logicates related via a bilogical morphism. We start with an analog of Proposition 1.5 of [12]. Note, however, that, as C and C' may not be closure operators, the collections $\mathcal{C}$ and $\mathcal{C}'$ may not be complete lattices. So we may only establish a poset isomorphism between them.

Proposition 21 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates and $h : \mathbf{A} \to \mathbf{A}'$ be an epimorphism. Then $h : \mathbb{L} \to_b \mathbb{L}'$ if and only if the posets $\langle \mathcal{C}, \subseteq\rangle$ and $\langle \mathcal{C}', \subseteq\rangle$ are isomorphic via the mapping induced by h.*

Proof: The conclusion follows directly from Proposition 20. ■

We conclude the section by showing that Tarski congruence systems are preserved under the action of inverse bilogical morphisms.

Proposition 22 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be algebraic logicates and $h : \mathbb{L} \to_b \mathbb{L}'$. Then*

$$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}')).$$

Proof: We have, for all $a, b \in A$,

$$\begin{array}{ll}
\langle a, b\rangle \in h^{-1}(\widetilde{\Omega}(\mathbb{L}')) & \\
\quad \text{iff} & \langle h(a), h(b)\rangle \in \widetilde{\Omega}(\mathbb{L}') \\
\quad \text{iff,} & \text{for all } X' \in \mathcal{C}', \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c}' \text{ in } A', \\
& \varphi^{\mathbf{A}'}(h(a), \overline{c}') \in X' \quad \text{iff} \quad \varphi^{\mathbf{A}'}(h(b), \overline{c}') \in X' \\
\quad \text{iff,} & \text{for all } X \in \mathcal{C}, \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \text{ in } A, \\
& \varphi^{\mathbf{A}'}(h(a), h(\overline{c})) \in h(X) \quad \text{iff} \quad \varphi^{\mathbf{A}'}(h(b), h(\overline{c})) \in h(X) \\
\quad \text{iff,} & \text{for all } X \in \mathcal{C}, \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \text{ in } A, \\
& h(\varphi^{\mathbf{A}}(a, \overline{c})) \in h(X) \quad \text{iff} \quad h(\varphi^{\mathbf{A}'}(b, \overline{c})) \in h(X) \\
\quad \text{iff,} & \text{for all } X \in \mathcal{C}, \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \text{ in } A, \\
& \varphi^{\mathbf{A}}(a, \overline{c}) \in X \quad \text{iff} \quad \varphi^{\mathbf{A}'}(b, \overline{c}) \in X \\
\quad \text{iff} & \langle a, b\rangle \in \widetilde{\Omega}(\mathbb{L}).
\end{array}$$

We conclude that $\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}'))$. ■

3.4 Quotients

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates. $\mathbb{L}$ and $\mathbb{L}'$ are **isomorphic**, written $\mathbb{L} \cong \mathbb{L}'$, if there exists a bijective logical morphism $h : \mathbb{L} \to \mathbb{L}'$, whose inverse is also a logical morphism. Equivalently, h is a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$, which happens to be an algebra isomorphism.

We need to make a comment here to prevent a possible misunderstanding. The term "isomorphic" is used because the definition is very similar (almost a repetition) of the one used for isomorphisms in the monotonic setting (see Page 21 of [12]). However, in the present context, because of lack of monotonicity, many properties are lacking and the term may be misconstrued as implying that an isomorphism is structure preserving, whereas it only preserves theories in both directions. So, perhaps, instead of the term "isomorphism", a better term would be "isopotency". In any case, we keep the term "isomorphism" for now, pretending that theories are the only objects that matter, since they are, after all, of utmost importance.

Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate and suppose $\theta \in \mathrm{Con}(\mathbf{A})$. Consider the quotient algebra $\mathbf{A}/\theta = \langle A/\theta, \mathcal{L}^{\mathbf{A}/\theta}\rangle$ and define a mapping

$$C^\theta := C/\theta : \mathcal{P}(A/\theta) \to \mathcal{P}(A/\theta)$$

by setting, for all $S \subseteq A/\theta$,

$$C^\theta(S) = \pi_\theta(C(\pi_\theta^{-1}(S))),$$

where $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ is the canonical projection homomorphism onto the quotient.

We show that, if θ happens to be a logical congruence, then the quotient is a legitimate logicate on the quotient algebra. For the corresponding result in the traditional framework, see Page 21 of [12].

Proposition 23 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and suppose* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then*

$$\mathbb{L}^\theta = \mathbb{L}/\theta := \langle \mathbf{A}/\theta, C^\theta\rangle$$

is also an algebraic logicate.

Proof: Let $S \subseteq A/\theta$. Then

$$\begin{array}{rcll} C^\theta(C^\theta(S)) & = & \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))))) & \text{(Definition of } C^\theta) \\ & = & \pi_\theta(C(C(\pi_\theta^{-1}(S)))) & (\theta \in \mathrm{Con}(\mathbb{L})) \\ & = & \pi_\theta(C(\pi_\theta^{-1}(S))) & \text{(Idempotency)} \\ & = & C^\theta(S). & \text{(Definition of } C^\theta) \end{array}$$

Thus, C^θ is idempotent and $\mathbb{L}^\theta$ is an algebraic logicate, as claimed. ∎

$\mathbb{L}^\theta = \mathbb{L}/\theta$ is called the **quotient logicate of $\mathbb{L}$ by θ.**

Now that we know that, in case θ is a logical congruence, $\mathbb{L}$ and its quotient $\mathbb{L}^\theta$ are logicates, we show that the canonical projection $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ becomes a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}/\theta$.

Proposition 24 *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ be an algebraic logicate and suppose $\theta \in \mathrm{Con}(\mathbb{L})$. Then $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ is a bilogical morphism.*

Proof: Given that π_θ is an epimorphism, it suffices to show that it projectively generates $\mathbb{L}$ from $\mathbb{L}/\theta$. Suppose, first $X \in \mathcal{C}$. Then

$$\begin{array}{rcl} \pi_\theta^{-1}(C^\theta(\pi_\theta(X))) & = & \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X))))) \quad (\text{Definition of } C^\theta) \\ & = & C(X) \quad (\theta \in \mathrm{Con}(\mathbb{L})) \\ & = & X. \quad (X \in \mathcal{C}) \end{array}$$

This shows that $\mathcal{C} \subseteq \pi_\theta^{-1}(\mathcal{C}^\theta)$. Conversely, let $S \in \mathcal{C}^\theta$. Then

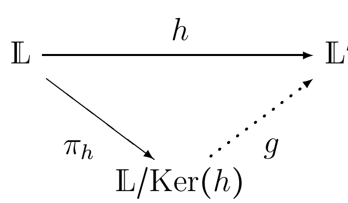
$$\begin{array}{rcl} \pi_\theta^{-1}(S) & = & \pi_\theta^{-1}(C^\theta(S)) \quad (S \in \mathcal{C}^\theta) \\ & = & \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))) \quad (\text{Definition of } C^\theta) \\ & = & C(\pi_\theta^{-1}(S)). \quad (\theta \in \mathrm{Con}(\mathbb{L})) \end{array}$$

Hence $\pi_\theta^{-1}(\mathcal{C}^\theta) \subseteq \mathcal{C}$. Thus, $\mathcal{C} = \pi_\theta^{-1}(\mathcal{C}^\theta)$, showing that π_θ projectively generates $\mathbb{L}$ from $\mathbb{L}/\theta$. So $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ is a bilogical morphism. ∎

We call $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ the **quotient morphism**, or the **natural projection morphism**, from $\mathbb{L}$ to its quotient $\mathbb{L}^\theta$ by θ.

Theorems 1.8, 1.9 and 1.10 of [12] adapt to the framework of abstract logics the well-known Homomorphism Theorems of universal algebra [5, 18, 1]. We undertake here their adaptation to the general nonmonotonic setting. The fundamentals, of course, remain the same.

Theorem 25 (Homomorphism Theorem) *Let $\mathbb{L} = \langle \mathbf{A}, C\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ be two algebraic logicates and $h : \mathbb{L} \to_b \mathbb{L}'$. Then $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$ via a unique isomorphism g, such that $h = g \circ \pi_h$,*



where $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ is the bilogical projection morphism.

Proof: By hypothesis, h is a bilogical morphism. Thus, by Proposition 20, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. It follows, by Proposition 24, that $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ is a bilogical morphism. By Universal Algebra, there exists a unique $g : \mathbf{A}/\theta \cong \mathbf{A}'$, such that $h = g \circ \pi_h$. So it suffices to show that this is a bilogical morphism. We have

$$\begin{array}{rcll} \mathcal{C}^{\mathrm{Ker}(h)} & = & \pi_h(\mathcal{C}) & (\pi_h : \mathbb{L} \to_b \mathbb{L}/\mathrm{Ker}(h)) \\ & = & g^{-1}(h(\mathcal{C})) & (h = g \circ \pi_h \text{ and } g : \mathbf{A}/\theta \cong \mathbf{A}') \\ & = & g^{-1}(\mathcal{C}'). & (h : \mathbb{L} \to_b \mathbb{L}') \end{array}$$

Thus, by definition, $g : \mathbb{L}/\mathrm{Ker}(h) \to \mathbb{L}'$ is a bilogical morphism. ∎

Theorem 26 (Second Isomorphism Theorem) *Let* $\mathbb{L} = \langle \mathbf{A}, C \rangle$ *be an algebraic logicate and* $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$*, such that* $\theta \subseteq \theta'$*. Then* $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$ *and*

$$(\mathbb{L}/\theta)/(\theta'/\theta) \cong \mathbb{L}/\theta',$$

where the isomorphism is given by

$$(a/\theta)/(\theta'/\theta) \mapsto a/\theta'.$$

Proof: By Universal Algebra, we know that

$$\begin{array}{rrcl} h: & (\mathbf{A}/\theta)/(\theta'/\theta) & \longrightarrow & \mathbf{A}/\theta'; \\ & (a/\theta)/(\theta'/\theta) & \longmapsto & a/\theta', \end{array}$$

is an isomorphism that makes the following rectangle commute,

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\pi_{\theta'}} & \mathbf{A}/\theta' \\ {\scriptstyle \pi_\theta}\downarrow & & \uparrow{\scriptstyle h} \\ \mathbf{A}/\theta & \xrightarrow[\pi_{\theta'/\theta}]{} & (\mathbf{A}/\theta)/(\theta'/\theta) \end{array}$$

where $\pi_\theta, \pi_{\theta'}$ and $\pi_{\theta'/\theta}$ are the natural projections. Since $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, by Proposition 24, π_θ and $\pi_{\theta'}$ are bilogical morphisms. We can also show that $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$. Let $a, b \in A$ and $S \subseteq A/\theta$, such that $\langle a/\theta, b/\theta \rangle \in \theta'/\theta$ and $a/\theta \in C^\theta(S)$. Then, by the definition of C^θ, $\langle a, b \rangle \in \theta'$ and $a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$. Hence, since $\theta \in \mathrm{Con}(\mathbb{L})$, $\langle a, b \rangle \in \theta'$ and $a \in C(\pi_\theta^{-1}(S))$. Since $\theta' \in \mathrm{Con}(\mathbb{L})$, this yields $b \in C(\pi_\theta^{-1}(S))$, whence $b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$, i.e., $b/\theta \in C^\theta(S)$. Hence $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. Now, again by Proposition 24, the projection $\pi_{\theta'/\theta}$ is also a bilogical morphism. Now we check that h, which is already known to be an algebraic isomorphism, is also a bilogical morphism. We have

$$\begin{array}{rcll} (\mathcal{C}^\theta)^{\theta'/\theta} & = & \pi_{\theta'/\theta}(\mathcal{C}^\theta) & (\pi_{\theta'/\theta} : \mathbb{L}^\theta \to_b (\mathbb{L}^\theta)^{\theta'/\theta}) \\ & = & \pi_{\theta'/\theta}(\pi_\theta(\mathcal{C})) & (\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta) \\ & = & h^{-1}(\pi_{\theta'}(\mathcal{C})) & (h \circ \pi_{\theta'/\theta} \circ \pi_\theta = \pi_{\theta'} \text{ and } h \text{ an iso}) \\ & = & h^{-1}(\mathcal{C}^{\theta'}). & (\pi_{\theta'} : \mathbb{L} \to_b \mathbb{L}^{\theta'}) \end{array}$$

Thus, $h : (\mathbb{L}/\theta)/(\theta'/\theta) \to \mathbb{L}/\theta'$ is indeed a bilogical morphism and, hence, an isomorphism. ∎

Theorem 27 (Correspondence Theorem) *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then the segment* $[\theta, \widetilde{\Omega}(\mathbb{L})]$ *of the lattice* $\mathbf{Con}(\mathbb{L})$ *is isomorphic to the lattice* $\mathbf{Con}(\mathbb{L}^\theta)$ *by the mapping* $\theta' \mapsto \theta'/\theta$.

Proof: By Theorem 26, if $\theta \subseteq \theta' \subseteq \widetilde{\Omega}(\mathbb{L})$, then $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. By Universal Algebra, it suffices to prove that, for all $\theta \subseteq \theta' \in \mathrm{Con}(\mathbf{A})$, if $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$, then $\theta' \in \mathrm{Con}(\mathbb{L})$. So let $a, b \in A$ and $X \in \mathcal{C}$, such that $\langle a, b\rangle \in \theta'$ and $a \in X$. As $\theta \in \mathrm{Con}(\mathbb{L})$, θ is compatible with X. Thus X is the union of θ-classes, that is $X = \pi_\theta^{-1}(\pi_\theta(X))$. Now, starting with the assumption, we get

$$\begin{array}{ll}\langle a,b\rangle \in \theta' \text{ and } a \in X & \\ \quad \text{iff} & \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X)))) \\ \quad \text{iff} & \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in C^\theta(\pi_\theta(X)) \\ \quad \text{implies} & b/\theta \in C^\theta(\pi_\theta(X)) \\ \quad \text{iff} & b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X)))) \\ \quad \text{iff} & b \in X.\end{array}$$

Therefore, $\theta' \in \mathrm{Con}(\mathbb{L})$ and the correspondence is established. ■

The Correspondence Theorem has a significant consequence in relation to the Tarski congruences.

Corollary 28 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then*

$$\widetilde{\Omega}(\mathbb{L}^\theta) = \widetilde{\Omega}(\mathbb{L})/\theta.$$

Proof: By definition, the largest element in $\mathrm{Con}(\mathbb{L}^\theta)$ is $\widetilde{\Omega}(\mathbb{L}^\theta)$, whereas the largest element in $[\theta, \widetilde{\Omega}(\mathbb{L})]$ is clearly $\widetilde{\Omega}(\mathbb{L})$. Since, under the established correspondence of Theorem 27, these two elements correspond, we get the conclusion. ■

It follows that, for any algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$,

$$\widetilde{\Omega}(\mathbb{L}/\widetilde{\Omega}(\mathbb{L})) = \widetilde{\Omega}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}/\widetilde{\Omega}(\mathbb{L})}.$$

This leads us to the core definition of *reduction* (see Definition 1.12 of [12]). We say that an algebraic logicate $\mathbb{L} = \langle \mathbf{A}, C\rangle$ is **reduced** when it has only one logical congruence, i.e., when $\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}}$. Given an arbitrary algebraic logicate $\mathbb{L}$, we define the **reduction** $\mathbb{L}^*$ **of** $\mathbb{L}$ by

$$\mathbb{L}^* = \mathbb{L}/\widetilde{\Omega}(\mathbb{L}).$$

If L is a class of algebraic logicates, then we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

If $\mathbb{L}$ is an algebraic logicate, then $\mathbb{L}^*$ is always reduced. Moreover, if $\mathbb{L}$ happens to already be reduced, then $\mathbb{L}$ and $\mathbb{L}^*$ are isomorphic and they may be identified.

We prove, next, some analogs of Propositions 1.13 and 1.14 of [12]. The first asserts that the reduction of a quotient of a logicate by a logical congruence is isomorphic to the reduction of the logicate itself. The second proves that the reductions of two logicates related via a bilogical morphism are isomorphic.

Proposition 29 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *be an algebraic logicate and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then*

$$(\mathbb{L}^\theta)^* \cong \mathbb{L}^*.$$

Proof: We have

$$\begin{array}{rcl}(\mathbb{L}^\theta)^* & = & \mathbb{L}^\theta/\widetilde{\Omega}(\mathbb{L}^\theta) \quad \text{(Definition of Reduction)}\\ & = & \mathbb{L}^\theta/(\widetilde{\Omega}(\mathbb{L})/\theta) \quad \text{(Corollary 28)}\\ & \cong & \mathbb{L}/\widetilde{\Omega}(\mathbb{L}) \quad \text{(Theorem 26)}\\ & = & \mathbb{L}^*. \quad \text{(Definition of Reduction)}\end{array}$$

The conclusion follows. ∎

Proposition 30 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ *be algebraic logicates and* $h : \mathbb{L} \to_b \mathbb{L}'$. *Then*

$$\mathbb{L}^* \cong \mathbb{L}'^*.$$

Proof: By Theorem 25, $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$. By Proposition 22,

$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}'^*.$$

Since, by Proposition 20, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$, by Proposition 29,

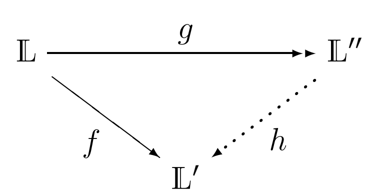
$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}^*.$$

Therefore, $\mathbb{L}'^* \cong \mathbb{L}^*$. ∎

We close the section with an analog of Proposition 1.15 of [12], a sort of "fill-in" theorem for arrows.

Proposition 31 *Let* $\mathbb{L} = \langle \mathbf{A}, C\rangle$, $\mathbb{L}' = \langle \mathbf{A}', C'\rangle$ *and* $\mathbb{L}'' = \langle \mathbf{A}'', C''\rangle$ *be algebraic logicates,* $f : \mathbb{L} \to \mathbb{L}'$ *a logical morphism and* $g : \mathbb{L} \to_b \mathbb{L}''$ *a bilogical morphism, such that* $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. *Then, there is a unique logical morphism* $h : \mathbb{L}'' \to \mathbb{L}'$, *such that*

$$h \circ g = f.$$



Moreover, f *projectively generates* $\mathbb{L}$ *from* $\mathbb{L}'$ *if and only if* h *projectively generates* $\mathbb{L}''$ *from* $\mathbb{L}'$.

Proof: Let $a'' \in A''$. By surjectivity of g, there exists $a \in A$, such that $g(a) = a''$. We define

$$h(a'') = f(a).$$

Immediately observe that, since $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$, this assignment is well defined, that is, independent of the choice of $a \in A$. Now we have

$$\begin{array}{rcl} g^{-1}(h^{-1}(\mathcal{C}')) & = & f^{-1}(\mathcal{C}') \quad (f = h \circ g) \\ & \subseteq & \mathcal{C} \quad (f : \mathbb{L} \to \mathbb{L}') \\ & = & g^{-1}(\mathcal{C}''). \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

We now get $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}''$. Thus, $h : \mathbb{L} \to \mathbb{L}'$ is a logical morphism. That it projectively generates $\mathbb{L}''$ from $\mathbb{L}'$ if f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ follows from the fact that, in that case, the inclusion becomes an equality. Conversely, assume that h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. Then we have

$$\begin{array}{rcl} f^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) \quad (f = h \circ g) \\ & = & g^{-1}(\mathcal{C}'') \quad \text{(Assumption)} \\ & = & \mathcal{C}. \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

So f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. ∎

3.5 Interpretations, Filters and Matrices

In this section, taking after the theory of logical matrices (see, e.g., [24, 3, 12, 8]), we present a similar theory suitable for algebraic logicates. Note, however, that, due to lack of structurality, one has to fix interpretations, i.e., homomorphisms onto which the underlying algebra of the logicate is interpreted. A model theory along similar lines was devised for π-institutions in [21] (again based on the work of Font and Jansana on abstract logics [12]). On the other hand, when structurality is added to the mix, as will be done in a short overview in the Addendum, then considering other models, resembling more closely the ordinary matrices in Algebraic Logic, also makes sense.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. This is thought of as the focal object of our study, for which models are to be devised. So it is referred to as a **base logicate**. This explains the use of $\mathbf{B}$ for its underlying algebra and $C^\flat$ for its consequence operator. The most appropriate notion of **interpretation** is that of a pair $\mathcal{A} = \langle \mathbf{A}, h \rangle$, where:

- $\mathbf{A}$ is an algebra of the same type as the base algebra $\mathbf{B}$;
- $h : \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism.

We say that $F \subseteq A$ is an $\mathbb{L}$-**filter** (or a **filter of** $\mathbb{L}$) **on** $\mathcal{A}$, if

$$h^{-1}(F) \in \mathcal{C}^\flat,$$

i.e., the inverse image under h of the $\mathbb{L}$-filter is a theory of the logicate. If this is the case, the pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is called a **matrix for** $\mathbb{L}$ or an **$\mathbb{L}$-matrix**. The class of all $\mathbb{L}$-matrices is denoted $\mathrm{Mat}(\mathbb{L})$. An $\mathbb{L}$-matrix $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is **reduced** if $\Omega_{\mathbf{A}}(F) = \Delta_{\mathbf{A}}$. The class of all reduced $\mathbb{L}$-matrices is denoted $\mathrm{Mat}^*(\mathbb{L})$. By $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is denoted the collection of all $\mathbb{L}$-filters on the interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$.

Among the most important features of interpretations is that, if the kernel of their interpretation morphisms is a congruence of the base logicate, then they induce an algebraic logicate on the algebra into which the interpretation takes place. Moreover, if this is the case, the mapping of the interpretation becomes a bilogical morphism from the base logicate into the induced logicate.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation. Define $C_{\mathcal{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $Y \subseteq A$,

$$C_{\mathcal{A}}(Y) = h(C^\flat(h^{-1}(Y))).$$

Proposition 32 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate and* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. *Then* $\mathbb{L}_{\mathcal{A}} = \langle \mathbf{A}, C_{\mathcal{A}} \rangle$ *is an algebraic logicate. Moreover,* $h : \mathbf{B} \to \mathbf{A}$ *is a bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$.

Proof: To see that $\mathbb{L}_{\mathcal{A}}$ is a logicate, we must show idempotence of $C_{\mathcal{A}}$. Let $Y \subseteq A$. Then, we have

$$\begin{aligned} C_{\mathcal{A}}(C_{\mathcal{A}}(Y)) &= h(C^\flat(h^{-1}(h(C^\flat(h^{-1}(Y)))))) \quad \text{(Definition of } C_{\mathcal{A}}) \\ &= h(C^\flat(C^\flat(h^{-1}(Y)))) \quad (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ &= h(C^\flat(h^{-1}(Y))) \quad (\mathbb{L} \text{ a logicate}) \\ &= C_{\mathcal{A}}(Y). \quad \text{(Definition of } C_{\mathcal{A}}) \end{aligned}$$

Thus, $\mathbb{L}_{\mathcal{A}}$ is a logicate. To see that h becomes a logical morphism, we must show that $h^{-1}(\mathcal{C}_{\mathcal{A}}) \subseteq \mathcal{C}^\flat$. So let $Y \in \mathcal{C}_{\mathcal{A}}$. Then, we have

$$\begin{aligned} h^{-1}(Y) &= h^{-1}(C_{\mathcal{A}}(Y)) \quad (Y \in \mathcal{C}_{\mathcal{A}}) \\ &= h^{-1}(h(C^\flat(h^{-1}(Y)))) \quad \text{(Definition of } C_{\mathcal{A}}) \\ &= C^\flat(h^{-1}(Y)) \quad (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ &\in \mathcal{C}^\flat. \quad \text{(Definition of } \mathcal{C}^\flat) \end{aligned}$$

To see that it is a bilogical morphism, let $X \in \mathcal{C}^\flat$. Then we have

$$\begin{aligned} h^{-1}(C_{\mathcal{A}}(h(X))) &= h^{-1}(h(C^\flat(h^{-1}(h(X))))) \quad \text{(Definition of } C_{\mathcal{A}}) \\ &= C^\flat(X) \quad (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ &= X. \quad (X \in \mathcal{C}^\flat) \end{aligned}$$

We conclude that $\mathcal{C}^\flat = h^{-1}(\mathcal{C}_{\mathcal{A}})$ and, as, by hypothesis, h is surjective, $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$ is a bilogical morphism. ■

An additional property of these algebraic logicates is that the theories of the algebraic logicate coincide with the $\mathcal{S}$-filters on the underlying interpretation.

Proposition 33 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate and* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$*. Then*

$$\mathcal{C}_{\mathcal{A}} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}).$$

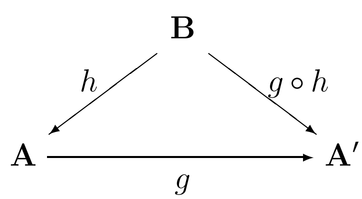
Proof: We have

$$\begin{array}{lll} \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & = & \{Y \subseteq A : h^{-1}(Y) \in \mathcal{C}^\flat\} \quad \text{(Definition of an } \mathbb{L}\text{-filter)} \\ & = & \{Y \subseteq A : h(h^{-1}(Y)) \in \mathcal{C}_{\mathcal{A}}\} \quad \text{(Proposition 32)} \\ & = & \{Y \subseteq A : Y \in \mathcal{C}_{\mathcal{A}}\} \quad (h \text{ Surjective}) \\ & = & \mathcal{C}_{\mathcal{A}}. \end{array}$$

So the displayed equality in the statements holds. ■

Next, we show that filters of interpretations that are related via surjective homomorphisms interact in a nice way.

Proposition 34 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate,* $\mathbf{A}$ *and* $\mathbf{A}'$ *be algebras and* $h : \mathbf{B} \to \mathbf{A}$ *and* $g : \mathbf{A} \to \mathbf{A}'$ *surjective homomorphisms and set* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$*.*



For all $G \subseteq A'$*,* $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ *if and only if* $g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*.*

Proof: We have

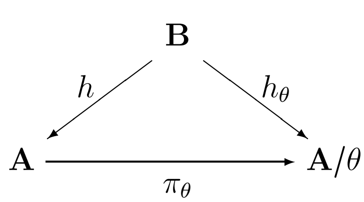
$$\begin{array}{lll} G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') & \text{iff} & (g \circ h)^{-1}(G) \in \mathcal{C}^\flat \quad (\text{Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')) \\ & \text{iff} & h^{-1}(g^{-1}(G)) \in \mathcal{C}^\flat \quad ((g \circ h)^{-1} = h^{-1} \circ g^{-1}) \\ & \text{iff} & g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}). \quad (\text{Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) \end{array}$$

■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $\theta \in \mathrm{Con}(\mathbf{A})$. Then we set

$$\mathcal{A}^\theta = \mathcal{A}/\theta = \langle \mathbf{A}/\theta, h_\theta \rangle,$$

where, $h_\theta : \mathbf{B} \to \mathbf{A}/\theta$ is defined by



$$h_\theta := \pi_\theta \circ h,$$

with $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ the natural projection homomorphism.

Proposition 35 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate,* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation,* $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *and* $\theta \in \mathrm{Con}(\mathbf{A})$*. Then* θ *is compatible with* F*, i.e.,* $\theta \subseteq \Omega_{\mathbf{A}}(F)$*, if and only if* $F = \pi_\theta^{-1}(G)$*, for some* $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$*.*

Proof: Suppose, first, that θ is compatible with F. Set $G = \pi_\theta(F)$. Then, we have

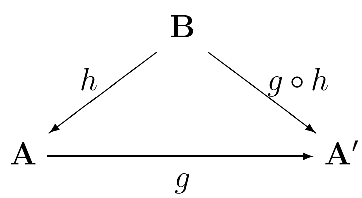
$$\begin{array}{rcll} h_\theta^{-1}(G) & = & (\pi_\theta \circ h)^{-1}(\pi_\theta(F)) & (h_\theta := \pi_\theta \circ h) \\ & = & h^{-1}(\pi_\theta^{-1}(\pi_\theta(F))) & ((\pi_\theta \circ h)^{-1} = h^{-1} \circ \pi_\theta^{-1}) \\ & = & h^{-1}(F). & (\theta \text{ compatible with } F) \end{array}$$

Since $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $h_\theta^{-1}(G) = h^{-1}(F) \in \mathcal{C}^\flat$ and, thus, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Moreover, by compatibility, $F = \pi_\theta^{-1}(\pi_\theta(F)) = \pi_\theta^{-1}(G)$.

Suppose, conversely, that $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Let $a, b \in A$, such that $\langle a, b \rangle \in \theta$ and $a \in F$. Then $a \in \pi_\theta^{-1}(G)$, whence $a/\theta \in G$. So $b/\theta = a/\theta \in G$. This gives $b \in \pi_\theta^{-1}(G) = F$. So θ is compatible with F. ■

Continuing from Proposition 34, we characterize the property of a connecting homomorphism between two interpretations being a bilogical morphism. In this, we are helped by Proposition 20, which contained a general characterization of bilogical morphisms of logicates. This is an "analog" in the present context of Proposition 1.21 of [12].

Proposition 36 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate,* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation and* $g : \mathbf{A} \to \mathbf{A}'$ *an epimorphism. Set* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$*.*



The following statements are equivalent:

(i) $g : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$*, with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$*, is a bilogical morphism;*

(ii) For all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*,* $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ *and* $\mathrm{Ker}(g) \in \mathrm{Con}(\langle \mathcal{A}, C \rangle)$*;*

(iii) g *induces an isomorphism between the poset* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \subseteq \rangle$ *and the poset* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'), \subseteq \rangle$*.*

Proof:

(i)⇒(ii) The implication (i)⇒(ii) follows by the hypothesis and Proposition 20.

(ii)⇒(iii) By hypothesis, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Conversely, if $Y \in \mathcal{C}'$, then by Proposition 34, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By surjectivity, $g(g^{-1}(Y)) = Y$, for all $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By compatibility, $g^{-1}(g(X)) = X$, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Since it is clear that both g and g' are order preserving, the conclusion follows.

(iii)⇒(i) By hypothesis, $g^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. So, by definition, $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$ is a bilogical morphism.

■

It turns out that a bilogical morphism between two models $\langle\langle \mathbf{A}, h\rangle, C\rangle$ and $\langle\langle \mathbf{A}', g \circ h\rangle, C'\rangle$ of a base logicate $\mathbb{L}$, where $g : \mathbf{A} \to \mathbf{A}'$ is an epimorphism, forces $\mathcal{C}'$ to be the full collection of $\mathbb{L}$-filters on $\mathcal{A}'$, provided that $\mathcal{C}$ is a full collection of $\mathbb{L}$-filters on $\mathcal{A}$.

Proposition 37 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a base logicate,* $\mathcal{A} = \langle \mathbf{A}, h\rangle$ *an interpretation and* $g : \mathbf{A} \to \mathbf{A}'$ *an epimorphism. Set* $\mathcal{A}' = \langle \mathbf{A}', g \circ h\rangle$. *If* $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$, *with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, *is a bilogical morphism, then*

$$\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}').$$

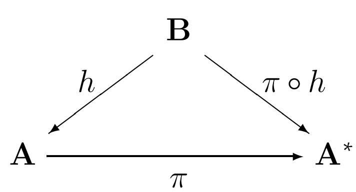
Proof: Suppose, first, that $Y \in \mathcal{C}'$. Then, since $g : \langle \mathcal{A}, C\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $g^{-1}(Y) \in \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 34, $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. So $\mathcal{C}' \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Assume, conversely, that $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Then, by Proposition 34, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) = \mathcal{C}$. Thus, since $g : \langle \mathcal{A}, C\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $Y = g(g^{-1}(Y)) \in \mathcal{C}'$. So $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') \subseteq \mathcal{C}'$. We conclude that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. ■

Corollaries 38 and 39 include consequences that constitute analogs in the context of logicates of the contents of Proposition 1.22 of [12].

Corollary 38 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a base logicate and* $\mathcal{A} = \langle \mathbf{A}, h\rangle$ *an interpretation. Then*

$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

Proof: One works with the diagram



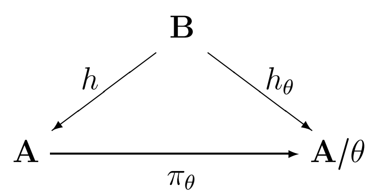
where $\pi : \mathbf{A} \to \mathbf{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))$ is the natural projection homomorphism. Recalling that, by Proposition 24, it is a bilogical morphism, we may apply Proposition 37 to get the conclusion. ■

Corollary 39 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a formula logicate,* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation and* $g : \mathbf{A} \to \mathbf{A}'$ *an isomorphism. Set* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$. *If* $g : \langle \mathcal{A}, C \rangle \cong \langle \mathcal{A}', C' \rangle$ *is an isomorphism, then*

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) \quad \textit{iff} \quad \mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}').$$

Proof: By Proposition 37. ∎

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Suppose $\mathfrak{A} = \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ and $\theta \in \mathrm{Con}(\mathfrak{A})$.



Then, using the compatibility of θ with F, we can see that $\mathfrak{A}/\theta = \langle \mathcal{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$. We call $\mathfrak{A}/\theta = \langle \mathcal{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$ the **quotient matrix of $\mathfrak{A}$ by** θ. In particular, $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathcal{A}}(F) \in \mathrm{Mat}^*(\mathbb{L})$. $\mathfrak{A}^*$ is the **reduction** of $\mathfrak{A}$. We let $\mathrm{Alg}^*(\mathbb{L})$ be the class of algebraic reducts of matrices in $\mathrm{Mat}^*(\mathbb{L})$.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and consider a class M of matrices. We say that $\mathbb{L}$ is **complete with respect to** M if

$$\mathcal{C}^\flat = \{h^{-1}(F) : \langle \langle \mathbf{A}, h \rangle, F \rangle \in \mathsf{M}\}.$$

In this case we say M is a **complete semantics for** $\mathbb{L}$. Observe that by definition of an $\mathbb{L}$-filter, the C-theories of a base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ are captured as the $\mathbb{L}$-filters on the interpretation $\langle \mathbf{B}, i_{\mathbf{B}} \rangle$. With this in mind, it is not difficult to see that, as is the case in the classical theory of logical matrices, $\mathbb{L}$ is complete both with respect to the class of all its matrices and with respect to the class of all its reduced matrices.

Proposition 40 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete both with respect to* $\mathrm{Mat}(\mathbb{L})$ *and with respect to* $\mathrm{Mat}^*(\mathbb{L})$.

Proof: By the definition of $\mathrm{Mat}(\mathbb{L})$, we have

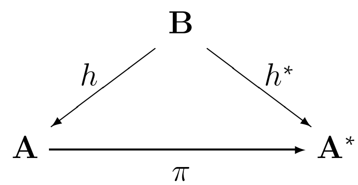
$$\{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\} \subseteq \mathcal{C}^\flat.$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then the pair $\langle \langle \mathbf{B}, i_{\mathbf{B}} \rangle, X \rangle \in \mathrm{Mat}(\mathbb{L})$ and $i_{\mathbf{B}}^{-1}(X) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}(\mathbb{L})$.

Let $\langle\langle \mathbf{A}^*, h^*\rangle, X^*\rangle \in \mathrm{Mat}^*(\mathbb{L})$. Then, by the definition of $\mathrm{Mat}^*(\mathbb{L})$,



$$\begin{array}{rcll}(h^*)^{-1}(X^*) & = & (\pi\circ h)^{-1}(\pi(X)) & \text{(Definition of } h^*)\\ & = & h^{-1}(\pi^{-1}(\pi(X))) & ((\pi\circ h)^{-1} = h^{-1}\circ\pi^{-1})\\ & = & h^{-1}(X) & \text{(Compatibility)}\\ & \in & \mathcal{C}^\flat. & (X\in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))\end{array}$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then, letting $\pi : \mathbf{B} \to \mathbf{B}^*$, where $\mathbf{B}^* = \mathbf{B}/\Omega_{\mathbf{B}}(X)$, the pair $\langle\langle \mathbf{B}^*, \pi\rangle, X^*\rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $\pi^{-1}(X^*) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle \mathcal{A}, F\rangle \in \mathrm{Mat}^*(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{L})$. ■

Given a class K of $\mathcal{L}$-algebras, an $\mathcal{L}$-algebra $\mathbf{A}$, not necessarily in the class, and a congruence $\theta \in \mathrm{Con}(\mathbf{A})$, one writes $\theta \in \mathrm{Con}_{\mathsf{K}}(\mathbf{A})$ to signify that the quotient algebra $\mathbf{A}/\theta \in \mathsf{K}$. In this case, θ is termed a **K-congruence**. So $\mathrm{Con}_{\mathsf{K}}(\mathbf{A})$ is the collection of all K-congruences on the algebra $\mathbf{A}$. Using a variant of this notation, we may write

$$\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}).$$

Given a base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ and a subset $X \subseteq B$, we have, for all $b, b' \in B$,

$$\begin{array}{lll}\langle b, b'\rangle \in \Omega_{\mathbf{B}}(X) & \text{iff} & \text{for all } \varphi(x,\overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c}\in B\\ & & \varphi^{\mathbf{B}}(b,\overline{c}) \in X \quad \text{iff} \quad \varphi^{\mathbf{B}}(b',\overline{c}) \in X.\end{array}$$

Similarly,

$$\begin{array}{lll}\langle b, b'\rangle \in \widetilde{\Omega}(\mathbb{L}) & \text{iff} & \text{for all } X\in\mathcal{C}^\flat,\ \varphi(x,\overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c}\in B\\ & & \varphi^{\mathbf{B}}(b,\overline{c}) \in X \quad \text{iff} \quad \varphi^{\mathbf{B}}(b',\overline{c}) \in X.\end{array}$$

The quotient algebra $\mathbf{B}^* \coloneqq \mathbf{B}/\widetilde{\Omega}(\mathbb{L})$ is called the **Lindenbaum-Tarski algebra** of $\mathbb{L}$. The quotient logicate $\mathbb{L}^* \coloneqq \langle \mathbf{B}^*, C^{\flat *}\rangle$ is called the **Lindenbaum-Tarski quotient** of $\mathbb{L}$. The variety generated by $\mathbf{B}^*$ is denoted by $\mathsf{K}_{\mathbb{L}}$.

Addendum: Structural Logicates

In this Addendum, we briefly overview an alternative formulation of filters and matrices applicable in case the consequence operator of the logicate, in addition to idempotency, satisfies also structurality. In that case, filters and matrices may be defined as in the traditional monotonic theory, without recourse to fixed interpretation morphisms.

A base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ is said to be **structural** if the consequence operator C, in addition to being idempotent, satisfies, for all $e : \mathbf{B} \to \mathbf{B}$,

(**Structurality**) $e^{-1}(\mathcal{C}^\flat) \subseteq \mathcal{C}^\flat$,

that is, $\mathcal{C}^\flat$ is closed under inverse endomorphisms.

For a given structural logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ it makes sense to consider another notion of *interpretation* that simulates more closely the classical treatment. Namely, we consider algebras $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ of the same type as the structural logicate (without specific morphisms attached). Let us call this a **structural interpretation**. We say that $F \subseteq A$ is an (**structural**) $\mathbb{L}$-**filter** on $\mathbf{A}$ if, for all $h : \mathbf{B} \to \mathbf{A}$,

$$h^{-1}(F) \in \mathcal{C}^\flat,$$

i.e., the inverse image of an $\mathbb{L}$-filter under all homomorphisms from the formula algebra into $\mathbf{A}$ is a theory of $\mathbb{L}$. If this is the case, the pair $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is called a (**structural**) **matrix for** $\mathbb{L}$ or a (**structural**) $\mathbb{L}$-**matrix**. The class of all structural $\mathbb{L}$-matrices is denoted $\mathrm{Mat}(\mathbb{L})$. An $\mathbb{L}$-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is **reduced** if $\Omega_{\mathbf{A}}(F) = \Delta_{\mathbf{A}}$. The class of all reduced $\mathbb{L}$-matrices is denoted $\mathrm{Mat}^*(\mathbb{L})$. By $\mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$ is denoted the collection of all $\mathbb{L}$-filters on the algebra $\mathbf{A}$.

Many of the results established previously continue to hold, in an appropriately recalibrated form, for structural filters and matrices. We present here some samples, pointing to corresponding results in Section 3.5 of which they form analogs. E.g., Proposition 41 is an analog of Proposition 34 for structural filters.

Proposition 41 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a structural logicate,* $\mathbf{A}$ *and* $\mathbf{A}'$ *be algebras,* $h : \mathbf{A} \to \mathbf{A}'$ *a homomorphism and* $G \subseteq A'$.

(a) If $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$, *then* $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$.

(b) If h *is surjective and* $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$, *then* $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$.

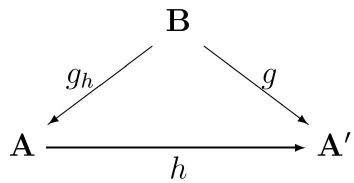
Proof:

(a) Suppose that $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$. Let $g : \mathbf{B} \to \mathbf{A}$. Then we have

$$g^{-1}(h^{-1}(G)) = (h \circ g)^{-1}(G) \in \mathcal{C}^\flat.$$

Thus, $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$.

(b) Suppose, now, that $h^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. Let $g : \mathbf{B} \to \mathbf{A}'$.



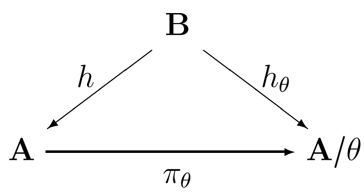
Then, there exists $g_h : \mathbf{B} \to \mathbf{A}$, such that $h \circ g_h = g$. So we have

$$g^{-1}(G) = (h \circ g_h)^{-1}(G) = g_h^{-1}(h^{-1}(G)) \in \mathcal{C}^\flat.$$

Thus, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$.

■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate, $\mathbf{A}$ an algebra and $\theta \in \mathrm{Con}(\mathbf{A})$. we write $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ for the canonical projection homomorphism and, given $h : \mathbf{B} \to \mathbf{A}$, we denote by h_θ the composition $h_\theta = \pi_\theta \circ h$.



Proposition 42 forms an analog of Proposition 35 for structural filters.

Proposition 42 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate, $\mathbf{A}$ an algebra, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{A})$. Then θ is compatible with F, i.e., $\theta \subseteq \Omega_{\mathbf{A}}(F)$, if and only if $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}/\theta)$.*

Proof: Suppose, first, that θ is compatible with F. Set $G = \pi_\theta(F)$. Then $\pi_\theta^{-1}(\pi_\theta(F)) = F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. By Proposition 41, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}/\theta)$. Since $F = \pi_\theta^{-1}(G)$, this proves necessity.

Suppose, conversely, that $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}/\theta)$. Let $a, b \in A$, such that $\langle a, b \rangle \in \theta$ and $a \in F$. Then $\pi_\theta(a) = \pi_\theta(b)$ and $a \in \pi_\theta^{-1}(G)$. Thus, $\pi_\theta(b) = \pi_\theta(a) \in G$, whence $b \in \pi_\theta^{-1}(G) = F$. So θ is compatible with F. ■

As far as an analog of Proposition 36, we get the following

Proposition 43 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate, $\mathbf{A}$, $\mathbf{A}'$ be algebras and $h : \mathbf{A} \to \mathbf{A}'$ an epimorphism. The following statements are equivalent:*

(i) $h : \langle \mathbf{A}, C \rangle \to \langle \mathbf{A}', C' \rangle$ is a bilogical morphism, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$ and $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$;

(ii) $h : \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}) \to \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$ is a bijection;

(ii) *For all* $F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$, $h(F) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$ *and* $\mathrm{Ker}(h) \in \mathrm{Con}(\langle \mathcal{A}, C\rangle)$.

Proof:

(i)⇒(ii) By Proposition 20.

(ii)⇒(iii) The first statement follows by hypothesis. For the second, suppose $\langle a, b\rangle \in \mathrm{Ker}(h)$ and $a \in F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. Then $h(a) = h(b)$ and $h(a) \in h(F)$. Thus, $h(b) \in h(F)$. Hence, $b \in h^{-1}(h(F)) = F$. This establishes that $\mathrm{Ker}(h) \in \mathrm{Con}(\langle \mathbf{A}, C\rangle)$.

(iii)⇒(i) By Proposition 20, we must show that

$$\mathcal{C}' = h(\mathcal{C}).$$

The right-to-left inclusion holds by hypothesis. Suppose, next, that $G \in \mathcal{C}'$. Let $F = h^{-1}(G)$. Then, by Proposition 41, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$. By surjectivity, $G = h(h^{-1}(G)) = h(F)$. This proves the left-to-right inclusion. Now Proposition 20 gives that $h : \langle \mathbf{A}, C\rangle \to \langle \mathbf{A}', C'\rangle$ is a bilogical morphism.

■

The next proposition is an analog of Proposition 37 for structural logicates.

Proposition 44 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a structural logicate,* $\mathbf{A}$, $\mathbf{A}'$ *be algebras and* $h : \mathbf{A} \to \mathbf{A}'$ *surjective. If* $h : \langle \mathbf{A}, C\rangle \to \langle \mathbf{A}', C'\rangle$ *is a bilogical morphism and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A})$, *then*

$$\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}').$$

Proof: By Proposition 41, $\mathcal{C}' \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$. Assume, conversely, that $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}')$. Then, by Proposition 41, $h^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}) = \mathcal{C}$. Thus, by hypothesis, $Y = h(h^{-1}(Y)) \in \mathcal{C}'$. Hence, $\mathrm{Fi}_{\mathbb{L}}(\mathbf{A}') \subseteq \mathcal{C}'$. ■

Corollary 45 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a structural logicate and* $\mathbf{A}$ *an algebra. Then*

$$\mathrm{Fi}_{\mathbb{L}}(\mathbf{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}^*).$$

Proof: By Proposition 44. ■

Corollary 46 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be a structural logicate,* $\mathbf{A}$, $\mathbf{A}'$ *algebras and and* $h : \langle \mathbf{A}, C\rangle \to \langle \mathbf{A}', C'\rangle$ *an isomorphism. Then*

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}) \quad \textit{iff} \quad \mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathbf{A}').$$

Proof: By Proposition 44. ■

Many of the definitions and results concerning classes of algebras also have counterparts for structural logicates.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate. Suppose $\mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ be a structural matrix and $\theta \in \mathrm{Con}(\mathfrak{A})$. Then $\mathfrak{A}/\theta = \langle \mathbf{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$. $\mathfrak{A}/\theta$ is called the **quotient structural matrix of $\mathfrak{A}$ by θ**. In particular, $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathbf{A}}(F) \in \mathrm{Mat}^*(\mathbb{L})$. $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathbf{A}}(F)$ is called the **reduction** of $\mathfrak{A}$. We let $\mathrm{Alg}^*(\mathbb{L})$ be the class of algebraic reducts of matrices in $\mathrm{Mat}^*(\mathbb{L})$. So we have $\Omega_{\mathbf{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathbf{A})$. In the structural case, $\mathcal{C}^\flat$ coincides with the set of $\mathbb{L}$-filters on $\mathbf{B}$. This fact allows us, here also, to obtain a completeness analog of Proposition 40, which more closely resembles the one from the traditional setting.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate and consider a class M of structural matrices. We say that $\mathbb{L}$ is **complete with respect to M** if

$$\mathcal{C}^\flat = \{h^{-1}(F) : \langle \mathbf{A}, F \rangle \in \mathsf{M} \text{ and } h : \mathbf{B} \to \mathbf{A}\}.$$

Proposition 47 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a structural logicate. $\mathbb{L}$ is complete both with respect to $\mathrm{Mat}(\mathbb{L})$ and with respect to $\mathrm{Mat}^*(\mathbb{L})$.*

Proof: By the definition of $\mathrm{Mat}(\mathbb{L})$, if $\langle \mathbf{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ and $h : \mathbf{B} \to \mathbf{A}$, then $h^{-1}(F) \in \mathcal{C}^\flat$. Conversely, if $T \in \mathcal{C}^\flat$, then, by structurality, $\langle \mathbf{B}, T \rangle \in \mathrm{Mat}(\mathbb{L})$ and $i_{\mathbf{B}}^{-1}(T) = T$. Hence, $\mathbb{L}$ is complete with respect to $\mathrm{Mat}(\mathbb{L})$.

By the definition of $\mathrm{Mat}^*(\mathbb{L})$, if $\mathfrak{A}^* = \langle \mathbf{A}^*, F^* \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $h : \mathbf{B} \to \mathbf{A}^*$, then there exists $h^* : \mathbf{B} \to \mathbf{A}$, such that

$$\pi \circ h^* = h,$$

where $\pi : \mathbf{A} \to \mathbf{A}^*$ is the canonical projection. Thus

$$\begin{array}{rcl} h^{-1}(F^*) & = & (\pi \circ h^*)^{-1}(F^*) \\ & = & (h^*)^{-1}(\pi^{-1}(F^*)) \\ & = & (h^*)^{-1}(F) \\ & \in & \mathcal{C}^\flat. \end{array}$$

Assume, conversely, that $T \in \mathcal{C}^\flat$. Then the pair $\langle \mathbf{B}^*, T^* \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $\pi^{-1}(T^*) = T$. This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{L})$. ■

We close this section with another result borrowed from the classical theory pertaining to the class $\mathrm{Alg}^*(\mathbb{L})$ of algebras of a structural logicate whose underlying algebra is a free algebra with countably many generators.

Proposition 48 *Let $\mathbb{L} = \langle \mathbf{Fm}_{\mathcal{L}}(V), C^\flat \rangle$ be a structural logicate. The class $\mathsf{K}_{\mathbb{L}}$ is the variety generated by the class $\mathrm{Alg}^*(\mathbb{L})$.*

Proof: Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. We have

$$\begin{array}{lll}
\mathsf{K}_{\mathbb{L}} \vDash \varphi \approx \psi & \text{iff} & \langle \varphi, \psi \rangle \in \widetilde{\Omega}(\mathbb{L}) \\
& \text{iff} & \text{for all } \chi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V) \text{ and } T \in \mathcal{C}^{\flat}, \\
& & \chi(\varphi, \overline{z}) \in T \quad \text{iff} \quad \chi(\psi, \overline{z}) \in T \\
& \text{iff} & \text{for all } \chi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V), \langle \mathbf{A}, F \rangle \in \mathrm{Mat}^*(\mathbb{L}), \overline{a}, \overline{c} \text{ in } A, \\
& & \chi^{\mathbf{A}}(\varphi^{\mathbf{A}}(\overline{a}), \overline{c}) \in F \quad \text{iff} \quad \chi^{\mathbf{A}}(\psi^{\mathbf{A}}(\overline{a}), \overline{c}) \in F \\
& \text{iff} & \text{for all } \langle \mathbf{A}, F \rangle \in \mathrm{Mat}^*(\mathbb{L}), \overline{a} \in A, \\
& & \langle \varphi^{\mathbf{A}}(\overline{a}), \psi^{\mathbf{A}}(\overline{a}) \rangle \in \Omega_{\mathbf{A}}(F) \\
& \text{iff} & \text{for all } \langle \mathbf{A}, F \rangle \in \mathrm{Mat}^*(\mathbb{L}), \overline{a} \in A, \\
& & \varphi^{\mathbf{A}}(\overline{a}) = \psi^{\mathbf{A}}(\overline{a}) \\
& \text{iff} & \mathrm{Mat}^*(\mathbb{L}) \vDash \varphi \approx \psi.
\end{array}$$

■

Chapter 4

Model Theory

4.1 Introduction

In this, third chapter, on logicates, we focus specifically on the role that algebraic logicates play as models of other logicates. Logicates are models more suitable for many purposes than simple logical matrices, even though logicate models can be viewed as bundles of matrices over the same underlying interpretation.

Our framework and starting point is the study in Chapter 3 of interpretations. We are assuming a given logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ which consists of an algebra and an idempotent operator, called a *consequence operator*, on its powerset. Logicates are supposed to represent logical systems for which inflationarity and monotonicity may fail. $\mathbb{L}$ is viewed as the focus of study and it is called, accordingly, a *base logicate*. An *interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ consists of a surjective homomorphism from the algebra $\mathbf{B}$ of $\mathbb{L}$ onto a similar algebra $\mathbf{A}$. If on the target algebra, there is given a logicate structure, say $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}$ induces a logicate on the base algebra. We say that $\mathbb{A}$ is a *model* of $\mathbb{L}$ if the inverse images under h of the theories of $\mathbb{A}$ form a subset of the theories of $\mathbb{L}$, written $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. Two logicates connected by a bilogical morphism that commutes with interpretations share the property of being simultaneously models or not being models.

One of the key constructions in our framework is passing from a model to its Tarski reduction. The *Tarski operator* was used as a key ingredient in the theory of Font and Jansana (Page 19 of [12]) and, as our work is based on theirs, it continues to play a crucial role here as well. Given a logicate model, one may construct its *reduction* by moding out both the interpretation and the idempotent operator by the Tarski congruence of the logicate. A first result is that a logicate is complete with respect to both its class of logicate models and its class of reduced logicate models. Completeness here simply means that collecting all inverse images of theories of the models of the class yields the full collection of theories of the base logicate.

Connecting the theory of logicate models with the theory of matrix models of the base logicate, which was detailed in Chapter 3, we obtain the fact, well-known in classical theory (Proposition 2.7 of [12]), that, a logicate, viewed as a bundle of matrices, is a model of $\mathbb{L}$ if and only if every member of the bundle is a matrix model of $\mathbb{L}$.

The next key concept adapted here from the theory of abstract logics of [12] is that of a *full model*. A *basic (full logicate) model* is a model whose collection of theories consists of all filters on its interpretation. A *full (logicate) model* is one whose Tarski reduction is a basic full logicate model. The terminology is justified by the fact that a basic model turns out to be a full model according to these definitions. It is shown here, in a result that parallels one pertaining to abstract logics, that logicate models connected via bilogical morphisms commuting with interpretations are either both full or both fail to be full. As a consequence the property of being full is also preserved and reflected by reductions. These results yield a characterization of the class of full models as the smallest class that contains all basic full models and is closed (in both directions) under bilogical morphisms.

Full models are the first ingredient in establishing a key *Isomorphism Theorem*, along the lines of the Isomorphism Theorem (Theorem 2.30) of Font and Jansana, which is one of the main results of the abstract treatment in the theory they present in [12]. The second ingredient relates to congruences whose quotients are algebras in $\mathrm{Alg}(\mathbb{L})$. The class $\mathrm{Alg}(\mathbb{L})$ consists of the underlying interpretations of reduced full models of $\mathbb{L}$. Another class of interpretations that is related to a class of algebras traditionally studied in algebraic logic is the class $\mathrm{Alg}^*(\mathbb{L})$. It consists of all underlying interpretations of reduced matrix models of $\mathbb{L}$. The tight connection, mentioned previously, between logicate models and matrix models, yields a (sort of induced) relationship between the two classes. In the traditional setting one class turns out to be the class of subdirect products of the other (see, e.g., Theorem 2.23 of [12]). In the present setting, because of the presence of fixed interpretation morphisms, we find it convenient (and perhaps necessary) to define a related but different operation on interpretations, called a *subdirect intersection*. It is shown that the class $\mathrm{Alg}(\mathbb{L})$ consists exactly of subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$.

Our work in this part culminates with proving an analog of the Isomorphism Theorem 2.30 of [12] for the present context. We view this as one of the main results of the work. The analog proven here has some significant deviations as compared to its predecessor. First, all parts are taken to be over fixed underlying interpretations. This is compelled by the absence of structurality for logicates. If one added structurality, then something closer, perhaps, to the original could be obtained. But this seemed rather restrictive and, in addition, the framework of π-institutions [21] has provided some experience in dealing with fixed interpretations. Second, one cannot expect to establish an isomorphism theorem dealing with all full models, since full models that are equipotent, have identical Tarski congruences. So one has, by necessity, to pass to equipotency classes of full models over fixed interpretations. Taking these comments into account, we establish an order isomorphism between the set of equipotency classes of full logicate models, ordered under the superset relation between sets of theories, and the set of $\mathrm{Alg}(\mathbb{L})$-congruences under inclusion. It is also shown that the latter poset is a complete lattice. As a consequence, one obtains that the former has the same structure as well.

In Section 4.2 we recall the notion of interpretation and use it to define *logicate interpretations*. These, in turn, serve in defining models of a logicate. Logicate models have a tight relationship with matrix models. We also define reductions. We show that, if two logicate interpretations are related via a bilogical morphism, then one is a model if and only if the other is. This implies that a given one is a model if and only if its reduction is. We also formulate analogs of the well-known Completeness Theorems of Algebraic

Logic both with respect to the class of all models and with respect to the class of all reduced models.

In Section 4.3, the notion of *full model* for logicates is introduced, taking after the corresponding notion for the monotonic framework (see Definition 2.8 of [12]). A model is a *basic (full) model* if the set of its theories coincides with the set of all filters on the underlying interpretation. A model is a *full model* if its reduction is a basic full model. Several properties, paralleling ones proved by Font and Jansana for sentential logics in [12], are adapted and proved in this setting. They culminate in two different characterizations of full models, which, as Font and Jansana explain, may be taken as justifications of the term "full". The class of full models is shown to be the smallest class that includes all basic full models and is closed under bilogical morphisms (see Corollary 2.13 of [12]). It is also the class of all models whose sets of theories consist of all preimages under canonical projections of all filters on the Tarski reduction of the model (see Theorem 2.14 of [12]).

In Section 4.4 we define the notion of $\mathbb{L}$-*algebra* for a given logicate $\mathbb{L}$. These parallel $\mathcal{S}$-algebras for a sentential logic [12]. In the present context, however, they should be referred to as $\mathbb{L}$-*interpretations*, since they are pairs consisting of an algebra together with a mapping from the base algebra of $\mathbb{L}$ onto the algebra. But the term "algebra" is retained because of the similarity of the role they play. Several results encapsulating the interaction of these algebras with full models and reduced full models are given. We also revisit the relation between the class of algebras which are reducts of reduced matrix models and the class of $\mathbb{L}$-algebras, which are reducts of reduced full models of $\mathbb{L}$. An operation, called *subdirect intersection*, paralleling that of subdirect product in the ordinary framework, is defined and comes in handy in this task. The result is an analog of Theorem 2.23 of [12].

In Section 4.5 the main goal is establishing an *Isomorphism Theorem*, along the lines of Theorem 2.30 of [12]. The Tarski operator over a fixed interpretation forms a mapping from logicates over that interpretation into congruences. Moreover, it is constant over equipotency classes of logicates. Thus, it may be viewed as an operator over equipotency classes of logicates to congruences. We introduce here an operator from $\mathrm{Alg}(\mathbb{L})$-congruences that is seeing as being the inverse of the restriction of the Tarski operator on equipotency classes of full models. Moreover, both operators are order preserving, when order is taken to be the one reflecting the superset relation between sets of theories. So they establish an isomorphism between equipotency classes of full models and $\mathbb{L}$-algebra congruences. Some consequences of this isomorphism are encountered here, among which is the fact that the equipotency classes of full models form a complete lattice. This is proven using the isomorphism theorem and a result showing that the collection of $\mathrm{Alg}(\mathbb{L})$-congruences under the subset relation form a complete lattice.

4.2 Models of Logicates

Let $\mathbf{B} = \langle B, \mathcal{L}^{\mathbf{B}} \rangle$ be an algebra, which, in this context, is termed **base algebra**. An **interpretation** is a pair $\mathcal{A} = \langle \mathbf{A}, h \rangle$, where

- $\mathbf{A}$ is an $\mathcal{L}$-algebra;
- $h : \mathbf{B} \twoheadrightarrow \mathbf{A}$ is an epimorphism from the base algebra onto $\mathbf{A}$.

A **logicate interpretation**, is a pair $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where:

- $\mathcal{A} = \langle \mathbf{A}, h \rangle$ is an interpretation;
- $\langle \mathbf{A}, C \rangle$ is an algebraic logicate on the algebra $\mathbf{A}$.

The logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ induces a function

$$C^{\mathbb{A}} : \mathcal{P}(B) \to \mathcal{P}(B),$$

where, for all $X \subseteq B$,

$$C^{\mathbb{A}}(X) = h^{-1}(C(h(X))).$$

We write

$$\mathbb{L}^{\mathbb{A}} := \langle \mathbf{B}, C^{\mathbb{A}} \rangle.$$

This construction forms an analog of the construction in Definition 2.1 of [12]. We show that $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicate on the base algebra and that the epimorphism h is a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{A}$.

Proposition 49 *Let* $\mathbf{B}$ *be a base algebra and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *a logicate interpretation, with* $\mathcal{A} = \langle \mathbf{A}, h \rangle$.

(a) $\mathbb{L}^{\mathbb{A}}$ *is an algebraic logicate.*

(b) $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{A}$ *is a bilogical morphism.*

Proof:

(a) We must show that $C^{\mathbb{A}}$ is idempotent. Let $X \subseteq B$. Then

$$\begin{array}{rcl} C^{\mathbb{A}}(C^{\mathbb{A}}(X)) & = & h^{-1}(C(h(h^{-1}(C(h(X)))))) \quad \text{(Definition of } C^{\mathbb{A}}) \\ & = & h^{-1}(C(C(h(X)))) \quad (h \text{ Surjective}) \\ & = & h^{-1}(C(h(X))) \quad (C \text{ Idempotent}) \\ & = & C^{\mathbb{A}}(X). \quad \text{(Definition of } C^{\mathbb{A}}) \end{array}$$

Thus, $C^{\mathbb{A}}$ is idempotent and, therefore, $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicate.

(b) By the surjectivity of h and the definition of $C^{\mathbb{A}}$, we have, for all $X \subseteq B$,

$$h(C^{\mathbb{A}}(X)) = h(h^{-1}(C(h(X)))) = C(h(X)).$$

Suppose $X \in \mathcal{C}^{\mathbb{A}}$. Then

$$h(X) = h(C^{\mathbb{A}}(X)) = C(h(X)).$$

Hence $h(X) \in \mathcal{C}$. Suppose, conversely, that $Y \in \mathcal{C}$. Consider $X \subseteq B$, such that $h(X) = Y$. Then

$$h(C^{\mathbb{A}}(X)) = C(h(X)) = Y.$$

Thus Y is the image of a $C^{\mathbb{A}}(X) \in \mathcal{C}^{\mathbb{A}}$. Finally, if $b, b' \in B$, such that $\langle b, b' \rangle \in \mathrm{Ker}(h)$ and $b \in C^{\mathbb{A}}(X)$, then $h(b) = h(b')$ and, by definition of $C^{\mathbb{A}}$, $b \in h^{-1}(C(h(X)))$, whence $h(b') = h(b) \in C(h(X))$, showing that $b' \in h^{-1}(C(h(X))) = C^{\mathbb{A}}(X)$. Hence $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L}^{\mathbb{A}})$. By Proposition 20, we get that $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{L}$ is a bilogical morphism.

■

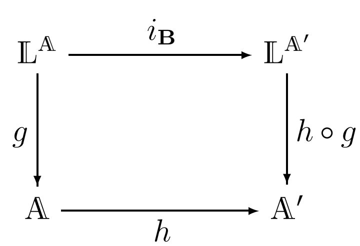
We call $\mathbb{L}^{\mathbb{A}}$ the **logicate induced on B** by $\mathbb{A}$.

An analog of Proposition 2.3 of [12] ensures that logicate interpretations related via "compatible" bilogical morphisms induce isomorphic logicates on the base algebra. Recall, however, the discussion on "isomorphisms" from Section 3.4 intended to thwart any misunderstandings concerning the term and its rather counterintuitive meaning. One may be better off as thinking of "isomorphisms" as "isopotent" bijections, meaning that they preserve and reflect theories without necessarily preserving consequences.

Proposition 50 *Let* $\mathbf{B}$ *be a base algebra and* $\mathbb{A} = \langle\langle \mathbf{A}, g\rangle, C\rangle$ *and* $\mathbb{A}' = \langle\langle \mathbf{A}', h \circ g\rangle, C'\rangle$ *two logicate interpretations, with* $h : \mathbb{A} \to \mathbb{A}'$ *a bilogical morphism. The identity* $i_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ *is an isomorphism*

$$i_{\mathbf{B}} : \mathbb{L}^{\mathbb{A}} \cong \mathbb{L}^{\mathbb{A}'}.$$

Proof: Using the diagram below, we have



$$\begin{aligned}
\mathcal{C}^{\mathbb{A}} &= g^{-1}(\mathcal{C}) \quad (g : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{A}) \\
&= g^{-1}(h^{-1}(\mathcal{C}')) \quad (h : \mathbb{A} \to_b \mathbb{A}') \\
&= (h \circ g)^{-1}(\mathcal{C}') \quad ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\
&= \mathcal{C}^{\mathbb{A}'}. \quad (h \circ g : \mathbb{L}^{\mathbb{A}'} \to_b \mathbb{A}')
\end{aligned}$$

So $i_{\mathbf{B}} : \mathbb{L}^{\mathbb{A}} \cong \mathbb{L}^{\mathbb{A}'}$ is an isomorphism. ∎

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate, perceived as constituting the main object of investigation. A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, h \rangle$, is called a **model of** $\mathbb{L}$ or an $\mathbb{L}$**-model** if

$$h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat.$$

We denote by $\text{Mod}(\mathbb{L})$ the class of all models of $\mathbb{L}$.

Recalling from Chapter 2 the ordering $\trianglelefteq$ on logicates, we can relate the notion of model with the construction of the logicate induced by a logicate interpretation.

Proposition 51 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, h \rangle$, a logicate interpretation. Then $\mathbb{A}$ is an $\mathbb{L}$-model if and only if $\mathbb{L} \trianglelefteq \mathbb{L}^{\mathbb{A}}$.*

Proof: Note that $\mathbb{A}$ being an $\mathbb{L}$-model means that $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$, whereas $\mathbb{L} \trianglelefteq \mathbb{L}^{\mathbb{A}}$ means that $\mathcal{C}^{\mathbb{A}} \subseteq \mathcal{C}^\flat$. Thus, to prove the result, it suffices to show that $\mathcal{C}^{\mathbb{A}} = h^{-1}(\mathcal{C})$.

Suppose, first, that $X \in \mathcal{C}^{\mathbb{A}}$. Consider $C(h(X)) \in \mathcal{C}$. We have

$$\begin{array}{rcll} h^{-1}(C(h(X))) & = & C^{\mathbb{A}}(X) & (\text{Definition of } C^{\mathbb{A}}) \\ & = & X. & (X \in \mathcal{C}^{\mathbb{A}}) \end{array}$$

This shows that $\mathcal{C}^{\mathbb{A}} \subseteq h^{-1}(\mathcal{C})$.

Assume, conversely, that $X \in h^{-1}(\mathcal{C})$. Then, $X = h^{-1}(Y)$, for some $Y \in \mathcal{C}$. Thus, we get

$$\begin{array}{rcll} C^{\mathbb{A}}(X) & = & h^{-1}(C(h(X))) & (\text{Definition of } C^{\mathbb{A}}) \\ & = & h^{-1}(C(h(h^{-1}(Y)))) & (X = h^{-1}(Y)) \\ & = & h^{-1}(C(Y)) & (h \text{ surjective}) \\ & = & h^{-1}(Y) & (Y \in \mathcal{C}) \\ & = & X. & (X = h^{-1}(Y)) \end{array}$$

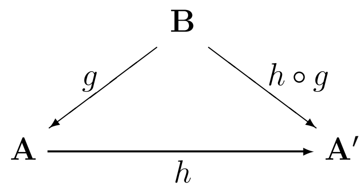
Hence, $X \in \mathcal{C}^{\mathbb{A}}$ and $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^{\mathbb{A}}$. ∎

Let L be a class of models of $\mathbb{L}$. $\mathbb{L}$ is said to be **complete with respect to** L if

$$\mathcal{C}^\flat = \bigcup \{h^{-1}(\mathcal{C}) : \langle \langle \mathbf{A}, h \rangle, C \rangle \in \mathsf{L}\}.$$

The following result is an analog of Part (1) of Proposition 2.5 of [12] for logicate interpretations.

Proposition 52 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate,* $\mathbb{A} = \langle \langle \mathbf{A}, g \rangle, C \rangle$, $\mathbb{A}' = \langle \langle \mathbf{A}', h \circ g \rangle, C' \rangle$ *be logicate interpretations, where* $h : \mathbb{A} \to \mathbb{A}'$ *be a bilogical morphism.*



Then $\mathbb{A}$ *is a model of* $\mathbb{L}$ *if and only if* $\mathbb{A}'$ *is a model of* $\mathbb{L}$.

Proof: Suppose $\mathbb{A}$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcll} (h \circ g)^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & = & g^{-1}(\mathcal{C}) & (h : \mathbb{A} \to_b \mathbb{A}') \\ & \subseteq & \mathcal{C}^\flat. & (\mathbb{A} \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}'$ is a model of $\mathbb{L}$. Assume, conversely, that $\mathbb{A}'$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcll} g^{-1}(\mathcal{C}) & = & g^{-1}(h^{-1}(\mathcal{C}')) & (h : \mathbb{A} \to_b \mathbb{A}') \\ & = & (h \circ g)^{-1}(\mathcal{C}') & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & \subseteq & \mathcal{C}^\flat. & (\mathbb{A}' \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}$ is a model of $\mathbb{L}$. ■

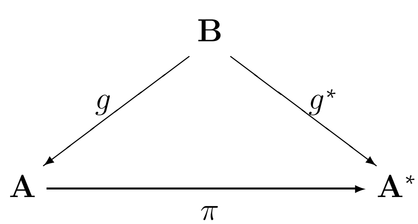
In order to formulate another Part (2) of Proposition 2.5 of [12], we need to define the Tarski reduction of a logicate interpretation.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an base logicate. Consider the pair $\mathbb{A} = \langle \langle \mathbf{A}, g \rangle, C \rangle$. Recall the Tarski congruence $\widetilde{\Omega}(\mathbb{A}) \coloneqq \widetilde{\Omega}_{\mathbf{A}}(C)$. We define the pair

$$\mathbb{A}^* = \langle \langle \mathbf{A}^*, g^* \rangle, C^* \rangle$$

by setting:

- $\mathbf{A}^* = \mathbf{A} / \widetilde{\Omega}_{\mathbf{A}}(C)$;
- $g^* = \pi \circ g$, where $\pi : \mathbf{A} \to \mathbf{A}^*$ is the natural projection;



- $C^* : \mathcal{P}(A / \widetilde{\Omega}_{\mathbf{A}}(C)) \to \mathcal{P}(A / \widetilde{\Omega}_{\mathbf{A}}(C))$, where, for all $S \subseteq A / \widetilde{\Omega}_{\mathbf{A}}(C)$,

$$C^*(S) = \pi(C(\pi^{-1}(S))).$$

Based on results already obtained, we may show that, if $\mathbb{A}$ is a model of $\mathbb{L}$, then so is $\mathbb{A}^*$.

Corollary 53 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. A pair $\mathbb{A} = \langle \langle \mathbf{A}, g \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if $\mathbb{A}^*$ is a model of $\mathbb{L}$.*

Proof: This follows directly from Proposition 52, since, by Proposition 24, the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism. ■

Corollary 54 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is complete with respect to a class L of models if and only if it is complete with respect to the class L^*.*

Proof: Let

$$\mathsf{L} = \{\langle \langle \mathbf{A}_i, g_i \rangle, C_i \rangle : i \in I\}.$$

Denote by $\pi_i : \mathbf{A}_i \to \mathbf{A}_i^*$, $i \in I$, the natural projections. By Proposition 24, for all $i \in I$, $\pi_i : \mathbb{A}_i \to \mathbb{A}_i^*$ is a bilogical morphism. So we have, for all $i \in I$,

$$\begin{array}{rcll} \bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i) & = & \bigcup_{i \in I} g_i^{-1}(\pi_i^{-1}(\mathcal{C}_i^*)) & (\pi_i : \mathbb{A}_i \to_b \mathbb{A}_i^*) \\ & = & \bigcup_{i \in I} (\pi_i \circ g_i)^{-1}(\mathcal{C}_i^*) & ((\pi_i \circ g_i)^{-1} = g_i^{-1} \circ \pi_i^{-1}) \\ & = & \bigcup_{i \in I} (g_i^*)^{-1}(\mathcal{C}_i^*). & (g_i^* = \pi_i \circ g_i) \end{array}$$

Thus, $\mathbb{L}$ is complete with respect to L if and only if, by definition, $\mathcal{C}^\flat = \bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i)$ if and only if, by the displayed equality, $\mathcal{C}^\flat = \bigcup_{i \in I} (g_i^*)^{-1}(\mathcal{C}_i^*)$ if and only if, by definition, $\mathbb{L}$ is complete with respect to L^*. ■

As far as completeness properties go, note that $\langle \langle \mathbf{B}, i_{\mathbf{B}} \rangle, C^\flat \rangle$ is a model of $\mathbb{L}$. This yields the following results, forming, together, an analog of Proposition 2.6 of [12].

Proposition 55 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is complete with respect to any class L of models that includes $\mathbb{A} = \langle \langle \mathbf{B}, i_{\mathbf{B}} \rangle, C^\flat \rangle$ or $\mathbb{A}^*$.*

Proof: Let $\mathsf{L} = \{\langle \langle \mathbf{A}_i, g_i \rangle, C_i \rangle : i \in I\}$. On the one hand, since $\mathsf{L} \subseteq \mathrm{Mod}(\mathbb{L})$, $\bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i) \subseteq \mathcal{C}^\flat$. On the other, since $\mathbb{A} \in \mathsf{L}$, $\mathcal{C}^\flat = i_{\mathbf{B}}^{-1}(\mathcal{C}^\flat) \subseteq \bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i)$. Thus, $\mathbb{L}$ is complete with respect to L.

Assume, now, that $\mathbb{A}^* \in \mathsf{L}$. The first inclusion is justified in the same way. For the second, letting $\pi : \mathbf{B} \to \mathbf{B}^*$ be the natural projection, we have

$$\begin{array}{rcll} \mathcal{C}^\flat & = & i_{\mathbf{B}}^{-1}(\mathcal{C}^\flat) & (i_{\mathbf{B}} \text{ identity}) \\ & = & i_{\mathbf{B}}^{-1}(\pi^{-1}(\mathcal{C}^{\flat *})) & (\pi : \mathbb{A} \to_b \mathbb{A}^*) \\ & = & (\pi \circ i_{\mathbf{B}})^{-1}(\mathcal{C}^{\flat *}) & ((\pi \circ i_{\mathbf{B}})^{-1} = i_{\mathbf{B}}^{-1} \circ \pi^{-1}) \\ & \subseteq & \bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i). & (\mathbb{A}^* \in \mathsf{L}) \end{array}$$

Thus, $\mathbb{L}$ is again complete with respect to L. ■

Corollary 56 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete with respect to the class of all its models and with respect to the class of all its reduced models.*

Proof: Clearly, $\mathbb{A} = \langle \langle \mathbf{B}, i_{\mathbf{B}} \rangle, C^\flat \rangle \in \mathrm{Mod}(\mathbb{L})$ and $\mathbb{A}^* \in \mathrm{Mod}^*(\mathbb{L})$. Thus, by Proposition 55, $\mathbb{L}$ is complete both with respect to $\mathrm{Mod}(\mathbb{L})$ and with respect to $\mathrm{Mod}^*(\mathbb{L})$. ■

Logicate models are closely connected with matrix models. The connection is given in the following proposition, which parallels Proposition 2.7 of [12].

Proposition 57 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. Then* $\langle \langle \mathbf{A}, h \rangle, C \rangle$ *is a model of* $\mathbb{L}$ *if and only if, for all* $Y \in \mathcal{C}$, $\langle \langle \mathbf{A}, h \rangle, Y \rangle$ *is an* $\mathbb{L}$*-matrix.*

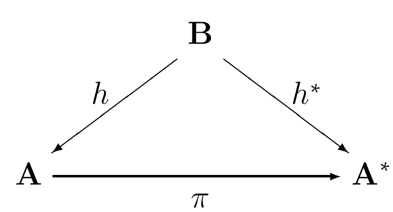
Proof: We have that $\langle \langle \mathbf{A}, h \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if, by definition, $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$ if and only if, for all $Y \in \mathcal{C}$, $h^{-1}(Y) \in \mathcal{C}^\flat$ if and only if, by definition, for all $Y \in \mathcal{C}$, $\langle \langle \mathbf{A}, h \rangle, Y \rangle$ is an $\mathbb{L}$-matrix. ■

Proposition 57 asserts that the weakest equipotency class of models of $\mathbb{L}$ on $\mathcal{A}$ with respect to the $\trianglelefteq$ relation on equipotency classes of logicates is the one determined by

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}).$$

4.3 Full Models

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, h \rangle$, is called a **full model of** $\mathbb{L}$ or a **full** $\mathbb{L}$**-model** (see Definition 2.8 of [12]) if



$$\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a **basic (full) model of** $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, rephrasing the definition, we may say that $\mathbb{A}$ is a full model of $\mathbb{L}$ if and only if its reduction is a basic full model of $\mathbb{L}$.

$\mathrm{FMod}(\mathbb{L})$ denotes the class of all full models of $\mathbb{L}$. $\mathrm{FMod}^*(\mathbb{L})$ is the class of all reduced full models of $\mathbb{L}$. Given an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$ is the class of all full models of $\mathbb{L}$ on $\mathcal{A}$.

An analog of Part (1) of Proposition 2.9 of [12] provides a justification for the use of the term "model" for full models.

Proposition 58 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and $\mathbb{A} = \langle \mathcal{A}, C \rangle$ a full model of $\mathbb{L}$. Then $\mathbb{A}$ is a model of $\mathbb{L}$.*

Proof: Suppose $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$. By definition, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, by Proposition 57, $\mathbb{A}^*$ is a model of $\mathbb{L}$. Thus, by Corollary 53, $\mathbb{A}$ is also a model of $\mathbb{L}$. ■

The next result, an analog of Proposition 2.10 of [12], asserts that every basic full model is actually a full model, justifying the "full" in the definition of basic (full) models.

Proposition 59 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. A logicate interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ on $\mathcal{A}$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a full model of $\mathbb{L}$ and is among the $\trianglelefteq$-weakest full models of $\mathbb{L}$ on $\mathcal{A}$.*

Proof: By Proposition 24, the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism. By Corollary 38, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$, i.e., $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, $\mathbb{A}$ is a full model of $\mathbb{L}$. It is among the $\trianglelefteq$-weakest full models since it is among the $\trianglelefteq$-weakest models, by Proposition 57. ■

Proposition 2.11 of [12], concerning closure of the class of full models under bilogical morphisms, has the following analog.

Proposition 60 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. The class $\mathrm{FMod}(\mathbb{L})$ is closed under bilogical morphisms, i.e., if $h : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$, where $\mathcal{A} = \langle \mathbf{A}, g \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', h \circ g \rangle$, is a bilogical morphism, then*

$$\langle \mathcal{A}, C \rangle \in \mathrm{FMod}(\mathbb{L}) \quad \textit{iff} \quad \langle \mathcal{A}', C' \rangle \in \mathrm{FMod}(\mathbb{L}).$$

Proof: Suppose $h : \mathbb{A} \to \mathbb{A}'$ is a bilogical morphism. By Proposition 30, there exists an isomorphism $h^* : \mathbb{A}^* \cong \mathbb{A}'^*$. Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. Then $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Thus, by Corollary 39, $\mathcal{C}'^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'^*)$. But $\mathbb{A}'^*$ is reduced, whence $\mathbb{A}'^*$ is a full model of $\mathbb{L}$. A similar reasoning yields the converse. ■

Corollary 61 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Then $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$ if and only if $\mathbb{A}^* \in \mathrm{FMod}(\mathbb{L})$.*

Proof: Directly from Proposition 60, since the natural projection $\pi : \mathbf{A} \to \mathbf{A}^*$ is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$. ■

Corollary 62 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. Then $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$ if and only if there exists a bilogical morphism from $\mathbb{A}$ onto a model $\langle \mathcal{A}', C' \rangle$, with $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$.*

Proof: The "only if" follows directly from the definition of full model, as the projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

Conversely, assume there is a bilogical morphism $h : \mathbb{A} \to_b \mathbb{A}'$, such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By Proposition 59, $\mathbb{A}'$ is a full model, whence, by Proposition 60, $\mathbb{A}$ is also a full model. ∎

Our work culminates in a characterization of the class of full models of a logicate $\mathbb{L}$ along the lines of Corollary 2.13 of [12].

Corollary 63 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. Then* $\mathrm{FMod}(\mathbb{L})$ *is the smallest class containing all* $\langle \mathcal{A}, C \rangle$*, with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, and closed under bilogical morphisms.*

Proof: Let L be the smallest class containing all $\langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, and closed under bilogical morphisms.

On the one hand, every $\langle \mathcal{A}, C \rangle$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a full model, by Proposition 59. Moreover, by Proposition 11, the class of full models is closed under bilogical morphisms. This shows that $\mathsf{L} \subseteq \mathrm{FMod}(\mathbb{L})$. The reverse inclusion is a direct consequence of Corollary 62. ∎

Corollary 63 provides one justification for the "fullness" property of full models. According to this justification, a full model is one that is obtained via a bilogical morphism by a model whose theories constitute a full set of $\mathbb{L}$-filters. A second justification is given in the following theorem. According to this, a model's "fullness" rests on the fact that its theories contain all possible $\mathbb{L}$-filters corresponding to $\mathbb{L}$-filters of the Tarski-reduction of the model.

Theorem 64 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be an algebraic logicate. Then* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *is a full model of* $\mathbb{L}$ *if and only if*

$$\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}.$$

Proof: Suppose, first, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$. Let $X \in \mathcal{C}$. By Proposition 57, $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By definition of $\widetilde{\Omega}(\mathbb{A})$, it always holds that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. For the reverse inclusion, let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. Then, by Proposition 35, there exists $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$, such that $X = \pi^{-1}(Y)$, where $\pi : \mathbf{A} \to \mathbf{A}/\widetilde{\Omega}(\mathbb{A})$ is the natural projection. But, by Proposition 24, $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism and, moreover, since $\mathbb{A}$ is full, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $X \in \mathcal{C}$.

Suppose, conversely, that $\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}$. Since the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $\mathbb{A}$ is a full model of $\mathbb{L}$. ∎

4.4 $\mathbb{L}$-Algebras

Reduced full models of $\mathbb{L}$ are those models of the form $\langle \mathcal{A}, C\rangle$, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, that are reduced. The interpretation reducts of such models are given a special name.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate. An interpretation $\mathcal{A} = \langle \mathbf{A}, h\rangle$ is an **$\mathbb{L}$-algebra** (see Definition 2.16 of [12]) if

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) = \Delta_{\mathbf{A}}.$$

The class of all $\mathbb{L}$-algebras is denoted by $\mathrm{Alg}(\mathbb{L})$. Perhaps, a more suitable term and notation here would have been $\mathbb{L}$-*interpretation* and $\mathrm{Int}(\mathbb{L})$, but we keep the ones used in the traditional theory, even though the entities differ, since they play analogous roles as the $\mathbb{L}$-algebras in the traditional theory.

The following characterization takes after Proposition 2.17 of [12].

Proposition 65 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathbb{A} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, h\rangle$. Then the following statements are equivalent:*

(i) $\mathbb{A} = \langle \mathcal{A}, C\rangle$ *is a reduced full model of* $\mathbb{L}$*;*

(ii) $\mathbb{A} = \langle \mathcal{A}, C\rangle$ *is reduced and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*;*

(iii) $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$ *and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) By the definition of a reduced full model.

(ii)⇒(iii) By the definition of an $\mathbb{L}$-algebra, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$.

(iii)⇒(i) Since $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, there exists $C' : \mathcal{P}(A) \to \mathcal{P}(A)$, such that $\mathbb{A} = \langle \mathcal{A}, C'\rangle$ is a reduced full model of $\mathbb{L}$. But then $\mathcal{C}' = \mathcal{C}$ and $\mathbb{A} = \langle \mathcal{A}, C\rangle$ is a reduced full model of $\mathbb{L}$.

■

Proposition 66 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathbb{A} = \langle \mathcal{A}, C\rangle$ a full model of $\mathbb{L}$. Then $\mathcal{A}^* := \mathcal{A}/\widetilde{\Omega}(\mathbb{A})$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$.*

Proof: By Corollary 61, $\mathbb{A}^*$ is a full model of $\mathbb{L}$ and it is clearly reduced. Hence, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra. This also yields the second statement using the definition of $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. ■

A characterization of $\mathbb{L}$-algebras, an analog of Proposition 2.19 of [12], shows that the notion of model, without reference to fullness, suffices to characterize the class $\mathrm{Alg}(\mathbb{L})$.

Proposition 67 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. The class* $\mathrm{Alg}(\mathbb{L})$ *is the class of algebraic reducts of all reduced models of* $\mathbb{L}$.

Proof: By definition, if $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, then $\mathcal{A}$ is the algebraic reduct of a reduced full model; in particular of a reduced model. Assume, conversely, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a reduced model of $\mathbb{L}$. Let $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$, be such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 57, $\mathbb{A}'$ is a model of $\mathbb{L}$ and, by Proposition 59, it is clearly full. It is also reduced, since

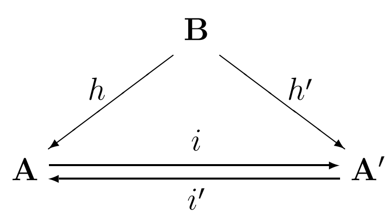
$$\widetilde{\Omega}(\mathbb{A}') \subseteq \widetilde{\Omega}(\mathbb{A}) = \Delta_{\mathbf{A}}.$$

Therefore, by definition, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ■

Closure under isomorphisms is guaranteed by the following proposition, an analog of Proposition 2.20 of [12].

Proposition 68 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. The class* $\mathrm{Alg}(\mathbb{L})$ *is closed under isomorphisms (commuting with the interpretations).*

Proof: Let $i : \mathbf{A} \cong \mathbf{A}'$. We have the following diagram.



Suppose that $\mathcal{A} = \langle \mathbf{A}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for some C, $\langle \mathcal{A}, C \rangle$ is a reduced full model of $\mathbb{L}$. Consider $\mathcal{A}' = \langle \mathbf{A}', h' \rangle = \langle \mathbf{A}', i \circ h \rangle$. We have, $\langle \mathcal{A}', C' \rangle$, with $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, is a reduced full model of $\mathbb{L}$. Thus, $\mathcal{A}' \in \mathrm{Alg}(\mathbb{L})$. The reverse implication can be proved similarly. ■

Putting together several of the previous results, we get the following alternative characterizations of full models involving $\mathbb{L}$-algebras.

Proposition 69 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$, *with* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *an interpretation. Then the following statements are equivalent.*

(i) $\mathbb{A}$ *is a full model of* $\mathbb{L}$*;*

(ii) $\mathcal{A}^*$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$*;*

(iii) There exists a bilogical morphism $g : \mathbb{A} \to \mathbb{A}'$, *with* $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$, *such that* $\mathcal{A}'$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$.

Proof:

(i)⇒(ii) Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. By definition, $\mathbb{A}^*$ is a basic full model of $\mathbb{L}$. Thus, by Proposition 65, $\mathcal{A}^* = \langle \mathbf{A}^*, h^* \rangle$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

(ii)⇒(iii) Assume $\mathcal{A}^* = \langle \mathbf{A}^*, h^* \rangle$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Then (iii) is immediate by considering the natural projection $\pi : \mathbf{A} \to \mathbf{A}^*$, which is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$ and such that $\mathbb{A}^*$ fulfills the required conditions by (ii).

(iii)⇒(i) By Proposition 65, $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$, is a reduced full model of $\mathbb{L}$. Therefore, by Corollary 62, $\mathbb{A}$ is a full model of $\mathbb{L}$.

■

An analog of the Completeness Theorem 2.22 of [12] asserts that the class of full models, the class of reduced full models, as well as the class of all basic full models of a logicate can serve as a complete semantics for the logicate.

Theorem 70 (Completeness) *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete with respect to the following classes of models:*

1. *The class* $\mathrm{FMod}(\mathbb{L})$ *of all full models of* $\mathbb{L}$*;*

2. *The class of all basic full models of* $\mathbb{L}$*;*

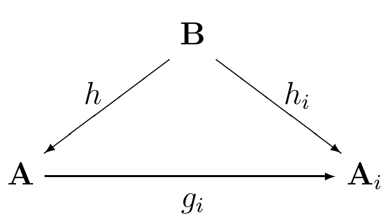
3. *The class* $\mathrm{FMod}^*(\mathbb{L})$ *of all reduced full models of* $\mathbb{L}$*.*

Proof: All three classes consist of models of $\mathbb{L}$. In addition each contains the model $\langle \langle \mathbf{B}^*, \pi \rangle, C^{\flat *} \rangle$, where $\pi : \mathbf{B} \to \mathbf{B}^*$ is the canonical projection. Thus, by Proposition 55, $\mathbb{L}$ is complete with respect to each of these three classes. ■

We now establish an analog of the well known theorem (Theorem 2.23 of [12]) relating the classes $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$. Recall that $\mathrm{Alg}^*(\mathbb{L})$ is the class of all algebraic (interpretation) reducts of reduced matrix models of $\mathbb{L}$, whereas $\mathrm{Alg}(\mathbb{L})$ is the class of all algebraic (interpretation) reducts of reduced full models of $\mathbb{L}$. In the present setting, however, due to the presence of morphisms from the base logicate in the interpretations involved, one has to replace subdirect products by a different operation, named *subdirect intersection*.

Let $\mathcal{A}_i = \langle \mathbf{A}_i, h_i \rangle$, $i \in I$, be a collection of interpretations. We say that an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ is a **subdirect intersection of** the $\mathcal{A}_i$ **relative to B** if:

- There exist surjective homomorphisms $g_i : \mathbf{A} \to \mathbf{A}_i$, $i \in I$, such that the following diagram commutes for all $i \in I$.

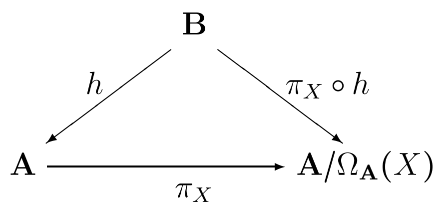


- $\bigcap_{i\in I}\mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$.

Since the role of **B** is going to be played by the base algebra, we usually omit the "relative to **B**" in the terminology.

Theorem 71 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate. The class* $\mathrm{Alg}(\mathbb{L})$ *is the class of all subdirect intersections of interpretations in* $\mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathcal{A} = \langle \mathbf{A}, h\rangle \in \mathrm{Alg}(\mathbb{L})$. Then there exists C, such that $\mathbb{A} = \langle \mathcal{A}, C\rangle$ is a reduced full model of $\mathbb{L}$. For every $X \in \mathcal{C}$, we form the commutative triangle of epimorphisms



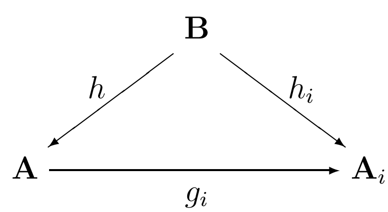
where $\pi_X : \mathbf{A} \to \mathbf{A}/\Omega_{\mathbf{A}}(X)$ denotes the canonical projection. Note that

$$\begin{array}{rcl}\bigcap_{X\in\mathcal{C}}\mathrm{Ker}(\pi_X) & = & \bigcap_{X\in\mathcal{C}}\Omega_{\mathbf{A}}(X) \quad (\pi_X : \mathbf{A} \to \mathbf{A}/\Omega_{\mathbf{A}}(X)) \\ & = & \widetilde{\Omega}(\mathbb{A}) \quad (\text{Definition of } \widetilde{\Omega}(\mathbb{A})) \\ & = & \Delta_{\mathbf{A}}. \quad (\mathbb{A} = \langle \mathcal{A}, C\rangle \text{ reduced})\end{array}$$

Hence **A** is a subdirect intersection of

$$\mathcal{A}/\Omega_{\mathcal{A}}(X) = \langle \mathbf{A}/\Omega_{\mathcal{A}}(X), \pi_X \circ h\rangle \in \mathrm{Alg}^*(\mathbb{L}), \quad X \in \mathcal{C}.$$

Assume, conversely, that $\mathcal{A} = \langle \mathbf{A}, h\rangle$ is a subdirect intersection of a collection $\mathcal{A}_i = \langle \mathbf{A}_i, h_i\rangle \in \mathrm{Alg}^*(\mathbb{L})$, $i \in I$. Then, by hypothesis, we have commutative diagrams of epimorphisms,



such that $\bigcap_{i\in I}\mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$. Moreover, since, for all $i \in I$, $\mathcal{A}_i \in \mathrm{Alg}^*(\mathbb{L})$, there exists $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, such that $\Omega_{\mathcal{A}_i}(X) = \Delta_{\mathbf{A}_i}$. Let $C : \mathcal{P}(A) \to \mathcal{P}(A)$, be such that

$$\mathcal{C} = \{g_i^{-1}(X_i) : i \in I\}$$

and set $\mathbb{A} = \langle \mathcal{A}, C\rangle$. Since $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, $i \in I$, we have, by Proposition 34, that $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover,

$$\begin{array}{rcl}\widetilde{\Omega}(\mathbb{A}) & = & \bigcap_{i\in I}\Omega_{\mathcal{A}}(g_i^{-1}(X_i)) \quad (\text{Definition of } \widetilde{\Omega}(\mathbb{A})) \\ & = & \bigcap_{i\in I} g_i^{-1}(\Omega_{\mathcal{A}_i}(X_i)) \quad (\text{Property of } \Omega) \\ & = & \bigcap_{i\in I} g_i^{-1}(\Delta_{\mathbf{A}_i}) \quad (\Omega_{\mathcal{A}_i}(X) = \Delta_{\mathbf{A}_i}) \\ & = & \bigcap_{i\in I}\mathrm{Ker}(g_i) \quad (\text{Definition of } \mathrm{Ker}(g_i)) \\ & = & \Delta_{\mathbf{A}}. \quad (\text{Assumption})\end{array}$$

Thus, by Proposition 67, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ■

Corollary 72 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. Then* $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$*. Moreover,* $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$ *if and only if* $\mathrm{Alg}^*(\mathbb{L})$ *is closed under subdirect intersections.*

Recall from Chapter 2 the preorder $\trianglelefteq$ on logicates over the same underlying set A. Here, of course, we understand it to be between logicates over the same underlying algebra. We write $\langle \mathbf{A}, C \rangle \trianglelefteq \langle \mathbf{A}, C' \rangle$ to signify that $\mathcal{C}' \subseteq \mathcal{C}$. This becomes a partial order on equipotency classes of logicates (equipotency is the relation of having identical sets of theories).

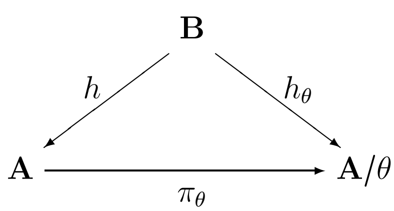
Proposition 73 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *and* $\mathbb{L}' = \langle \mathbf{B}, C'^\flat \rangle$ *be algebraic logicates over the same algebra, such that* $\mathbb{L} \trianglelefteq \mathbb{L}'$*. Then* $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$ *and* $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$*.*

Proof: Suppose $\mathbb{L} \trianglelefteq \mathbb{L}'$. Then, for all interpretations $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}'}(\mathcal{A}) \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$. By Theorem 71, we also have $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$. ■

4.5 The Lattice of Full Models

We fix a base logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ and an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$. Let $\theta \in \mathrm{Con}(\mathbf{A})$. Recall that

$$\mathcal{A}/\theta = \langle \mathbf{A}/\theta, h_\theta \rangle,$$



where $h_\theta = \pi_\theta \circ h$, with $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ being the canonical projection. Consider the equipotency class of algebraic logicates (here meant over the same fixed interpretation $\mathcal{A}$) represented by $\langle \mathcal{A}/\theta, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Define

$$\widetilde{h}_{\mathcal{A}}(\theta) \coloneqq \langle \mathcal{A}, C_\theta \rangle,$$

where $\langle \mathcal{A}, C_\theta \rangle$ is the algebraic logicate induced by $\langle \langle \mathbf{A}/\theta, \pi_\theta \rangle, C \rangle$ on $\mathbf{A}$ (viewing $\mathbf{A}$ as the base algebra momentarily). The induced logicate $\langle \mathcal{A}, C_\theta \rangle$ depends on the choice of representative $\langle \mathcal{A}/\theta, C \rangle$ from the equipotency class on $\mathcal{A}/\theta$. However, any two representatives result in equipotent logicates on $\mathcal{A}$. So, it makes sense to define the function

$$\begin{array}{rrcl} \widetilde{H}_{\mathcal{A}}(\theta) : & \mathrm{Con}(\mathbf{A}) & \longrightarrow & \mathrm{Log}(\mathcal{A})/\triangleq; \\ & \theta & \longmapsto & \langle \mathcal{A}, C_\theta \rangle/\triangleq. \end{array}$$

Note that, restricting to representatives, we have, by Proposition 49, that

$$\pi_\theta : \widetilde{h}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C\rangle$$

is a bilogical morphism.

Lemma 74 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ *be an algebraic logicate,* $\mathcal{A} = \langle \mathbf{A}, h\rangle$ *a fixed interpretation and* $\theta \in \mathrm{Con}(\mathbf{A})$.

(a) $\theta \in \mathrm{Con}(\widetilde{h}_{\mathcal{A}}(\theta))$;

(b) $\widetilde{h}_{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}/\theta, C\rangle$;

(c) $\widetilde{h}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$;

(d) $\theta \mapsto \widetilde{H}_{\mathcal{A}}(\theta)$ *is order preserving, i.e., if* $\theta \subseteq \theta'$, *then* $\widetilde{H}_{\mathcal{A}}(\theta) \trianglelefteq \widetilde{H}_{\mathcal{A}}(\theta')$.

Proof:

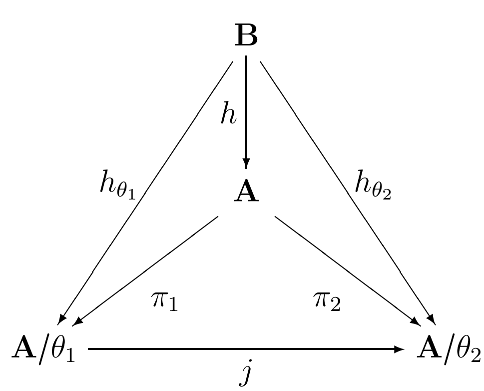
(a) By Proposition 49, $\pi_\theta : \widetilde{h}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C\rangle$ is a bilogical morphism. By Proposition 20, $\theta \in \mathrm{Con}(\widetilde{h}_{\mathcal{A}}(\theta))$.

(b) We have

$$\pi_\theta(C_\theta(X)) = \pi_\theta(\pi_\theta^{-1}(C(\pi_\theta(X)))) = C(\pi_\theta(X)).$$

(c) By hypothesis, $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Thus, by Proposition 59, $\langle \mathcal{A}/\theta, C\rangle$ is a full model of $\mathbb{L}$. Thus, by Proposition 60, $\widetilde{h}_{\mathcal{A}}(\theta)$ is also a full model of $\mathbb{L}$.

(d) Let $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$, such that $\theta_1 \subseteq \theta_2$. Let $\pi_1 : \mathcal{A} \to \mathcal{A}/\theta_1$ and $\pi_2 : \mathcal{A} \to \mathcal{A}/\theta_2$ be the canonical projections. Let, also, $j : \mathcal{A}/\theta_1 \to \mathcal{A}/\theta_2$ be the map given by $a/\theta_1 \mapsto a/\theta_2$, which is well defined due to the inclusion $\theta_1 \subseteq \theta_2$. In addition, we have the following commutative diagram.



Now we get

$$\begin{array}{rcll} \mathcal{C}_{\theta_2} & = & \pi_2^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2)) & (\pi_2 : \widetilde{h}_{\mathcal{A}}(\theta_2) \to_b \langle \mathcal{A}/\theta_2, C_2\rangle) \\ & = & \pi_1^{-1}(j^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2))) & (\pi_2 = j \circ \pi_1) \\ & \subseteq & \pi_1^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_1)) & (\text{Proposition 34}) \\ & = & \mathcal{C}_{\theta_1}. & (\pi_1 : \widetilde{h}_{\mathcal{A}}(\theta_1) \to_b \langle \mathcal{A}/\theta_1, C_1\rangle) \end{array}$$

This shows that $\widetilde{H}_{\mathcal{A}}(\theta_1) \le \widetilde{H}_{\mathcal{A}}(\theta_2)$.

■

Now we are in a position to prove a general analog of the Isomorphism Theorem of Font and Jansana (Theorem 2.30 of [12]) which is applicable even in contexts involving non-monotonicity. Of course, one has to pay the price that, instead of individual full models, equipotency classes of full models are considered. This has the undesirable effect that consequence is taken into account only as a determinator of the set of theories. On the other hand, this is to be expected and it creates an advantage. It is to be expected, since the Tarski operator is constant on equipotency classes and, hence, only restricting to equipotency classes is there a possibility of it becoming injective. The advantage created is that the result becomes applicable more widely, while its restriction to closure operators (i.e., inflationary and monotone consequence operators) is an Isomorphism Theorem very much resembling the one of Font and Jansana modulo the introduction of fixed interpretations. In this latter respect, we follow more closely the generalization of Font and Jansana's result that was presented as Theorem 13 of [21] for logical systems formalized as π-institutions.

Theorem 75 (Isomorphism) *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ a fixed interpretation. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is an order isomorphism between the ordered set $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ of equipotency classes of full models of $\mathbb{L}$ and the ordered set $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ of $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$, ordered under inclusion. Moreover, the mapping $\widetilde{H}_{\mathcal{A}}$ is its inverse.*

Proof: By Proposition 66, if $\mathbb{A} \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$, then $\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By Lemma 74, if $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$, then $\widetilde{H}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$. So it suffices to show that $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse mappings and that they are both order preserving.

Let $\mathbb{A} = \langle \mathcal{A}, C\rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. By Proposition 66, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}_{\mathcal{A}}(C) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. As $\mathbb{A}$ is induced by its reduction $\mathbb{A}^* = \langle \mathcal{A}^*, C^*\rangle$, with $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, along the natural projection $\pi : \mathcal{A} \to \mathcal{A}^*$, we get, by definition, that $\mathbb{A}/\triangleq = \widetilde{H}_{\mathcal{A}}(\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}))$.

Suppose, conversely, that $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Consider $\mathbb{A}^\theta = \langle \mathcal{A}^\theta, C\rangle$, where $\mathcal{A}^\theta = \langle \mathbf{A}/\theta, \pi_\theta \circ h\rangle$ and $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^\theta)$. Then, since $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$, $\widetilde{\Omega}_{\mathcal{A}^\theta}(\mathcal{C}) =$

$\Delta_{\mathbf{A}/\theta}$. Thus, by Proposition 34,

$$\begin{array}{rcll}
\widetilde{\Omega}_{\mathcal{A}}(\widetilde{H}_{\mathcal{A}}(\theta)) & = & \widetilde{\Omega}_{\mathcal{A}}(\mathcal{C}_\theta) & (\text{Definition of } \widetilde{H}_{\mathcal{A}}(\theta)) \\
& = & \widetilde{\Omega}_{\mathcal{A}}(\pi_\theta^{-1}(\mathcal{C})) & (\text{Definition of } \mathcal{C}_\theta) \\
& = & \pi_\theta^{-1}(\widetilde{\Omega}_{\mathcal{A}^\theta}(\mathcal{C})) & (\text{Proposition 22}) \\
& = & \pi_\theta^{-1}(\Delta_{\mathbf{A}/\theta}) & (\widetilde{\Omega}_{\mathcal{A}^\theta}(\mathcal{C}) = \Delta_{\mathbf{A}/\theta}) \\
& = & \theta. & (\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta \text{ natural projection})
\end{array}$$

Hence, $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse bijections. $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving by definition. Finally, by Lemma 74, $\widetilde{H}_{\mathcal{A}}$ is also order preserving. This shows that

$$\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq \rangle \underset{\widetilde{H}_{\mathcal{A}}}{\overset{\widetilde{\Omega}_{\mathcal{A}}}{\rightleftarrows}} \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle$$

are inverse order isomorphisms. ■

The structure of $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle$ is not very hard to obtain. This is done in the following analog of Theorem 2.31 of [12].

Theorem 76 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation. Then $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle$ is a complete lattice with meet coinciding with intersection.*

Proof: Let $\varnothing \neq \{\theta_i : i \in I\} \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Set $\theta = \bigcap_{i \in I} \theta_i$. We must show that $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Note that, for all $a \in A$,

$$a/\theta = a/\left(\bigcap_{i\in I} \theta_i\right) = \bigcap_{i \in I} a/\theta_i.$$

Let $h_i : \mathcal{A}/\theta \to \mathcal{A}/\theta_i$ be the mapping $a/\theta \mapsto a/\theta_i$. By hypothesis, $\mathbb{A}_i = \langle \mathcal{A}/\theta_i, C_i \rangle$, with $\mathcal{C}_i = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_i)$, is reduced. We must show that $\mathbb{A} = \langle \mathcal{A}/\theta, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$, is also reduced. By Proposition 34, $h_i^{-1}(\mathcal{C}_i) \subseteq \mathcal{C}$, for all $i \in I$. Hence, for all $i \in I$,

$$\begin{array}{rcl}
\widetilde{\Omega}(\mathbb{A}) & \subseteq & \widetilde{\Omega}_{\mathcal{A}/\theta}(h_i^{-1}(\mathcal{C}_i)) \\
& = & h_i^{-1}(\widetilde{\Omega}_{\mathcal{A}/\theta_i}(\mathcal{C}_i)) \\
& = & h_i^{-1}(\Delta_{\mathbf{A}/\theta_i}).
\end{array}$$

Thus, $\langle a, b \rangle \in \widetilde{\Omega}(\mathbb{A})$ implies $h_i(a) = h_i(b)$, for all $i \in I$, whence, $a/\theta = b/\theta$. So $\mathbb{A}$ is reduced and $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$.

We still need to investigate what happens when we take the collection of congruences to be empty. We then get $\bigcap \varnothing = \nabla_{\mathbf{A}}$. Thus, $\mathcal{A}/\theta = \langle \mathbf{1}, 1 \rangle$, where $\mathbf{1}$ is the trivial one-element algebra and $1 : \mathbf{B} \to \mathbf{1}$ denotes the only available homomorphism. In this case, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ can be any of the sets $\{\varnothing\}$, $\{A\}$ or $\{\varnothing, A\}$ depending on the theories of $\mathbb{L}$. In all three cases, $\langle \mathcal{A}, C \rangle$, with

$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is reduced. Hence, it is a full model of $\mathbb{L}$ on $\mathcal{A}$ and, therefore, $\nabla^{\mathbf{A}} \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. ∎

Based on the Isomorphism Theorem 75, Theorem 76 helps us determine the structure of the partially ordered set $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$.

Corollary 77 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate and $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation. The ordered set $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ is a complete lattice and the Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is a lattice isomorphism from $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ onto $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$.*

Proof: This is an immediate consequence of the Theorem 75. ∎

Note that, via the isomorphism established in Theorem 75, we can say that, given a collection of equipotency classes $\{\mathbb{A}_i/\triangleq : i \in I\}$ of full models on $\mathcal{A}$, the meet is obtained by first taking the $\mathrm{Alg}(\mathbb{L})$-congruence $\theta = \bigcap_{i\in I} \widetilde{\Omega}(\mathbb{A}_i)$ and, then constructing the class represented by the full model induced by $\langle \mathcal{A}/\theta, C\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$, along the canonical projection $\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta$.

Proposition 78 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, $\mathbb{A} = \langle \mathcal{A}, C\rangle$ and $\mathbb{A}' = \langle \mathcal{A}', C'\rangle$ be two full models of $\mathbb{L}$, with $h : \mathbb{A} \to \mathbb{A}'$ a bilogical morphism between them. Then*

$$\mathbb{B} \stackrel{h}{\longmapsto} \mathbb{B}'$$

establishes an isomorphism between the lattice of equipotency classes of all full models of $\mathbb{L}$ on $\mathcal{A}$ $\trianglelefteq$-extending $\mathbb{A}$ and the lattice of equipotency classes of all full models of $\mathbb{L}$ on $\mathcal{A}'$ $\trianglelefteq$-extending $\mathbb{A}'$. Moreover, the principal filters of $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}')$ determined by $\widetilde{\Omega}(\mathbb{A})$ and $\widetilde{\Omega}(\mathbb{A}')$, respectively, are isomorphic.

Proof: By Proposition 21, h establishes an isomorphism between the lattice of equivalence classes of logicates on $\mathcal{A}$ extending $\mathbb{A}$ and the lattice of equivalence classes of logicates on $\mathcal{A}'$ extending $\mathbb{A}'$. The mapping establishes a bilogical morphism between representatives of each of the corresponding classes. Hence, by Proposition 60, one of those is full if and only if the other is. ∎

Corollary 79 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, $\mathcal{A} = \langle \mathbf{A}, h\rangle$ an interpretation and $g : \mathbf{A} \to \mathbf{A}'$ and epimorphism, such that $g : \langle \mathcal{A}, C\rangle \to \langle\langle \mathbf{A}, g \circ h\rangle, C'\rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\langle \mathbf{A}, g \circ h\rangle)$, is a bilogical morphism. Then g induces an isomorphism between the complete lattices $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$ and $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A}')/\triangleq, \trianglelefteq\rangle$, where $\mathcal{A}' = \langle \mathbf{A}, g \circ h\rangle$. Further, the complete lattices $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ and $\langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}'), \subseteq\rangle$ are isomorphic.*

Proof: By Proposition 59, for any interpretation $\mathcal{A}$, the equipotency class of $\langle \mathcal{A}, C\rangle$ is the least element in $\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq, \trianglelefteq\rangle$. Thus, the result follows by combining Proposition 36 and Proposition 78. ∎

Chapter 5

Aspects of the Hierarchy

5.1 Introduction

One of the main achievements of the abstract theory of Algebraic Logic is the classification of logics in an algebraic hierarchy. A logic is represented by a structural closure operator on the algebra of terms (or formulas) over an algebraic type generated by countably many variables. The theory prescribes a method (or, rather, methods) one may follow to select a particular class of algebras over the same signature as the logic to associate with the logic. The higher the logic is classified in the hierarchy, the closer the ties between the logic and its associated class of algebras. Because of their clarity and comprehensiveness, but, also because they were written by pioneers, two monographs [3, 12], a survey [14], a book [8] and a textbook [10] have been used for many years as guides in being introduced to, in understanding and in delving deeper into the theory.

Since the *algebraic hierarchy* is one of the crowns (and jewels) of the traditional theory, it is only fair to, at least start to, investigate and give a first idea of how one could attempt to keep alive aspects of the theory in a rougher terrain. This is the effort we expend in the present and last chapter of Part I.

Among the major, perhaps most important, classes in the traditional hierarchy are protoalgebraic logics [2] (see, also, [8, 14, 10]). These are the logics in which, roughly speaking, indistinguishability modulo a theory implies interderivability modulo the theory. Another important characterization asserts that they are the logics on whose lattices of theories, the Leibniz operator is monotone. In Section 5.2, we use the definition from the classical framework to define *protoalgebraic logicates* and try to establish some equivalent conditions, some with and some without extra assumptions.

One of the key consequences of protoalgebraicity, which forms an important feature in their study, is the so-called *Correspondence Theorem*. This result is partly the reason why Blok and Pigozzi declared that protoalgebraic logics form the widest class of logics amenable to algebraic techniques of study, even though they are not "algebraizable", i.e., do not belong to the highest step in the hierarchy but are, rather, located near the bottom. The Correspondence Theorem establishes an isomorphism between the lattice of filters of the logic on a given algebra including a fixed filter and the lattice of filters on the quotient algebra, formed by dividing out by the Leibniz congruence of the fixed filter including the quotient of the fixed filter. We discuss this result and some of its consequences in Section 5.3. Again our focus remains to safeguard some of the result from the traditional theory, with or without provisos, in this less robust environment.

As is the case in the traditional theory [12], and as was shown provisionally in Chapter 4, full models play a key role in the investigation of the logical structure. In the context of protoalgebraic logics, full models are inextricably connected to, so-called, *Leibniz filters* [12, 17]. So, in Section

5.4, we investigate the relation between full models and Leibniz filters in the context of protoalgebraic logicates. Their study naturally segues into the study of *weakly algebraizable logicates* in Section 5.5. These are defined by analogy with the corresponding class in the monotonic framework [9] (see, also, [8, 14, 10]). Several characterizations paralleling the ones from the traditional setting are provided, but, generally, they require the additional hypothesis that the set of theories be closed under intersection.

As is well known in the ordinary setting, weak algebraizability [9] results by simultaneously insisting that a logic be protoalgebraic [2] and truth equational [19]. So in the last section, Section 5.6, we use the original definition to identify the class of *truth equational logicates*. They are characterized by the Leibniz operator on their theories being monotone and injective. Again assuming closure of the set of theories under intersection, we prove that, for logicates also, weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

5.2 Protoalgebraic Logicates

Recall that an algebraic logicate $\langle \mathbf{A}, C\rangle$ consists of an algebra $\mathbf{A}$ and an idempotent operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Recall, also, that we use $\mathcal{C}$ to denote the collection of all fixed points or theories of C.

In the abstract study of logicates, a particular fixed logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ is at the focus of investigations and it is called the *base logicate*. Both *matrix* (Chapter 3) and *logicate* (Chapter 4) *models* of the base logicate are based on *interpretations* $\mathcal{A} = \langle \mathbf{A}, h\rangle$, which are epimorphisms from the base algebra $\mathbf{B}$ onto a similar algebra $\mathbf{A}$.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be an algebraic logicate. We say that $\mathbb{L}$ is **protoalgebraic** (see [2] and, also, [8, 12, 10]) if, for all $a, b \in B$ and all $X \in \mathcal{C}^\flat$,

$$\langle a, b\rangle \in \Omega_{\mathbf{B}}(X) \quad \text{implies,} \quad \begin{array}{l} \text{for all } X \subseteq X' \in \mathcal{C}^\flat, \\ a \in X' \text{ iff } b \in X'. \end{array}$$

We make an observation and then introduce some notation that will help abbreviate the definition.

Observe that protoalgebraicity depends only on the collection of theories of a logicate. This is commensurate with the monotonic theory, where protoalgebraicity depends only on the theory lattice of the sentential logic, even though, in contrast with the present framework, in the monotonic framework the theory lattice fully determines the logic itself. Thus, if two logicates are equipotent (see Chapter 2), then they are either both protoalgebraic or none of the two is. This implies that protoalgebraicity may be viewed, without harm, as a property of equipotency classes instead of individual logicates. We may implicitly use this fact whenever convenient.

Given a logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ and $X \subseteq B$, we define a **logical indistinguishability relation** $\Lambda_{\mathbb{L}}(X)$ on B with the goal of capturing the defining property of protoalgebraicity. We set, for all $X \subseteq B$ and all $a, b \in B$,

$$\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X) \quad \text{iff,} \quad \begin{array}{l} \text{for all } X \subseteq X' \in \mathcal{C}^\flat, \\ a \in X' \text{ iff } b \in X'. \end{array}$$

With this definition available, we may rephrase the definition of protoalgebraicity. Clearly, $\mathbb{L}$ is **protoalgebraic** if and only if, for all $X \in \mathcal{C}^\flat$,

$$\Omega_{\mathbf{B}}(X) \subseteq \Lambda_{\mathbb{L}}(X).$$

It is well known that, in the traditional framework, protoalgebraicity is tantamount to the monotonicity of the Leibniz operator (see [3]) on the lattice of theories of the logic (see, e.g., [8, 12, 10]). The following proposition revisits this characterization in the context of logicates.

Proposition 80 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. $\mathbb{L}$ is protoalgebraic if and only if $\Omega_{\mathbf{B}}$ is monotone on $\mathcal{C}^\flat$.*

Proof: Suppose $\mathbb{L}$ is protoalgebraic and let $X, X' \in \mathcal{C}^\flat$, such that $X \subseteq X'$. Let $a, b \in B$, such that $\langle a, b \rangle \in \Omega_{\mathbf{B}}(X)$ and $a \in X'$. By protoalgebraicity, $\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X)$ and $a \in X'$. Since $X \subseteq X'$, $b \in X'$. This shows that $\Omega_{\mathbf{B}}(X)$ is compatible with X'. By the maximality property of $\Omega_{\mathbf{B}}(X')$ with respect to compatibility with X', we conclude that $\Omega_{\mathbf{B}}(X) \subseteq \Omega_{\mathbf{B}}(X')$. Thus, $\Omega_{\mathbf{B}}$ is monotone on $\mathcal{C}^\flat$.

Suppose, conversely, that $\Omega_{\mathbf{B}}$ is monotone on $\mathcal{C}^\flat$. Let $a, b \in B$, $X \in \mathcal{C}^\flat$, such that $\langle a, b \rangle \in \Omega_{\mathbf{B}}(X)$ and $X' \in \mathcal{C}^\flat$, with $X \subseteq X'$. Then $\langle a, b \rangle \in \Omega_{\mathbf{B}}(X')$. So, by the compatibility of $\Omega_{\mathbf{B}}(X')$ with X', $a \in X'$ iff $b \in X'$. Thus, $\mathbb{L}$ is protoalgebraic. ■

The characterization extends to the monotonicity of the Leibniz operator on the $\mathbb{L}$-filters of any interpretation. This is one of many results of this type in the abstract theory of Algebraic Logic. They are collectively known as "transfer theorems" and assert that a certain property that holds on the theories of a logic, also holds, more generally, on the filters of the logic on any interpretation structure.

Proposition 81 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is protoalgebraic if and only if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_{\mathcal{A}}$ is monotone on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: Suppose that $\mathbb{L}$ is protoalgebraic. Let $Y_1, Y_2 \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $Y_1 \subseteq Y_2$, and $a_1, a_2 \in A$. Note that, since $h : \mathbf{B} \to \mathbf{A}$ is surjective, there exist

$b_1, b_2 \in B$, such that $a_1 = h(b_1)$ and $a_2 = h(b_2)$. Now we have

$$\begin{array}{rcl}\langle a_1, a_2\rangle \in \Omega_{\mathcal{A}}(Y_1) & \text{iff} & \langle h(b_1), h(b_2)\rangle \in \Omega_{\mathcal{A}}(Y_1)\\ & \text{iff} & \langle b_1, b_2\rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_1))\\ & \text{iff} & \langle b_1, b_2\rangle \in \Omega_{\mathbf{B}}(h^{-1}(Y_1))\\ & \text{implies} & \langle b_1, b_2\rangle \in \Omega_{\mathbf{B}}(h^{-1}(Y_2))\\ & \text{iff} & \langle b_1, b_2\rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_2))\\ & \text{iff} & \langle h(b_1), h(b_2)\rangle \in \Omega_{\mathcal{A}}(Y_2)\\ & \text{iff} & \langle a_1, a_2\rangle \in \Omega_{\mathcal{A}}(Y_2).\end{array}$$

Note that we have used the well-known property that the Leibniz operator commutes with inverse surjective homomorphisms. Even though this fact was not proven here, the proof closely parallels the one included in showing that the Tarski operator satisfies the same property in Proposition 22. We have now shown that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, if the condition in the statement holds, then the monotonicity of $\Omega_{\mathbf{B}}$ on $\mathcal{C}^\flat$ follows by taking $\mathcal{A} = \langle \mathbf{B}, i_{\mathbf{B}}\rangle$. Then the conclusion follows from Proposition 80 and the observation that $\mathcal{C}^\flat = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. ■

One may also devise a slightly different characterization in the special case in which the set of theories is closed under intersections. We show, first, that, in this case, the collection of all $\mathbb{L}$-filters on any interpretation is also closed under intersections.

Lemma 82 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate, such that its set $\mathcal{C}^\flat$ of theories is closed under arbitrary (respectively, nonempty, binary) intersections. Then, for any interpretation $\mathcal{A} = \langle \mathbf{A}, h\rangle$, the set $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ of $\mathbb{L}$-filters on $\mathcal{A}$ is also closed under arbitrary (respectively, nonempty, binary) intersections.*

Proof: All three statements are proven similarly. E.g., for the first one, consider an interpretation $\mathcal{A} = \langle \mathbf{A}, h\rangle$ and let $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ be a collection of $\mathbb{L}$-filters on $\mathcal{A}$. Then

$$h^{-1}\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} h^{-1}(X_i) \in \mathcal{C}^\flat,$$

where membership follows by the definition of $\mathbb{L}$-filter and the hypothesis. This shows that $\bigcap_{i\in I} X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. ■

Now we show that, if the set of theories of the base logicate is closed under intersections, then protoalgebraicity is equivalent to the property that the Leibniz operator on the filters of any interpretation commutes with arbitrary intersections.

Proposition 83 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat\rangle$ be a base logicate.*

(i) *If the set* $\mathcal{C}^\flat$ *of theories is closed under arbitrary intersections and* $\mathbb{L}$ *is protoalgebraic, then, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_{\mathcal{A}}$ *commutes with arbitrary intersections on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, *i.e., for all* $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\Omega_{\mathcal{A}}\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i).$$

(ii) *If the set* $\mathcal{C}^\flat$ *of theories is closed under binary intersection and, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_{\mathcal{A}}$ *commutes with binary intersections on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, *then* $\mathbb{L}$ *is protoalgebraic.*

Proof: By Proposition 81, protoalgebraicity is equivalent to the monotonicity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$.

(i) Suppose, first, that $\mathcal{C}^\flat$ is closed under arbitrary intersection and $\Omega_{\mathcal{A}}$ is monotone. Let $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Lemma 82, $\bigcap_{i\in I} X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By monotonicity, $\Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i) \subseteq \Omega_{\mathcal{A}}(X_i)$, for all $i \in I$. Thus, $\Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i)$. On the other hand, $\bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i)$ is compatible with $\bigcap_{i\in I} X_i$. Thus, by the maximality property of $\Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i)$ with respect to compatibility with $\bigcap_{i\in I} X_i$, we obtain $\bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i) \subseteq \Omega_{\mathcal{A}}(\bigcap_{i\in I} X_i)$.

(ii) Suppose, next, that $\mathcal{C}^\flat$ is closed under binary intersection and, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_{\mathcal{A}}$ commutes with binary intersections on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \subseteq X'$. Then $X \cap X' = X$ and we have

$$\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X \cap X') = \Omega_{\mathcal{A}}(X) \cap \Omega_{\mathcal{A}}(X').$$

Therefore, $\Omega_{\mathcal{A}}(X) \subseteq \Omega_{\mathcal{A}}(X')$, showing that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and, hence, $\mathbb{L}$ is protoalgebraic.

■

Corollary 84 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate, such that its set* $\mathcal{C}^\flat$ *of theories is closed under arbitrary intersection.* $\mathbb{L}$ *is protoalgebraic if and only if, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\Omega_{\mathcal{A}}$ *commutes with arbitrary intersections on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

5.3 Correspondence Theorem

Given a logicate $\mathbb{L}$, an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ and a filter $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we write

$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F := \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : F \subseteq X\}.$$

Parts of the well-known Correspondence Theorem for protoalgebraic logics (see, e.g., Theorem 6.19 of [10]) may also be retained in the present context, since they are concerned solely with the structure of theories and filters. Note, however, that what is an isomorphism between complete lattices in the monotonic framework, has to be replaced here by, merely, an isomorphism between ordered sets.

Proposition 85 (Correspondence) *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. If* $\mathbb{L}$ *is protoalgebraic, then, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *and every* $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, letting* $\pi : \mathbf{A} \to \mathbf{A}/\Omega_{\mathcal{A}}(F)$ *be the natural projection,*

$$\begin{array}{rrcl} \pi : & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F & \longrightarrow & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}; \\ & X & \longmapsto & \pi(X), \end{array}$$

establishes an isomorphism between the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \subseteq \rangle$ *and the ordered set* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \subseteq \rangle$*.*

Proof: Let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $F \subseteq X$. By protoalgebraicity, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(X)$. Hence $\Omega_{\mathcal{A}}(F)$ is compatible with X. It follows that, for $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$, $X = \pi^{-1}(\pi(X))$. Thus, by Proposition 34, we obtain $\pi(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Clearly, since $F \subseteq X$, $\pi(F) \subseteq \pi(X)$. On the other hand, if $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, then, again by Proposition 34, $\pi^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover, $\pi(F) \subseteq Y$ implies $F = \pi^{-1}(\pi(F)) \subseteq \pi^{-1}(Y)$. Thus, π establishes an isomorphism between the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \subseteq \rangle$ and the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \subseteq \rangle$, as claimed. ■

Note that, for any algebra $\mathbf{A}$, we have $\Omega_{\mathbf{A}}(\varnothing) = \nabla_{\mathbf{A}}$. Thus, by the definition of protoalgebraicity, the only protoalgebraic logicates $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ for which $\varnothing$ is a theory are the ones with $\mathcal{C}^\flat = \{\varnothing\}$ or $\mathcal{C}^\flat = \{\varnothing, B\}$.

We now provide some additional characterizations of protoalgebraicity in terms of the Tarski operator. Given an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, a logicate $\mathbb{A} = \langle \mathcal{A}, C \rangle$ and a theory F of C, we shall write $\mathbb{A}^F = \langle \mathcal{A}, C^F \rangle$ for a logicate with

$$\mathcal{C}^F = \{X \in \mathcal{C} : F \subseteq X\},$$

assuming that only the theories matter and that the exact consequence structure is irrelevant, i.e., thinking of $\mathbb{A}^F$ as a representative of its equipotency class.

Proposition 86 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate. The following statements are equivalent.*

(i) $\mathbb{L}$ *is protoalgebraic;*

(ii) For any $\mathbb{L}$*-model* $\mathbb{A} = \langle \mathcal{A}, C \rangle$*, if* $\mathcal{C}$ *has a minimum element, then* $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$*;*

(iii) For any $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $Y \in \mathcal{C}$, $\widetilde{\Omega}(\mathbb{A}^Y) = \Omega_{\mathcal{A}}(Y)$;

(iv) For any $X \in \mathcal{C}^\flat$, $\widetilde{\Omega}(\mathbb{L}^X) = \Omega_{\mathbf{B}}(X)$.

Proof:

(i)⇒(ii) Suppose $\mathbb{A} \in \mathrm{Mod}(\mathbb{L})$. Then, by Proposition 57, $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 81, $\Omega_{\mathcal{A}}$ is order preserving on $\mathcal{C}$. So we get

$$\widetilde{\Omega}(\mathbb{A}) = \bigcap_{X \in \mathcal{C}} \Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(\min \mathcal{C}).$$

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Trivial.

(iv)⇒(i) Let $X, X' \in \mathcal{C}^\flat$, such that $X \subseteq X'$. Then $X' \in \mathcal{C}^{\flat X}$. Thus, we get

$$\begin{array}{rcll} \Omega_{\mathbf{B}}(X) & = & \widetilde{\Omega}(\mathbb{L}^X) & \text{(Hypothesis (iv))} \\ & = & \bigcap_{Y \in \mathcal{C}^{\flat X}} \Omega_{\mathbf{B}}(Y) & \text{(Tarski Congruence)} \\ & \subseteq & \Omega_{\mathbf{B}}(X'). & (X' \in \mathcal{C}^{\flat X}) \end{array}$$

So $\Omega_{\mathbf{B}}$ is monotone on $\mathcal{C}^\flat$, showing that $\mathbb{L}$ is protoalgebraic.

■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate and consider an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$. If $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ has a minimum element, which, e.g., is the case when $\mathcal{C}^\flat$ is closed under intersections, then, by Proposition 86, for $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})).$$

The following proposition is an analog of Proposition 3.2 of [12] in the present setting.

Proposition 87 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate, such that, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ has a minimum element. Then*

$$\mathrm{Alg}(\mathbb{L}) = \mathrm{Alg}^*(\mathbb{L}).$$

Proof: By Corollary 72, we have $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$, without any preconditions. Suppose, conversely, that $\mathcal{A} = \langle \mathbf{A}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $\widetilde{\Omega}(\langle \mathcal{A}, C \rangle) = \Delta_{\mathbf{A}}$. By hypothesis and Proposition 86,

$$\Omega_{\mathcal{A}}(\min \mathcal{C}) = \widetilde{\Omega}(\langle \mathcal{A}, C \rangle) = \Delta_{\mathbf{A}}.$$

This shows that $\mathbf{A} \in \mathrm{Alg}^*(\mathbb{L})$. ■

Lemma 3.3 of [12] asserts that two full models of a protoalgebraic logic that share the same sets of theorems are identical. The following analog requires the two logicate models compared to share the same minimum theories and, in that case, asserts that the logicates in question must be equipotent.

Lemma 88 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a protoalgebraic logicate and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$ *two full models of* $\mathbb{L}$ *over the same interpretation that have minimum theories* X_0, X_0', *respectively. If* $X_0 = X_0'$, *then* $\mathcal{C} = \mathcal{C}'$.

Proof: By hypothesis and Proposition 86,

$$\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(X_0) = \Omega_{\mathcal{A}}(X_0') = \widetilde{\Omega}(\mathbb{A}').$$

Thus, by the Isomorphism Theorem 75, $\mathbb{A}$ and $\mathbb{A}'$ are in the same equipotency class, i.e., $\mathcal{C} = \mathcal{C}'$. ■

From the previous few results, it has become apparent that full models that have minimum theories play a somewhat important role. So one may use, if needed, special notation, such as $\mathrm{FMod}^m(\mathbb{L})$ for the class of all full models of $\mathbb{L}$ with a minimum theory and $\mathrm{FMod}^m_{\mathbb{L}}(\mathcal{A})$ for the class of all full models of $\mathbb{L}$ on $\mathcal{A}$ with a minimum theory.

Protoalgebraicity in the monotonic theory was characterized in terms of full models in Theorem 3.4 of [12]. A similar characterization is possible here, provided that all full models of a logicate have minimum theories.

Theorem 89 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a protoalgebraic logicate all of whose full models have a minimum theory. Then* $\mathbb{L}$ *is protoalgebraic if and only if all full models of* $\mathbb{L}$ *are of the form* $\langle \mathcal{A}, C^F \rangle$, *with* $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, *for some interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$ *and some* $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: We work, first, to prove the "only if". Let $\mathbb{A} = \langle \mathcal{A}, C \rangle$ be a full model of $\mathbb{L}$, with $F = \min \mathcal{C}$. By protoalgebraicity and Proposition 86, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(F)$. Hence, the natural projection $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \langle \mathcal{A}, C \rangle \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), C^{\Omega_{\mathcal{A}}(F)} \rangle.$$

By hypothesis, $\mathbb{A}$ is a full model of $\mathbb{L}$. It follows that $\mathcal{C}^{\Omega_{\mathcal{A}}(F)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Consider $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Then $F \subseteq X$ and, by protoalgebraicity, $\Omega_{\mathcal{A}}(F)$ is compatible with X. Thus, $X = \pi^{-1}(\pi(X))$. By Proposition 34, $\pi(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Hence, since $X = \pi^{-1}(\pi(X))$, $X \in \mathcal{C}$. This proves that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$.

We turn, next, to the "if". Suppose that all models of $\mathbb{L}$ have the indicated form. Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \subseteq X'$. Consider $\Omega_{\mathcal{A}}(X)$. By Corollary 72, $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$. Hence $\Omega_{\mathcal{A}}(X) \in \mathrm{Alg}(\mathbb{L})$. By the Isomorphism Theorem 75, there exists a full model $\mathbb{A} = \langle \mathcal{A}, C \rangle$ of $\mathbb{L}$, such that $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(X)$. Moreover, by hypothesis, there exists $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Since $\mathbb{A}$ is full, the natural projection $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(X)$ is a bilogical morphism

$$\pi : \mathbb{A} \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(X), \mathcal{C}^{\Omega_{\mathcal{A}}(X)} \rangle,$$

where $\mathcal{C}^{\Omega_{\mathcal{A}}(X)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(X))$. Moreover, as $X = \pi^{-1}(\pi(X))$, we get $X \in \mathcal{C}$. Hence, $F \subseteq X \subseteq X'$, whence, $X' \in \mathcal{C}$. Now we get

$$\Omega_{\mathcal{A}}(X) = \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X'),$$

i.e., $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 81, $\mathbb{L}$ is protoalgebraic. ■

5.4 Leibniz Filters

Leibniz filters were introduced by Font and Jansana in [12] (see Page 63), extensively studied in [11] and [17], and used further in applications of the theory in [13]. Here we define an analog in the framework of logicates.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation. We define

$$\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}) = \{F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \text{if } \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \text{ then } \langle \mathcal{A}, C \rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})\}.$$

The elements in $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ are called **Leibniz filters of $\mathbb{L}$ on $\mathcal{A}$**.

Proposition 90 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation over which all full models have a minimum theory. Then $\Omega_{\mathcal{A}}$ is a lattice isomorphism*

$$\Omega_{\mathcal{A}} : \langle \mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}), \subseteq \rangle \cong \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle = \langle \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}), \subseteq \rangle.$$

Proof: Consider the mapping

$$F \longmapsto \langle \mathcal{A}, C^F \rangle,$$

where $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. By the definition of $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$, this is a mapping from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$. It is injective and it is order preserving and order reflecting. By protoalgebraicity and Theorem 89, it is also surjective. So it is an order isomorphism from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$. By the Isomorphism Theorem 75, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$ is isomorphic to $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$ via the Tarski operator. Thus, the composition

$$F \longmapsto \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F\right)$$

establishes an order isomorphism between $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 86, $\widetilde{\Omega}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$. By protoalgebraicity and Proposition 87, $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}) = \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. Therefore, the Leibniz operator is an order isomorphism from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. ■

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate. In case $\mathcal{C}^\flat$ is closed under intersections, the $\mathbb{L}$-filters in $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ on a given interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$

may be characterized without reference to full models. To show this, we consider a binary relation $\sim_\Omega$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ defined as the kernel of the Leibniz operator on $\mathcal{A}$, i.e., for all $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$X \sim_\Omega X' \quad \text{iff} \quad \Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X').$$

Under the assumption that $\mathcal{C}^\flat$ is closed under intersection, we get, by Lemma 82, that $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is also closed under intersection. A fortiori, every full model of $\mathbb{L}$ on $\mathcal{A}$ has a minimum filter. Thus, if $\mathbb{L}$ is protoalgebraic, by Proposition 90, at most one $\mathbb{L}$-filter in each $\sim_\Omega$-equivalence class is in $\mathrm{Fi}^\bigstar_{\mathbb{L}}(\mathcal{A})$. As in Proposition 3.6 of [12], it is possible in this setting as well to characterize this filter.

Suppose $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ is protoalgebraic, with $\mathcal{C}^\flat$ closed under intersection, and $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation. Then each $\sim_\Omega$-class in $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ has a minimum element. In fact, if $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $[X]_\Omega$ denotes its $\sim_\Omega$-class, then, by hypothesis, $\bigcap [X]_\Omega \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and

$$\begin{aligned} \Omega_{\mathcal{A}}(\textstyle\bigcap [X]_\Omega) &= \Omega_{\mathcal{A}}(\textstyle\bigcap \{Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \Omega_{\mathcal{A}}(Y) = \Omega_{\mathcal{A}}(X)\} \\ &\quad \text{(Definition of } [X]_\Omega) \\ &= \textstyle\bigcap \{\Omega_{\mathcal{A}}(Y) : Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \Omega_{\mathcal{A}}(Y) = \Omega_{\mathcal{A}}(X)\} \\ &\quad \text{(Proposition 83)} \\ &= \Omega_{\mathcal{A}}(X). \end{aligned}$$

Hence, $\bigcap [X]_\Omega \in [X]_\Omega$. The identification of $\bigcap [X]_\Omega$ as the least member in the $\sim_\Omega$-equivalence class of X is the key in helping us characterize the class of Leibniz filters of $\mathbb{L}$ on $\mathcal{A}$. This is the promised analog of Proposition 3.6 of [12] for logicates. The proof remains virtually the same.

Proposition 91 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate, with $\mathcal{C}^\flat$ closed under intersection, $\mathcal{A} = \langle \mathbf{A}, h \rangle$ an interpretation and $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. The following statements are equivalent:*

(i) $F \in \mathrm{Fi}^\bigstar_{\mathbb{L}}(\mathcal{A})$, i.e., $\langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of $\mathbb{L}$;

(ii) F is the minimum element in its $\sim_\Omega$-equivalence class;

(iii) $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Proof:

(ii)⇒(iii) Suppose $F = \min [F]_\Omega$ and let $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Our goal is to show that $F/\Omega_{\mathcal{A}}(F) \subseteq G$. Let $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ be the natural projection and set $F' = \pi^{-1}(G) \cap F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, where membership is due to Proposition 34, the hypothesis and Lemma 82. Then

$$\begin{aligned} F' &= \pi^{-1}(G) \cap \pi^{-1}(\pi(F)) \quad \text{(Compatibility of } \Omega_{\mathcal{A}}(F) \text{ with } F) \\ &= \pi^{-1}(G \cap \pi(F)). \quad \text{(Set Theoretical)} \end{aligned}$$

Hence, F' is a union of $\Omega_{\mathcal{A}}(F)$-classes, i.e., $\Omega_{\mathcal{A}}(F)$ is compatible with F'. By the maximality property of the Leibniz congruence, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(F')$. As, by definition, $F' \subseteq F$, by protoalgebraicity, $\Omega_{\mathcal{A}}(F') \subseteq \Omega_{\mathcal{A}}(F)$. Consequently, $\Omega_{\mathcal{A}}(F') = \Omega_{\mathcal{A}}(F)$, i.e., $F \sim_{\Omega} F'$. By hypothesis, $F \subseteq F'$ and, since, by definition, $F' \subseteq F$, $F = F'$. Thus, $F \subseteq \pi^{-1}(G)$. This yields

$$F/\Omega_{\mathcal{A}}(F) = \pi(F) \subseteq \pi(\pi^{-1}(G)) = G.$$

Therefore, $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

(iii)⇒(i) Assume $F/\Omega_{\mathcal{A}}(F) = \min \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. By protoalgebraicity and the correspondence established in Proposition 85, the natural projection $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ gives an order isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ and $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{F/\Omega_{\mathcal{A}}(F)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Also by protoalgebraicity and Proposition 86,

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F).$$

Hence $\langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of $\mathbb{L}$.

(i)⇒(ii) Suppose $F \in \mathrm{Fi}^{\bigstar}_{\mathbb{L}}(\mathcal{A})$ and $G = \min [F]_{\Omega}$. By (ii)⇒(iii)⇒(i), $\mathbb{A}^G = \langle \mathcal{A}, C^G \rangle$, with $\mathcal{C}^G = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^G$, is a full model of $\mathbb{L}$. By hypothesis, $\mathbb{A}^F = \langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is also a full model of $\mathbb{L}$. But

$$\begin{array}{rcll} \widetilde{\Omega}(\mathbb{A}^F) & = & \Omega_{\mathcal{A}}(F) & \text{(Proposition 86)} \\ & = & \Omega_{\mathcal{A}}(G) & (G \sim_{\Omega} F) \\ & = & \widetilde{\Omega}(\mathbb{A}^G). & \text{(Proposition 86)} \end{array}$$

Thus, by the Isomorphism Theorem 75, $\mathcal{C}^F = \mathcal{C}^G$, showing that $F = G$. Therefore, F is the minimum element in the class $[F]_{\Omega}$.

■

5.5 Weak Algebraizability

In [3], Blok and Pigozzi introduced the notion of *algebraizable logic*. As they explain, the notion was a natural abstraction from many well-known examples, the most prototypical ones, perhaps, being that of classical propositional logic, of intuitionistic logic and the various implicative logics of Rasiowa [20]. Making an exact notion of algebraizability precise had, besides unification and clarification, the advantage of being able to show, for the first time, that logics that were known not to be amenable to algebraizability techniques, were somehow intrinsically non-algebraizable, since they did not fall under the scope of Blok and Pigozzi's definition. Blok and Pigozzi worked with finitary sentential logics, but their results were soon generalized further to cover many additional systems. One of the earliest generalizations was by

Herrmann [15, 16] to cover infinitary logics. Algebraizability was shown to be equivalent to the conjunction of equivalentiality [6, 7] and of truth equationality [19]. Equivalentiality is a stronger property than protoalgebraicity, since it requires that the Leibniz operator be both monotone and commute with substitutions. If equivalentiality is weakened to protoalgebraicity, that is, if one requires that the logic be protoalgebraic and truth equational, then weak algebraizability [9] is obtained. All these properties and their characterizations and interconnections are studied in surveys on abstract Algebraic Logic, e.g., [8, 12, 14]. *Weak algebraizability* is the property studied here in the context of logicates.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. We say that $\mathbb{L}$ is **weakly algebraizable** [9] (see, also, [12, 8]) if the Leibniz operator is monotone and injective on $\mathcal{C}^\flat$.

Proposition 92 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a base logicate. $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, the Leibniz operator $\Omega_\mathcal{A}$ on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$ is injective and monotone.*

Proof: First, by Proposition 81, monotonicity of $\Omega_\mathbf{B}$ on $\mathcal{C}^\flat$ is equivalent to monotonicity of $\Omega_\mathcal{A}$ on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$, for every interpretation $\mathcal{A}$. So it suffices to see that injectivity of $\Omega_\mathbf{B}$ on $\mathcal{C}^\flat$ is equivalent to injectivity of $\Omega_\mathcal{A}$ on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$, for every interpretation $\mathcal{A}$.

Assume, first, that $\Omega_\mathbf{B}$ is injective on $\mathcal{C}^\flat$. Consider an interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$ and let $X, X' \in \mathrm{Fi}_\mathbb{L}(\mathcal{A})$, such that $\Omega_\mathcal{A}(X) = \Omega_\mathcal{A}(X')$. Applying h^{-1}, we get $h^{-1}(\Omega_\mathcal{A}(X)) = h^{-1}(\Omega_\mathcal{A}(X'))$. By commutativity of the Leibniz operator with inverse surjective homomorphisms, $\Omega_\mathbf{B}(h^{-1}(X)) = \Omega_\mathbf{B}(h^{-1}(X'))$. By hypothesis, $h^{-1}(X) = h^{-1}(X')$. By the surjectivity of h, $X = X'$. Thus $\Omega_\mathcal{A}$ is injective on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$.

Conversely, if $\Omega_\mathcal{A}$ on $\mathrm{Fi}_\mathbb{L}(\mathcal{A})$ is injective, for every interpretation $\mathcal{A}$, then, by considering $\mathcal{A} = \langle \mathbf{B}, i_\mathbf{B} \rangle$, we get that $\Omega_\mathbf{B}$ is injective on $\mathcal{C}^\flat$. ■

The work of the preceding section on characterizing Leibniz filters of $\mathbb{L}$ on an interpretation $\mathcal{A}$ comes in handy in case one wants to provide a characterization of weakly algebraizable logicates inside the class of protoalgebraic logicates (at least in some better behaved cases).

Proposition 93 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate, with $\mathcal{C}^\flat$ closed under intersection. Then $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A}$, $\mathrm{Fi}_\mathbb{L}^\bigstar(\mathcal{A}) = \mathrm{Fi}_\mathbb{L}(\mathcal{A})$, i.e., for all $F \in \mathrm{Fi}_\mathbb{L}(\mathcal{A})$, $\mathbb{A} = \langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_\mathbb{L}(\mathcal{A})^F$, is a full model of $\mathbb{L}$.*

Proof: We have

$$\begin{array}{rll} \mathrm{Fi}_\mathbb{L}^\bigstar(\mathcal{A}) = \mathrm{Fi}_\mathbb{L}(\mathcal{A}) & \text{iff,} & \text{for all } F \in \mathrm{Fi}_\mathbb{L}(\mathcal{A}), [F]_\Omega = \{F\} \\ & \text{iff,} & \text{for all } F, G \in \mathrm{Fi}_\mathbb{L}(\mathcal{A}), \\ & & \Omega_\mathcal{A}(F) = \Omega_\mathcal{A}(G) \text{ implies } F = G \\ & \text{iff} & \Omega_\mathcal{A} \text{ is injective on } \mathrm{Fi}_\mathbb{L}(\mathcal{A}) \\ & \text{iff} & \mathbb{L} \text{ is weakly algebraizable,} \end{array}$$

the last equivalence since, by hypothesis, $\mathbb{L}$ is protoalgebraic. ∎

This leads to several additional characterizations of weak algebraizability in the special case in which the set of theories of the logicate is closed under intersection.

Theorem 94 *Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be a protoalgebraic logicate, with $\mathcal{C}^\flat$ closed under intersection. The following statements are equivalent:*

(i) $\mathbb{L}$ *is weakly algebraizable;*

(ii) For every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is monotone and injective on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*;*

(iii) $\mathbb{L}$ *is protoalgebraic and, for every interpretation $\mathcal{A}$ and every filter $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $F/\Omega_{\mathcal{A}}(F)$ is the least filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$;*

(iv) For every interpretation $\mathcal{A}$, the mapping $F \mapsto \langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a bijection between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})/\triangleq$;

(v) For every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$;

(vi) For every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$.

Proof:

(i)⇔(ii) By Proposition 92.

(ii)⇔(iii) By Propositions 91 and 93.

(iii)⇒(iv) Consider $F \mapsto \langle \mathcal{A}, C^F \rangle$, viewed as a map into equipotency classes. It is injective. By Proposition 91 and the hypothesis, it is well defined. By Theorem 89, it is also surjective. Thus, it is a bijection. Since it is clearly order preserving and order reflecting, we get that it is a lattice isomorphism.

(iv)⇒(v) Consider $F \mapsto \langle \mathcal{A}, C^F \rangle$, again viewed as a map into equipotency classes. Since, by hypothesis, it is onto, by Theorem 89, $\mathbb{L}$ is protoalgebraic. Further, the composition of this mapping with the mapping $\widetilde{\Omega}_{\mathcal{A}}$ from the Isomorphism Theorem 75 gives an isomorphism from $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 86, the mapping is identical to $F \mapsto \widetilde{\Omega}(\langle \mathcal{A}, C^F \rangle) = \Omega_{\mathcal{A}}(F)$.

(v)⇒(vi) In general, $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}) \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Also in general, $\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. By hypothesis, each $\mathrm{Alg}(\mathbb{L})$-congruence is of the form $\Omega_{\mathcal{A}}(F)$, for some interpretation $\mathcal{A}$ and some $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus,

$$\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}) = \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}).$$

This yields (vi).

(vi)⇒(i) Trivial.

■

For a weakly algebraizable logicate $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$, we call a class K of $\mathcal{L}$-algebras an **equivalent algebraic semantics** for $\mathbb{L}$ if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$, there exists an isomorphism

$$\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \subseteq \rangle \cong \langle \mathrm{Con}_{\mathsf{K}}(\mathcal{A}), \subseteq \rangle.$$

Proposition 95 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a weakly algebraizable logicate. Then* $\mathrm{Alg}^*(\mathbb{L})$ *is an equivalent algebraic semantics for* $\mathbb{L}$.

Proof: Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation. Define

$$\begin{array}{cccc} \Omega_{\mathcal{A}}: & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & \longrightarrow & \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}); \\ & X & \longmapsto & \Omega_{\mathcal{A}}(X). \end{array}$$

This mapping is well defined since $\langle \mathcal{A}/\Omega_{\mathcal{A}}(X), X/\Omega_{\mathcal{A}}(X) \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and, hence, $\mathbf{A}/\Omega_{\mathcal{A}}(X) \in \mathrm{Alg}^*(\mathbb{L})$. By weak algebraizability and Proposition 92, it is injective and monotone. So it suffices to show that it is surjective.

Let $\theta \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{Alg}^*(\mathbb{L})$, that is, there exists $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) = \theta$. Hence, $\Omega_{\mathcal{A}}$ is surjective. ■

Corollary 96 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be an algebraizable logicate, such that, for every interpretation* $\mathcal{A}$, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *has a minimum element. Then* $\mathrm{Alg}(\mathbb{L})$ *is an equivalent algebraic semantics for* $\mathbb{L}$.

Proof: By hypothesis and Proposition 95, $\mathrm{Alg}^*(\mathbb{L})$ is an equivalent algebraic semantics for $\mathbb{L}$. Also by hypothesis and Proposition 87, $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$. Therefore, $\mathrm{Alg}(\mathbb{L})$ is an equivalent algebraic semantics for $\mathbb{L}$. ■

5.6 Truth Equationality

Recall that weak algebraizability [9] is the combination of protoalgebraicity [2] and truth equationality [19]. Since we briefly studied both protoalgebraicity and weak algebraizability in the context of logicates, it is only fair that we, at least briefly, look also at truth equationality as a property on its own. We introduce a definition adapted from [19], we show that it transfers and then prove the main result that weak algebraizability is indeed the conjunction of protoalgebraicity and truth equationality.

Let $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ be an algebraic logicate. $\mathbb{L}$ is **truth equational** if the Leibniz operator $\Omega_{\mathbf{B}}$ is **completely order reflecting on** $\mathcal{C}^\flat$, i.e., if for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$, such that $\bigcap_{i \in I} X_i \in \mathcal{C}^\flat$,

$$\bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \subseteq \Omega_{\mathbf{B}}(X) \quad \text{implies} \quad \bigcap_{i \in I} X_i \subseteq X.$$

Lemma 97 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is truth equational if and only if, for every interpretation* $\mathcal{A} = \langle \mathbf{A}, h \rangle$*, the Leibniz operator on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *is completely order reflecting, i.e., for all* $\{Y_i : i \in I\} \cup \{Y\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, such that* $\bigcap_{i \in I} Y_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*,*

$$\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) \quad \textit{implies} \quad \bigcap_{i \in I} Y_i \subseteq Y.$$

Proof: The right to left implication is again obtained by applying the hypothesis to the interpretation $\langle \mathbf{B}, i_{\mathbf{B}} \rangle$. For the left to right implication, let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation and $\{Y_i : i \in I\} \cup \{Y\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, with $\bigcap_{i \in I} Y_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Then we have

$$\begin{array}{rcl}
\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) & \text{iff} & h^{-1}(\bigcap_{i \in I} \Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\
& \text{iff} & \bigcap_{i \in I} h^{-1}(\Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\
& \text{iff} & \bigcap_{i \in I} \Omega_{\mathbf{B}}(h^{-1}(Y_i)) \subseteq \Omega_{\mathbf{B}}(h^{-1}(Y)) \\
& \text{implies} & \bigcap_{i \in I} h^{-1}(Y_i) \subseteq h^{-1}(Y) \\
& \text{iff} & h^{-1}(\bigcap_{i \in I} Y_i) \subseteq h^{-1}(Y) \\
& \text{iff} & \bigcap_{i \in I} Y_i \subseteq Y.
\end{array}$$

Hence, the Leibniz operator on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is completely order reflecting. ■

And, finally the equivalence of weak algebraizability with protoalgebraicity and truth equationality for logicates whose theory sets are closed under intersection.

Theorem 98 *Let* $\mathbb{L} = \langle \mathbf{B}, C^\flat \rangle$ *be a base logicate, with* $\mathcal{C}^\flat$ *closed under intersection. Then* $\mathbb{L}$ *is weakly algebraizable if and only if it is protoalgebraic and truth equational.*

Proof: Suppose $\mathbb{L}$ is weakly algebraizable. Since, by definition $\Omega_{\mathbf{B}}$ is monotone, $\mathbb{L}$ is certainly protoalgebraic. To show that it is also truth-equational, consider $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$, with $\bigcap_{i \in I} X_i \in \mathcal{C}^\flat$, such that $\bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \subseteq \Omega_{\mathbf{B}}(X)$. Then

$$\begin{array}{rcl}
\Omega_{\mathbf{B}}(\bigcap_{i \in I} X_i \cap X) & = & \bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \cap \Omega_{\mathbf{B}}(X) \quad \text{(Corollary 84)} \\
& = & \bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \quad (\bigcap_{i \in I} \Omega_{\mathbf{B}}(X_i) \subseteq \Omega_{\mathbf{B}}(X)) \\
& = & \Omega_{\mathbf{B}}(\bigcap_{i \in I} X_i). \quad \text{(Corollary 84)}
\end{array}$$

As $\Omega_{\mathbf{B}}$ is injective, $\bigcap_{i \in I} X_i \cap X = \bigcap_{i \in I} X_i$. So $\bigcap_{i \in I} X_i \subseteq X$. This shows that $\Omega_{\mathbf{B}}$ is completely order reflecting and, therefore, $\mathbb{L}$ is also truth equational.

Suppose, conversely, that $\mathbb{L}$ is protoalgebraic and truth equational. By protoalgebraicity, $\Omega_{\mathbf{B}}$ is monotone. So it suffices to show that it is injective. Let $X, Y \in \mathcal{C}^\flat$, such that $\Omega_{\mathbf{B}}(X) = \Omega_{\mathbf{B}}(Y)$. Then, as $\Omega_{\mathbf{B}}$ is, by hypothesis, completely order reflective, we get $X \subseteq Y$ and $Y \subseteq X$, which yields $X = Y$. Hence $\Omega_{\mathbf{B}}$ is also injective, showing that $\mathbb{L}$ is weakly algebraizable. ■

Part II

Logicoids

Chapter 6

General Theory

6.1 Introduction

In our work in Part I, we already glimpsed, at least twice, how, in studying a logicate, having an underlying order on the subsets of its universe A may be beneficial. E.g., in Chapter 2, when we looked at linearized consequences, we saw that artificially linearizing allow us to study instead of an arbitrary idempotent operator, an operator that also satisfies all three properties of an ordinary closure operator. Furthermore, in Chapter 5, we saw how many of the results related to classes in the algebraic hierarchy required that the set of theories was closed under intersections or has a minimum element. These observations lead us in Part II to look at structures in which order plays a role from the get go. To take advantage of the powerful machinery of the traditional framework of monotonic logics [12], without, however, losing sight of the fact that we are dealing with, possibly, nonmonotonic systems, we introduce a complete lattice ordering on the powerset of the underlying set A.

Comparing to the development in Part I, we could say that, in Part I, we took the logical notion of consequence operator as foundational and constructed, based on it, an "ordered" consequence, which involved a type of imposed ordering, either "artificial", e.g., a linearization, or "natural", e.g., based on $\subseteq$, reflecting, necessarily in a rather loose way, to the extent possible the "chaotic" logical structure. On the other hand, in Part II, a reversal of roles occurs. More precisely, we presume an underlying order on the powerset $\mathcal{P}(A)$ of the set A and then build a logical structure that is, in some way, commensurate with the underlying ordering. We visualize the presumed preexisting ordering as an artificially created "molecular" shape and, since the logic is developed on that construct, it is termed a "logicoid". This approach imitates more closely, and captures more accurately, many of the features of more traditional logical systems. On the other hand, expectations must be tempered, since the ordering is one among many that could possibly be chosen, and as such, its role is not quite natural. As noted, also, in comments in Part I,, we attempt to do what we can in a challenging setting, among rather adverse features as compared with those naturally available in the monotonic framework.

In Section 6.2, we introduce the notion of a *grid*, which consists of an underlying set A (viewed as a set of abstract sentences), together with an arbitrary complete lattice ordering on the powerset of A. The fact that this ordering is arbitrary and not the "subset" ordering is what permits accommodating nonmonotonicity and make the framework suitable for our purposes, while still maintaining many of the advantages afforded by the complete lattice structure. Naturally enough, we then introduce *grid morphisms* that connect grids. They are surjective mapping between the underlying sets that make their induced inverse powerset mappings complete lattice embeddings. Continuing, we define *closure operators* as ones that are inflationary, monotone and idempotent, but not with respect to the natural subset ordering, but, rather, with respect to the "artificial" ordering of the grid. We also define *closure systems* and, using the notion of *theory*, we show that, as in the ordinary framework, losure operators and closure systems (on the grid, as it were) are still in one to one correspondence and, thus, interchangeable.

In Section 6.3, we introduce the "*weaker than*" and "*finer than*" relations to compare closure operators of logicoids and closure systems on the underlying grids, respectively. These relationships parallel the ones in the classical (monotonic) framework, except that, instead of being with respect to the subset relation, they are based on the grid ordering.

In Section 6.4, we look at *boosting* for logicoids by a chosen set of axioms, which corresponds to taking the axiomatic extension of a sentential logic in the ordinary monotonic context. We saw the difficulties inherent in defining such an operation for logicates in Section 2.4. Here, the presence of a complete lattice ordering in the grid on which a logicoid is based, creates an environment in which some of the nice features may be recovered, albeit with respect to the $\leq$ ordering of the grid rather than the natural subset ordering that serves the same purpose in the monotonic framework.

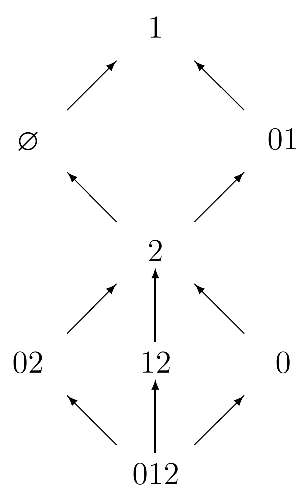
Our main interest is in what we call *algebraic logicoids*, which are logicoids built on *algebraic grids*, that is, grids on sets having an algebraic structure. Naturally enough, treating them algebraically requires having some algebraic fundamentals available for handling them. This is precisely the purpose that Section 6.5 is supposed to fulfill. Here, we formally define *algebraic grids*, which consist of an algebra together with a complete lattice ordering on its powerset. We also define *grid morphisms* and *grid congruences*. We show that these constructs interact as expected. We then employ them to develop analogs of the fundamental Homomorphism Theorems of Universal Algebra for algebraic grids, their homomorphisms and their congruences.

6.2 Logicoids

In this section, we introduce the basic notion of *logicoid* which forms the underlying object of study throughout.

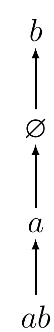
Let A be a set. Let $\mathcal{P}(A)$ denote the powerset of A. A **grid** is a pair $\hat{A} = \langle A, \leq \rangle$, where $\leq$ is a complete lattice ordering on $\mathcal{P}(A)$.

Example 99 *Consider the set* $A = \{0, 1, 2\}$. *Let* $\leq$ *be the ordering on its* *powerset $\mathcal{P}(A)$ shown in the diagram.*



Then $\hat{A} = \langle A, \leq \rangle$ is a grid.

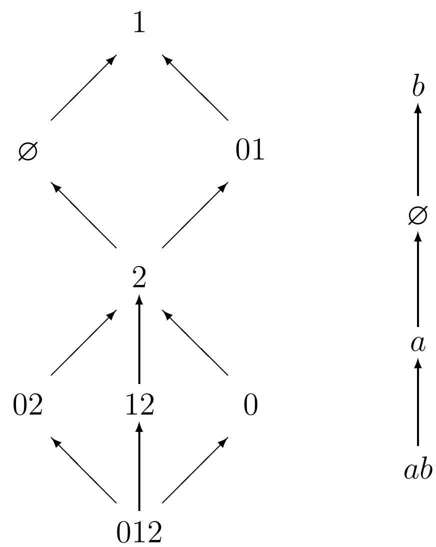
Example 100 *Consider the set $A' = \{a, b\}$. Let $\leq'$ be the ordering on its powerset shown in the diagram.*



Then $\hat{A}' = \langle A', \leq' \rangle$ is a grid.

A **grid morphism** $h : \hat{A} \to \hat{A}'$ is a surjective mapping $h : A \twoheadrightarrow A'$, such that $h^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \to \langle \mathcal{P}(A), \leq \rangle$ is a complete lattice embedding.

Example 101 *Consider, again, the grids $\hat{A}$ and $\hat{A}'$ of the preceding two* *examples.*



The mapping $h: A \to A'$*, with* $0 \mapsto a$*,* $1 \mapsto b$ *and* $2 \mapsto a$ *is a grid morphism as can be checked quickly by hand. We have*

$$012 \le 02 \le \varnothing \le 1 \quad \textit{iff} \quad ab \le' a \le' \varnothing \le' b.$$

A **closure operator on** $\hat{A}$ is a mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for all $X, Y \subseteq A$,

(**Inflationarity**) $X \le C(X)$;

(**Monotonicity**) $X \le Y$ implies $C(X) \le C(Y)$;

(**Idempotency**) $C(C(X)) = C(X)$.

A **logicoid** is a pair $\mathbb{L} = \langle \hat{A}, C \rangle$, where:

- $\hat{A} = \langle A, \le \rangle$ is a grid;
- C a closure operator on $\hat{A}$.

We denote by $\mathrm{Lgcd}(\hat{A})$ the collection of all logicoids on $\hat{A}$.

A **closure system on** a grid $\hat{A} = \langle A, \le \rangle$ is a pair $\hat{\mathcal{X}} = \langle \mathcal{X}, \le \rangle$, where $\mathcal{X} \subseteq \mathcal{P}(A)$ is closed under arbitrary grid meets and $\le$ is the grid order. We denote the collection of all closure systems on the grid $\hat{A}$ by $\mathrm{Clos}(\hat{A})$.

Let $\hat{A} = \langle A, \le \rangle$ be a grid and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ a closure operator on $\hat{A}$. We consider the set of its **fixed points** or **theories**

$$\mathcal{C} = \{X \subseteq A : C(X) = X\}.$$

We define the partially ordered set

$$\hat{\mathcal{C}} = \langle \mathcal{C}, \le \rangle,$$

where $\leq$ is the grid order.

On the other hand, let $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$ be a closure system on a grid $\hat{A}$. Then define an operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $X \subseteq A$,

$$C(X) = \bigwedge \{Y \in \mathcal{C} : X \leq Y\}.$$

One can prove immediately the following properties paralleling properties that hold in the monotonic setting.

Proposition 102 *Let $\hat{A} = \langle A, \leq \rangle$ be a grid.*

(a) If C is a closure operator on $\hat{A}$, then $\hat{\mathcal{C}}$ is a closure system on $\hat{A}$;

(b) If $\hat{\mathcal{C}}$ is a closure system on $\hat{A}$, then C is a closure operator on $\hat{A}$;

(c) The two mappings are inverses of one another, whence closure operators on $\hat{A}$ are in one-to-one correspondence with closure systems on $\hat{A}$.

Proof:

(a) Let C be a closure operator and $\{X_i : i \in I\} \subseteq \mathcal{C}$. By Inflationarity, $\bigwedge_{i \in I} X_i \leq C(\bigwedge_{i \in I} X_i)$. On the other hand, for all $i \in I$, by Monotonicity and the fact that $X_i \in \mathcal{C}$, $C(\bigwedge_{i \in I} X_i) \leq C(X_i) = X_i$, whence $C(\bigwedge_{i \in I} X_i) \leq \bigwedge_{i \in I} X_i$. Hence, $\bigwedge_{i \in I} X_i \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary meets, i.e., $\hat{\mathcal{C}}$ is a closure system on $\hat{A}$.

(b) Suppose $\hat{\mathcal{C}}$ is a closure system on $\hat{A}$. Then, we have the following:

- For all $X \subseteq A$, $X \leq \bigwedge \{Y \in \mathcal{C} : X \leq Y\} = C(X)$.
- For all $X, Y \subseteq A$, such that $X \leq Y$,

$$C(X) = \bigwedge \{Z \in \mathcal{C} : X \leq Z\} \leq \bigwedge \{Z \in \mathcal{C} : Y \leq Z\} = C(Y).$$

- For all $X \subseteq A$,

$$\begin{array}{rcl} C(C(X)) & = & \bigwedge \{Z \in \mathcal{C} : C(X) \leq Z\} \\ & = & \bigwedge \{Z \in \mathcal{C} : \bigwedge \{Y \in \mathcal{C} : X \leq Y\} \leq Z\} \\ & = & \bigwedge \{Z \in \mathcal{C} : X \leq Z\} \\ & = & C(X). \end{array}$$

Thus, C is a closure operator on $\hat{A}$.

(c) Suppose, first, that C is a closure operator. Consider the closure operator C' associated with the closure system $\hat{C}$. Then, for all $X \subseteq A$,

$$\begin{array}{rcll} C'(X) & = & \bigwedge \{Y \in \mathcal{C} : X \leq Y\} & \text{(Definition of } C'\text{)} \\ & = & \bigwedge \{Y \in \mathcal{C} : C(X) \leq Y\} & \text{(Monotonicity of } C\text{)} \\ & = & C(X). & \end{array}$$

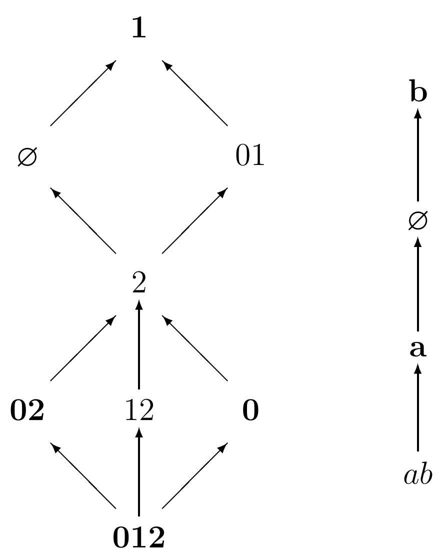
Moreover, if $\hat{\mathcal{C}}$ is a closure system, let $\hat{\mathcal{C}}'$ be the closure system associated with its closure operator C. Then, we have, for all $X \subseteq A$,

$$\begin{array}{lll} X \in \mathcal{C}' & \text{iff} & C(X) = X \quad (\text{Definition of } \mathcal{C}') \\ & \text{iff} & \bigwedge\{Y \in \mathcal{C} : X \leq Y\} = X \quad (\text{Definition of } C) \\ & \text{iff} & \bigwedge\{Y \in \mathcal{C} : C(X) \leq Y\} = X \quad (\text{Monotonicity of } C) \\ & \text{iff} & C(X) = X \\ & \text{iff} & X \in \mathcal{C}. \end{array}$$

This shows that closure operators and closure systems on a grid $\hat{A}$ are in one-to-one correspondence.

■

Example 103 *Consider the grids $\hat{A}$ and $\hat{A}'$ seen previously.*



Take $\mathcal{C} = \{\{0,1,2\},\{0,2\},\{0\},\{1\}\}$ and $\mathcal{C}' = \{\{a\},\{b\}\}$. Denote by C and C' the corresponding closure operators on $\hat{A}$ and $\hat{A}'$, respectively. Then $\mathbb{L} = \langle \hat{A}, C\rangle$ and $\mathbb{L}' = \langle \hat{A}', C'\rangle$ are logicoids on $\hat{A}$ and $\hat{A}'$, respectively.

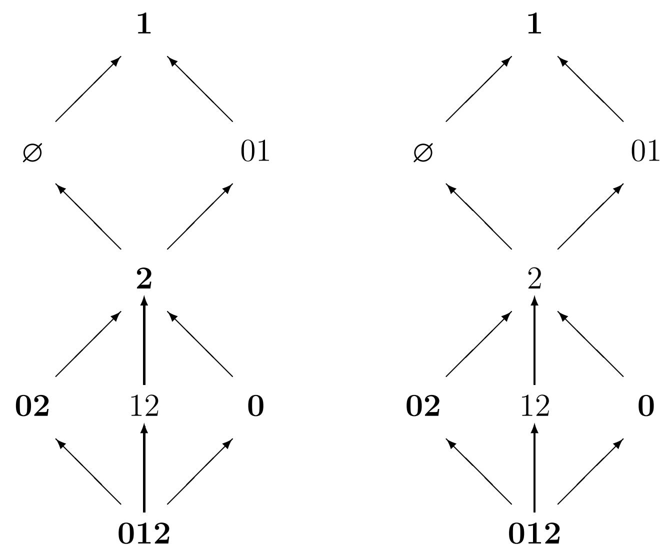
6.3 Comparing Logicoids

Let $\hat{A} = \langle A, \leq\rangle$ be a grid. Given two closure operators C and C' on $\hat{A}$, we say C is **weaker than** C' or C' is **stronger than** C, written $C \leq C'$, if, for all $X \subseteq A$,

$$C(X) \leq C'(X).$$

We extend the same terminology for the two logicoids $\mathbb{L} = \langle \hat{A}, C\rangle$ and $\mathbb{L}' = \langle \hat{A}, C'\rangle$, that is, we set $\mathbb{L} \leq \mathbb{L}'$ if and only if $C \leq C'$.

Example 104 *Consider the grid $\hat{A}$ seen previously.*



Take $\mathcal{C} = \{\{0,1,2\},\{0,2\},\{0\},\{2\},\{1\}\}$ and $\mathcal{C}' = \{\{0,1,2\},\{0,2\},\{0\},\{1\}\}$. Denote by C and C' the corresponding closure operators on $\hat{A}$. Then $\mathbb{L} = \langle \hat{A}, C\rangle$ and $\mathbb{L}' = \langle \hat{A}, C'\rangle$ are such that $\mathbb{L} \leq \mathbb{L}'$.

Proposition 105 *Let $\hat{A} = \langle A, \leq\rangle$ be a grid. Then* $\mathbf{Lgcd}(\hat{A}) = \langle \mathrm{Lgcd}(\hat{A}), \leq\rangle$ *is a complete lattice.*

Proof: Let $\mathbb{L}_i = \langle \hat{A}, C_i\rangle$, $i \in I$, be a collection of logicoids on $\hat{A}$. Define $C : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $X \subseteq A$,

$$C(X) = \bigwedge_{i\in I} C_i(X).$$

It can be shown that C is a closure operator on $\hat{A}$.

- For all $X \subseteq A$, $X \leq C_i(X)$, for all $i \in I$, whence, $X \leq \bigwedge_{i\in I} C_i(X) = C(X)$.
- For all $X, Y \subseteq A$, such that $X \leq Y$, $C_i(X) \leq C_i(Y)$, for all $i \in I$, whence $C(X) = \bigwedge_{i\in I} C_i(X) \leq \bigwedge_{i\in I} C_i(Y) = C(Y)$.
- Finally, let $X \subseteq A$. We have, for all $i \in I$,

  $$C(C(X)) = \bigwedge_{i\in I} C_i\left(\bigwedge_{i\in I} C_i(X)\right) \leq C_i(C_i(X)) = C_i(X).$$

  Thus, $C(C(X)) \leq \bigwedge_{i\in I} C_i(X) = C(X)$. Since the reverse inclusion holds by Inflationarity, we get that C is also idempotent.

Moreover, $C = \bigwedge_{i\in I} C_i$, where the meet is with respect to the $\leq$ ordering on logicoids on $\hat{A}$. Clearly, by definition, for all X, $C(X) = \bigwedge_{i\in I} C_i(X) \leq C_i(X)$. Thus, C is a $\leq$-lower bound of the C_i, $i \in I$. Further, if C' is also a lower bound, one has, for all $X \subseteq A$, $C'(X) \leq C_i(X)$, whence, $C'(X) \leq \bigwedge_{i \in I} C_i(X) = C(X)$, i.e., C is the meet of the C_i in $\leq$. Therefore, $\mathbf{Lgcd}(\hat{A})$ is a complete lattice. ■

A similar comparison applies to closure systems. Given two closure systems $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$ and $\hat{\mathcal{C}}' = \langle \mathcal{C}', \leq \rangle$ on $\hat{A}$, we say that $\mathcal{C}$ is **finer than** $\mathcal{C}'$ or that $\mathcal{C}'$ is **coarser than** $\mathcal{C}$ if

$$\mathcal{C}' \subseteq \mathcal{C}.$$

Proposition 106 *Let $\hat{A} = \langle A, \leq \rangle$ be a grid and $\mathbb{L} = \langle \hat{A}, C \rangle$, $\mathbb{L}' = \langle \hat{A}, C' \rangle$ be logicoids on $\hat{A}$. Then*

$$\mathbb{L} \leq \mathbb{L}' \quad \text{iff} \quad \mathcal{C}' \subseteq \mathcal{C}.$$

Proof: Suppose, first, that $C \leq C'$. Let $X \in \mathcal{C}'$. Then $C(X) \leq C'(X) = X$, whence, since, by Inflationarity, $X \leq C(X)$, $C(X) = X$ and $X \in \mathcal{C}$. Thus, $\mathcal{C}' \subseteq \mathcal{C}$. Assume, conversely, that $\mathcal{C}' \subseteq \mathcal{C}$ and let $X \subseteq A$. Then

$$C(X) = \bigwedge\{Y \in \mathcal{C} : X \leq Y\} \leq \bigwedge\{Y \in \mathcal{C}' : X \leq Y\} = C'(X).$$

Hence $C \leq C'$. ■

Proposition 107 *Let $\hat{A} = \langle A, \leq \rangle$ be a grid. Then the partially ordered sets $\mathbf{Lgcd}(\hat{A}) = \langle \mathrm{Lgcd}(\hat{A}), \leq \rangle$ and $\mathbf{Clos}(\hat{A}) = \langle \mathrm{Clos}(\hat{A}), \supseteq \rangle$ are isomorphic.*

Proof: The correspondence established in Proposition 102 is, by Proposition 106, order preserving and order reflecting, whence it establishes the required isomorphism. ■

Corollary 108 *Let $\hat{A} = \langle A, \leq \rangle$ be a grid. Then $\mathbf{Clos}(\hat{A}) = \langle \mathrm{Clos}(\hat{A}), \subseteq \rangle$ is a complete lattice with meet being set intersection.*

Proof: By Proposition 107, using Proposition 105. ■

6.4 Boosting

In this section we build operations that are, in the nonmonotonic context, "parallel" to axiomatic extensions in the traditional monotonic framework. We could call this operation a "nonmonotonic axiomatic extension", but, in analogy with the operation introduced in the context of logicates, we call it *boosting* instead.

Let $\hat{A} = \langle A, \leq \rangle$ be a grid, $\mathbb{L} = \langle \hat{A}, C \rangle$ a logicoid on $\hat{A}$ and $T \subseteq A$. The **boosting of** C **by** T is the operator $C^T : \mathcal{P}(A) \to \mathcal{P}(A)$ that is defined, for all $X \subseteq A$, by

$$C^T(X) = C(T \vee X).$$

We show that this recipe gives a bona fide consequence operator.

Proposition 109 *Let $\hat{A} = \langle A, \leq \rangle$ be a grid, $\mathbb{L} = \langle \hat{A}, C \rangle$ a logicoid and $T \subseteq A$. Then $C^T : \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operator on $\hat{A}$.*

Proof: Let $X \subseteq A$. Then

$$X \leq X \vee T \leq C(X \vee T) = C^T(X).$$

Let $X, Y \subseteq A$, such that $X \leq Y$. Then $X \vee T \leq Y \vee T$. By Monotonicity, $C(X \vee T) \leq C(Y \vee T)$ and, therefore, $C^T(X) \leq C^T(Y)$. Finally, let $X \subseteq A$. Then

$$\begin{aligned} C^T(C^T(X)) &= C(C^T(X) \vee T) \\ &= C(C(X \vee T) \vee T) \\ &= C(C(X \vee T)) \\ &= C(X \vee T) \\ &= C^T(X). \end{aligned}$$

So C^T satisfies Inflationarity, Monotonicity and Idempotency and is, therefore, a closure operator on $\hat{A}$. ■

We call the logicoid $\mathbb{L}^T = \langle \hat{A}, C^T \rangle$ the **boosting of $\mathbb{L}$ by T**.

We also describe the way the two closure systems are related. Consider a closure system $\hat{C} = \langle \mathcal{C}, \leq \rangle$ on $\hat{A}$ and let $T \subseteq A$. Define the **boosted closure system $\hat{\mathcal{C}}^T = \langle \mathcal{C}^T, \leq \rangle$ of $\hat{\mathcal{C}}$ by T** on $\hat{A}$, by setting

$$\mathcal{C}^T = \{X \in \mathcal{C} : T \leq X\}$$

and taking $\leq$ be the restriction of $\leq$ on $\mathcal{C}^T$.

Proposition 110 *Let $\hat{A} = \langle A, \leq \rangle$ be a grid, $\mathbb{L} = \langle \hat{A}, C \rangle$ a logicoid and $T \subseteq A$. If $\hat{\mathcal{C}}$ is the closure system corresponding to $\mathbb{L}$, then $\hat{\mathcal{C}}^T$ is the closure system corresponding to $\mathbb{L}^T$.*

Proof: To prove the claim, we must show that, for all $X \subseteq A$,

$$C^T(X) = X \quad \text{iff} \quad X \in \hat{\mathcal{C}}^T.$$

Suppose $C^T(X) = X$. By definition, $C(X \vee T) = X$. On the one hand, $C(X) \leq C(X \vee T) = X$, whence $X \in \mathcal{C}$. On the other, $T \leq X \vee T \leq C(X \vee T) = C^T(X) = X$. Thus, $X \in \hat{\mathcal{C}}^T$.

Assume, conversely, that $X \in \hat{\mathcal{C}}^T$. Then $C(X) = X$ and $T \leq X$. It follows that $X = C(X)$ and $T \leq C(X)$. Hence, $X \vee T \leq C(X)$, which yields $C(X \vee T) \leq C(X)$. Thus, we get

$$C^T(X) = C(X \vee T) \leq C(X) = X,$$

whence $C^T(X) = X$. ■

Finally, we obtain the following characterization of boosting.

Proposition 111 *Let $\hat{A} = \langle A, \leq\rangle$ be a grid, $\mathbb{L} = \langle \hat{A}, C\rangle$ a logicoid and $T \subseteq A$. Then $\mathbb{L}^T$ is the weakest logicoid in $\mathbf{Lgd}(\hat{A})$, satisfying*

- $C \leq C^T$;
- $T \leq C^T(X)$, *for all* $X \subseteq A$.

Proof: That C^T satisfies the two statement is shown as follows:

- $C(X) \leq C(X \vee T) = C^T(X)$;
- $T \leq X \vee T \leq C(X \vee T) = C^T(X)$.

Suppose that C' is a closure operator on $\hat{A}$, such that $C \leq C'$ and $T \leq C'(X)$, for all $X \subseteq A$. Note that these give, for all $X \subseteq A$, $X \vee T \leq C'(X)$. Then, we get

$$\begin{array}{rcl} C^T(X) & = & C(X \vee T) \\ & \leq & C(C'(X)) \\ & \leq & C'(C'(X)) \\ & = & C'(X). \end{array}$$

This shows that $C^T \leq C'$ and, hence, C^T is the weakest closure operator satisfying the two conditions. ■

6.5 Algebraic Grids and Logicoids

In the sequel, we will focus on logicoids built not simply on an underlying set A, but on an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ of an arbitrary, but fixed, type $\mathcal{L}$. We call those *algebraic logicoids* and start their study per se in Chapter 7. However, since there is substantial interaction between the algebraic structure $\mathbf{A}$ and the complete lattice ordering $\leq$ on the powerset $\mathcal{P}(A)$, we need to delve a little into universal algebra. This we do in this section as preparation for what is to follow.

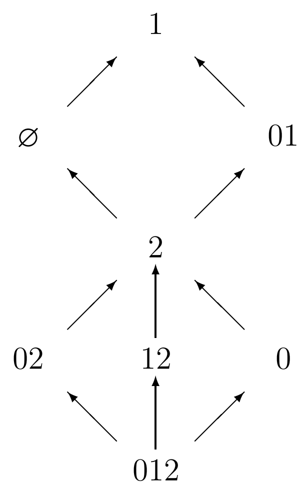
Fix an algebraic type $\mathcal{L}$. An (**algebraic**) **grid** is a pair $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$, where:

- $\mathbf{A}$ is an $\mathcal{L}$-algebra;
- $\leq$ is a complete lattice ordering on $\mathcal{P}(A)$.

Example 112 *Let $\mathcal{L} = \{*\}$, where $*$ is a binary operation symbol. Consider the $\mathcal{L}$-algebra $\mathbf{A} = \langle A, *^{\mathbf{A}}\rangle$, where $A = \{0, 1, 2\}$ and $*^{\mathbf{A}}$ is given by the following table.*

$*^{\mathbf{A}}$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	0

Let $\leq$ be the ordering on its powerset $\mathcal{P}(A)$ shown in the diagram.



Then $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ is an algebraic grid.

Example 113 *Let $\mathcal{L} = \{*\}$, where $*$ is a binary operation symbol. Consider the $\mathcal{L}$-algebra $\mathbf{A}' = \langle A', *^{\mathbf{A}'} \rangle$, where $A' = \{a, b\}$ and $*^{\mathbf{A}'}$ is given by the following table.*

$*^{\mathbf{A}'}$	a	b
a	a	b
b	b	a

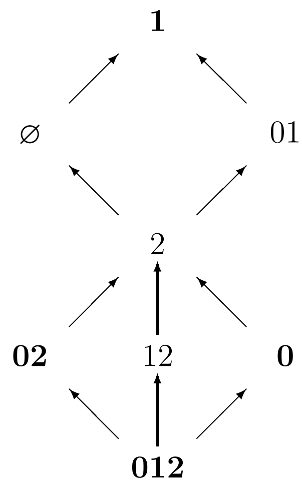
Let $\leq'$ be the ordering on its powerset shown in the diagram.

Then $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$ is an algebraic grid.

An **algebraic logicoid** is a pair $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$, where:

- $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ is an algebraic grid;
- C is a closure operator on $\hat{\mathbf{A}}$.

Example 114 *Consider the algebraic grid* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *constructed above.*



Let

$$\mathcal{C} = \{\{0,1,2\},\{0,2\},\{0\},\{1\}\}.$$

Then $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *is an algebraic logicoid.*

Example 115 *Consider the algebraic grid* $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$ *constructed above.*

Let $\mathcal{C}' = \{\{a\},\{b\}\}$. *Then* $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ *is an algebraic logicoid.*

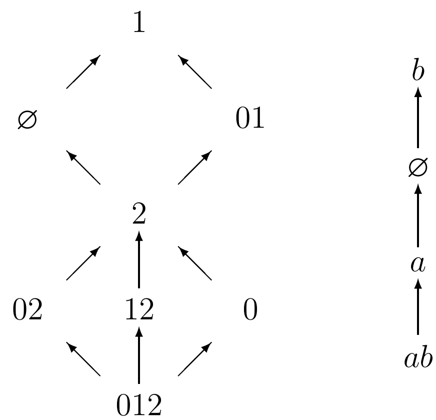
Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$ be two algebraic grids. An (**algebraic**) **grid morphism** $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ is a surjective (algebra) homomorphism $h : \mathbf{A} \to \mathbf{A}'$, such that

$$h^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \to \langle \mathcal{P}(A), \leq \rangle$$

is a complete lattice embedding.

Example 116 *Consider the algebraic grids* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *and* $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$

constructed above.



The mapping $h : A \to A'$*, determined by* $0 \mapsto a$*,* $1 \mapsto b$ *and* $2 \mapsto a$ *is a homomorphism* $h : \mathbf{A} \to \mathbf{A}'$*. Moreover, as can be checked by hand, it is a grid morphism.*

To define *grid congruences*, we look at some preliminary concepts. Let A be a set and θ be an equivalence relation on A. We say that θ is **compatible with** a subset $X \subseteq A$, if, for all $a, b \in A$,

$$\langle a, b\rangle \in \theta \quad \text{and} \quad a \in X \quad \text{imply} \quad b \in X.$$

We denote by $\mathrm{Cmp}(\theta)$ the set of all subsets of A that are compatible with θ. These are characterized in various ways.

Lemma 117 *Let A be a set, θ an equivalence relation on A and $X \subseteq A$. The following statements are equivalent.*

(i) $X \in \mathrm{Cmp}(\theta)$*;*

(ii) X *is a union of* θ*-equivalence classes;*

(iii) $X = \pi_\theta^{-1}(Y)$*, for some* $Y \subseteq A/\theta$*.*

Proof:

(i)⇒(ii) Let $X \subseteq A$ be compatible with θ. Suppose $x \in X$ and $a \in A$, such that $\langle x, a\rangle \in \theta$. Since $x \in X$, by compatibility, $a \in X$. Thus, X is a union of θ-equivalence classes.

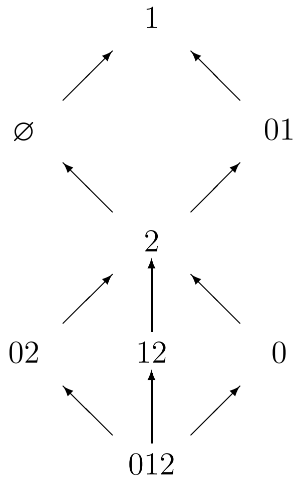
(ii)⇒(iii) Let X be a union of θ-equivalence classes. Set $Y = \pi_\theta(X) \in A/\theta$. Then $X = \pi_\theta^{-1}(\pi_\theta(X)) = \pi_\theta^{-1}(Y)$, where the hypothesis is used to ensure that the first equality holds.

(iii)⇒(i) Let $X = \pi_\theta^{-1}(Y)$, for some $Y \subseteq A/\theta$. Consider $a, b \in A$, such that $\langle a, b\rangle \in \theta$ and $a \in X$. Then $\pi_\theta(a) \in Y$. Hence, $\pi_\theta(b) = \pi_\theta(a) \in Y$. So $b \in \pi_\theta^{-1}(Y) = X$. This shows that θ is compatible with X.

■

Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ be an algebraic grid. A **grid congruence** θ is a congruence on the algebra $\mathbf{A}$, such that $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of the complete lattice $\langle \mathcal{P}(A), \leq \rangle$.

Example 118 *Consider again the algebraic grid* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$.



Since $*^{\mathbf{A}}$ *is given by*

$*^{\mathbf{A}}$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	0

the equivalence θ *whose classes are* $\{0, 2\}$ *and* $\{1\}$ *is a grid congruence on* $\hat{\mathbf{A}}$.

We denote by $\mathrm{Con}(\hat{\mathbf{A}})$ the set of all grid congruences on the grid $\hat{\mathbf{A}}$. We show that $\mathrm{Con}(\hat{\mathbf{A}})$, ordered by $\subseteq$, forms a complete lattice $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \mathrm{Con}(\hat{\mathbf{A}}), \subseteq \rangle$, whose join coincides with the join in $\mathbf{Con}(\mathbf{A})$.

Lemma 119 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *be an algebraic grid. The poset* $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \mathrm{Con}(\hat{\mathbf{A}}), \subseteq \rangle$ *is closed under arbitrary joins in* $\mathbf{Con}(\mathbf{A})$.

Proof: Let $\{\theta_i : i \in I\} \subseteq \mathrm{Con}(\hat{\mathbf{A}})$. Consider the join $\bigvee_{i \in I} \theta_i$ in $\mathbf{A}$. It suffices to show that $\langle \mathrm{Cmp}(\bigvee_{i \in I} \theta_i), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. Let $X \subseteq A$. We show

$$X \in \mathrm{Cmp}\left(\bigvee_{i \in I} \theta_i\right) \quad \text{iff} \quad X \in \bigcap_{i \in I} \mathrm{Cmp}(\theta_i).$$

For the left-to-right implication, let $a, b \in A$, such that $\langle a, b \rangle \in \theta_i$ and $a \in X$. Then $\langle a, b \rangle \in \bigvee_{i \in I} \theta_i$ and $a \in X$. By hypothesis, $b \in X$. Hence, X is compatible with θ_i, for all $i \in I$, i.e., $X \in \bigcap_{i \in I} \mathrm{Cmp}(\theta_i)$.

For the right-to-left implication, suppose $\langle a,b\rangle \in \bigvee_{i\in I}\theta_i$ and $a \in X$. Then, there exist $k \in \omega$ and $i_0, i_1, \dots, i_{k-1} \in I$, such that

$$a = c_0\ \theta_{i_0}\ c_1\ \theta_{i_1}\ \cdots\ \theta_{i_{k-2}}\ c_{k-1}\ \theta_{i_{k-1}} c_k = b,$$

for some $c_0, c_1, \dots, c_k \in A$. Since, by hypothesis, $X \in \bigcap_{i\in I}\mathrm{Cmp}(\theta_i)$, we get

$$a = c_0, c_1, c_2, \dots, c_{k-1}, c_k = b \in X.$$

This shows that $X \in \mathrm{Cmp}(\bigvee_{i\in I}\theta_i)$.

Since $\langle \mathrm{Cmp}(\theta_i), \leq\rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$, for all $i \in I$, we have that $\langle \bigcap_{i\in I}\mathrm{Cmp}(\theta_i), \leq\rangle$ is also a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$, which shows that $\langle \mathrm{Cmp}(\bigvee_{i\in I}\theta_i), \leq\rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$. ∎

Proposition 120 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$ *be an algebraic grid. The poset* $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \mathrm{Con}(\hat{\mathbf{A}}), \subseteq\rangle$ *forms a complete lattice, with joins coinciding with those in* $\mathbf{Con}(\mathbf{A})$.

Proof: Apply Lemma 119. ∎

We may now establish a correspondence between grid morphisms and grid congruences, taking after the one between homomorphisms and congruences in Universal Algebra. For the kernel of a homomorphism $h : \mathbf{A} \to \mathbf{A}'$ we use the notation

$$\mathrm{Ker}(h) = \{\langle a,b\rangle \in A^2 : h(a) = h(b)\}.$$

Lemma 121 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$, $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq'\rangle$ *be algebraic grids and* $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ *be a grid morphism. Then* $\mathrm{Ker}(h)$ *is a grid congruence on* $\hat{\mathbf{A}}$.

Proof: First, since $h : \mathbf{A} \to \mathbf{A}'$ is a homomorphism, its kernel is a congruence on $\mathbf{A}$. In addition, by hypothesis, $h^{-1} : \langle \mathcal{P}(A'), \leq'\rangle \to \langle \mathcal{P}(A), \leq\rangle$ is a complete lattice embedding. By Lemma 117, the image of h^{-1} in $\mathcal{P}(A)$ is exactly $\mathrm{Cmp}(\mathrm{Ker}(h))$. Thus, $\langle \mathrm{Cmp}(\mathrm{Ker}(h)), \leq\rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$. ∎

Let, now $\hat{\mathbf{A}}$ be a grid and $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Construct the pair

$$\hat{\mathbf{A}}^\theta = \hat{\mathbf{A}}/\theta := \langle \mathbf{A}/\theta, \leq^\theta\rangle$$

by setting, for all $Y, Y' \in \mathcal{P}(A/\theta)$,

$$Y \leq^\theta Y' \quad \text{iff} \quad \pi_\theta^{-1}(Y) \leq \pi_\theta^{-1}(Y').$$

Lemma 122 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$ *be an algebraic grid and* $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$.

(a) $\hat{\mathbf{A}}/\theta = \langle \mathbf{A}/\theta, \leq^\theta\rangle$ *is an algebraic grid.*

(b) $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ *is a grid morphism.*

Proof:

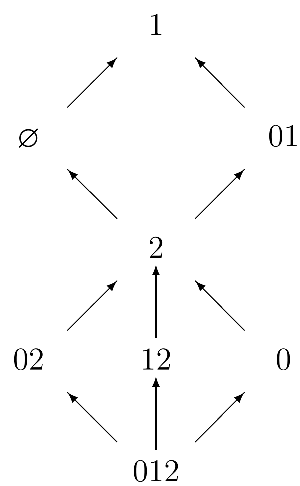
(a) Since θ is a congruence, $\mathbf{A}/\theta$ is an algebra. So to prove that the pair forms a grid, it suffices to show that $\leq^\theta$ is a complete lattice ordering on $\mathcal{P}(A/\theta)$. But, θ is also a grid congruence, which means that $\langle \mathrm{Cmp}(\theta), \leq\rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$. By Lemma 117, $\langle \pi_\theta^{-1}(A/\theta), \leq\rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$. Hence, by definition of $\leq^\theta$, $\leq^\theta$ is a complete lattice ordering on $\mathcal{P}(A/\theta)$.

(b) We know that π_θ is a homomorphism. By the proof of Part (a), π_θ^{-1} is also a complete lattice embedding of $\langle \mathcal{P}(A/\theta), \leq^\theta\rangle$ into $\langle \mathcal{P}(A), \leq\rangle$. Therefore, $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ is a grid morphism.

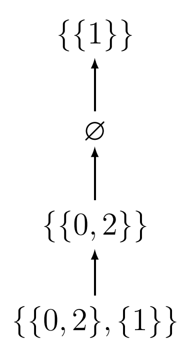
■

We call $\hat{\mathbf{A}}^\theta = \hat{\mathbf{A}}/\theta := \langle \mathbf{A}/\theta, \leq^\theta\rangle$ the **quotient (algebraic) grid** of $\hat{\mathbf{A}}$ by θ and π_θ the **quotient (algebraic) grid morphism** or the **canonical projection grid morphism**.

Example 123 *Consider, once more, the algebraic grid* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$.



In a previous example, we looked at the grid congruence $\theta = \{\{0,2\},\{1\}\}$. *The quotient grid* $\hat{\mathbf{A}}/\theta$ *consists of the algebra* $\mathbf{A}/\theta$ *and the complete lattice ordering* $\leq^\theta$ *on* $\mathcal{P}(A/\theta)$, *given in the diagram.*



Finally, we define isomorphisms. Consider two algebraic grids $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$. A bijection $h : A \to A'$ is an **isomorphism** between $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$ if both $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ and $h^{-1} : \hat{\mathbf{A}}' \to \hat{\mathbf{A}}$ are grid homomorphisms. We then write $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$.

Lemma 124 *Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$ be algebraic grids and $h : A \to A'$ a bijection. $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ if and only if $h : \mathbf{A} \cong \mathbf{A}'$ and $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$.*

Proof: The left to right implication is trivial. For the right to left, it suffices to show that, for all $X, Y \subseteq A$,

$$X \leq Y \quad \text{iff} \quad h(X) \leq' h(Y).$$

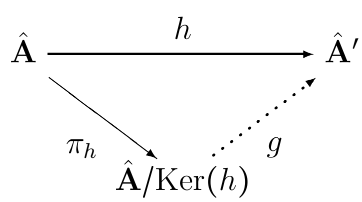
Since h is a bijection, for the subsets $X' = h(X)$ and $Y' = h(Y)$ of A', we have $X = h^{-1}(X')$ and $Y = h^{-1}(Y')$. Thus, we get

$$\begin{array}{lll} X \leq Y & \text{iff} & h^{-1}(X') \leq h^{-1}(Y') \quad (X = h^{-1}(X') \text{ and } Y = h^{-1}(Y')) \\ & \text{iff} & X' \leq' Y' \quad (h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}') \\ & \text{iff} & h(X) \leq h(Y). \quad (X' = h(X) \text{ and } Y' = h(Y)) \end{array}$$

Thus, $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$. ■

Once we have this machinery available, we can work out versions of the Homomorphism Theorems of Universal Algebra for grid morphisms.

Theorem 125 (Homomorphism) *Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq' \rangle$ be algebraic grids and $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ be a grid morphism. Then, there exists a unique grid isomorphism $g : \hat{\mathbf{A}}/\mathrm{Ker}(h) \cong \hat{\mathbf{A}}'$, such that $h = g \circ \pi_h$,*



where $\pi_h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\mathrm{Ker}(h)$ is the quotient grid morphism.

Proof: By the Homomorphism Theorem of Universal Algebra, we know that there exists a unique $g : \mathbf{A}/\mathrm{Ker}(h) \cong \mathbf{A}'$. Thus, by Lemma 124, we must show that $g^{-1} : \langle \mathcal{P}(A'), \leq' \rangle \to \langle \mathcal{P}(A/\mathrm{Ker}(h)), \leq^{\mathrm{Ker}(h)} \rangle$ is a complete lattice embedding. For this, it suffices to show that, for all $Y, Y' \in \mathcal{P}(A')$,

$$g^{-1}(Y) \leq^{\mathrm{Ker}(h)} g^{-1}(Y') \quad \text{iff} \quad Y \leq' Y'.$$

Indeed, we have, for all $Y, Y' \in \mathcal{P}(A')$,

$$\begin{array}{lll} g^{-1}(Y) \leq^{\mathrm{Ker}(h)} g^{-1}(Y') & \text{iff} & \pi_h^{-1}(g^{-1}(Y)) \leq \pi_h^{-1}(g^{-1}(Y')) \quad \text{(Lemma 122)} \\ & \text{iff} & h^{-1}(Y) \leq h^{-1}(Y') \quad (h = g \circ \pi_h) \\ & \text{iff} & Y \leq' Y'. \quad (h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}') \end{array}$$

We conclude that $g : \hat{\mathbf{A}}/\mathrm{Ker}(h) \cong \hat{\mathbf{A}}'$. ■

We continue with a version of the Second Isomorphism Theorem.

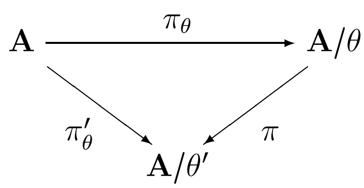
Theorem 126 (Second Isomorphism Theorem) *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *be an algebraic logicate and* $\theta, \theta' \in \mathrm{Con}(\hat{\mathbf{A}})$*, such that* $\theta \subseteq \theta'$*. Then* $\theta'/\theta \in \mathrm{Con}(\hat{\mathbf{A}}/\theta)$ *and*

$$(\hat{\mathbf{A}}/\theta)/(\theta'/\theta) \cong \hat{\mathbf{A}}/\theta',$$

where the isomorphism is given by

$$(a/\theta)/(\theta'/\theta) \mapsto a/\theta'.$$

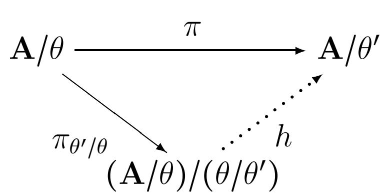
Proof: We consider the commutative diagram of natural quotient homomorphisms.



By Lemma 122, $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ and $\pi_{\theta'} : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta'$ are grid morphisms. We can show that $\pi : \hat{\mathbf{A}}/\theta \to \hat{\mathbf{A}}/\theta'$ is also a grid morphism. E.g., we have, for all $\{Y'_i : i \in I\} \subseteq \mathcal{P}(A/\theta')$,

$$\begin{array}{rcl}\pi^{-1}(\bigwedge^{\theta'}_{i\in I} Y'_i) & = & \pi_\theta(\pi^{-1}_{\theta'}(\bigwedge^{\theta'}_{i\in I} Y'_i)) \\ & = & \pi_\theta(\bigwedge_{i\in I} \pi^{-1}_{\theta'}(Y'_i)) \\ & = & \bigwedge^{\theta}_{i\in I}(\pi_\theta(\pi^{-1}_{\theta'}(Y'_i))) \\ & = & \bigwedge^{\theta}_{i\in I}(\pi^{-1}(Y'_i)).\end{array}$$

We know that $\mathrm{Ker}(\pi) = \theta'/\theta$, which, by Theorem 26, is a grid congruence on $\hat{\mathbf{A}}/\theta$. So we consider the diagram



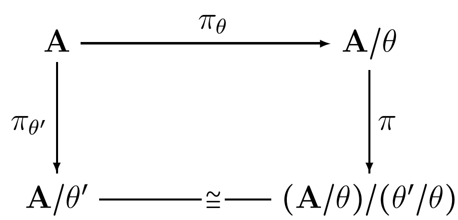
where h is the unique grid isomorphism given by Theorem 26. Thus, $h : (\hat{\mathbf{A}}/\theta)/(\theta'/\theta) \cong \hat{\mathbf{A}}/\theta'$. ∎

We finally prove an analog of the Correspondence Theorem.

Theorem 127 (Correspondence Theorem) *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *be an algebraic grid and* $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$*. Then the segment* $[\theta, \nabla^{\mathbf{A}}]$ *of the poset* $\mathbf{Con}(\hat{\mathbf{A}})$ *is isomorphic to the poset* $\mathbf{Con}(\hat{\mathbf{A}}/\theta)$ *by the mapping* $\theta' \mapsto \theta'/\theta$.

Proof: By Theorem 26, if $\theta \subseteq \theta'$ are grid congruences on $\hat{\mathbf{A}}$, then $\theta'/\theta \in \mathrm{Con}(\hat{\mathbf{A}}^\theta)$. By Universal Algebra, it suffices to prove that, for all $\theta \subseteq \theta' \in$

$\mathrm{Con}(\mathbf{A})$, if $\theta'/\theta \in \mathrm{Con}(\hat{\mathbf{A}}^\theta)$, then $\theta' \in \mathrm{Con}(\hat{\mathbf{A}})$.



Since, by hypothesis, $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ and $\theta'/\theta \in \mathrm{Con}(\hat{\mathbf{A}}^\theta)$, we have π_θ and π are grid morphisms. Identifying elements corresponding under the bottom isomorphism, we have, for all $X \subseteq A$,

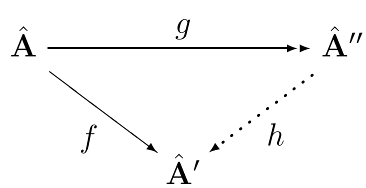
$$\begin{array}{lll} X \in \mathrm{Cmp}(\mathrm{Ker}(\pi \circ \pi_\theta)) & \text{iff} & X = \pi_\theta^{-1}(\pi^{-1}(Y)), \text{ for } Y \subseteq \mathcal{P}((A/\theta)/(\theta'/\theta)), \\ & \text{iff} & X = \pi_{\theta'}^{-1}(Y), \text{ for } Y \subseteq \mathcal{P}(A/\theta'), \\ & \text{iff} & X \in \mathrm{Cmp}(\theta'). \end{array}$$

Thus, since $\langle \mathrm{Cmp}(\mathrm{Ker}(\pi \circ \pi_\theta)), \leq\rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq\rangle$, then so is $\langle \mathrm{Cmp}(\theta'), \leq\rangle$, showing that $\theta' \in \mathrm{Con}(\hat{\mathbf{A}})$. ∎

We close the section with a "fill-in" lemma for arrows, which will play a role later in establishing for logicoids an analog of Proposition 1.15 of [12] (see, also, Proposition 31 for logicates).

Lemma 128 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$, $\hat{\mathbf{A}}' = \langle \mathbf{A}', \leq'\rangle$ *and* $\hat{\mathbf{A}}'' = \langle \mathbf{A}'', \leq''\rangle$ *be algebraic grids,* $f : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ *and* $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}''$ *be grid morphisms, such that* $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. *Then, there exists a unique grid morphism* $h : \hat{\mathbf{A}}'' \to \hat{\mathbf{A}}'$*, such that*

$$h \circ g = f.$$



Proof: We know from Universal Algebra that there exists a unique $h : \mathbf{A}'' \to \mathbf{A}'$, such that $h \circ g = f$. Moreover, by the definition of grid morphism, the following mappings are complete lattice embeddings.

$$\begin{array}{rrcl} g^{-1} : & \langle \mathcal{P}(A''), \leq''\rangle & \to & \langle \mathcal{P}(A), \leq\rangle, \\ f^{-1} : & \langle \mathcal{P}(A'), \leq'\rangle & \to & \langle \mathcal{P}(A), \leq\rangle. \end{array}$$

These suffice to show that

$$h^{-1} : \langle \mathcal{P}(A'), \leq'\rangle \to \langle \mathcal{P}(A''), \leq''\rangle$$

is also a complete lattice embedding. Indeed, for all $X', Y' \subseteq A'$, we have

$$\begin{array}{lll} X' \leq' Y' & \text{iff} & f^{-1}(X') \leq f^{-1}(Y') \quad (f : \hat{\mathbf{A}} \to \hat{\mathbf{A}}') \\ & \text{iff} & g^{-1}(h^{-1}(X')) \leq g^{-1}(h^{-1}(X')) \quad (h \circ g = f) \\ & \text{iff} & h^{-1}(X') \leq'' h^{-1}(Y'). \quad (g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'') \end{array}$$

Therefore, $h : \hat{\mathbf{A}}'' \to \hat{\mathbf{A}}'$ is a grid morphism. ∎

Chapter 7

Algebraic Theory

7.1 Introduction

In Chapter 7, we develop the rudiments of the algebraic theory of logicoids with an eye towards developing, in Chapter 8, a model theory, paralleling the one in [12] and that developed for logicates in Part I. We first introduce the key concept of *logical grid congruence*. Based on those, we define the *Leibniz grid congruence* of a logical matrix and the *Tarski grid congruence* of a logicoid. We then study *bilogical morphisms* between logicoids, which, unlike those used for logicates, respect the logical consequence and not merely the theories of the structure. So, in that respect, they resemble more closely those introduced by Font and Jansana [12]. We then look at *quotient logicoids* and prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. This gives us the chance to look closely at *reductions* and at *reduced logicoids*. We then turn to analogs of *interpretations*, *filters* and *matrix models* and study many of their properties, including the way they interact with grid morphisms, their interplay with closure systems and their transformations via bilogical morphisms.

In more detail, Section 7.2 undertakes the study of *logical grid congruences*. Recall that, given an algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, a congruence θ on $\mathbf{A}$ is called a *grid congruence* on $\hat{\mathbf{A}}$ if $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ based on the grid $\hat{\mathbf{A}}$, θ is a *logical grid congruence* of $\mathbb{L}$ if it is a grid congruence on $\hat{\mathbf{A}}$, such that $\mathcal{C} \subseteq \mathrm{Cmp}(\theta)$. It is shown that the collection of all logical grid congruences of $\mathbb{L}$ forms a principal ideal of the complete lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\widetilde{\Omega}(\mathbb{L})$ is called the *Tarski grid congruence* of $\mathbb{L}$. An analogous situation occurs if one considers logical matrices $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$ based on an algebraic grid $\hat{\mathbf{A}}$. Here a *matrix grid congruence* is a grid congruence θ on $\hat{\mathbf{A}}$, such that $X \in \mathrm{Cmp}(\theta)$. Again, the collection of all matrix grid congruences of $\mathfrak{A}$ forms a principal ideal of the lattice of all grid congruences on $\hat{\mathbf{A}}$ and its generator $\Omega(\mathfrak{A})$ is called the *Leibniz grid congruence* of $\mathfrak{A}$. Two of the most useful observations related to these concepts are that $\widetilde{\Omega}$ is monotone on logicoids over the same grid and that, given a logicoid, its Tarski grid congruence is the intersection of all Leibniz grid congruences of those logical matrices formed by each of its theories.

In Section 7.3, we introduce and study *logical* and *bilogical morphisms* between logicoids. Since logicoids are based on algebraic grids, all these morphisms are algebraic grid morphisms, which were studied extensively in Section 6.5, and we rely quite heavily on that machinery. A *logical morphism* $h : \mathbb{L} \to \mathbb{L}'$ from a logicoid $\mathbb{L}$ based on $\hat{\mathbf{A}}$ to a logicoid $\mathbb{L}'$ based on $\hat{\mathbf{A}}'$ is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$. In case $h^{-1}(\mathcal{C}') = \mathcal{C}$ we say that $\mathbb{L}$ is *projectively generated from* $\mathbb{L}'$ *by* h. A logical morphism $h : \mathbb{L} \to \mathbb{L}'$ is a *bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}'$ between $\mathbb{L}$ and $\mathbb{L}'$ if it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. We provide a characterization theorem for bilogical morphisms along the lines of Proposition 1.4 of [12] and we show that, if $h : \mathbb{L} \to_b \mathbb{L}'$, then

$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}'))$. Finally, the notion of *isomorphism* between logicoids is introduced as a bijective mapping $h : A \to A'$ for which both $h : \mathbb{L} \to \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to \mathbb{L}$ are logical. It is shown that this is tantamount to requiring that $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ and $h : \mathbb{L} \to_b \mathbb{L}'$.

In Section 7.4, given an algebraic grid $\hat{\mathbf{A}}$ and a grid congruence θ on $\hat{\mathbf{A}}$, we define the *quotient closure operator* C^θ on the quotient grid $\hat{\mathbf{A}}/\theta$ of an operator C on $\hat{\mathbf{A}}$. This gives rise to the *quotient logicoid* $\mathbb{L}^\theta = \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$ of a given logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and, moreover, makes the quotient grid morphism $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ into a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta$. Quotient logicoids are important because, among other things, they allow us to prove analogs of the Homomorphism Theorems of Universal Algebra for logicoids. We prove analogs of the Homomorphism Theorem, of the Second Isomorphism Theorem and of the Correspondence Theorem. The latter, in particular, enables us to show that the Tarski grid congruence of a quotient logicoid is the quotient of the Tarski grid congruence of the parent. We also look at *reductions* of logicoids. We show the important results that the reduction of a quotient logicoid coincides with the reduction of its parent and that the reductions of two logicoids related via a bilogical morphism are isomorphic logicoids.

In Section 7.5, the goal is to develop a theory of matrix models for logicoids along the lines of the traditional theory for monotonic logics and the theory developed in Section 3.5 for logicates. We start with a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ over a base algebraic grid $\hat{\mathbf{B}}$. Lack of structurality compels us to consider structures over fixed *interpretations*. These are pairs $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where $\hat{\mathbf{A}}$ is an algebraic grid and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism. A *matrix* is a pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, where $F \subseteq A$. If F is an $\mathbb{L}$-*filter*, i.e., if $h^{-1}(F) \in \mathcal{C}^\flat$, then $\mathfrak{A}$ is called an $\mathbb{L}$-*matrix*. Moreover, $\mathfrak{A}$ is *reduced* if $\Omega_{\mathcal{A}}(F) = \Delta_{\mathbf{A}}$. On any interpretation $\mathcal{A}$, there is, induced by $\mathbb{L}$ and h, a closure operator $C_{\mathcal{A}}$. In case $\mathrm{Ker}(h)$ is a logical grid congruence of $\mathbb{L}$, the induced structure $\mathbb{L}_{\mathcal{A}} = \langle \hat{\mathbf{A}}, C_{\mathcal{A}} \rangle$ is a logicoid and, moreover, the mapping $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ becomes a bilogical morphism $h : \mathbb{L} \to_b \mathbb{L}_{\mathcal{A}}$. Further, it can be shown that the theories of $\mathbb{L}_{\mathcal{A}}$ coincide with the $\mathbb{L}$-filters of $\mathbb{L}$ on $\mathcal{A}$.

In the remainder of the section we look at ways grid morphisms interact with filters. For instance, we show that, for two interpretations connected by a grid morphism, inverse images of $\mathbb{L}$-filters are $\mathbb{L}$-filters and conversely. Considering quotient interpretations, it is shown that for an $\mathbb{L}$-filter F on $\mathcal{A}$ to be the inverse image under a quotient morphism π_θ of an $\mathbb{L}$-filter on $\mathcal{A}/\theta$ it is necessary and sufficient that θ be compatible with F. Two interpretations that are related by a grid morphism may, under certain circumstances, establish very close ties between corresponding $\mathbb{L}$-filters. The closest connection occurs when the grid morphism in question is a bilogical morphism between the filter structures. It then establishes an isomorphism between the two posets of filters under the corresponding grid orderings. If this happens between two closure structures, one in the source interpretation and another in the target, and the structure in the source interpretation consists of all $\mathbb{L}$-filters, then so does the one in the target. This yields that the $\mathbb{L}$-filters on a reduced interpretation coincide with the reductions of the $\mathbb{L}$-filters on the parent interpretation. At the end of the section, we present an analog of a standard result asserting that a base logicoid is complete with respect to all its matrix models as well as with respect to all its reduced matrix models.

7.2 Logical Congruences

Let $\mathcal{L}$ be a logical (or algebraic) language. That is, $\mathcal{L}$ is a set of connectives (or operation symbols) of finite arities. We consider algebras of type $\mathcal{L}$, or $\mathcal{L}$-algebras, $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$. If, in addition, a complete lattice order $\leq$ on $\mathcal{P}(A)$ is given, then the structure $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ is called an (*algebraic*) *grid*, in analogy with the *grids* $\hat{A} = \langle A, \leq \rangle$ on sets. Both structures were introduced and studied in Chapter 6. Continuing our review from Chapter 6, a *closure operator* on $\hat{\mathbf{A}}$ is a function

$$C : \mathcal{P}(A) \to \mathcal{P}(A),$$

satisfying, for all $X, Y \subseteq A$,

(Inflationarity) $X \leq C(X)$;

(Monotonicity) $X \leq Y$ implies $C(X) \leq C(Y)$;

(Idempotency) $C(C(X)) = C(X)$.

An *algebraic logicoid* is a pair $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$, where:

- $\hat{\mathbf{A}}$ is an algebraic grid;
- C is a closure operator on $\hat{\mathbf{A}}$.

We use $\mathcal{C}$ for the set of its *theories*,

$$\mathcal{C} = \{X \subseteq A : C(X) = X\}.$$

Recall from Proposition 102 that, with the order inherited from the grid, $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq \rangle$ form a complete lattice, with meets identical with those in the grid.

Let $\theta \in \mathrm{Con}(\mathbf{A})$ be a congruence on the $\mathcal{L}$-algebra $\mathbf{A}$. We call θ *compatible with* a set $X \subseteq A$ if, for all $a, b \in A$,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in X \quad \text{imply} \quad b \in X.$$

Compatibility of θ with X is tantamount to X being a union of θ-congruence classes. We denoted by $\mathrm{Cmp}(\theta)$ the set of all $X \subseteq A$ with which θ is compatible.

Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ be an algebraic grid. A congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is called a *grid congruence* of $\hat{\mathbf{A}}$, written $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ if $\langle \mathrm{Cmp}(\theta), \leq \rangle$ is a complete sublattice of $\langle \mathcal{P}(A), \leq \rangle$. It was shown in Proposition 120 that $\mathbf{Con}(\hat{\mathbf{A}}) = \langle \mathrm{Con}(\hat{\mathbf{A}}), \subseteq \rangle$ is a complete lattice whose joins coincide with those of $\mathbf{Con}(\mathbf{A})$.

Let $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ be a grid and $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ an algebraic logicoid on $\hat{\mathbf{A}}$. We say that $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ is a **logical grid congruence of** C, or **of** $\mathbb{L}$, if θ is compatible with every theory of C, i.e.,

$$\mathcal{C} \subseteq \mathrm{Cmp}(\theta).$$

We write $\mathrm{Con}(\mathbb{L})$ for the collection of all logical congruences of $\mathbb{L}$. Moreover, $\mathbf{Con}(\mathbb{L}) = \langle \mathrm{Con}(\mathbb{L}), \subseteq \rangle$ denotes the collection of logical congruences, ordered by the subset relation between congruences. We show that this partially ordered set has a maximum element.

Proposition 129 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *be an algebraic logicoid. Then* $\mathbf{Con}(\mathbb{L})$ *is a principal ideal of* $\mathbf{Con}(\hat{\mathbf{A}})$.

Proof: We use elements of the proofs of Lemma 119 and Proposition 120. First note that $\Delta_{\mathbf{A}} \in \mathrm{Con}(\mathbb{L})$, whence $\mathrm{Con}(\mathbb{L}) \neq \varnothing$. Now consider $\theta := \bigvee \mathrm{Con}(\mathbb{L})$. By Proposition 120, $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Moreover, since, for all $\eta \in \mathrm{Con}(\mathbb{L})$, $\mathcal{C} \subseteq \mathrm{Cmp}(\eta)$, we get, using the equivalence in the proof of Lemma 119,

$$\mathcal{C} \subseteq \bigcap_{\eta \in \mathrm{Con}(\mathbb{L})} \mathrm{Cmp}(\eta) = \mathrm{Cmp}\left(\bigvee_{\eta \in \mathrm{Con}(\mathbb{L})} \eta \right) = \mathrm{Cmp}(\theta).$$

Hence, $\theta \in \mathrm{Con}(\mathbb{L})$ and, therefore, θ is the maximum element in $\mathrm{Con}(\mathbb{L})$. ∎

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid on the grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$. The **Tarski grid congruence of** $\mathbb{L}$ (Definition 1.1 of [12]) is

$$\widetilde{\Omega}(\mathbb{L}) = \max \mathbf{Con}(\mathbb{L}),$$

that is, $\widetilde{\Omega}(\mathbb{L})$ is the largest logical grid congruence of $\mathbb{L}$. The **Tarski operator on** $\hat{\mathbf{A}}$ is the mapping

$$\widetilde{\Omega}_{\hat{\mathbf{A}}} : \mathbb{L} \longmapsto \widetilde{\Omega}(\mathbb{L}),$$

i.e., it is the mapping $\mathbb{L} \mapsto \widetilde{\Omega}(\mathbb{L})$ restricted to algebraic logicoids over the same underlying grid $\hat{\mathbf{A}}$. This notation can be extended by writing $\widetilde{\Omega}_{\hat{\mathbf{A}}}(C)$ for the Tarski congruence of the algebraic logicate $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$. It follows from the definition of $\widetilde{\Omega}(\mathbb{L})$ that

$$\mathrm{Con}(\mathbb{L}) = \{\theta \in \mathrm{Con}(\hat{\mathbf{A}}) : \theta \subseteq \widetilde{\Omega}(\mathbb{L})\}.$$

To obtain a better understanding of the Tarski congruence, we consider **logical matrices** over $\hat{\mathbf{A}}$, i.e., pairs $\mathfrak{A} = \langle \hat{\mathbf{A}}, X \rangle$, where $X \subseteq A$ (see, e.g., Section

1.4 of [3] or Page 16 of [12]). We say that a grid congruence $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ is a **grid congruence of** $\mathfrak{A}$, written $\theta \in \mathrm{Con}(\mathfrak{A}) = \mathrm{Con}(\langle \hat{\mathbf{A}}, X\rangle)$, if θ is compatible with X. Moreover, we use $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}), \subseteq\rangle$ for the corresponding partially ordered set.

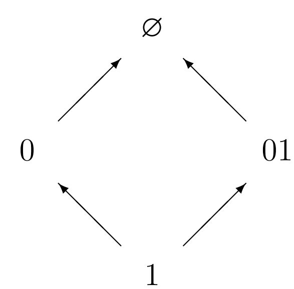
Corollary 130 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$ *be a grid and* $\mathfrak{A} = \langle \hat{\mathbf{A}}, X\rangle$ *a logical matrix over* $\hat{\mathbf{A}}$. *Then* $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}), \subseteq\rangle$ *is a complete lattice and a principal ideal of* $\mathbf{Con}(\mathbf{A})$.

Proof: This is a direct consequence of Proposition 129, since, given $\mathfrak{A} = \langle \hat{\mathbf{A}}, X\rangle$, one can construct an algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ whose only theories are X and $M := \max\langle \mathcal{P}(A), \leq\rangle$, i.e., set, for all $Y \subseteq A$,

$$C(Y) = \begin{cases} X, & \text{if } Y \leq X, \\ M, & \text{otherwise.} \end{cases}$$

Note that $M \in \mathrm{Cmp}(\theta)$, for all $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. It follows that $\mathbf{Con}(\mathfrak{A}) = \mathbf{Con}(\mathbb{L})$. ∎

The construction in the proof of Corollary 130 is very similar to the traditional one, but the nontraditional grid introduces potential nonmonotonic inferences. E.g., consider the set $A = \{0, 1\}$ and the grid shown below, where we use $0, 1$ and 01 as abbreviations for the subsets $\{0\}, \{1\}$ and $\{0, 1\}$ of $A = \{0, 1\}$, respectively.



Suppose that $X = 01$. Then, the closure operator C defined in the proof is

$$C(0) = C(\varnothing) = \varnothing, \ C(1) = C(01) = 01.$$

Notice the nonmonotonic inference $C(0) = \varnothing$. Notice also that, in the closure system $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq\rangle$, with the order inherited from the grid (see Chapter 6), the theory $\varnothing$ is greater than the theory 01.

Corollary 130 permits us to define the **Leibniz grid congruence of** $\mathfrak{A}$ or the **Leibniz grid congruence of** X **on** $\hat{\mathbf{A}}$, written $\Omega(\mathfrak{A}) = \Omega_{\hat{\mathbf{A}}}(X)$, as the largest grid congruence on $\hat{\mathbf{A}}$ that is compatible with X. Then, given an algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$, it is clear, by the definition of $\widetilde{\Omega}(\mathbb{L})$, that

$$\widetilde{\Omega}(\mathbb{L}) = \bigcap\{\Omega_{\hat{\mathbf{A}}}(X) : X \in \mathcal{C}\}.$$

A consequence is that the Tarski operator on a grid $\hat{\mathbf{A}}$ is monotone.

Lemma 131 *Let* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ *be a grid. Then, for all logicoids* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}, C' \rangle$ *on* $\hat{\mathbf{A}}$,

$$\mathbb{L} \leq \mathbb{L}' \quad \textit{implies} \quad \widetilde{\Omega}(\mathbb{L}) \subseteq \widetilde{\Omega}(\mathbb{L}').$$

Proof: We have that

$$\begin{array}{rcl} \mathbb{L} \leq \mathbb{L}' & \text{iff} & \mathcal{C}' \subseteq \mathcal{C} \quad \text{(Proposition 106)} \\ & \text{implies} & \widetilde{\Omega}(\mathbb{L}) \subseteq \widetilde{\Omega}(\mathbb{L}'), \end{array}$$

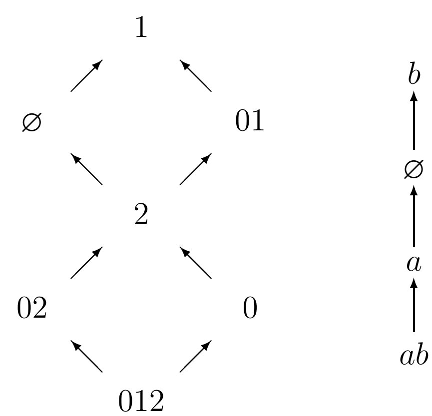
where the last inclusion follows directly from the definition of Tarski congruence. ■

7.3 Bilogical Morphisms

Given two algebraic logicoids $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ on the grids $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$, respectively, a **logical morphism** $h : \mathbb{L} \to \mathbb{L}'$ is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$, such that $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$, i.e., such that, for all $X' \subseteq A'$,

$$C'(X') = X' \quad \text{implies} \quad C(h^{-1}(X')) = h^{-1}(X').$$

Consider, e.g., the sets $A = \{0, 1, 2\}$ and $A' = \{a, b\}$, which form two grids when the following two orderings are applied on their powersets.



The mapping $h : A \to A'$, with $0 \mapsto a$, $1 \mapsto b$ and $2 \mapsto a$ is a grid morphism as can be checked quickly by hand. We have

$$012 \leq 02 \leq \varnothing \leq 1 \quad \text{iff} \quad ab \leq' a \leq' \varnothing \leq' b.$$

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ be the two logicoids determined by

$$\mathcal{C} = \{\{0, 2\}, \{1\}\}, \quad \mathcal{C}' = \{\{a\}, \{b\}\}.$$

Then $h : \mathbb{L} \to \mathbb{L}'$ becomes a logical morphism.

We say that $\mathbb{L}$ is **projectively generated from $\mathbb{L}'$ by h** if

$$\mathcal{C} = h^{-1}(\mathcal{C}').$$

Finally, h is a **bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$**, or a **bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$**, written $h : \mathbb{L} \to_b \mathbb{L}'$, if it is a grid morphism $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ and it projectively generates $\mathbb{L}$ from $\mathbb{L}'$. For these definitions, as applied to the traditional setting, see Page 20 of [12] (also [4] for the original notions). The following proposition is an analog for logicoids of Proposition 1.4 of [12]. Note that, contrary to the case for logicates (see Chapter 3), we can recover the six equivalent conditions, due to the presence of grids and grid morphisms.

Proposition 132 *Let $\mathbb{L} = \langle\hat{\mathbf{A}}, C\rangle$ and $\mathbb{L}' = \langle\hat{\mathbf{A}}', C'\rangle$ be algebraic logicoids and $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ a grid morphism. The following conditions are equivalent.*

(i) h *is a bilogical morphism from* $\mathbb{L}$ *onto* $\mathbb{L}'$;

(ii) *For all* $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$, $C(X) = h^{-1}(C'(h(X)))$;

(iii) *For all* $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$, $h(C(X)) = C'(h(X))$ *and* $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$;

(iv) *For all* $Y \subseteq A'$, $C'(Y) = h(C(h^{-1}(Y)))$ *and* $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$;

(v) $\mathcal{C}' = h(\mathcal{C})$ *and* $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$;

(vi) $\mathcal{C} = h^{-1}(\mathcal{C}')$.

Proof:

(i)⇒(ii) Suppose that $h : \mathbb{L} \to_b \mathbb{L}'$ and $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$. Since C' is $\le$-inflationary, $h(X) \le' C'(h(X))$. Since h^{-1} is a complete lattice embedding, $X \le h^{-1}(C'(h(X)))$. By $\le$-monotonicity of C and the hypothesis, $C(X) \le h^{-1}(C'(h(X)))$. Conversely, by $\le$-inflationarity of C, $X \le C(X)$. By the fact that h^{-1} is a complete lattice embedding, $h(X) \le' h(C(X))$. By hypothesis and the $\le'$-monotonicity of C', $C'(h(X)) \le' h(C(X))$. Since h^{-1} is a complete lattice embedding, $h^{-1}(C'(h(X))) \le C(X)$. We have shown that, for all $X \in \mathrm{Cmp}(\mathrm{Ker}(h))$, $C(X) = h^{-1}(C'(h(X)))$.

(ii)⇒(iii) Since h is surjective, the first equality follows immediately from the hypothesis. The second statement also follows from the hypothesis and Lemma 117.

(iii)⇒(iv) Let $Y \subseteq A'$. Then, there exists $X = h^{-1}(Y) \in \mathrm{Cmp}(\mathrm{Ker}(h))$, such that $Y = h(X)$. Thus, by assumption,

$$C'(Y) = C'(h(X)) = h(C(X)) = h(C(h^{-1}(Y))).$$

(iv)⇒(v) Both statements follow from the hypothesis.

(v)⇒(vi) Since $\mathcal{C}' = h(\mathcal{C})$ and $\mathcal{C} \subseteq \mathrm{Cmp}(\mathrm{Ker}(h))$, we get $h^{-1}(\mathcal{C}') = \mathcal{C}$.

(iv)⇒(i) By the definition of bilogical morphism.

■

Proposition 132 yields immediately the following results revealing a very tight relation between theories of two algebraic logicoids related via a bilogical morphism. We start with an analog of Proposition 1.5 of [12].

Proposition 133 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ be two algebraic logicoids and $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ be a grid morphism. Then $h : \mathbb{L} \to_b \mathbb{L}'$ if and only if the closure systems $\hat{\mathcal{C}} = \langle \mathcal{C}, \leq\rangle$ and $\hat{\mathcal{C}} = \langle \mathcal{C}', \leq\rangle$ are isomorphic via the mapping induced by h.*

Proof: By Proposition 132,

$$\begin{array}{rccc} h: & \mathcal{C} & \longrightarrow & \mathcal{C}'; \\ & X & \longmapsto & h(X), \end{array}$$

is a bijection. By the definition of grid morphisms, ir preserves and reflects order. Thus, $h : \hat{\mathcal{C}} \to \hat{\mathcal{C}}'$ is an isomorphism. Conversely, the hypothesis yields that $\mathcal{C} = h^{-1}(\mathcal{C}')$. Thus, by Proposition 132, $h : \mathbb{L} \to_b \mathbb{L}'$. ■

We now show that logical congruence systems correspond under the action of inverse bilogical morphisms. Let us write

$$\mathrm{Con}(\mathbb{L})^\eta := \{\theta \in \mathrm{Con}(\mathbb{L}) : \eta \subseteq \theta\}.$$

Proposition 134 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ be algebraic logicoids and $h : \mathbb{L} \to_b \mathbb{L}'$. Then*

$$\mathrm{Con}(\mathbb{L})^{\mathrm{Ker}(h)} = h^{-1}(\mathrm{Con}(\mathbb{L}')).$$

Proof: We consider the mapping

$$\mathrm{Con}(\mathbb{L})^{\mathrm{Ker}(h)} \underset{h^{-1}}{\overset{h}{\rightleftarrows}} \mathrm{Con}(\mathbb{L}')$$

Note that, if $\theta' \in \mathrm{Con}(\mathbf{A}')$, then $h^{-1}(\theta') \in \mathrm{Con}(\mathbf{A})$ and, clearly, since θ' is reflexive, $\mathrm{Ker}(h) \subseteq h^{-1}(\theta')$. Conversely, if $\theta \in \mathrm{Con}(\mathbf{A})$, such that $\mathrm{Ker}(h) \subseteq \theta$, it is not difficult to see that $h(\theta) \in \mathrm{Con}(\mathbf{A}')$. The hypothesis $\mathrm{Ker}(h) \subseteq \theta$ is used in showing the transitivity property of $h(\theta)$.

Next, we use the fact that $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ is a grid morphism to show that the mappings are well defined between grid congruences. If $X \subseteq A$ and $\theta' \in \mathrm{Con}(\mathbf{A}')$, we have

$$X \in \mathrm{Cmp}(h^{-1}(\theta')) \quad \text{iff} \quad h(X) \in \mathrm{Cmp}(\theta').$$

So the fact that, if $\theta' \in \mathrm{Con}(\hat{\mathbf{A}}')$, then $h^{-1}(\theta) \in \mathrm{Con}(\hat{\mathbf{A}})$ is established by looking at the following diagram.

$$\begin{array}{ccc}
\langle \mathrm{Cmp}(\theta'), \leq' \rangle & \hookrightarrow & \langle \mathcal{P}(A'), \leq' \rangle \\
h \uparrow & & \downarrow h^{-1} \\
\langle \mathrm{Cmp}(h^{-1}(\theta')), \leq \rangle & & \langle \mathcal{P}(A), \leq \rangle
\end{array}$$

A consequence of the preceding equivalence is that, if $X' \subseteq A'$ and $\theta \in \mathrm{Con}(\mathbf{A})$, such that $\mathrm{Ker}(h) \subseteq \theta$, then

$$X' \in \mathrm{Cmp}(h(\theta)) \quad \text{iff} \quad h^{-1}(X') \in \mathrm{Cmp}(\theta).$$

So the fact that, if $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$, with $\mathrm{Ker}(h) \subseteq \theta$, then $h(\theta) \in \mathrm{Con}(\hat{\mathbf{A}}')$ is established by looking at the following diagram.

$$\begin{array}{ccc}
\langle \mathrm{Cmp}(h(\theta)), \leq' \rangle & & \langle \mathcal{P}(A'), \leq' \rangle \\
h^{-1} \downarrow & & \downarrow h^{-1} \\
\langle \mathrm{Cmp}(\theta), \leq \rangle & \hookrightarrow & \langle \mathcal{P}(A), \leq \rangle
\end{array}$$

Finally, since $h : \mathbb{L} \to_b \mathbb{L}'$ is a bilogical morphism, the preceding mappings are also well defined between logical congruences. ∎

Proposition 135 *Let* $\mathbb{L} = \langle \mathbf{A}, C \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', C' \rangle$ *be algebraic logicates and* $h : \mathbb{L} \to_b \mathbb{L}'$. *Then*

$$\widetilde{\Omega}(\mathbb{L}) = h^{-1}(\widetilde{\Omega}(\mathbb{L}')).$$

Proof: This is a consequence of Proposition 134, the fact that $h : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ implies that h^{-1} is order preserving and order reflecting and the definition of Tarski congruence. ∎

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ be two algebraic logicoids. A bijection $h : A \to A'$ is an **isomorphism** between $\mathbb{L}$ and $\mathbb{L}'$, written $h : \mathbb{L} \cong \mathbb{L}'$, if both $h : \mathbb{L} \to \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to \mathbb{L}$ are logical morphisms.

Lemma 136 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ *be two algebraic logicoids and* $h : A \to A'$ *a bijection.* $h : \mathbb{L} \cong \mathbb{L}'$ *if and only if* $h : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$ *and* $h : \mathbb{L} \to_b \mathbb{L}'$.

Proof: To establish the equivalence, it suffices to show that

$$(h^{-1}(\mathcal{C}') \subseteq \mathcal{C} \quad \text{and} \quad h(\mathcal{C}) \subseteq \mathcal{C}') \quad \text{iff} \quad \mathcal{C} = h^{-1}(\mathcal{C}').$$

Suppose the conjunction in the parenthesis holds and let $X \in \mathcal{C}$. Then, by hypothesis, $X' = h(X) \in \mathcal{C}'$. Thus, $X = h^{-1}(h(X)) \in h^{-1}(\mathcal{C}')$. This proves that $\mathcal{C} \subseteq h^{-1}(\mathcal{C}')$. Since the reverse inclusion is part of the hypothesis, the conclusion follows.

Suppose, conversely, that $\mathcal{C} = h^{-1}(\mathcal{C}')$. Then, clearly, $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}$. So, suppose, $X \in \mathcal{C}$. By hypothesis, there exists $X' \in \mathcal{C}'$, such that $X = h^{-1}(X')$. Thus, $h(X) = h(h^{-1}(X')) = X' \in \mathcal{C}'$. This proves that $h(\mathcal{C}) \subseteq \mathcal{C}'$. ■

7.4 Quotients

Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ be an algebraic logicoid on a grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and suppose $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Consider the quotient grid $\hat{\mathbf{A}}/\theta = \langle \mathbf{A}/\theta, \leq^\theta \rangle$ and define a mapping

$$C^\theta := C/\theta : \mathcal{P}(A/\theta) \to \mathcal{P}(A/\theta)$$

by setting, for all $S \subseteq A/\theta$,

$$C^\theta(S) = \pi_\theta(C(\pi_\theta^{-1}(S))),$$

where $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ is the quotient grid morphism.

We show that, if θ happens to be a logical congruence, then the quotient is a legitimate logicoid on the quotient grid. For the corresponding result in the traditional framework, see Page 21 of [12]. For the one related to logicates, see Proposition 23.

Proposition 137 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *be an algebraic logicoid and suppose* $\theta \in \mathrm{Con}(\mathbb{L})$*. Then*

$$\mathbb{L}^\theta = \mathbb{L}/\theta := \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$$

is also an algebraic logicoid.

Proof: Since $\theta \in \mathrm{Con}(\mathbb{L})$, a fortiori $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Hence, by Lemma 122, $\hat{\mathbf{A}}/\theta$ is a legitimate grid and $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ becomes a grid morphism. Let $Y, Y' \subseteq \mathcal{P}(A/\theta)$.

- Since $C^\theta(Y) = \pi_\theta(C(\pi_\theta^{-1}(Y)))$, we get

  $$\pi_\theta^{-1}(Y) \leq C(\pi_\theta^{-1}(Y)) = \pi_\theta^{-1}(C^\theta(Y)).$$

  Hence, by definition, $Y \leq^\theta C^\theta(Y)$.

- Suppose, next, that $Y \leq^\theta Y'$. By definition of $\leq^\theta$, $\pi_\theta^{-1}(Y) \leq \pi_\theta^{-1}(Y')$. By monotonicity of C on $\hat{\mathbf{A}}$, $C(\pi_\theta^{-1}(Y)) \leq C(\pi_\theta^{-1}(Y'))$. By definition of $\leq^\theta$ and the fact that $\theta \in \mathrm{Con}(\mathbb{L})$, $\pi_\theta(C(\pi_\theta^{-1}(Y))) \leq^\theta \pi_\theta(C(\pi_\theta^{-1}(Y)))$. Thus, by definition of C^θ, $C^\theta(Y) \leq^\theta C^\theta(Y')$.

- Finally, we have

$$\begin{array}{rcl} C^\theta(C^\theta(S)) & = & \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))))) \quad \text{(Definition of } C^\theta\text{)} \\ & = & \pi_\theta(C(C(\pi_\theta^{-1}(S)))) \quad (\theta \in \mathrm{Con}(\mathbb{L})) \\ & = & \pi_\theta(C(\pi_\theta^{-1}(S))) \quad \text{(Idempotency)} \\ & = & C^\theta(S). \quad \text{(Definition of } C^\theta\text{)} \end{array}$$

Thus, C^θ is inflationary, monotone and idempotent with respect to $\leq^\theta$, showing that $\mathbb{L}^\theta$ is an algebraic logicoid on $\hat{\mathbf{A}}^\theta$. ■

We call $\mathbb{L}^\theta = \mathbb{L}/\theta \coloneqq \langle \hat{\mathbf{A}}/\theta, C^\theta \rangle$ the **quotient logicoid** of $\mathbb{L}$ by the logical grid congruence θ.

Now that we know that $\mathbb{L}$ and its quotient $\mathbb{L}^\theta$ are logicoids, we show that the quotient grid morphism $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ becomes a bilogical morphism $\pi_\theta : \mathbb{L} \to_b \mathbb{L}/\theta$.

Proposition 138 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *be an algebraic logicoid and suppose* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then* $\pi_\theta : \mathbb{L} \to \mathbb{L}^\theta$ *is a bilogical morphism.*

Proof: By Lemma 122, $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ is a grid morphism. So it suffices to show that it projectively generates $\mathbb{L}$ from $\mathbb{L}^\theta$. Suppose, first $X \in \mathcal{C}$. Then

$$\begin{array}{rcl} \pi_\theta^{-1}(C^\theta(\pi_\theta(X))) & = & \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X))))) \quad \text{(Definition of } C^\theta\text{)} \\ & = & C(X) \quad (\theta \in \mathrm{Con}(\mathbb{L})) \\ & = & X. \quad (X \in \mathcal{C}) \end{array}$$

This shows that $\mathcal{C} \subseteq \pi_\theta^{-1}(\mathcal{C}^\theta)$. Conversely, let $S \in \mathcal{C}^\theta$. Then

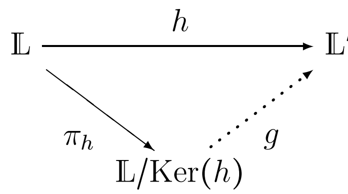
$$\begin{array}{rcl} \pi_\theta^{-1}(S) & = & \pi_\theta^{-1}(C^\theta(S)) \quad (S \in \mathcal{C}^\theta) \\ & = & \pi_\theta^{-1}(\pi_\theta(C(\pi_\theta^{-1}(S)))) \quad \text{(Definition of } C^\theta\text{)} \\ & = & C(\pi_\theta^{-1}(S)). \quad (\theta \in \mathrm{Con}(\mathbb{L})) \end{array}$$

Hence $\pi_\theta^{-1}(\mathcal{C}^\theta) \subseteq \mathcal{C}$. Thus, $\mathcal{C} = \pi_\theta^{-1}(\mathcal{C}^\theta)$, showing that π_θ projectively generates $\mathbb{L}$ from $\mathbb{L}/\theta$. So $\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta$ is a bilogical morphism. ■

$\pi_\theta : \mathbb{L} \to_b \mathbb{L}/\theta$ is called the **quotient (bilogical) morphism** or the **canonical projection (bilogical) morphism**.

Theorems 1.8, 1.9 and 1.10 of [12] adapt to the framework of abstract logics the well-known Homomorphism Theorems of universal algebra [5, 18, 1]. Theorems 25, 26 and 27 further reformulated those results to fit in the framework of logicates. We undertake here a similar adaptation for logicoids. The flavor is still the same.

Theorem 139 (Homomorphism) *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}', C'\rangle$ *be two algebraic logicoids and* $h : \mathbb{L} \to_b \mathbb{L}'$. *Then* $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$ *via a unique isomorphism* g*, such that* $h = g \circ \pi_h$,



where $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ *is the bilogical projection morphism.*

Proof: By hypothesis, h is a bilogical morphism. Thus, by Proposition 132, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$. It follows, by Proposition 138, that $\pi_h : \mathbb{L} \to \mathbb{L}/\mathrm{Ker}(h)$ is a bilogical morphism. By Theorem 125, there exists a unique $g : \hat{\mathbf{A}}/\theta \cong \hat{\mathbf{A}}'$, such that $h = g \circ \pi_h$. So it suffices to show that this is a bilogical morphism. We have

$$\begin{array}{lll}
\mathcal{C}^{\mathrm{Ker}(h)} & = & \pi_h(\mathcal{C}) \quad (\pi_h : \mathbb{L} \to_b \mathbb{L}/\mathrm{Ker}(h)) \\
& = & g^{-1}(h(\mathcal{C})) \quad (h = g \circ \pi_h \text{ and } g : \hat{\mathbf{A}}/\theta \cong \hat{\mathbf{A}}') \\
& = & g^{-1}(\mathcal{C}'). \quad (h : \mathbb{L} \to_b \mathbb{L}')
\end{array}$$

Thus, by definition, $g : \mathbb{L}/\mathrm{Ker}(h) \to \mathbb{L}'$ is a bilogical morphism. ∎

Theorem 140 (Second Isomorphism) *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ *be an algebraic logicoid and* $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, *such that* $\theta \subseteq \theta'$. *Then* $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$ *and*

$$(\mathbb{L}/\theta)/(\theta'/\theta) \cong \mathbb{L}/\theta',$$

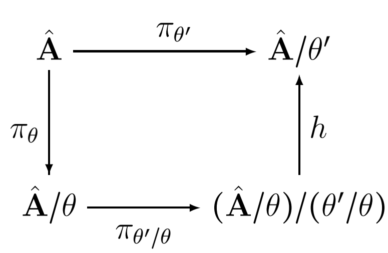
where the isomorphism is given by

$$(a/\theta)/(\theta'/\theta) \mapsto a/\theta'.$$

Proof: By Theorem 126, we know that

$$\begin{array}{rlll}
h: & (\hat{\mathbf{A}}/\theta)/(\theta'/\theta) & \longrightarrow & \hat{\mathbf{A}}/\theta'; \\
& (a/\theta)/(\theta'/\theta) & \longmapsto & a/\theta',
\end{array}$$

is an isomorphism that makes the following rectangle commute,



where $\pi_\theta, \pi_{\theta'}$ and $\pi_{\theta'/\theta}$ are the natural projections. Since $\theta, \theta' \in \mathrm{Con}(\mathbb{L})$, by Proposition 138, π_θ and $\pi_{\theta'}$ are bilogical morphisms. We can also show that $\theta'/\theta \in \mathrm{Con}(\mathbb{L}/\theta)$. Let $a, b \in A$ and $S \subseteq A/\theta$, such that $\langle a/\theta, b/\theta\rangle \in \theta'/\theta$ and $a/\theta \in C^\theta(S)$. Then, by the Definition of C^θ, $\langle a, b\rangle \in \theta'$ and $a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$. Hence, since $\theta \in \mathrm{Con}(\mathbb{L})$, $\langle a, b\rangle \in \theta'$ and $a \in C(\pi_\theta^{-1}(S))$. Since $\theta' \in \mathrm{Con}(\mathbb{L})$, this yields $b \in C(\pi_\theta^{-1}(S))$, whence $b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(S)))$, i.e., $b/\theta \in C^\theta(S)$. Hence $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. Now, again by Proposition 138, the projection $\pi_{\theta'/\theta}$ is also a bilogical morphism. We check that h, which is already known to be a grid isomorphism, is also a bilogical morphism. We have

$$\begin{array}{rcll} (\mathcal{C}^\theta)^{\theta'/\theta} & = & \pi_{\theta'/\theta}(\mathcal{C}^\theta) & (\pi_{\theta'/\theta} : \mathbb{L}^\theta \to_b (\mathbb{L}^\theta)^{\theta'/\theta}) \\ & = & \pi_{\theta'/\theta}(\pi_\theta(\mathcal{C})) & (\pi_\theta : \mathbb{L} \to_b \mathbb{L}^\theta) \\ & = & h^{-1}(\pi_{\theta'}(\mathcal{C})) & (h \circ \pi_{\theta'/\theta} \circ \pi_\theta = \pi_{\theta'} \text{ and } h \text{ an iso}) \\ & = & h^{-1}(\mathcal{C}^{\theta'}). & (\pi_{\theta'} : \mathbb{L} \to_b \mathbb{L}^{\theta'}) \end{array}$$

Thus, $h : (\mathbb{L}/\theta)/(\theta'/\theta) \to \mathbb{L}/\theta'$ is indeed a bilogical morphism and, hence, an isomorphism. ∎

To formulate an analog of the Correspondence Theorem (see Theorem 1.10 of [12] for abstract logics and Theorem 27 for logicates), recall that $\widetilde{\Omega}(\mathbb{L})$ denotes the Tarski congruence of $\mathbb{L}$, i.e., the largest grid congruence on $\hat{\mathbf{A}}$ that is compatible with all theories of $\mathbb{L}$.

Theorem 141 (Correspondence) *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ be an algebraic logicoid and $\theta \in \mathrm{Con}(\mathbb{L})$. Then the segment $[\theta, \widetilde{\Omega}(\mathbb{L})]$ of the lattice $\mathbf{Con}(\hat{\mathbf{A}})$ is isomorphic to the lattice $\mathbf{Con}(\mathbb{L}^\theta)$ by the mapping $\theta' \mapsto \theta'/\theta$.*

Proof: By Theorem 140, if $\theta \subseteq \theta' \in \mathrm{Con}(\mathbb{L})$, then $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$. By Theorem 126, it suffices to prove that, for all $\theta \subseteq \theta' \in \mathrm{Con}(\hat{\mathbf{A}})$, if $\theta'/\theta \in \mathrm{Con}(\mathbb{L}^\theta)$, then $\theta' \in \mathrm{Con}(\mathbb{L})$. So let $a, b \in A$ and $X \in \mathcal{C}$, such that $\langle a, b\rangle \in \theta'$ and $a \in X$. As $\theta \in \mathrm{Con}(\mathbb{L})$, θ is compatible with X. Thus X is the union of θ-classes, that is $X = \pi_\theta^{-1}(\pi_\theta(X))$. Now, starting with the assumption, we get

$$\begin{array}{ll} \langle a, b\rangle \in \theta' \text{ and } a \in X & \\ \quad\text{iff} & \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X)))) \\ \quad\text{iff} & \langle a/\theta, b/\theta\rangle \in \theta'/\theta \text{ and } a/\theta \in C^\theta(\pi_\theta(X)) \\ \quad\text{implies} & b/\theta \in C^\theta(\pi_\theta(X)) \\ \quad\text{iff} & b/\theta \in \pi_\theta(C(\pi_\theta^{-1}(\pi_\theta(X)))) \\ \quad\text{iff} & b \in X. \end{array}$$

Therefore, $\theta' \in \mathrm{Con}(\mathbb{L})$ and the correspondence is established. ∎

The Correspondence Theorem has a significant consequence in relation to the Tarski congruences.

Corollary 142 *Let $\mathbb{L} = \langle \hat{\mathbf{A}}, C\rangle$ be an algebraic logicoid and $\theta \in \mathrm{Con}(\mathbb{L})$. Then*

$$\widetilde{\Omega}(\mathbb{L}^\theta) = \widetilde{\Omega}(\mathbb{L})/\theta.$$

Proof: By definition, the largest element in $\mathrm{Con}(\mathbb{L}^\theta)$ is $\widetilde{\Omega}(\mathbb{L}^\theta)$, whereas the largest element in $[\theta, \widetilde{\Omega}(\mathbb{L})]$ is clearly $\widetilde{\Omega}(\mathbb{L})$. Since, under the established correspondence of Theorem 141, these two elements correspond, we get the conclusion. ■

It follows that, for any algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$,

$$\widetilde{\Omega}(\mathbb{L}/\widetilde{\Omega}(\mathbb{L})) = \widetilde{\Omega}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}/\widetilde{\Omega}(\mathbb{L})}.$$

This leads us to the definition of reduction (see Definition 1.12 of [12]). We say that an algebraic logicoid $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ is **reduced** when it has only one logical congruence, i.e., when $\widetilde{\Omega}(\mathbb{L}) = \Delta_{\mathbf{A}}$. Given an algebraic logicoid $\mathbb{L}$, we define the **reduction** $\mathbb{L}^*$ **of** $\mathbb{L}$ by

$$\mathbb{L}^* = \mathbb{L}/\widetilde{\Omega}(\mathbb{L}).$$

If L is a class of algebraic logicoids, then we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

If $\mathbb{L}$ is an algebraic logicoid, then $\mathbb{L}^*$ is always reduced. Moreover, if $\mathbb{L}$ happens to already be reduced, then $\mathbb{L}$ and $\mathbb{L}^*$ are isomorphic and they may be identified.

We prove, next, some analogs of Propositions 1.13 and 1.14 of [12] (see, also, Propositions 29 and 30 for the case of logicates). The first asserts that the reduction of a quotient of a logicoid by a logical morphism is isomorphic to the reduction of the logicoid itself. The second proves that the reductions of two logicoids related via a bilogical morphism are isomorphic.

Proposition 143 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *be an algebraic logicoid and* $\theta \in \mathrm{Con}(\mathbb{L})$. *Then*

$$(\mathbb{L}^\theta)^* \cong \mathbb{L}^*.$$

Proof: We have

$$\begin{aligned} (\mathbb{L}^\theta)^* &= \mathbb{L}^\theta/\widetilde{\Omega}(\mathbb{L}^\theta) \quad \text{(Definition of Reduction)} \\ &= \mathbb{L}^\theta/(\widetilde{\Omega}(\mathbb{L})/\theta) \quad \text{(Corollary 142)} \\ &\cong \mathbb{L}/\widetilde{\Omega}(\mathbb{L}) \quad \text{(Theorem 140)} \\ &= \mathbb{L}^*. \quad \text{(Definition of Reduction)} \end{aligned}$$

The conclusion follows. ■

Proposition 144 *Let* $\mathbb{L} = \langle \hat{\mathbf{A}}, C \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{A}}', C' \rangle$ *be algebraic logicoids and* $h : \mathbb{L} \to_b \mathbb{L}'$ *a bilogical morphism. Then*

$$\mathbb{L}^* \cong \mathbb{L}'^*.$$

Proof: By Theorem 139, $\mathbb{L}/\mathrm{Ker}(h) \cong \mathbb{L}'$. By Proposition 135,

$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}'^*.$$

Since, by Proposition 132, $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$, by Proposition 143,

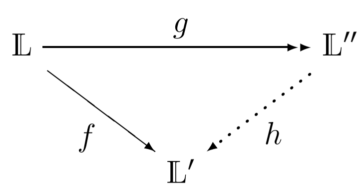
$$(\mathbb{L}/\mathrm{Ker}(h))^* \cong \mathbb{L}^*.$$

Therefore, $\mathbb{L}'^* \cong \mathbb{L}^*$. ■

We close the section with an analog of Proposition 1.15 of [12] (Proposition 31 for logicates), a sort of a "fill-in" theorem for arrows.

Proposition 145 *Let* $\mathbb{L}$, $\mathbb{L}'$ *and* $\mathbb{L}''$ *be algebraic logicoids,* $f : \mathbb{L} \to \mathbb{L}'$ *a logical morphism and* $g : \mathbb{L} \to \mathbb{L}''$ *a bilogical morphism, such that* $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. *Then, there is a unique logical morphism* $h : \mathbb{L}'' \to \mathbb{L}'$, *such that*

$$h \circ g = f.$$



Moreover, f *projectively generates* $\mathbb{L}$ *from* $\mathbb{L}'$ *if and only if* h *projectively generates* $\mathbb{L}''$ *from* $\mathbb{L}'$.

Proof: By Lemma 128, there exists a unique $h : \hat{\mathbf{A}}'' \to \hat{\mathbf{A}}'$, such that $h \circ g = f$. Now we have

$$\begin{array}{rcl} g^{-1}(h^{-1}(\mathcal{C}')) & = & f^{-1}(\mathcal{C}') \quad (f = h \circ g) \\ & \subseteq & \mathcal{C} \quad (f : \mathbb{L} \to \mathbb{L}') \\ & = & g^{-1}(\mathcal{C}''). \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

We now get $h^{-1}(\mathcal{C}') \subseteq \mathcal{C}''$. Thus, $h : \mathbb{L} \to \mathbb{L}'$ is a logical morphism. That it projectively generates $\mathbb{L}''$ from $\mathbb{L}'$ if f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ follows from the fact that, in that case, the inclusion becomes an equality. Conversely, assume that h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. Then we have

$$\begin{array}{rcl} f^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) \quad (f = h \circ g) \\ & = & g^{-1}(\mathcal{C}'') \quad (\text{Assumption}) \\ & = & \mathcal{C}. \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

So f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. ■

7.5 Interpretations, Filters and Matrices

In this section, taking after the theory of logical matrices (see, e.g., [24, 3, 12, 8]), we present a similar theory suitable for algebraic logicoids along the lines of the theory developed for logicates in Section 3.5. Here, also, because of lack of structurality, one has to fix interpretations, i.e., grid morphisms that help interpret the underlying grid of the logicoid. A model theory along similar lines was devised for π-institutions in [21].

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, with $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$, be an algebraic logicoid. This is thought of as the focal object of our study, for which models are to be devised. So it is referred to as a **base (algebraic) logicoid** and its underlying algebraic grid $\hat{\mathbf{B}}$ as the **base (algebraic) grid**. The most appropriate notion of **(grid) interpretation** is that of a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where:

- $\hat{\mathbf{A}}$ is a grid algebra of the same type as the base grid $\hat{\mathbf{B}}$;
- $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism.

We say that $F \subseteq A$ is an $\mathbb{L}$-**filter** on $\hat{\mathcal{A}}$, if

$$h^{-1}(F) \in \mathcal{C}^\flat,$$

i.e., the inverse image under h of the $\mathbb{L}$-filter is a theory of the logicoid. By $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is denoted the collection of all $\mathbb{L}$-filters on the grid interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. If F is an $\mathbb{L}$-filter on $\mathcal{A}$, the pair $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is called a **(grid) matrix for** $\mathbb{L}$ or a **(grid)** $\mathbb{L}$-**matrix**. The class of all grid $\mathbb{L}$-matrices is denoted $\mathrm{Mat}(\mathbb{L})$.

A grid congruence $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$ is called a **(grid) matrix congruence** of $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ if $F \in \mathrm{Cmp}(\theta)$, i.e., if θ is compatible with F. The **Leibniz grid congruence of** $\mathfrak{A}$ or the **Leibniz grid congruence of** F **on** $\mathcal{A}$, denoted $\Omega(\mathfrak{A})$ or $\Omega_{\mathcal{A}}(F)$ is the largest grid matrix congruence of $\mathfrak{A}$, provided such a congruence exists. An grid $\mathbb{L}$-matrix $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is **reduced** if $\Omega_{\mathcal{A}}(F) = \Delta_{\mathbf{A}}$. The class of all reduced grid $\mathbb{L}$-matrices is denoted $\mathrm{Mat}^*(\mathbb{L})$.

Among the most important features of interpretations is that, if the kernel of their interpretation morphism is a grid congruence of the base logicoid, then they induce an algebraic logicoid on the algebra into which the interpretation takes place. Moreover, if this is the case, the mapping of the interpretation becomes a bilogical morphism from the original logicoid into the induced logicoid. This is similar to the situation encountered in the case of logicates (Proposition 32).

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. Define $C_{\mathcal{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$ by setting, for all $Y \subseteq A$,

$$C_{\mathcal{A}}(Y) = h(C^\flat(h^{-1}(Y))).$$

Proposition 146 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$*, with* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$*, an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$*. Then* $\mathbb{L}_\mathcal{A} = \langle \hat{\mathbf{A}}, C_\mathcal{A} \rangle$ *is an algebraic logicoid. Moreover, the grid morphism* $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ *is a bilogical morphism* $h : \mathbb{L} \to_b \mathbb{L}_\mathcal{A}$*.*

Proof: To see that $\mathbb{L}_\mathcal{A}$ is a logicoid, we must show inflationarity, monotonicity and idempotence of $C_\mathcal{A}$ with respect to $\leq$. Let $Y \subseteq A$. Since $C^\flat$ is inflationary, $h^{-1}(Y) \leq^\flat C^\flat(h^{-1}(Y))$. Since h^{-1} is a complete lattice embedding, $Y \leq h(C^\flat(h^{-1}(Y)))$. Thus, by definition of $C_\mathcal{A}$, $Y \leq C_\mathcal{A}(Y)$ and $C_\mathcal{A}$ is inflationary.

Next, let $Y, Y' \subseteq A$, such that $Y \leq Y'$. Then, since h^{-1} is a complete lattice embedding, $h^{-1}(Y) \leq^\flat h^{-1}(Y')$. By monotonicity of $C^\flat$ with respect to $\leq^\flat$, we get $C^\flat(h^{-1}(Y)) \leq^\flat C^\flat(h^{-1}(Y'))$. Therefore, again by the embedding property of h^{-1}, $h(C^\flat(h^{-1}(Y))) \leq h(C^\flat(h^{-1}(Y')))$, which, by definition of $C_\mathcal{A}$, amounts to $C_\mathcal{A}(Y) \leq C_\mathcal{A}(Y')$, showing that $C_\mathcal{A}$ is also monotone.

Finally, suppose $Y \subseteq A$. Then

$$\begin{array}{rcll} C_\mathcal{A}(C_\mathcal{A}(Y)) & = & h(C^\flat(h^{-1}(h(C^\flat(h^{-1}(Y)))))) & \text{(Definition of } C_\mathcal{A}) \\ & = & h(C^\flat(C^\flat(h^{-1}(Y)))) & (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ & = & h(C^\flat(h^{-1}(Y))) & (\mathbb{L} \text{ a logicoid}) \\ & = & C_\mathcal{A}(Y). & \text{(Definition of } C_\mathcal{A}) \end{array}$$

Thus, $C_\mathcal{A}$ is a closure operator with respect to $\leq$ and, hence, $\mathbb{L}_\mathcal{A}$ is a logicoid.

To see that h becomes a logical morphism, we must show that $h^{-1}(\mathcal{C}_\mathcal{A}) \subseteq \mathcal{C}^\flat$. So let $Y \in \mathcal{C}_\mathcal{A}$. Then, we have

$$\begin{array}{rcll} h^{-1}(Y) & = & h^{-1}(C_\mathcal{A}(Y)) & (Y \in \mathcal{C}_\mathcal{A}) \\ & = & h^{-1}(h(C^\flat(h^{-1}(Y)))) & \text{(Definition of } C_\mathcal{A}) \\ & = & C^\flat(h^{-1}(Y)) & (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ & \in & \mathcal{C}^\flat. & \text{(Definition of } \mathcal{C}^\flat) \end{array}$$

To see that it is a bilogical morphism, let $X \in \mathcal{C}^\flat$. Then we have

$$\begin{array}{rcll} h^{-1}(C_\mathcal{A}(h(X))) & = & h^{-1}(h(C^\flat(h^{-1}(h(X))))) & \text{(Definition of } C_\mathcal{A}) \\ & = & C^\flat(X) & (\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})) \\ & = & X. & (X \in \mathcal{C}^\flat) \end{array}$$

We conclude that $\mathcal{C}^\flat = h^{-1}(\mathcal{C}_\mathcal{A})$ and, therefore, $h : \mathbb{L} \to \mathbb{L}_\mathcal{A}$ is a bilogical morphism. ■

We call $\mathbb{L}_\mathcal{A} = \langle \mathcal{A}, C_\mathcal{A} \rangle$ the **logicoid induced on $\mathcal{A}$ by $\mathbb{L}$**.

An additional property of these algebraic logicoids is that the theories of the algebraic logicoid coincide with the $\mathbb{L}$-filters on the underlying interpretation.

Proposition 147 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$*, with* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$*, an interpretation, such that* $\mathrm{Ker}(h) \in \mathrm{Con}(\mathbb{L})$*. Then*

$$\mathcal{C}_\mathcal{A} = \mathrm{Fi}_\mathbb{L}(\mathcal{A}).$$

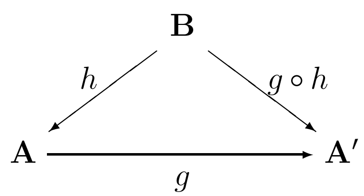
Proof: We have

$$\begin{array}{rcll}\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & = & \{Y \subseteq A : h^{-1}(Y) \in \mathcal{C}^\flat\} & \text{(Definition of an } \mathbb{L}\text{-filter)}\\ & = & \{Y \subseteq A : h(h^{-1}(Y)) \in \mathcal{C}_{\mathcal{A}}\} & \text{(Proposition 146)}\\ & = & \{Y \subseteq A : Y \in \mathcal{C}_{\mathcal{A}}\} & (h \text{ Surjective})\\ & = & \mathcal{C}_{\mathcal{A}}. & \end{array}$$

So the displayed equality in the statements holds. ■

We embark, next, on a series of results that clarify the interaction between filterhood and morphisms and, in particular, between filters and quotients. The next proposition shows that, for two interpretations, one of which results from the other by composition with a grid morphism, inverse images of filters are filters and conversely (see Proposition 34 for logicates).

Proposition 148 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$ be algebraic grids and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ and $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ grid morphisms. Setting $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h \rangle$, we have,*



for all $G \subseteq A'$, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ if and only if $g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: We have

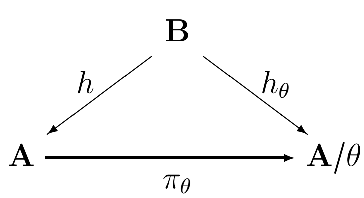
$$\begin{array}{rll}G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') & \text{iff} & (g \circ h)^{-1}(G) \in \mathcal{C}^\flat \quad (\text{Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'))\\ & \text{iff} & h^{-1}(g^{-1}(G)) \in \mathcal{C}^\flat \quad ((g \circ h)^{-1} = h^{-1} \circ g^{-1})\\ & \text{iff} & g^{-1}(G) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}). \quad (\text{Definition of } \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))\end{array}$$

■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Then we set

$$\mathcal{A}^\theta = \mathcal{A}/\theta = \langle \hat{\mathbf{A}}/\theta, h_\theta \rangle,$$

where, $h_\theta : \hat{\mathbf{B}} \to \hat{\mathbf{A}}/\theta$ is defined by



$$h_\theta := \pi_\theta \circ h,$$

with $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ the quotient grid morphism.

We next show that a necessary and sufficient condition for an $\mathbb{L}$-filter F on $\mathcal{A}$ to be the inverse image under the quotient mapping of an $\mathbb{L}$-filter on the quotient $\mathcal{A}/\theta$ is that θ be compatible with F (see Proposition 35 for logicates).

Proposition 149 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation, $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Then θ is compatible with F if and only if $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$.*

Proof: Suppose, first, that θ is compatible with F. Set $G = \pi_\theta(F)$. Then, we have

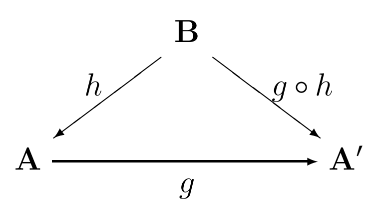
$$\begin{array}{rcll} h_\theta^{-1}(G) & = & (\pi_\theta \circ h)^{-1}(\pi_\theta(F)) & (h_\theta := \pi_\theta \circ h) \\ & = & h^{-1}(\pi_\theta^{-1}(\pi_\theta(F))) & ((\pi_\theta \circ h)^{-1} = h^{-1} \circ \pi_\theta^{-1}) \\ & = & h^{-1}(F). & \text{(Compatibility)} \end{array}$$

Since $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $h^{-1}(F) \in \mathcal{C}^\flat$ and, thus, $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Moreover, by compatibility, $F = \pi_\theta^{-1}(\pi_\theta(F)) = \pi_\theta^{-1}(G)$.

Suppose, conversely, that $F = \pi_\theta^{-1}(G)$, for some $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Let $a, b \in A$, such that $\langle a, b \rangle \in \theta$ and $a \in F$. Then $a \in \pi_\theta^{-1}(G)$, whence $a/\theta \in G$. So $b/\theta = a/\theta \in G$. This gives $b \in \pi_\theta^{-1}(G) = F$. So θ is compatible with F. ■

Next, we investigate conditions under which the $\mathbb{L}$-filters on two interpretations that are related via a grid morphism are in correspondence. This result forms an analog for logicoids of Proposition 36.

Proposition 150 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation and $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ a grid morphism. Set $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h \rangle$.*



The following statements are equivalent:

(i) $g : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$, *with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, *is a bilogical morphism;*

(ii) *For all* $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$ *and* $\mathrm{Ker}(g) \in \mathrm{Con}(\langle \mathcal{A}, C \rangle)$;

(iii) *g induces an isomorphism between the poset* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \le \rangle$ *and the poset* $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'), \le' \rangle$.

Proof:

(i)⇒(ii) The implication (i)⇒(ii) follows by the hypothesis and Proposition 132.

(ii)⇒(iii) By hypothesis, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $g(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Conversely, if $Y \in \mathcal{C}'$, then by Proposition 148, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By surjectivity, $g(g^{-1}(Y)) = Y$, for all $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By compatibility, $g^{-1}(g(X)) = X$, for all $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Since g is a grid morphism, g^{-1} is both order preserving and order reflecting with respect to $\leq'$ and $\leq$. So the conclusion follows.

(iii)⇒(i) By hypothesis, $g^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. So, by definition, $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$ is a bilogical morphism.

■

If two interpretations are related via a grid morphism and the morphism happens to be a bilogical morphism between two closure structures, one on each interpretation, then it turns out that, if the structure on the source interpretation consists of the entire collection of $\mathbb{L}$-filters, then so does the structure on the target interpretation. Again, for logicates, see Proposition 37.

Proposition 151 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ *be a base logicoid,* $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ *an interpretation and* $g : \hat{\mathbf{A}} \to \hat{\mathbf{A}}'$ *a grid morphism. Set* $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h\rangle$. *If* $g : \langle \mathcal{A}, C\rangle \to \langle \mathcal{A}', C'\rangle$, *with* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, *is a bilogical morphism, then*

$$\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}').$$

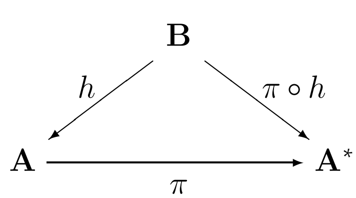
Proof: Suppose, first, that $Y \in \mathcal{C}'$. Then, since $g : \langle \mathcal{A}, C_{\mathcal{A}}\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $g^{-1}(Y) \in \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 148, $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. So $\mathcal{C}' \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Assume, conversely, that $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. Then, by Proposition 148, $g^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) = \mathcal{C}$. Thus, since $g : \langle \mathcal{A}, C\rangle \to_b \langle \mathcal{A}', C'\rangle$, we get $Y = g(g^{-1}(Y)) \in \mathcal{C}'$. So $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}') \subseteq \mathcal{C}'$. We conclude that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. ■

As a corollary we get that the collection of $\mathbb{L}$-filters on a quotient structure coincides with the reductions of the $\mathbb{L}$-filters on the original structure.

Corollary 152 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ *be a base logicoid and* $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ *an interpretation. Then*

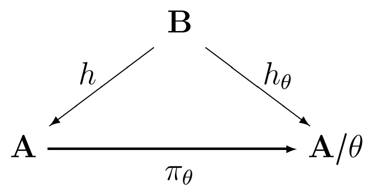
$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

Proof: One works with the diagram



where $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\widetilde{\Omega}_{\mathcal{A}}(\mathcal{C}_{\mathcal{A}})$ is the quotient grid morphism. Recalling that, by Proposition 138, it is a bilogical morphism, we may apply Proposition 151 to get the conclusion. ∎

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. Suppose $\mathfrak{A} = \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})$ and $\theta \in \mathrm{Con}(\mathfrak{A})$.



Then, using the compatibility of θ with F, we can see that $\mathfrak{A}/\theta = \langle \mathcal{A}/\theta, F/\theta \rangle \in \mathrm{Mat}(\mathbb{L})$. $\mathfrak{A}/\theta$ is called the **quotient (grid) matrix** of $\mathfrak{A}$ by θ. In particular, $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathcal{A}}(F) \in \mathrm{Mat}^*(\mathbb{L})$. $\mathfrak{A}^* = \mathfrak{A}/\Omega_{\mathcal{A}}(F)$ is called the **reduction** of $\mathfrak{A}$. We let $\mathrm{Alg}^*(\mathbb{L})$ be the class of algebraic (grid interpretation) reducts of matrices in $\mathrm{Mat}^*(\mathbb{L})$.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid and consider a class M of matrices. We say that $\mathbb{L}$ is **complete with respect to** M if

$$\mathcal{C}^\flat = \{h^{-1}(F) : \langle \langle \hat{\mathbf{A}}, h \rangle, F \rangle \in \mathsf{M}\}.$$

Observe that by definition of an $\mathbb{L}$-filter, the theories of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ are captured as the $\mathbb{L}$-filters on the interpretation $\langle \hat{\mathbf{B}}, i_{\mathbf{B}} \rangle$. With this in mind, it is not difficult to see that, as is the case in the classical theory of logical matrices, $\mathbb{L}$ is complete both with respect to the class of all its grid matrices and with respect to the class of all its reduced grid matrices.

Proposition 153 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete both with respect to $\mathrm{Mat}(\mathbb{L})$ and with respect to $\mathrm{Mat}^*(\mathbb{L})$.*

Proof: By the definition of $\mathrm{Mat}(\mathbb{L})$, we have

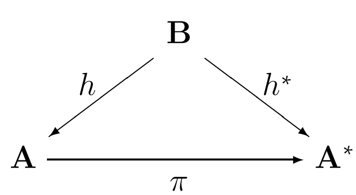
$$\{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\} \subseteq \mathcal{C}^\flat.$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then the pair $\langle \langle \hat{\mathbf{B}}, i_{\mathbf{B}} \rangle, X \rangle \in \mathrm{Mat}(\mathbb{L})$ and $i_{\mathbf{B}}^{-1}(X) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle \mathcal{A}, F \rangle \in \mathrm{Mat}(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}(\mathbb{L})$.

Let $\langle \langle \hat{\mathbf{A}}^*, h^* \rangle, X^* \rangle \in \mathrm{Mat}^*(\mathbb{L})$. Then, by the definition of $\mathrm{Mat}^*(\mathbb{L})$,



$$\begin{array}{rcl} (h^*)^{-1}(X^*) & = & (\pi \circ h)^{-1}(\pi(X)) \quad \text{(Definition of } h^*) \\ & = & h^{-1}(\pi^{-1}(\pi(X))) \quad ((\pi \circ h)^{-1} = h^{-1} \circ \pi^{-1}) \\ & = & h^{-1}(X) \quad \text{(Compatibility)} \\ & \in & \mathcal{C}^\flat. \quad (X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) \end{array}$$

Assume, conversely, that $X \in \mathcal{C}^\flat$. Then, letting $\pi : \hat{\mathbf{B}} \to \hat{\mathbf{B}}^*$, where $\hat{\mathbf{B}}^* = \hat{\mathbf{B}}/\Omega_{\hat{\mathbf{B}}}(X)$, the pair $\langle\langle\hat{\mathbf{B}}^*, \pi\rangle, X^*\rangle \in \mathrm{Mat}^*(\mathbb{L})$ and $\pi^{-1}(X^*) = X$. Therefore,

$$\mathcal{C}^\flat \subseteq \{h^{-1}(F) : \langle\mathcal{A}, F\rangle \in \mathrm{Mat}^*(\mathbb{L})\}.$$

This proves that $\mathbb{L}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{L})$. ∎

Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid. Given a class K of grid interpretations, a grid interpretation $\mathcal{A} = \langle\hat{\mathbf{A}}, h\rangle$, not necessarily in the class K, and a congruence $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$, one writes $\theta \in \mathrm{Con}_{\mathsf{K}}(\mathcal{A})$ to signify that the quotient interpretation $\mathcal{A}/\theta \in \mathsf{K}$. In this case, θ is termed a **grid K-congruence**. So $\mathrm{Con}_{\mathsf{K}}(\mathcal{A})$ is the collection of all grid K-congruences on the grid interpretation $\mathcal{A}$. Using a variant of this notation, we may write

$$\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}).$$

The quotient grid $\hat{\mathbf{B}}^* := \hat{\mathbf{B}}/\widetilde{\Omega}(\mathbb{L})$ is called the **Lindenbaum-Tarski grid** of $\mathbb{L}$. The quotient logicoid $\mathbb{L}^* := \langle\hat{\mathbf{B}}^*, C^{\flat *}\rangle$ is called the **Lindenbaum-Tarski quotient** of $\mathbb{L}$.

Chapter 8

Model Theory

8.1 Introduction

This chapter discusses *logicoid models* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$. They are based on grid interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, which consist of a grid $\hat{\mathbf{A}}$ together with a grid morphism h from the base grid $\hat{\mathbf{B}}$ onto $\hat{\mathbf{A}}$. We also define and study the reduction $\mathbb{A}^*$ of such a logicoid interpretation $\mathbb{A}$. Among those models, we single out the *full models*, which are the ones whose reductions are *basic full models*, i.e., consist of all possible $\mathbb{L}$-filters on their underlying interpretations. We characterize this class of models. Moreover, we show that it consists of those models that have all possible filters corresponding to filters on the reduced interpretations. Reduced $\mathbb{L}$-models give rise to $\mathbb{L}$-*algebras*, i.e., interpretations that are reducts of reduced $\mathbb{L}$-models. Their class is shown to be the class of subdirect intersections of interpretations in $\mathrm{Alg}^*(\mathbb{L})$, which consists of all interpretation reducts of reduced $\mathbb{L}$-matrices. Our study culminates with an Isomorphism Theorem for logicoids asserting that the Tarksi operator on a fixed interpretation $\mathcal{A}$ is an isomorphism between the ordered set of full $\mathbb{L}$-models on $\mathcal{A}$ and the partially ordered set of grid $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$.

In Section 8.2, we introduce the notion of an *interpretation* of a base algebraic grid $\hat{\mathbf{B}}$ and that of a *logicoid interpretation*. The first is a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, consisting of an algebraic grid $\hat{\mathbf{A}}$ and a grid morphism $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$. The second consists of an algebraic logicoid $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is an interpretation and $\langle \hat{\mathbf{A}}, C \rangle$ is a logicoid based on $\hat{\mathbf{A}}$. A logicoid interpretation of $\hat{\mathbf{B}}$ induces a logicoid structure $\mathbb{L}^{\mathbb{A}} = \langle \hat{\mathbf{B}}, C^{\mathbb{A}} \rangle$ on $\hat{\mathbf{B}}$ in such a way that h becomes a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{A}$. Further, if two logicoid interpretations are related via a bilogical morphism, then they induce identical logicoids on the base grid $\hat{\mathbf{B}}$. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a *model* of a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ if $\mathbb{L} \leq \mathbb{L}^{\mathbb{A}}$ or, equivalently, if $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$. The section continues with a discussion of completeness of a base logicoid with respect to a class of models. In this context, *reductions* of models and *reduced models* are discussed and analogs of classical completeness results with respect to the class of all models and with respect to the class of all reduced models are formulated. The section closes by connecting the notion of logicoid model with that of a grid matrix model, introduced and studied in Section 7.5.

In Section 8.3, we study *full models* of logicoids. Given a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, a logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a *basic full model* of $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., if its set of theories is the entire collection of $\mathbb{L}$-filters on its underlying interpretation. A *full model* of $\mathbb{L}$ is one whose reduction is a basic full model. Full models are indeed models and basic full models are indeed full models. So the terminology chosen is sound. It turns out that bilogical morphisms preserve fullness in both directions, which implies that $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if its reduction $\mathbb{A}^*$ is also a full $\mathbb{L}$-model. Additionally, $\mathbb{A}$ is a full $\mathbb{L}$-model if and only if there exists a bilogical morphism from it onto a basic full $\mathbb{L}$-model. As a consequence we get that the class of full $\mathbb{L}$-models is the smallest class containing all basic full $\mathbb{L}$-models and closed under bilogical morphisms (in both directions). The section concludes with a result providing an additional justification of the term "full". It shows that full $\mathbb{L}$-models are those whose collection of $\mathbb{L}$-filters consists of all possible ones corresponding to $\mathbb{L}$-filters on the reduced interpretation.

Section 8.4 introduces $\mathbb{L}$-algebras (more accurately $\mathbb{L}$-interpretations) for a logicoid $\mathbb{L}$. These are interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, such that $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))$ is the identity grid congruence on $\hat{\mathbf{A}}$. Some results relating $\mathbb{L}$-algebras with full $\mathbb{L}$-models and with their theories are provided. It is shown that, for every full $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, C \rangle$, the reduction $\mathcal{A}^*$ of the interpretation $\mathcal{A}$ is an $\mathbb{L}$-algebra. Additionally, the class $\mathrm{Alg}(\mathbb{L})$ of $\mathbb{L}$-algebras is characterized as the class of interpretation reducts of reduced $\mathbb{L}$-models. Moreover, it is shown that $\mathrm{Alg}(\mathbb{L})$ is the class of all subdirect intersections of interpretations in the class $\mathrm{Alg}^*(\mathbb{L})$ of all interpretation reducts of reduced grid matrix models of $\mathbb{L}$. This characterization yields that $\mathrm{Alg}^*(\mathbb{L})$ is contained in $\mathrm{Alg}(\mathbb{L})$ and, also, that given logicoids $\mathbb{L}$, $\mathbb{L}'$ over the same base grid $\hat{\mathbf{B}}$, such that $\mathbb{L} \leq^\flat \mathbb{L}'$, we have that $\mathrm{Alg}(\mathbb{L}')$ is contained in the class $\mathrm{Alg}(\mathbb{L})$.

In Section 8.5, the last section of the chapter, we prove an analog of the Isomorphism Theorem 13 of [12] for logicoids. The result parallels the Isomorphism Theorem for logicates (Theorem 75) and the proof is similar.

8.2 Models of Logicoids

We consider a **base algebraic grid** $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq \rangle$, where $\mathbf{B} = \langle B, \mathcal{L}^{\mathbf{B}} \rangle$ is an algebra, which, in this context, is termed the **base algebra** and $\leq$ is a complete lattice order on $\mathcal{P}(B)$. An **interpretation** is a pair $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, where $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ is an algebraic grid and $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism from the base algebra onto $\hat{\mathbf{A}}$. An *algebraic logicoid* $\langle \hat{\mathbf{A}}, C \rangle$ consists of an algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ and a $\leq$-closure operator C on $\hat{\mathbf{A}}$. A **logicoid interpretation** is a pair $\mathbb{A} = \langle \mathcal{A}, C \rangle$, where:

- $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is an interpretation;
- $\langle \hat{\mathbf{A}}, C \rangle$ is an algebraic logicoid on the grid $\hat{\mathbf{A}}$.

The logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ induces a function

$$C^{\mathbb{A}} : \mathcal{P}(B) \to \mathcal{P}(B),$$

defined, for all $X \subseteq B$, by

$$C^{\mathbb{A}}(X) = \bigwedge\nolimits^{\flat} h^{-1}(\mathcal{C})^X,$$

where

$$h^{-1}(\mathcal{C})^X = \{h^{-1}(Y) : Y \in \mathcal{C} \text{ and } X \leq^{\flat} h^{-1}(Y)\}.$$

We write

$$\mathbb{L}^{\mathbb{A}} := \langle \hat{\mathbf{B}}, C^{\mathbb{A}} \rangle.$$

This construction forms an analog of the construction in Definition 2.1 of [12]. We show that $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicoid on the base grid and that the epimorphism h is a bilogical morphism $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{L}$.

Proposition 154 *Let $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ be a base grid and $\mathbb{A} = \langle \mathcal{A}, C \rangle$ a logicoid interpretation, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$.*

(a) $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicoid.

(b) $h : \mathbb{L}^{\mathbb{A}} \to \mathbb{L}$ is a bilogical morphism.

Proof:

(a) We must show that $C^{\mathbb{A}}$ is inflationary, monotone and idempotent with respect to $\leq^\flat$. First, let $X \subseteq B$. By definition, $X \leq^\flat \bigwedge^\flat h^{-1}(\mathcal{C})^X = C^{\mathbb{A}}(X)$. So $C^{\mathbb{A}}$ is inflationary. Next, let $X, Y \subseteq B$, such that $X \leq^\flat Y$. Then

$$C^{\mathbb{A}}(X) = \bigwedge\nolimits^\flat h^{-1}(\mathcal{C})^X \leq^\flat \bigwedge\nolimits^\flat h^{-1}(\mathcal{C})^Y = C^{\mathbb{A}}(Y).$$

So $C^{\mathbb{A}}$ is also monotone. Finally, let $X \subseteq B$. Then, taking into account that $\mathcal{C}$ is closed under intersections and that h^{-1} is a complete lattice embedding,

$$\begin{aligned} C^{\mathbb{A}}(C^{\mathbb{A}}(X)) &= \textstyle\bigwedge^\flat h^{-1}(\mathcal{C})^{C^{\mathbb{A}}(X)} \\ &= \textstyle\bigwedge^\flat h^{-1}(\mathcal{C})^{\bigwedge^\flat h^{-1}(\mathcal{C})^X} \\ &= \textstyle\bigwedge^\flat h^{-1}(\mathcal{C})^X \\ &= C^{\mathbb{A}}(X). \end{aligned}$$

Thus, $C^{\mathbb{A}}$ is inflationary, monotone and idempotent with respect to $\leq^\flat$ and, therefore, $\mathbb{L}^{\mathbb{A}}$ is an algebraic logicoid.

(b) We show that $\mathcal{C}^{\mathbb{A}} = h^{-1}(\mathcal{C})$. Then, by Proposition 132, it will follow that $h : \mathbb{L}^{\mathbb{A}} \to_b \mathbb{L}$ is a bilogical morphism. If $Y \in \mathcal{C}$, then

$$C^{\mathbb{A}}(h^{-1}(Y)) = \bigwedge\nolimits^\flat h^{-1}(\mathcal{C})^{h^{-1}(Y)} = h^{-1}(Y).$$

Hence $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^{\mathbb{A}}$. Assume, conversely, that $X \in \mathcal{C}^{\mathbb{A}}$. Then

$$X = C^{\mathbb{A}}(X) = \bigwedge\nolimits^\flat h^{-1}(\mathcal{C})^X \in h^{-1}(\mathcal{C}),$$

where membership follows from closure of $\mathcal{C}$ under meets and the fact that h^{-1} is a complete lattice embedding.

■

We call $\mathbb{L}^{\mathbb{A}}$ the **logicoid induced on B by** $\mathbb{A}$.

An analog of Proposition 2.3 of [12] ensures that logicoid interpretations related via "compatible" bilogical morphisms induce the same logicoid on the base grid.

Proposition 155 *Let $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ be a base grid and $\mathbb{A} = \langle \langle \hat{\mathbf{A}}, g \rangle, C \rangle$ and $\mathbb{A}' = \langle \langle \hat{\mathbf{A}}', h \circ g \rangle, C' \rangle$ two logicoid interpretations, with $h : \mathbb{A} \to_b \mathbb{A}'$ a bilogical morphism. Then $\mathbb{L}^{\mathbb{A}} = \mathbb{L}^{\mathbb{A}'}$.*

Proof: Using the diagram below, we have, for all $X \subseteq B$,

$$\begin{array}{ccc} \mathbb{L}^{\mathbb{A}} & \xrightarrow{i_{\mathbf{B}}} & \mathbb{L}^{\mathbb{A}'} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h \circ g} \\ \mathbb{A} & \xrightarrow[h]{} & \mathbb{A}' \end{array}$$

$$\begin{array}{rcll} C^{\mathbb{A}}(X) & = & \bigwedge^\flat g^{-1}(\mathcal{C})^X & \text{(Definition of } C^{\mathbb{A}}) \\ & = & \bigwedge^\flat g^{-1}(h^{-1}(\mathcal{C}'))^X & (h : \mathbb{A} \to_b \mathbb{A}') \\ & = & \bigwedge^\flat (h \circ g)^{-1}(\mathcal{C}')^X & ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & = & C^{\mathbb{A}'}(X). & \text{(Definition of } C^{\mathbb{A}'}) \end{array}$$

So $\mathbb{L}^{\mathbb{A}} = \mathbb{L}^{\mathbb{A}'}$. ∎

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, with $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$, be a base logicoid, perceived as constituting the main object of investigation. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, is called a **model of** $\mathbb{L}$ or an $\mathbb{L}$-**model** if, $C^\flat \leq^\flat C^{\mathbb{A}}$, i.e., for all $X \subseteq B$,

$$C^\flat(X) \leq^\flat C^{\mathbb{A}}(X).$$

We denote by $\mathrm{Mod}(\mathbb{L})$ the class of all models of $\mathbb{L}$.

Lemma 156 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is a model of $\mathbb{L}$ if and only if*

$$h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat.$$

Proof: We have the following equivalences

$$\begin{array}{rcll} C^\flat \leq^\flat C^{\mathbb{A}} & \text{iff} & \mathcal{C}^{\mathbb{A}} \subseteq \mathcal{C}^\flat & \text{(Proposition 106)} \\ & \text{iff} & h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat. & \text{(Part (b) of Proposition 154)} \end{array}$$

∎

Let L be a class of models of $\mathbb{L}$. $\mathbb{L}$ is said to be **complete with respect to** L if, for all $X \subseteq B$,

$$C^\flat(X) = \bigwedge\nolimits^\flat_{\mathbb{A} \in \mathsf{L}} C^{\mathbb{A}}(X).$$

Lemma 157 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid and L be a class of models of $\mathbb{L}$. $\mathbb{L}$ is complete with respect to L if and only if $\mathcal{C}^\flat$ is the $\leq^\flat$-closure system generated by $\bigcup \{h^{-1}(\mathcal{C}) : \langle \langle \hat{\mathbf{A}}, h \rangle, C \rangle \in \mathsf{L}\}$.*

Proof: For the "if" direction, let $X \subseteq B$. We then have

$$\begin{array}{rcl} \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} C^{\mathbb{A}}(X) & = & \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} \bigwedge^\flat h^{-1}(\mathcal{C})^X \quad (\text{Definition of } C^{\mathbb{A}}) \\ & = & \bigwedge^\flat \{T \in \mathcal{C}^\flat : X \leq^\flat T\} \quad (\text{Hypothesis}) \\ & = & C^\flat(X). \quad (\text{Proposition 102}) \end{array}$$

For the "only if", assume that, for all $X \subseteq B$, $C^\flat(X) = \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} C^{\mathbb{A}}(X)$. By Lemma 156, we know that $\bigcup\{h^{-1}(\mathcal{C}) : \langle\langle\hat{\mathbf{A}}, h\rangle, C\rangle \in \mathsf{L}\} \subseteq \mathcal{C}^\flat$. Conversely, if $X \in \mathcal{C}^\flat$, then

$$X = C^\flat(X) = \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} C^{\mathbb{A}}(X) = \bigwedge^\flat_{\mathbb{A}\in\mathsf{L}} \bigwedge^\flat h^{-1}(\mathcal{C})^X.$$

So X is in the $\leq^\flat$-closure system generated by $\bigcup\{h^{-1}(\mathcal{C}) : \langle\langle\hat{\mathbf{A}}, h\rangle, C\rangle \in \mathsf{L}\}$. ■

The following result is an analog of part of Proposition 2.5 of [12] for logicoid interpretations.

Proposition 158 *Let* $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ *be a base logicoid,* $\mathbb{A} = \langle\langle\hat{\mathbf{A}}, g\rangle, C\rangle$*,* $\mathbb{A}' = \langle\langle\hat{\mathbf{A}}', h \circ g\rangle, C'\rangle$ *be logicoid interpretations and* $h : \mathbb{A} \to \mathbb{A}'$ *be a bilogical morphism. Then* $\mathbb{A}$ *is a model of* $\mathbb{L}$ *if and only if* $\mathbb{A}'$ *is a model of* $\mathbb{L}$.

Proof: Suppose $\mathbb{A}$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcl} (h \circ g)^{-1}(\mathcal{C}') & = & g^{-1}(h^{-1}(\mathcal{C}')) \quad ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & = & g^{-1}(\mathcal{C}) \quad (h : \mathbb{A} \to_b \mathbb{A}') \\ & \subseteq & \mathcal{C}^\flat. \quad (\mathbb{A} \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}'$ is a model of $\mathbb{L}$. Assume, conversely, that $\mathbb{A}'$ is a model of $\mathbb{L}$. Then

$$\begin{array}{rcl} g^{-1}(\mathcal{C}) & = & g^{-1}(h^{-1}(\mathcal{C}')) \quad (h : \mathbb{A} \to_b \mathbb{A}') \\ & = & (h \circ g)^{-1}(\mathcal{C}') \quad ((h \circ g)^{-1} = g^{-1} \circ h^{-1}) \\ & \subseteq & \mathcal{C}^\flat. \quad (\mathbb{A}' \text{ an } \mathbb{L}\text{-model}) \end{array}$$

Hence, $\mathbb{A}$ is a model of $\mathbb{L}$. ■

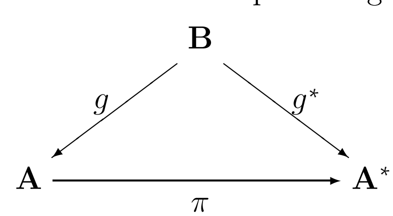
In order to formulate another part of Proposition 2.5 of [12], we need to define the Tarski reduction of a logicoid interpretation.

Let $\mathbb{L} = \langle\mathbf{B}, C^\flat\rangle$ be an base logicoid. Consider the pair $\mathbb{A} = \langle\langle\hat{\mathbf{A}}, g\rangle, C\rangle$. Recall the Tarski congruence $\widetilde{\Omega}(\mathbb{A}) \coloneqq \widetilde{\Omega}_{\hat{\mathbf{A}}}(C)$. We define the pair

$$\mathbb{A}^* = \langle\langle\hat{\mathbf{A}}^*, g^*\rangle, C^*\rangle$$

by setting:

- $\hat{\mathbf{A}}^* = \hat{\mathbf{A}}/\widetilde{\Omega}_{\hat{\mathbf{A}}}(C)$;
- $g^* = \pi \circ g$, where $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}^*$ is the quotient grid morphism;



- $C^* : \mathcal{P}(A/\widetilde{\Omega}_{\mathbf{A}}(C)) \to \mathcal{P}(A/\widetilde{\Omega}_{\mathbf{A}}(C))$, where, for all $S \subseteq A/\widetilde{\Omega}_{\mathbf{A}}(C)$,
$$C^*(S) = \pi(C(\pi^{-1}(S))).$$

Based on results already obtained, we may show that, if $\mathbb{A}$ is a model of $\mathbb{L}$, then so is $\mathbb{A}^*$.

Corollary 159 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. A pair $\mathbb{A} = \langle \langle \hat{\mathbf{A}}, g \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if $\mathbb{A}^*$ is a model of $\mathbb{L}$.*

Proof: This follows directly from Proposition 158, since, by Proposition 138, the natural projection $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism. ■

Corollary 160 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete with respect to a class L of models if and only if it is complete with respect to the class L^*.*

Proof: Let
$$\mathsf{L} = \{\langle \langle \hat{\mathbf{A}}_i, g_i \rangle, C_i \rangle : i \in I\}.$$
Denote by $\pi_i : \hat{\mathbf{A}}_i \to \hat{\mathbf{A}}_i^*$, $i \in I$, the quotient grid morphisms. By Proposition 138, for all $i \in I$, $\pi_i : \mathbb{A}_i \to_b \mathbb{A}_i^*$ is a bilogical morphism. So we have, for all $i \in I$,
$$\bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i) = \bigcup_{i \in I} g_i^{-1}(\pi_i^{-1}(\mathcal{C}_i^*)) = \bigcup_{i \in I} (\pi_i \circ g_i)^{-1}(\mathcal{C}_i^*) = \bigcup_{i \in I} (g_i^*)^{-1}(\mathcal{C}_i^*).$$
Thus, $\mathbb{L}$ is complete with respect to L if and only if, by definition, $\mathcal{C}^\flat$ is $\leq^\flat$-generated by $\bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i)$ if and only if, by the displayed equality, $\mathcal{C}^\flat$ is $\leq^\flat$-generated by $\bigcup_{i \in I} (g_i^*)^{-1}(\mathcal{C}_i^*)$ if and only if, by definition, $\mathbb{L}$ is complete with respect to L^*. ■

As far as completeness properties go, note that $\langle \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle, C^\flat \rangle$ is a model of $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$. This yields the following results, forming together an analog of Proposition 2.6 of [12].

Proposition 161 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is complete with respect to any class L of models that includes $\mathbb{A} = \langle \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle, C^\flat \rangle$ or $\mathbb{A}^*$.*

Proof: Let $\mathsf{L} = \{\langle \langle \hat{\mathbf{A}}_i, g_i \rangle, C_i \rangle : i \in I\}$. On the one hand, since $\mathsf{L} \subseteq \mathrm{Mod}(\mathbb{L})$, $\bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i) \subseteq \mathcal{C}^\flat$. On the other, since $\mathbb{A} \in \mathsf{L}$, $\mathcal{C}^\flat = i_{\hat{\mathbf{B}}}^{-1}(\mathcal{C}^\flat) \subseteq \bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i)$. Thus, $\mathbb{L}$ is complete with respect to L.

Assume, now, that $\mathbb{A}^* \in \mathsf{L}$. The first inclusion is justified in the same way. For the second, letting $\pi : \hat{\mathbf{B}} \to \hat{\mathbf{B}}^*$ be the quotient grid morphism, we have
$$\mathcal{C}^\flat = i_{\hat{\mathbf{B}}}^{-1}(\mathcal{C}^\flat) = i_{\hat{\mathbf{B}}}^{-1}(\pi^{-1}(\mathcal{C}^{\flat *})) = \pi^{-1}(\mathcal{C}^{\flat *}) \subseteq \bigcup_{i \in I} g_i^{-1}(\mathcal{C}_i).$$
Thus, $\mathbb{L}$ is again complete with respect to L. ■

Corollary 162 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicate.* $\mathbb{L}$ *is complete with respect to the class of all its models and with respect to the class of all its reduced models.*

Proof: Clearly, $\mathbb{A} = \langle \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle, C^\flat \rangle \in \mathrm{Mod}(\mathbb{L})$ and $\mathbb{A}^* \in \mathrm{Mod}^*(\mathbb{L})$. Thus, by Proposition 161, $\mathbb{L}$ is complete both with respect to $\mathrm{Mod}(\mathbb{L})$ and with respect to $\mathrm{Mod}^*(\mathbb{L})$. ■

Logicoid models are closely connected with matrix models. The connection is given in the following proposition, which parallels Proposition 2.7 of [12].

Proposition 163 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\langle \langle \hat{\mathbf{A}}, h \rangle, C \rangle$ *is a model of* $\mathbb{L}$ *if and only if, for all* $Y \in \mathcal{C}$, $\langle \langle \hat{\mathbf{A}}, h \rangle, Y \rangle$ *is an* $\mathbb{L}$*-matrix.*

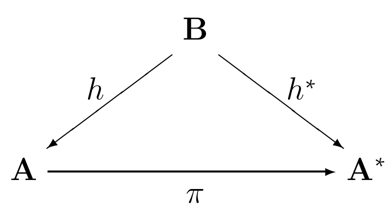
Proof: We have that $\langle \langle \hat{\mathbf{A}}, h \rangle, C \rangle$ is a model of $\mathbb{L}$ if and only if, by Lemma 156, $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}^\flat$ if and only if, for all $Y \in \mathcal{C}$, $h^{-1}(Y) \in \mathcal{C}^\flat$ if and only if, by definition, for all $Y \in \mathcal{C}$, $\langle \langle \hat{\mathbf{A}}, h \rangle, Y \rangle$ is an $\mathbb{L}$-matrix. ■

Proposition 163 asserts that the weakest model of $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ on an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is the one determined by

$$\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}).$$

8.3 Full Models

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, is called a **full model of** $\mathbb{L}$ or a **full** $\mathbb{L}$**-model** (see Definition 2.8 of [12]) if



$$\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*).$$

A logicoid interpretation $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is called a **basic full model of** $\mathbb{L}$ if $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, rephrasing the definition, we may say that $\mathbb{A}$ is a full model of $\mathbb{L}$ if and only if its reduction is a basic full model of $\mathbb{L}$.

$\mathrm{FMod}(\mathbb{L})$ denotes the class of all full models of $\mathbb{L}$. $\mathrm{FMod}^*(\mathbb{L})$ is the class of all reduced full models of $\mathbb{L}$. Given an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$ is the class of all full models of $\mathbb{L}$ on $\mathcal{A}$.

An analog of Part (1) of Proposition 2.9 of [12] and of Proposition 58 for logicates provides a justification for the use of the term "model" for full models.

Proposition 164 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *a full model of* $\mathbb{L}$*. Then* $\mathbb{A}$ *is a model of* $\mathbb{L}$*.*

Proof: Suppose $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$. By definition, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, by Proposition 163, $\mathbb{A}^*$ is a model of $\mathbb{L}$. Thus, by Corollary 159, $\mathbb{A}$ is also a model of $\mathbb{L}$. ■

The next result, an analog of Proposition 2.10 of [12] and of Proposition 59 for logicates, asserts that every basic full model is actually a full model, justifying the "full" in the definition of basic full models.

Proposition 165 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. A logicoid interpretation* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *on* $\mathcal{A}$*, such that* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*, is a full model of* $\mathbb{L}$ *and is the weakest full model of* $\mathbb{L}$ *on* $\mathcal{A}$*.*

Proof: By Proposition 138, the natural projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism. By Corollary 152, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$, i.e., $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Hence, $\mathbb{A}$ is a full model of $\mathbb{L}$. Taking into account Proposition 164, it is the weakest full model on $\mathcal{A}$, since it is the weakest model on $\mathcal{A}$, by Proposition 163. ■

Proposition 2.11 of [12], concerning closure of the class of full models under bilogical morphisms, has the following analog (see, also, Proposition 60 for logicates).

Proposition 166 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. The class* $\mathrm{FMod}(\mathbb{L})$ *is closed under bilogical morphisms, i.e., if* $h : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}', C' \rangle$*, where* $\mathcal{A} = \langle \hat{\mathbf{A}}, g \rangle$ *and* $\mathcal{A}' = \langle \hat{\mathbf{A}}', h \circ g \rangle$*, is a bilogical morphism, then*

$$\langle \mathcal{A}, C \rangle \in \mathrm{FMod}(\mathbb{L}) \quad \textit{iff} \quad \langle \mathcal{A}', C' \rangle \in \mathrm{FMod}(\mathbb{L}).$$

Proof: Suppose $h : \mathbb{A} \to \mathbb{A}'$ is a bilogical morphism. By Proposition 144, there exists an isomorphism $h^* : \mathbb{A}^* \cong \mathbb{A}'^*$. Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. Then $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Thus, by Proposition 151, $\mathcal{C}'^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}'^*)$. But $\mathbb{A}'^*$ is reduced, whence $\mathbb{A}'^*$ is a full model of $\mathbb{L}$. A similar reasoning yields the converse. ■

Corollary 167 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\mathbb{A} \in \mathrm{FMod}(\mathbb{L})$ *if and only if* $\mathbb{A}^* \in \mathrm{FMod}(\mathbb{L})$*.*

Proof: Directly from Proposition 166, since, by Proposition 138, the quotient grid morphism $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}^*$ is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$. ■

Corollary 168 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. Then* $\mathbb{A} = \langle \mathcal{A}, C \rangle$ *is a full model of* $\mathbb{L}$ *if and only if there exists a bilogical morphism from* $\mathbb{A}$ *onto a model* $\langle \mathcal{A}', C' \rangle$*, with* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$*.*

Proof: The "only if" follows directly from the definition of full model, as the projection $\pi : \mathbb{A} \to \mathbb{A}^*$ is a bilogical morphism and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

Conversely, assume there is a bilogical morphism $h : \mathbb{A} \to \mathbb{A}'$, such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$. By Proposition 165, $\mathbb{A}'$ is a full model, whence, by Proposition 166, $\mathbb{A}$ is also a full model. ∎

Our work culminates in a characterization of the class of full models of a logicate $\mathbb{L}$ along the lines of Corollary 2.13 of [12] (Corollary 63 for logicates).

Corollary 169 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. Then $\mathrm{FMod}(\mathbb{L})$ is the smallest class containing all $\langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, and closed under bilogical morphisms.*

Proof: Let L be the smallest class containing all $\langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, and closed under bilogical morphisms.

On the one hand, every $\langle \mathcal{A}, C \rangle$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, is a full model, by Proposition 165. Moreover, by Proposition 166, the class of full models is closed under bilogical morphisms. This shows that $\mathsf{L} \subseteq \mathrm{FMod}(\mathbb{L})$. The reverse inclusion is a direct consequence of Corollary 168. ∎

Corollary 169 provides one justification for the "fullness" property of full models. According to this justification, a full model is one that is obtained via a bilogical morphism by a model whose theories constitute a full set of $\mathbb{L}$-filters. A second justification is given in the following theorem. According to this, a model's "fullness" rests on the fact that its theories contain all possible $\mathbb{L}$-filters corresponding to $\mathbb{L}$-filters of the reduction of the model.

Theorem 170 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be an base logicoid. Then $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$ if and only if*

$$\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}.$$

Proof: Suppose, first, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a full model of $\mathbb{L}$. Let $X \in \mathcal{C}$. By Proposition 163, $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By definition of $\widetilde{\Omega}(\mathbb{A})$, it always holds that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. For the reverse inclusion, let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)$. Then, by Proposition 149, there exists $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$, such that $X = \pi^{-1}(Y)$, where $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\widetilde{\Omega}(\mathbb{A})$ is the quotient grid morphism. But, by Proposition 138, $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism and, moreover, since $\mathbb{A}$ is full, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $X \in \mathcal{C}$.

Suppose, conversely, that $\mathcal{C} = \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X)\}$. Since the natural projection $\pi : \mathbb{A} \to_b \mathbb{A}^*$ is a bilogical morphism, $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{A}))$. Thus, $\mathbb{A}$ is a full model of $\mathbb{L}$. ∎

8.4 $\mathbb{L}$-Algebras

Reduced full models of $\mathbb{L}$ are those models of the form $\langle\mathcal{A}, C\rangle$, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, that are reduced. The interpretation reducts of such models are given a special name.

Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid. An interpretation $\mathcal{A} = \langle\hat{\mathbf{A}}, h\rangle$ is an **$\mathbb{L}$-algebra** (see Definition 2.16 of [12]) if

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})) = \Delta_{\mathbf{A}}.$$

The class of all $\mathbb{L}$-algebras is denoted by $\mathrm{Alg}(\mathbb{L})$.

The following characterization takes after Proposition 2.17 of [12] (see, also, Proposition 65 for logicates).

Proposition 171 *Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathbb{A} = \langle\mathcal{A}, C\rangle$, with $\mathcal{A} = \langle\hat{\mathbf{A}}, h\rangle$. Then the following statements are equivalent:*

(i) $\mathbb{A} = \langle\mathcal{A}, C\rangle$ *is a reduced full model of* $\mathbb{L}$*;*

(ii) $\mathbb{A} = \langle\mathcal{A}, C\rangle$ *is reduced and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$*;*

(iii) $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$ *and* $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof:

(i)⇒(ii) By the definition of a reduced full model.

(ii)⇒(iii) By the definition of an $\mathbb{L}$-algebra, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$.

(iii)⇒(i) Since $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, there exists $C' : \mathcal{P}(A) \to \mathcal{P}(A)$, such that $\mathbb{A} = \langle\mathcal{A}, C'\rangle$ is a reduced full model of $\mathbb{L}$. But then $\mathcal{C}' = \mathcal{C}$ and $\mathbb{A} = \langle\mathcal{A}, C\rangle$ is a reduced full model of $\mathbb{L}$.

■

Proposition 172 *Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathbb{A} = \langle\mathcal{A}, C\rangle$ a full model of $\mathbb{L}$. Then $\mathcal{A}^* := \mathcal{A}/\widetilde{\Omega}(\mathbb{A})$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$.*

Proof: By Corollary 167, $\mathbb{A}^*$ is a full model of $\mathbb{L}$ and it is clearly reduced. Hence, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra. This also yields the second statement using the definition of $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. ■

A characterization of $\mathbb{L}$-algebras, an analog of Proposition 2.19 of [12] and of Proposition 67 for logicates, shows that the notion of model, without reference to fullness, suffices to characterize the class $\mathrm{Alg}(\mathbb{L})$.

Proposition 173 *Let $\mathbb{L} = \langle\hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid. The class $\mathrm{Alg}(\mathbb{L})$ is the class of algebraic reducts of all reduced models of $\mathbb{L}$.*

Proof: By definition, if $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$, then $\mathcal{A}$ is the algebraic reduct of a reduced full model; in particular of a reduced model. Assume, conversely, that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a reduced model of $\mathbb{L}$. Let $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$, be such that $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 163, $\mathbb{A}'$ is a model of $\mathbb{L}$ and, by Proposition 165, it is clearly full. It is also reduced, since

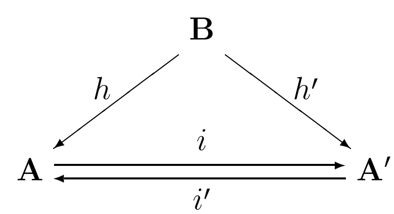
$$\widetilde{\Omega}(\mathbb{A}') \subseteq \widetilde{\Omega}(\mathbb{A}) = \Delta_{\mathbf{A}}.$$

Therefore, by definition, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ∎

Closure under grid isomorphisms is guaranteed by the following proposition, an analog of Proposition 2.20 of [12] (see Proposition 68 for logicates).

Proposition 174 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. The class* $\mathrm{Alg}(\mathbb{L})$ *is closed under grid isomorphisms (commuting with the interpretations).*

Proof: Let $i : \hat{\mathbf{A}} \cong \hat{\mathbf{A}}'$. We have the following diagram.



Suppose that $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for some C, $\langle \mathcal{A}, C \rangle$ is a reduced full model of $\mathbb{L}$. Consider $\mathcal{A}' = \langle \hat{\mathbf{A}}', h' \rangle = \langle \hat{\mathbf{A}}', i \circ h \rangle$. We have, $\langle \mathcal{A}', C' \rangle$, with $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$, is a reduced full model of $\mathbb{L}$. Thus, $\mathcal{A}' \in \mathrm{Alg}(\mathbb{L})$. The reverse implication can be proved similarly. ∎

Putting together several of the previous results, we get the following alternative characterizations of full models involving $\mathbb{L}$-algebras, an analog of Proposition 69 regarding logicates.

Proposition 175 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid and* $\mathbb{A} = \langle \mathcal{A}, C \rangle$*, with* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ *an interpretation. Then the following statements are equivalent.*

(i) $\mathbb{A}$ *is a full model of* $\mathbb{L}$*;*

(ii) $\mathcal{A}^*$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$*;*

(iii) There exists a bilogical morphism $g : \mathbb{A} \to \mathbb{A}'$*, with* $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', g \circ h \rangle$*, such that* $\mathcal{A}'$ *is an* $\mathbb{L}$*-algebra and* $\mathcal{C}' = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}')$*.*

Proof:

(i)⇒(ii) Suppose $\mathbb{A}$ is a full model of $\mathbb{L}$. By definition, $\mathbb{A}^*$ is a basic full model of $\mathbb{L}$. Thus, by Proposition 171, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$.

(ii)⇒(iii) Assume $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^*)$. Then (iii) is immediate by considering the quotient grid morphism $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}^*$, which is a bilogical morphism $\pi : \mathbb{A} \to_b \mathbb{A}^*$ and such that $\mathbb{A}^*$ fulfills the required conditions by (ii).

(iii)⇒(i) By Proposition 171, $\mathbb{A}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \hat{\mathbf{A}}', g \circ h \rangle$, is a reduced full model of $\mathbb{L}$. Therefore, by Corollary 168, $\mathbb{A}$ is a full model of $\mathbb{L}$.

■

An analog of the Completeness Theorem 2.22 of [12], and also of Theorem 70, asserts that the class of full models, the class of reduced full models, as well as the class of all basic full models of a logicoid can serve as a complete semantics for the logicoid.

Theorem 176 (Completeness) *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid.* $\mathbb{L}$ *is complete with respect to the following classes of models:*

1. *The class* $\mathrm{FMod}(\mathbb{L})$ *of all full models of* $\mathbb{L}$*;*

2. *The class of all basic full models of* $\mathbb{L}$*;*

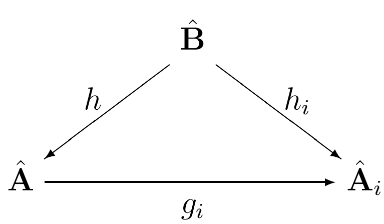
3. *The class* $\mathrm{FMod}^*(\mathbb{L})$ *of all reduced full models of* $\mathbb{L}$*.*

Proof: All three classes consist of models of $\mathbb{L}$. In addition each contains the model $\langle \langle \hat{\mathbf{B}}^*, \pi \rangle, C^{\flat *} \rangle$, where $\pi : \hat{\mathbf{B}} \to \hat{\mathbf{B}}/\widetilde{\Omega}(\mathbb{L})$ is the quotient grid morphism. Thus, by Proposition 161, $\mathbb{L}$ is complete with respect to each of these three classes. ■

We now establish an analog of the well known theorem (Theorem 2.23 of [12]) relating the classes $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$. Recall that $\mathrm{Alg}^*(\mathbb{L})$ is the class of all algebraic reducts of reduced matrix models of $\mathbb{L}$, whereas $\mathrm{Alg}(\mathbb{L})$ is the class of all algebraic reducts of reduced full models of $\mathbb{L}$. In the present setting, however, due to the presence of morphisms from the base logicoid in the interpretations involved, one has to replace subdirect products by a different operation, named subdirect intersection.

Let $\mathcal{A}_i = \langle \hat{\mathbf{A}}_i, h_i \rangle$, $i \in I$, be a collection of interpretations. We say that an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is a **subdirect intersection of** the $\mathcal{A}_i$ **relative to** $\hat{\mathbf{B}}$ if:

- There exist grid morphisms $g_i : \hat{\mathbf{A}} \to \hat{\mathbf{A}}_i$, $i \in I$, such that the following diagram commutes for all $i \in I$.

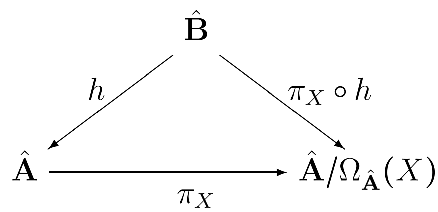


- $\bigcap_{i\in I}\mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$.

Since the role of $\hat{\mathbf{B}}$ is going to be played by the base algebraic grid (the algebraic grid reduct of the base logicoid), we usually omit the "relative to $\hat{\mathbf{B}}$" in the terminology.

Theorem 177 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid. The class* $\mathrm{Alg}(\mathbb{L})$ *is the class of all subdirect intersections of interpretations in* $\mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. Then there exist C, such that $\mathbb{A} = \langle \mathcal{A}, C \rangle$ is a reduced full model of $\mathbb{L}$. We form the commutative triangle of grid morphisms



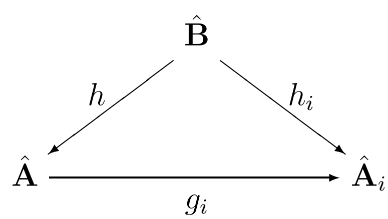
where $\pi_X : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\Omega_{\hat{\mathbf{A}}}(X)$ denotes the canonical projection. Note that

$$\bigcap_{X\in\mathcal{C}} \mathrm{Ker}(\pi_X) = \bigcap_{X\in\mathcal{C}} \Omega_{\hat{\mathbf{A}}}(X) = \widetilde{\Omega}(\mathbb{A}) = \Delta_{\mathbf{A}}.$$

Hence $\mathcal{A}$ is a subdirect intersection of

$$\mathcal{A}/\Omega_{\mathcal{A}}(X) = \langle \hat{\mathbf{A}}/\Omega_{\hat{\mathbf{A}}}(X), \pi_X \circ h \rangle \in \mathrm{Alg}^*(\mathbb{L}), \quad X \in \mathcal{C}.$$

Assume, conversely, that $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ is a subdirect intersection of a collection $\mathcal{A}_i = \langle \hat{\mathbf{A}}_i, h_i \rangle \in \mathrm{Alg}^*(\mathbb{L})$, $i \in I$. Then, by hypothesis, we have commutative diagrams of grid morphisms,



such that $\bigcap_{i\in I}\mathrm{Ker}(g_i) = \Delta_{\mathbf{A}}$. Moreover, since, for all $i \in I$, $\mathcal{A}_i \in \mathrm{Alg}^*(\mathbb{L})$, there exists $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, such that $\Omega_{\hat{\mathbf{A}}_i}(X) = \Delta_{\mathbf{A}_i}$. Let $C : \mathcal{P}(A) \to \mathcal{P}(A)$, be such that

$$\mathcal{C} = \{g_i^{-1}(X_i) : i \in I\}$$

and set $\mathbb{A} = \langle \mathcal{A}, C \rangle$. Since $X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}_i)$, $i \in I$, we have that $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover,

$$\begin{array}{rcll}
\widetilde{\Omega}(\mathbb{A}) & = & \bigcap_{i\in I}\Omega_{\hat{\mathbf{A}}}(g_i^{-1}(X_i)) & (\text{Definition of } \widetilde{\Omega}(\mathbb{A}))\\
& = & \bigcap_{i\in I} g_i^{-1}(\Omega_{\hat{\mathbf{A}}_i}(X_i)) & (\text{Property of } \Omega)\\
& = & \bigcap_{i\in I} g_i^{-1}(\Delta_{\mathbf{A}_i}) & (\Omega_{\mathcal{A}_i}(X) = \Delta_{\mathbf{A}_i})\\
& = & \bigcap_{i\in I}\mathrm{Ker}(g_i) & (\text{Definition of } \mathrm{Ker}(g_i))\\
& = & \Delta_{\mathbf{A}}. & (\text{Assumption})
\end{array}$$

Thus, $\mathcal{A} \in \mathrm{Alg}(\mathbb{L})$. ∎

Corollary 178 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be an algebraic logicoid. Then* $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$. *Moreover,* $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$ *if and only if* $\mathrm{Alg}^*(\mathbb{L})$ *is closed under subdirect intersections.*

We may also relate the classes of algebras of two logicoids over the same grid that are themselves related by the $\leq^\flat$ relation. Recall that, given an algebraic grid $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ and two logicoids $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ and $\mathbb{L}' = \langle \hat{\mathbf{B}}, C'^\flat \rangle$, we write $\mathbb{L} \leq^\flat \mathbb{L}'$ if and only if, for all $X \subseteq B$,

$$C'^\flat(X) \leq^\flat C^\flat(X).$$

Recall, also, that this is equivalent to $\mathcal{C}'^\flat \subseteq \mathcal{C}^\flat$. Proposition 179 is an analog of Proposition 73 in the context of logicoids.

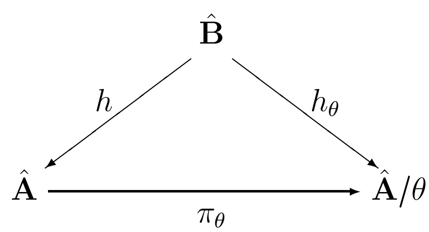
Proposition 179 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *and* $\mathbb{L}' = \langle \hat{\mathbf{B}}, C'^\flat \rangle$ *be algebraic logicoids over the same grid* $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$*, such that* $\mathbb{L} \leq^\flat \mathbb{L}'$. *Then* $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$ *and* $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$.

Proof: Suppose $\mathbb{L} \leq^\flat \mathbb{L}'$. Then, for all interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}'}(\mathcal{A}) \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus, $\mathrm{Alg}^*(\mathbb{L}') \subseteq \mathrm{Alg}^*(\mathbb{L})$. By Theorem 177, we also have $\mathrm{Alg}(\mathbb{L}') \subseteq \mathrm{Alg}(\mathbb{L})$. ∎

8.5 An Isomorphism Theorem

We fix a base logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ and an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. Let $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$. Recall that

$$\mathcal{A}/\theta = \langle \hat{\mathbf{A}}/\theta, h_\theta \rangle,$$



where $h_\theta = \pi_\theta \circ h$, with $\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta$ being the quotient grid morphism. Consider the model $\langle \mathcal{A}/\theta, C \rangle$, where $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Define

$$\widetilde{H}_{\mathcal{A}}(\theta) := \langle \mathcal{A}, C_\theta \rangle,$$

where $\langle \mathcal{A}, C_\theta \rangle$ is the algebraic logicoid induced by $\langle \langle \hat{\mathbf{A}}/\theta, \pi_\theta \rangle, C \rangle$ on $\mathbf{A}$. This defines a function

$$\begin{array}{rrcl} \widetilde{H}_{\mathcal{A}}(\theta) : & \mathrm{Con}(\hat{\mathbf{A}}) & \longrightarrow & \mathrm{Lgcd}(\mathcal{A}); \\ & \theta & \longmapsto & \langle \mathcal{A}, C_\theta \rangle. \end{array}$$

Note that, by Proposition 154, we have that

$$\pi_\theta : \widetilde{H}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C\rangle$$

is a bilogical morphism.

Lemma 180 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ *be a base logicoid,* $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ *a fixed interpretation and* $\theta \in \mathrm{Con}(\hat{\mathbf{A}})$*.*

(a) $\theta \in \mathrm{Con}(\widetilde{H}_{\mathcal{A}}(\theta))$*;*

(b) $\widetilde{H}_{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}/\theta, C\rangle$*;*

(c) $\widetilde{H}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$*;*

(d) $\theta \mapsto \widetilde{H}_{\mathcal{A}}(\theta)$ *is order preserving, i.e., if* $\theta \subseteq \theta'$*, then* $\widetilde{H}_{\mathcal{A}}(\theta) \leq^\flat \widetilde{H}_{\mathcal{A}}(\theta')$*.*

Proof:

(a) By Proposition 154, $\pi_\theta : \widetilde{H}_{\mathcal{A}}(\theta) \to_b \langle \mathcal{A}/\theta, C\rangle$ is a bilogical morphism. By Proposition 132, $\theta \in \mathrm{Con}(\widetilde{H}_{\mathcal{A}}(\theta))$.

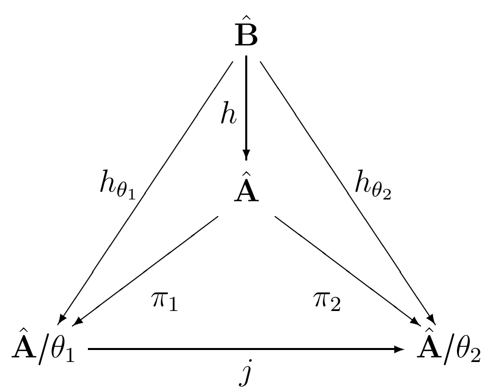
(b) We have, for all $S \subseteq A/\theta$,

$$\begin{array}{rcll} (C_\theta/\theta)(S) & = & \pi_\theta(C_\theta(\pi_\theta^{-1}(S))) & \text{(Definition of } C_\theta/\theta) \\ & = & \pi_\theta\left(\bigwedge \pi_\theta^{-1}(\mathcal{C})^{\pi_\theta^{-1}(S)}\right) & \text{(Definition of } C_\theta) \\ & = & \pi_\theta(\pi_\theta^{-1}(C(S))) & (\pi_\theta : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\theta) \\ & = & C(S). & \text{(Surjectivity)} \end{array}$$

Thus, $\widetilde{H}_{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}/\theta, C\rangle$.

(c) By hypothesis, $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta)$. Thus, by Proposition 165, $\langle \mathcal{A}/\theta, C\rangle$ is a full model of $\mathbb{L}$. Thus, by Proposition 166, $\widetilde{H}_{\mathcal{A}}(\theta)$ is also a full model of $\mathbb{L}$.

(d) Let $\theta_1, \theta_2 \in \mathrm{Con}(\hat{\mathbf{A}})$, such that $\theta_1 \subseteq \theta_2$. Let $\pi_1 : \mathcal{A} \to \mathcal{A}/\theta_1$ and $\pi_2 : \mathcal{A} \to \mathcal{A}/\theta_2$ be the canonical projections. Let, also, $j : \mathcal{A}/\theta_1 \to \mathcal{A}/\theta_2$ be the map given by $a/\theta_1 \mapsto a/\theta_2$, which is well defined due to the inclusion $\theta_1 \subseteq \theta_2$. In addition, we have the following commutative diagram.



Now we get

$$\begin{array}{rcl}\mathcal{C}_{\theta_2} & = & \pi_2^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2)) \quad (\pi_2 : \widetilde{H}_{\mathcal{A}}(\theta_2) \to_b \langle \mathcal{A}/\theta_2, C_2\rangle) \\ & = & \pi_1^{-1}(j^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_2)) \quad (\pi_2 = j \circ \pi_1) \\ & \subseteq & \pi_1^{-1}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\theta_1)) \quad (\text{Proposition } 148) \\ & = & \mathcal{C}_{\theta_1}. \quad (\pi_1 : \widetilde{H}_{\mathcal{A}}(\theta_1) \to_b \langle \mathcal{A}/\theta_1, C_1\rangle)\end{array}$$

This shows that $\widetilde{H}_{\mathcal{A}}(\theta_1) \leq \widetilde{H}_{\mathcal{A}}(\theta_2)$.

■

Now we are in a position to prove a general analog of the Isomorphism Theorem of Font and Jansana (Theorem 2.30 of [12]) which is applicable even in contexts involving non-monotonicity. This result becomes applicable very widely, while its restriction to traditional $\subseteq$-closure operators is an Isomorphism Theorem very much resembling the one of Font and Jansana modulo the introduction of fixed interpretations. In this latter respect, we follow more closely the generalization of Font and Jansana's result that was presented as Theorem 13 of [21] for logical systems formalized as π-institutions. The Isomorphism Theorem 75 for logicates, proved in Chapter 4, is also a result in the same tradition.

Theorem 181 (Isomorphism) *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat\rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$ a fixed interpretation. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is an order isomorphism between the ordered set $\mathbf{FMod}_{\mathbb{L}}(\mathcal{A}) = \langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A}), \leq\rangle$ of full models of $\mathbb{L}$ on $\mathcal{A}$ and the ordered set $\mathbf{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}) = \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq\rangle$ of $\mathrm{Alg}(\mathbb{L})$-congruences on $\mathcal{A}$, ordered under inclusion. The mapping $\widetilde{H}_{\mathcal{A}}$ is its inverse.*

Proof: By Proposition 172, if $\mathbb{A} \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$, then $\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By Lemma 180, if $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$, then $\widetilde{H}_{\mathcal{A}}(\theta) \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. So it suffices to show that $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse mappings and that they are both order preserving.

Let $\mathbb{A} = \langle \mathcal{A}, C\rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. By Proposition 172, $\mathcal{A}^*$ is an $\mathbb{L}$-algebra and $\widetilde{\Omega}_{\mathcal{A}}(C) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. As $\mathbb{A}$ is induced by its reduction $\mathbb{A}^* = \langle \mathcal{A}^*, C^*\rangle$, with $\mathcal{C}^* = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, along the quotient grid morphism $\pi : \mathcal{A} \to \mathcal{A}^*$, we get, by definition, that $\mathbb{A} = \widetilde{H}_{\mathcal{A}}(\widetilde{\Omega}_{\mathcal{A}}(\mathbb{A}))$.

Suppose, conversely, that $\theta \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Consider $\mathbb{A}^\theta = \langle \mathcal{A}^\theta, C\rangle$, where $\mathcal{A}^\theta = \langle \hat{\mathbf{A}}/\theta, \pi_\theta \circ h\rangle$ and $C = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}^\theta)$. Then $\widetilde{\Omega}_{\mathcal{A}^\theta}(C) = \Delta_{\mathbf{A}/\theta}$. Thus, by Proposition 148,

$$\begin{array}{rcl}\widetilde{\Omega}_{\mathcal{A}}(\widetilde{H}_{\mathcal{A}}(\theta)) & = & \widetilde{\Omega}_{\mathcal{A}}(\pi_\theta^{-1}(C)) \\ & = & \pi_\theta^{-1}(\widetilde{\Omega}_{\mathcal{A}^\theta}(C)) \\ & = & \pi_\theta^{-1}(\Delta_{\mathbf{A}/\theta}) \\ & = & \theta.\end{array}$$

Hence, $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse bijections. $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving by definition. Finally, by Lemma 180, $\widetilde{H}_{\mathcal{A}}$ is also order preserving. This shows that

$$\langle \mathrm{FMod}_{\mathbb{L}}(\mathcal{A}), \leq \rangle \underset{\widetilde{H}_{\mathcal{A}}}{\overset{\widetilde{\Omega}_{\mathcal{A}}}{\rightleftarrows}} \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle$$

are inverse order isomorphisms. ■

Chapter 9

Aspects of the Hierarchy

9.1 Introduction

Chapter 8 is intended to only provide a relatively superficial flavor of a semantically defined algebraic hierarchy of classes of logicoids based on properties of their Leibniz operator, paralleling the classical one for monotonic logics (see, e.g., [8, 14, 10]). The best part of it (Sections 9.2-9.4) is dedicated to the study of *protoalgebraicity*, Section 9.5 looks briefly at *weak algebraizability* and Section 9.6 looks even more briefly at *truth equationality*. We give several characterizations of *protoalgebraicity*, which is the property of having a monotone Leibniz operator, and we study the *Correspondence Theorem* and several of its consequences. This segues nicely into the introduction of *Leibniz filters* and some of their properties. *Weak algebraizability* is the property of having both a monotone and an order reflecting Leibniz operator, whereas *truth equationality* is the property of having a completely order reflecting Leibniz operator. As in the traditional monotonic case, it turns out that weak algebraizability is the conjunction of protoalgebraicity and truth equationality.

In Section 9.2, we introduce *protoalgebraic logicoids*. A logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ is *protoalgebraic* if, for all theories X and all $a, b \in B$, $\langle a, b\rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ implies that $\langle a, b\rangle \in \Lambda_{\mathbb{L}}(X)$, where $\Lambda_{\mathbb{L}}(X)$ is the relation holding if, for every theory X', with $X \leq^\flat X'$, $a \in X'$ iff $b \in X'$. We show that protoalgebraicity is equivalent to the monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on the theories of $\mathbb{L}$. Moreover, $\mathbb{L}$ is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. An additional characterization asserts that $\mathcal{L}$ is protoalgebraic if and only if $\Omega_{\mathcal{A}}$ is submeetive on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, meaning that, for all $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $\Omega_{\mathcal{A}}(\bigwedge_{i\in I} X_i) \subseteq \bigcap_{i\in I} \Omega_{\mathcal{A}}(X_i)$.

In Section 9.3, the central task is proving a Correspondence Theorem, an analog of Theorem 6.19 of [10], for protoalgebraic logicoids. After doing this, we explore some of its consequences. Among these are some additional characterizations of protoalgebraicity using the Tarski operator. We show, e.g., that $\mathbb{L}$ is protoalgebraic if and only if, for every $\mathbb{L}$-model $\mathbb{A} = \langle \mathcal{A}, \mathcal{C}\rangle$, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$, where $\min \mathcal{C}$ is the least theory of $\mathbb{A}$ (or the set of theorems of $\mathbb{A}$). Another consequence is that, if the logicoid $\mathbb{L}$ happens to be protoalgebraic, then the classes of interpretations $\mathrm{Alg}^*(\mathbb{L})$ and $\mathrm{Alg}(\mathbb{L})$ coincide. In addition, if $\mathbb{L}$ is protoalgebraic, then any two logicoid models $\mathbb{A}$ and $\mathbb{A}'$ over the same underlying interpretation that share the same minimum theories must be identical. The last result of the section is a theorem characterizing the full models of a protoalgebraic logicoid, while, at the same time providing yet another characterization of protoalgebraicity. It asserts that $\mathbb{L}$ is protoalgebraic if and only if its full models are of the form $\langle \mathcal{A}, \mathcal{C}^F\rangle$, where $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, for some interpretation $\mathcal{A}$ and some filter F in $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Here, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ denotes the collection of all $\mathbb{L}$-filters on $\mathcal{A}$ dominating F in the $\leq$ ordering of the subsets of A in the underlying algebraic grid $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq\rangle$ of $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$.

Section 9.4 considers a question that arises naturally from the characterization of full models of protoalgebraic logicoids. More precisely, it attempts to characterize those $\mathbb{L}$-filters F on an interpretation $\mathcal{A}$ for which $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ is a full $\mathbb{L}$-model. To do this, we form the subset of such filters $\mathrm{Fi}^{\star}_{\mathbb{L}}(\mathcal{A})$. These filters are termed *Leibniz filters*. If $\mathbb{L}$ is protoalgebraic, $\Omega_{\mathcal{A}}$ turns out to be an order isomorphism from $\mathrm{Fi}^{\star}_{\mathbb{L}}(\mathcal{A})$, ordered by $\leq$ onto $\mathrm{Con}_{\mathrm{Alg}^*}(\mathcal{A})$, ordered by $\subseteq$. Further, we introduce an equivalence $\sim_\Omega$ between $\mathbb{L}$-filters on an interpretation $\mathcal{A}$ that "identifies" two filters if they have the same Leibniz grid congruence. The $\sim_\Omega$-class of an $\mathbb{L}$-filter F is denoted by $[F]_\Omega$. If $\mathbb{L}$ is protoalgebraic, then F is the minimum element in the $\leq$ ordering in $[F]_\Omega$. This affords the characterization of Leibniz filters as those $\mathbb{L}$-filters on an interpretation that are minimum in their $\sim_\Omega$-equivalence classes. Equivalently, they are the $\mathbb{L}$-filters F, whose quotients $F/\Omega_{\mathcal{A}}(F)$ are minimum $\mathbb{L}$-filters in the $\leq^{\Omega_{\mathcal{A}}(F)}$ ordering on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Section 9.5 deals with a second question that may be seen to arise from the characterization of full models of a protoalgebraic logicoid. Namely, identify those situations for which the collection of Leibniz filters is the entire collection of filters on an interpretation. We call a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *weakly algebraizable* if the Leibniz operator is order preserving and order reflecting on $\mathcal{C}^\flat$. This is equivalent to the Leibniz operator $\Omega_{\mathcal{A}}$ being order preserving and order reflecting on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. Furthermore, it turns out that $\mathbb{L}$ is weakly algebraizable if and only if $\mathrm{Fi}^{\star}_{\mathbb{L}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$ of $\mathbb{L}$. Thus, weak algebraizability settles the initial problem of discovering a property under which the collection of the Leibniz $\mathbb{L}$-filters on any interpretation coinciding with the entire collection of $\mathbb{L}$-filters. This characterization, combined with the results of Section 9.4, provides several additional characterizations of weak algebraizability. E.g., we get that $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A}$ and all $\mathbb{L}$-filters F on $\mathcal{A}$, $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on the quotient interpretation $\mathcal{A}/\Omega_{\mathcal{A}}(F)$ and that $\mathbb{L}$ is weakly algebraizable if and only if $\Omega_{\mathcal{A}}$ is a lattice isomorphism from $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$.

In Section 9.6, we briefly introduce the property of *truth equationality*. We say that a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ is *truth equational* if $\Omega_{\hat{\mathbf{B}}}$ is completely order reflecting on $\mathcal{C}^\flat$, that is, if, for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$,

$$\bigcap_{i\in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies} \quad \bigwedge_{i\in I}^{\flat} X_i \leq^\flat X.$$

We show that this is equivalent to the complete order reflectivity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for all interpretations $\mathcal{A}$ of $\mathbb{L}$. Finally, we prove that weak algebraizability is characterized as the conjunction of protoalgebraicity and truth equationality.

9.2 Protoalgebraic Logicoids

Recall that an *algebraic grid* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$ consists of an algebra $\mathbf{A}$ and a complete lattice ordering $\leq$ on $\mathcal{P}(A)$. Recall, further, that an *algebraic logicoid* $\langle \hat{\mathbf{A}}, C \rangle$ consists of an algebraic grid $\hat{\mathbf{A}}$ and a $\leq$-closure operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$. That is, an operator that is inflationary, monotone and idempotent with respect to $\leq$. We use $\mathcal{C}$ to denote the collection of all *theories* of C, i.e., subsets $X \subseteq A$, such that $C(X) = X$.

In the abstract study of logicoids, a particular fixed logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, with $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$, is at the focus of investigations and it is called the *base logicoid*. Both matrix (Chapter 7) and logicoid (Chapter 8) models of the base logicoid are based on interpretations $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, which are grid morphisms from the base grid $\hat{\mathbf{B}}$ onto a grid $\hat{\mathbf{A}}$ over a similar algebra. A *grid morphism* $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a surjective algebra homomorphism $h : \mathbf{B} \to \mathbf{A}$, such that $h^{-1} : \langle \mathcal{P}(A), \leq \rangle \to \langle \mathcal{P}(B), \leq^\flat \rangle$ is a complete lattice embedding.

Let $\hat{\mathbf{B}} = \langle \mathbf{B}, \leq^\flat \rangle$ be an algebraic grid. A *grid congruence* θ on $\hat{\mathbf{B}}$ is a congruence on $\mathbf{B}$, such that $\langle \mathrm{Cmp}(\theta), \leq^\flat \rangle$ is a complete sublattice of $\langle \mathcal{P}(B), \leq^\flat \rangle$. Here $\mathrm{Cmp}(\theta)$ denotes the set of all subsets of B with which θ is compatible. Given an $X \subseteq B$, the *Leibniz congruence* $\Omega_{\hat{\mathbf{B}}}(X)$ of the logical matrix $\mathfrak{A} = \langle \hat{\mathbf{B}}, X \rangle$ is the largest grid congruence on $\hat{\mathbf{B}}$ that is compatible with X. Given a $\leq$-closure operator C on $\hat{\mathbf{B}}$, the *Tarski congruence* $\widetilde{\Omega}_{\hat{\mathbf{B}}}(C)$ of the logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C \rangle$ is the largest grid congruence on $\hat{\mathbf{B}}$ that is compatible with $\mathcal{C}$.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be an algebraic logicoid. We say that $\mathbb{L}$ is **protoalgebraic** (see [2] and, also, [8, 12, 10]) if, for all $a, b \in B$ and all $X \in \mathcal{C}^\flat$,

$$\langle a, b \rangle \in \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies,} \quad \begin{array}{l} \text{for all } X \leq^\flat X' \in \mathcal{C}^\flat, \\ a \in X' \text{ iff } b \in X'. \end{array}$$

For the corresponding definition for logicates, see Section 5.2. We make an observation and then introduce some notation that will help abbreviate the definition.

Observe that protoalgebraicity depends only on the collection of theories of a logicoid. This is commensurate with the monotonic theory, where protoalgebraicity depends only on the theory lattice of a sentential logic. Note, also, that, as in the monotonic framework, in the framework of logicoids, the theory lattice fully determines the logicoid itself, due to the presence of the underlying grid.

Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ and $X \subseteq B$, we define a relation $\Lambda_{\mathbb{L}}(X)$ on B with the goal of capturing the defining property of protoalgebraicity. We set, for all $X \subseteq B$ and all $a, b \in B$,

$$\langle a, b \rangle \in \Lambda_{\mathbb{L}}(X) \quad \text{iff,} \quad \begin{array}{l} \text{for all } X \leq^\flat X' \in \mathcal{C}^\flat, \\ a \in X' \text{ iff } b \in X'. \end{array}$$

With this definition available, we may rephrase the definition of protoalgebraicity. Clearly, $\mathbb{L}$ is **protoalgebraic** if and only if, for all $X \in \mathcal{C}^\flat$,

$$\Omega_{\hat{\mathbf{B}}}(X) \subseteq \Lambda_{\mathbb{L}}(X).$$

It is well known that, in the traditional framework, protoalgebraicity is tantamount to the monotonicity of the Leibniz operator (see [3]) on the lattice of theories of the logic (see, e.g., [8, 12, 10]). An analogous result was proven for protoalgebraic logicates in Proposition 80. The following proposition revisits this characterization in the context of logicoids.

Proposition 182 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be an algebraic logicoid. $\mathbb{L}$ is protoalgebraic if and only if $\Omega_{\hat{\mathbf{B}}}$ is monotone on $\mathcal{C}^\flat$.*

Proof: Suppose $\mathbb{L}$ is protoalgebraic and let $X, X' \in \mathcal{C}^\flat$, such that $X \leq^\flat X'$. Let $a, b \in B$, such that $\langle a, b\rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ and $a \in X'$. By protoalgebraicity, $\langle a, b\rangle \in \Lambda_{\mathbb{L}}(X)$ and $a \in X'$. Since $X \leq^\flat X'$, $b \in X'$. This shows that $\Omega_{\hat{\mathbf{B}}}(X)$ is compatible with X'. By the maximality property of $\Omega_{\hat{\mathbf{B}}}(X')$ with respect to compatibility with X', we conclude that $\Omega_{\hat{\mathbf{B}}}(X) \subseteq \Omega_{\hat{\mathbf{B}}}(X')$. Thus, $\Omega_{\hat{\mathbf{B}}}$ is monotone on $\mathcal{C}^\flat$.

Suppose, conversely, that $\Omega_{\hat{\mathbf{B}}}$ is monotone on $\mathcal{C}^\flat$. Let $a, b \in B$, $X \in \mathcal{C}^\flat$, such that $\langle a, b\rangle \in \Omega_{\hat{\mathbf{B}}}(X)$ and $X' \in \mathcal{C}^\flat$, with $X \leq^\flat X'$. Then $\langle a, b\rangle \in \Omega_{\hat{\mathbf{B}}}(X')$. So, by the compatibility of $\Omega_{\hat{\mathbf{B}}}(X')$ with X', $a \in X'$ iff $b \in X'$. Thus, $\mathbb{L}$ is protoalgebraic. ■

Protoalgebraicity of logicoids offers an opportunity of formulating another transfer theorem, analogous to Proposition 81. It extends monotonicity of the Leibniz operator to the monotonicity of the Leibniz operator on the $\mathbb{L}$-filters of any interpretation.

Proposition 183 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is protoalgebraic if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h\rangle$, $\Omega_{\mathcal{A}}$ is monotone on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Proof: Suppose that $\mathbb{L}$ is protoalgebraic. Let $Y_1, Y_2 \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $Y_1 \leq Y_2$, and $a_1, a_2 \in A$. Note that, since $h : \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ is a grid morphism, there exist $b_1, b_2 \in B$, such that $a_1 = h(b_1)$ and $a_2 = h(b_2)$. Now we have

$$\begin{array}{rll}
\langle a_1, a_2\rangle \in \Omega_{\mathcal{A}}(Y_1) & \text{iff} & \langle h(b_1), h(b_2)\rangle \in \Omega_{\mathcal{A}}(Y_1) \\
& \text{iff} & \langle b_1, b_2\rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_1)) \\
& \text{iff} & \langle b_1, b_2\rangle \in \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y_1)) \\
& \text{implies} & \langle b_1, b_2\rangle \in \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y_2)) \\
& \text{iff} & \langle b_1, b_2\rangle \in h^{-1}(\Omega_{\mathcal{A}}(Y_2)) \\
& \text{iff} & \langle h(b_1), h(b_2)\rangle \in \Omega_{\mathcal{A}}(Y_2) \\
& \text{iff} & \langle a_1, a_2\rangle \in \Omega_{\mathcal{A}}(Y_2).
\end{array}$$

Note that we have used the property that the Leibniz operator commutes with inverse grid morphisms. We have now shown that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, if the condition in the statement holds, then the monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on $\mathcal{C}^\flat$ follows by taking $\mathcal{A} = \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle$. Then the conclusion follows from Proposition 182 and the observation that $\mathcal{C}^\flat = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. ■

One may also devise a slightly different characterization involving meets. Let us show, first, that, for every interpretation, the collection of $\mathbb{L}$-filters on the interpretation is closed under arbitrary meets.

Lemma 184 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. Then, for any interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, the set $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ of $\mathbb{L}$-filters on $\mathcal{A}$ is closed under arbitrary meets.*

Proof: Let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ be a collection of $\mathbb{L}$-filters on $\mathcal{A}$. Then, since h^{-1} is a complete lattice embedding,

$$h^{-1}\left(\bigwedge_{i \in I} X_i\right) = \bigwedge_{i \in I}^{\flat} h^{-1}(X_i) \in \mathcal{C}^\flat,$$

where membership follows by the definition of $\mathbb{L}$-filter and Proposition 102. This shows that $\bigwedge_{i \in I} X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. ■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. We say that the Leibniz operator is **submeetive** on $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, if, for all $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\Omega_{\mathcal{A}}\left(\bigwedge_{i \in I} X_i\right) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i).$$

We show that protoalgebraicity is equivalent to the Leibniz operator on the filters of any interpretation being submeetive. This property is an adaptation of the "difficult half" of the well-known property of "commuting with intersections". Except that, in the present setting, since in $\mathbf{Fi}_{\mathbb{L}}(\mathcal{A})$, $\subseteq$ has been replaced by (an arbitrary complete lattice ordering) $\leq$, it may be that the "easy half" may not hold.

Proposition 185 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is protoalgebraic if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\Omega_{\mathcal{A}}$ is submeetive on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., for all $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,*

$$\Omega_{\mathcal{A}}\left(\bigwedge_{i \in I} X_i\right) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i).$$

Proof: By Proposition 183, protoalgebraicity is equivalent to the monotonicity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$.

Suppose, first, that $\Omega_{\mathcal{A}}$ is monotone. Let $\{X_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Lemma 184, $\bigwedge_{i \in I} X_i \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By monotonicity, $\Omega_{\mathcal{A}}(\bigwedge_{i \in I} X_i) \subseteq \Omega_{\mathcal{A}}(X_i)$, for all $i \in I$. Thus, $\Omega_{\mathcal{A}}(\bigwedge_{i \in I} X_i) \subseteq \bigcap_{i \in I} \Omega_{\mathcal{A}}(X_i)$.

Suppose, next, that , for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\Omega_{\mathcal{A}}$ is submeetive on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \leq X'$. Then $X \wedge X' = X$ and we have

$$\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X \wedge X') \subseteq \Omega_{\mathcal{A}}(X) \cap \Omega_{\mathcal{A}}(X') \subseteq \Omega_{\mathcal{A}}(X').$$

So $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and, hence, $\mathbb{L}$ is protoalgebraic. ■

9.3 Correspondence Theorem

Given a logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, \leq^\flat \rangle$, an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, with $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, and a filter $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we write

$$\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F := \{X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : F \leq X\}.$$

The well-known Correspondence Theorem for protoalgebraic logics (see, e.g., Theorem 6.19 of [10] and Proposition 85 for a logicate version) dealing with the structure of theories and filters may be adapted to the present context. Here, it establishes an isomorphism between complete lattices of filters on an interpretation and on the quotient of an interpretation by a Leibniz congruence.

Proposition 186 (Correspondence) *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. If $\mathbb{L}$ is protoalgebraic, then, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and every $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, letting $\pi : \hat{\mathbf{A}} \to \hat{\mathbf{A}}/\Omega_{\mathcal{A}}(F)$ be the quotient grid morphism,*

$$\begin{array}{rrcl} \pi : & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F & \longrightarrow & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}; \\ & X & \longmapsto & \pi(X), \end{array}$$

establishes an isomorphism between the complete lattice $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \leq \rangle$ and the complete lattice $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \leq^{\Omega_{\mathcal{A}}(F)} \rangle$.

Proof: Let $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $F \leq X$. By protoalgebraicity, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(X)$. Hence $\Omega_{\mathcal{A}}(F)$ is compatible with X. It follows that, for $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$, $X = \pi^{-1}(\pi(X))$. Thus, by Proposition 148, we obtain $\pi(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Clearly, since $F \leq X$ and π^{-1} is a complete lattice embedding, $\pi(F) \leq^{\Omega_{\mathcal{A}}(F)} \pi(X)$. On the other hand, if $Y \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, then, again by Proposition 148, $\pi^{-1}(Y) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Moreover, $\pi(F) \leq^{\Omega_{\mathcal{A}}(F)} Y$ and π^{-1} a complete lattice embedding imply $F = \pi^{-1}(\pi(F)) \leq \pi^{-1}(Y)$. Thus, π establishes an isomorphism between the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \leq \rangle$ and the ordered set $\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{\pi(F)}, \leq^{\Omega_{\mathcal{A}}(F)} \rangle$, as claimed. ■

We now provide some additional characterizations of protoalgebraicity in terms of the Tarski operator. Given an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, with $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, a logicoid $\mathbb{A} = \langle \hat{\mathcal{A}}, C \rangle$ and a theory F of C, we shall write $\mathbb{A}^F = \langle \mathcal{A}, C^F \rangle$ for the logicoid with

$$\mathcal{C}^F = \{X \in \mathcal{C} : F \leq X\}.$$

Proposition 187 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicate. The following statements are equivalent.*

(i) $\mathbb{L}$ *is protoalgebraic;*

(ii) *For any* $\mathbb{L}$*-model* $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle$, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$;

(iii) *For any* $\mathbb{L}$*-model* $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle$, *with* $Y \in \mathcal{C}$, $\widetilde{\Omega}(\mathbb{A}^Y) = \Omega_{\mathcal{A}}(Y)$;

(iv) *For any* $X \in \mathcal{C}^\flat$, $\widetilde{\Omega}(\mathbb{L}^X) = \Omega_{\hat{\mathbf{B}}}(X)$.

Proof:

(i)⇒(ii) Suppose $\mathbb{A} = \langle \mathcal{A}, \mathcal{C} \rangle \in \mathrm{Mod}(\mathbb{L})$, where $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. Then, by Proposition 163, $\mathcal{C} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Hence, by Proposition 183, $\Omega_{\mathcal{A}}$ is order preserving on $\mathcal{C}$. By definition of the Leibniz congruence and its monotonicity, $\Omega_{\mathcal{A}}(\min \mathcal{C})$ is a grid congruence on $\hat{\mathbf{A}}$ compatible with all $X \in \mathcal{C}$. Hence, by the definition of the Tarski congruence, $\Omega_{\mathcal{A}}(\min \mathcal{C}) \subseteq \widetilde{\Omega}(\mathbb{A})$. On the other hand, $\widetilde{\Omega}(\mathbb{A})$ is a grid congruence on $\hat{\mathbf{A}}$, compatible with all $X \in \mathcal{C}$ and, thus, in particular with $\min \mathcal{C}$. Hence, by the definition of the Leibniz congruence, $\widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(\min \mathcal{C})$. So we get $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C})$.

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Trivial.

(iv)⇒(i) Let $X, X' \in \mathcal{C}^\flat$, such that $X \subseteq X'$. Then $X' \in \mathcal{C}^{\flat X}$. Thus, we get

$$\begin{array}{rcll} \Omega_{\hat{\mathbf{B}}}(X) & = & \widetilde{\Omega}(\mathbb{L}^X) & \text{(Hypothesis (iv))} \\ & \subseteq & \bigcap_{Y \in \mathcal{C}^{\flat X}} \Omega_{\hat{\mathbf{B}}}(Y) & \text{(Tarski Congruence)} \\ & \subseteq & \Omega_{\hat{\mathbf{B}}}(X'). & (X' \in \mathcal{C}^{\flat X}) \end{array}$$

So $\Omega_{\hat{\mathbf{B}}}$ is monotone on $\mathcal{C}^\flat$, showing that $\mathbb{L}$ is protoalgebraic.

■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid and consider an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$. By Proposition 187, for $\mathbb{A} = \langle \mathcal{A}, C \rangle$, with $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})).$$

The following proposition is an analog of Proposition 3.2 of [12] in the present setting (see, also Proposition 87 for logicates).

Proposition 188 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a protoalgebraic logicoid. Then*

$$\mathrm{Alg}(\mathbb{L}) = \mathrm{Alg}^*(\mathbb{L}).$$

Proof: By Corollary 178, we have $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$, without any preconditions. Suppose, conversely, that $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle \in \mathrm{Alg}(\mathbb{L})$. Then, for $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, we have $\widetilde{\Omega}(\langle \mathcal{A}, C \rangle) = \Delta_{\mathbf{A}}$. By hypothesis and Proposition 187,

$$\Omega_{\mathcal{A}}(\min \mathcal{C}) = \widetilde{\Omega}(\langle \mathcal{A}, C \rangle) = \Delta_{\mathbf{A}}.$$

This shows that $\mathcal{A} \in \mathrm{Alg}^*(\mathbb{L})$. ∎

Lemma 3.3 of [12] asserts that two protoalgebraic logics over the same signature that share the same sets of theorems are identical. The following analog requires the two protoalgebraic logicoids compared to have identical minimum theories and, in that case, asserts that the logicoids in question coincide.

Lemma 189 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid and $\mathbb{A} = \langle \mathcal{A}, C \rangle$, $\mathbb{A}' = \langle \mathcal{A}, C' \rangle$ be two full models of $\mathbb{L}$ over the same interpretation. Then*

$$\min \mathcal{C} = \min \mathcal{C}' \quad \textit{implies} \quad \mathcal{C} = \mathcal{C}'.$$

Proof: By hypothesis and Proposition 187,

$$\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(\min \mathcal{C}) = \Omega_{\mathcal{A}}(\min \mathcal{C}') = \widetilde{\Omega}(\mathbb{A}').$$

Thus, by the Isomorphism Theorem 181, $\mathcal{C} = \mathcal{C}'$. ∎

Protoalgebraicity in the monotonic theory was characterized in terms of full models in Theorem 3.4 of [12]. A similar characterization is possible in the case of logicoids. This forms a logicoid analog of Theorem 89 applicable to logicates.

Theorem 190 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is protoalgebraic if and only if all full models of $\mathbb{L}$ are of the form $\langle \mathcal{A}, C^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, for some interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and some $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.*

Proof: We work, first, to prove the "only if". Let $\mathbb{A} = \langle \mathcal{A}, C \rangle$ be a full model of $\mathbb{L}$, with $F = \min \mathcal{C}$. By protoalgebraicity and Proposition 187, $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(F)$. Hence, the quotient grid morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \langle \mathcal{A}, C \rangle \to \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), C^{\Omega_{\mathcal{A}}(F)} \rangle.$$

Since, by hypothesis, $\mathbb{A}$ is a full model of $\mathbb{L}$, $\mathcal{C}^{\Omega_{\mathcal{A}}(F)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Consider $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Then $F \leq X$ and, by protoalgebraicity, $\Omega_{\mathcal{A}}(F)$ is compatible with X. Thus, $X = \pi^{-1}(\pi(X))$. By Proposition 148, $\pi(X) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Hence, since $X = \pi^{-1}(\pi(X))$, $X \in \mathcal{C}$. This proves that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$.

We turn, next, to the "if". Suppose that all models of $\mathbb{L}$ have the indicated form. Let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $X \leq X'$. Consider $\Omega_{\mathcal{A}}(X)$. By Corollary 178, $\mathrm{Alg}^*(\mathbb{L}) \subseteq \mathrm{Alg}(\mathbb{L})$. Hence $\Omega_{\mathcal{A}}(X) \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By the Isomorphism Theorem 181, there exists a full model $\mathbb{A} = \langle \mathcal{A}, C \rangle$ of $\mathbb{L}$, such that $\widetilde{\Omega}(\mathbb{A}) = \Omega_{\mathcal{A}}(X)$. Moreover, by hypothesis, there exists $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. Since $\mathbb{A}$ is full, the quotient grid morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(X)$ is a bilogical morphism

$$\pi : \mathbb{A} \to \langle \mathcal{A}/\Omega_{\mathcal{A}}(X), \mathcal{C}^{\Omega_{\mathcal{A}}(X)} \rangle,$$

where $\mathcal{C}^{\Omega_{\mathcal{A}}(X)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(X))$. Moreover, as $X = \pi^{-1}(\pi(X))$, we get $X \in \mathcal{C}$. Hence, $F \le X \le X'$, whence, $X' \in \mathcal{C}$. Now we get

$$\Omega_{\mathcal{A}}(X) = \widetilde{\Omega}(\mathbb{A}) \subseteq \Omega_{\mathcal{A}}(X'),$$

i.e., $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. By Proposition 183, $\mathbb{L}$ is protoalgebraic. ■

9.4 Leibniz Filters

Leibniz filters were introduced by Font and Jansana in [12] (see Page 63), extensively studied in [11] and [17], and used further in applications of the theory in [13] and, in the case of logicates, in Section 5.4. Here we define an analog in the framework of logicoids.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. We define

$$\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}) = \{F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) : \text{if } \mathcal{C} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F, \text{ then } \langle \mathcal{A}, C \rangle \in \mathrm{FMod}_{\mathbb{L}}(\mathcal{A})\}.$$

The elements in $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ are called **Leibniz filters of $\mathbb{L}$ on $\mathcal{A}$**.

Proposition 191 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. Then $\Omega_{\mathcal{A}}$ is a poset isomorphism*

$$\Omega_{\mathcal{A}} : \langle \mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}), \le \rangle \cong \langle \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}), \subseteq \rangle = \langle \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}), \subseteq \rangle.$$

Proof: Consider the mapping

$$F \longmapsto \langle \mathcal{A}, C^F \rangle,$$

where $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$. By the definition of $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$, this is a mapping from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. It is injective and it is $\le$-order preserving and order reflecting. By protoalgebraicity and Theorem 190, it is also surjective. So it is an order isomorphism from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$. By the Isomorphism Theorem 181, $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$ is isomorphic to $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$ via the Tarski operator. Thus, the composition

$$F \longmapsto \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F\right)$$

establishes an order isomorphism between $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 187, $\widetilde{\Omega}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$. By protoalgebraicity and Proposition 188, $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}) = \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. Therefore, the Leibniz operator is an order isomorphism from $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A})$ to $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. ■

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid. The $\mathbb{L}$-filters in $\mathrm{Fi}^\bigstar_{\mathbb{L}}(\mathcal{A})$ on a given interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ may be characterized without reference to full models. To show this, we consider a binary relation $\sim_\Omega$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ defined as the kernel of the Leibniz operator on $\mathcal{A}$, i.e., for all $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$X \sim_\Omega X' \quad \text{iff} \quad \Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(X').$$

We write $[X]_\Omega$ for the $\sim_\Omega$-equivalence class of an $\mathbb{L}$-filter X. By Lemma 184, $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is closed under arbitrary meets. A fortiori, every full model of $\mathbb{L}$ on $\mathcal{A}$ has a minimum filter. Thus, if $\mathbb{L}$ is protoalgebraic, by Proposition 191, at most one $\mathbb{L}$-filter in each $\sim_\Omega$-equivalence class is in $\mathrm{Fi}^\bigstar_{\mathbb{L}}(\mathcal{A})$. As in Proposition 3.6 of [12], it is possible to characterize this filter. However, since in the present setting $\subseteq$-monotonicity is lost, it is not necessarily the case that this filter is the intersection of all filters in the class, as in Page 64 of [12] (see paragraph before Proposition 3.6). Instead, we need to use a slightly different argument taking into account the definition of a Leibniz filter.

Lemma 192 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid and $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation. If $F \in \mathrm{Fi}^\bigstar_{\mathbb{L}}(\mathcal{A})$, then $F = \min [F]_\Omega$.*

Proof: Suppose $F \in \mathrm{Fi}^\bigstar_{\mathbb{L}}(\mathcal{A})$. By definition of Leibniz filters, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F \rangle \in \mathrm{FMod}(\mathbb{L})$. By protoalgebraicity, Proposition 187 and the definition of a full model, the quotient grid morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F \rangle \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F)) \rangle.$$

Now consider $X \in [F]_\Omega$. Then

$$\begin{array}{rcl}(\pi \circ h)^{-1}(X/\Omega_{\mathcal{A}}(F)) & = & h^{-1}(\pi^{-1}(X/\Omega_{\mathcal{A}}(F))) \quad ((\pi \circ h)^{-1} = h^{-1} \circ \pi^{-1}) \\ & = & h^{-1}(X) \quad (\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{A}}(F)) \\ & \in & \mathcal{C}^\flat. \quad (X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}))\end{array}$$

Hence, by definition of a filter, $X/\Omega_{\mathcal{A}}(F) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Then, since π is a bilogical morphism,

$$X = \pi^{-1}(X/\Omega_{\mathcal{A}}(F)) \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F.$$

This shows that $F \le X$. Since X was an arbitrary element in $[F]_\Omega$, we conclude that $F = \min [F]_\Omega$. ■

Now we may characterize the class of Leibniz filters of $\mathbb{L}$ on $\mathcal{A}$. This is an analog for logicoids of Proposition 3.6 of [12] and of Proposition 91.

Proposition 193 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicate, $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ an interpretation and $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. The following statements are equivalent:*

(i) $F \in \mathrm{Fi}^{\bigstar}_{\mathbb{L}}(\mathcal{A})$*, i.e.,* $\langle \mathcal{A}, C^F\rangle$*, with* $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$*, is a full model of* $\mathbb{L}$*;*

(ii) F *is the minimum element in its* $\sim_\Omega$*-equivalence class;*

(iii) $F/\Omega_{\mathcal{A}}(F)$ *is the least* $\mathbb{L}$*-filter on* $\mathcal{A}/\Omega_{\mathcal{A}}(F)$*.*

Proof:

(i)⇒(ii) This is the content of Lemma 192.

(ii)⇒(iii) Suppose $F = \min[F]$ and let $G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Our goal is to show that $F/\Omega_{\mathcal{A}}(F) \leq^{\Omega_{\mathcal{A}}(F)} G$. Let $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ be the quotient grid morphism and set $F' = \pi^{-1}(G) \wedge F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, where membership is due to Proposition 148, the hypothesis and Lemma 184. Then

$$\begin{array}{rcll} F' & = & \pi^{-1}(G) \wedge \pi^{-1}(\pi(F)) & \text{(Compatibility of } \Omega_{\mathcal{A}}(F) \text{ with } F) \\ & = & \pi^{-1}(G \wedge^{\Omega_{\mathcal{A}}(F)} \pi(F)). & (\pi \text{ grid morphism}) \end{array}$$

Hence, F' is a union of $\Omega_{\mathcal{A}}(F)$-classes, i.e., $\Omega_{\mathcal{A}}(F)$ is compatible with F'. By the maximality property of the Leibniz congruence, $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(F')$. As, by definition, $F' \leq F$, by protoalgebraicity, $\Omega_{\mathcal{A}}(F') \subseteq \Omega_{\mathcal{A}}(F)$. Consequently, $\Omega_{\mathcal{A}}(F') = \Omega_{\mathcal{A}}(F)$, i.e., $F \sim_\Omega F'$. By hypothesis, $F \leq F'$ and, since, by definition, $F' \leq F$, $F = F'$. Thus, $F \leq \pi^{-1}(G)$. This yields

$$F/\Omega_{\mathcal{A}}(F) = \pi(F) \leq^{\Omega_{\mathcal{A}}(F)} \pi(\pi^{-1}(G)) = G.$$

Therefore, $F/\Omega_{\mathcal{A}}(F)$ is the least $\mathbb{L}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

(iii)⇒(i) Assume $F/\Omega_{\mathcal{A}}(F) = \min \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. By protoalgebraicity and the correspondence established in Proposition 186, the quotient grid morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ gives an order isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$ and $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{F/\Omega_{\mathcal{A}}(F)} = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Also by protoalgebraicity and Proposition 187,

$$\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F).$$

Hence $\langle \mathcal{A}, C^F\rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of $\mathbb{L}$.

■

9.5 Weak Algebraizability

In [3], Blok and Pigozzi introduced the notion of algebraizable logic. As they explain, the notion was a natural abstraction from many well-known examples, the most prototypical, perhaps, being that of classical propositional logic, of intuitionistic logic and the various implicative logics of Rasiowa [20]. Making an exact notion of algebraizability precise had, besides unification and clarification, the advantage of being able to show, for the first time, that logics that were known not to be amenable to algebraizability techniques, were somehow intrinsically non-algebraizable, since they did not fall under the scope of Blok and Pigozzi's definition. Blok and Pigozzi worked with finitary sentential logics, but their results were soon generalized further to cover many additional systems. One of the earliest generalizations was by Herrmann [15, 16] to cover infinitary logics. Algebraizability was shown to be equivalent to the conjunction of equivalentiality [6, 7] and of truth equationality [19]. Equivalentiality is a stronger property than protoalgebraicity, since it requires that the Leibniz operator be both monotone and commute with substitutions. If equivalentiality is weakened to protoalgebraicity, that is, if one requires that the logic be protoalgebraic and truth equational, then weak algebraizability [9] is obtained. All these properties and their characterizations and interconnections are studied, e.g., in the surveys [8, 12, 14]. In Section 5.5, we studied weak algebraizability in the context of logicates. We study, next, weak algebraizability in the context of logicoids.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be an algebraic logicoid. We say that $\mathbb{L}$ is **weakly algebraizable** [9] (see, also, [12, 8]) if the Leibniz operator is monotone (order preserving) and order reflecting on $\mathcal{C}^\flat$.

Note that the fact that in the traditional framework, in which the Leibniz operator commutes with intersections, monotonicity and injectivity are together equivalent to the property of the Leibniz operator being order preserving and order reflecting. So, as far as the traditional framework is concerned, postulating order reflectivity instead of injectivity, together with monotonicity, neither adds nor subtracts to the power of weak algebraizability. Recall, however, that, in the present setting, the Leibniz operator is only submeetive (and does not necessarily commute with meets). So order reflectivity postulated on top of monotonicity here adds more power than simply imposing injectivity in addition to monotonicity.

Proposition 194 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is order preserving and order reflecting.*

Proof: First, by Proposition 183, monotonicity of $\Omega_{\hat{\mathbf{B}}}$ on $\mathcal{C}^\flat$ is equivalent to monotonicity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$. So it suffices to see that order reflectivity of $\Omega_{\hat{\mathbf{B}}}$ on $\mathcal{C}^\flat$ is equivalent to order reflectivity of $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, for every interpretation $\mathcal{A}$.

Assume, first, that $\Omega_{\hat{\mathbf{B}}}$ is order reflective on $\mathcal{C}^\flat$. Consider an interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ and let $X, X' \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) \subseteq \Omega_{\mathcal{A}}(X')$. Applying h^{-1}, we get $h^{-1}(\Omega_{\mathcal{A}}(X)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(X'))$. By commutativity of the Leibniz operator with inverse grid morphisms, $\Omega_{\hat{\mathbf{B}}}(h^{-1}(X)) \subseteq \Omega_{\hat{\mathbf{B}}}(h^{-1}(X'))$. By hypothesis, $h^{-1}(X) \leq^\flat h^{-1}(X')$. Since h is a grid morphism, $X \leq X'$. Thus $\Omega_{\mathcal{A}}$ is order reflecting on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Conversely, suppose $\Omega_{\mathcal{A}}$ on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is order reflecting, for every interpretation $\mathcal{A}$. Then, by considering $\mathcal{A} = \langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle$, we get that $\Omega_{\hat{\mathbf{B}}}$ is order reflecting on $\mathcal{C}^\flat$. ■

The work of the preceding section on characterizing Leibniz filters of $\mathbb{L}$ on an interpretation $\mathcal{A}$ comes in handy in case one wants to provide a characterization of weakly algebraizable logicoids inside the class of protoalgebraic logicoids.

Proposition 195 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid. $\mathbb{L}$ is weakly algebraizable if and only if, for every interpretation $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, i.e., for all $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $\mathbb{A} = \langle \mathcal{A}, \mathcal{C}^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a full model of $\mathbb{L}$.*

Proof: Suppose, first, that $\mathbb{L}$ is weakly algebraizable. Then, for every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is order preserving and order reflecting and, hence, a fortiori, $\Omega_{\mathcal{A}}$ is injective. Thus, for all $\mathcal{A}$ and all $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $[F]_\Omega = \{F\}$. Hence, by Proposition 193, $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$.

Assume, conversely, that $\mathrm{Fi}_{\mathbb{L}}^{\bigstar}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and let $F, G \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(F) \subseteq \Omega_{\mathcal{A}}(G)$. Thus, $\Omega_{\mathcal{A}}(F)$ is compatible with G. This implies that $G/\Omega_{\mathcal{A}}(F)$ is an $\mathbb{L}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$. By Proposition 193, $F/\Omega_{\mathcal{A}}(F) \leq^{\Omega_{\mathcal{A}}(F)} G/\Omega_{\mathcal{A}}(F)$. Since π is a grid morphism, we get that $F \leq G$. This proves that $\Omega_{\mathcal{A}}$ is also order reflecting and, hence, $\mathbb{L}$ is weakly algebraizable. ■

This leads to several additional characterizations of weak algebraizability.

Theorem 196 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a protoalgebraic logicoid. The following statements are equivalent:*

(i) *$\mathbb{L}$ is weakly algebraizable;*

(ii) *For every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is monotone and injective on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$;*

(iii) *$\mathbb{L}$ is protoalgebraic and, for every interpretation $\mathcal{A}$ and every filter $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, $F/\Omega_{\mathcal{A}}(F)$ is the least filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$;*

(iv) *For every interpretation $\mathcal{A}$, the mapping $F \mapsto \langle \mathcal{A}, \mathcal{C}^F \rangle$, with $\mathcal{C}^F = \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})^F$, is a bijection between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathrm{FMod}_{\mathbb{L}}(\mathcal{A})$;*

(v) *For every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$;*

(vi) *For every interpretation $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ and $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$.*

Proof:

(i)⇔(ii) By Proposition 194.

(ii)⇔(iii) By Propositions 193 and 195.

(iii)⇒(iv) Consider the mapping $F \mapsto \langle \mathcal{A}, C^F \rangle$. It is injective. By Proposition 193 and the hypothesis, it is well defined. By Theorem 190, it is also surjective. Thus, it is a bijection. Since it is clearly order preserving and order reflecting, we get that it is a lattice isomorphism.

(iv)⇒(v) Consider again the mapping $F \mapsto \langle \mathcal{A}, C^F \rangle$. Since, by hypothesis, it is onto, by Theorem 190, $\mathbb{L}$ is protoalgebraic. Further, the composition of this mapping with the mapping $\widetilde{\Omega}_{\mathcal{A}}$ from the Isomorphism Theorem 181 gives an isomorphism from $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ onto $\mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. By protoalgebraicity and Proposition 187, the mapping is identical to $F \mapsto \widetilde{\Omega}(\langle \mathcal{A}, C^F \rangle) = \Omega_{\mathcal{A}}(F)$.

(v)⇒(vi) In general, $\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}) \subseteq \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A})$. Also in general, $\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. By hypothesis, each $\mathrm{Alg}(\mathbb{L})$-congruence is of the form $\Omega_{\mathcal{A}}(F)$, for some interpretation $\mathcal{A}$ and some $F \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Thus,

$$\mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}) = \mathrm{Con}_{\mathrm{Alg}(\mathbb{L})}(\mathcal{A}).$$

This yields (vi).

(vi)⇒(i) Trivial.

■

For a weakly algebraizable logicoid $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$, we call a class K of interpretations an **equivalent algebraic semantics** for $\mathbb{L}$ if, for every interpretation $\mathcal{A} = \langle \mathbf{A}, h \rangle$,

$$\langle \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}), \leq \rangle \cong \langle \mathrm{Con}_{\mathsf{K}}(\mathcal{A}), \subseteq \rangle.$$

Proposition 197 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a weakly algebraizable logicoid. Then* $\mathrm{Alg}^*(\mathbb{L})$ *is an equivalent algebraic semantics for* $\mathbb{L}$.

Proof: Let $\mathcal{A} = \langle \mathbf{A}, h \rangle$ be an interpretation. Define

$$\begin{array}{rrcl} \Omega_{\mathcal{A}}: & \mathrm{Fi}_{\mathbb{L}}(\mathcal{A}) & \longrightarrow & \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A}); \\ & X & \longmapsto & \Omega_{\mathcal{A}}(X). \end{array}$$

This mapping is well defined since $\langle \mathcal{A}/\Omega_{\mathcal{A}}(X), X/\Omega_{\mathcal{A}}(X) \rangle \in \mathrm{Mat}^*(\mathbb{L})$ and, hence, $\mathcal{A}/\Omega_{\mathcal{A}}(X) \in \mathrm{Alg}^*(\mathbb{L})$. By weak algebraizability and Proposition 194, it is both order preserving and order reflecting and, hence, a fortiori, injective. So it suffices to show that it is surjective.

Let $\theta \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{L})}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{Alg}^*(\mathbb{L})$, that is, there exists $X \in \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$, such that $\Omega_{\mathcal{A}}(X) = \theta$. Hence, $\Omega_{\mathcal{A}}$ is surjective. ■

Corollary 198 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a weakly algebraizable logicoid. Then* $\mathrm{Alg}(\mathbb{L})$ *is an equivalent algebraic semantics for* $\mathbb{L}$.

Proof: By hypothesis and Proposition 197, $\mathrm{Alg}^*(\mathbb{L})$ is an equivalent algebraic semantics for $\mathbb{L}$. Also by hypothesis and Proposition 188, $\mathrm{Alg}^*(\mathbb{L}) = \mathrm{Alg}(\mathbb{L})$. Therefore, $\mathrm{Alg}(\mathbb{L})$ is an equivalent algebraic semantics for $\mathbb{L}$. ■

9.6 Truth Equationality

Recall that weak algebraizability [9] is the combination of protoalgebraicity [2] and truth equationality [19]. Having studied both protoalgebraicity and weak algebraizability in the context of logicoids, we, now, look briefly at truth equationality. We introduce a definition adapted from [19], we show that it transfers and then prove the main result that weak algebraizability is indeed the conjunction of protoalgebraicity and truth equationality.

Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be an algebraic logicoid. $\mathbb{L}$ is **truth equational** if the Leibniz operator $\Omega_{\hat{\mathbf{B}}}$ is **completely order reflecting on** $\mathcal{C}^\flat$, i.e., if for all $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$,

$$\bigcap_{i\in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X) \quad \text{implies} \quad \bigwedge\nolimits^\flat_{i\in I} X_i \leq^\flat X.$$

Lemma 199 *Let* $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ *be a base logicoid.* $\mathbb{L}$ *is truth equational if and only if, for every interpretation* $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$, *with* $\hat{\mathbf{A}} = \langle \mathbf{A}, \leq \rangle$, *the Leibniz operator on* $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ *is completely order reflecting, i.e., for all* $\{Y_i : i \in I\} \cup \{Y\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$,

$$\bigcap_{i\in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) \quad \textit{implies} \quad \bigwedge_{i\in I} Y_i \leq Y.$$

Proof: The right to left implication is again obtained by applying the hypothesis to the interpretation $\langle \hat{\mathbf{B}}, i_{\hat{\mathbf{B}}} \rangle$. For the left to right implication, let $\mathcal{A} = \langle \hat{\mathbf{A}}, h \rangle$ be an interpretation and $\{Y_i : i \in I\} \cup \{Y\} \subseteq \mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$. Then we have

$$\begin{array}{rcl}
\bigcap_{i\in I} \Omega_{\mathcal{A}}(Y_i) \subseteq \Omega_{\mathcal{A}}(Y) & \text{iff} & h^{-1}(\bigcap_{i\in I} \Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\
& \text{iff} & \bigcap_{i\in I} h^{-1}(\Omega_{\mathcal{A}}(Y_i)) \subseteq h^{-1}(\Omega_{\mathcal{A}}(Y)) \\
& \text{iff} & \bigcap_{i\in I} \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y_i)) \subseteq \Omega_{\hat{\mathbf{B}}}(h^{-1}(Y)) \\
& \text{implies} & \bigwedge^\flat_{i\in I} h^{-1}(Y_i) \leq^\flat h^{-1}(Y) \\
& & \quad \text{(Truth Equationality)} \\
& \text{iff} & h^{-1}(\bigwedge_{i\in I} Y_i) \subseteq h^{-1}(Y) \\
& & \quad (h \text{ a grid morphism}) \\
& \text{iff} & \bigwedge_{i\in I} Y_i \subseteq Y.
\end{array}$$

Hence, the Leibniz operator on $\mathrm{Fi}_{\mathbb{L}}(\mathcal{A})$ is completely order reflecting. ■

Finally, we prove the equivalence of weak algebraizability with protoalgebraicity and truth equationality for logicoids.

Theorem 200 *Let $\mathbb{L} = \langle \hat{\mathbf{B}}, C^\flat \rangle$ be a base logicoid. Then $\mathbb{L}$ is weakly algebraizable if and only if it is protoalgebraic and truth equational.*

Proof: Suppose $\mathbb{L}$ is weakly algebraizable. Since, by definition $\Omega_{\hat{\mathbf{B}}}$ is monotone, $\mathbb{L}$ is certainly protoalgebraic. To show that it is also truth equational, consider $\{X_i : i \in I\} \cup \{X\} \subseteq \mathcal{C}^\flat$, such that $\bigcap_{i\in I} \Omega_{\hat{\mathbf{B}}}(X_i) \subseteq \Omega_{\hat{\mathbf{B}}}(X)$. Then

$$\begin{array}{rcl}\Omega_{\hat{\mathbf{B}}}(\bigwedge^\flat_{i\in I} X_i) & \subseteq & \bigcap_{i\in I} \Omega_{\hat{\mathbf{B}}}(X_i) \quad \text{(Corollary 185)}\\ & \subseteq & \Omega_{\hat{\mathbf{B}}}(X). \quad \text{(Hypothesis)}\end{array}$$

As $\Omega_{\hat{\mathbf{B}}}$ is order reflecting, $\bigwedge^\flat_{i\in I} X_i \leq^\flat X$. This shows that $\Omega_{\hat{\mathbf{B}}}$ is completely order reflecting and, therefore, $\mathbb{L}$ is also truth equational.

Suppose, conversely, that $\mathbb{L}$ is protoalgebraic and truth equational. By protoalgebraicity, $\Omega_{\hat{\mathbf{B}}}$ is monotone. So it suffices to show that it is order reflecting. Let $X, Y \in \mathcal{C}^\flat$, such that $\Omega_{\hat{\mathbf{B}}}(X) \subseteq \Omega_{\hat{\mathbf{B}}}(Y)$. As $\Omega_{\hat{\mathbf{B}}}$ is, by hypothesis, completely order reflective, we get $X \leq^\flat Y$. Hence $\Omega_{\hat{\mathbf{B}}}$ is also order reflective, showing that $\mathbb{L}$ is weakly algebraizable. ∎

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