Algebraic Logic for Graded Truth

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Chapter 1

Introduction

In their pioneering "Memoirs monograph" [6], Blok and Pigozzi, based on Czelakowski's [13, 14] and their own previous work [5], made, for the first time, precise the notion of *algebraizable logic*. Many researchers followed in their footsteps and an explosion of research activity ensued, culminating in a beautiful and fascinating body of work that came to be known as *Abstract Algebraic Logic*. Under its roof it houses various directions of study, but its crown jewel, and, perhaps, its best known accomplishment, is the *Leibniz* or *algebraic hierarchy* of classes of logical systems, see, e.g., the surveys [15, 29, 28, 27]. The higher in the hierarchy a class is located, the more intimate the connection of the logics in that class with their algebraic counterparts.

Blok and Pigozzi study *sentential logics* or *deductive systems*. These are pairs $\mathcal{S} = \langle \mathcal{L}, \vdash \rangle$ consisting of a logical (or algebraic, depending on the point of view) language $\mathcal{L}$ and a structural consequence relation

$$\vdash \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$$

on the set $\mathrm{Fm}_{\mathcal{L}}(V)$ of $\mathcal{L}$-formulas constructed using variables in a countably infinite set V. A consequence relation is one that satisfies inflationarity, monotonicity and idempotency. Structurality means that the consequence is invariant under the application of substitutions. Since structural consequence relations turn out to be equivalent to structural closure operators, a logic may be equivalently presented as a pair $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where

$$C : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$$

is the structural closure operator associated with $\vdash$, given, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$C(\Gamma) = \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \Gamma \vdash \varphi\}.$$

In this formulation, the axioms governing the logic are:

(Inflationarity) $\Gamma \subseteq C(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$;

(Monotonicity) $\Gamma \subseteq \Delta$ implies $C(\Gamma) \subseteq C(\Delta)$, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$;

(Idempotency) $C(C(\Gamma)) = C(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$;

(Structurality) $\sigma(C(\Gamma)) \subseteq C(\sigma(\Gamma))$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Blok and Pigozzi [6] deal exclusively with *finitary* deductive systems, that is, those satisfying the additional axiom that, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

(Finitarity) $C(\Gamma) = \bigcup_{\Gamma_0 \subseteq_f \Gamma} C(\Gamma_0)$,

where $\subseteq_f$ denotes the finite subset relation. The majority of their results and almost all their methodology were later generalized and shown to hold for all sentential logics (see. e.g., [33, 34]).

Given two logics $\mathcal{S} = \langle \mathcal{L}, C\rangle$ and $\mathcal{S}' = \langle \mathcal{L}, C'\rangle$, we say that $\mathcal{S}$ is *weaker than* $\mathcal{S}'$ or that $\mathcal{S}'$ is *stronger than* $\mathcal{S}$ and write $\mathcal{S} \le \mathcal{S}'$ or $C \le C'$ to signify that $C(\Gamma) \subseteq C'(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$.

A *theory* T of a logic $\mathcal{S}$ is a closed set of formulas, that is, such that $C(T) = T$. The set of all theories of $\mathcal{S}$ is denoted $\mathrm{Th}(\mathcal{S})$. Ordered by the subset relation, it forms a complete lattice $\mathbf{Th}(\mathcal{S}) = \langle \mathrm{Th}(\mathcal{S}), \subseteq\rangle$. It turns out that a closure operator is completely specified by its set of theories and, hence, a third equivalent presentation of a logic is as a pair $\mathcal{S} = \langle \mathcal{L}, \mathrm{Th}(\mathcal{S})\rangle$.

To study the semantics of logics one uses logical matrices. A *logical matrix* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ consists of an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ and a set $F \subseteq A$ of designated elements, called the *filter* of the matrix. Given a class M of $\mathcal{L}$-matrices, M induces a logic $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}}\rangle$ by setting, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll}\varphi \in C_{\mathsf{M}}(\Gamma) & \text{iff} & \text{for all } \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A},\\ & & h(\Gamma) \subseteq F \text{ implies } h(\varphi) \in F.\end{array}$$

A matrix $\mathfrak{A}$ is a *matrix model* of $\mathcal{S}$ or an *$\mathcal{S}$-matrix* if

$$C \le C_{\mathfrak{A}} := C_{\{\mathfrak{A}\}}.$$

In this case F is called an *$\mathcal{S}$-filter*. It is well-known that the collection $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ of all $\mathcal{S}$-filters on $\mathbf{A}$, ordered by $\subseteq$, forms a complete lattice $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \subseteq\rangle$. Further, one can show that $\mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V)) = \mathrm{Th}(\mathcal{S})$. If, for a class M of matrices, we have $C = C_{\mathsf{M}}$, then M is called a *matrix semantics* for $\mathcal{S}$.

One of the goals of algebraization is to connect a deductive system with a class K of algebras. Such a connection is possible when a correspondence may be established between the lattice of theories of the deductive system and the lattice of equational theories, i.e., of congruences, associated with the equational logic induced by the class of algebras. This necessitates the study of equational consequences, which are consequences over the set $\mathrm{Eq}_{\mathcal{L}}(V) = \mathrm{Fm}^2_{\mathcal{L}}(V)$ of *$\mathcal{L}$-equations*. The *equational logic* $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ associated with a class K of $\mathcal{L}$-algebras consists of a language $\mathcal{L}$ and a closure operator

$$C_{\mathsf{K}} : \mathcal{P}(\mathrm{Eq}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}}(V)),$$

defined by setting, for all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$,

$$\begin{array}{lll}\varphi \approx \psi \in C_{\mathsf{K}}(E) & \text{iff} & \text{for all } \mathbf{A} \in \mathsf{K} \text{ and all } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A},\\ & & h(\varepsilon_1) = h(\varepsilon_2), \text{ for all } \varepsilon_1 \approx \varepsilon_2 \in E,\\ & & \qquad \text{implies } h(\varphi) = h(\psi).\end{array}$$

A *theory* θ of $\mathcal{S}_{\mathsf{K}}$ is a set $\theta \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$, such that $C_{\mathsf{K}}(\theta) = \theta$. Theories coincide with K-*congruences*, i.e., congruences on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\mathbf{Fm}_{\mathcal{L}}(V)/\theta \in \mathsf{K}$.

In Abstract Algebraic Logic the association of a deductive system with a class of algebras that serves as its algebraic semantics is done via the Leibniz operator, a tool as important as the hierarchy itself, since it constitutes its building foundation. Given an algebra $\mathbf{A}$ and a subset $F \subseteq A$, i.e., a matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$, a congruence θ on $\mathbf{A}$ is *compatible with* F, or is a *congruence of* $\mathfrak{A}$, if, for all $a, b \in A$,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in F \quad \text{imply} \quad b \in F.$$

This is equivalent to saying that F is a union of θ-congruence classes. We denote the collection of all such congruences by $\mathrm{Con}(\mathfrak{A})$. It turns out that it forms a principal ideal of the lattice of all congruences on $\mathbf{A}$, ordered by $\subseteq$, denoted $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}), \subseteq \rangle$. Its generator, i.e., the largest congruence on $\mathbf{A}$ compatible with F, is called the *Leibniz congruence* of F on $\mathbf{A}$, or the *Leibniz congruence* of $\mathfrak{A}$ and is denoted by $\Omega_{\mathbf{A}}(F)$ or $\Omega(\mathfrak{A})$. The mapping $F \mapsto \Omega_{\mathbf{A}}(F)$ on $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is termed the *Leibniz operator* of $\mathcal{S}$ on $\mathbf{A}$ (Definition 1.4 and Theorem 1.5 of [6]). Blok and Pigozzi showed that $\Omega_{\mathbf{A}}(F)$ admits the following characterization:

$$\begin{aligned} \Omega_{\mathbf{A}}(F) = \{ \langle a, b \rangle \in A^2 : & \text{ for all } \varphi \in \mathrm{Fm}_{\mathcal{L}}(V) \text{ and all } \overline{c} \in A, \\ & \varphi^{\mathbf{A}}(a, \overline{c}) \in F \text{ iff } \varphi^{\mathbf{A}}(b, \overline{c}) \in F \}. \end{aligned}$$

Close to the bottom of the algebraic hierarchy (see, e.g., [15, 29, 27]) lies the class of *protoalgebraic logics* [5]. They are characterized by the fact that any two sentences which are equivalent modulo the Leibniz congruence of a theory are also interderivable modulo that same theory. This idea is roughly expressed by the motto "indistinguishability implies interderivability". According to Blok and Pigozzi, this class is the widest class of logics on which the powerful methods of algebra can be brought to bear in the study of their properties. Protoalgebraic logics may alternatively be characterized as the ones on whose lattice of theories the Leibniz operator is monotone or, equivalently, meet continuous.

To define algebraizability, Blok and Pigozzi introduced translations and interpretations between formulas and equations (see Definitions 2.2 and 2.8 of [6]). From a slightly more contemporary point of view, a *translation* from formulas to equations is a set

$$\delta(x) \approx \varepsilon(x) = \{ \delta_i(x) \approx \varepsilon_i(x) : i \in I \}$$

of equations in a single variable. Dually a *translation* from equations to formulas is a set

$$\Delta(x, y) = \{ \Delta_j(x, y) : j \in J \}$$

of formulas in two variables. Blok and Pigozzi considered finitary deductive systems only and so took translations to be finite, i.e., both sets of indices I

and J were taken to be finite. Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a logic and K be a class of $\mathcal{L}$-algebras. A translation $\delta \approx \varepsilon$ from formulas to equations is an *interpretation* from $\mathcal{S}$ to $\mathcal{S}_\mathsf{K}$, written $\delta \approx \varepsilon : \mathcal{S} \to \mathcal{S}_\mathsf{K}$, if, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}(V)$,

$$\varphi \in C(\Gamma) \quad \text{iff} \quad \delta(\varphi) \approx \varepsilon(\varphi) \subseteq C_\mathsf{K}(\delta(\Gamma) \approx \varepsilon(\Gamma)),$$

where we adopt the relatively intuitive notation

$$\delta(\varphi) \approx \varepsilon(\varphi) \coloneqq \{\delta_i(\varphi) \approx \varepsilon_i(\varphi) : i \in I\}$$

and

$$\delta(\Gamma) \approx \varepsilon(\Gamma) \coloneqq \bigcup_{\gamma \in \Gamma} \delta(\gamma) \approx \varepsilon(\gamma).$$

Dually, a translation Δ from equations to formulas is an *interpretation* from $\mathcal{S}_\mathsf{K}$ to $\mathcal{S}$, written $\Delta : \mathcal{S}_\mathsf{K} \to \mathcal{S}$, if, for all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_\mathcal{L}(V)$,

$$\varphi \approx \psi \in C_\mathsf{K}(E) \quad \text{iff} \quad \Delta(\varphi, \psi) \subseteq C(\Delta(E)),$$

where similar conventions as before are assumed regarding the notation.

Given a deductive system $\mathcal{S} = \langle \mathcal{L}, C \rangle$, a class of $\mathcal{L}$-algebras K is called an *algebraic semantics* for $\mathcal{S}$ (Definition 2.2 of [6]) if there exists an interpretation $\delta \approx \varepsilon : \mathcal{S} \to \mathcal{S}_\mathsf{K}$. In this case $\delta \approx \varepsilon$ are called the (system of) *defining equations*. Due to finitarity, it turns out that, if K is an algebraic semantics for $\mathcal{S}$, so is the quasivariety $\mathbb{Q}(\mathsf{K})$ generated by the class K. There is a close relationship between algebraic semantics and matrix semantics. Given a translation $\delta \approx \varepsilon$ from formulas to equations and an algebra $\mathbf{A}$, one may define a filter on $\mathbf{A}$ by setting

$$F_\mathbf{A}^{\delta \approx \varepsilon} = \left\{ a \in A : \delta^\mathbf{A}(a) = \varepsilon^\mathbf{A}(a) \right\}.$$

Moreover, given a class K of $\mathcal{L}$-algebras, one may construct an associated class of matrices

$$\mathsf{M} = \left\{ \langle \mathbf{A}, F_\mathbf{A}^{\delta \approx \varepsilon} \rangle : \mathbf{A} \in \mathsf{K} \right\}.$$

In Theorem 2.4 of [6], Blok and Pigozzi prove that a class K is an algebraic semantics for $\mathcal{S}$ if and only if M is a matrix semantics for $\mathcal{S}$.

The usefulness of possessing an algebraic semantics rests with the fact that the deductive system apparatus of the logic is reflected via the defining equations into the algebraic deductive apparatus induced by the class of algebras. This forces the logic to exhibit some of the characteristics of equational consequences. E.g., in Theorem 2.7 of [6], it is shown that, if $\mathcal{S}$ has an algebraic semantics with defining equations $\delta \approx \varepsilon$, then $\mathcal{S}$ must satisfy the deduction

$$\varepsilon_i(x) \in C(x, \delta_i(x)), \quad i \in I.$$

This is a consequence of the fact that, regardless of the class K, one has equationally

$$\delta(\varepsilon_i(x)) \approx \varepsilon(\varepsilon_i(x)) \subseteq C_\mathsf{K}(\delta(x) \approx \varepsilon(x), \delta(\delta_i(x)) \approx \varepsilon(\delta_i(x))).$$

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a logic and K an algebraic semantics with defining equations $\delta \approx \varepsilon$. K is said to be an *equivalent algebraic semantics* for $\mathcal{S}$ and $\mathcal{S}$ is then called *algebraizable* if there exists also a translation $\Delta : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$, such that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$C_{\mathsf{K}}(\delta(\Delta(\varphi,\psi)) \approx \varepsilon(\Delta(\varphi,\psi))) = C_{\mathsf{K}}(\varphi \approx \psi).$$

In Corollary 2.9 of [6], it is shown that the roles of $\delta \approx \varepsilon$ and Δ are completely symmetric, in the sense that K is an equivalent algebraic semantics of $\mathcal{S}$ if and only if $\Delta : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ is an interpretation and, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$C(\Delta(\delta(\varphi), \varepsilon(\varphi))) = C(\varphi).$$

If any of these equivalent pairs of conditions hold (and, therefore, all four), then $\delta \approx \varepsilon$ and Δ are said to be *inverses* of one another. Blok and Pigozzi proceed to show (Theorem 2.15 of [6]) that, if a deductive system is algebraizable, it is algebraizable in an essentially unique way, in the sense that all translations used (in either direction) are interderivable and the equational consequences generated by the classes of algebras used are identical.

Besides a plenitude of examples to support the usefulness of the definition and the accompanying theory (see Chapter 5 of [6]), Blok and Pigozzi provide alternative characterizations of algebraizability, which further attest to the naturalness of the notion. The first characterization assumes that a class of algebras K is given and it is to be tested for whether it forms an equivalent algebraic semantics of the logic $\mathcal{S}$ at hand. Because K is external information (to the logic) this characterization is a, so called, *extrinsic* characterization. In Theorem 3.7 of [6], it is shown that K is the equivalent algebraic semantics for $\mathcal{S}$ iff there is an isomorphism between the theory lattice of $\mathcal{S}$ and the equational theory lattice of K that commutes with the substitution operators. The other two characterizations they provide are *intrinsic*, in the sense that no data external to the given logic are involved. In Theorem 4.2 of [6] Blok and Pigozzi show that a deductive system $\mathcal{S}$ is algebraizable iff the Leibniz operator is injective and order-preserving on the lattice $\mathbf{Th}(\mathcal{S})$ of theories and (due to finitarity) preserves unions of directed subsets of $\mathrm{Th}(\mathcal{S})$. The second intrinsic characterization is of a more syntactic nature. It asserts (Theorem 4.7 of [6]) that $\mathcal{S}$ is algebraizable if and only if there exist a system Δ of formulas in two variables and a system $\delta \approx \varepsilon$ of equations in a single variable, such that, for all $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$, every n-ary λ in $\mathcal{L}$ and all $\varphi_1, \psi_1, \dots, \varphi_n, \psi_n \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(i) $\Delta(\varphi, \varphi) \subseteq C(\varnothing)$;

(ii) $\Delta(\psi, \varphi) \subseteq C(\Delta(\varphi, \psi))$;

(iii) $\Delta(\varphi, \chi) \subseteq C(\Delta(\varphi, \psi), \Delta(\psi, \chi))$;

(iv) $\Delta(\lambda(\varphi_1,\dots,\varphi_n),\lambda(\psi_1,\dots,\psi_n)) \subseteq C(\Delta(\varphi_1,\psi_1),\dots,\Delta(\varphi_n,\psi_n))$;

(v) $C(\varphi) = C(\Delta(\delta(\varphi),\varepsilon(\varphi)))$.

As we saw, Blok and Pigozzi [6] made for the first time precise the notion of algebraizable logic. Roughly speaking, a logic is algebraizable if its consequence relation is interpretable, in an invertible way, in the equational consequence of a class of algebras of the same type as that of the logic. They characterized algebraizability by showing that a logic is algebraizable if and only if there is an isomorphism from the complete lattice of the theories of the logic onto the complete lattice of the equational theories associated with the algebraizing class of algebras which commutes with substitutions (see Theorem 3.7 of [6]). This characterization is extrinsic in the sense that, apart from the logic, one needs the class of algebras to identify the lattice of equational theories in order to be able to apply it. In what was, perhaps, the most important result in [6], which inspired much of subsequent work, Blok and Pigozzi gave an intrinsic characterization of algebraizability. They showed (Theorem 4.2 of [6]) that a logic is algebraizable if and only if the Leibniz operator, which can be computed only with knowledge of the theories of the logic, is order preserving, injective and commutes with unions of directed sets of theories. Subsequent work focused on introducing and investigating new classes of logics, mostly weaker than the algebraizable ones, that can also be characterized using properties of the Leibniz operator on the lattice of theories. The totality of those classes constitute the algebraic or Leibniz hierarchy of logics (see, e.g., [15, 29, 27]), a cornerstone of Algebraic Logic and one of the most beautiful parts of the theory.

Among the most important contributors to the field have been the members of the Barcelona Group of Algebraic Logic. Their work over many decades has been summarized and explained in a unified, coherent and elegant way in the seminal monograph of Font and Jansana [28]. Our focus in Chapter 3 is on one of their main and most beautiful abstract results, the Isomorphism Theorem 2.30 of [28]. Of course to be able to adapt their theory to our purposes and reformulate and prove a version of the Isomorphism Theorem in the context of graded logics, we have to develop the necessary machinery.

Their starting point is a version of the aforementioned characterization of Blok and Pigozzi asserting that a logic $\mathcal{S}$ is algebraizable if and only if, for every algebra $\mathbf{A}$, the Leibniz operator on $\mathbf{A}$ is an isomorphism between the lattice of $\mathcal{S}$-filters on $\mathbf{A}$ and the lattice of K-congruences on $\mathbf{A}$, where K is the class of algebras serving as the algebraic counterpart of the logic (see Theorem 5.1 of [6]). Font and Jansana recognize, based on previous experience acquired by the Barcelona Group, that a plethora of concrete logics of interest are not algebraizable in the sense of Blok and Pigozzi. Thus, they embark on a quest to generalize the characterization of algebraizable logics in order to obtain an isomorphism theorem between logical objects (such as theories) and congruences that would be valid for arbitrary logics. To achieve this goal, and in quite an ingenious move, they change the types of objects used as models of the logical systems.

In contrast to the theory of Blok and Pigozzi, where the focus is on logical matrices as models of the sentential logics under study, Font and Jansana consider abstract logics or generalized matrices. Whereas logical matrices consist of a single filter on the underlying algebra, abstract logics consist of collections of filters forming a closed set system. This change in focus requires also passing from the Leibniz operator to the Tarski operator. The Tarski operator associates to an abstract logic the largest congruence on its underlying algebra that is compatible with all filters of the abstract logic. As such, it is obtained as the intersection of the Leibniz congruences of the matrices obtained by the abstract logic by considering each of its filters individually. Font and Jansana prove that, for any logic $\mathcal{S}$ (not necessarily algebraizable) and any algebra $\mathbf{A}$, the Tarski operator on $\mathbf{A}$ is an isomorphism from the lattice of full models of $\mathbf{A}$ onto the lattice of $\mathrm{Alg}(\mathcal{S})$-congruences on $\mathbf{A}$ (Theorem 2.30 of [28]).

We look, next, at the influence of the theory to model theoretic investigations. In classical Model Theory (see, e.g., [10, 35]) one establishes results that characterize via operations on classes of models, their definability via syntactic means. Among such results are, e.g., those characterizing elementary classes, universal classes, universal Horn classes and universal atomic classes of structures. In addition, these results encompass some of the signature results in Universal Algebra, such as Birkhoff's result [3] characterizing varieties and Mal'vev's result [36] characterizing (generalized) quasivarieties of algebras. All these results, in their classical form, assume the presence of an equality predicate in the language. On the other hand, the model theory of logical matrices, interpreting sentential logics in the abstract theory of Algebraic Logic [15] is encompassed by the equality free fragment of first order logic with a single unary predicate [8] (see Section 1.3 of [6]). So the accumulation of results in this field during the '80s, with the pioneering work of Blok and Pigozzi, Czelakowski and Font and Jansana, among others, led to a corresponding increase in interest in the equality free aspects of model theory of first order languages. Here the Leibniz congruence of a structure plays the role of equality, since it represents indiscernibility, which is akin to equality in logics defined without equality. As a result, it was only natural that, under the guidance of Pigozzi and Font and Jansana, Elgueta [22] and Dellunde [17] in Barcelona developed the machinery needed for formulating analogs of some of the most important characterization results in Model Theory concerning definability of classes of structures in the context of equality free first order logic.

In Section 1 of [23], Elgueta revisits classical constructions in Model Theory in the context of languages without equality. He introduces substructures, filter extensions and discusses elementarity. He delves into homomorphisms of structures and related constructions of image and preimage structures and discusses some special types of homomorphisms. Further, he defines products, filtered products (and ultraproducts) and subdirect products. Section 2 is dedicated to the introduction of congruences of structures and the definition of Leibniz equality, or indiscernibility relation, in this context, which "stands in" for equality in the absence of an equality predicate. Section 3 recalls quotient structures, formulates an analog of the Homomorphism Theorem and, in addition, asserts the validity of analogs of the remaining homomorphism theorems, including the Correspondence Theorem.

Section 4 introduces operators on classes of structures defined without equality. These form the cornerstone of the characterizations of definability of those classes by various syntactic means. Very briefly, and, mainly, in order to have the relevant notation at hand, the list of operators includes the operator S of taking isomorphic copies of substructures, the operator S_e of taking isomorphic copies of elementary substructures, the operator F of taking isomorphic copies of filter extensions, the operator R of taking isomorphic copies of reductions (images under strict surjective homomorphisms), the operator E of taking isomorphic copies of extensions (preimages under strict surjective homomorphisms) and, finally, the operators P, P_f, P_u of taking isomorphic copies of direct products, filtered products and ultraproducts and P_{sd} of taking isomorphic copies of subdirect products. Elgueta studies in a systematic way, reminiscent of the corresponding methodology of Universal Algebra, the properties that hold when these operators are composed. Of equal importance are an analog devised in this context of the Diagram Lemma of Model Theory and the Reduction Operator Lemma. The latter asserts that, given an operator $\mathcal{O} \in \{S, P, P_f, P_u, P_{sd}\}$, $L\mathcal{O} = L\mathcal{O}L$, where L is the class operator of taking Leibniz reductions of structures in a given class. Applying L after an operator $\mathcal{O}$, that is, the composed operator $L\mathcal{O}$ is denoted by $\mathcal{O}^*$. Thus, the assertion above is equivalent to $\mathcal{O}^* = \mathcal{O}^*L$. The second part of the Reduction Operator Lemma asserts that $LS_e = LS_eL = S_eL$.

Section 5 is the main section of [23] and contains the main characterization theorems. In Subsection 5.1, it is shown that K is an elementary class of structures defined without equality if and only if it is closed under E, R, S_e and $\bar{P}_u$, where $\bar{P}_u$ is P_u applied only on nonempty collections of structures. In a companion result for reductions, a class K of reduced structures is a reduced elementary class if and only if it is closed under S_e and $\bar{P}_u^*$. In Subsection 5.2, it is shown that K is a universal class of structures defined without equality if and only if it is closed under E, R, S and $\bar{P}_u$. As for reduced structures, a class K of reduced structures is a reduced universal class if and only if it is closed under S^* and $\bar{P}_u^*$. Subsection 5.3 turns to universal Horn classes of structures defined without equality. It is shown that K is such a class if and only if it is closed under E, R, S and P_f. As for reduced structures, a class K of reduced structures is a reduced universal Horn class if and only if it is closed under S^* and P_f^*. Finally Subsection 5.4 deals with universal atomic classes of structures defined without equality. It is shown that K is a universal atomic class if and only if it is closed under H, E, S and P. Moreover, a class K of reduced structures is a reduced universal atomic class if and only if it is closed under F^*, E, S and P.

In a different direction, and motivated mainly by applied considerations, many scientists have advocated the use of multi-valued logics (and sets) to model various phenomena. The study of such logics (and sets) has given rise to a huge body of work. Let us mention, here, the pioneering works of Zadeh [41], Goguen [32] and Pavelka [39]. Let us, also, point to some, among many available, reference works on the topic which contain material which is closer to the level of abstraction that we are aiming for in this study, e.g., [11], [1], [12]. In these references the reader can find further pointers to the extensive literature available on the topic. Of the last three, Cintula and Noguera's work [12] is very closely related to the material studied in Chapters 2 and 3 (on the logical side), whereas Bělohlávek and Vychodil's work [1] is the one that covers much of the material on the algebraic and the equational logical foundations that is used throughout this work.

In Chapter 2, we present an abstract theory for the algebraization of multi-valued logics (G-logics, as we call them) analogous to the theory of Blok and Pigozzi [6]. For a more detailed summary of the contents by section, see Section 2.1.

In Chapter 3, we present a theory of general algebraic semantics for G-logics analogous to the theory of Font and Jansana [28]. For a more detailed summary of the contents by section, see Section 3.1.

The third part of the work, Chapter 4, develops some model theory by generalizing the structures that are developed in Chapter 3. We follow here the work of Elgueta [23] (see, also, [22] and the work of Dellunde [17] and of Dellunde and Jansana [20]) in developing (up to a point) a model theory of equality free structures for structures in which sentences take truth values in a complete Boolean algebra, rather than being either true or false. For a more detailed summary of the contents by section, see Section 4.1.

Chapter 2

Algebraizable Graded Logics

2.1 Introduction

Our goal in this chapter is to generalize the framework of Blok and Pigozzi outlined in Chapter 1, to the extent possible, to cover consequences that deal with sentences whose truth values are drawn from a complete lattice. On some contexts, the lattice may be assumed to have additional properties, e.g., being completely distributive or a Boolean algebra. In any case, all results are abstractions of those dealing with the classical framework, where a formula assumes two possible truth values.

We outline the contents of the chapter by section.

In Section 2.2, we introduce the central structure of *G-logic*. Roughly speaking, it is a structural closure operator on *G-sets*, which are mappings of the set of formulas $\mathrm{Fm}_{\mathcal{L}}(V)$ into a fixed complete lattice $\mathbf{G} = \langle G, \leq\rangle$ to be thought of as the lattice of truth values. Under a natural definition of an ordering $\leq$ on G-logics, induced in a "pointwise" manner by the $\leq$ ordering of $\mathbf{G}$, we show that the class $\mathrm{Log}_{\mathbf{G}}(\mathcal{L})$ of G-logics forms a complete lattice.

In Section 2.3, we look at *G-theories*. These are G-sets that are closed sets of the closure operator of the G-logic. They form a complete lattice under $\leq$. We also define a notion of *finitarity* for G-logics. We finally establish some properties of G-theories. We show that, due to structurality, for every G-theory T and all substitutions σ, $T\circ\sigma$ is also a G-theory. Moreover, meets and joins of G-theories commute with substitutions, in the sense that, for every collection $\{T_i : i \in I\}$ of G-theories and all substitutions σ, $\bigwedge_{i\in I}(T_i \circ \sigma) = (\bigwedge_{i\in I} T_i) \circ \sigma$ and similarly for joins.

In Section 2.4, we look at *G-matrices*. These are pairs $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is a G-set on A, i.e., $F : A \to G$. Any given class of G-matrices M induces a G-logic $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}}\rangle$ on $\mathrm{Fm}_{\mathcal{L}}(V)$. We say that $\mathfrak{A}$ is a *matrix of $\mathcal{S}$* or an *$\mathcal{S}$-matrix* is $\mathcal{S} \leq \mathcal{S}_{\mathsf{M}}$, that is, if $C \leq C_{\mathsf{M}}$. In this case F is called an *$\mathcal{S}$-filter*. The collection of all $\mathcal{S}$-filters on an algebra $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Ordered by $\leq$ it forms a complete lattice. Furthermore, $\mathcal{S}$-filters on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ coincide with $\mathcal{S}$-theories. Finally, a class M of G-matrices constitutes a *G-matrix semantics* for a G-logic $\mathcal{S}$ if $\mathcal{S} = \mathcal{S}_{\mathsf{M}}$ and, in that case, M is said to be *strongly adequate for $\mathcal{S}$*.

In Section 2.5, we define *G-congruences*. These are G-sets of equations on $\mathbf{A}$, that is, mappings $\Theta : A^2 \to G$, that satisfy reflexivity, symmetry, transitivity and congruence. Reflexivity means that $\Theta(a, a) = \top$, for all $a \in A$, symmetry means that $\Theta(a, b) = \Theta(b, a)$, for all $a, b \in A$, transitivity that $\Theta(a, b) \wedge \Theta(b, c) \leq \Theta(a, c)$, for all $a, b, c \in A$ and, similarly, congruence means that, for all n-ary $\lambda \in \mathcal{L}$ and all $\bar{a}, \bar{b} \in A^n$, $\bigwedge_{i=1}^n \Theta(a_i, b_i) \leq \Theta(\lambda^{\mathbf{A}}(\bar{a}), \lambda^{\mathbf{A}}(\bar{b}))$. We show that G-congruences, ordered by $\leq$, form a complete lattice. We also introduce *stratified congruences*. These are families $\theta = \{\theta_g : g \in G\}$ indexed by the elements of the complete lattice $\mathbf{G} = \langle G, \leq\rangle$ that, in addition, satisfy $\theta_{g_2} \subseteq \theta_{g_1}$, for all $g_1, g_2 \in G$, such that $g_1 \leq g_2$. We show that, under certain conditions, the two mappings

$$\begin{array}{ll}\Theta \mapsto \hat{\Theta}; & \hat{\Theta}_g = \{\langle a,b\rangle : \Theta(a,b) \geq g\},\ g \in G,\\ \theta \mapsto \check{\theta}; & \check{\theta}(a,b) = \bigvee\{g : \langle a,b\rangle \in \theta_g\},\ a,b \in A,\end{array}$$

are inverse mappings from G-congruences to stratified congruences on $\mathbf{A}$.

Section 2.6 discusses *compatibility* of G-congruences with G-filters and the existence of the *Leibniz G-congruence*. We say that a G-congruence Θ is *compatible with* a G-filter F, or that Θ is a *G-congruence of the G-matrix* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, if, for all $a,b \in A$, $\Theta(a,b) \wedge F(a) \leq F(b)$. In case $\mathbf{G}$ has an implication $\rightarrow$, such that, for all $g,g',g'' \in G$, $g \wedge g' \leq g''$ if and only if $g \leq g' \rightarrow g''$, then the condition above becomes equivalent to $\Theta(a,b) \leq F(a) \leftrightarrow F(b)$, where $\leftrightarrow$ is the biconditional corresponding to $\rightarrow$. We denote by $\mathrm{Gon}(\mathfrak{A})$ the collection of all G-congruences on $\mathbf{A}$ that are compatible with F. In some special cases, it can be shown that, regardless of $\mathfrak{A}$, $\mathrm{Gon}(\mathfrak{A})$ is a principal ideal of the complete lattice $\mathbf{Gon(A)}$ of all G-congruences on $\mathbf{A}$. For us, this is a firm desideratum. So we call a complete lattice $\mathbf{G}$ *Leibniz permitting* if it is such that, for all $\mathfrak{A}$, $\mathrm{Gon}(\mathfrak{A})$ is a principal ideal in $\mathbf{Gon(A)}$ and we restrict attention, throughout our work, to such complete lattices. The generator of this principal ideal, i.e., the largest G-congruence on $\mathbf{A}$ that is compatible with the G-filter F, is called the *Leibniz G-congruence of* $\mathfrak{A}$, or the *Leibniz G-congruence of F on* $\mathbf{A}$. We provide a Blok-Pigozzi style characterization of Leibniz G-congruences. Namely, we show that, for all $a,b \in A$,

$$\Omega_{\mathbf{A}}(F)(a,b) = \bigwedge\{F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \in A\}.$$

The section concludes with a result of a rather technical nature. We say that a G-set Θ of equations is *definable* (*with parameters*) *in* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ if there exist a formula $\varphi(x,y,\bar{z})$ and parameters $\bar{c} \in A$, such that, for all $a,b \in A$, $\Theta(a,b) = F(\varphi^{\mathbf{A}}(a,b,\bar{c}))$. We show that, if Θ is a G-congruence on $\mathbf{A}$, definable in $\mathfrak{A}$, and compatible with F, then $\Theta = \Omega_{\mathbf{A}}(F)$.

Section 2.7 is parenthetical, briefly introducing and giving a characterization of *protoalgebraic G-logics*. This topic will be discussed more extensively in Chapter 3, together with other classes of G-logics in the algebraic hierarchy, when additional machinery will be at our disposal.

In Section 2.8, we introduce and study *G-2-logics*. These are analogs of G-logics, but act on G-sets of equations rather than on G-sets of formulas. They are needed, in the same way that 2-deductive systems are needed, to formalize a Blok-Pigozzi style theory of algebraizability of G-logics. A *G-2-logic* is a mapping $C : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \rightarrow G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ that satisfies appropriate versions of inflationarity, monotonicity, idempotency and structurality. Perhaps the most distinguishing and important feature of G-2-logics is that they can be used to express equational logics of specific classes of G-algebras, i.e., pairs $\mathcal{A} = \langle \mathbf{A}, \Theta\rangle$, where $\Theta : A^2 \rightarrow G$ is a G-congruence on $\mathbf{A}$. Each such class K gives rise to a G-2-logic $\mathcal{S}_{\mathsf{K}}$. The theories of this logic are G-congruences on $\mathbf{Fm}_{\mathcal{L}}(V)$ and, conversely, each G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta \rangle \in \mathsf{K}$ is an $\mathcal{S}_{\mathsf{K}}$-theory. The collection $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, ordered by $\le$, forms a complete lattice $\mathbf{Th}(\mathcal{S}_{\mathsf{K}}) = \langle \mathrm{Th}(\mathcal{S}_{\mathsf{K}}), \le \rangle$.

In Section 2.9, we define *translations* between G-logics and G-2-logics and vice-versa. These form analogs of the ordinary translations that form the building blocks of the theory of algebraizable logics of Blok and Pigozzi [6]. A *translation from* G-formulas *to* G-equations is a join preserving mapping $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. Dually, a *translation from* G-equations *to* G-formulas is a join preserving mapping $\mathcal{F}$ in the opposite direction. Consider, now, a G-logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$, a class K of $\mathcal{L}$-algebras and the G-2-logic $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}} \rangle$ associated with K. A translation $\mathcal{E}$ from G-formulas to G-equations is an *interpretation* $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ if, for all $\Gamma, \Phi \in G^{\mathrm{Fm}_{\mathcal{L}}(V)}$,

$$\Phi \le C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

Dually, an interpretation from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$ is a translation $\mathcal{F}$ from G-equations to G-formulas, such that, for all $\Theta, E \in G^{\mathrm{Eq}_{\mathcal{L}}(V)}$,

$$E \le C_{\mathsf{K}}(\Theta) \quad \text{iff} \quad \mathcal{F}(E) \le C(\mathcal{F}(\Theta)).$$

We provide characterizations of when a given translation $\mathcal{E}$ or $\mathcal{F}$ is an interpretation. E.g., $\mathcal{E}$ is an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ if and only if, for all $\Gamma \in G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(C(\Gamma)))$ and $C(\Gamma) = \bigwedge \{C(\Phi) : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(\Phi))\}$, and, dually for $\mathcal{F}$.

In Section 2.10, we give an alternative, but equivalent, representation of translations. A *hybrid translation* from formulas to equations is a mapping $E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$, such that, for all G-sets of formulas $\{\Gamma_i : i \in I\} \cup \{\Gamma\}$ and all formulas φ, $E(\bot, \varphi) = \bot$, $E(\Gamma, \varphi) = E(\Gamma^\varphi, \varphi)$ and $E(\bigvee_i \Gamma_i, \varphi) = \bigvee_i E(\Gamma_i, \varphi)$, where Γ^φ is the G-set of formulas that picks the value $\Gamma(\varphi)$ at φ and assigns $\bot$ to all other formulas. The definition of *hybrid translations* from equations to formulas is defined dually. It turns out that translations and hybrid translations are two faces of the same coin. Given a translation $\mathcal{E}$ from G-formulas to G-equations, we define the hybrid translation $\mathcal{E}^h$ by setting, for all G-sets of equations Γ and all formulas φ, $\mathcal{E}^h(\Gamma, \varphi) = \mathcal{E}(\Gamma^\varphi)$. Conversely, given a hybrid translation E, we define E^t by setting, for all G-sets Γ of formulas, $E^t(\Gamma) = \bigvee_\varphi E(\Gamma^\varphi, \varphi)$. Then $\mathcal{E}^h$ is a hybrid translation, E^t is a translation and, further, $\mathcal{E}^{ht} = \mathcal{E}$ and $E^{th} = E$. The same situation occurs, in a completely dual fashion, when translations and hybrid translations from equations to formulas are treated.

Section 2.11 continues the work started in Section 2.10. Here, it is shown that the equivalence between translations and hybrid translations can be restricted to *structural translations* and *structural hybrid translations*. To give the main idea, a translation $\mathcal{E}$ from G-formulas to G-equations is *structural* if, for all G-sets of formulas Γ, all formulas φ and all substitutions σ, $\mathcal{E}(\Gamma^{\sigma(\varphi)}) = \mathcal{E}(\Gamma^\varphi) \circ \sigma$. Similarly, a hybrid translation E from formulas to equations is *structural* if, for all G-sets of formulas Γ, all formulas φ and all substitutions σ, $E(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) = E(\Gamma^\varphi, \varphi) \circ \sigma$. We show that the mappings $\mathcal{E} \mapsto \mathcal{E}^h$ and $E \mapsto E^t$ of Section 2.10 are inverse mappings between structural translations and structural hybrid translations. The dual equivalence between translations from G-equations to G-formulas and hybrid translations from equations to formulas yields an equivalence between structural translations and structural hybrid translations as well.

In Section 2.12, we introduce the dual notions of *G-algebraic semantics* for a G-logic $\mathcal{S}$ and of *G-logical semantics* for a class K of G-algebras. We say that a class K of G-algebras is a *G-algebraic semantics* for a G-logic $\mathcal{S}$ if there exists an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_\mathsf{K}$. Dually, a G-logic $\mathcal{S}$ is a *G-logical semantics* for a class K of G-algebras if there exists an interpretation $\mathcal{F} : \mathcal{S}_\mathsf{K} \to \mathcal{S}$. Under certain conditions on the witnessing interpretations, the notions of G-algebraic semantics and of G-matrix semantics are related. The same happens, dually, with the notions of G-logical semantics and of G-2-matrix semantics, a G-matrix semantics applicable to G-2-logics. We say that a translation $\mathcal{E}$ from G-formulas to G-equations is *order reflecting* if, for all G-sets of formulas Γ, Γ',

$$\mathcal{E}(\Gamma) \le \mathcal{E}(\Gamma') \quad \text{implies} \quad \Gamma \le \Gamma'.$$

Dually for a translation $\mathcal{F}$ from G-equations to G-formulas. Furthermore, we say that $\mathcal{E}$ is *reflectively structural* if, for every algebra $\mathbf{A}$ and all G-congruences Θ on $\mathbf{A}$, there exists a G-filter F on $\mathbf{A}$, such that, for every $h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A}$,

$$\mathcal{E}(F \circ h) = \Theta \circ h^2.$$

Dually for a translation $\mathcal{F}$ from G-equations to G-formulas. Reflective structurality of $\mathcal{E}$ allows one to construct, given a class K of G-algebras, a corresponding class $\mathsf{K}^\mathcal{E}$ of G-matrices. Dually, reflective structurality of $\mathcal{F}$ allows one to construct, given a class M of G-matrices, a corresponding class $\mathsf{M}^\mathcal{F}$ of G-algebras. We show that, if $\mathcal{E}$ is order reflecting and reflectively structural, then K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$ if and only if $\mathsf{K}^\mathcal{E}$ is a G-matrix semantics for $\mathcal{S}$ and, dually, if $\mathcal{F}$ is order reflecting and reflectively structural, then, for a class M of G-matrices, $\mathcal{S}_\mathsf{M}$ is a G-logical semantics for K via $\mathcal{F}$ if and only if $\mathsf{M}^\mathcal{F}$ is a G-2-matrix semantics for $\mathcal{S}_\mathsf{K}$.

In Section 2.13, starting from interpretations, we define the concept of an *equivalent G-algebraic semantics* for a G-logic $\mathcal{S}$. A class K of G-algebras is an *equivalent G-algebraic semantics* for $\mathcal{S} = \langle \mathcal{L}, C \rangle$ if there exist two translations $\mathcal{E}$ from G-formulas to G-equations and $\mathcal{F}$ from G-equations to G-formulas, such that, for all G-sets of formulas Γ, Φ and all G-sets of equations Θ,

(i) $\Phi \le C(\Gamma)$ iff $\mathcal{E}(\Phi) \le C_\mathsf{K}(\mathcal{E}(\Gamma))$, i.e., $\mathcal{E}$ is an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_\mathsf{K}$;

(ii) $C_\mathsf{K}(\Theta) = C_\mathsf{K}(\mathcal{E}(\mathcal{F}(\Theta)))$.

As in the theory of Blok and Pigozzi (see Corollary 2.9 of [6]), it turns out that Conditions (i) and (ii) together are equivalent to their dual statements, namely, that, for all G-sets of equations E, Θ and for all G-sets of formulas Γ,

(iii) $E \leq C_{\mathsf{K}}(\Theta)$ iff $\mathcal{F}(E) \leq C(\mathcal{F}(\Theta))$, i.e., $\mathcal{F}$ is an interpretation $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$;

(iv) $C(\Gamma) = C(\mathcal{F}(\mathcal{E}(\Gamma)))$.

Thus, the roles played by the interpretations $\mathcal{E}$ and $\mathcal{F}$ are completely symmetric. We say that $\mathcal{E}$ and $\mathcal{F}$ are *inverse interpretations* when Conditions (i)-(iv) hold. In the context of G-logics, we are not able to replicate the Uniqueness Theorem 2.15 of [6]. However, we are able to show uniqueness in case the interpretations are of a special type, which allows us to emulate very closely the framework of sentential logics. We describe this partial result briefly. If we assume that $\mathcal{S}$ has two equivalent G-algebraic semantics K via $\mathcal{E}, \mathcal{F}$ and K' via $\mathcal{E}', \mathcal{F}'$ and that, for every G-set of equations Θ, it holds that $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$, then we can show that $\mathcal{S}_{\mathsf{K}} = \mathcal{S}_{\mathsf{K}'}$ and that, for every G-set of formulas Γ, $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma))$. But the hypothesis of this implication does not seem to be valid in the context of G-logics. We call a translation $\mathcal{E}$ from G-formulas to G-equations *standard* if, roughly speaking, it is induced by a set $\delta \approx \varepsilon$ of equations in a single variable. Dually, we call $\mathcal{F}$ *standard* if it is induced by a set Δ of formulas in two variables. We can now show that if K via standard $\mathcal{E}, \mathcal{F}$ and K' via standard $\mathcal{E}', \mathcal{F}'$ are two equivalent G-algebraic semantics for $\mathcal{S}$, then the equality $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$ is guaranteed and, thus, we recover the Uniqueness Theorem of an equational G-algebraic semantics and of the associated interpretations up to G-consequence.

Section 2.14 is the main (and longest) section of Chapter 2. It corresponds to Chapter 3 of [6], culminating in a Characterization Theorem, Theorem 42, of algebraizability of G-logics in terms of isomorphisms between lattices of theories, satisfying additional conditions. It forms an analog of the well known Theorem 3.7 of Blok and Pigozzi [6]. The section starts with given a G-logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ which has a G-algebraic semantics K via an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$. Based on these data, two mappings $H_{\mathsf{K}} : \mathrm{Th}(\mathcal{S}_{\mathsf{K}}) \to \mathrm{Th}(\mathcal{S})$ and $\Omega_{\mathsf{K}} : \mathrm{Th}(\mathcal{S}) \to \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ are defined. More concretely, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $T \in \mathrm{Th}(\mathcal{S})$, we set

$$\begin{array}{rcl} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma \in G^{\mathrm{Fm}_{\mathcal{L}}(V)} : \mathcal{E}(\Gamma) \leq \Theta\}; \\ \Omega_{\mathsf{K}}(T) & = & C_{\mathsf{K}}(\mathcal{E}(T)). \end{array}$$

We show that $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is a join continuous mapping and, for all $T \in \mathrm{Th}(\mathcal{S})$, $H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T)) = T$, whereas, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, $\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) \leq \Theta$, with equality holding if and only if $\Theta \in \Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$. These observations allow us to show that, if $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$, then Ω_{K} maps $\mathbf{Th}(\mathcal{S})$ isomorphically onto a join complete subsemilattice of $\mathbf{Th}(\mathcal{S}_\mathsf{K})$ and that, moreover, if K is equivalent to $\mathcal{S}$ via $\mathcal{E}$, then $\Omega_\mathsf{K} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$ is an isomorphism, with $\mathcal{E}$ *invertible*, in the sense that, for some translation $\mathcal{F}$ from G-equations to G-formulas, $H_\mathsf{K}(C_\mathsf{K}(\Theta)) = C(\mathcal{F}(\Theta))$, for every G-set of equations Θ.

To abstract the framework, we look at an arbitrary join complete embedding $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$. We say that Ξ is *$\mathcal{E}$-regular* if it is induced by a some translation $\mathcal{E}$, in the sense that, for all G-sets of formulas Γ, $\Xi(C(\Gamma)) = C_\mathsf{K}(\mathcal{E}(\Gamma))$. Dually, for $Z : \mathbf{Th}(\mathcal{S}_\mathsf{K}) \to \mathbf{Th}(\mathcal{S})$, Z is *$\mathcal{F}$-regular* if it is induced by some translation $\mathcal{F}$ in the opposite direction. It is shown that, if $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$ is an $\mathcal{E}$-regular order isomorphism, then $\mathcal{E}$ is invertible via $\mathcal{F}$ if and only if Ξ^{-1} is $\mathcal{F}$-regular. This enables us to prove that a class K of G-algebras is a G-algebraic semantics for $\mathcal{S}$ if and only if, there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \Xi(\mathbf{Th}(\mathcal{S}))$, where $\Xi(\mathbf{Th}(\mathcal{S}))$ is a join complete subsemilattice of $\mathbf{Th}(\mathcal{S}_\mathsf{K})$. Further, K is equivalent to $\mathcal{S}$ if and only if there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$, with $\mathcal{E}$ invertible.

In Section 2.15, the final section of Chapter 2, we deal exclusively with algebraization via standard interpretations. The developments here are rather technical, but, in a nutshell, this allows us to recover many of the distinctive features of Blok and Pigozzi's theory [6], perhaps the most striking among them the fact that Ω_K must coincide with the Leibniz operator Ω. As a result, one may provide in this setting an intrinsic characterization of standard algebraizability. We show that $\mathcal{S}$ is algebraizable via standard interpretations if and only if the Leibniz operator Ω is $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, injective and join continuous on $\mathbf{Th}(\mathcal{S})$.

2.2 Graded Logics

Let $\mathcal{L}$ be a **logical language**, i.e., a set of logical connectives, or an algebraic signature, i.e., a set of operation symbols, depending on the point of view taken. Let V be a countably infinite set of variables. Denote by $\mathrm{Fm}_\mathcal{L}(V)$ the set of **$\mathcal{L}$-formulas** or **$\mathcal{L}$-terms** with variables in V and by $\mathbf{Fm}_\mathcal{L}(V) = \langle \mathrm{Fm}_\mathcal{L}(V), \mathcal{L}\rangle$ the corresponding absolutely free algebra generated by V.
A **substitution** $\sigma : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{Fm}_\mathcal{L}(V)$ is an endomorphism of $\mathbf{Fm}_\mathcal{L}(V)$, which is completely determined by the values it assigns to the variables in V.

Let $\mathbf{G} = \langle G, \leq\rangle$ be a poset, which, often, will be assumed to have additional structure, e.g., be a complete lattice. Given any set X and functions $f, g : X \to G$, also written $f, g \in G^X$, we define

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x), \text{ for all } x \in X.$$

A **G-set of formulas** is a function

$$\Gamma : \mathrm{Fm}_\mathcal{L}(V) \to G.$$

A G-**logic** is a pair $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where

$$C : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

satisfies the following axioms, for all $\Gamma, \Delta : \mathrm{Fm}_{\mathcal{L}}(V) \to G$.

(**Inflationarity**) $\Gamma \le C(\Gamma)$;

(**Monotonicity**) $\Gamma \le \Delta$ implies $C(\Gamma) \le C(\Delta)$;

(**Idempotency**) $C(C(\Gamma)) = C(\Gamma)$;

(**Structurality**) $C(\Gamma \circ \sigma) \le C(\Gamma) \circ \sigma$, for all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ and $\mathcal{S}' = \langle \mathcal{L}, C' \rangle$ be two G-logics over the same signature $\mathcal{L}$. We say $\mathcal{S}'$ is an **extension** of $\mathcal{S}$ and that $\mathcal{S}$ is a **sublogic** of $\mathcal{S}'$, written $\mathcal{S} \le \mathcal{S}'$, if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C(\Gamma) \le C'(\Gamma).$$

We denote by $\mathsf{Log}_{\mathbf{G}}(\mathcal{L})$ be the collection of all G-logics over $\mathcal{L}$.

Proposition 1 *If* $\mathbf{G} = \langle G, \le \rangle$ *is a complete lattice, then* $\mathsf{Log}_{\mathbf{G}}(\mathcal{L})$*, ordered by* $\le$*, becomes a complete lattice.*

Proof: Let $\top$ be the top element in $\mathbf{G}$. Define $C_\top : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C_\top(\Gamma)(\varphi) = \top, \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

Note that this defines a G-logic $\mathcal{S}_\top = \langle \mathcal{L}, C_\top \rangle$, which is obviously a top element in $\mathsf{Log}_{\mathbf{G}}(\mathcal{L})$ under $\le$.

Next consider a collection $\mathcal{S}_i = \langle \mathcal{L}, C_i \rangle$, $i \in I$. Define $\bigwedge_i \mathcal{S}_i = \langle \mathcal{L}, \bigwedge_i C_i \rangle$ by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\left(\bigwedge_i C_i\right)(\Gamma)(\varphi) = \bigwedge_i C_i(\Gamma)(\varphi), \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

We show that $\bigwedge_i \mathcal{S}_i$ is a G-logic.

- First, by Inflationarity, $\Gamma \le C_i(\Gamma)$, for all $i \in I$ and all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Thus, $\Gamma \le \bigwedge_i C_i(\Gamma)$. Hence, by definition, $\Gamma \le (\bigwedge_i C_i)(\Gamma)$.

- Next, for Monotonicity, suppose $\Gamma \le \Delta$. By Monotonicity, $C_i(\Gamma) \le C_i(\Delta)$, for all $i \in I$. Thus, $\bigwedge_{i \in I} C_i(\Gamma) \le \bigwedge_{i \in I} C_i(\Delta)$. Hence, by definition, $(\bigwedge_i C_i)(\Gamma) \le (\bigwedge_i C_i)(\Delta)$.

- For Idempotency, note, first, that, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\left(\bigwedge_i C_i\right)\left(\left(\bigwedge_i C_i\right)(\Gamma)\right) \le C_i(C_i(\Gamma)) = C_i(\Gamma).$$

  This gives $(\bigwedge_i C_i)((\bigwedge_i C_i)(\Gamma)) \le (\bigwedge_i C_i)(\Gamma)$. The reverse inequality is assured by Inflationarity.

- Finally, for Structurality, let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and σ be a substitution. Then we have

$$\begin{array}{rcl}(\bigwedge_i C_i)(\Gamma \circ \sigma) & = & \bigwedge_i C_i(\Gamma \circ \sigma) \\ & \le & \bigwedge_i (C_i(\Gamma) \circ \sigma) \\ & = & (\bigwedge_i C_i(\Gamma)) \circ \sigma \\ & = & ((\bigwedge_i C_i)(\Gamma)) \circ \sigma.\end{array}$$

$\bigwedge_i \mathcal{S}_i$ is clearly a lower bound of $\{\mathcal{S}_i\}_{i \in I}$ under $\le$. Finally, it follows directly from the definition that $\bigwedge_i \mathcal{S}_i$ is the greatest lower bound of $\{\mathcal{S}_i\}_{i \in I}$.

A dual reasoning shows that $\bigvee_i \mathcal{S}_i = \langle \mathcal{L}, \bigvee_i C_i \rangle$, defined dually, forms a least upper bound of $\{\mathcal{S}_i\}_{i \in I}$ under $\le$. Thus, $\langle \mathsf{Log}_{\mathbf{G}}(\mathcal{L}), \le \rangle$ is a complete lattice, as asserted. ∎

The complete lattice of Proposition 1 is denoted by

$$\mathbf{Log}_{\mathbf{G}}(\mathcal{L}) = \langle \mathsf{Log}_{\mathbf{G}}(\mathcal{L}), \le \rangle.$$

2.3 Graded Theories

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. A G-set of formulas T is called a G-**theory** of $\mathcal{S}$ if

$$C(T) = T.$$

For any $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, $C(\Gamma)$ is the smallest G-theory of $\mathcal{S}$, such that $\Gamma \le C(\Gamma)$. We say that Γ **generates** the G-theory $C(\Gamma)$. The collection of all G-theories of $\mathcal{S}$ is denoted by $\mathrm{Th}(\mathcal{S})$.

Proposition 2 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic, with $\mathbf{G} = \langle G, \le \rangle$ a complete lattice. Then $\mathrm{Th}(\mathcal{S})$, ordered by $\le$, forms a complete lattice.*

Proof: Consider an arbitrary collection $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$. We must show that $\bigwedge_{i \in I} T_i \in \mathrm{Th}(\mathcal{S})$. Note that

$$\begin{array}{rcll}C(\bigwedge_{i \in I} T_i) & \le & C(T_i) & \text{(Monotonicity)} \\ & = & T_i. & (T_i \text{ a } G\text{-theory})\end{array}$$

Thus, we get $C(\bigwedge_{i\in I} T_i) \leq \bigwedge_{i\in I} T_i$. Since the reverse inequality holds by Inflationarity, we get that $C(\bigwedge_{i\in I} T_i) = \bigwedge_{i\in I} T_i$, i.e., $\bigwedge_{i\in I} T_i \in \mathrm{Th}(\mathcal{S})$. Hence, $\mathrm{Th}(\mathcal{S})$ forms a complete lattice under $\leq$. ∎

We denote the complete lattice of G-theories of a G-logic $\mathcal{S}$ by $\mathbf{Th}(\mathcal{S}) = \langle \mathrm{Th}(\mathcal{S}), \leq\rangle$. Its largest element is the constant function $\top : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, with

$$\top(\varphi) = \top, \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

Its smallest element is $C(\bot)$, where $\bot : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ is given by

$$\bot(\varphi) = \bot, \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ be a G-set of formulas. We say that Γ is **finite** if, for all but finitely many formulas φ,

$$\Gamma(\varphi) = \bot.$$

Given G-sets of formulas Γ, Δ, we write

$$\Gamma \leq_f \Delta$$

to signify that $\Gamma \leq \Delta$ and Γ is finite.

We say a G-logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$ is **finitary** if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C(\Gamma) = \bigvee_{\Gamma_0 \leq_f \Gamma} C(\Gamma_0).$$

The following lemmas give some properties of the G-theories of $\mathcal{S}$.

Lemma 3 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. Then, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$T \circ \sigma \in \mathrm{Th}(\mathcal{S}).$$

Proof: Using Structurality and the fact that T is a G-theory, we get

$$C(T \circ \sigma) \leq C(T) \circ \sigma = T \circ \sigma.$$

Since the reverse inclusion always holds, $C(T \circ \sigma) = T \circ \sigma$. Therefore, $T \circ \sigma$ is a G-theory. ∎

Lemma 4 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. Then, for all $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$\bigwedge_{i\in I}(T_i \circ \sigma) = \left(\bigwedge_{i\in I} T_i\right) \circ \sigma \quad \text{and} \quad \bigvee_{i\in I}(T_i \circ \sigma) = \left(\bigvee_{i\in I} T_i\right) \circ \sigma.$$

Proof: This follows directly from the definitions involved. E.g., we have, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcl}\bigvee_{i\in I}(T_i \circ \sigma)(\varphi) & = & \bigvee_{i\in I} T_i(\sigma(\varphi)) \\ & = & (\bigvee_{i\in I} T_i)(\sigma(\varphi)) \\ & = & ((\bigvee_{i\in I} T_i) \circ \sigma)(\varphi).\end{array}$$

Therefore, $\bigvee_{i\in I}(T_i \circ \sigma) = (\bigvee_{i\in I} T_i) \circ \sigma$. Similarly for meet. ∎

2.4 Graded Matrix Semantics

A **graded $\mathcal{L}$-matrix** or simply G-**matrix** is a pair $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, where:

- $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ is an $\mathcal{L}$-algebra;
- $F : A \to G$ is a function, called the **graded set of designated elements**, the **graded filter** or, simply, the G-**filter** of the G-matrix.

Given a G-matrix $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, we define a mapping

$$C_{\mathfrak{A}} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by, setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C_{\mathfrak{A}}(\Gamma) = \bigwedge\{F \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \le F \circ h\}.$$

More generally, given a class M of G-matrices, we define a mapping

$$C_{\mathsf{M}} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by, setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C_{\mathsf{M}}(\Gamma) = \bigwedge\{C_{\mathfrak{A}} : \mathfrak{A} \in \mathsf{M}\}.$$

Proposition 5 *Let M be a class of G-matrices, with $\mathbf{G} = \langle G, \le\rangle$ a complete lattice. Then $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}}\rangle$ is a G-logic.*

Proof: We fist check Inflationarity. Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then, for all $\mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}$, we get

$$\Gamma \le \bigwedge\{F \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \le F \circ h\} = C_{\mathfrak{A}}(\Gamma).$$

Therefore, $\Gamma \le \bigwedge_{\mathfrak{A}\in\mathsf{M}} C_{\mathfrak{A}}(\Gamma) = C_{\mathsf{M}}(\Gamma)$.

We turn, next, to Monotonicity. Suppose $\Gamma, \Delta : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ are such that $\Gamma \le \Delta$. Then we have

$$\begin{array}{rcl} C_{\mathsf{M}}(\Gamma) & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \le F \circ h\} \\ & \le & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Delta \le F \circ h\} \\ & = & C_{\mathsf{M}}(\Delta). \end{array}$$

Next, we look at Idempotency. For all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, we get

$$\begin{array}{rcl} C_{\mathsf{M}}(C_{\mathsf{M}}(\Gamma)) & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ & & \qquad C_{\mathsf{M}}(\Gamma) \le F \circ h\} \\ & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \le F \circ h\} \\ & = & C_{\mathsf{M}}(\Gamma). \end{array}$$

Finally, for Structurality, let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and σ be a substitution. Then we get

$$\begin{array}{rcl} C_{\mathsf{M}}(\Gamma \circ \sigma) & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \circ \sigma \leq F \circ h\} \\ & \leq & \bigwedge\{F \circ h \circ \sigma : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ & & \qquad\qquad\qquad\qquad \Gamma \circ \sigma \leq F \circ h \circ \sigma\} \\ & \leq & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \leq F \circ h\} \circ \sigma \\ & = & C_{\mathsf{M}}(\Gamma) \circ \sigma. \end{array}$$

Thus, $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}}\rangle$ is a G-logic. ■

$\mathcal{S}_{\mathsf{M}}$ is called the G-**logic determined by** or **induced by** the class M of G-matrices. If $\mathsf{M} = \{\mathfrak{A}\}$, then we write $C_{\mathfrak{A}}$ instead of $C_{\{\mathfrak{A}\}}$.

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. A G-matrix $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ is called a G-**matrix of** $\mathcal{S}$ or an $\mathcal{S}$-**matrix** if

$$C \leq C_{\mathfrak{A}}.$$

In this case, the G-filter F of $\mathfrak{A}$ is called an $\mathcal{S}$-**filter** on $\mathbf{A}$. The collection of all $\mathcal{S}$-filters on $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. We now look at the structure of this set. We have the following technical lemma.

Lemma 6 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, with $\mathbf{G} = \langle G, \leq\rangle$ a complete lattice, $\mathbf{A}$ an $\mathcal{L}$-algebra and $\{F_i : i \in I\}$ a collection of G-filters on $\mathbf{A}$. Then*

$$\bigwedge_{i\in I} C_{\langle \mathbf{A}, F_i\rangle} = C_{\langle \mathbf{A}, \bigwedge_{i\in I} F_i\rangle}.$$

Proof: We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcl} (\bigwedge_{i\in I} C_{\langle \mathbf{A}, F_i\rangle})(\Gamma) & = & \bigwedge_i \bigwedge_h \{F_i \circ h : \Gamma \leq F_i \circ h\} \\ & = & \bigwedge_h \bigwedge_i \{F_i \circ h : \Gamma \leq F_i \circ h\} \\ & = & \bigwedge_h \{(\bigwedge_i F_i) \circ h : \Gamma \leq (\bigwedge_i F_i) \circ h\} \\ & = & C_{\langle \mathbf{A}, \bigwedge_i F_i\rangle}(\Gamma). \end{array}$$

■

Lemma 7 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, with $\mathbf{G} = \langle G, \leq\rangle$ a complete lattice, and $\mathbf{A}$ an $\mathcal{L}$-algebra. Then*

$$\mathbf{Fi}_{\mathcal{S}}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \leq\rangle$$

is a complete lattice.

Proof: It is clear that the constant function $\top : A \to G$, with

$$\top(a) = \top, \quad a \in A,$$

is a maximum element in $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Moreover, given $F_i \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, $i \in I$, we have, by definition, $C \leq C_{\langle \mathbf{A}, F_i \rangle}$, for all $i \in I$, whence, by Lemma 6,

$$C \leq \bigwedge_{i \in I} C_{\langle \mathbf{A}, F_i \rangle} = C_{\langle \mathbf{A}, \bigwedge_{i \in I} F_i \rangle}.$$

Thus, $\bigwedge F_i$ is also an $\mathcal{S}$-filter on $\mathbf{A}$. ■

In closing, we show that the collection of $\mathcal{S}$-filters on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ coincides with the collection of G-theories of $\mathcal{S}$.

Lemma 8 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. Then*

$$\mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V)) = \mathrm{Th}(\mathcal{S}).$$

Proof: Suppose, first, that $T \in \mathrm{Th}(\mathcal{S})$. Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$, such that $\Gamma \leq T \circ \sigma$. Then, we have

$$C(\Gamma) \leq C(T \circ \sigma) \leq C(T) \circ \sigma = T \circ \sigma.$$

Therefore, $T \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V))$.

Suppose, conversely, that $T \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V))$. Clearly, $T \leq T \circ i$, where i is the identity homomorphism on $\mathbf{Fm}_{\mathcal{L}}(V)$. Thus, by hypothesis,

$$C(T) \leq T \circ i = T.$$

This shows that $T = C(T)$ and, hence, $T \in \mathrm{Th}(\mathcal{S})$. ■

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and M a class of G-matrices. M is called a G-**matrix semantics** of (or for) $\mathcal{S}$ if

$$C = C_{\mathsf{M}}.$$

In this case it is said that M is **strongly adequate for** $\mathcal{S}$. Both the class of all $\mathcal{S}$-models and the class of all $\mathcal{S}$-models on the formula algebra are strongly adequate for $\mathcal{S}$.

2.5 G-Congruences and Congruences

Since most of our results require that G be a complete lattice, we make this assumption from now own, even if it is not explicitly mentioned.

We denote by $\mathrm{Eq}_{\mathcal{L}}(V)$ the set of all $\mathcal{L}$-**equations**, i.e.,

$$\mathrm{Eq}_{\mathcal{L}}(V) = \mathrm{Fm}^2_{\mathcal{L}}(V).$$

For $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, such an equation may be denoted by $\langle \varphi, \psi \rangle$ or $\varphi \approx \psi$.

Let $\mathbf{A}$ be an algebra. A G-**congruence** on $\mathbf{A}$ is a mapping $\Theta : A^2 \to G$, such that, for all $\lambda \in \mathcal{L}$ and all $a, b, c, \overline{a}, \overline{b} \in A$, (**Reflexivity**) $\Theta(a,a) = \top$;

(**Symmetry**) $\Theta(a,b) = \Theta(b,a)$;

(**Transitivity**) $\Theta(a,b) \wedge \Theta(b,c) \leq \Theta(a,c)$;

(**Congruence**) $\bigwedge_i \Theta(a_i,b_i) \leq \Theta(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b}))$.

Let $\mathrm{Gon}(\mathbf{A})$ denote the collection of all G-congruences on $\mathbf{A}$.

Proposition 9 *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra and* $\mathbf{G} = \langle G, \leq\rangle$ *a complete lattice. The collection* $\mathrm{Gon}(\mathbf{A})$ *of* G*-congruences on* $\mathbf{A}$*, ordered by* $\leq$*, forms a complete lattice.*

Proof: First, note that $\top : A^2 \to G$, with

$$\top(a,b) = \top, \text{ for all } a,b \in A,$$

is a G-congruence on $\mathbf{A}$ and is clearly the largest G-congruence under $\leq$.

Next consider the collection Θ_i, $i \in I$, of G-congruences on $\mathbf{A}$. Define $\bigwedge_i \Theta_i : A^2 \to G$ by setting, for all $a,b \in A$,

$$\left(\bigwedge_i \Theta_i\right)(a,b) = \bigwedge_i \Theta_i(a,b).$$

We show that $\bigwedge_i \Theta_i$ is a G-congruence.

- First, for all $a \in A$, $(\bigwedge_i \Theta_i)(a,a) = \bigwedge_i \Theta_i(a,a) = \bigwedge_i \top = \top$.
- Next, for all $a,b \in A$,

$$\left(\bigwedge_i \Theta_i\right)(a,b) = \bigwedge_i \Theta_i(a,b) = \bigwedge_i \Theta_i(b,a) = \left(\bigwedge_i \Theta_i\right)(b,a).$$

- Next, for all $a,b,c \in A$,

$$\begin{array}{rcl}(\bigwedge_i \Theta_i)(a,b) \wedge (\bigwedge_i \Theta_i)(b,c) & = & \bigwedge_i \Theta_i(a,b) \wedge \bigwedge_i \Theta_i(b,c) \\ & = & \bigwedge_i (\Theta_i(a,b) \wedge \Theta_i(b,c)) \\ & \leq & \bigwedge_i \Theta_i(a,c) \\ & = & (\bigwedge_i \Theta_i)(a,c).\end{array}$$

- Finally, for every n-ary $\lambda \in \mathcal{L}$ and all $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$, we have

$$\begin{array}{rcl}\bigwedge_{j=1}^n (\bigwedge_i \Theta_i)(a_j,b_j) & = & \bigwedge_{j=1}^n \bigwedge_i \Theta_i(a_j,b_j) \\ & = & \bigwedge_i (\bigwedge_{j=1}^n \Theta_i(a_j,b_j)) \\ & \leq & \bigwedge_i \Theta_i(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})) \\ & = & (\bigwedge_i \Theta_i)(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})).\end{array}$$

$\bigwedge_i \Theta_i$ is clearly a lower bound of $\{\Theta_i\}_{i\in I}$ under $\leq$. Finally, it follows directly from the definition that $\bigwedge_i \Theta_i$ is the greatest lower bound of $\{\Theta_i\}_{i\in I}$.

This proves that $\langle \mathrm{Gon}(\mathcal{L}), \leq\rangle$ is a complete lattice, as asserted. ∎

The lattice of G-congruences on $\mathbf{A}$, given by Proposition 9, is denoted by

$$\mathbf{Gon}(\mathbf{A}) = \langle \mathrm{Gon}(\mathbf{A}), \leq\rangle.$$

G-congruences are closely associated with *families of ordinary congruences*. We give this correspondence in detail because, on the one hand, it illuminates the concept of G-congruence by relating it to a more familiar concept and, on the other, passing back and forth between G-congruences and congruences is very useful in applying the concepts, especially in reductions.

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra and $\Theta : A^2 \to G$ be a G-congruence. Let $g \in G$. Define a binary relation

$$\hat{\Theta}_g \subseteq A \times A$$

by setting, for all $a, b \in A$,

$$\langle a, b\rangle \in \hat{\Theta}_g \quad \text{iff} \quad \Theta(a, b) \geq g.$$

Proposition 10 *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra,* $\Theta : A^2 \to G$ *be a* G*-congruence on* $\mathbf{A}$ *and* $g \in G$*. Then* $\hat{\Theta}_g$ *is a congruence on* $\mathbf{A}$*. Moreover, for all* $g_1, g_2 \in G$*,*

$$g_1 \leq g_2 \quad \textit{implies} \quad \hat{\Theta}_{g_2} \subseteq \hat{\Theta}_{g_1}.$$

Proof: For all $a \in A$, we have $\Theta(a, a) = \top \geq g$. Hence, $\langle a, a\rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is reflexive. For symmetry, let $a, b \in A$. Then we have

$$\langle a, b\rangle \in \hat{\Theta}_g \Rightarrow \Theta(a, b) \geq g \Rightarrow \Theta(b, a) \geq g \Rightarrow \langle b, a\rangle \in \hat{\Theta}_g.$$

For transitivity, let $a, b, c \in A$. Then

$$\begin{aligned}\langle a, b\rangle \in \hat{\Theta}_g \text{ and } \langle b, c\rangle \in \hat{\Theta}_g &\Rightarrow \Theta(a, b) \geq g \text{ and } \Theta(b, c) \geq g\\ &\Rightarrow \Theta(a, b) \wedge \Theta(b, c) \geq g\\ &\Rightarrow \Theta(a, c) \geq g\\ &\Rightarrow \langle a, c\rangle \in \hat{\Theta}_g.\end{aligned}$$

Thus, $\hat{\Theta}_g$ is an equivalence relation. To finish the demonstration that it is a congruence, let $\lambda \in \mathcal{L}$ be n-ary and suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. Then

$$\begin{aligned}\langle a_i, b_i\rangle \in \hat{\Theta}_g,\ i \in I, &\Rightarrow \Theta(a_i, b_i) \geq g,\ i \in I,\\ &\Rightarrow \textstyle\bigwedge_i \Theta(a_i, b_i) \geq g\\ &\Rightarrow \Theta(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})) \geq g\\ &\Rightarrow \langle \lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})\rangle \in \hat{\Theta}_g.\end{aligned}$$

Finally, note that, if $g_1, g_2 \in G$, such that $g_1 \leq g_2$, we have, for all $a, b \in A$,

$$\langle a,b\rangle \in \hat{\Theta}_{g_2} \;\Rightarrow\; \Theta(a,b) \geq g_2 \;\Rightarrow\; \Theta(a,b) \geq g_1 \;\Rightarrow\; \langle a,b\rangle \in \hat{\Theta}_{g_1}.$$

This shows that the displayed antimonotonicity property holds. ■

We set

$$\hat{\Theta} = \{\hat{\Theta}_g : g \in G\}$$

and call $\hat{\Theta}$ the **stratified congruence associated with** the G-congruence Θ. Further, for a fixed $g \in G$, we call $\hat{\Theta}_g$ the g-**stratum** of Θ.

Conversely, let us call a family $\theta = \{\theta_g : g \in G\}$ of congruences on $\mathbf{A}$ a **stratified congruence** if it satisfies

$$\theta_{g_2} \subseteq \theta_{g_1}, \text{ for all } g_1 \leq g_2.$$

Define the mapping $\check{\theta} : A^2 \to G$ by setting, for all $a, b \in A$,

$$\check{\theta}(a,b) = \bigvee\{g \in G : \langle a,b\rangle \in \theta_g\}.$$

Proposition 11 *Suppose* $\mathbf{G} = \langle G, \leq\rangle$ *satisfies complete distributivity. Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra and* $\theta = \{\theta_g : g \in G\}$ *a stratified congruence on* $\mathbf{A}$*. Then* $\check{\theta}$ *is a* G*-congruence on* $\mathbf{A}$*.*

Proof: For all $a \in A$, we have

$$\check{\theta}(a,a) = \bigvee\{g : \langle a,a\rangle \in \theta_g\} = \bigvee G = \top.$$

For all $a, b \in A$,

$$\check{\theta}(a,b) = \bigvee\{g : \langle a,b\rangle \in \theta_g\} = \bigvee\{g : \langle b,a\rangle \in \theta_g\} = \check{\theta}(b,a).$$

For all $a, b, c \in A$,

$$\begin{array}{rcl} \check{\theta}(a,b) \wedge \check{\theta}(b,c) & = & \bigvee\{g : \langle a,b\rangle \in \theta_g\} \wedge \bigvee\{h : \langle b,c\rangle \in \theta_h\} \\ & = & \bigvee\{g \wedge h : \langle a,b\rangle \in \theta_g \text{ and } \langle b,c\rangle \in \theta_h\} \\ & \leq & \bigvee\{g \wedge h : \langle a,b\rangle \in \theta_{g\wedge h} \text{ and } \langle b,c\rangle \in \theta_{g\wedge h}\} \\ & \leq & \bigvee\{g \wedge h : \langle a,c\rangle \in \theta_{g\wedge h}\} \\ & \leq & \check{\theta}(a,c). \end{array}$$

Similarly, for all $\lambda \in \mathcal{L}$ n-ary and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$\begin{array}{rcl} \bigwedge_{i=1}^n \check{\theta}(a_i,b_i) & = & \bigwedge_{i=1}^n \bigvee\{g_i : \langle a_i,b_i\rangle \in \theta_{g_i}\} \\ & = & \bigvee\{\bigwedge_{i=1}^n g_i : \langle a_i,b_i\rangle \in \theta_{g_i}, i \in I\} \\ & \leq & \bigvee\{\bigwedge_{i=1}^n g_i : \langle a_i,b_i\rangle \in \theta_{\wedge_i g_i}\} \\ & \leq & \bigvee\{g : \langle \lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})\rangle \in \theta_g\} \\ & = & \check{\theta}(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})). \end{array}$$

This proves that $\check{\theta}$ is a G-congruence on $\mathbf{A}$. ∎

We call $\check{\theta}$ the G-**congruence associated with** the stratified congruence $\theta = \{\theta_g : g \in G\}$.

Finally, we show that, under special circumstances, e.g., when G is completely distributive and every element in G is a (possibly infinite) join of completely join-irreducibles, then the correspondences established via $\hat{}$ and $\check{}$ are inverses of one another. In such cases, therefore, the point of view taken, G-congruence versus stratified congruence, is a matter of preference and/or convenience.

Proposition 12 *Suppose* $\mathbf{G} = \langle G, \leq\rangle$ *satisfies complete distributivity. Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra. Then the following statements hold.*

(a) $\check{\hat{\Theta}} = \Theta$*, for every* G*-congruence* Θ *on* $\mathbf{A}$*;*

(b) $\hat{\check{\theta}}_g = \theta_g$*, for every stratified congruence* $\theta = \{\theta_g : g \in G\}$ *on* $\mathbf{A}$*, provided* g *is completely join irreducible in* $\mathbf{G}$*.*

Proof: For Part (a), suppose $a, b \in A$. Then we have

$$\check{\hat{\Theta}}(a,b) = \bigvee\{g : \langle a,b\rangle \in \hat{\Theta}_g\} = \bigvee\{g : \Theta(a,b) \geq g\} = \Theta(a,b).$$

For Part (b), we have, for all $a, b \in A$,

$$\begin{array}{rll} \langle a,b\rangle \in \hat{\check{\theta}}_g & \text{iff} & \check{\theta}(a,b) \geq g \\ & \text{iff} & \bigvee\{g' : \langle a,b\rangle \in \theta_{g'}\} \geq g \\ & \text{iff} & \langle a,b\rangle \in \theta_{g'}, \text{ for some } g' \geq g, \\ & \text{iff} & \langle a,b\rangle \in \theta_g. \end{array}$$

The one equivalence before the last is a consequence of complete distributivity and complete join irreducibility. ∎

Corollary 13 *Let* $\mathbf{G} = \langle G, \leq\rangle$ *be such that, for every algebra* $\mathbf{A}$*, the mappings* $\Theta \mapsto \hat{\Theta}$ *and* $\theta \mapsto \check{\theta}$ *are inverses of one another. Then, for all algebras* $\mathbf{A}$*, all* G*-congruences* $\Theta', \Theta'' \in \mathrm{Gon}(\mathbf{A})$*, and all* $a, b \in A$*,*

$$(\Theta' \vee^{\mathbf{Gon(A)}} \Theta'')(a,b) = \bigvee\{g : \langle a,b\rangle \in \hat{\Theta}'_g \vee^{\mathbf{Con(A)}} \hat{\Theta}''_g\}.$$

Proof: We have

$$\begin{array}{rcl} (\Theta' \vee^{\mathbf{Gon(A)}} \Theta'')(a,b) & = & \bigvee\{g : \langle a,b\rangle \in \widehat{\Theta' \vee^{\mathbf{Gon(A)}} \Theta''}_g\} \\ & = & \bigvee\{g : \langle a,b\rangle \in \hat{\Theta}'_g \vee^{\mathbf{Con(A)}} \hat{\Theta}''_g\}. \end{array}$$

This proves the statement. ∎

2.6 Compatibility and Leibniz Congruences

Let $\mathbf{A}$ be an algebra and $F : A \to G$ a G-filter. A G-congruence Θ on $\mathbf{A}$ is said to be **compatible with** F if, for all $a, b \in A$,

$$\Theta(a,b) \wedge F(a) \leq F(b).$$

Suppose $\mathbf{G}$ has an implication $\to$, satisfying

$$g_1 \wedge g_2 \leq g_3 \quad \text{iff} \quad g_2 \leq g_1 \to g_3,$$

and denote

$$g_1 \leftrightarrow g_2 := (g_1 \to g_2) \wedge (g_2 \to g_1).$$

Then Θ is compatible with F if and only if, for all $a, b \in A$,

$$\Theta(a,b) \leq F(a) \leftrightarrow F(b).$$

If Θ is compatible with F, we also say that Θ is a G-**congruence of** the G-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$. By $\mathrm{Gon}(\mathfrak{A})$ we denote the collection of all G-congruences of $\mathfrak{A}$.

We show that, under some hypotheses, $\mathrm{Gon}(\mathfrak{A})$, ordered by $\leq$, forms a principal ideal of the complete lattice $\mathbf{Gon(A)} = \langle \mathrm{Gon}(\mathbf{A}), \leq \rangle$ of all G-congruences on $\mathbf{A}$. From then on, we shall assume that G satisfies those hypotheses and take the conclusion for granted. We start with a lemma.

Lemma 14 *Suppose $\mathbf{G} = \langle G, \leq \rangle$ is a complete lattice with an implication $\to$. Let $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ be a G-matrix. The G-relation $R_{\mathfrak{A}} : A^2 \to G$, defined, for all $a, b \in A$, by*

$$R_{\mathfrak{A}}(a,b) = \bigwedge \{F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \in A\}$$

is a G-congruence on $\mathbf{A}$ compatible with F, i.e., $R_{\mathfrak{A}} \in \mathrm{Gon}(\mathfrak{A})$.

Proof: It is clear that $R_{\mathfrak{A}}$ satisfies Reflexivity and Symmetry. For Transitivity, let $a, b, c \in A$. Then we have

$$\begin{aligned}
&R_{\mathfrak{A}}(a,b) \wedge R_{\mathfrak{A}}(b,c)\\
&\quad = [\textstyle\bigwedge_{\varphi,\overline{d}} F(\varphi^{\mathbf{A}}(a,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{d}))]\\
&\qquad\qquad \wedge [\textstyle\bigwedge_{\psi,\overline{e}} F(\psi^{\mathbf{A}}(b,\overline{e})) \leftrightarrow F(\psi^{\mathbf{A}}(c,\overline{e}))]\\
&\quad \leq \textstyle\bigwedge_{\varphi,\overline{d}}[(F(\varphi^{\mathbf{A}}(a,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{d})))\\
&\qquad\qquad \wedge (F(\varphi^{\mathbf{A}}(b,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\overline{d})))]\\
&\quad \leq \textstyle\bigwedge_{\varphi,\overline{d}} F(\varphi^{\mathbf{A}}(a,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\overline{d}))\\
&\quad = R_{\mathfrak{A}}(a,c).
\end{aligned}$$

For the Congruence property, let $\lambda \in \mathcal{L}$ be n-ary and $a_i, b_i \in A$, $i < n$. Then, for all $\varphi(x,\overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\overline{c} \in A$,

$$\begin{aligned}
R_{\mathfrak{A}}(a_i,b_i) \leq{}& F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(a_0,\ldots,a_{i-1},a_i,b_{i+1},\ldots,b_{n-1}),\overline{c}))\\
&\leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(a_0,\ldots,a_{i-1},b_i,b_{i+1},\ldots,b_{n-1}),\overline{c})).
\end{aligned}$$

Thus, by Transitivity,

$$\bigwedge_i R_{\mathfrak{A}}(a_i, b_i) \le F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(\overline{a}), \overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(\overline{a}), \overline{c})),$$

i.e.,

$$\bigwedge_i R_{\mathfrak{A}}(a_i, b_i) \le R_{\mathfrak{A}}(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})).$$

So $R_{\mathfrak{A}}$ is a G-congruence on $\mathbf{A}$. Compatibility with F is straightforward by taking $\varphi = x$ in the defining property of $R_{\mathfrak{A}}$. ■

Next, we prove that $\mathrm{Gon}(\mathfrak{A})$ forms a principal ideal of the complete lattice $\mathbf{Gon(A)} = \langle \mathrm{Gon}(\mathbf{A}), \le \rangle$ of all G-congruences on $\mathbf{A}$.

Proposition 15 *Suppose* $\mathbf{G} = \langle G, \le \rangle$ *is a complete lattice with an implication* $\to$*. Let* $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ *be a* G*-matrix. Then* $\mathrm{Gon}(\mathfrak{A})$ *is a principal ideal in* $\mathbf{Gon(A)}$.

Proof: First, suppose $\Theta, \Theta' \in \mathrm{Gon}(\mathbf{A})$, such that $\Theta \le \Theta' \in \mathrm{Gon}(\mathfrak{A})$. Then, for all $a, b \in A$,

$$\Theta(a, b) \wedge F(a) \le \Theta'(a, b) \wedge F(a) \le F(b).$$

Thus, $\Theta \in \mathrm{Gon}(\mathfrak{A})$. So $\mathrm{Gon}(\mathfrak{A})$ is a downset in $\mathbf{Gon(A)}$.

Suppose, next, that $\Theta, \Theta' \in \mathrm{Gon}(\mathfrak{A})$. One may show by induction on the structure of an $\mathcal{L}$-term φ that, for all $a, b, \bar{c} \in A$,

$$\Theta(a, b) \le \Theta(\varphi^{\mathbf{A}}(a, \bar{c}), \varphi^{\mathbf{A}}(b, \bar{c})).$$

By compatibility of Θ with F, it follows that

$$\Theta(a, b) \le F(\varphi^{\mathbf{A}}(a, \bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b, \bar{c})).$$

Therefore, $\Theta(a, b) \le R_{\mathfrak{A}}(a, b)$. Similarly, $\Theta'(a, b) \le R_{\mathfrak{A}}(a, b)$. This yields

$$(\Theta \bigvee^{\mathbf{Gon(A)}} \Theta')(a, b) \wedge F(a) \le R_{\mathfrak{A}}(a, b) \wedge F(a) \le F(b).$$

Therefore, $\Theta \bigvee^{\mathbf{Gon(A)}} \Theta' \in \mathrm{Gon}(\mathfrak{A})$. This shows that $\mathrm{Gon}(\mathfrak{A})$ is an ideal in $\mathbf{Gon(A)}$. Finally, to see that it is principal, it suffices to show that it has a maximal element. This can be shown using Zorn's Lemma. Indeed every chain in $\mathrm{Gon}(\mathfrak{A})$ is upper bounded by its join, which is also compatible with F. ■

We shall restrict attention to lattices $\mathbf{G} = \langle G, \le \rangle$ for which the conclusion of Proposition 15 holds. This is due to the fact that our theory, attempting to emulate the main points of the theory of Blok and Pigozzi [6] and of Font and Jansana [28], requires the existence of a maximum element in $\mathbf{Gon}(\mathfrak{A})$. Let us call such lattices **Leibniz permitting**. Some of the techniques and results presented, however, may be transferrable to more relaxed settings.

The generator of $\mathrm{Gon}(\mathfrak{A})$, i.e., the largest G-congruence on $\mathbf{A}$ compatible with F, is called the **Leibniz G-congruence of F on A** or the **Leibniz G-congruence of $\mathfrak{A}$**. It is denoted by $\Omega_{\mathbf{A}}(F)$ or $\Omega(\mathcal{A})$. The operator

$$\Omega_{\mathbf{A}} : G^A \to \mathrm{Gon}(\mathbf{A})$$

is called the **Leibniz operator** on $\mathbf{A}$. In the special case in which $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}}(V)$, we write Ω for $\Omega_{\mathbf{Fm}_{\mathcal{L}}(V)}$ to simplify notation.

Based on bits and pieces of preceding work, we can provide a characterization of the Leibniz G-congruence of a G-filter on an algebra in terms of indistinguishability, paralleling the one given by Blok and Pigozzi to explain the name "Leibniz congruence" for the original notion they introduced (Page 11 of [6]).

Theorem 16 *Suppose $\mathbf{G} = \langle G, \leq\rangle$ is a complete lattice with an implication $\to$. Let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be a G-matrix. Then $\Omega_{\mathbf{A}}(F) = R_{\mathfrak{A}}$.*

Proof: Suppose, first, that $a, b \in A$. Then, since $\Omega_{\mathbf{A}}(F)$ is a G-congruence, we have, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\overline{c} \in A$,

$$\Omega_{\mathbf{A}}(F)(a,b) \leq \Omega_{\mathbf{A}}(F)(\varphi^{\mathbf{A}}(a,\overline{c}), \varphi^{\mathbf{A}}(b,\overline{c})).$$

Finally, by compatibility of $\Omega_{\mathbf{A}}(F)$ with F, we get

$$\begin{array}{rcl}\Omega_{\mathbf{A}}(F)(a,b) \wedge F(\varphi^{\mathbf{A}}(a,\overline{c})) & \leq & \Omega_{\mathbf{A}}(F)(\varphi^{\mathbf{A}}(a,\overline{c}), \varphi^{\mathbf{A}}(b,\overline{c})) \wedge F(\varphi^{\mathbf{A}}(a,\overline{c})) \\ & \leq & F(\varphi^{\mathbf{A}}(b,\overline{c})).\end{array}$$

Thus, we conclude that

$$\Omega_{\mathbf{A}}(F)(a,b) \leq \bigwedge_{\varphi,\overline{c}} F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})),$$

i.e., $\Omega_{\mathbf{A}}(F) \leq R_{\mathfrak{A}}$. For the reverse inequality, because of the definition of $\Omega_{\mathbf{A}}(F)$, it suffices to show that $R_{\mathfrak{A}}$ is a G-congruence on $\mathbf{A}$ compatible with F. This, however, holds by Lemma 14. Since $\Omega_{\mathbf{A}}(F)$ is, by definition, the largest congruence on $\mathbf{A}$ compatible with F, we get $R_{\mathfrak{A}} \leq \Omega_{\mathbf{A}}(F)$. ■

We close the section with a result of a slightly more technical nature pertaining to the identification of the Leibniz G-congruence of a G-filter. Let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be a G-matrix. We say that $\Theta : A^2 \to G$ is **definable (with parameters) in $\mathfrak{A}$** if there exist $\varphi(x,y,\overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\overline{c} \in A$, such that, for all $a, b \in A$,

$$\Theta(a,b) = F(\varphi^{\mathbf{A}}(a,b,\overline{c})).$$

Theorem 17 *Suppose $\mathbf{G} = \langle G, \leq\rangle$ is a complete lattice with $\to$. Let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be a G-matrix and $\Theta : A^2 \to G$ definable in $\mathfrak{A}$.*

(a) *If* $\Theta(a,a) = \top$, *for all* $a \in A$, *then* $\Omega_{\mathbf{A}}(F) \leq \Theta$.

(b) *If, in addition,* Θ *is a* G*-congruence on* $\mathbf{A}$ *compatible with* F, *then* $\Omega_{\mathbf{A}}(F) = \Theta$.

Proof:

(a) Let $a,b \in A$. Suppose Θ is definable by $\psi(x,y,\overline{z})$, with parameters $\overline{d}$. Then we have

$$\begin{array}{rcl}
\Omega_{\mathbf{A}}(F)(a,b) & = & \bigwedge_{\varphi,\overline{c}} F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})) \quad \text{(Theorem 16)} \\
& \leq & F(\psi^{\mathbf{A}}(a,b,\overline{d})) \leftrightarrow F(\psi^{\mathbf{A}}(b,b,\overline{d})) \quad \text{(Instantiation)} \\
& = & \Theta(a,b) \leftrightarrow \Theta(b,b) \quad \text{(Definability of } \Theta \text{ in } \mathfrak{A}) \\
& = & \Theta(a,b) \leftrightarrow \top \quad (\Theta(b,b) = \top) \\
& = & \Theta(a,b). \quad \text{(Property of } \rightarrow)
\end{array}$$

Hence, $\Omega_{\mathbf{A}}(F) \leq \Theta$.

(b) By hypothesis, $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Since $\Omega_{\mathbf{A}}(F)$ is the largest G-congruence in $\mathrm{Gon}(\mathfrak{A})$, we get $\Theta \leq \Omega_{\mathbf{A}}(F)$. Therefore, by Part (a), $\Omega_{\mathbf{A}}(F) = \Theta$.

■

2.7 Protoalgebraic Graded Logics

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. We say that $\mathcal{S}$ is **protoalgebraic** if, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\varphi,\psi) \wedge T'(\varphi) \leq T'(\psi), \quad \text{for all } T \leq T' \in \mathrm{Th}(\mathcal{S}).$$

In implication form, should such a connective exist, we can reformulate the defining condition equivalently as

$$\Omega(T)(\varphi,\psi) \leq T'(\varphi) \leftrightarrow T'(\psi), \quad \text{for all } T \leq T' \in \mathrm{Th}(\mathcal{S}).$$

Protoalgebraic G-logics are characterized by the monotonicity of the Leibniz operator on the complete lattice of their G-theories.

Theorem 18 *Let* $\mathcal{S} = \langle \mathcal{L}, C \rangle$ *be a* G*-logic.* $\mathcal{S}$ *is protoalgebraic if and only if, for all* $T, T' \in \mathrm{Th}(\mathcal{S})$,

$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

Proof: Suppose, first, that $\mathcal{S}$ is protoalgebraic. Let $T, T' \in \mathrm{Th}(\mathcal{S})$, such that $T \leq T'$. To see that $\Omega(T) \leq \Omega(T')$, it suffices to show, by the maximality property of the Leibniz G-congruence, that $\Omega(T)$ is compatible with T'. Let $\varphi,\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then, by G-protoalgebraicity,

$$\Omega(T)(\varphi,\psi) \wedge T'(\varphi) \le T'(\psi).$$

Hence $\Omega(T)$ is compatible with T' and $\Omega(T) \le \Omega(T')$.

Assume, conversely, that, for all $T, T' \in \mathrm{Th}(\mathcal{S})$, with $T \le T'$, we have $\Omega(T) \le \Omega(T')$. Thus, $\Omega(T)$ is a G-congruence that is compatible with T'. By compatibility, for all $\varphi,\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\varphi,\psi) \wedge T'(\varphi) \le T'(\psi).$$

This proves that $\mathcal{S}$ is protoalgebraic. ■

2.8 Graded 2-Logics

Let $\mathrm{Eq}_{\mathcal{L}}(V)$ be the set of all $\mathcal{L}$-equations and $\mathbf{G} = \langle G, \le\rangle$ a complete lattice. A G-**set of equations** is a mapping

$$\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G.$$

A G-2-**logic** is a pair $\mathcal{S} = \langle \mathcal{L}, C\rangle$, where

$$C : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

is a mapping that satisfies the following axioms, for all $\Theta, \Theta' : \mathrm{Eq}_{\mathcal{L}}(V) \to G$.

(**Inflationarity**) $\Theta \le C(\Theta)$;

(**Monotonicity**) $\Theta \le \Theta'$ implies $C(\Theta) \le C(\Theta')$;

(**Idempotency**) $C(C(\Theta)) = C(\Theta)$;

(**Structurality**) $C(\Theta \circ \sigma) \le C(\Theta) \circ \sigma$, for all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Structurality would be more accurately defined by the inequality

$$C(\Theta \circ \langle \sigma,\sigma\rangle) \le C(\Theta) \circ \langle \sigma,\sigma\rangle,$$

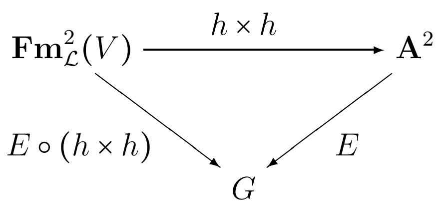
but it is very common to overload notation and apply a substitution σ on equations by applying the substitution to each side of the equation. A G-**algebra** is a pair $\mathcal{A} = \langle \mathbf{A}, E\rangle$, where $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ is an $\mathcal{L}$-algebra and $E : A^2 \to G$ is a G-congruence on $\mathbf{A}$. In the context of G-2-logics, G-algebras play the role that G-matrices play in the context of G-logics.

A key concept from the point of view of algebraic logic is that of the G-2-logic determined by a given class of G-algebras. This parallels, for equations, the concept of a G-logic determined by a G-matrix, defined at the beginning of Section 2.4.

Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra. Define the mapping

$$C_{\mathcal{A}} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

by setting, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,



$$C_{\mathcal{A}}(\Theta) = \bigwedge\{E \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}.$$

Extending this definition, for a class K of G-algebras, we define

$$C_{\mathsf{K}} = \bigwedge\{C_{\mathcal{A}} : \mathcal{A} \in \mathsf{K}\}.$$

Proposition 19 *Let K be a class of G-algebras. Then $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ is a G-2-logic.*

Proof: The proof is similar to the proof of Proposition 5. We must show that C_{K} satisfies Inflationarity, Monotonicity, Idempotency and Structurality. For Inflationarity, let $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{array}{rcl} \Theta & \leq & \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\} \\ & = & C_{\mathsf{K}}(\Theta). \end{array}$$

For Monotonicity, suppose $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and $\Theta' : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, such that $\Theta \leq \Theta'$. Then we have

$$\begin{array}{rcl} C_{\mathsf{K}}(\Theta) & = & \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\} \\ & \leq & \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta' \leq E \circ h\} \\ & = & C_{\mathsf{K}}(\Theta'). \end{array}$$

For Idempotency, let $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{array}{l} C_{\mathsf{K}}(C_{\mathsf{K}}(\Theta)) \\ \quad = \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, C_{\mathsf{K}}(\Theta) \leq E \circ h\} \\ \quad = \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\} \\ \quad = C_{\mathsf{K}}(\Theta). \end{array}$$

Finally, for Structurality, suppose $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and let σ be a substitution. Then

$$\begin{aligned}
&C_{\mathsf{K}}(\Theta \circ \sigma)\\
&\quad = \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \circ \sigma \leq E \circ h\}\\
&\quad \leq \bigwedge\{E \circ h \circ \sigma : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \circ \sigma \leq E \circ h \circ \sigma\}\\
&\quad \leq \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\} \circ \sigma\\
&\quad = C_{\mathsf{K}}(\Theta) \circ \sigma.
\end{aligned}$$

We have now shown that $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ is a G-2-logic. ■

$\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ is called the G-2-**logic determined by** or **induced by** the class K of G-algebras.

A G-set of equations $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ is said to be a G-2-**theory** or a G-**congruence** of $\mathcal{S}_{\mathsf{K}}$ if

$$C_{\mathsf{K}}(\Theta) = \Theta.$$

We denote the collection of all G-2-theories of $\mathcal{S}_{\mathsf{K}}$ by $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$.

The next lemma provides a reassurance that the name G-congruence for G-2-theories is well-deserved, since it does not conflict with previous terminology concerning G-congruences on $\mathcal{L}$-algebras.

Lemma 20 *Let K be a class of G-algebras.*

(a) The G-2-theories of $\mathcal{S}_{\mathsf{K}}$ are G-congruences on $\mathbf{Fm}_{\mathcal{L}}(V)$.

(b) Every G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K}$, is a G-2-theory of $\mathcal{S}_{\mathsf{K}}$.

Proof: Let $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then

$$\begin{aligned}
\Theta(\varphi, \varphi) &= C_{\mathsf{K}}(\Theta)(\varphi, \varphi)\\
&= \bigwedge\{E(h(\varphi), h(\varphi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K},\\
&\qquad\qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= \top.
\end{aligned}$$

Let $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then

$$\begin{aligned}
\Theta(\varphi, \psi) &= C_{\mathsf{K}}(\Theta)(\varphi, \psi)\\
&= \bigwedge\{E(h(\varphi), h(\psi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K},\\
&\qquad\qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= \bigwedge\{E(h(\psi), h(\varphi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K},\\
&\qquad\qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= C_{\mathsf{K}}(\Theta)(\psi, \varphi)\\
&= \Theta(\psi, \varphi).
\end{aligned}$$

Let $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then

$$\begin{aligned}
&\Theta(\varphi, \psi) \wedge \Theta(\psi, \chi)\\
&= C_{\mathsf{K}}(\Theta)(\varphi, \psi) \wedge C_{\mathsf{K}}(\Theta)(\psi, \chi)\\
&= \bigwedge\{E(h(\varphi), h(\psi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\}\\
&\qquad \wedge \bigwedge\{E(h(\psi), h(\chi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\}\\
&\le \bigwedge\{E(h(\varphi), h(\psi)) \wedge E(h(\psi), h(\chi)) :\\
&\qquad\qquad\qquad \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\}\\
&\le \bigwedge\{E(h(\varphi), h(\chi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\}\\
&= C_{\mathsf{K}}(\Theta)(\varphi, \chi)\\
&= \Theta(\varphi, \chi).
\end{aligned}$$

The Congruence property can be demonstrated similarly.

Suppose, that Θ is a G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K}$. Then

$$\begin{aligned}
C_{\mathsf{K}}(\Theta) &= \textstyle\bigwedge_{\mathcal{A}=\langle \mathbf{A}, E\rangle, h}\{E \circ h : \Theta \le E \circ h\}\\
&\le \Theta. \quad (\mathcal{A} = \langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K} \text{ and } h = i_{\mathbf{Fm}_{\mathcal{L}}(V)})
\end{aligned}$$

Hence Θ is a G-2-theory of $\mathcal{S}_{\mathsf{K}}$. ∎

The collection $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ of all G-congruences of $\mathcal{S}_{\mathsf{K}}$ forms a complete lattice when ordered by $\le$. It is denoted by $\mathbf{Th}(\mathcal{S}_{\mathsf{K}}) = \langle \mathrm{Th}(\mathcal{S}_{\mathsf{K}}), \le\rangle$.

Proposition 21 *Let* K *be a class of* G*-algebras. Then* $\mathbf{Th}(\mathcal{S}_{\mathsf{K}}) = \langle \mathrm{Th}(\mathcal{S}_{\mathsf{K}}), \le\rangle$ *is a complete lattice.*

Proof: Note, again, that $\top : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, with $\top(\varphi, \psi) = \top$, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, is an $\mathcal{S}_{\mathsf{K}}$-theory. Clearly, it is the largest $\mathcal{S}_{\mathsf{K}}$-theory under $\le$. We show closure under $\bigwedge$. Let $\{\Theta_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. We have

$$\begin{aligned}
C_{\mathsf{K}}(\textstyle\bigwedge \Theta_i) &= \textstyle\bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \bigwedge \Theta_i \le E \circ h\}\\
&\le \textstyle\bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta_i \le E \circ h\}\\
&= C_{\mathsf{K}}(\Theta_i)\\
&= \Theta_i.
\end{aligned}$$

Thus, $C_{\mathsf{K}}(\bigwedge \Theta_i) \le \bigwedge \Theta_i$. The reverse inclusion always holds. So $C_{\mathsf{K}}(\bigwedge \Theta_i) = \bigwedge \Theta_i$, showing that $\bigwedge \Theta_i \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. ∎

2.9 Graded Translations and Interpretations

A **translation** from G-formulas to G-equations is a join preserving mapping

$$\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}.$$

A **translation** from G-equations to G-formulas is a join preserving mapping

$$\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}.$$

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras.

An **interpretation** from $\mathcal{S}$ to $\mathcal{S}_{\mathsf{K}}$ is a translation

$$\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)},$$

such that, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Phi \leq C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

An **interpretation** from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$ is a translation

$$\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)},$$

such that, for all $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$E \leq C_{\mathsf{K}}(\Theta) \quad \text{iff} \quad \mathcal{F}(E) \leq C(\mathcal{F}(\Theta)).$$

The following propositions provide useful characterizations of interpretations.

Proposition 22 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras. Suppose $\mathcal{E}$ is a translation from G-formulas to G-equations. Then $\mathcal{E}$ is an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ if and only if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,*

$$C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))$$

and

$$C(\Gamma) = \bigwedge\{C(\Phi) : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(\Phi))\}.$$

Proof: Suppose that $\mathcal{E}$ is an interpretation. Then, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, we have

$$\begin{array}{rll}
\Gamma \leq C(C(\Gamma)) & \text{implies} & \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \\
& \text{implies} & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \leq C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))); \\
C(\Gamma) \leq C(\Gamma) & \text{implies} & \mathcal{E}(C(\Gamma)) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \\
& \text{implies} & C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)).
\end{array}$$

This proves the first displayed condition. For the second, we have

$$\begin{array}{rcl}
C(\Gamma) & = & \bigwedge\{C(\Phi) : \Gamma \leq C(\Phi)\} \\
& = & \bigwedge\{C(\Phi) : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(\Phi))\}.
\end{array}$$

Next, we turn to the converse. Suppose that the two displayed conditions hold and let $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{array}{lll} \Phi \le C(\Gamma) & \text{implies} & \mathcal{E}(\Phi) \le \mathcal{E}(C(\Gamma)) \\ & \text{implies} & \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \\ & \text{implies} & \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)). \end{array}$$

Further, if $\mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma))$, then

$$\begin{array}{lll} C(\Phi) & = & \bigwedge\{C(\Phi') : \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Phi'))\} \\ & \le & \bigwedge\{C(\Phi') : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(\Phi'))\} \\ & = & C(\Gamma). \end{array}$$

This shows that $\Phi \le C(\Gamma)$. Hence, the two displayed conditions are necessary and sufficient for a translation $\mathcal{E}$ to be an interpretation. ■

The dual statement is formalized as

Proposition 23 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras. Suppose $\mathcal{F}$ is a translation from G-equations to G-formulas. Then $\mathcal{F}$ is an interpretation $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ if and only if, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,*

$$C(\mathcal{F}(\Theta)) = C(\mathcal{F}(C_{\mathsf{K}}(\Theta))$$

and

$$C_{\mathsf{K}}(\Theta) = \bigwedge\{C_{\mathsf{K}}(E) : \mathcal{F}(\Theta) \le C(\mathcal{F}(E))\}.$$

Proof: The proof is dual to that of Proposition 22. ■

2.10 Slicing

Given a G-set of formulas $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and formula $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$, we define the G-set $\Gamma^\varphi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ by setting, for all $\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma^\varphi(\psi) = \begin{cases} \Gamma(\varphi), & \text{if } \psi = \varphi, \\ \bot, & \text{otherwise.} \end{cases}$$

Γ^φ is called the **instantiation of Γ to φ**. Similarly, given a G-set of equations $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and an equation $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}(V)$, we define the G-set $E^{\varphi,\psi} : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ by setting, for all $\delta \approx \varepsilon \in \mathrm{Eq}_{\mathcal{L}}(V)$,

$$E^{\varphi,\psi}(\delta, \varepsilon) = \begin{cases} E(\varphi, \psi), & \text{if } \langle \delta, \varepsilon \rangle = \langle \varphi, \psi \rangle, \\ \bot, & \text{otherwise.} \end{cases}$$

$E^{\varphi,\psi}$ is called the **instantiation of E to $\varphi \approx \psi$**.

Related to this notation, we also have the following.

Given a function $\Gamma : X \to G$, where $X \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, define $\widehat{\Gamma} : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ by setting, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\widehat{\Gamma}(\varphi) = \begin{cases} \Gamma(\varphi), & \text{if } \varphi \in X, \\ \bot, & \text{otherwise.} \end{cases}$$

$\widehat{\Gamma}$ is called the **lifting of** Γ.

Given a function $E : Y \to G$, where $Y \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$, define $\widehat{E} : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ by setting, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\widehat{E}(\varphi, \psi) = \begin{cases} E(\varphi, \psi), & \text{if } \langle \varphi, \psi \rangle \in Y, \\ \bot, & \text{otherwise.} \end{cases}$$

$\widehat{E}$ is called the **lifting of** E.

E.g., identifying the function $\Gamma = \{\langle \varphi, g \rangle\}$ with the pair it contains, we have, for all $\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\widehat{\langle \varphi, g \rangle}(\psi) = \begin{cases} g, & \text{if } \psi = \varphi, \\ \bot, & \text{otherwise.} \end{cases}$$

A **hybrid translation** from formulas to equations is a function

$$E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

that satisfies the following conditions, for all $\Gamma, \Gamma_i \in G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(**Bottom**) $E(\bot, \varphi) = \bot$;

(**Slicing**) $E(\Gamma, \varphi) = E(\Gamma^{\varphi}, \varphi)$;

(**Join Continuity**) $E(\bigvee_{i \in I} \Gamma_i, \varphi) = \bigvee_{i \in I} E(\Gamma_i, \varphi)$.

A **hybrid translation** from equations to formulas is a function

$$F : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

that satisfies the following conditions, for all $E, E_i \in G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(**Bottom**) $F(\bot, \langle \varphi, \psi \rangle) = \bot$;

(**Slicing**) $F(E, \langle \varphi, \psi \rangle) = F(E^{\varphi, \psi}, \langle \varphi, \psi \rangle)$;

(**Join Continuity**) $F(\bigvee_{i \in I} E_i, \langle \varphi, \psi \rangle) = \bigvee_{i \in I} F(E_i, \langle \varphi, \psi \rangle)$.

There is a close connection between translations and hybrid translations, which we now describe. We do this in full detail for translations from formulas to equations. The treatment for translations from equations to formulas, which is dual, is then only briefly described.

First, consider a translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. We define a function

$$\mathcal{E}^h : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{E}^h(\Gamma, \varphi) = \mathcal{E}(\Gamma^\varphi).$$

We show that $\mathcal{E}^h$ is a hybrid translation. First, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll} \mathcal{E}^h(\bot, \varphi) & = & \mathcal{E}(\bot^\varphi) & (\text{Definition of } \mathcal{E}^h) \\ & = & \mathcal{E}(\bot) & (\bot^\varphi = \bot) \\ & = & \bot. & (\mathcal{E} \text{ join continuous}) \end{array}$$

Second, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll} \mathcal{E}^h(\Gamma, \varphi) & = & \mathcal{E}(\Gamma^\varphi) & (\text{Definition of } \mathcal{E}^h) \\ & = & \mathcal{E}((\Gamma^\varphi)^\varphi) & ((\Gamma^\varphi)^\varphi = \Gamma^\varphi) \\ & = & \mathcal{E}(\Gamma^\varphi, \varphi). & (\text{Definition of } \mathcal{E}^h) \end{array}$$

Finally, for all $\Gamma_i : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll} \mathcal{E}^h(\bigvee_{i \in I} \Gamma_i, \varphi) & = & \mathcal{E}((\bigvee_{i \in I} \Gamma_i)^\varphi) & (\text{Definition of } \mathcal{E}^h) \\ & = & \mathcal{E}(\bigvee_{i \in I} \Gamma_i^\varphi) & ((\bigvee_{i \in I} \Gamma_i)^\varphi = \bigvee_{i \in I} \Gamma_i^\varphi) \\ & = & \bigvee_{i \in I} \mathcal{E}(\Gamma_i^\varphi) & (\mathcal{E} \text{ join continuous}) \\ & = & \bigvee_{i \in I} \mathcal{E}^h(\Gamma_i, \varphi). & (\text{Definition of } \mathcal{E}^h) \end{array}$$

Conversely, consider a hybrid translation $E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. Define a function

$$E^t : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$E^t(\Gamma) = \bigvee \{E(\Gamma^\varphi, \varphi) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V)\}.$$

We show that E^t is join continuous and, therefore, a translation. We have, for all $\Gamma_i : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll} E^t(\bigvee_{i \in I} \Gamma_i) & = & \bigvee_\varphi E((\bigvee_{i \in I} \Gamma_i)^\varphi, \varphi) & (\text{Definition of } E^t) \\ & = & \bigvee_\varphi E(\bigvee_{i \in I} \Gamma_i^\varphi, \varphi) & ((\bigvee_{i \in I} \Gamma_i)^\varphi = \bigvee_{i \in I} \Gamma_i^\varphi) \\ & = & \bigvee_\varphi \bigvee_{i \in I} E(\Gamma_i^\varphi, \varphi) & (E \text{ join continuous}) \\ & = & \bigvee_{i \in I} \bigvee_\varphi E(\Gamma_i^\varphi, \varphi) & (\text{Identical joins}) \\ & = & \bigvee_{i \in I} E^t(\Gamma_i). & (\text{Definition of } E^t) \end{array}$$

Finally, we show that the two processes are inverses of each other. We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll}\mathcal{E}^{ht}(\Gamma) & = & \bigvee_\psi \mathcal{E}^h(\Gamma^\psi, \psi) & (\text{Definition of } \mathcal{E}^{ht}) \\ & = & \bigvee_\psi \mathcal{E}((\Gamma^\psi)^\psi) & (\text{Definition of } \mathcal{E}^h) \\ & = & \bigvee_\psi \mathcal{E}(\Gamma^\psi) & ((\Gamma^\psi)^\psi = \Gamma^\psi) \\ & = & \mathcal{E}(\bigvee_\psi \Gamma^\psi) & (\mathcal{E} \text{ join continuous}) \\ & = & \mathcal{E}(\Gamma). & (\bigvee_\psi \Gamma^\psi = \Gamma)\end{array}$$

Finally, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll}E^{th}(\Gamma, \varphi) & = & E^t(\Gamma^\varphi) & (\text{Definition of } E^{th}) \\ & = & \bigvee_\psi E((\Gamma^\varphi)^\psi, \psi) & (\text{Definition of } E^t) \\ & = & E(\Gamma^\varphi, \varphi) & (\text{Cases and Bottom}) \\ & = & E(\Gamma, \varphi). & (\text{Slicing})\end{array}$$

We close the section by describing briefly the dual scenario concerning translations from equations to formulas.

Suppose we are given a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ from G-equations to G-formulas. Then we define a hybrid translation

$$\mathcal{F}^h : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by setting, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{F}^h(E, \langle \varphi, \psi \rangle) = \mathcal{F}(E^{\varphi,\psi}).$$

Conversely, given a hybrid translation $F : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, we define a translation

$$F^t : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by setting, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$F^t(E) = \bigvee_{\varphi,\psi} F(E^{\varphi,\psi}, \langle \varphi, \psi \rangle).$$

Then, as before, $\mathcal{F}^h$ is a hybrid translation, F^t is a translation and

$$\mathcal{F}^{ht} = \mathcal{F} \quad \text{and} \quad F^{th} = F,$$

that is, the two processes are inverses of one another.

We conclude that considering translations or their corresponding hybrid versions is only a matter of convenience (and/or taste) depending on context, since they are interchangeable.

2.11 Structural Translations and Slicing

A translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ from G-formulas to G-equations is said to be **structural** if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{E}(\Gamma^{\sigma(\varphi)}) = \mathcal{E}(\Gamma^{\varphi}) \circ \sigma^2.$$

A translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ from G-equations to G-formulas is said to be **structural** if, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, all $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{F}(E^{\sigma(\varphi),\sigma(\psi)}) = \mathcal{F}(E^{\varphi,\psi}) \circ \sigma.$$

A hybrid translation from formulas to equations

$$E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

is **structural** if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

(**Structurality**) $E(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) = E(\Gamma^{\varphi}, \varphi) \circ \sigma^2$.

A hybrid translation from equations to formulas

$$F : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

is **structural** if, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

(**Structurality**) $F(E^{\sigma(\varphi),\sigma(\psi)}, \langle \sigma(\varphi), \sigma(\psi) \rangle) = F(E^{\varphi,\psi}, \langle \varphi, \psi \rangle) \circ \sigma$.

Let us note, here, that in the preceding definitions, one could relax structurality to surjective structurality without losing any power. Surjective structurality stipulates that the structurality condition hold for all surjective substitutions $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

We want to show that the correspondence established in Section 2.10 between translations and hybrid translations extends to one between structural translations and structural hybrid translations. We deal again only with translations from formulas to equations, since translations from equations to formulas are handled dually. Moreover, we only check structurality, since all other properties were checked in detail in Section 2.10.

Consider a structural translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. We show that the hybrid translation

$$\mathcal{E}^h : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

defined, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\mathcal{E}^h(\Gamma, \varphi) = \mathcal{E}(\Gamma^\varphi),$$

is also structural. We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll}
\mathcal{E}^h(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) & = & \mathcal{E}((\Gamma^{\sigma(\varphi)})^{\sigma(\varphi)}) & \text{(Definition of } \mathcal{E}^h) \\
& = & \mathcal{E}(\Gamma^{\sigma(\varphi)}) & ((\Gamma^{\sigma(\varphi)})^{\sigma(\varphi)} = \Gamma^{\sigma(\varphi)}) \\
& = & \mathcal{E}(\Gamma^\varphi) \circ \sigma^2 & (\mathcal{E} \text{ structural}) \\
& = & \mathcal{E}((\Gamma^\varphi)^\varphi) \circ \sigma^2 & ((\Gamma^\varphi)^\varphi = \Gamma^\varphi) \\
& = & \mathcal{E}^h(\Gamma^\varphi, \varphi) \circ \sigma^2. & \text{(Definition of } \mathcal{E}^h)
\end{array}$$

Consider, next, a structural hybrid translation $E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. We show that the translation

$$E^t : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

defined, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, by

$$E^t(\Gamma) = \bigvee \{E(\Gamma^\varphi, \varphi) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V)\}$$

is structural. We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll}
E^t(\Gamma^\varphi) \circ \sigma^2 & = & \bigvee_\psi E((\Gamma^\varphi)^\psi, \psi) \circ \sigma^2 & \text{(Definition of } E^t) \\
& = & E(\Gamma^\varphi, \varphi) \circ \sigma^2 & \text{(Cases and Bottom)} \\
& = & E(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) & (E \text{ structural}) \\
& = & \bigvee_\psi E((\Gamma^{\sigma(\varphi)})^\psi, \psi) & \text{(Cases and Bottom)} \\
& = & E^t(\Gamma^{\sigma(\varphi)}). & \text{(Definition of } E^t)
\end{array}$$

Since the definitions of ^{h} and of ^{t} are the same as in the preceding section, we know, based on our work there, that the two processes are inverses of each other.

2.12 Algebraic and Matrix Semantics

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras.

We say that K is a G-**algebraic semantics** for $\mathcal{S}$ if there exists an interpretation $\mathcal{E}$ from $\mathcal{S}$ to $\mathcal{S}_{\mathsf{K}}$.

We say that $\mathcal{S}$ is a G-**logical semantics** for K if there exists an interpretation $\mathcal{F}$ from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$.

For specific types of translations, these notions are connected with some concepts encountered earlier.

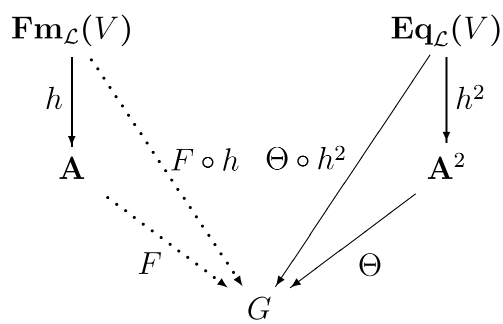
We say that a translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ from G-formulas to G-equations is **order reflecting** if, for all $\Gamma, \Gamma' : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\mathcal{E}(\Gamma) \leq \mathcal{E}(\Gamma') \quad \text{implies} \quad \Gamma \leq \Gamma'.$$

Since, by definition, $\mathcal{E}$ is join continuous, it is, a fortiori, order preserving, whence this condition may be equivalently expressed as a biconditional, i.e., for all $\Gamma, \Gamma' : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Gamma \leq \Gamma' \quad \text{iff} \quad \mathcal{E}(\Gamma) \leq \mathcal{E}(\Gamma').$$

Further, the G-translation $\mathcal{E}$ is called **reflectively structural** if, for every algebra $\mathbf{A}$ and every G-congruence $\Theta : A^2 \to G$ on $\mathbf{A}$, there exists a G-filter $F : A \to G$ on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,



$$\mathcal{E}(F \circ h) = \Theta \circ h^2.$$

This property, when present, allows one to construct, given a G-congruence on an algebra $\mathbf{A}$, a "corresponding" G-filter F on $\mathbf{A}$, where "corresponding" here means that they are connected via the displayed equation. Exploiting this, given a reflectively structural translation $\mathcal{E}$ and a G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$, we define the G-matrix

$$\mathcal{A}^{\mathcal{E}} = \langle \mathbf{A}, E^{\mathcal{E}} \rangle,$$

where $E^{\mathcal{E}}$ is the G-filter on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\mathcal{E}(E^{\mathcal{E}} \circ h) = E \circ h^2.$$

Then, given a class K of G-algebras, we define the class $\mathsf{K}^{\mathcal{E}}$ of G-matrices by

$$\mathsf{K}^{\mathcal{E}} = \{\mathcal{A}^{\mathcal{E}} : \mathcal{A} \in \mathsf{K}\}.$$

Theorem 24 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic, K be a class of G-algebras and $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ a reflectively structural, order reflecting G-translation. K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$ if and only if $\mathsf{K}^{\mathcal{E}}$ is a G-matrix semantics for $\mathcal{S}$.*

Proof: We have the following equivalences, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll}
\mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)) & \text{iff} & \mathcal{E}(\Phi) \le \bigwedge_{\mathcal{A},h}\{E \circ h^2 : \mathcal{E}(\Gamma) \le E \circ h^2\} \\
& & \text{(Definition of } C_{\mathsf{K}}) \\
& \text{iff} & \mathcal{E}(\Phi) \le \bigwedge_{\mathcal{A},h}\{\mathcal{E}(E^{\mathcal{E}} \circ h) : \mathcal{E}(\Gamma) \le \mathcal{E}(E^{\mathcal{E}} \circ h)\} \\
& & \text{(Refl. Struct. and Definition of } E^{\mathcal{E}}) \\
& \text{iff} & \Phi \le \bigwedge_{\mathcal{A},h}\{E^{\mathcal{E}} \circ h : \Gamma \le E^{\mathcal{E}} \circ h\} \\
& & \text{(Order Reflectivity)} \\
& \text{iff} & \Phi \le C_{\mathsf{K}^{\mathcal{E}}}(\Gamma). \quad \text{(Definition of } C_{\mathsf{K}^{\mathcal{E}}})
\end{array}$$

We now obtain that $\mathsf{K}^{\mathcal{E}}$ is a G-matrix semantics for $\mathcal{S}$ if and only if $C = C_{\mathsf{K}^{\mathcal{E}}}$ if and only if, using the preceding equalities and the definition, K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$. ■

Of course, a dual treatment leads to a dual theorem. Let us briefly recount the basic conditions and steps for the sake of completeness.

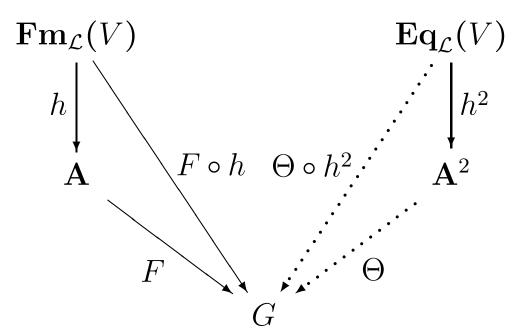
We say that a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ from G-equations to G-formulas is **order reflecting** if, for all $E, E' : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$\mathcal{F}(E) \le \mathcal{F}(E') \quad \text{implies} \quad E \le E'.$$

Again, taking into account the join continuity of $\mathcal{F}$, we may equivalently write, for all $\Gamma, \Gamma' : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$E \le E' \quad \text{iff} \quad \mathcal{F}(E) \le \mathcal{F}(E').$$

Further, the translation $\mathcal{F}$ is called **reflectively structural** if, for every algebra $\mathbf{A}$ and every G-filter $F : A \to G$ on $\mathbf{A}$, there exists a G-congruence $\Theta : A^2 \to G$ on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,



$$\mathcal{F}(\Theta \circ h^2) = F \circ h.$$

If $\mathcal{F}$ is reflectively structural, given a G-filter on an algebra $\mathbf{A}$, there exists a "corresponding" G-congruence Θ on $\mathbf{A}$ satisfying the displayed equation. Exploiting this, given a reflectively structural translation $\mathcal{F}$ and a G-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$, we define the G-algebra

$$\mathfrak{A}^{\mathcal{F}} = \langle \mathbf{A}, F^{\mathcal{F}} \rangle,$$

where $F^{\mathcal{F}}$ is the G-congruence on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\mathcal{F}(F^{\mathcal{F}} \circ h^2) = F \circ h.$$

Then, given a class M of G-matrices, we define the class $\mathsf{M}^{\mathcal{F}}$ of G-algebras by

$$\mathsf{M}^{\mathcal{F}} = \{\mathfrak{A}^{\mathcal{F}} : \mathfrak{A} \in \mathsf{M}\}.$$

Theorem 25 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, K be a class of G-algebras and $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ a reflectively structural, order reflecting translation. C_{M} is a G-logical semantics for K via $\mathcal{F}$ if and only if $\mathsf{M}^{\mathcal{F}}$ is a G-2-matrix semantics for $\mathcal{S}_{\mathsf{K}}$.*

Proof: Dual to the proof of Theorem 24. ■

It can be seen that, as in the classical framework, if the G-logic has a G-algebraic semantics, then $\mathcal{S}$ must exhibit some special characteristics inherited from the G-2-logic $\mathcal{S}_{\mathsf{K}}$ of the G-algebraic semantics K. In the classical framework of Blok and Pigozzi, this property is exploited, e.g., in Theorem 2.7 of [6]. After some deliberation, its statement and proof may be seen to be a special case of the following result.

Theorem 26 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via the interpretation $\mathcal{E}$. Then, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,*

$$\bigvee\{\Phi : \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma))\} \leq C(\Gamma).$$

Proof: Suppose $\mathcal{S} = \langle \mathcal{L}, C\rangle$ is a G-logic and K a G-algebraic semantics for $\mathcal{S}$ via the interpretation $\mathcal{E}$. Fix $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and let $\Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ be arbitrary, such that $\mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma))$. Since K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$, we get $\Phi \leq C(\Gamma)$. Hence, taking joins over all such Φ, we get

$$\bigvee\{\Phi : \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma))\} \leq C(\Gamma),$$

which is the inequality in the statement. ■

2.13 Equivalent Graded Algebraic Semantics

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K a G-algebraic semantics for $\mathcal{S}$, via the interpretation $\mathcal{E}$, that is, we have, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Phi \leq C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

K is said to be **equivalent to** $\mathcal{S}$ if there exists a translation $\mathcal{F}$ from G-equations to G-fomulas, such that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$C_{\mathsf{K}}(\Theta) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))).$$

In this case, $\mathcal{F}$ is said to be **inverse** to $\mathcal{E}$.

We have the following characterization of an equivalent G-algebraic semantics, paralleling the one proved in the classical case in Corollary 2.9 of [6]. Among other things, it shows that the roles of the interpretation $\mathcal{E}$ and the translation $\mathcal{F}$, establishing the equivalence, are completely symmetric.

Proposition 27 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K an equivalent G-algebraic semantics for $\mathcal{S}$, via an interpretation $\mathcal{E}$ and a translation $\mathcal{F}$. Then, for all $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, and all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,*

(i) $E \leq C_{\mathsf{K}}(\Theta)$ *iff* $\mathcal{F}(E) \leq C(\mathcal{F}(\Theta))$;

(ii) $C(\Gamma) = C(\mathcal{F}(\mathcal{E}(\Gamma)))$.

Conversely, if there exist translations $\mathcal{E}$ and $\mathcal{F}$ satisfying Conditions (i) and (ii), then K is equivalent to $\mathcal{S}$ via the interpretations $\mathcal{E}$ and $\mathcal{F}$.

Proof: Suppose, first, that K is equivalent to $\mathcal{S}$ via an interpretation $\mathcal{E}$ and a translation $\mathcal{F}$. Then, for all $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, we have

$$\begin{array}{lll} E \leq C_{\mathsf{K}}(\Theta) & \text{iff} & \mathcal{E}(\mathcal{F}(E)) \leq C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) \\ & \text{iff} & \mathcal{F}(E) \leq C(\mathcal{F}(\Theta)). \end{array}$$

Moreover, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, we have

$$\begin{array}{lll} C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(\Gamma)) & \text{iff} & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\mathcal{E}(\Gamma)))) \\ & \text{iff} & C(\Gamma) = C(\mathcal{F}(\mathcal{E}(\Gamma))). \end{array}$$

Conversely, if Conditions (i) and (ii) hold, then one can prove similarly that, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Phi \in C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma))$$

and, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$C_{\mathsf{K}}(\Theta) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))).$$

Thus, K is equivalent to $\mathcal{S}$ via $\mathcal{E}$ and $\mathcal{F}$ iff Conditions (i) and (ii) hold. ■

We say that a G-logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$ is **algebraizable** if it has an equivalent G-algebraic semantics.

Lemma 2.13 of [6] shows how one may take advantage of equivalence to translate properties of a class of algebras into properties of the logic. Its abstraction to the graded setting yields a kind of dual result to Theorem 26.

Lemma 28 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K an equivalent G-algebraic semantics of $\mathcal{S}$ via the interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for every G-congruence Θ of $\mathcal{S}_{\mathsf{K}}$,*

$$\bigvee\{E : \mathcal{F}(E) \leq C(\mathcal{F}(\Theta))\} = \Theta.$$

Proof: Let Θ be a G-2-theory of $\mathcal{S}_{\mathsf{K}}$. Then, for any $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, such that $\mathcal{F}(E) \leq C(\mathcal{F}(\Theta))$, we have $E \leq C_K(\Theta) = \Theta$. Therefore,

$$\bigvee\{E : \mathcal{F}(E) \leq C(\mathcal{F}(\Theta))\} \leq \Theta.$$

On the other hand, observe that Θ is a member of the set on the left, since $\mathcal{F}(\Theta) \leq C(\mathcal{F}(\Theta)$. Thus, the displayed equality holds. ∎

We now specialize this result to get an analog of Lemma 2.13 of [6].

Corollary 29 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K an equivalent G-algebraic semantics via interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for every G-congruence Θ on $\mathbf{Fm}_{\mathcal{L}}(V)$, all $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\xi(x) \in \mathrm{Fm}_{\mathcal{L}}(V)$:*

(a) $\mathcal{F}(\Theta^{\varphi,\varphi}) \leq C(\bot)$;

(b) $\mathcal{F}(\Theta^{\psi,\varphi}) \leq C(\mathcal{F}(\Theta^{\varphi,\psi}))$;

(c) $\mathcal{F}(\Theta^{\varphi,\chi}) \leq C(\mathcal{F}(\Theta^{\varphi,\psi}) \vee \mathcal{F}(\Theta^{\psi,\chi}))$;

(d) $\mathcal{F}(\Theta^{\xi(\varphi),\xi(\psi)}) \leq C(\mathcal{F}(\Theta^{\varphi,\psi}))$.

Proof: Let $\mathsf{A}_{\mathcal{L}}$ be the class of all G-algebras of type $\mathcal{L}$ and consider $\mathcal{S}_{\mathsf{A}_{\mathcal{L}}} = \langle \mathcal{L}, C_{\mathsf{A}_{\mathcal{L}}} \rangle$. Then, it can be shown that, for all $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\xi(x) \in \mathrm{Fm}_{\mathcal{L}}(V)$:

(a) $\Theta^{\varphi,\varphi} \leq C_{\mathsf{A}_{\mathcal{L}}}(\bot)$;

(b) $\Theta^{\psi,\varphi} \leq C_{\mathsf{A}_{\mathcal{L}}}(\Theta^{\varphi,\psi})$;

(c) $\Theta^{\varphi,\chi} \leq C_{\mathsf{A}_{\mathcal{L}}}(\Theta^{\varphi,\psi} \vee \Theta^{\psi,\chi})$;

(d) $\Theta^{\xi(\varphi),\xi(\psi)} \leq C_{\mathsf{A}_{\mathcal{L}}}(\Theta^{\varphi,\psi})$.

Taking, first, into account that $C_{\mathsf{A}_{\mathcal{L}}} \leq C_{\mathsf{K}}$ and then the fact that $\mathcal{F}$ is a G-interpretation, we obtain the conclusions in the statement. ∎

For the next two lemmas, we must specialize to specific types of interpretations that mimic the ones employed in the deductive system framework by Blok and Pigozzi [6].

Let us call an interpretation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ **standard** if, there exists a set

$$\delta(x) \approx \varepsilon(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \in I\}$$

of equations in a single variable x, such that, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g \in G$,

$$\mathcal{E}(\widehat{\langle\varphi, g\rangle}) = \widehat{\langle\delta(\varphi) \approx \varepsilon(\varphi), g\rangle},$$

where

$$\widehat{\langle\delta(\varphi) \approx \varepsilon(\varphi), g\rangle} := \widehat{\{\langle\langle\delta_i(\varphi), \varepsilon_i(\varphi)\rangle, g\rangle : i \in I\}}.$$

Similarly, a G-interpretation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ is called **standard** if, there exists a set

$$\Delta(x, y) = \{\Delta_j(x, y) : j \in J\}$$

of formulas in two variables x and y, such that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g \in G$,

$$\mathcal{F}(\widehat{\langle\varphi \approx \psi, g\rangle}) = \widehat{\langle\Delta(\varphi, \psi), g\rangle},$$

where

$$\widehat{\langle\Delta(\varphi, \psi), g\rangle} := \widehat{\{\langle\Delta_j(\varphi, \psi), g\rangle : j \in J\}}.$$

Lemma 30 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic and K an equivalent G-algebraic semantics via standard interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g, g' \in G$,*

$$\widehat{\langle\psi, g \wedge g'\rangle} \le C(\widehat{\langle\Delta(\varphi, \psi), g\rangle, \langle\varphi, g'\rangle}).$$

Proof: First, note that by properties of equational consequence, we get

$$\widehat{\langle\delta(\psi) \approx \varepsilon(\psi), g \wedge g'\rangle} \le C_{\mathsf{K}}(\widehat{\langle\varphi \approx \psi, g\rangle, \langle\delta(\varphi) \approx \varepsilon(\varphi), g'\rangle}).$$

Therefore, interpreting through the standard $\mathcal{F}$,

$$\widehat{\langle\Delta(\delta(\psi), \varepsilon(\psi)), g \wedge g'\rangle} \le C(\widehat{\langle\Delta(\varphi, \psi), g\rangle, \langle\Delta(\delta(\varphi), \varepsilon(\varphi)), g'\rangle}).$$

Hence, by the equivalence of the semantics,

$$\widehat{\langle\psi, g \wedge g'\rangle} \le C(\widehat{\langle\Delta(\varphi, \psi), g\rangle, \langle\varphi, g'\rangle}).$$

This proves the displayed equation of the statement. ∎

Lemma 31 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic. Suppose K is an equivalent G-algebraic semantics for $\mathcal{S}$ via standard interpretations $\mathcal{E}$ and $\mathcal{F}$, and K' is an equivalent G-algebraic semantics for $\mathcal{S}$ via standard interpretations $\mathcal{E}'$ and $\mathcal{F}'$. Then, all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g \in G$,*

$$C(\mathcal{F}(\widehat{\langle\varphi \approx \psi, g\rangle})) = C(\mathcal{F}'(\widehat{\langle\varphi \approx \psi, g\rangle})).$$

Proof: Using Part (d) of Corollary 29, we have

$$\widehat{\langle\Delta'(\varphi,\varphi)\approx\Delta'(\varphi,\psi),g\rangle}\leq C_{\mathsf{K}}(\widehat{\langle\varphi\approx\psi,g\rangle}).$$

By applying the standard interpretation $\mathcal{F}$, we get

$$\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle}\leq C(\widehat{\langle\Delta(\varphi,\psi),g\rangle}).$$

By Lemma 30,

$$\widehat{\langle\Delta'(\varphi,\psi),g\rangle}\leq C(\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle,\langle\Delta'(\varphi,\varphi),\top\rangle}).$$

By Part (a) of Corollary 29,

$$\widehat{\langle\Delta'(\varphi,\varphi),\top\rangle}\leq C(\bot).$$

Putting all these pieces together, we get

$$\begin{array}{rcl}\widehat{\langle\langle\Delta'(\varphi,\psi),g\rangle\rangle} & \leq & C(\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle,\langle\Delta'(\varphi,\varphi),\top\rangle})\\ & \leq & C(\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle})\\ & \leq & C(\widehat{\langle\Delta(\varphi,\psi),g\rangle}).\end{array}$$

Translating in the language of interpretations, we get that

$$\mathcal{F}'(\widehat{\langle\varphi\approx\psi,g\rangle})\leq C(\mathcal{F}(\widehat{\langle\varphi\approx\psi,g\rangle}).$$

By symmetry, we obtain the required equality. ∎

It turns out, that, for any G-logic $\mathcal{S}$ for which any two G-equivalent algebraic semantics K via interpretations $\mathcal{E}$, $\mathcal{F}$ and K' via interpretations $\mathcal{E}'$, $\mathcal{F}'$ satisfy, for all $\Theta:\mathrm{Eq}_{\mathcal{L}}(V)\to G$,

$$C(\mathcal{F}(\Theta))=C(\mathcal{F}'(\Theta)),$$

can be algebraized in an essentially unique way. It is conjectured that unique algebraization without proviso, as proved by Blok and Pigozzi in Theorem 2.15 of [6] for ordinary deductive systems, does not extend to arbitrary algebraizable G-logics.

Proposition 32 *Let $\mathcal{S}=\langle\mathcal{L},C\rangle$ be a G-logic. Suppose K is an equivalent G-algebraic semantics of $\mathcal{S}$, via interpretations $\mathcal{E}$ and $\mathcal{F}$, and K' is an equivalent G-algebraic semantics of $\mathcal{S}$, via interpretations $\mathcal{E}'$ and $\mathcal{F}'$, such that, for all $\Theta:\mathrm{Eq}_{\mathcal{L}}(V)\to G$, and all $\varphi,\psi\in\mathrm{Fm}_{\mathcal{L}}(V)$,*

$$C(\mathcal{F}(\Theta^{\varphi,\psi}))=C(\mathcal{F}'(\Theta^{\varphi,\psi})).$$

Then the following hold:

(i) $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$, *for all* $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$;

(ii) $\mathcal{S}_{\mathsf{K}} = \mathcal{S}_{\mathsf{K}'}$;

(iii) $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma))$, *for all* $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$.

Proof: First, let $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then we obtain

$$\begin{array}{rcll}
C(\mathcal{F}(\Theta)) & = & C(\mathcal{F}(\bigvee_{\varphi,\psi} \Theta^{\varphi,\psi})) & (\Theta = \bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}) \\
& = & C(\bigvee_{\varphi,\psi}(\mathcal{F}(\Theta^{\varphi,\psi}))) & (\mathcal{F} \text{ join continuous}) \\
& = & \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S})} C(\mathcal{F}(\Theta^{\varphi,\psi})) & (\text{Property of theories}) \\
& = & \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S})} C(\mathcal{F}'(\Theta^{\varphi,\psi})) & (\text{Hypothesis}) \\
& = & C(\bigvee_{\varphi,\psi}(\mathcal{F}'(\Theta^{\varphi,\psi}))) & (\text{Property of theories}) \\
& = & C(\mathcal{F}'(\bigvee_{\varphi,\psi} \Theta^{\varphi,\psi})) & (\mathcal{F}' \text{ join continuous}) \\
& = & C(\mathcal{F}'(\Theta)). & (\Theta = \bigvee_{\varphi,\psi} \Theta^{\varphi,\psi})
\end{array}$$

Now, let $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then, we have

$$\begin{array}{rll}
E \leq C_{\mathsf{K}}(\Theta) & \text{iff} & \mathcal{F}(E) \leq C(\mathcal{F}(\Theta)) \quad (\mathcal{F} \text{ an interpretation}) \\
& \text{iff} & \mathcal{F}'(E) \leq C(\mathcal{F}'(\Theta)) \quad (\text{Part (i)}) \\
& \text{iff} & E \leq C_{\mathsf{K}'}(\Theta). \quad (\mathcal{F}' \text{ an interpretation})
\end{array}$$

Next, suppose $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then

$$\begin{array}{rll}
C(\Gamma) = C(\Gamma) & \text{iff} & C(\mathcal{F}(\mathcal{E}(\Gamma))) = C(\mathcal{F}'(\mathcal{E}'(\Gamma))) \\
& & (\mathcal{E}, \mathcal{F} \text{ and } \mathcal{E}', \mathcal{F}' \text{ inverse interpretations}) \\
& \text{iff} & C(\mathcal{F}(\mathcal{E}(\Gamma))) = C(\mathcal{F}(\mathcal{E}'(\Gamma))) \quad (\text{Part (i)}) \\
& \text{iff} & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma)). \quad (\mathcal{F} \text{ an interpretation})
\end{array}$$

■

Lemma 31 and Proposition 32 ensure that, if a G-logic is algebraizable solely via standard interpretations, then it is algebraizable in an essentially unique way.

Theorem 33 (Special Uniqueness) *Let* $\mathcal{S} = \langle \mathcal{L}, C \rangle$ *be a* G*-logic,* K *an equivalent* G*-algebraic semantics for* $\mathcal{S}$ *via standard interpretations* $\mathcal{E}, \mathcal{F}$ *and* K' *an equivalent* G*-algebraic semantics for* $\mathcal{S}$ *via standard interpretations* $\mathcal{E}', \mathcal{F}'$*. Then the following hold:*

(i) $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$, *for all* $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$;

(ii) $\mathcal{S}_{\mathsf{K}} = \mathcal{S}_{\mathsf{K}'}$;

(iii) $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma))$, *for all* $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$.

Proof: Statement (i) is given by Lemma 31. Then, Statements (ii) and (iii) are given by Proposition 32. ■

2.14 The Lattice of Theories

Recall that a given class K of G-algebras induces a G-2-logic $\mathcal{S}_{\mathsf{K}}$. Finitarity is defined by analogy with G-logics, that is, $\mathcal{S}_{\mathsf{K}}$ is **finitary** if, for every $\Theta : \mathrm{Eq}_{\mathcal{L}}(V)$,

$$C_{\mathsf{K}}(\Theta) = \bigvee_{Z \leq_f \Theta} C_{\mathsf{K}}(Z).$$

We also have the following analogs of Lemmas 3 and 4 for the case of a G-2-logic $\mathcal{S}_{\mathsf{K}}$ induced by a class K of G-algebras.

Lemma 34 *Let K be a class of G-algebras. Then, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$\Theta \circ \sigma \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}}).$$

Proof: Using Structurality of $\mathcal{S}_{\mathsf{K}}$ (Proposition 19) and the fact that $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, we get

$$C_{\mathsf{K}}(\Theta \circ \sigma) \leq C_{\mathsf{K}}(\Theta) \circ \sigma = \Theta \circ \sigma.$$

Since the reverse inclusion always holds, $C_{\mathsf{K}}(\Theta \circ \sigma) = \Theta \circ \sigma$. Therefore, $\Theta \circ \sigma$ is a theory of $\mathcal{S}_{\mathsf{K}}$. ∎

Lemma 35 *Let K be a class of G-algebras. Then, for all $\{\Theta_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$\bigvee_{i \in I} (\Theta_i \circ \sigma) = \left(\bigvee_{i \in I} \Theta_i \right) \circ \sigma.$$

Proof: This follows directly from the definitions involved. We have, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{aligned}
&(\textstyle\bigvee_{i \in I}(\Theta_i \circ \sigma))(\varphi, \psi) \\
&\quad = \textstyle\bigvee_{i \in I}((\Theta_i \circ \sigma)(\varphi, \psi)) \quad (\text{Definition of } \textstyle\bigvee_{i \in I}(\Theta_i \circ \sigma)) \\
&\quad = \textstyle\bigvee_{i \in I} \Theta_i(\sigma(\varphi), \sigma(\psi)) \quad (\text{Definition of } \circ) \\
&\quad = (\textstyle\bigvee_{i \in I} \Theta_i)(\sigma(\varphi), \sigma(\psi)) \quad (\text{Definition of } \textstyle\bigvee_{i \in I} \Theta_i) \\
&\quad = ((\textstyle\bigvee_{i \in I} \Theta_i) \circ \sigma)(\varphi, \psi). \quad (\text{Definition of } \circ)
\end{aligned}$$

Therefore, $\bigvee_{i \in I}(\Theta_i \circ \sigma) = (\bigvee_{i \in I} \Theta_i) \circ \sigma$. ∎

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$. We define two functions

$$\begin{array}{rccc}
H_{\mathsf{K}} : & \mathrm{Th}(\mathsf{S}_{\mathsf{K}}) & \to & \mathrm{Th}(\mathcal{S}), \\
\Omega_{\mathsf{K}} : & \mathrm{Th}(\mathcal{S}) & \to & \mathrm{Th}(\mathcal{S}_{\mathsf{K}}).
\end{array}$$

First, regarding H_{K}, we define, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$,

$$H_{\mathsf{K}}(\Theta) = \bigvee \{\Gamma \in G^{\mathrm{Fm}_{\mathcal{L}}(V)} : \mathcal{E}(\Gamma) \leq \Theta\}.$$

We show that $H_{\mathsf{K}}(\Theta) \in \mathrm{Th}(\mathcal{S})$. Indeed, we have

$$\begin{array}{rcl} C(H_{\mathsf{K}}(\Theta)) & = & \bigvee\{\Gamma : \Gamma \leq C(H_{\mathsf{K}}(\Theta))\} \quad \text{(Definition of join)} \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(H_{\mathsf{K}}(\Theta)))\} \quad (\mathcal{E} \text{ an interpretation}) \\ & \leq & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\Theta)\} \quad (\mathcal{E}(H_{\mathsf{K}}(\Theta)) \leq \Theta) \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Theta\} \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \\ & = & H_{\mathsf{K}}(\Theta). \quad \text{(Definition of } H_{\mathsf{K}}(\Theta)) \end{array}$$

Let us also show that, equivalently, $H_{\mathsf{K}}(\Theta)$ may be defined by

$$H_{\mathsf{K}}(\Theta) = \bigvee^{\mathbf{Th}(\mathcal{S})}\{T \in \mathrm{Th}(\mathcal{S}) : \mathcal{E}(T) \leq \Theta\}.$$

To see this, first note that, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rll} \mathcal{E}(\Gamma) \leq \Theta & \text{iff} & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \leq \Theta \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \\ & \text{iff} & C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \leq \Theta \quad \text{(Proposition 22)} \\ & \text{iff} & \mathcal{E}(C(\Gamma)) \leq \Theta. \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \end{array}$$

Therefore, we have

$$\begin{array}{rcl} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Theta\} \quad \text{(Definition of } H_{\mathsf{K}}(\Theta)) \\ & \leq & \bigvee\{C(\Gamma) : \mathcal{E}(C(\Gamma)) \leq \Theta\} \quad \text{(Inflationarity)} \\ & \leq & C(\bigvee\{C(\Gamma) : \mathcal{E}(C(\Gamma)) \leq \Theta\}) \quad \text{(Inflationarity)} \\ & \leq & C(\bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Theta\}) \quad \text{(Monotonicity)} \\ & = & H_{\mathsf{K}}(\Theta). \quad \text{(Definition of } H_{\mathsf{K}}(\Theta) \in \mathrm{Th}(\mathcal{S})) \end{array}$$

Thus,

$$H_{\mathsf{K}}(\Theta) = C(\bigvee\{C(\Gamma) : \mathcal{E}(C(\Gamma)) \leq \Theta\}) = \bigvee^{\mathbf{Th}(\mathcal{S})}\{T : \mathcal{E}(T) \leq \Theta\}.$$

Next, regarding Ω_{K}, define, for all $T \in \mathrm{Th}(\mathcal{S})$,

$$\Omega_{\mathsf{K}}(T) = C_{\mathsf{K}}(\mathcal{E}(T)).$$

We have shown that H_{K} is well defined and it is clear that Ω_{K} is also well defined, that is, they map theories into theories. Moreover, both H_{K} and Ω_{K} depend on $\mathcal{E}$. On the other hand, Theorem 33 partially (but, unfortunately, slightly) alleviates the pain, as it shows that for algebraizability via standard interpretations, all possible standard interpretations result in the same operators H_{K} and Ω_{K}.

Note that, (almost) by definition, each of H_{K} and Ω_{K} is order preserving. We have, for all $\Theta, \Theta' \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, such that $\Theta \leq \Theta'$, and all $T, T' \in \mathrm{Th}(\mathcal{S})$, such that $T \leq T'$,

$$\begin{array}{rcl} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Theta\} \quad \text{(Definition of } H_{\mathsf{K}}) \\ & \leq & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Theta'\} \quad (\Theta \leq \Theta') \\ & = & H_{\mathsf{K}}(\Theta') \quad \text{(Definition of } H_{\mathsf{K}}) \end{array}$$

and, further,

$$\begin{array}{rcll}\Omega_{\mathsf{K}}(T) & = & C_{\mathsf{K}}(\mathcal{E}(T)) & \text{(Definition of } \Omega_{\mathsf{K}}) \\ & \leq & C_{\mathsf{K}}(\mathcal{E}(T')) & (T \leq T', \text{ Join Continuity of } \mathcal{E}, \\ & & & \text{Monotonicity of } C_{\mathsf{K}}) \\ & = & \Omega_{\mathsf{K}}(T'). & \text{(Definition of } \Omega_{\mathsf{K}})\end{array}$$

The next lemma gives an alternative way for computing Ω_{K} based on slicing and, in addition, proves a continuity property regarding Ω_{K}.

Lemma 36 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$.*

(a) For every $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Omega_{\mathsf{K}}(C(\Gamma)) = C_{\mathsf{K}}\left(\bigvee\nolimits_{\varphi\in\mathrm{Fm}_{\mathcal{L}}(V)} \mathcal{E}(\Gamma^\varphi)\right).$$

(b) Ω_{K} is a join continuous map from $\mathbf{Th}(\mathcal{S})$ into $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, i.e., for all $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$,

$$\Omega_{\mathsf{K}}\left(\bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right) = \bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(T_i).$$

Proof:

(a) Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{array}{rcll}\Omega_{\mathsf{K}}(C(\Gamma)) & = & \Omega_{\mathsf{K}}(C(\bigvee_\varphi \Gamma^\varphi)) & (\Gamma = \bigvee_\varphi \Gamma^\varphi) \\ & = & C_{\mathsf{K}}(\mathcal{E}(C(\bigvee_\varphi \Gamma^\varphi))) & \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(\bigvee_\varphi \Gamma^\varphi)) & \text{(Proposition 22)} \\ & = & C_{\mathsf{K}}(\bigvee_\varphi \mathcal{E}(\Gamma^\varphi)). & (\mathcal{E} \text{ join continuous})\end{array}$$

(b) By the definition of $\bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}$, for all $\Gamma_i : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, $i \in I$,

$$C_{\mathsf{K}}\left(\bigvee\nolimits_i \mathcal{E}(\Gamma_i)\right) = \bigvee\nolimits_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(\Gamma_i)).$$

The inequality $\leq$ is from the fact that $\bigvee_i \mathcal{E}(\Gamma_i) \leq \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(\Gamma_i))$. The reverse inequality follows from $C_{\mathsf{K}}(\mathcal{E}(\Gamma_i)) \leq C_{\mathsf{K}}(\bigvee_i \mathcal{E}(\Gamma_i))$, for all i. Now we have

$$\begin{array}{rcll}\Omega_{\mathsf{K}}\left(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right) & = & C_{\mathsf{K}}\left(\mathcal{E}\left(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right)\right) & \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & C_{\mathsf{K}}\left(\mathcal{E}\left(C\left(\bigvee_{i\in I} T_i\right)\right)\right) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S})}) \\ & = & C_{\mathsf{K}}\left(\mathcal{E}\left(\bigvee_{i\in I} T_i\right)\right) & \text{(Proposition 22)} \\ & = & C_{\mathsf{K}}\left(\bigvee_i \mathcal{E}(T_i)\right) & (\mathcal{E} \text{ join continuous}) \\ & = & \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(T_i)) & \text{(Displayed Formula)} \\ & = & \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(T_i). & \text{(Definition of } \Omega_{\mathsf{K}})\end{array}$$

■

We now prove an analog of Lemma 3.4 of [6], which begins the quest for a characterization of G-algebraic semantics and of equivalent G-algebraic semantics in terms of correspondences between lattices of theories.

Lemma 37 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$.*

(a) $H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T)) = T$, for every $T \in \mathrm{Th}(\mathcal{S})$;

(b) $\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) \leq \Theta$, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, and $\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) = \Theta$ just in case $\Theta \in \Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$, i.e., Θ is the image of some theory of $\mathcal{S}$ under Ω_{K}.

Proof: Let $T \in \mathrm{Th}(\mathcal{S})$. Then

$$\begin{array}{rcl} H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T)) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Omega_{\mathsf{K}}(T)\} \quad \text{(Definition of } H_{\mathsf{K}}) \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(T))\} \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & \bigvee\{\Gamma : \Gamma \leq C(T)\} \quad (\mathcal{E} \text{ an interpretation}) \\ & = & C(T) \quad \text{(Property of join)} \\ & = & T. \quad (T \in \mathrm{Th}(\mathcal{S})) \end{array}$$

Next, suppose $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$.

$$\begin{array}{rcl} \Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) & = & \Omega_{\mathsf{K}}(\bigvee^{\mathbf{Th}(\mathcal{S})}\{T : \mathcal{E}(T) \leq \Theta\}) \quad \text{(Definition of } H_{\mathsf{K}}) \\ & = & \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}\{\Omega_{\mathsf{K}}(T) : \mathcal{E}(T) \leq \Theta\} \quad \text{(Lemma 36)} \\ & = & \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}\{C_{\mathsf{K}}(\mathcal{E}(T)) : \mathcal{E}(T) \leq \Theta\} \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & \leq & C_{\mathsf{K}}(\Theta) \quad \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \\ & & \qquad\qquad \text{and Monotonicity of } C_{\mathsf{K}}) \\ & = & \Theta. \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \end{array}$$

Finally, assume $\Theta = \Omega_{\mathsf{K}}(T)$, for some $T \in \mathrm{Th}(\mathcal{S})$. Then, we get

$$\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) = \Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T))) = \Omega_{\mathsf{K}}(T) = \Theta,$$

the second equality holding by Part (a). ■

By Lemma 37, Ω_{K} is a bijection from $\mathrm{Th}(\mathcal{S})$ to $\Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S})) \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. Since Ω_{K} is order preserving, $\Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$ forms a complete lattice under the order relation inherited by $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$. The corresponding lattice is denoted by $\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))$. In general, $\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))$ may not be a sublattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, since $\Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$ may fail to be closed under intersections. On the other hand we show that the join operations in the two lattices coincide.

To formulate Part (b) of the following result, concerning equivalence of the G-algebraic semantics, we introduce the notion of an *invertible interpretation* $\mathcal{E}$. We say that an interpretation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ is **invertible** if there exists a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, such that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

We say that the interpretation $\mathcal{E}$ is **invertible via $\mathcal{F}$** or **$\mathcal{F}$-invertible**.

Lemma 38 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$.*

(a) Ω_{K} maps $\mathbf{Th}(\mathcal{S})$ isomorphically onto a join-complete subsemilattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$.

(b) K is equivalent to $\mathcal{S}$ via the interpretation $\mathcal{E}$ if and only if $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism and $\mathcal{E}$ is invertible.

Proof:

(a) By Lemma 37, it suffices to show that, for all $\{\Theta_i : i \in I\} \subseteq \Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$,

$$\bigvee_{i\in I}^{\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))} \Theta_i = \bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Theta_i.$$

For every $i \in I$, there exists $T_i \in \mathrm{Th}(\mathcal{S})$, such that $\Omega_{\mathsf{K}}(T_i) = \Theta_i$. Thus, we have

$$\begin{array}{lll} \bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Theta_i & = & \bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(T_i) \\ & = & \Omega_{\mathsf{K}}\left(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right) \quad \text{(Lemma 36)} \\ & = & \bigvee_{i\in I}^{\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))} \Omega_{\mathsf{K}}(T_i) \quad \text{(Lemma 36 and definition of } \Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))) \\ & = & \bigvee_{i\in I}^{\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))} \Theta_i. \end{array}$$

(b) Suppose, first, that K is equivalent to $\mathcal{S}$ via interpretations $\mathcal{E}$ and $\mathcal{F}$. First, observe that, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$,

$$\begin{array}{lll} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le \Theta\} \quad \text{(Definition of } H_{\mathsf{K}}) \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta)))\} \quad \text{(Equivalence)} \\ & = & \bigvee\{\Gamma : \Gamma \le C(\mathcal{F}(\Theta))\} \quad (\mathcal{E} \text{ an interpretation}) \\ & = & C(\mathcal{F}(\Theta)). \quad \text{(Property of join)} \end{array}$$

Now we obtain, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$,

$$\begin{array}{lll} \Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) & = & C_{\mathsf{K}}(\mathcal{E}(H_{\mathsf{K}}(\Theta))) \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(C(\mathcal{F}(\Theta)))) \quad \text{(Preceding deduction)} \\ & = & C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) \quad \text{(Proposition 22)} \\ & = & C_{\mathsf{K}}(\Theta) \quad \text{(Equivalence)} \\ & = & \Theta. \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \end{array}$$

This shows that $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism. To finish the "only if", we must also show that $\mathcal{E}$ is invertible. In fact, its inverse is $\mathcal{F}$, since, as was shown above, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta)) = C(\mathcal{F}(C_{\mathsf{K}}(\Theta))) = C(\mathcal{F}(\Theta))).$$

Suppose, conversely, that $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism and that the interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ is invertible via $\mathcal{F}$. By Lemma 37, $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism with inverse H_{K}. To show that K is equivalent to $\mathcal{S}$ via $\mathcal{E}$ and $\mathcal{F}$, let $\Theta : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. We have

$$\begin{array}{rcll} C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) & = & C_{\mathsf{K}}(\mathcal{E}(C(\mathcal{F}(\Theta)))) & \text{(Proposition 22)} \\ & = & \Omega_{\mathsf{K}}(C(\mathcal{F}(\Theta))) & \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & \Omega_{\mathsf{K}}(C(\mathcal{F}(\bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}))) & (\Theta = \bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}) \\ & = & \Omega_{\mathsf{K}}(C(\bigvee_{\varphi,\psi} \mathcal{F}(\Theta^{\varphi,\psi}))) & (\mathcal{F} \text{ join continuous}) \\ & = & \Omega_{\mathsf{K}}(\bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S})} C(\mathcal{F}(\Theta^{\varphi,\psi}))) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S})}) \\ & = & \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(C(\mathcal{F}(\Theta^{\varphi,\psi}))) & \text{(Hypothesis)} \\ & = & \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi}))) & (\mathcal{F} \text{ inverse of } \mathcal{E}) \\ & = & \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\Theta^{\varphi,\psi}) & (H_{\mathsf{K}} \text{ inverse of } \Omega_{\mathsf{K}}) \\ & = & C_{\mathsf{K}}(\bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}) \\ & = & C_{\mathsf{K}}(\Theta). & (\Theta = \bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}) \end{array}$$

Thus, K is equivalent to $\mathcal{S}$ via the invertible interpretation $\mathcal{E}$.

■

Lemma 38 hints at the requirements that one needs to postulate on an isomorphism between theory lattices so that a (equivalent) G-algebraic semantics be obtained for a given G-logic. Clinically, these properties are chosen so that they are both necessary and sufficient. Thus, no unnecessary restrictions on the framework under consideration are imposed. We define the properties precisely, tie them to the property of invertibility of an interpretation introduced earlier and then formulate one of our main theorems.

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K a class of G-algebras. A join complete embedding $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is called **regular** if there exists a translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$, such that, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Xi(C(\Gamma^\varphi)) = C_{\mathsf{K}}(\mathcal{E}(\Gamma^\varphi)).$$

In this case, we say that Ξ is **regular via** $\mathcal{E}$ or $\mathcal{E}$-**regular**. Let us see that the condition above is equivalent to the statement that, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Xi(C(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

We have

$$\begin{array}{rcll} \Xi(C(\Gamma)) & = & \Xi(C(\bigvee_\varphi \Gamma^\varphi)) & (\Gamma = \bigvee_\varphi \Gamma^\varphi) \\ & = & \Xi(\bigvee_\varphi^{\mathbf{Th}(\mathcal{S})} C(\Gamma^\varphi)) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S})}) \\ & = & \bigvee_\varphi^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Xi(C(\Gamma^\varphi)) & (\Xi \text{ join continuous}) \\ & = & \bigvee_\varphi^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(\Gamma^\varphi)) & (\Xi \text{ regular via } \mathcal{E}) \\ & = & C_{\mathsf{K}}(\bigvee_\varphi \mathcal{E}(\Gamma^\varphi)) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(\bigvee_\varphi \Gamma^\varphi)) & (\mathcal{E} \text{ join continuous}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(\Gamma)). & (\Gamma = \bigvee_\varphi \Gamma^\varphi) \end{array}$$

We show, first, that, if Ξ is regular via $\mathcal{E}$, then $\mathcal{E}$ is an interpretation from $\mathcal{S}$ into $\mathcal{S}_{\mathsf{K}}$.

Lemma 39 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic,* K *a class of G-algebras and* $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ *a join complete embedding. If Ξ is regular via $\mathcal{E}$, then $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ is an interpretation.*

Proof: We have, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} \Phi \le C(\Gamma) & \text{iff} & C(\Phi) \le C(\Gamma) \quad \text{(Property of } C\text{)} \\ & \text{iff} & \Xi(C(\Phi)) \le \Xi(C(\Gamma)) \quad (\Xi \text{ join complete embedding}) \\ & \text{iff} & C_{\mathsf{K}}(\mathcal{E}(\Phi)) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \quad (\Xi \text{ regular via } \mathcal{E}) \\ & \text{iff} & \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)). \quad \text{(Property of } C_{\mathsf{K}}\text{)} \end{array}$$

This shows that $\mathcal{E}$ is an interpretation. ■

Naturally, if $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism, then we say that $\Xi^{-1} : \mathbf{Th}(\mathcal{S}_{\mathsf{K}}) \to \mathbf{Th}(\mathcal{S})$ is **regular** if there exists a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, such that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Xi^{-1}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

We also use the term **regular via** $\mathcal{F}$ or $\mathcal{F}$-**regular** if this situation obtains. In the same way as above, it can be seen that this is equivalent to declaring that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$\Xi^{-1}(C_{\mathsf{K}}(\Theta)) = C(\mathcal{F}(\Theta)).$$

Further, if Ξ^{-1} is regular via $\mathcal{F}$, then, as in Lemma 39, it may be shown that $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ is an interpretation.

Lemma 40 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic,* K *a class of G-algebras and* $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ *an isomorphism. If Ξ^{-1} is regular via $\mathcal{F}$, then $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ is an interpretation.*

Proof: Dual to the proof of Lemma 39. ■

Recall that an interpretation $\mathcal{E}$ is said to be *invertible* in case there exists an interpretation $\mathcal{F}$, such that

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

If $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism that is regular via $\mathcal{E}$, it turns out that the regularity via $\mathcal{F}$ of Ξ^{-1} is closely related to the invertibility of $\mathcal{E}$.

Lemma 41 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic,* K *a class of G-algebras and* $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ *an $\mathcal{E}$-regular order isomorphism. $\mathcal{E}$ is invertible via $\mathcal{F}$ if and only if $\Xi^{-1} : \mathbf{Th}(\mathcal{S}_{\mathsf{K}}) \to \mathbf{Th}(\mathcal{S})$ is $\mathcal{F}$-regular.*

Proof: By hypothesis, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll} \Xi(C(\Gamma)) & = & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) & (\Xi \text{ regular via } \mathcal{E}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) & (\text{Lemma 39} \\ & & & \text{and Proposition 22}) \\ & = & \Omega_{\mathsf{K}}(C(\Gamma)). & (\text{Definition of } \Omega_{\mathsf{K}}) \end{array}$$

Thus, since Ξ is invertible, by Lemma 37, $\Xi^{-1} = H_{\mathsf{K}}$. Thus, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})) \quad \text{iff} \quad \Xi^{-1}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

This proves that $\mathcal{E}$ is invertible via $\mathcal{F}$ iff Ξ^{-1} is regular via F. ■

Now we present the main characterization theorem, which we view, in the present context, as an analog of the celebrated Characterization Theorem 3.7 of [6], which has triggered several generalizations in various directions, e.g., [40],[4] and [31], with [30] being the most definitive among them, encompassing all its predecessors.

Theorem 42 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K a class of G-algebras.*

(a) K is a G-algebraic semantics for $\mathcal{S}$ if and only if there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \Xi(\mathbf{Th}(\mathcal{S}))$, where $\Xi(\mathbf{Th}(\mathcal{S}))$ is a join-complete subsemilattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$.

(b) K is equivalent to $\mathcal{S}$ if and only if there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, with $\mathcal{E}$ invertible.

Proof:

(a) Part (a) of Lemma 38 proves necessity. For sufficiency, suppose

$$\Xi : \mathbf{Th}(\mathcal{S}) \to \Xi(\mathbf{Th}(\mathcal{S}))$$

is an $\mathcal{E}$-regular isomorphism of $\mathbf{Th}(\mathcal{S})$ onto a join-complete subsemilattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$. By Lemma 39, $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ is a G-interpretation. Hence, K is a G-algebraic semantics for $\mathcal{S}$.

(b) If K is equivalent to $\mathcal{S}$, then by Part (b) of Lemma 38, there exists an $\mathcal{E}$-regular isomorphism $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, with $\mathcal{E}$ invertible.

Assume, conversely, that $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an $\mathcal{E}$-regular isomorphism, with $\mathcal{E}$ $\mathcal{F}$-invertible. Then, by Lemma 41, Ξ^{-1} is $\mathcal{F}$-regular. Therefore, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll} C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) & = & \Xi(C(\mathcal{F}(\Theta)) & (\Xi \text{ regular via } \mathcal{E}) \\ & = & \Xi(\Xi^{-1}(C_{\mathsf{K}}(\Theta))) & (\Xi^{-1} \text{ regular via } \mathcal{F}) \\ & = & C_{\mathsf{K}}(\Theta). & \end{array}$$

Thus, K is an equivalent G-algebraic semantics for $\mathcal{S}$.

■

If $\mathcal{S}$ is algebraizable and K is its equivalent G-algebraic semantics, then Ω_{K} has a simple characterization in terms of the interpretation $\mathcal{F}:\mathcal{S}_{\mathsf{K}}\to\mathcal{S}$. Note that this makes Ω_{K} appear completely analogous to H_{K}, since H_{K} was defined exactly in the dual way in terms of $\mathcal{E}$.

Lemma 43 *Let $\mathcal{S}=\langle\mathcal{L},C\rangle$ be a G-logic and K an equivalent G-algebraic semantics for $\mathcal{S}$ via interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for every $T\in\mathrm{Th}(\mathcal{S})$,*

$$\Omega_{\mathsf{K}}(T)=\bigvee\{\Theta\in G^{\mathrm{Eq}_{\mathcal{L}}(V)}:\mathcal{F}(\Theta)\le T\}.$$

Proof: We have, for all $T\in\mathrm{Th}(\mathcal{S})$,

$$\begin{array}{rcll}\Omega_{\mathsf{K}}(T) & = & C_{\mathsf{K}}(\mathcal{E}(T)) & \text{(Definition of } \Omega_{\mathsf{K}})\\ & = & \bigvee\{\Theta:\Theta\le C_{\mathsf{K}}(\mathcal{E}(T))\} & \text{(Property of join)}\\ & = & \bigvee\{\Theta:\mathcal{F}(\Theta)\le C(\mathcal{F}(\mathcal{E}(T)))\} & (\mathcal{F}\text{ an interpretation})\\ & = & \bigvee\{\Theta:\mathcal{F}(\Theta)\le C(T)\} & \text{(Equivalence)}\\ & = & \bigvee\{\Theta:\mathcal{F}(\Theta)\le T\}. & (T\in\mathrm{Th}(\mathcal{S}))\end{array}$$

This proves the statement. ■

2.15 The Leibniz Operator

In this section we start working with algebraization in the standard sense. In other words, we assume that the interpretations involved are standard (see Section 2.13). In this case, Theorem 33 ensures that the equational G-2-logic $\mathcal{S}_{\mathsf{K}}$ is unique and that the interpretations involved are essentially unique, i.e., they are interderivable modulo the equational and logical entailments. The following result asserts that the isomorphism $\Omega_{\mathsf{K}}:\mathbf{Th}(\mathcal{S})\to\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, induced by the standard interpretation $\mathcal{E}:\mathcal{S}\to\mathcal{S}_{\mathsf{K}}$, coincides with the Leibniz operator Ω on $\mathbf{Th}(\mathcal{S})$. This forms an analog of Theorem 4.1 of Blok and Pigozzi [6].

Theorem 44 *Let $\mathcal{S}=\langle\mathcal{L},C\rangle$ be an algebraizable G-logic and K an equivalent G-algebraic semantics via standard interpretations $\mathcal{E}$ and $\mathcal{F}$. Then $\Omega_{\mathsf{K}}:\mathbf{Th}(\mathcal{S})\to\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ coincides with the Leibniz operator, i.e.,*

$$\Omega_{\mathsf{K}}(T)=\Omega(T),\ \textit{for all } T\in\mathrm{Th}(\mathcal{S}).$$

Proof: Let $T\in\mathrm{Th}(\mathcal{S})$. First, by definition, $\Omega_{\mathsf{K}}(T)\in\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. By Lemma 20, $\Omega_{\mathsf{K}}(T)$ is a G-congruence. By Lemma 43 and the fact that $\mathcal{E}$ and $\mathcal{F}$ are standard, for all $T\in\mathrm{Th}(\mathcal{S})$ and all $\varphi,\psi\in\mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega_{\mathsf{K}}(T)(\varphi,\psi)=\bigwedge_{j\in J}T(\Delta_j(\varphi,\psi)).$$

Further, by Lemma 30, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)) \wedge T(\varphi) \leq T(\psi).$$

Thus, we have

$$\Omega_{\mathsf{K}}(T)(\varphi, \psi) \wedge T(\varphi) = \bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)) \wedge T(\varphi) \leq T(\psi).$$

We conclude that $\Omega_{\mathsf{K}}(T)$ is a G-congruence compatible with T. But $\Omega(T)$ is the largest G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ compatible with T, whence $\Omega_{\mathsf{K}}(T) \leq \Omega(T)$.

Conversely, by the fact that $\Omega(T)$ is a G-congruence compatible with T, we get that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\varphi, \psi) \leq T(\Delta_j(\varphi, \varphi)) \leftrightarrow T(\Delta_j(\varphi, \psi)), \quad j \in J.$$

As, for all $j \in J$, $T(\Delta_j(\varphi, \varphi)) = \top$, this gives that

$$\Omega(T)(\varphi, \psi) \leq \bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)) = \Omega_{\mathsf{K}}(T)(\varphi, \psi).$$

We conclude that, for all $T \in \mathrm{Th}(\mathcal{S})$, $\Omega_{\mathsf{K}}(T) = \Omega(T)$. ■

In Lemma 46, we show that, if Ω is order preserving, then it is meet continuous and surjectively structural. For the latter property, we must have available a technical lemma on the behavior of surjective substitutions.

Lemma 45 *Suppose $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$ is a surjective substitution. Then, for all $\vartheta \in \mathrm{Fm}_{\mathcal{L}}(V)$ and every variable x occurring in ϑ, there exists $\vartheta' \in \mathrm{Fm}_{\mathcal{L}}(V)$ and a variable y, such that, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,*

$$\sigma(\vartheta'(\varphi/y)) = \vartheta(\sigma(\varphi)/x).$$

Proof: An inverse image of a variable under any substitution must also be a variable. Thus, since σ is surjective, there exists, for each variable z, another variable z', such that $\sigma(z') = z$. Let ϑ' be obtained from ϑ by simultaneously replacing each variable z different from x by z' and x by any variable y different from all the z'. Then, we can see that, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\sigma(\vartheta'(\varphi/y)) = \vartheta(\sigma(\varphi)/x)$. ■

Lemma 46 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and suppose Ω is order preserving on $\mathbf{Th}(\mathcal{S})$.*

(a) For all $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$,

$$\Omega\left(\bigwedge_{i \in I} T_i\right) = \bigwedge_{i \in I} \Omega(T_i),$$

Hence, $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under meets.

(b) *For all* $T \in \mathrm{Th}(\mathcal{S})$ *and all* $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T) \circ \sigma = \Omega(T \circ \sigma).$$

Hence, $\Omega(\mathrm{Th}(\mathcal{S}))$ *is "surjectively structural", a property akin to closure under inverse surjective substitutions.*

Proof:

(a) By hypothesis, Ω is order preserving. Thus, for all $i \in I$, $\Omega(\bigwedge_{i\in I} T_i) \leq \Omega(T_i)$. Therefore, $\Omega(\bigwedge_{i\in I} T_i) \leq \bigwedge_{i\in I} \Omega(T_i)$. Conversely, note that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\bigwedge_{i\in I} \Omega(T_i)(\varphi,\psi) \wedge \bigwedge_{i\in I} T_i(\varphi) \leq \Omega(T_i)(\varphi,\psi) \wedge T_i(\varphi) \leq T_i(\psi).$$

Thus,

$$\bigwedge_{i\in I} \Omega(T_i)(\varphi,\psi) \wedge \bigwedge_{i\in I} T_i(\varphi) \leq \bigwedge_{i\in I} T_i(\psi).$$

This shows that $\bigwedge_{i\in I} \Omega(T_i)$ is a G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ compatible with $\bigwedge_{i\in I} T_i$ and, hence, by the maximality property of the Leibniz G-congruence, $\bigwedge_{i\in I} \Omega(T_i) \leq \Omega(\bigwedge_{i\in I} T_i)$.

(b) First, by the compatibility of $\Omega(T)$ with T, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\sigma(\varphi),\sigma(\psi)) \wedge T(\sigma(\varphi)) \leq T(\sigma(\psi)).$$

Hence, $\Omega(T) \circ \sigma$ is compatible with $T \circ \sigma$. Thus, by the maximality property of the Leibniz G-congruence, $\Omega(T) \circ \sigma \leq \Omega(T \circ \sigma)$. For the reverse inequality, suppose for the sake of obtaining a contradiction, that $\Omega(T \circ \sigma) \nleq \Omega(T) \circ \sigma$. Thus, there exist $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, such that

$$\Omega(T \circ \sigma)(\varphi,\psi) \nleq \Omega(T)(\sigma(\varphi),\sigma(\psi)).$$

Recalling Theorem 16, there exists $\vartheta(x) \in \mathrm{Fm}_{\mathcal{L}}(V)$, such that

$$\Omega(T \circ \sigma)(\varphi,\psi) \nleq T(\vartheta(\sigma(\varphi))) \leftrightarrow T(\vartheta(\sigma(\psi))).$$

So, by Lemma 45, there exists ϑ', such that

$$\Omega(T \circ \sigma)(\varphi,\psi) \nleq T(\sigma(\vartheta'(\varphi))) \leftrightarrow T(\sigma(\vartheta'(\psi))).$$

This contradicts the compatibility of $\Omega(T \circ \sigma)$ with $T \circ \sigma$.

∎

Since $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under arbitrary intersections, it forms a complete lattice which is denoted by $\Omega(\mathbf{Th}(\mathcal{S}))$. If Ω is injective, then it is an isomorphism from $\mathbf{Th}(\mathcal{S})$ onto $\Omega(\mathbf{Th}(\mathcal{S}))$. Our goal is to be able to apply

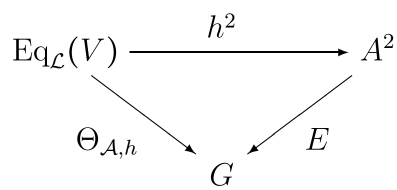
Theorem 42. This requires showing that $\Omega(\mathbf{Th}(\mathcal{S}))$ coincides with $\mathbf{Th}(\mathcal{S}_\mathsf{K})$ for some class K of G-algebras. More precisely, we aim to show that $\mathbf{Th}(\mathcal{S}_\mathsf{K})$ and $\Omega(\mathbf{Th}(\mathcal{S}))$ are isomorphic under the identity mapping.

For a G-congruence Θ on $\mathbf{Fm}_\mathcal{L}(V)$, we define a G-algebra

$$\mathcal{F}^\Theta = \langle \mathbf{Fm}_\mathcal{L}(V), \Theta\rangle.$$

Given any G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and a homomorphism $h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A}$, we define the G-**kernel** $\Theta_{\mathcal{A},h} : \mathrm{Eq}_\mathcal{L}(V) \to G$ by setting

$$\Theta_{\mathcal{A},h} = E \circ h^2,$$



i.e., we have, for all $\varphi, \psi \in \mathrm{Fm}_\mathcal{L}(V)$,

$$\Theta_{\mathcal{A},h}(\varphi,\psi) = E(h(\varphi), h(\psi)).$$

$\Theta_{\mathcal{A},h}$ is a G-congruence. Moreover, note that, given a G-congruence Θ,

$$\Theta_{\mathcal{F}^\Theta, i} = \Theta,$$

where $i : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{Fm}_\mathcal{L}(V)$ is the identity homomorphism.

Now let K be a class of G-algebras and suppose that $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}$. Then $\Theta_{\mathcal{A},h} \in \mathrm{Th}(\mathcal{S}_\mathsf{K})$. To see this, it suffices to show that $C_\mathsf{K}(\Theta_{\mathcal{A},h}) \leq \Theta_{\mathcal{A},h}$. We have

$$\begin{array}{rcl}
C_\mathsf{K}(\Theta_{\mathcal{A},h}) & = & \bigwedge_{\substack{\mathcal{B}=\langle \mathbf{B},F\rangle\in\mathsf{K} \\ g:\mathbf{Fm}_\mathcal{L}(V)\to\mathbf{B}}} \{F \circ g^2 : \Theta_{\mathcal{A},h} \leq F \circ g^2\} \\
& & \quad \text{(Definition of } C_\mathsf{K}) \\
& = & \bigwedge_{\substack{\mathcal{B}=\langle \mathbf{B},F\rangle\in\mathsf{K} \\ g:\mathbf{Fm}_\mathcal{L}(V)\to\mathbf{B}}} \{F \circ g^2 : E \circ h^2 \leq F \circ g^2\} \\
& & \quad \text{(Definition of } \Theta_{\mathcal{A},h}) \\
& \leq & E \circ h^2 \quad (\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K} \text{ and } h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A}) \\
& = & \Theta_{\mathcal{A},h}. \quad \text{(Definition of } \Theta_{\mathcal{A},h})
\end{array}$$

More generally, by definition, given a G-set of equations $Z : \mathrm{Eq}_\mathcal{L}(V) \to G$, the G-congruence in $\mathrm{Th}(\mathcal{S}_\mathsf{K})$ generated by Z is given by

$$C_\mathsf{K}(Z) = \bigwedge_{\substack{\mathcal{A}=\langle \mathbf{A},E\rangle\in\mathsf{K} \\ h:\mathbf{Fm}_\mathcal{L}(V)\to\mathbf{A}}} \{E \circ h^2 : Z \leq E \circ h^2\}.$$

Our next goal is to show that, if the Leibniz operator Ω is join continuous on the theories of a G-logic, one can construct a class K of G-algebras, such that the image $\Omega(\mathrm{Th}(\mathcal{S}))$ coincides with $\mathrm{Th}(\mathcal{S}_\mathsf{K})$. However, we precede this by a technical result which is needed for the proof of the main lemma.

Lemma 47 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ a homomorphism, with the property that each formula is the image under h of infinitely many variables. Then, for all $T \in \mathrm{Th}(\mathcal{S})$, there exists a surjective $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$, such that*

$$\Theta_{\mathcal{F}^{\Omega(T)},h} = \Omega(T) \circ \sigma.$$

Proof: Let σ be any substitution, such that $\sigma(v_i) = h(v_i)$, for $i = 1, 2, \ldots$, and furthermore, such that each v_i is the image under σ of some v_j. Such a σ exists because of the assumption that each formula is the image under h of an infinite number of variables. Then σ is surjective and we have $h(v_i) = \sigma(v_i)$, for all $i = 1, 2, \ldots$. Now, using the definition of the G-kernel $\Theta_{\mathcal{F}^{\Omega(T)},h}$, we get

$$\Theta_{\mathcal{F}^{\Omega(T)},h} = \Omega(T) \circ h = \Omega(T) \circ \sigma.$$

So the conclusion holds. ■

Lemma 48 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and set*

$$\mathsf{K} = \{\mathcal{F}^{\Theta} : \Theta \in \Omega(\mathrm{Th}(\mathcal{S}))\}.$$

If Ω is join continuous on $\mathbf{Th}(\mathcal{S})$ then, we have $\Omega(\mathrm{Th}(\mathcal{S})) = \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$.

Proof: We begin with some observations. Since, by hypothesis, for every $\{T_i :\in I\} \subseteq \mathrm{Th}(\mathcal{S})$,

$$\Omega\Big(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\Big) = \bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega(T_i),$$

we infer that $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under joins and, also, that Ω is order preserving. Thus, by Lemma 46, it is also closed under meets and it is surjectively structural.

We look, first, at the inclusion $\Omega(\mathrm{Th}(\mathcal{S})) \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. Suppose $T \in \mathrm{Th}(\mathcal{S})$ and let $\Theta = \Omega(T)$. Then, by the definition of K, we get $\mathcal{F}^{\Theta} \in \mathsf{K}$. Thus, $\Omega(T) = \Theta = \Theta_{\mathcal{F}^{\Theta},i} \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ (see work preceding Lemma 47).

Next, we turn to the reverse inclusion. Suppose $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. Consider, first, the case where Θ is finitely generated. Then, there exists finite $Q : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, such that $\Theta = C_{\mathsf{K}}(Q)$. We have

$$\begin{aligned}\Theta &= \bigwedge_{\substack{\mathcal{A}=\langle \mathbf{A},E\rangle\in\mathsf{K}\\ h:\mathbf{Fm}_{\mathcal{L}}(V)\to\mathbf{A}}} \{E \circ h^2 : Q \le F \circ h^2\}\\ &= \bigwedge_{\substack{T\in\mathrm{Th}(\mathcal{S})\\ h:\mathbf{Fm}_{\mathcal{L}}(V)\to\mathbf{Fm}_{\mathcal{L}}(V)}} \{\Omega(T) \circ h^2 : Q \le \Omega(T) \circ h^2\},\end{aligned}$$

the last equality following by the definition of K. Now note that $Q(\varphi, \psi) \ne \bot$ only for finitely many formulas. So all h's can be taken to satisfy the hypothesis of Lemma 47. Thus, we obtain

$$\begin{aligned}\Theta &= \bigwedge_{\substack{T\in\mathrm{Th}(\mathcal{S})\\ \sigma \text{ surjective}}} \{\Omega(T) \circ \sigma^2 : Q \le \Omega(T) \circ \sigma^2\}\\ &= \bigwedge_{\substack{T\in\mathrm{Th}(\mathcal{S})\\ \sigma \text{ surjective}}} \{\Omega(T \circ \sigma) : Q \le \Omega(T \circ \sigma)\} \quad \text{(Lemma 46(b))}\\ &\in \Omega(\mathrm{Th}(\mathcal{S})). \quad \text{(Lemma 46(a))}\end{aligned}$$

Finally, we turn to the general case. Consider an arbitrary $\Theta \in \mathrm{Th}(\mathcal{S}_\mathsf{K})$. Then, we have

$$\Theta = \bigvee^{\mathbf{Th}(\mathcal{S}_\mathsf{K})}\{C_\mathsf{K}(\Gamma) : \Gamma \leq \Theta \text{ finite}\}.$$

By what was shown above, for all finite $\Gamma \leq \Theta$, $C_\mathsf{K}(\Gamma) \in \Omega(\mathrm{Th}(\mathcal{S}))$. By hypothesis, $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under joins. Thus $\Theta \in \Omega(\mathrm{Th}(\mathcal{S}))$. Thus, we conclude that $\mathrm{Th}(\mathcal{S}_\mathsf{K}) \subseteq \Omega(\mathrm{Th}(\mathcal{S}))$. ■

Theorem 49 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. $\mathcal{S}$ is algebraizable via standard interpretations if and only if the Leibniz operator is $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, injective and join continuous on $\mathbf{Th}(\mathcal{S})$.*

Proof: By Theorem 44, $\Omega_\mathsf{K} = \Omega$. By Lemma 38, Ω is an $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, isomorphism. Thus, in particular, it is injective and join continuous.

Suppose, conversely, that Ω is $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, injective and join continuous. Then, by Lemma 48, it is an isomorphism $\Omega : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$, where

$$\mathsf{K} = \{\langle \mathbf{Fm}_\mathcal{L}(V), \Omega(T)\rangle : T \in \mathrm{Th}(\mathcal{S})\}.$$

By Theorem 42, Part (b), K is equivalent to $\mathcal{S}$ and $\mathcal{S}$ is algebraizable. ■

Chapter 3

Graded Algebraic Semantics for Graded Logics

3.1 Introduction

Our goal in this chapter is to extend the results of Font and Jansana [28] in order to apply to arbitrary graded logics. The setting is inspired by the work of Diaconescu on the formalization of many-valued logics as institutions [21]. In order to accomplish this, we first develop a machinery involving graded models (G-models) and graded congruences (G-congruences). An attempt at a theory on the same premises, paralleling the work of Blok and Pigozzi rather than the one of Font and Jansana was presented in Chapter 2.

We now provide an outline of the contents section-by section.

In Section 3.2, we define *G-algebras* and develop the machinery needed to deal with those algebras, much like the basic machinery of Universal Algebra that allows manipulating ordinary algebras. More precisely, we introduce *G-morphisms* and *G-congruences* and prove analogs of the Homomorphism, the Second Isomorphism and the Correspondence Theorems. We also introduce *subalgebras* and *direct products* and define *subdirect products*. We conclude with a characterization theorem for subdirect products involving quotient G-algebras, akin to the well known characterization of subdirect products of ordinary algebras involving quotient algebras.

In Section 3.3, we adapt the notion of logical filter and logical matrix to the graded context, thus obtaining the notions of *G-filter* and of *G-matrix*, respectively. We also define the key concept of the *Leibniz G-congruence* of a G-matrix and prove an analog of the well known characterization theorem of Blok and Pigozzi (Theorem 1.5 of [6]). We show that Leibniz G-congruences commute with inverse surjective G-morphisms. We define *reduced G-matrices*. The Leibniz G-congruence of a G-matrix allows passing from a given G-matrix to its *reduction*. Similarly, a *reduced class* is a class of G-matrices obtained by another class by applying the process of reduction to all its members.

In Section 3.4, *graded G-logics* are introduced, essentially as closure-like operators on the set of all functions from an underlying G-algebra to the grade set G. That is, they are operators satisfying the properties of inflationarity, monotonicity, idempotency and translation (an analog of structurality) with respect to the ordering induced by G. As with ordinary closure operators, they turn out to be in one-to-one correspondence with the corresponding collections of *closed functions* from the underlying G-algebra into G. One may also define an ordering on G-logics over the same underlying G-algebra by comparing the action of their corresponding G-operators or, equivalently, by comparing (in the reverse inclusion ordering) the collections of their closed functions.

In Section 3.5, *logical G-congruences* are introduced. A *logical G-congruence* on a G-logic is a G-congruence on the underlying G-algebra that is compatible with every closed G-set, or function, of the G-logic. The greatest such congruence is called the *Tarski G-congruence* of the G-logic. Tarski

G-congruences have a characterization induced by the characterization of Leibniz G-congruences and by the close relationship that governs Leibniz and Tarski G-congruences, both paralleling the standard ones in the context of classical Algebraic Logic. This characterization allows one to immediately conclude that the Tarski operator is monotone on G-logics over the same underlying G-algebra.

In Section 3.6, we discuss *logical G-morphisms*. These are G-morphisms between underlying G-algebras that preserve the logical structure. If the morphism is surjective and both preserves and reflects the logical structure, it is termed a *bilogical G-morphism*. We provide several characterizing conditions for a surjective G-morphism to be a bilogical G-morphism along the lines of the ones presented by Font and Jansana in Proposition 1.4 of [28]. From the characterizing properties one may infer that bilogical G-morphisms establish a correspondence of closed G-sets of the bimorphic G-logics and also isomorphisms of the lattices of corresponding G-logics over the underlying G-algebras. These are also properties known to hold in the traditional context (Proposition 1.5 and Corollary 1.6 of [28]). Also known in the conventional framework is the property that, roughly speaking, Tarski G-congruences are preserved under inverse bilogical G-morphisms. The section closes with a quick look at (*logical*) *G-isomorphisms*, which are bijective logical G-morphisms whose inverses are also logical G-morphisms. They are characterized as being bilogical G-morphisms which are G-algebra isomorphisms.

In Section 3.7, we use logical G-congruences to define *quotients* of G-logics. These quotients and accompanying *quotient G-morphisms* are used to establish analogs of the Homomorphism Theorems of Universal Algebra for G-logics. Analogs are obtained of the *Homomorphism*, of the *Second Isomorphism* and of the *Correspondence Theorems*. We also define *reduced G-logics* and the *reduction* of a G-logic, which is the quotient of the G-logic by its Tarski G-congruence. Several results involving reductions are established. E.g., it is shown that the reduction of a quotient G-logic is isomorphic to the reduction of the original G-logic and that two G-logics related via a bilogical G-morphism have isomorphic reductions.

In Section 3.8, we introduce *sentential G-logics*, that is, G-logics whose underlying G-algebras are formula algebras. These can be interpreted, as in the ordinary theory, either via G-matrices or via G-logic models (G-models, for simplicity). We explore both types and study some basic properties, in particular interactions with bilogical G-morphisms and the effect of reductions. Finally we discuss *completeness* of a sentential G-logic with respect to special classes of G-models.

Section 3.9 is devoted to a more in-depth study of *G-models* of a sentential G-logic. These models play the role that abstract logics, serving as models of sentential logics, play in the traditional setting of [28]. The property of being a G-model is preserved under bilogical G-morphisms. Several completeness results are revisited. The section closes with a characterization of G-models as those G-logics whose families of closed G-sets consist entirely of filters of the G-logic. This is a standard result, well known in the context of sentential logics (Proposition 2.7 of [28]), and serves as the main connecting thread between G-matrix models and G-logic models of a given sentential G-logic.

Section 3.10 focuses on a special class of G-models, named *full G-models*. These are G-models whose reductions have as collection of closed G-sets the entire collection of G-filters on the quotient G-algebra. It is shown that any G-model whose collection of closed G-sets are of this form is in fact a full G-model. Such models are called *basic full G-models*. The class of all full G-models is closed in both the forward and backward directions (that is, both under direct and inverse images) under bilogical G-morphisms. Exploiting these results, the class of all full G-models is characterized as the smallest class of G-logics containing all basic full G-models and closed under both images and preimages of bilogical G-morphisms. This parallels the situation in the traditional setting (Corollary 2.13 of [28]).

In Section 3.11, $\mathbb{S}$-*algebras* are introduced. They constitute the G-algebraic reducts of reduced full models of a sentential G-logic $\mathbb{S}$. The class of $\mathbb{S}$-algebras may be characterized, without reference to full models, as the class of G-algebraic reducts of all reduced G-models of $\mathbb{S}$. Moreover, it is an abstract class, i.e., closed under isomorphisms. Other characterizations are also provided, relating, e.g., $\mathbb{S}$-algebras with full G-models and bilogical G-morphisms. The section revisits the issue of completeness. It is shown that a sentential G-logic $\mathbb{S}$ is complete with respect to the classes of full G-models, basic full G-models and reduced full G-models. These completeness results refine the standard coarser results asserting completeness with respect to the classes of all G-models and of all reduced G-models. The section concludes by explicating the relations between the classes $\mathrm{Alg}(\mathbb{S})$ of $\mathbb{S}$-algebras and of $\mathrm{Alg}^*(\mathbb{S})$ of G-algebraic reducts of reduced G-matrix models of $\mathbb{S}$ and by comparing corresponding classes for sentential G-logics over the same logical signature.

Section 3.12 is the one detailing the work specifically targeted towards obtaining an analog of the Isomorphism Theorem 2.30 of Font and Jansana [28]. A sentential G-logic $\mathbb{S}$ is given and a G-algebra $\mathcal{A}$ is fixed. For any G-congruence Θ on $\mathcal{A}$, one considers the quotient G-morphism $\mathcal{A} \to \mathcal{A}/\Theta$ and the full G-model $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$, whose closed functions consist of all $\mathbb{S}$-filters on the quotient G-algebra. This model generates via the quotient morphism a full G-model $\widetilde{H}_{\mathcal{A}}(\Theta)$ on $\mathcal{A}$. The Isomorphism Theorem establishes an isomorphism between the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}$ and the lattice of all $\mathrm{Alg}(\mathbb{S})$-G-congruences on $\mathcal{A}$. Moreover, this isomorphism is given by the Tarski operator and its inverse is the operator $\Theta \mapsto \widetilde{H}_{\mathcal{A}}(\Theta)$, described above.

In the last three sections, Sections 3.13, 3.14 and 3.15, we introduce specific classes of G-logics in the algebraic hierarchy. Section 3.13 deals with *protoalgebraic G-logics*, which were cursorily introduced in Section 2.7 and are characterized by the monotonicity of the Leibniz operator. They form the class corresponding to the protoalgebraic sentential logics of Blok and Pigozzi [5] (see also [15]). Section 3.15, on the other hand, introduces *weakly algebraizable* and *algebraizable G-logics*, corresponding to weakly algebraizable [16] and algebraizable [6, 33, 34] deductive systems. They form a narrower class than protoalgebraic logics and require that, in addition to being monotone, the Leibniz operator be injective and injective plus join continuous, respectively. The study of injectivity in the context of monotonicity, which passes through the notion of a *Leibniz G-filter* of a protoalgebraic logic, is the subject of Section 3.14. The study in Sections 3.13 and 3.15 of the few classes of the Leibniz hierarchy of G-logics does not go in depth, since they are only meant to indicate how the machinery developed in earlier sections may be used in this setting. Only a few of their basic properties, closely related or inferred by the definitions and known from the traditional context, are revisited and adapted to the setting of G-logics.

3.2 Graded Algebras

We fix again a complete lattice $\mathbf{G} = \langle G, \leq \rangle$ (sometimes with additional structure, as needed). Recall that, for any set A, the set G^A of functions $X : A \to G$, is ordered by

$$X \leq Y \quad \text{iff} \quad X(a) \leq Y(a), \text{ for all } a \in A.$$

As was the case in Chapter 2, this ordering will also play a critical role in the sequel.

Let $\mathcal{L}$ be a fixed (but arbitrary) language, consisting of logical connectives or algebraic operation symbols, depending on the point of view, with attached arities. We work with algebras $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ of similarity type $\mathcal{L}$. The set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ is denoted $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$. We write $h : \mathbf{A} \to \mathbf{B}$ to signify that $h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$.

Recall that a *graded congruence* or *G-congruence* Θ on $\mathbf{A}$ is a function

$$\Theta : A^2 \to G,$$

such that, for all $a, b, c \in A$, all operation symbols $\lambda \in \mathcal{L}$, of arity n, and all $a_1, b_1, \dots, a_n, b_n \in A$,

(**Reflexivity**) $\Theta(a, a) = \top$;

(**Symmetry**) $\Theta(a, b) = \Theta(b, a)$;

(**Transitivity**) $\Theta(a, b) \wedge \Theta(b, c) \leq \Theta(a, c)$;

(**Congruence**) $\Theta(a_1, b_1) \wedge \cdots \wedge \Theta(a_n, b_n) \leq \Theta(\lambda^{\mathbf{A}}(a_1, \dots, a_n), \lambda^{\mathbf{A}}(b_1, \dots, b_n))$.

The set of all G-congruences of $\mathbf{A}$ is denoted by $\mathrm{Gon}(\mathbf{A})$. It is naturally ordered by setting, for all $\Theta, \Theta' \in \mathrm{Gon}(\mathbf{A})$,

$$\Theta \le \Theta' \quad \text{iff} \quad \Theta(a,b) \le \Theta'(a,b), \text{ for all } a,b \in A.$$

The set $\mathrm{Gon}(\mathbf{A})$, equipped with $\le$, forms a complete lattice, denoted

$$\mathbf{Gon}(\mathbf{A}) = \langle \mathrm{Gon}(\mathbf{A}), \le \rangle.$$

Its least element is the G-congruence $\Delta_{\mathbf{A}} : A^2 \to G$, with

$$\Delta_{\mathbf{A}}(a,b) = \begin{cases} \top, & \text{if } a = b, \\ \bot, & \text{if } a \ne b, \end{cases}$$

and its largest element is the G-congruence $\nabla_{\mathbf{A}} : A^2 \to G$, with

$$\nabla_{\mathbf{A}}(a,b) = \top, \text{ for all } a,b \in A.$$

A G-congruence Θ on $\mathbf{A}$ is called **reduced** if, for all $a,b \in A$,

$$\Theta(a,b) = \top \quad \text{if and only if} \quad a = b.$$

Consider an arbitrary G-congruence Θ on $\mathbf{A}$. Recall, from Section 2.5, the stratified congruence $\hat{\Theta} = \{\hat{\Theta}_g : g \in G\}$ associated with G. Here, we use extensively the $\top$-stratum $\hat{\Theta}_\top$ of $\hat{\Theta}$. So by slightly abusing (or, rather, overloading) notation, we set

$$\hat{\Theta} := \hat{\Theta}_\top = \{\langle a,b\rangle \in A^2 : \Theta(a,b) = \top\}.$$

By Proposition 10, $\hat{\Theta}$ is a congruence on $\mathbf{A}$. Construct the quotient $\mathbf{A}/\hat{\Theta}$ and define on it a function

$$\bar{\Theta} : (A/\hat{\Theta})^2 \to G$$

by

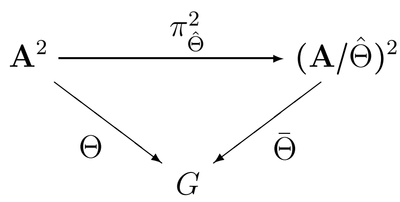
$$\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta(a,b).$$

The function $\bar{\Theta}$ is well defined. Consider $a,a',b,b' \in A$, such that $\langle a,a'\rangle \in \hat{\Theta}$ and $\langle b,b'\rangle \in \hat{\Theta}$. Then

$$\begin{array}{rcl} \Theta(a',b') & = & \top \wedge \Theta(a',b') \wedge \top \\ & = & \Theta(a,a') \wedge \Theta(a',b') \wedge \Theta(b',b) \\ & & (\langle a,a'\rangle \in \hat{\Theta} \text{ and } \langle b,b'\rangle \in \hat{\Theta}) \\ & \le & \Theta(a,b), \quad \text{(Transitivity)} \end{array}$$

whence, by symmetry, $\Theta(a,b) = \Theta(a',b')$.

Proposition 50 *Let* $\mathbf{A}$ *be an algebra and* Θ *a* G*-congruence on* $\mathbf{A}$*. Then* $\bar{\Theta}$ *is a reduced* G*-congruence on* $\mathbf{A}/\hat{\Theta}$ *and*



$$\Theta = \bar{\Theta} \circ \pi^2_{\hat{\Theta}},$$

where $\pi_{\hat{\Theta}} : \mathbf{A} \to \mathbf{A}/\hat{\Theta}$ *is the natural quotient homomorphism.*

Proof: That $\bar{\Theta}$ is a G-congruence is a consequence of the fact that Θ is a G-congruence. E.g., for Transitivity, given $a, b, c \in A$, we get

$$\begin{aligned} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{\Theta}(b/\hat{\Theta}, c/\hat{\Theta}) &= \Theta(a,b) \wedge \Theta(b,c) \quad \text{(Definition of } \bar{\Theta}) \\ &\le \Theta(a,c) \quad \text{(Transitivity)} \\ &= \bar{\Theta}(a/\hat{\Theta}, c/\hat{\Theta}). \quad \text{(Definition of } \bar{\Theta}) \end{aligned}$$

To see that $\bar{\Theta}$ is reduced, let $a, b \in A$. Then

$$\begin{aligned} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \top \quad &\text{iff} \quad \Theta(a,b) = \top \quad \text{(Definition of } \bar{\Theta}) \\ &\text{iff} \quad \langle a, b\rangle \in \hat{\Theta} \quad \text{(Definition of } \hat{\Theta}) \\ &\text{iff} \quad a/\hat{\Theta} = b/\hat{\Theta}. \end{aligned}$$

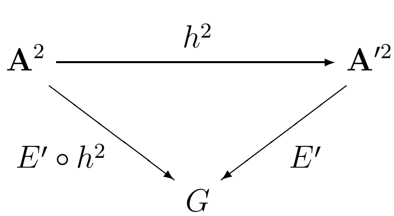
Finally, for all $a, b \in A$, $\bar{\Theta}(\pi_{\hat{\Theta}}(a), \pi_{\hat{\Theta}}(b)) = \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta(a, b)$. ■

A **reduced graded algebra** or **reduced** G**-algebra** is a pair $\mathcal{A} = \langle \mathbf{A}, E\rangle$, where:

- $\mathbf{A}$ is an $\mathcal{L}$-algebra;
- E is a reduced G-congruence on $\mathbf{A}$.

Since in this chapter, as opposed to Chapter 2, we are going to be dealing exclusively with reduced G-algebras, to simplify discussion we again (hopefully not perilously) overload terminology and refer to reduced G-algebras simply as G**-algebras**. Note that every $\mathcal{L}$-algebra $\mathbf{A}$ may be viewed as a G-algebra, namely, the G-algebra $\mathcal{A} = \langle \mathbf{A}, \Delta_{\mathbf{A}}\rangle$.

Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ be G-algebras. A G**-algebra morphism** or G**-morphism** $h : \mathcal{A} \to \mathcal{A}'$ is an algebra homomorphism $h : \mathbf{A} \to \mathbf{A}'$, such that



$$E \leq E' \circ h^2.$$

Note that $E' \circ h^2$ is a G-congruence on $\mathbf{A}$. E.g., for Transitivity, for all $a, b, c \in A$,

$$E'(h(a), h(b)) \wedge E'(h(b), h(c)) \leq E'(h(a), h(c))$$

follows by the transitivity of E' on $\mathbf{A}'$.

A G-congruence Θ on $\mathbf{A}$ is said to be a G-**congruence** on $\mathcal{A} = \langle \mathbf{A}, E \rangle$ if

$$E \leq \Theta.$$

We denote the set of all G-congruences on $\mathcal{A}$ by $\mathrm{Gon}(\mathcal{A})$.

Proposition 51 *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra. The set $\mathrm{Gon}(\mathcal{A})$ is a principal filter of $\mathbf{Gon}(\mathbf{A})$ and, hence, $\mathbf{Gon}(\mathcal{A}) = \langle \mathrm{Gon}(\mathcal{A}), \leq \rangle$ is a complete lattice.*

Proof: By definition

$$\mathrm{Gon}(\mathcal{A}) = \{\Theta \in \mathrm{Gon}(\mathbf{A}) : E \leq \Theta\}.$$

Thus, $\mathrm{Gon}(\mathcal{A})$ is clearly a principal filter of $\mathbf{Gon}(\mathbf{A})$. ∎

Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. The **quotient** G-**algebra** $\mathcal{A}/\Theta$ of $\mathcal{A}$ by Θ is defined by

$$\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta} \rangle.$$

Proposition 50 ensures that this is a well-defined G-algebra. Moreover, it shows that

$$\pi_\Theta : \mathcal{A} \to \mathcal{A}/\Theta$$

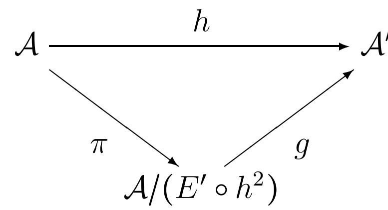
is a G-morphism. It is called the **quotient** G-**morphism**.

The preceding machinery allows us to obtain, for G-algebras, G-congruences and G-morphisms analogs of the well known Homomorphism Theorems of Universal Algebra [9, 38, 2]. First, we revisit the Homomorphism Theorem.

Theorem 52 (Homomorphism) *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{A}'$ a surjective G-morphism. Then $E' \circ h^2$ is a G-congruence on $\mathcal{A}$ and*

$$\mathcal{A}/(E' \circ h^2) \cong \mathcal{A}'$$

by means of a unique G-isomorphism g satisfying commutativity of



where $\pi : \mathcal{A} \to \mathcal{A}/(E' \circ h^2)$ denotes the quotient G-morphism.

Proof: We have, for all $a,b\in A$,

$$\begin{array}{rll}(E'\circ h^2)(a,b)=\top & \text{iff} & E'(h(a),h(b))=\top \quad \text{(Composition)}\\ & \text{iff} & h(a)=h(b) \quad (E' \text{ reduced})\\ & \text{iff} & \langle a,b\rangle\in \mathrm{Ker}(h). \quad (\text{Definition of } \mathrm{Ker}(h))\end{array}$$

Therefore, by the Homomorphism Theorem of Universal Algebra, there exists a unique isomorphism

$$g: \begin{array}{rcl}\mathbf{A}/\widehat{E'\circ h^2} & \longrightarrow & \mathbf{A}';\\ a/\widehat{E'\circ h^2} & \longmapsto & h(a).\end{array}$$

It suffices now to show that this algebra isomorphism is also a G-algebra isomorphism, i.e., that it satisfies $E'\circ g^2=\overline{E'\circ h^2}$. We have, for all $a,b\in A$,

$$\begin{array}{rcl}(E'\circ g^2)(a/\widehat{E'\circ h^2},b/\widehat{E'\circ h^2}) & = & E'(g(a/\widehat{E'\circ h^2}),g(b/\widehat{E'\circ h^2}))\\ & & \quad \text{(Composition)}\\ & = & E'(h(a),h(b))\\ & & \quad (\text{Definition of } g)\\ & = & \overline{E'\circ h^2}(a/\widehat{E'\circ h^2},b/\widehat{E'\circ h^2}).\\ & & \quad (\text{Definition of } \overline{E'\circ h^2})\end{array}$$

Therefore, $g:\mathcal{A}/(E'\circ h^2)\cong\mathcal{A}'$. ■

The analog of the Second Isomorphism Theorem comes next. Suppose $\mathcal{A}=\langle\mathbf{A},E\rangle$ is a G-algebra and $\Theta,\Theta'\in\mathrm{Gon}(\mathcal{A})$, such that $\Theta\le\Theta'$. On $\mathcal{A}/\Theta=\langle\mathbf{A}/\hat{\Theta},\bar{\Theta}\rangle$, we define the function

$$\Theta'/\Theta:(A/\hat{\Theta})^2\to G,$$

given, for all $a,b\in A$, by

$$\Theta'/\Theta(a/\hat{\Theta},b/\hat{\Theta})=\Theta'(a,b).$$

This is well defined, since, for all $a,a',b,b'\in A$, such that $\langle a,a'\rangle\in\hat{\Theta}$ and $\langle b,b'\rangle\in\hat{\Theta}$, we have $\Theta(a,a')=\top$ and $\Theta(b,b')=\top$, and, hence,

$$\begin{array}{rcl}\Theta'(a,b) & = & \top\wedge\Theta'(a,b)\wedge\top\\ & = & \Theta(a',a)\wedge\Theta'(a,b)\wedge\Theta(b,b')\\ & \le & \Theta'(a',a)\wedge\Theta'(a,b)\wedge\Theta'(b,b')\\ & \le & \Theta'(a',b').\end{array}$$

Thus, by symmetry, $\Theta'(a,b)=\Theta'(a',b')$.

Theorem 53 (Second Isomorphism) *Let $\mathcal{A}=\langle\mathbf{A},E\rangle$ be a G-algebra and $\Theta,\Theta'\in\mathrm{Gon}(\mathcal{A})$, such that $\Theta\le\Theta'$. Then $\Theta'/\Theta\in\mathrm{Gon}(\mathcal{A}/\Theta)$ and*

$$(\mathcal{A}/\Theta)/(\Theta'/\Theta)\cong\mathcal{A}/\Theta'$$

via $(a/\hat{\Theta})/\widehat{\Theta'/\Theta}\mapsto a/\hat{\Theta}'$.

Proof: We first show that Θ'/Θ is a G-congruence on $\mathcal{A}/\Theta$. This is essentially a direct consequence of the fact that Θ' is a G-congruence on $\mathcal{A}$. We have, for all $a, b, c \in A$, all n-ary $\lambda \in \mathcal{L}$ and all $a_1, b_1, \ldots, a_n, b_n \in A$:

- For Reflexivity,

$$\begin{array}{rcl} \Theta'/\Theta(a/\hat{\Theta}, a/\hat{\Theta}) & = & \Theta'(a,a) \quad (\text{Definition of } \Theta'/\Theta) \\ & = & \top. \quad (\Theta' \in \mathrm{Gon}(\mathcal{A})) \end{array}$$

- For Symmetry,

$$\begin{array}{rcl} \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta'(a,b) \quad (\text{Definition of } \Theta'/\Theta) \\ & = & \Theta'(b,a) \quad (\Theta' \in \mathrm{Gon}(\mathcal{A})) \\ & = & \Theta'/\Theta(b/\hat{\Theta}, a/\hat{\Theta}). \quad (\text{Definition of } \Theta'/\Theta) \end{array}$$

- For Transitivity,

$$\begin{array}{rcl} \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \Theta'/\Theta(b/\hat{\Theta}, c/\hat{\Theta}) & = & \Theta'(a,b) \wedge \Theta'(b,c) \\ & & \quad (\text{Definition of } \Theta'/\Theta) \\ & \leq & \Theta'(a,c) \\ & & \quad (\Theta' \in \mathrm{Gon}(\mathcal{A})) \\ & = & \Theta'/\Theta(a/\hat{\Theta}, c/\hat{\Theta}). \\ & & \quad (\text{Definition of } \Theta'/\Theta) \end{array}$$

- Finally, for Congruence,

$$\begin{array}{l} \bigwedge_{i=1}^n \Theta'/\Theta(a_i/\hat{\Theta}, b_i/\hat{\Theta}) \\ \qquad = \bigwedge_{i=1}^n \Theta'(a_i, b_i) \\ \qquad\quad (\text{Definition of } \Theta'/\Theta) \\ \qquad \leq \Theta'(\lambda^{\mathbf{A}}(a_1,\ldots,a_n), \lambda^{\mathbf{A}}(b_1,\ldots,b_n)) \\ \qquad\quad (\Theta' \in \mathrm{Gon}(\mathcal{A})) \\ \qquad = \Theta'/\Theta(\lambda^{\mathbf{A}}(a_1,\ldots,a_n)/\hat{\Theta}, \lambda^{\mathbf{A}}(b_1,\ldots,b_n)/\hat{\Theta}) \\ \qquad\quad (\text{Definition of } \Theta'/\Theta) \\ \qquad = \Theta'/\Theta(\lambda^{\mathbf{A}/\hat{\Theta}}(a_1/\hat{\Theta},\ldots,a_n/\hat{\Theta}), \lambda^{\mathbf{A}/\hat{\Theta}}(b_1/\hat{\Theta},\ldots,b_n/\hat{\Theta})). \\ \qquad\quad (\text{Definition of } \lambda^{\mathbf{A}/\hat{\Theta}}) \end{array}$$

Additionally, for all $a, b \in A$,

$$\begin{array}{rcl} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a,b) \quad (\text{Definition of } \bar{\Theta}) \\ & \leq & \Theta'(a,b) \quad (\text{Hypothesis}) \\ & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}). \quad (\text{Definition of } \Theta'/\Theta) \end{array}$$

Now we conclude that $\Theta'/\Theta \in \mathrm{Gon}(\mathcal{A}/\Theta)$. Next, for all $a, b \in A$, we have

$$\begin{array}{rl} \langle a/\hat{\Theta}, b/\hat{\Theta}\rangle \in \widehat{\Theta'/\Theta} & \text{iff} \quad \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) = \top \quad (\text{Definition of } \widehat{\Theta'/\Theta}) \\ & \text{iff} \quad \Theta'(a,b) = \top \quad (\text{Definition of } \Theta'/\Theta) \\ & \text{iff} \quad \langle a,b\rangle \in \hat{\Theta}' \quad (\text{Definition of } \hat{\Theta}') \\ & \text{iff} \quad \langle a/\hat{\Theta}, b/\hat{\Theta}\rangle \in \hat{\Theta}'/\hat{\Theta}. \quad (\text{Definition of } \hat{\Theta}'/\hat{\Theta}) \end{array}$$

This proves that $\widehat{\Theta'/\Theta} = \hat{\Theta}'/\hat{\Theta}$. Thus, by the Second Isomorphism Theorem of Universal Algebra,

$$(a/\hat{\Theta})(\hat{\Theta}'/\hat{\Theta}) \mapsto a/\hat{\Theta}'$$

is an isomorphism $(\mathbf{A}/\hat{\Theta})/\widehat{\Theta'/\Theta} \cong \mathbf{A}/\hat{\Theta}'$.

It only remains to show that this algebra isomorphism is actually a G-algebra isomorphism. We have, for all $a, b \in A$,

$$\begin{array}{rcl}\overline{\Theta'/\Theta}((a/\hat{\Theta})/\widehat{\Theta'/\Theta}, (b/\hat{\Theta})/\widehat{\Theta'/\Theta}) & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) \\ & = & \Theta'(a, b) \\ & = & \bar{\Theta}'(a/\hat{\Theta}', b/\hat{\Theta}').\end{array}$$

Hence $(\mathcal{A}/\Theta)/(\Theta'/\Theta) \cong \mathcal{A}/\Theta'$ via $(a/\hat{\Theta})(\hat{\Theta}'/\hat{\Theta}) \mapsto a/\hat{\Theta}'$. ∎

We may now formulate an analog of the Correspondence Theorem.

Theorem 54 (Correspondence Theorem) *Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. The segment $[\Theta, \nabla_{\mathcal{A}}]$ of $\mathbf{Gon}(\mathcal{A})$ is isomorphic to the lattice $\mathbf{Gon}(\mathcal{A}/\Theta)$ by the mapping $\Theta' \mapsto \Theta'/\Theta$.*

Proof: Suppose $\Theta' \in \mathrm{Gon}(\mathcal{A})$ is such that $\Theta \le \Theta'$. By the Second Isomorphism Theorem 53, we have $\Theta'/\Theta \in \mathrm{Gon}(\mathcal{A}/\Theta)$. Conversely, consider $\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)$. Define $\tilde{\Theta}'' : A^2 \to G$ by setting, for all $a, b \in A$,

$$\tilde{\Theta}''(a, b) = \Theta''(a/\hat{\Theta}, b/\hat{\Theta}).$$

We show that $\tilde{\Theta}'' \in \mathrm{Gon}(\mathbf{A})$. We have, for all $a, b, c \in A$, all n-ary $\lambda \in \mathcal{L}$ and all $a_1, b_1, \ldots, a_n, b_n \in A$:

- For Reflexivity,

$$\begin{array}{rcl}\tilde{\Theta}''(a, a) & = & \Theta''(a/\hat{\Theta}, a/\hat{\Theta}) \quad \text{(Definition of } \tilde{\Theta}''\text{)} \\ & = & \top. \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta))\end{array}$$

- For Symmetry,

$$\begin{array}{rcl}\tilde{\Theta}''(a, b) & = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}) \quad \text{(Definition of } \tilde{\Theta}''\text{)} \\ & = & \Theta''(b/\hat{\Theta}, a/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)) \\ & = & \tilde{\Theta}''(b, a). \quad \text{(Definition of } \tilde{\Theta}''\text{)}\end{array}$$

- For Transitivity,

$$\begin{array}{rcl}\tilde{\Theta}''(a, b) \wedge \tilde{\Theta}''(b, c) & = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \Theta''(b/\hat{\Theta}, c/\hat{\Theta}) \\ & & \quad \text{(Definition of } \tilde{\Theta}''\text{)} \\ & \le & \Theta''(a/\hat{\Theta}, c/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)) \\ & = & \tilde{\Theta}''(a, c). \quad \text{(Definition of } \tilde{\Theta}''\text{)}\end{array}$$

- For Congruence,

$$\begin{array}{rcl}
\bigwedge_{i=1}^n \tilde{\Theta}''(a_i, b_i) & = & \bigwedge_{i=1}^n \Theta''(a_i/\hat{\Theta}, b_i/\hat{\Theta}) \quad (\text{Definition of } \tilde{\Theta}'') \\
& \leq & \Theta''(\lambda^{\mathbf{A}/\hat{\Theta}}(a_1/\hat{\Theta}, \dots, a_n/\hat{\Theta}), \lambda^{\mathbf{A}/\hat{\Theta}}(b_1/\hat{\Theta}, \dots, b_n/\hat{\Theta})) \\
& & \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)) \\
& = & \Theta''(\lambda^{\mathbf{A}}(a_1, \dots, a_n)/\hat{\Theta}, \lambda^{\mathbf{A}}(b_1, \dots, b_n)/\hat{\Theta}) \\
& & \quad (\text{Definition of } \lambda^{\mathbf{A}/\hat{\Theta}}) \\
& = & \tilde{\Theta}''(\lambda^{\mathbf{A}}(a_1, \dots, a_n), \lambda^{\mathbf{A}}(b_1, \dots, b_n)). \\
& & \quad (\text{Definition of } \tilde{\Theta}'')
\end{array}$$

Next we show $\Theta' \in \mathrm{Gon}(\mathcal{A})$. We have, for all $a, b \in A$,

$$\begin{array}{rcl}
\Theta(a, b) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad (\text{Definition of } \bar{\Theta}) \\
& \leq & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)) \\
& = & \tilde{\Theta}''(a, b). \quad (\text{Definition of } \tilde{\Theta}'')
\end{array}$$

The two mapping above are easily seen to be order preserving and inverses of one another. Indeed, for all $\Theta' \in \mathrm{Gon}(\mathcal{A})$, with $\Theta \leq \Theta'$, and all $a, b \in A$,

$$\begin{array}{rcl}
\widetilde{\Theta'/\Theta}(a, b) & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) \quad (\text{Definition of } \widetilde{\Theta'/\Theta}) \\
& = & \Theta'(a, b). \quad (\text{Definition of } \Theta'/\Theta)
\end{array}$$

And, for all $\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)$ and all $a, b \in A$,

$$\begin{array}{rcl}
\tilde{\Theta}''/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) & = & \tilde{\Theta}''(a, b) \quad (\text{Definition of } \tilde{\Theta}''/\Theta) \\
& = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}). \quad (\text{Definition of } \tilde{\Theta}'')
\end{array}$$

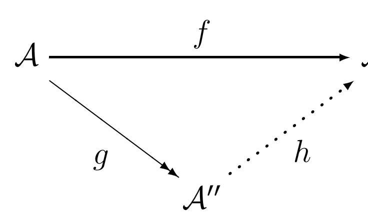
This establishes the claimed correspondence. ■

We also present a fill-in lemma for G-algebras that will come in handy in the sequel.

Lemma 55 *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$, $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$ and $\mathcal{A}'' = \langle \mathbf{A}'', E'' \rangle$ be G-algebras, $f : \mathcal{A} \to \mathcal{A}'$ a G-morphism and $g : \mathcal{A} \to \mathcal{A}''$ a surjective G-morphism, such that*

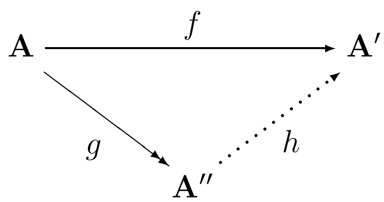
$$E'' \circ g^2 \leq E' \circ f^2.$$

Then, there exists a unique G-morphism $h : \mathcal{A}'' \to \mathcal{A}'$, such that

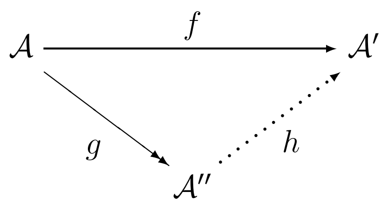


$$h \circ g = f.$$

Proof: Notice that $E'' \circ g^2 \leq E' \circ f^2$ implies $\widehat{E'' \circ g^2} \subseteq \widehat{E' \circ f^2}$, i.e., $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. Thus, there exists a unique homomorphism $h : \mathbf{A}'' \to \mathbf{A}'$ which makes the following triangle commute.



This gives rise to a commutative diagram of G-algebras.



In fact, for all $a'', b'' \in A''$, we have

$$\begin{array}{rcll} E''(a'',b'') & = & E''(g(a),g(b)) & (g \text{ surjective}) \\ & \leq & E'(f(a),f(b)) & (\text{Hypothesis}) \\ & = & E'(h(g(a)),h(g(b))) & (h \circ g = f) \\ & = & E'(h(a''),h(b'')). & (\text{Choice of } a,b) \end{array}$$

Hence $h : \mathcal{A}'' \to \mathcal{A}'$ is a G-morphism. ■

Another operation on G-algebras that will be needed later is that of taking subdirect products. Let $\mathcal{A}_i = \langle \mathbf{A}_i, E_i \rangle$, $i \in I$, be a collection of G-algebras. The **direct product** of the $\mathcal{A}_i$, $i \in I$, is the G-algebra

$$\prod_{i \in I} \mathcal{A}_i = \left\langle \prod_{i \in I} \mathbf{A}_i, \bigwedge_{i \in I} E_i \right\rangle,$$

where $\prod_{i \in I} \mathbf{A}_i$ is the product $\mathcal{L}$-algebra of the $\mathbf{A}_i$, $i \in I$, and $\bigwedge_{i \in I} E_i$ is the function $\bigwedge_{i \in I} E_i : \prod_{i \in I} A_i \to G$ defined, for all $a_i, b_i \in A_i$, $i \in I$, by

$$\bigwedge_{i \in I} E_i(\bar{a}, \bar{b}) = \bigwedge_{i \in I} E_i(a_i, b_i).$$

We note that this is well defined, i.e., $\bigwedge_{i \in I} E_i$ is a reduced G-congruence on $\prod_{i \in I} \mathbf{A}_i$. E.g., to show Transitivity, let $a_i, b_i, c_i \in A_i$, $i \in I$. Then

$$\begin{array}{rcl} \bigwedge_{i \in I} E_i(\bar{a}, \bar{b}) \wedge \bigwedge_{i \in I} E_i(\bar{b}, \bar{c}) & = & \bigwedge_{i \in I} E_i(a_i, b_i) \wedge \bigwedge_{i \in I} E_i(b_i, c_i) \\ & = & \bigwedge_{i \in I}(E_i(a_i, b_i) \wedge E_i(b_i, c_i)) \\ & \leq & \bigwedge_{i \in I} E_i(a_i, c_i) \\ & = & \bigwedge_{i \in I} E_i(\bar{a}, \bar{c}). \end{array}$$

To show it is reduced, let $a_i, b_i \in A_i$, $i \in I$. Then

$$\begin{array}{rll}\bigwedge_{i\in I} E_i(\bar{a},\bar{b}) = \top & \text{iff} & \bigwedge_{i\in I} E_i(a_i,b_i) = \top \\ & \text{iff} & E_i(a_i,b_i) = \top,\ i \in I, \\ & \text{iff} & a_i = b_i,\ i \in I, \\ & \text{iff} & \bar{a} = \bar{b}.\end{array}$$

Note that the projection $\pi_i : \prod_{i\in I} \mathbf{A}_i \to \mathbf{A}_i$ becomes a G-morphism

$$\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i.$$

Indeed, for all $a_i, b_i \in A_i$, $i \in I$,

$$\bigwedge_{i\in I} E_i(\bar{a},\bar{b}) = \bigwedge_{i\in I} E_i(a_i,b_i) \le E_i(a_i,b_i) = (E_i \circ \pi_i)(\bar{a},\bar{b}).$$

$\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i$ is called the **i-th projection G-morphism**.

Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ be G-algebras. $\mathcal{A}$ is said to be a **subalgebra** of $\mathcal{A}'$, written $\mathcal{A} \le \mathcal{A}'$, if $\mathbf{A}$ is a subalgebra of $\mathbf{A}'$ and E is the restriction of E' on A^2, i.e., for all $a, b \in A$,

$$E(a,b) = E'(a,b).$$

A G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is called a **subdirect product** of the G-algebras $\mathcal{A}_i$, $i \in I$, written $\mathcal{A} \subseteq_{\mathsf{sd}} \prod_{i\in I} \mathcal{A}_i$, if it is a subalgebra of $\prod_{i\in I} \mathcal{A}_i$, such that, for all $i \in I$, $\pi_i(A)$ is surjective, where $\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i$ denotes the i-th projection G-morphism. An embedding $j : \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i$ is a **subdirect embedding**, written $j : \mathcal{A} \hookrightarrow_{\mathsf{sd}} \prod_{i\in I} \mathcal{A}_i$, if $j(\mathcal{A})$ is a subdirect product of the $\mathcal{A}_i$, $i \in I$.

We have the following result paralleling the one applicable to subdirect products of ordinary algebras.

Proposition 56 *Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra and $\Theta, \Theta_i \in \mathrm{Gon}(\mathcal{A})$, $i \in I$. Then*

$$\begin{array}{rrcl} j: & \mathcal{A}/\Theta & \longrightarrow & \prod_{i\in I} \mathcal{A}/\Theta_i; \\ & a/\hat{\Theta} & \longmapsto & \langle a/\hat{\Theta}_i : i \in I\rangle,\end{array}$$

is a subdirect embedding if and only if $\Theta = \bigwedge_{i\in I} \Theta_i$.

Proof: Suppose, first, that $\Theta = \bigwedge_{i\in I} \Theta_i$. Then, for all $a, b \in A$,

$$\begin{array}{rll}\langle a,b\rangle \in \hat{\Theta} & \text{iff} & \Theta(a,b) = \top \quad \text{(Definition of } \hat{\Theta}) \\ & \text{iff} & \bigwedge_{i\in I} \Theta_i(a,b) = \top \quad \text{(Hypothesis)} \\ & \text{iff} & \Theta_i(a,b) = \top,\ i \in I, \quad \text{(Property of } \bigwedge) \\ & \text{iff} & \langle a,b\rangle \in \hat{\Theta}_i,\ i \in I. \quad \text{(Definition of } \hat{\Theta}_i)\end{array}$$

Thus, j is an algebra embedding. Moreover, for all $a, b \in A$,

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a,b) \quad \text{(Definition of } \bar{\Theta}) \\ & = & \bigwedge_{i\in I} \Theta_i(a,b) \quad \text{(Hypothesis)} \\ & = & \bigwedge_{i\in I} \bar{\Theta}_i(a/\hat{\Theta}_i, b/\hat{\Theta}_i) \quad \text{(Definition of } \bar{\Theta}_i) \\ & = & \bigwedge_{i\in I} \bar{\Theta}_i \circ j(a/\hat{\Theta}, b/\hat{\Theta}). \quad \text{(Definition of } j)\end{array}$$

Hence, j is a G-algebra embedding. Finally, it is clear that $\pi_i \circ j$ is surjective, for all $i \in I$. So j is a subdirect embedding.

Suppose, conversely, that j is a subdirect embedding. It follows that, for all $a, b \in A$, we have

$$\begin{array}{c}\langle a,b\rangle \in \hat{\Theta} \quad \text{iff} \quad \langle a,b\rangle \in \hat{\Theta}_i,\ i \in I, \\ \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \bigwedge_{i\in I} \bar{\Theta}_i(a/\hat{\Theta}_i, b/\hat{\Theta}_i).\end{array}$$

It is easy to see that these two conditions ensure that $\Theta = \bigwedge_{i\in I} \Theta_i$. For all $a, b \in A$,

$$\begin{array}{rcl}\Theta(a,b) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad \text{(Definition of } \bar{\Theta}) \\ & = & \bigwedge_{i\in I} \bar{\Theta}_i(a/\hat{\Theta}_i, b/\hat{\Theta}_i) \quad \text{(Display above)} \\ & = & \bigwedge_{i\in I} \Theta_i(a,b). \quad \text{(Definition of } \bar{\Theta}_i)\end{array}$$

Thus, j is a subdirect embedding if and only if $\Theta = \bigwedge \Theta_i$. ■

Given a class K of G-algebras and a G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$ not necessarily in K, a G-congruence $\Theta \in \mathrm{Gon}(\mathcal{A})$ is called a **K-G-congruence** if $\mathcal{A}/\Theta \in \mathsf{K}$. We let $\mathrm{Gon}_{\mathsf{K}}(\mathcal{A})$ denote the set of all K-G-congruences on the G-algebra $\mathcal{A}$. This set ordered by $\leq$ is not generally a lattice.

The absolutely free $\mathcal{L}$-algebra is denoted by $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V), \mathcal{L}\rangle$, where V is a fixed countably infinite set of variables. This algebra may also be represented as a G-algbera $\mathcal{F}m_{\mathcal{L}}(V) = \langle \mathbf{Fm}_{\mathcal{L}}(V), \Delta_{\mathbf{Fm}_{\mathcal{L}}(V)}\rangle$ by attaching the reduced G-congruence that has value $\top$ on the diagonal and $\bot$ elsewhere. Formulas are denoted by $\varphi, \psi, \chi, \ldots$ and equations, formally pairs of formulas, by $\langle \varphi, \psi\rangle$ or $\varphi \approx \psi$.

Given a G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, an **interpretation** in $\mathcal{A}$ is a G-morphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$. Note that such morphisms coincide with homomorphisms $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, because of the G-congruence of $\mathcal{F}m_{\mathcal{L}}(V)$. They are completely specified by their values on the variables. Accordingly, given a formula φ, we write $\varphi(\bar{x})$ to mean that the variables appearing in φ are among those listed in $\bar{x}$. Moreover, we write $\varphi^{\mathbf{A}}(\bar{a})$ to denote the element of A resulting by interpreting the variables in $\bar{x}$ by the corresponding elements in $\bar{a}$. This notation is naturally extended to sets of formulas, which are denoted by $\Phi, \Psi, \Gamma, \ldots$.

A substitution $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ is an endomorphism of $\mathbf{Fm}_{\mathcal{L}}(V)$.

3.3 Graded Matrices

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra. We saw that a *graded filter* or *G-filter* on $\mathbf{A}$ is a function

$$F : A \to G.$$

Recall from Section 2.6 that, given a G-congruence E on $\mathbf{A}$ and a G-filter F on $\mathbf{A}$, E is said to be *compatible with* F, written E comp F, if, for all $a, b \in A$,

$$E(a,b) \wedge F(a) \leq F(b).$$

If E is compatible with F, then F is said to be a *G-filter of* the G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$. In this chapter, we enhance slightly the notion of G-matrix in comparison to the definition we gave in Section 2.4. Here a **graded matrix** or G-**matrix** $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is a pair consisting of:

- A G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$;
- A G-filter F on $\mathcal{A}$.

A G-**congruence on** $\mathfrak{A}$ is a G-congruence Θ on $\mathcal{A}$ that is compatible with F. That is, it has to satisfy:

- $\Theta \geq E$;
- Θ comp F.

We write $\mathrm{Gon}(\mathfrak{A})$ for the set of all G-congruences on the G-matrix $\mathfrak{A}$.

Let $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-matrix and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Consider the quotient G-algebra $\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta} \rangle$ and on $\mathcal{A}/\Theta$ define the function $\bar{F} : \mathcal{A}/\hat{\Theta} \to G$ by setting, for all $a \in A$,

$$\bar{F}(a/\hat{\Theta}) = F(a).$$

This function is well defined. If $a, b \in A$, such that $\langle a, a' \rangle \in \hat{\Theta}$, then

$$\begin{array}{rcll} F(a) & = & \top \wedge F(a) & \\ & = & \Theta(a,a') \wedge F(a) & (\langle a, a' \rangle \in \hat{\Theta}) \\ & \leq & F(a'), & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \end{array}$$

whence, by symmetry, $F(a) = F(a')$.

The **quotient** G-**matrix** $\mathfrak{A}/\Theta$ **of** $\mathfrak{A}$ **by** Θ is the G-matrix

$$\mathfrak{A}/\Theta = \langle \mathcal{A}/\Theta, \bar{F} \rangle.$$

To see that this is well defined it must be shown that $\bar{\Theta}$ comp $\bar{F}$. Suppose $a, b \in A$. Then

$$\begin{array}{rcll} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{F}(a/\hat{\Theta}) & = & \Theta(a,b) \wedge F(a) & (\text{Definitions of } \bar{\Theta} \text{ and } \bar{F}) \\ & \leq & F(b) & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{F}(b/\hat{\Theta}). & (\text{Definition of } \bar{F}) \end{array}$$

We can show that, under certain conditions on G, the collection $\mathrm{Gon}(\mathfrak{A})$ of all G-congruences on $\mathfrak{A}$ is a principal ideal of $\mathbf{Gon}(\mathcal{A})$. This was essentially done in Proposition 15. We shall restrict attention to those lattices for which this is the case and, hence, we shall always assume the existence of a largest G-congruence in $\mathrm{Gon}(\mathfrak{A})$. This maximum element is called the **Leibniz G-congruence** of the G-matrix $\mathfrak{A}$, or of the G-filter F on $\mathcal{A}$, and is denoted by $\Omega(\mathfrak{A})$ or $\Omega_{\mathcal{A}}(F)$. The well-known characterization of Blok and Pigozzi (see Page 11 of [6]) of the ordinary Leibniz congruence is generalized as follows. Note that this is a slight improvement over Theorem 16, since our G-matrices here are over G-algebras and not simply over algebras, as was the case in Chapter 2. However, the statement and the proof technique, modulo some details, are almost identical. We give it again for the sake of completeness.

Theorem 57 *Let $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ be a G-matrix. We have, for all $a, b \in A$,*

$$\Omega_{\mathcal{A}}(F)(a,b) = \bigwedge_{\substack{\varphi(x,\bar{z})\in \mathrm{Fm}_{\mathcal{L}}(V)\\ \bar{c}\in A}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Proof: Define, for all $a, b \in A$,

$$\Theta(a,b) = \bigwedge_{\substack{\varphi(x,\bar{z})\in \mathrm{Fm}_{\mathcal{L}}(V)\\ \bar{c}\in A}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Our goal is to show that $\Omega_{\mathcal{A}}(F) = \Theta$. For the inequality left to right, suppose $a, b \in A$, $\varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\bar{c} \in A$. Then

$$\begin{array}{rcl}\Omega_{\mathcal{A}}(F)(a,b) & \leq & \Omega_{\mathcal{A}}(F)(\varphi^{\mathbf{A}}(a,\bar{c}), \varphi^{\mathbf{A}}(b,\bar{c}))\\ & & \quad (\Omega_{\mathcal{A}}(F) \text{ a } G\text{-congruence})\\ & \leq & F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})).\\ & & \quad (\Omega_{\mathcal{A}}(F) \text{ compatible with } F)\end{array}$$

Since $\varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\bar{c} \in A$ were arbitrary,

$$\Omega_{\mathcal{A}}(F)(a,b) \leq \bigwedge_{\substack{\varphi(x,\bar{z})\in \mathrm{Fm}_{\mathcal{L}}(V)\\ \bar{c}\in A}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})) = \Theta(a,b).$$

By definition, E is also a G-congruence on $\mathbf{A}$ that is compatible with F. Hence, using the same reasoning, with E in place of $\Omega_{\mathcal{A}}(F)$, we obtain that $E \leq \Theta$.

For the reverse inequality, taking into account the property of $\Omega_{\mathcal{A}}(F)$ as the largest G-congruence on $\mathcal{A}$ compatible with F, it suffices to show that Θ is a G-congruence on $\mathcal{A}$ compatible with F. We know that $E \leq \Theta$.

- For all $a \in A$,

$$\Theta(a,a) = \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(a,\bar{c})) = \top.$$

- For all $a, b \in A$,

$$\begin{aligned} \Theta(a,b) &= \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})) \\ &= \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(b,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(a,\bar{c})) \\ &= \Theta(b,a). \end{aligned}$$

- For all $a, b, c \in A$,

$$\begin{aligned} \Theta(a,b) \wedge \Theta(b,c) &= \bigwedge_{\varphi,\bar{e}} F(\varphi^{\mathbf{A}}(a,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{e})) \\ &\quad \wedge \bigwedge_{\varphi,\bar{e}} F(\varphi^{\mathbf{A}}(b,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\bar{e})) \\ &\le \bigwedge_{\varphi,\bar{e}} (F(\varphi^{\mathbf{A}}(a,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{e})) \\ &\qquad \wedge F(\varphi^{\mathbf{A}}(b,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\bar{e}))) \\ &\le \bigwedge_{\varphi,\bar{e}} F(\varphi^{\mathbf{A}}(a,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\bar{e})) \\ &= \Theta(a,c). \end{aligned}$$

- Suppose λ is an n-ary operation symbol in $\mathcal{L}$ and $a_1, b_1, \ldots, a_n, b_n \in \mathcal{L}$. Let us write, for convenience and brevity, $a_{i\ldots j}$ so signify the tuple $a_i, a_{i+1}, \ldots, a_j$. Then we have

$$\begin{aligned} \bigwedge_{i=1}^n \Theta(a_i,b_i) &= \bigwedge_{i=1}^n \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(a_i,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b_i,\bar{c})) \\ &\le \bigwedge_{i=1}^n \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i-1}, a_{i\ldots n}),\bar{c})) \\ &\qquad \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i}, a_{i+1\ldots n}),\bar{c})) \\ &\le \bigwedge_{\varphi,\bar{c}} \bigwedge_{i=1}^n F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i-1}, a_{i\ldots n}),\bar{c})) \\ &\qquad \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i}, a_{i+1\ldots n}),\bar{c})) \\ &\le \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(a_1,\ldots,a_n),\bar{c})) \\ &\qquad \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_1,\ldots,b_n),\bar{c})) \\ &= \Theta(\lambda^{\mathbf{A}}(a_1,\ldots,a_n), \lambda^{\mathbf{A}}(b_1,\ldots,b_n)). \end{aligned}$$

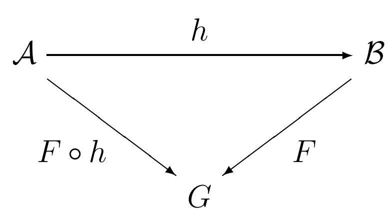
Finally, taking $x \in \mathrm{Fm}_{\mathcal{L}}(V)$ for φ in the definition of Θ, we have, for all $a, b \in A$,

$$\Theta(a,b) \le F(a) \leftrightarrow F(b),$$

whence Θ is compatible with F. We conclude that $\Theta \le \Omega_{\mathcal{A}}(F)$. ■

The following result is an analog of the well known property of the traditional theory asserting that the Leibniz operator commutes with inverse surjective homomorphisms.

Theorem 58 *Let $\mathcal{A} = \langle \mathbf{A}, D \rangle$ and $\mathcal{B} = \langle \mathbf{B}, E \rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{B}$ a surjective G-morphism. For every G-filter F on $\mathcal{B}$, $F \circ h$ is a G-filter on $\mathcal{A}$ and*



$$\Omega_{\mathcal{B}}(F) \circ h^2 = \Omega_{\mathcal{A}}(F \circ h).$$

Proof: We must show that D comp $F \circ h$. We have, for all $a, b \in A$,

$$\begin{array}{rcll} D(a,b) \wedge F(h(a)) & \leq & E(h(a),h(b)) \wedge F(h(a)) & (h : \mathcal{A} \to \mathcal{B}) \\ & \leq & F(h(b)). & (F\ G\text{-filter on } \mathcal{B}) \end{array}$$

For the last equality, let $a, b \in A$. Then, we have

$$\begin{array}{rcl} \Omega_{\mathcal{B}}(F)(h(a),h(b)) & = & \bigwedge_{\varphi,\bar{d}} F(\varphi^{\mathbf{B}}(h(a),\bar{d})) \leftrightarrow F(\varphi^{\mathbf{B}}(h(b),\bar{d})) \\ & & \text{(Theorem 57)} \\ & = & \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{B}}(h(a),h(\bar{c}))) \leftrightarrow F(\varphi^{\mathbf{B}}(h(b),h(\bar{c}))) \\ & & (h \text{ surjective}) \\ & = & \bigwedge_{\varphi,\bar{c}} F(h(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(h(\varphi^{\mathbf{A}}(b,\bar{c}))) \\ & & (h : \mathbf{A} \to \mathbf{B}) \\ & = & \Omega_{\mathcal{A}}(F \circ h)(a,b). \quad \text{(Theorem 57)} \end{array}$$

Therefore, $\Omega_{\mathcal{B}}(F) \circ h^2 = \Omega_{\mathcal{A}}(F \circ h)$. ∎

The mapping $F \mapsto \Omega_{\mathcal{A}}(F)$ is called the **Leibniz operator** of the G-algebra $\mathcal{A}$.

We say a G-matrix $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, is **reduced** when its only G-matrix congruence is E.

Proposition 59 *Let $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-matrix. Then $\mathfrak{A}^* := \mathfrak{A}/\Omega_{\mathcal{A}}(F)$ is reduced.*

Proof: To prove that $\mathfrak{A}^*$ is reduced, we need to show that $\Omega_{\mathcal{A}/\Omega_{\mathcal{A}}(F)}(\bar{F})$ is equal to $\bar{\Omega}_{\mathcal{A}}(F)$. Let $a, b \in A$. We have

$$\begin{array}{l} \Omega_{\mathcal{A}/\Omega_{\mathcal{A}}(F)}(\bar{F})(a/\hat{\Omega}_{\mathcal{A}}(F), b/\hat{\Omega}_{\mathcal{A}}(F)) \\ \quad = (\Omega_{\mathcal{A}/\Omega_{\mathcal{A}}(F)}(\bar{F}) \circ \pi^2_{\Omega_{\mathcal{A}}(F)})(a,b) \quad (\text{Definition of } \pi_{\Omega_{\mathcal{A}}(F)}) \\ \quad = \Omega_{\mathcal{A}}(\bar{F} \circ \pi_{\Omega_{\mathcal{A}}(F)})(a,b) \quad \text{(Theorem 58)} \\ \quad = \Omega_{\mathcal{A}}(F)(a,b) \quad (\text{Definition of } \bar{F}) \\ \quad = \bar{\Omega}_{\mathcal{A}}(F)(a/\hat{\Omega}_{\mathcal{A}}(F), b/\hat{\Omega}_{\mathcal{A}}(F)). \quad \text{(Proposition 50)} \end{array}$$

This shows that $\mathfrak{A}^*$ is reduced. ∎

$\mathfrak{A}^*$ is called the **reduction** of $\mathfrak{A}$. Given a class M of G-matrices, we let

$$\mathsf{M}^* = \{\mathfrak{A}^* : \mathfrak{A} \in \mathsf{M}\}.$$

3.4 Graded Logics

Recall the notion of *G-logic* from Section 2.2. Here, we generalize this notion to include logics that are not necessarily over (G-sets on) the formula algebra but on an arbitrary G-algebra. This comes in handy when discussing models, which is the central topic of this chapter.

Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra. By a (**closure**) G-**operator** on $\mathcal{A}$, we mean a mapping $C : G^A \to G^A$, such that the following axioms are satisfied:

Inflationarity $X \leq C(X)$, for all $X : A \to G$;

Monotonicity $C(X) \leq C(Y)$, for all $X, Y : A \to G$, with $X \leq Y$;

Idempotency $C(C(X)) = C(X)$, for all $X : A \to G$;

Translation $C(X \circ h) \leq C(X) \circ h$, for all $X : A \to G$ and all $h : \mathbf{A} \to \mathbf{A}$.

Given a G-operator C on $\mathcal{A}$, we say that a function $X : A \to G$, or a G-set on A, is **closed**, or that it is a **theory** of C, if

$$C(X) = X.$$

By a **graded logic**, or a G-**logic**, we mean a pair $\mathbb{L} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A} = \langle \mathbf{A}, E \rangle$ is a G-algebra and C is a G-operator on $\mathbf{A}$, such that E is compatible with every closed function X. We denote by $\mathcal{C} = \mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathrm{C})$ the collection of all closed functions, or closed G-sets, of $\mathbb{L}$. Note that the partial order $\leq$ on G^A is inherited by the set of closed functions of $\mathbb{L}$.

Lemma 60 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic. Then, for all $X : A \to G$,*

$$C(X) = \bigwedge \{Y : X \leq Y \in \mathrm{Cl}(\mathbb{L})\}.$$

Proof: If $X \leq Y \in \mathrm{Cl}(\mathbb{L})$, then, by Monotonicity and closure,

$$C(X) \leq C(Y) = Y.$$

Hence, we get $C(X) \leq \bigwedge \{Y : X \leq Y \in \mathrm{Cl}(\mathbb{L})\}$.

Conversely, since, by Inflationarity and Idempotency, $X \leq C(X) \in \mathrm{Cl}(\mathbb{L})$, we get

$$\bigwedge \{Y : X \leq Y \in \mathrm{Cl}(\mathbb{L})\} \leq C(X).$$

Therefore, $C(X) = \bigwedge \{Y : X \leq Y \in \mathrm{Cl}(\mathbb{L})\}$. ■

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$ and $\mathbb{L}' = \langle \mathcal{A}, C' \rangle$ be two G-logics over the same G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$. We say that $\mathbb{L}$ is **weaker than** $\mathbb{L}'$ or that $\mathbb{L}'$ is **stronger than** $\mathbb{L}$ or that $\mathbb{L}'$ is an **extension** of $\mathbb{L}$, written $\mathbb{L} \leq \mathbb{L}'$, if, for all $X \in G^A$,

$$C(X) \leq C'(X).$$

We denote the collection of all G-logics on a G-algebra $\mathcal{A}$ by $\mathrm{Log}_{\mathbf{G}}(\mathcal{A})$. This set forms the ordered structure $\mathbf{Log}_{\mathbf{G}}(\mathcal{A}) = \langle \mathrm{Log}_{\mathbf{G}}(\mathcal{A}), \leq \rangle$.

As is the case with ordinary logics, the extension relation between G-logics is characterized by the reverse containment relation between their closed G-sets.

Proposition 61 *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra and $\mathbb{L} = \langle \mathcal{A}, C \rangle$ and $\mathbb{L}' = \langle \mathcal{A}, C' \rangle$ be G-logics over $\mathcal{A}$. Then,*

$$\mathbb{L} \leq \mathbb{L}' \quad \textit{iff} \quad \mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L}).$$

Proof: Suppose, first, that $\mathbb{L} \le \mathbb{L}'$ and let $X \in \mathrm{Cl}(\mathbb{L}')$. To see that $X \in \mathrm{Cl}(\mathbb{L})$, compute

$$C(X) \le C'(X) = X.$$

Hence, $\mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L})$.

Suppose, conversely, that $\mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L})$ and let $X : A \to G$. Then, using Lemma 60, we get

$$C(X) = \bigwedge\{Y : X \le Y \in \mathrm{Cl}(\mathbb{L})\} \le \bigwedge\{Y : X \le Y \in \mathrm{Cl}(\mathbb{L}')\} = C'(X).$$

Therefore, $\mathbb{L} \le \mathbb{L}'$. ∎

3.5 Logical Congruences

Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic. A **logical G-congruence** of $\mathbb{L}$ is a G-congruence Θ on $\mathcal{A}$, such that, for all $X : A \to G$ and all $a, b \in A$,

$$\Theta(a,b) \wedge C(X)(a) \le C(X)(b).$$

This condition is equivalent to the assertion that Θ is compatible with every closed function in $\mathbb{L}$, i.e., it satisfies, for every $X \in \mathrm{Cl}(\mathbb{L})$ and all $a, b \in A$,

$$\Theta(a,b) \wedge X(a) \le X(b).$$

The collection of all logical G-congruences of $\mathbb{L}$ is denoted by $\mathrm{Gon}(\mathbb{L})$. From the characterization above, we see that

$$\mathrm{Gon}(\mathbb{L}) = \bigcap\{\mathrm{Gon}(\langle \mathcal{A}, \mathrm{X}\rangle) : \mathrm{X} \in \mathrm{Cl}(\mathbb{L})\}.$$

The set $\mathrm{Gon}(\mathbb{L})$, ordered by $\le$, is a complete lattice, $\mathbf{Gon}(\mathbb{L}) = \langle \mathrm{Gon}(\mathbb{L}), \le\rangle$, and a principal ideal of the lattice $\mathbf{Gon}(\mathcal{A})$. We term its generator the **Tarski G-congruence** of $\mathbb{L}$ and denote it by $\widetilde{\Omega}(\mathbb{L})$ or $\widetilde{\Omega}_{\mathcal{A}}(C)$,

$$\widetilde{\Omega}(\mathbb{L}) = \max \mathrm{Gon}(\mathbb{L}).$$

The **Tarski operator on $\mathcal{A}$** is the mapping that associates $\widetilde{\Omega}_{\mathcal{A}}(C)$ to a G-logic $\mathbb{L} = \langle \mathcal{A}, C\rangle$ on $\mathcal{A}$,

$$\widetilde{\Omega}_{\mathcal{A}} : \langle \mathcal{A}, C\rangle \mapsto \widetilde{\Omega}_{\mathcal{A}}(C).$$

From the definition, we have

$$\mathrm{Gon}(\mathbb{L}) = \{\Theta \in \mathrm{Gon}(\mathcal{A}) : \Theta \le \widetilde{\Omega}(\mathbb{L})\}.$$

Furthermore, for any $\mathbb{L} = \langle \mathcal{A}, C\rangle$, we have

$$\widetilde{\Omega}(\mathbb{L}) = \bigwedge \{\Omega_{\mathcal{A}}(X) : X \in \mathrm{Cl}(\mathbb{L})\}.$$

Directly from Theorem 57 we get the following characterization of the Tarski G-congruence of a G-logic.

Proposition 62 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic. Then, for all $a, b \in A$,*

$$\widetilde{\Omega}(\mathbb{L})(a,b) = \bigwedge_{\substack{X \in \mathrm{Cl}(\mathbb{L}) \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow X(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Proof: Indeed, for every $a, b \in A$, we have

$$\begin{array}{rcl} \widetilde{\Omega}(\mathbb{L})(a,b) & = & \bigwedge_{X \in \mathrm{Cl}(\mathbb{L})} \Omega_{\mathcal{A}}(X)(a,b) \\ & & \text{(Definition of } \widetilde{\Omega}(\mathbb{L})) \\ & = & \bigwedge_{X \in \mathrm{Cl}(\mathbb{L})} \bigwedge_{\substack{\varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow X(\varphi^{\mathbf{A}}(b,\bar{c})). \\ & & \text{(Theorem 57)} \end{array}$$

■

From Proposition 62, it is easily inferred that the Tarski operator on a given G-algebra is monotone with respect to extension ($\le$) on G-logics and $\le$ on G-congruences.

Proposition 63 *Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving on $\mathcal{A}$, i.e., if $\mathbb{L} = \langle \mathcal{A}, C\rangle$ and $\mathbb{L}' = \langle \mathcal{A}, C'\rangle$ are two G-logics on $\mathcal{A}$,*

$$\mathbb{L} \le \mathbb{L}' \quad \textit{implies} \quad \widetilde{\Omega}(\mathbb{L}) \le \widetilde{\Omega}(\mathbb{L}').$$

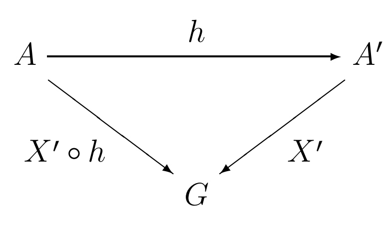
Proof: We have

$$\begin{array}{rcll} \mathbb{L} \le \mathbb{L}' & \text{iff} & \mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L}) & \text{(Proposition 61)} \\ & \text{implies} & \widetilde{\Omega}(\mathbb{L}) \le \widetilde{\Omega}(\mathbb{L}'). & \text{(Proposition 62)} \end{array}$$

■

3.6 Logical Morphisms

Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics. A **logical G-morphism** $h : \mathbb{L} \to \mathbb{L}'$ is a G-morphism $h : \mathcal{A} \to \mathcal{A}'$, such that, for all $X' \in G^{A'}$,



$$X' \in \mathrm{Cl}(\mathbb{L}') \quad \text{implies} \quad X' \circ h \in \mathrm{Cl}(\mathbb{L}).$$

This may be equivalently written as

$$\mathrm{Cl}(\mathbb{L}') \circ h = \{X' \circ h : X' \in \mathrm{Cl}(\mathbb{L}')\} \subseteq \mathrm{Cl}(\mathbb{L}).$$

Note that this definition makes sense, since for all $a, b \in A$,

$$\begin{array}{rcll} E(a,b) \wedge X'(h(a)) & \leq & E'(h(a),h(b)) \wedge X'(h(a)) & (E \leq E' \circ h^2) \\ & \leq & X'(h(b)), & (E' \text{ comp } X') \end{array}$$

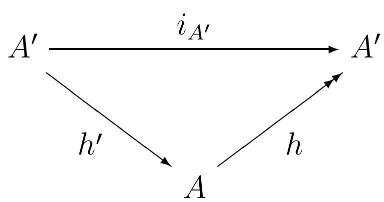
that is E is compatible with $X' \circ h$, for all $X' \in \mathrm{Cl}(\mathbb{L}')$.

Given a logical G-morphism $h : \mathbb{L} \to \mathbb{L}'$, we say that $\mathbb{L}$ is **projectively generated from $\mathbb{L}'$ by h** if

$$\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h.$$

We call $h : \mathbb{L} \to \mathbb{L}'$ a **bilogical G-morphism**, written $h : \mathbb{L} \to_b \mathbb{L}'$, if it is surjective and projectively generates $\mathbb{L}$ from $\mathbb{L}'$.

Recall that given a surjective mapping $h : A \to A'$, there exists a (non uniquely defined, in general) mapping $h' : A' \to A$, such that $h \circ h' = i_{A'}$.



Such a mapping $h' : A' \to A$ is called a **section of** h.

The following proposition, an analog of Proposition 1.4 of [28], provides several characterizations of bilogical G-morphisms.

Proposition 64 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathcal{A} \to \mathcal{A}'$ a surjective morphism, with a section $h' : A' \to A$. Then the following are equivalent:*

(i) $h : \mathbb{L} \to_b \mathbb{L}'$ *is a bilogical G-morphism;*

(ii) $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}') \circ h$ *and for all $X' \in G^{A'}$, $C(X' \circ h) = C'(X') \circ h$;*

(iii) For all $X' \in G^{A'}$, $C'(X') = C(X' \circ h) \circ h'$ and $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$;

(iv) $\mathrm{Cl}(\mathbb{L}') = \mathrm{Cl}(\mathbb{L}) \circ h'$ *and $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$;*

(v) $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h$.

Proof:

(i)⇒(ii) By definition $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h$. In particular, $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}') \circ h$. Moreover, for all $X' \in G^{A'}$, we have

$$\begin{aligned} C'(X') \circ h &= \bigwedge\{Y' : X' \leq Y' \in \mathrm{Cl}(\mathbb{L}')\} \circ h \\ &\quad \text{(Lemma 60)} \\ &= \bigwedge\{Y' \circ h : X' \circ h \leq Y' \circ h \text{ and } Y' \in \mathrm{Cl}(\mathbb{L}')\} \\ &\quad (h \text{ surjective}) \\ &= \bigwedge\{Y : X' \circ h \leq Y \in \mathrm{Cl}(\mathbb{L})\} \\ &\quad (\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h) \\ &= C(X' \circ h). \quad \text{(Lemma 60)} \end{aligned}$$

(ii)⇒(iii) By hypothesis, for all $X' \in G^{A'}$, $C(X' \circ h) = C'(X') \circ h$. Composing with h' on the right, we get $C'(X') = C(X' \circ h) \circ h'$. Next, since E' is compatible with every closed G-set in $\mathbb{L}'$, we have that, for all $a, b \in A$ and all $X' \in G^{A'}$,

$$E'(h(a), h(b)) \wedge C'(X')(h(a)) \leq C'(X')(h(b)).$$

Hence, by hypothesis, for all $X \in G^A$,

$$E'(h(a), h(b)) \wedge C(X)(a) \leq C(X)(b).$$

This proves that $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$.

(iii)⇒(iv) Suppose, first, that $X' \in \mathrm{Cl}(\mathbb{L}')$. Then

$$X' = C'(X') = C(X' \circ h) \circ h' \in \mathrm{Cl}(\mathbb{L}) \circ h'.$$

Conversely, observe that, for all $a \in A$, if $b = h'(h(a))$, then $h(a) = h(b)$ and, hence, $E'(h(a), h(b)) = \top$. Thus, since $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$, for all $X \in \mathrm{Cl}(\mathbb{L})$, $X(a) = X(b) = X(h'(h(a)))$, i.e., $X \circ h' \circ h = X$. Thus, if $X \in \mathrm{Cl}(\mathbb{L})$, then

$$\begin{aligned} X \circ h' &= C(X) \circ h' \quad \text{(Idempotency)} \\ &= C(X \circ h' \circ h) \circ h' \quad (X = X \circ h' \circ h) \\ &= C'(X \circ h') \quad \text{(Hypothesis)} \\ &\in \mathrm{Cl}(\mathbb{L}'). \end{aligned}$$

(iv)⇒(v) Suppose $X' \in \mathrm{Cl}(\mathbb{L}')$. Then we have, by hypothesis,

$$\begin{aligned} X' \circ h &= (X \circ h') \circ h, \text{ for some } X \in \mathrm{Cl}(\mathbb{L}), \\ &= X \in \mathrm{Cl}(\mathbb{L}). \quad (E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})) \end{aligned}$$

If, conversely, $X \in \mathrm{Cl}(\mathbb{L})$, then $X \circ h' \in \mathrm{Cl}(\mathbb{L}')$ and, again, taking into account that $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$,

$$X = (X \circ h') \circ h \in \mathrm{Cl}(\mathbb{L}') \circ h.$$

(v)⇒(i) By definition.

■

Statements (iv) and (v) of Proposition 64 allow us to establish an order isomorphism between lattices of closed G-sets of two logically bimorphic G-logics.

Proposition 65 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathcal{A} \to \mathcal{A}'$ a surjective morphism. Then $h : \mathbb{L} \to \mathbb{L}'$ is a bilogical G-morphism if and only if the latices $\langle \mathrm{Cl}(\mathbb{L}), \le \rangle$ and $\langle \mathrm{Cl}(\mathbb{L}'), \le \rangle$ of closed functions of $\mathbb{L}$ and $\mathbb{L}'$, respectively, are isomorphic under*

$$X \mapsto X \circ h' \quad \text{and} \quad Y \mapsto Y \circ h,$$

where $h' : A' \to A$ is a section of h.

Proof: The fact that the two maps are bijections is given by Proposition 64. That they are order preserving is clear from their definitions. ■

Moreover, the isomorphism of Proposition 65 extends to isomorphisms between lattices of logics on the two underlying G-algebras containing logics whose sets of closed functions correspond under the bilogical G-morphism.

Corollary 66 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then h induces an isomorphism*

$$\mathbf{Log_G}(\mathcal{A})^{\mathbb{L}} \cong \mathbf{Log_G}(\mathcal{A}')^{\mathbb{L}'}$$

between the lattice of all G-logics on $\mathcal{A}$ extending $\mathbb{L}$ and the lattice of all G-logics on $\mathcal{A}'$ extending $\mathbb{L}'$.

Proof: Directly from the isomorphism of Proposition 65. ■

Another property of bilogical G-morphisms is that they establish a close connection between the Tarski G-congruences of the bimorphically related G-logics. Roughly speaking, this is the property asserting that Tarski congruences are invariant under inverse bilogical morphisms, well-known in the traditional framework (see, e.g., Proposition 1.7 of [28]).

Proposition 67 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then*

$$\widetilde{\Omega}(\mathbb{L}) = \widetilde{\Omega}(\mathbb{L}') \circ h^2.$$

Proof: Using Proposition 62, we have, for all $a, b \in A$,

$$\begin{aligned}
&\widetilde{\Omega}(\mathbb{L}')(h(a), h(b)) \\
&\quad = \bigwedge_{\substack{X' \in \mathrm{Cl}(\mathbb{L}') \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c}' \in A'}} X'(\varphi^{\mathbf{A}'}(h(a), \bar{c}')) \leftrightarrow X'(\varphi^{\mathbf{A}'}(h(b), \bar{c}')) \\
&\qquad \text{(Proposition 62)} \\
&\quad = \bigwedge_{\substack{X' \in \mathrm{Cl}(\mathbb{L}') \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X'(\varphi^{\mathbf{A}'}(h(a), h(\bar{c}))) \leftrightarrow X'(\varphi^{\mathbf{A}'}(h(b), h(\bar{c}))) \\
&\qquad (h \text{ surjective}) \\
&\quad = \bigwedge_{\substack{X' \in \mathrm{Cl}(\mathbb{L}') \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X'(h(\varphi^{\mathbf{A}}(a, \bar{c}))) \leftrightarrow X'(h(\varphi^{\mathbf{A}}(b, \bar{c}))) \\
&\qquad (h : \mathbf{A} \to \mathbf{A}') \\
&\quad = \bigwedge_{\substack{X \in \mathrm{Cl}(\mathbb{L}) \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X(\varphi^{\mathbf{A}}(a, \bar{c})) \leftrightarrow X(\varphi^{\mathbf{A}}(b, \bar{c})) \quad (h : \mathbb{L} \to_b \mathbb{L}') \\
&\quad = \widetilde{\Omega}(\mathbb{L})(a, b). \quad \text{(Proposition 62)}
\end{aligned}$$

Therefore $\widetilde{\Omega}(\mathbb{L}') \circ h^2 = \widetilde{\Omega}(\mathbb{L})$. ∎

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics. A logical G-morphism $h : \mathbb{L} \to \mathbb{L}'$ is a (**logical**) G-**isomorphism**, written $h : \mathbb{L} \cong \mathbb{L}'$, if it is bijective and its inverse h^{-1} is also a logical G-morphism $h^{-1} : \mathbb{L}' \to \mathbb{L}$.

Proposition 68 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathcal{A} \to \mathcal{A}'$ a G-morphism. Then the following are equivalent:*

(i) $h : \mathbb{L} \cong \mathbb{L}'$ *is a G-isomorphism;*

(ii) $h : \mathbb{L} \to_b \mathbb{L}'$, *with $h : \mathcal{A} \to \mathcal{A}'$ an isomorphism;*

(iii) $h : \mathcal{A} \to \mathcal{A}'$ *an isomorphism, with $h^{-1} : \mathbb{L}' \to_b \mathbb{L}$.*

Proof:

(i)⇒(ii) By definition, $\mathrm{Cl}(\mathbb{L}') \circ h \subseteq \mathrm{Cl}(\mathbb{L})$. On the other hand, if $X \in \mathrm{Cl}(\mathbb{L})$, then $X = (X \circ h^{-1}) \circ h$, where $X \circ h^{-1} \in \mathrm{Cl}(\mathbb{L}')$. So $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}') \circ h$. Thus, $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h$ and $h : \mathbb{L} \to \mathbb{L}'$ is a bilogical G-morphism.

(ii)⇒(iii) This is (i)⇒(iv) of Proposition 64.

(iii)⇒(i) Notice that, if $h : \mathcal{A} \to \mathcal{A}'$ is an isomorphism, then

$$E \circ (h^{-1})^2 \in \mathrm{Gon}(\mathbb{L}') \quad \text{and} \quad E' \circ h^2 \in \mathrm{Gon}(\mathbb{L}).$$

E.g., for all $a', b' \in A'$ and all $X' : A' \to G$,

$$\begin{aligned} & E(h^{-1}(a'), h^{-1}(b')) \wedge C'(X')(a') \\ & \quad \le E'(h(h^{-1}(a')), h(h^{-1}(b'))) \wedge C'(X')(a') \quad (h : \mathcal{A} \to \mathcal{A}') \\ & \quad = E'(a', b') \wedge C'(X')(a') \quad (h \circ h^{-1} = i_{\mathcal{A}'}) \\ & \quad \le C'(X')(b'). \quad (E' \in \mathrm{Gon}(\mathbb{L}')) \end{aligned}$$

Hence, this part reduces to (iv)⇒(v) of Proposition 64.

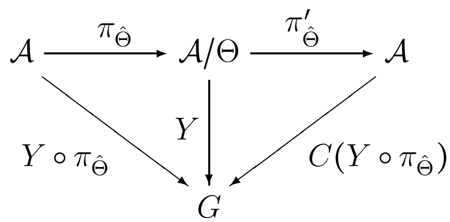
■

3.7 Quotients

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathcal{A})$. If $\Theta \in \mathrm{Gon}(\mathbb{L})$ it is reasonable to define a G-logic structure on the quotient G-algebra $\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta} \rangle$. We simply set

$$\mathbb{L}/\Theta = \langle \mathcal{A}/\Theta, C/\Theta \rangle,$$

where, for all $Y : A/\hat{\Theta} \to G$,



$$(C/\Theta)(Y) = C(Y \circ \pi_{\hat{\Theta}}) \circ \pi'_{\hat{\Theta}},$$

with $\pi'_{\hat{\Theta}}$ being a section of $\pi_{\hat{\Theta}}$. This is well defined, since, regardless of the choice of section, we have, for all $a, b \in A$ and all $Y : A/\hat{\Theta} \to G$,

$$\begin{aligned} & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge (C/\Theta)(Y)(a/\hat{\Theta}) \\ & \quad = \Theta(a, b) \wedge C(Y \circ \pi_{\hat{\Theta}})(\pi'_{\hat{\Theta}}(a/\hat{\Theta})) \quad (\text{Definitions of } \bar{\Theta} \text{ and } C/\Theta) \\ & \quad = \Theta(a, b) \wedge C(Y \circ \pi_{\hat{\Theta}})(a) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \quad \le C(Y \circ \pi_{\hat{\Theta}})(b) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \quad = C(Y \circ \pi_{\hat{\Theta}})(\pi'_{\hat{\Theta}}(b/\hat{\Theta})) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \quad = (C/\Theta)(Y)(b/\hat{\Theta}). \quad (\text{Definition of } C/\Theta) \end{aligned}$$

Hence $\bar{\Theta} \in \mathrm{Gon}(\mathbb{L}/\Theta)$. The structure $\mathbb{L}/\Theta$ is termed the **quotient G-logic** of $\mathbb{L}$ by Θ.

We show that, under this definition of $\mathbb{L}/\Theta$, the projection $\pi_{\hat{\Theta}} : \mathbb{L} \to \mathbb{L}/\Theta$ becomes a bilogical G-morphism.

Lemma 69 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathbb{L})$. The mapping $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$ is a bilogical G-morphism $\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta$.*

Proof: We must show that $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}}$. Let us denote again by $\pi'_{\hat{\Theta}}$ a section of $\pi_{\hat{\Theta}}$. Suppose, first, that $Y \in \mathrm{Cl}(\mathbb{L}/\Theta)$. Then

$$\begin{array}{rcll} Y \circ \pi_{\hat{\Theta}} & = & (C/\Theta)(Y) \circ \pi_{\hat{\Theta}} & (Y \in \mathrm{Cl}(\mathbb{L}/\Theta)) \\ & = & C(Y \circ \pi_{\hat{\Theta}}) \circ \pi'_{\hat{\Theta}} \circ \pi_{\hat{\Theta}} & (\text{Definition of } C/\Theta) \\ & = & C(Y \circ \pi_{\hat{\Theta}}) & (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \in & \mathrm{Cl}(\mathbb{L}). & \end{array}$$

Thus, $\mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}} \subseteq \mathrm{Cl}(\mathbb{L})$. Assume, conversely, that $X \in \mathrm{Cl}(\mathbb{L})$. Consider the mapping $X \circ \pi'_{\hat{\Theta}} : A/\hat{\Theta} \to G$. We have

$$\begin{array}{rcll} (C/\Theta)(X \circ \pi'_{\hat{\Theta}}) & = & C(X \circ \pi'_{\hat{\Theta}} \circ \pi_{\hat{\Theta}}) \circ \pi'_{\hat{\Theta}} & (\text{Definition of } C/\Theta) \\ & = & C(X) \circ \pi'_{\hat{\Theta}} & (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & = & X \circ \pi'_{\hat{\Theta}}. & (X \in \mathrm{Cl}(\mathbb{L})) \end{array}$$

So $X \circ \pi'_{\hat{\Theta}} \in \mathrm{Cl}(\mathbb{L}/\Theta)$ and we obtain

$$X = (X \circ \pi'_{\hat{\Theta}}) \circ \pi_{\hat{\Theta}} \in \mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}}.$$

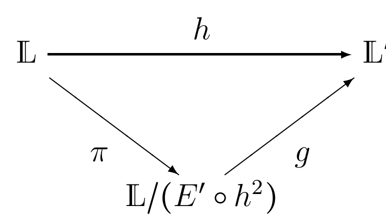
We conclude that $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}}$. Thus, $\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta$. ■

The next goal is to further abstract the Homomorphism Theorems of Universal Algebra, which were presented for G-algebras in Section 3.2, to G-logics and their quotients. These results form an extension of the corresponding results of Font and Jansana covering quotients of abstract logics in the traditional framework (see Theorems 1.8-1.10 of [28]).

Theorem 70 (Homomorphism) *Let* $\mathbb{L} = \langle \mathcal{A}, C\rangle$, *with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$, *and* $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, *with* $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, *be G-logics and* $h : \mathbb{L} \to_b \mathbb{L}'$ *a bilogical G-morphism. Then* $E' \circ h^2$ *is a logical G-congruence of* $\mathbb{L}$ *and*

$$\mathbb{L}/(E' \circ h^2) \cong \mathbb{L}'$$

by means of a unique G-isomorphism g satisfying commutativity of



where $\pi : \mathbb{L} \to \mathbb{L}/(E' \circ h^2)$ *denotes the quotient G-morphism.*

Proof: First, by Theorem 52, we know that there exists a unique isomorphism $g : \mathcal{A}/(E' \circ h^2) \to \mathcal{A}'$, such that $g \circ \pi = h$, defined by

$$g(a/\widehat{E' \circ h^2}) = h(a), \quad a \in A.$$

By Proposition 64, $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$. Let $\pi' : A/\widehat{E' \circ h^2} \to A$ be a section of π and note that, since $g \circ \pi = h$, we get $g = h \circ \pi'$. We now calculate

$$\begin{array}{rcll} \mathrm{Cl}(\mathbb{L}/(E' \circ h^2)) & = & \mathrm{Cl}(\mathbb{L}) \circ \pi' & \text{(Lemma 69)} \\ & = & \mathrm{Cl}(\mathbb{L}') \circ h \circ \pi' & (h : \mathbb{L} \to_b \mathbb{L}') \\ & = & \mathrm{Cl}(\mathbb{L}') \circ g. & (g = h \circ \pi') \end{array}$$

Therefore, by Proposition 68, g is a G-isomorphism. ∎

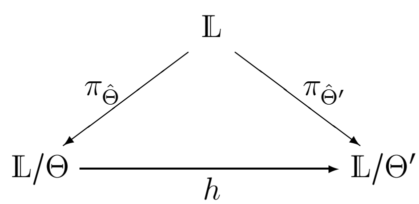
We turn, next, to an analog of the Second Isomorphism Theorem for G-logics. Our intention is to prove it using the Homomorphism Theorem 70.

Theorem 71 (Second Isomorphism) *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta, \Theta' \in \mathrm{Gon}(\mathbb{L})$, such that $\Theta \leq \Theta'$. Then $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$ and*

$$(\mathbb{L}/\Theta)/(\Theta'/\Theta) \cong \mathbb{L}/\Theta'$$

via $(a/\hat{\Theta})/\widehat{\Theta'/\Theta} \mapsto a/\hat{\Theta}'$.

Proof: Let us start with the following diagram.



By Lemma 69, $\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta$ and $\pi_{\hat{\Theta}'} : \mathbb{L} \to_b \mathbb{L}/\Theta'$ are bilogical morphisms. It is not difficult to show that

$$\begin{array}{rrcl} h: & \mathbb{L}/\Theta & \longrightarrow & \mathbb{L}/\Theta'; \\ & a/\hat{\Theta} & \longmapsto & a/\hat{\Theta}' \end{array}$$

is also a bilogical morphism. Denoting by $\pi'_{\hat{\Theta}}$ a section of $\pi_{\hat{\Theta}}$, we have

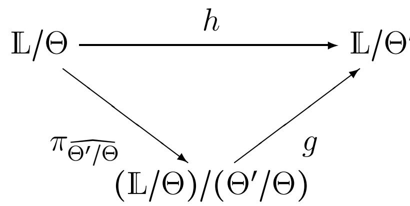
$$\begin{array}{rcll} \mathrm{Cl}(\mathbb{L}/\Theta') \circ h & = & \mathrm{Cl}(\mathbb{L}/\Theta') \circ \pi_{\hat{\Theta}'} \circ \pi'_{\hat{\Theta}} & (h = \pi_{\hat{\Theta}'} \circ \pi'_{\hat{\Theta}}) \\ & = & \mathrm{Cl}(\mathbb{L}) \circ \pi'_{\hat{\Theta}} & (\pi_{\hat{\Theta}'} : \mathbb{L} \to_b \mathbb{L}/\Theta') \\ & = & \mathrm{Cl}(\mathbb{L}/\Theta). & (\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta) \end{array}$$

Thus, $h : \mathbb{L}/\Theta \to_b \mathbb{L}/\Theta'$ is indeed a bilogical G-morphism.

Next we show that $\bar{\Theta}' \circ h^2 = \Theta'/\Theta$. We have, for all $a, b \in A$,

$$\begin{array}{rcll} \bar{\Theta}'(h(a/\hat{\Theta}), h(b/\hat{\Theta})) & = & \bar{\Theta}'(a/\hat{\Theta}', b/\hat{\Theta}') & \text{(Definition of } h) \\ & = & \Theta'(a, b) & \text{(Definition of } \bar{\Theta}') \\ & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}). & \text{(Definition of } \Theta'/\Theta) \end{array}$$

Finally, we look at the following diagram.



It conforms with the hypotheses of Theorem 70. So we may conclude that $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$ and that

$$\begin{array}{rrcl} g: & (\mathbb{L}/\Theta)/(\Theta'/\Theta) & \longrightarrow & \mathbb{L}/\Theta'; \\ & (a/\hat{\Theta})/\widehat{\Theta'/\Theta} & \longmapsto & a/\hat{\Theta}' \end{array}$$

is the unique G-isomorphism making the diagram commute. ■

The last theorem in this series that we establish is the analog for G-logics of the Correspondence Theorem of Universal Algebra. A similar generalization for abstract logics is given in Theorem 1.10 of [28].

Theorem 72 (Correspondence Theorem) *Let* $\mathbb{L} = \langle \mathcal{A}, C\rangle$, *with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$, *be a* G*-logic and* $\Theta \in \mathrm{Gon}(\mathbb{L})$. *The segment* $[\Theta, \widetilde{\Omega}(\mathbb{L})]$ *of* $\mathrm{Gon}(\mathbb{L})$ *is isomorphic to the lattice* $\mathrm{Gon}(\mathbb{L}/\Theta)$ *by the mapping* $\Theta' \mapsto \Theta'/\Theta$.

Proof: Suppose $\Theta' \in \mathrm{Gon}(\mathbb{L})$ is such that $\Theta \le \Theta' \le \widetilde{\Omega}(\mathbb{L})$. By the Second Isomorphism Theorem 71, we have $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$. By the Second Isomorphism Theorem 53 for G-algebras, we must show that, if $\Theta \le \Theta' \in \mathrm{Gon}(\mathcal{A})$, such that $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$, then $\Theta' \in \mathrm{Gon}(\mathbb{L})$. Denoting, again, by $\pi'_{\hat{\Theta}}$ a section of $\pi_{\hat{\Theta}}$, we have, for all $X \in \mathrm{Cl}(\mathbb{L})$ and all $a, b \in A$,

$$\begin{array}{rcl} \Theta'(a,b) \wedge X(a) & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) \wedge (X \circ \pi'_{\hat{\Theta}})(a/\hat{\Theta}) \\ & & \quad (\text{Definition of } \Theta'/\Theta \text{ and } \Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \le & (X \circ \pi'_{\hat{\Theta}})(b/\hat{\Theta}) \quad (\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)) \\ & = & X(b). \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \end{array}$$

Hence, $\Theta' \in \mathrm{Gon}(\mathbb{L})$. ■

We now show that the Tarski G-congruence of a quotient $\mathbb{L}/\Theta$ of a G-logic $\mathbb{L}$ by a logical G-congruence Θ coincides with the quotient by Θ of the Tarski G-congruence of $\mathbb{L}$ itself.

Corollary 73 *Let* $\mathbb{L} = \langle \mathcal{A}, C\rangle$, *with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$, *be a* G*-logic and* $\Theta \in \mathrm{Gon}(\mathbb{L})$. *Then*

$$\widetilde{\Omega}(\mathbb{L}/\Theta) = \widetilde{\Omega}(\mathbb{L})/\Theta.$$

Proof: Immediate by the Correspondence Theorem 72. ∎

Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic. It is called **reduced** if it has only one logical G-congruence, i.e., if $\widetilde{\Omega}(\mathbb{L}) = E$. Given a G-logic $\mathbb{L}$, we set

$$\mathbb{L}^* = \mathbb{L}/\widetilde{\Omega}(\mathbb{L})$$

and call $\mathbb{L}^*$ the **reduction** of $\mathbb{L}$. Further, given a class L of G-logics, we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

If $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, is a G-logic, then, by the preceding corollary, $\mathbb{L}^*$ is always reduced. Moreover, if $\mathbb{L}$ is reduced, it is isomorphic to its reduction and the two G-logics $\mathbb{L}$ and $\mathbb{L}^*$ may be identified. If $\mathbb{L} = \langle \mathcal{A}, C\rangle$, we use the notation

$$\begin{array}{rcl} \mathcal{A}^* & = & \mathbf{A}/\widetilde{\Omega}(\mathbb{L}), \\ C^* & = & C/\widetilde{\Omega}(\mathbb{L}); \\ \mathrm{Cl}(\mathbb{L})^* & = & \mathrm{Cl}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}). \end{array}$$

We prove that the reduction of a quotient $\mathbb{L}/\Theta$ of a G-logic $\mathbb{L}$ by a logical G-congruence Θ is G-isomorphic with the reduction of $\mathbb{L}$ itself. This forms an analog of Proposition 1.13 of [28].

Proposition 74 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathbb{L})$. Then*

$$(\mathbb{L}/\Theta)^* \cong \mathbb{L}^*.$$

Proof: We have

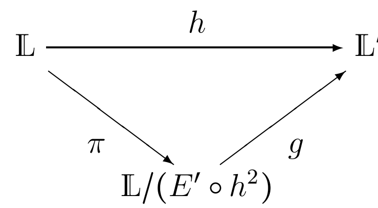
$$\begin{array}{rcl} (\mathbb{L}/\theta)^* & = & (\mathbb{L}/\theta)/\widetilde{\Omega}(\mathbb{L}/\theta) \quad \text{(Definition)} \\ & = & (\mathbb{L}/\theta)/(\widetilde{\Omega}(\mathbb{L})/\theta) \quad \text{(Corollary 73)} \\ & \cong & \mathbb{L}/\widetilde{\Omega}(\mathbb{L}) \quad \text{(Theorem 72)} \\ & = & \mathbb{L}^*. \quad \text{(Definition)} \end{array}$$

∎

More generally, we can show that if two G-logics are related by a bilogical G-morphism, then their corresponding reductions are G-isomorphic. This forms an analog of Proposition 1.14 of [28].

Proposition 75 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics. If there is a bilogical G-morphism $h : \mathbb{L} \to_b \mathbb{L}'$, then $\mathbb{L}^* \cong \mathbb{L}'^*$.*

Proof: Suppose $h : \mathbb{L} \to_b \mathbb{L}'$ is a bilogical G-morphism. By the Homomorphism Theorem 70, there exists a unique G-isomorphism $g : \mathbb{L}/(E' \circ h^2) \to \mathbb{L}'$ that makes the following triangle commute,



where $\pi : \mathbb{L} \to_b \mathbb{L}/(E' \circ h^2)$ is the quotient G-morphism. By Proposition 67,

$$\widetilde{\Omega}(\mathbb{L}/(E' \circ h^2)) = \widetilde{\Omega}(\mathbb{L}') \circ g^2.$$

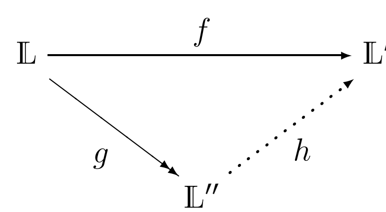
Therefore, as g is a G-isomorphism,

$$(\mathbb{L}/(E' \circ h^2))^* \cong \mathbb{L}'^*.$$

By Proposition 74, $(\mathbb{L}/(E' \circ h^2))^* \cong \mathbb{L}^*$. We conclude that $\mathbb{L}^* \cong \mathbb{L}'^*$. ∎

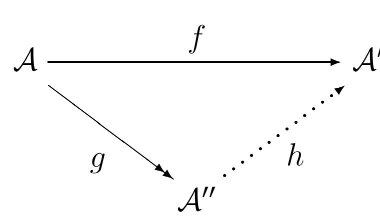
Taking advantage of Lemma 55, a fill-in type of result for G-algebras, we establish a similar result for G-logics, forming an analog of Proposition 1.15 of [28].

Proposition 76 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, and $\mathbb{L}'' = \langle \mathcal{A}'', C'' \rangle$, with $\mathcal{A}'' = \langle \mathbf{A}'', E'' \rangle$, be G-logics, such that $f : \mathbb{L} \to \mathbb{L}'$ is a logical G-morphism and $g : \mathbb{L} \to_b \mathbb{L}''$ is a bilogical G-morphism, such that $E'' \circ g^2 \le E' \circ f^2$. Then, there exists a unique logical G-morphism $h : \mathbb{L}'' \to \mathbb{L}'$, such that $h \circ g = f$.*



Moreover, f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ if and only if h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$.

Proof: By Lemma 55, there exists a commutative diagram of G-algebras.



We show that $h : \mathbb{L}'' \to \mathbb{L}'$ is a logical G-morphism. Let g' be a section of g. We have

$$\begin{array}{rcl}\mathrm{Cl}(\mathbb{L}') \circ h & = & \mathrm{Cl}(\mathbb{L}') \circ f \circ g' \quad (E'' \circ g^2 \leq E' \circ f^2) \\ & \subseteq & \mathrm{Cl}(\mathbb{L}) \circ g' \quad (f : \mathbb{L} \to \mathbb{L}') \\ & = & \mathrm{Cl}(\mathbb{L}''). \quad (g : \mathbb{L} \to_b \mathbb{L}'')\end{array}$$

If f projectively generates $\mathbb{L}$ from $\mathbb{L}'$, then the inclusion may be replaced by an equality. So, h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$.

Suppose, conversely, that h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. Then, we have

$$\begin{array}{rcl}\mathrm{Cl}(\mathbb{L}') \circ f & = & \mathrm{Cl}(\mathbb{L}') \circ h \circ g \quad (f = h \circ g) \\ & = & \mathrm{Cl}(\mathbb{L}'') \circ g \quad (\text{Hypothesis}) \\ & = & \mathrm{Cl}(\mathbb{L}). \quad (g : \mathbb{L} \to_b \mathbb{L}'')\end{array}$$

Therefore, f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. ■

3.8 Sentential Graded Logics

Recall that, in Section 2.2, we introduced G-logics as G-logics acting on G-sets of formulas. Then, in Section 3.4, we generalized this notion to include G-logics over arbitrary G-algebras. Since we now want to refocus on the special G-logics that act on G-sets of formulas, we reintroduce them under the special name of *sentential G-logic* to specifically emphasize the fact the their underlying G-algebra is the G-algebra of formulas over a fixed algebraic type.

A **sentential G-logic** is a pair $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$, where

$$\mathcal{F}m_{\mathcal{L}}(V) = \langle \mathbf{Fm}_{\mathcal{L}}(V), \Delta_{\mathbf{Fm}_{\mathcal{L}}(V)} \rangle$$

is the formula G-algebra and $C : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ a G-operator on $\mathcal{F}m_{\mathcal{L}}(V)$, that is, one that satisfies Inflationarity, Monotonicity, Idempotency and Translation. These are specializations of the corresponding axioms itemized at the beginning of Section 3.4. Even though they were also given explicitly in Section 2.2, we relist them here as well.

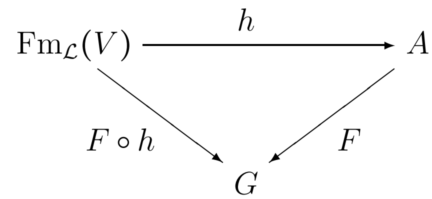
Inflationarity $X \leq C(X)$, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$;

Monotonicity $C(X) \leq C(Y)$, for all $X, Y : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, with $X \leq Y$;

Idempotency $C(C(X)) = C(X)$, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$;

Translation $C(X \circ h) \leq C(X) \circ h$, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

A G-matrix $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, is called an $\mathbb{S}$-**matrix** if, for all $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$ and all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,



$$X \leq F \circ h \quad \text{implies} \quad C(X) \leq F \circ h.$$

If $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is an $\mathbb{S}$-matrix, we call $F : A \to G$ an $\mathbb{S}$-**filter**.

Note that, due to the fact that the reduced G-congruence component of the formula G-algebra is the identity on $\mathbf{Fm}_{\mathcal{L}}(V)$, a G-morphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$ may be identified with a homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$. Consequently, the universal quantification "for all $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$" above may be equivalently replaced by the universal quantification "for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$". Note, additionally, that the condition defining an $\mathbb{S}$-matrix coincides with the condition $C \leq C_{\mathfrak{A}}$, which was used in Section 2.4 to define an $\mathcal{S}$-matrix.

The following is a characterization of $\mathbb{S}$-filters on a G-algebra by means of the closed sets of $\mathbb{S}$.

Lemma 77 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic and $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ a G-matrix. $\mathfrak{A}$ is an $\mathbb{S}$-matrix if and only if for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,*

$$F \circ h \in \mathrm{Cl}(\mathbb{S}).$$

Proof: First assume that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, $F \circ h \in \mathrm{Cl}(\mathbb{S})$. Then, we have, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ and all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} X \leq F \circ h & \text{implies} & C(X) \leq C(F \circ h) \quad \text{(Monotonicity)} \\ & \text{iff} & C(X) \leq F \circ h. \quad (F \circ h \in \mathrm{Cl}(\mathbb{S})) \end{array}$$

Hence, F is an $\mathbb{S}$-filter.

Suppose, conversely, that F is an $\mathbb{S}$-filter. Then, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, we have $F \circ h \leq F \circ h$, whence, by hypothesis, $C(F \circ h) \leq F \circ h$. The reverse inequality always holds. So $C(F \circ h) = F \circ h$ and, hence, $F \circ h \in \mathrm{Cl}(\mathbb{S})$. ■

We write $\mathrm{Mat}(\mathbb{S})$ for the collection of all $\mathbb{S}$-matrices. We use $\mathrm{Mat}^*(\mathbb{S})$ for the class $\mathrm{Mat}(\mathbb{S})^*$, i.e., the class of all reductions of members of $\mathrm{Mat}(\mathbb{S})$. We also use $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ to denote the family of $\mathbb{S}$-filters on a G-algebra $\mathcal{A}$.

We have the following characterization of the $\mathbb{S}$-filters on the formula algebra. Here, Translation plays a critical role.

Lemma 78 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic and $\mathfrak{A} = \langle \mathcal{F}m_{\mathcal{L}}(V), F \rangle$ a G-matrix. $\mathfrak{A}$ is an $\mathbb{S}$-matrix if and only if $F \in \mathrm{Cl}(\mathbb{S})$.*

Proof: First, if $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V))$, then, taking the identity homomorphism on $\mathbf{Fm}_{\mathcal{L}}(V)$ for h in Lemma 77, we get $F \in \mathrm{Cl}(\mathbb{S})$.

Suppose, conversely, that $F \in \mathrm{Cl}(\mathbb{S})$. Then, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ and all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} X \le F \circ h & \text{implies} & C(X) \le C(F \circ h) \quad \text{(Monotonicity)} \\ & \text{implies} & C(X) \le C(F) \circ h \quad \text{(Translation)} \\ & \text{implies} & C(X) \le F \circ h. \quad (F \in \mathrm{Cl}(\mathbb{S})) \end{array}$$

Therefore, $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V))$. ∎

We call the closed G-sets of $\mathbb{S}$, which, by Lemma 78, coincide with the $\mathbb{S}$-filters on the formula G-algebra $\mathcal{F}m_{\mathcal{L}}(V)$, the **theories of** $\mathbb{S}$ or $\mathbb{S}$-**theories**. The collection of $\mathbb{S}$-theories is denoted by

$$\mathrm{Th}(\mathbb{S}) := \mathrm{Cl}(\mathbb{S}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V))$$

and the complete lattice they form under $\le$ by $\mathbf{Th}(\mathbb{S}) = \langle \mathrm{Th}(\mathbb{S}), \le \rangle$.

The following proposition details how $\mathbb{S}$-filters and G-morphisms interact. It forms an analog of Proposition 1.19 of [28].

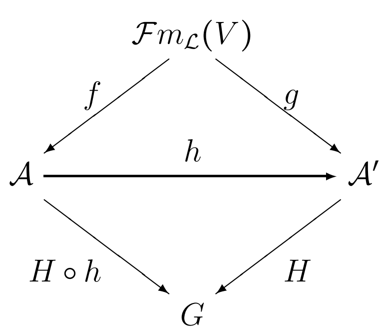
Proposition 79 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{A}'$ a G-morphism. Suppose $H : A' \to G$.*

(a) If $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$, then $H \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(b) If h is surjective and $H \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, then $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Proof:

(a) We do a little diagram chasing around the diagram shown.



$$\begin{array}{lll} H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') & \text{iff} & (\forall g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}')(H \circ g \in \mathrm{Cl}(\mathbb{S})) \\ & & \quad \text{(Lemma 77)} \\ & \text{implies} & (\forall f : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A})(H \circ h \circ f \in \mathrm{Cl}(\mathbb{S})) \\ & \text{iff} & H \circ f \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}). \quad \text{(Lemma 77)} \end{array}$$

(b) For this part, observe that, if $h : \mathcal{A} \to \mathcal{A}'$ is surjective, then, for all $g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}'$, there exists $\bar{g} : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ (playing the role of f in the diagram), such that $h \circ \bar{g} = g$. Thus, we get

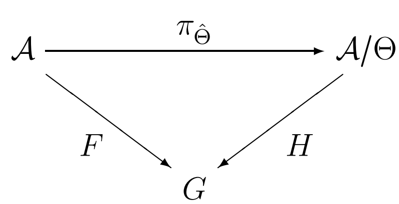
$$\begin{array}{lll} H \circ h \in \mathrm{Fi}_{\$}(\mathcal{A}) & \text{iff} & (\forall f : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A})(H \circ h \circ f \in \mathrm{Cl}(\$)) \\ & & \quad \text{(Lemma 77)} \\ & \text{implies} & (\forall g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}')(H \circ h \circ \bar{g} \in \mathrm{Cl}(\$)) \\ & \text{iff} & (\forall g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}')(H \circ g \in \mathrm{Cl}(\$)) \\ & \text{iff} & H \in \mathrm{Fi}_{\$}(\mathcal{A}'). \quad \text{(Lemma 77)} \end{array}$$

■

Proposition 79 implies, in particular, that, given a G-algebra $\mathcal{A}$ and $\Theta \in \mathrm{Gon}(\mathcal{A})$, if $\pi : \mathcal{A} \to \mathcal{A}/\Theta$ is the quotient G-morphism, then, for all $H : A/\hat{\Theta} \to G$, $H \circ \pi \in \mathrm{Fi}_{\$}(\mathcal{A})$ if and only if $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$. A partial result in the opposite direction, i.e., starting with $F : A \to G$, which forms an analog of Proposition 1.20 of [28], is given below.

Proposition 80 *Let $\$ = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra, $F \in \mathrm{Fi}_{\$}(\mathcal{A})$ and $\Theta \in \mathrm{Gon}(\mathcal{A})$. Then Θ is compatible with F, i.e., $\Theta \leq \Omega_{\mathcal{A}}(F)$, if and only if $F = H \circ \pi_{\hat{\Theta}}$, for some $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$, where $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$ is the quotient G-morphism.*

Proof: Let $F \in \mathrm{Fi}_{\$}(\mathcal{A})$. We work with the accompanying diagram.



Suppose, first, that Θ is compatible with F. By compatibility, it is reasonable to define $H : \mathcal{A}/\Theta \to G$ by setting, for all $a \in A$,

$$H(a/\hat{\Theta}) = F(a).$$

Indeed, if $\langle a, b \rangle \in \hat{\Theta}$, then $\Theta(a, b) = \top$, whence,

$$F(a) = \top \wedge F(a) = \Theta(a, b) \wedge F(a) \leq F(b)$$

and, by symmetry, $F(a) = F(b)$. Hence H is well defined. Moreover, clearly, $H \circ \pi_{\hat{\Theta}} = F$. Thus, since $\pi_{\hat{\Theta}}$ is surjective, by Proposition 79, Part (b), we get that $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$.

Assume, conversely, that $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$, such that $F = H \circ \pi_{\hat{\Theta}}$. Then, for all $a, b \in A$,

$$\begin{array}{lll} \Theta(a, b) \wedge F(a) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge H(a/\hat{\Theta}) \quad \text{(Definitions of } \bar{\Theta} \text{ and } F) \\ & \leq & H(b/\hat{\Theta}) \quad (H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)) \\ & = & F(b). \quad \text{(Definition of } F) \end{array}$$

Hence, Θ is compatible with F. ∎

We look, next, at bilogical G-morphisms between G-logics consisting of full collections of $\mathbb{S}$-filters on corresponding G-algebras. Proposition 81 forms an analog of Proposition 1.21 of [28] for G-logics.

Proposition 81 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{A}'$ an epimorphism. Then the following are equivalent:*

(i) $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ *is a bilogical G-morphism;*

(ii) *h induces an isomorphism between the lattices $\mathbf{Fi}_{\mathbb{S}}(\mathcal{A})$ and $\mathbf{Fi}_{\mathbb{S}}(\mathcal{A}')$;*

(iii) *For all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and section h' of h, $F \circ h' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ and $E' \circ h^2 \in \mathrm{Gon}(\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle)$.*

Proof:

(i)⇒(ii) Suppose $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ is a bilogical G-morphism. Then, by definition, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') \circ h$. So $Y \mapsto Y \circ h$ is a bijection from $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ to $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Clearly it is also order preserving and reflecting and, hence, a lattice isomorphism.

(ii)⇒(iii) Let $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. By hypothesis, there exists $Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$, such that $F = Y \circ h$. Thus, $F \circ h' = Y \circ h \circ h' = Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Assume, next, that $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and $a, b \in A$. Let $Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ be such that $F = Y \circ h$. Then

$$\begin{array}{rcl}(E' \circ h^2)(a,b) \wedge F(a) & = & E'(h(a),h(b)) \wedge Y(h(a)) \quad (F = Y \circ h)\\ & \leq & Y(h(b)) \quad (Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}'))\\ & = & F(b). \quad (F = Y \circ h)\end{array}$$

Therefore, $E' \circ h^2 \in \mathrm{Gon}(\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle)$.

(iii)⇒(i) By hypothesis, $E' \circ h^2 \in \mathrm{Gon}(\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle)$. So, by Proposition 64, it suffices to show that $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \circ h'$. First, if $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, then, by hypothesis, $F \circ h' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$. On the other hand, if $G \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$, then, by Proposition 79, $G \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and, moreover,

$$G = (G \circ h) \circ h' \in \mathrm{Fi}_{\mathbb{S}}(\mathbf{A}) \circ h'.$$

Thus, by (iv)⇒(i) of Proposition 64, we get the conclusion.

∎

The next result shows that, if a G-logic is the bilogical image of a G-logic whose closed G-sets consist of all $\mathbb{S}$-filters on the underlying G-algebra, then its own set of closed G-sets must be of the same type. This result will be important when we define and discuss full G-models in Section 3.10. It forms an analog of Proposition 1.22 of [28].

Proposition 82 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic and* $\mathcal{A} = \langle \mathbf{A}, E\rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ *be* G*-algebras. If* $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathcal{X}\rangle$ *is a bilogical* G*-morphism, then* $\mathcal{X} = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$*. As a consequence, we have:*

- $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$*;*
- *If* $\mathbb{L} = \langle \mathcal{A}, \mathcal{X}\rangle$ *and* $\mathbb{L}' = \langle \mathcal{A}', \mathcal{X}'\rangle$ *are* G*-isomorphic, then* $\mathcal{X} = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ *if and only if* $\mathcal{X}' = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$*.*

Proof: If $G \in \mathcal{X}$, then, by hypothesis, $G \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Thus, since h is surjective, by Proposition 79, $G \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$. Thus, $\mathcal{X} \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Suppose, conversely, that $G \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$. Then, by Proposition 79, $G \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Hence, by hypothesis, $G = (G \circ h) \circ h' \in \mathcal{X}$. Hence, we obtain $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') \subseteq \mathcal{X}$.

The first consequence is obtained by applying the main statement to the bilogical G-morphism

$$\pi : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to \langle \mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})), \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \circ \pi'\rangle,$$

where π' is a section of the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}))$. It gives $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. Finally, the second consequence results by applying the main statement to both bilogical G-morphisms $h : \mathbb{L} \to_b \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to_b \mathbb{L}$, where $h : \mathbb{L} \cong \mathbb{L}'$. ■

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Let $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ be an $\mathbb{S}$-matrix. If Θ is a G-congruence on $\mathcal{A}$ compatible with F, then

$$\mathfrak{A}/\Theta = \langle \mathcal{A}/\Theta, F \circ \pi'_{\hat{\Theta}}\rangle,$$

where $\pi'_{\hat{\Theta}}$ is a section of $\pi_{\hat{\Theta}}$, is also an $\mathbb{S}$-matrix. This follows from the equality $(F \circ \pi'_{\hat{\Theta}}) \circ \pi_{\hat{\Theta}} = F$. Therefore, $\mathfrak{A}^* = \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), F \circ \pi'_{\Omega_{\mathcal{A}}(F)}\rangle \in \mathrm{Mat}^*(\mathbb{S})$. So $\mathrm{Mat}^*(\mathbb{S})$ is the class of all reduced $\mathbb{S}$-matrices. Define $\mathrm{Alg}^*(\mathbb{S})$ as the class of all G-algebraic reducts of all G-matrices in $\mathrm{Mat}^*(\mathbb{S})$. We note that, by definition, for every G-algebra $\mathcal{A}$ and any $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, we have $\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A})$.

By Lemma 78, $\mathbb{S}$-filters on the formula algebra coincide with $\mathbb{S}$-theories. Moreover, in the case of the formula algebra, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, we have

$$\Omega_{\mathbf{Fm}_{\mathcal{L}}(V)}(X)(\varphi, \psi) = \bigwedge_{\gamma(x) \in \mathrm{Fm}_{\mathcal{L}}(V)} X(\gamma(\varphi)) \leftrightarrow X(\gamma(\psi)).$$

Therefore, we get

$$\widetilde{\Omega}(\mathbb{S})(\varphi, \psi) = \bigwedge_{\substack{X \in \mathrm{Th}(\mathbb{S}) \\ \gamma(x) \in \mathrm{Fm}_{\mathcal{L}}(V)}} X(\gamma(\varphi)) \leftrightarrow X(\gamma(\psi)).$$

The G-logic $\mathbb{S}^* = \mathbb{S}/\widetilde{\Omega}(\mathbb{S})$, obtained from $\mathbb{S}$ by taking the quotient modulo the Tarski G-congruence of $\mathbb{S}$, is called the **Lindenbaum-Tarski quotient of** $\mathbb{S}$. Its G-algebra reduct $\mathcal{F}m^*_{\mathcal{L}}(V) = \mathcal{F}m_{\mathcal{L}}(V)/\widetilde{\Omega}(\mathbb{S})$ is called the **Lindenbaum-Tarski algebra of** $\mathbb{S}$.

We say that a sentential G-logic is **complete with respect to** a class M of G-matrices if

$$\mathrm{Th}(\mathbb{S}) = \{F \circ h : \langle \mathcal{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}\}.$$

In particular, since, by Lemma 77, for all $\langle \mathcal{A}, F\rangle \in \mathsf{M}$, it holds that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, $F \circ h \in \mathrm{Th}(\mathbb{S})$, we must have $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Hence, any class of G-matrices with respect to which $\mathbb{S}$ is complete must be a subclass of $\mathrm{Mat}(\mathbb{S})$. Additionally, since, by Lemma 78, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V)) = \mathrm{Th}(\mathbb{S})$, we have that $\mathbb{S}$ is complete with respect to $\mathrm{Mat}(\mathbb{S})$. Moreover, it can be shown that $\mathbb{S}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{S})$.

3.9 Graded Models

Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic, that is $D : G^A \to G^A$ is a G-operator on $\mathcal{A}$. Define the sentential G-logic $\mathbb{S}_{\mathbb{L}} = \langle \mathcal{F}m_{\mathcal{L}}(V), C_{\mathbb{L}}\rangle$ by setting, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{aligned} C_{\mathbb{L}}(X) &= \bigwedge\{D(X') \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, X' : A \to G, \\ &\qquad \text{such that } X \leq D(X') \circ h\} \\ &= \bigwedge\{X' \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, X' \in \mathrm{Cl}(\mathbb{L}), \\ &\qquad \text{such that } X \leq X' \circ h\} \end{aligned}$$

Moreover, given a family $\mathsf{L} = \{\mathbb{L}_i : i \in I\}$ of G-logics $\mathbb{L}_i = \langle \mathcal{A}_i, D_i\rangle$, we set

$$C_{\mathsf{L}} = \bigwedge\{C_{\mathbb{L}} : \mathbb{L} \in \mathsf{L}\}.$$

Proposition 83 *Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}_i = \langle \mathcal{A}_i, D_i\rangle$, with $\mathcal{A}_i = \langle \mathbf{A}_i, E_i\rangle$, $i \in I$, be G-logics. Set $\mathsf{L} = \{\mathbb{L}_i : i \in I\}$. Then, $C_{\mathbb{L}}$ and C_{L} are G-operators on $\mathcal{F}m_{\mathcal{L}}(V)$.*

Proof: We show the details for $C_{\mathbb{L}}$. Starting from Inflationarity, we have, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$X \leq \bigwedge_{X',h}\{X' \circ h : X \leq X' \circ h\} = C_{\mathbb{L}}(X).$$

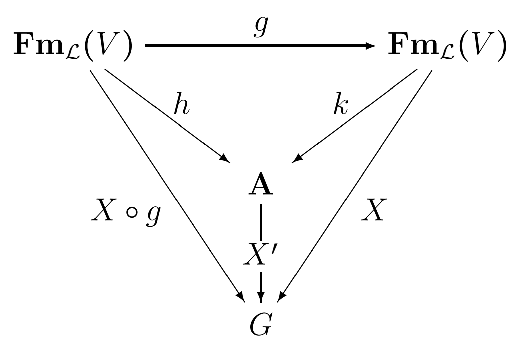
Continuing with Monotonicity, we have, for all $X, Y : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, with $X \leq Y$,

$$C_{\mathbb{L}}(X) = \bigwedge_{X',h}\{X' \circ h : X \leq X' \circ h\} \leq \bigwedge_{X',h}\{X' \circ h : Y \leq X' \circ h\} = C_{\mathbb{L}}(Y).$$

Next, for Idempotency, let $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and compute

$$\begin{array}{rcl} C_{\mathbb{L}}(C_{\mathbb{L}}(X)) & = & \bigwedge_{X',h}\{X' \circ h : C_{\mathbb{L}}(X) \le X' \circ h\} \\ & = & \bigwedge_{X',h}\{X' \circ h : X \le X' \circ h\} \quad (X' \circ h \in \mathrm{Cl}(\mathbb{S}_{\mathbb{L}})) \\ & = & C_{\mathbb{L}}(X). \end{array}$$

Finally, for Translation, let $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and $g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.



We have

$$\begin{array}{rcl} C_{\mathbb{L}}(X \circ g) & = & \bigwedge_{X',h}\{X' \circ h : X \circ g \le X' \circ h\} \\ & & \quad \text{(Definition of } C_{\mathbb{L}}\text{)} \\ & \le & \bigwedge_{X',k}\{X' \circ k \circ g : X \circ g \le X' \circ k \circ g\} \\ & & \quad \text{(Second set smaller)} \\ & \le & \bigwedge_{X',k}\{X' \circ k : X \le X' \circ k\} \circ g \\ & & \quad \text{(Second set smaller)} \\ & = & C_{\mathbb{L}}(X) \circ g. \quad \text{(Definition of } C_{\mathbb{L}}\text{)} \end{array}$$

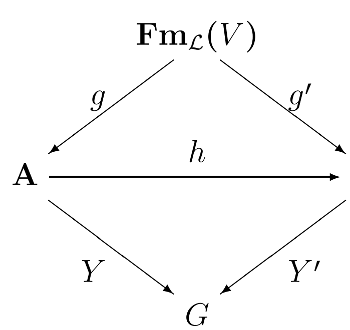
Thus, $C_{\mathbb{L}}$ is a G-operator on $\mathcal{F}m_{\mathcal{L}}(V)$. A similar proof applies to C_{L}. ■

We call $\mathbb{S}_{\mathsf{L}} = \langle \mathcal{F}m_{\mathcal{L}}(V), C_{\mathsf{L}} \rangle$ the (**sentential**) **G-logic induced**, or **generated, by** the class L of G-logics (or by the G-logic $\mathbb{L}$ if $\mathsf{L} = \{\mathbb{L}\}$).

In an analog of Proposition 2.3 of [28], we show that if two G-logics are related via a bilogical G-morphism, then the sentential G-logics that they induce on the formula G-algebra are identical.

Proposition 84 *Let $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then $\mathbb{S}_{\mathbb{L}} = \mathbb{S}_{\mathbb{L}'}$.*

Proof: We show that $\mathrm{Cl}(\mathbb{S}_{\mathbb{L}}) = \mathrm{Cl}(\mathbb{S}_{\mathbb{L}'})$. Note that, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,



$$\begin{array}{rcll} C_{\mathbb{L}'}(X) & = & \bigwedge_{\substack{Y' \in \mathrm{Cl}(\mathbb{L}') \\ g':\mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}'}} \{Y' \circ g' : X \leq Y' \circ g'\} & \text{(Definition of } C_{\mathbb{L}'}) \\ & = & \bigwedge_{\substack{Y' \in \mathrm{Cl}(\mathbb{L}') \\ g:\mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}}} \{Y' \circ h \circ g : X \leq Y' \circ h \circ g\} & (h \text{ surjective}) \\ & = & \bigwedge_{\substack{Y \in \mathrm{Cl}(\mathbb{L}) \\ g:\mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}}} \{Y \circ g : X \leq Y \circ g\} & (h : \mathbb{L} \to_b \mathbb{L}') \\ & = & C_{\mathbb{L}}(X). \quad \text{(Definition of } C_{\mathbb{L}}) & \end{array}$$

Hence $\mathbb{S}_{\mathbb{L}} = \{X : C_{\mathbb{L}}(X) = X\} = \{X : C_{\mathbb{L}'}(X) = X\} = \mathbb{S}_{\mathbb{L}'}$. ∎

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, a G-logic. We say $\mathbb{L}$ is a G-**model** of $\mathbb{S}$ if

$$C \leq C_{\mathbb{L}}.$$

We let $\mathrm{Mod}(\mathbb{S})$ denote the class of all G-models of $\mathbb{S}$.

We say that a sentential G-logic $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ is **complete with respect to** a class L of G-logics when

$$C = C_{\mathsf{L}}.$$

We now show that modelhood is preserved by bilogical G-morphisms and obtain, as a corollary, that a sentential G-logic is complete with respect to a class of G-logics if and only if it is complete with respect to the class of their reductions. This is an analog of Proposition 2.5 of [28] for G-logics.

Proposition 85 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L} = \langle \mathcal{A}', D' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics.*

(a) If there exists a bilogical G-morphism from $\mathbb{L}$ to $\mathbb{L}'$, then $\mathbb{L}$ is a G-model of $\mathbb{S}$ if and only if $\mathbb{L}'$ is a G-model of $\mathbb{S}$. In particular, $\mathbb{L}$ is a G-model of $\mathbb{S}$ if and only if $\mathbb{L}^*$ is.

(b) If $\mathbb{S}$ is complete with respect to a class L of G-logics, then it is also complete with respect to the class L^*.

Proof:

(a) Suppose there exists a bilogical G-morphism from $\mathbb{L}$ to $\mathbb{L}'$. Then, by Proposition 84, $C_{\mathbb{L}} = C_{\mathbb{L}'}$. Now we get

$$\begin{array}{rll} \mathbb{L} \in \mathrm{Mod}(\mathbb{S}) & \text{iff} & C \leq C_{\mathbb{L}} \\ & \text{iff} & C \leq C_{\mathbb{L}'} \\ & \text{iff} & \mathbb{L}' \in \mathrm{Mod}(\mathbb{S}). \end{array}$$

The second statement follows from the fact that, by Lemma 69, the quotient G-morphism $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism.

(b) This is similar. We get

$$\begin{array}{lll} \mathbb{S} \text{ complete with respect to } \mathsf{L} & \text{iff} & C = C_{\mathsf{L}} \\ & \text{iff} & C = C_{\mathsf{L}^*} \quad \text{(Part (a))} \\ & \text{iff} & \mathbb{S} \text{ complete with respect to } \mathsf{L}^*. \end{array}$$

■

If a class of models L contains the sentential G-logic $\mathbb{S}$ itself or its reduction, then $\mathbb{S}$ is complete with respect to L. This enables us to conclude that a sentential G-logic is complete with respect to both the class of its G-models and the class of its reduced G-models. The statement forms an analog of Proposition 2.6 of [28].

Proposition 86 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is complete with respect to any class L of its G-models that includes $\mathbb{S}$ or $\mathbb{S}^*$, and also with respect to the corresponding reduced class L^*. In particular, $\mathbb{S}$ is complete with respect to the class of all its G-models and with respect to the class of all its reduced G-models.*

Proof: First, by definition, for any $\mathsf{L} \subseteq \mathrm{Mod}(\mathbb{S})$, we have $C \le C_{\mathsf{L}}$. Next, suppose $\mathbb{S} \in \mathsf{L}$. It suffices to show that $C_{\mathsf{L}} \le C$. This is clear, since, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcl} C_{\mathsf{L}}(X) & = & \bigwedge\{X' \circ h : \mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathsf{L}, X' \in \mathrm{Cl}(\mathbb{L}), \\ & & \qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, X \le X' \circ h\} \\ & & \text{(Definition of } C_{\mathsf{L}}) \\ & \le & C(X), \end{array}$$

where the last inequality follows from the fact that $\mathbb{S} \in \mathsf{L}$, $C(X) \in \mathrm{Cl}(\mathbb{S})$ and the identity $i : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ satisfy the conditions for membership in the set appearing on the left of the inequality. The statement concerning reductions follows from the one proven above and Proposition 85. ■

We close the section by characterizing those G-logics that constitute G-models of a given sentential G-logic $\mathbb{S}$. They are exactly the ones all of whose closed G-sets are $\mathbb{S}$-filters on the underlying G-algebra. This constitutes an analog of Proposition 2.7 of [28].

Proposition 87 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. $\mathbb{L}$ is a G-model of $\mathbb{S}$ if and only if, for all $F \in \mathrm{Cl}(\mathbb{L})$, $\langle \mathbf{A}, F\rangle$ is an $\mathbb{S}$-matrix, i.e., $F \in \mathrm{Fi}_{\mathbb{S}}(\mathbf{A})$.*

Proof: Suppose, first, that, for all $F \in \mathrm{Cl}(\mathbb{L})$, we have $\langle \mathbf{A}, F\rangle \in \mathrm{Mat}(\mathbb{S})$, i.e., that, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$X \le F \circ h \quad \text{implies} \quad C(X) \le F \circ h.$$

Thus, we get, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C(X) \leq \bigwedge_{F,h} \{F \circ h : X \leq F \circ h\} = C_{\mathbb{L}}(X).$$

Hence, $\mathbb{L}$ is a G-model of $\mathbb{S}$.

Conversely, suppose that $\mathbb{L}$ is a G-model of $\mathbb{S}$ and let $F \in \mathrm{Cl}(\mathbb{L})$, $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, such that $X \leq F \circ h$. Then

$$\begin{array}{rcll} C(X) & \leq & C(F \circ h) & \text{(Monotonicity)} \\ & \leq & C_{\mathbb{L}}(F \circ h) & (\mathbb{L} \in \mathrm{Mod}(\mathbb{S})) \\ & = & F \circ h. & \text{(Definition of } C_{\mathbb{L}} \text{ and } F \in \mathrm{Cl}(\mathbb{L})) \end{array}$$

Therefore, $F \in \mathrm{Fi}_{\mathbb{S}}(\mathbf{A})$. ■

3.10 Full Graded Models

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, a G-logic. We say that $\mathbb{L}$ is a **full G-model** of $\mathbb{S}$ if

$$\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*) \rangle,$$

that is, if the closed functions of the reduction of $\mathbb{L}$ consist of all $\mathbb{S}$-filters on the quotient G-algebra. The class of all full G-models of $\mathbb{S}$ is denoted $\mathrm{FMod}(\mathbb{S})$. The class of all reduced full G-models of $\mathbb{S}$ is denoted $\mathrm{FMod}^*(\mathbb{S})$. For a given G-algebra $\mathcal{A}$, we write $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ for the class of full G-models of $\mathbb{S}$ on $\mathcal{A}$.

The following result justifies the word "model" in the term "full G-model". It is an analog of Part (1) of Proposition 2.9 of [28] for G-models.

Proposition 88 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic. If a G-logic $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, is a full G-model of $\mathbb{S}$, then it is a G-model of $\mathbb{S}$.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a full G-model of $\mathbb{S}$. By definition, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*) \rangle$. Hence, by Proposition 87, $\mathbb{L}^*$ is a G-model of $\mathbb{S}$. Therefore, by Proposition 85, $\mathbb{L}$ is a G-model of $\mathbb{S}$. ■

It turns out that every model whose collection of closed sets consists of all $\mathbb{S}$-filters on the underlying G-algebra is a full G-model of $\mathbb{S}$. Proposition 89 lifts to G-models Proposition 2.10 of [28].

Proposition 89 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E \rangle$ a G-algebra. The G-logic $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \rangle$ is a full G-model of $\mathbb{S}$. It is the weakest full G-model of $\mathbb{S}$ on the G-algebra $\mathcal{A}$.*

Proof: The first statement follows from Proposition 82, since we have

$$\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^* \cong \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*).$$

The second statement is obvious, since, by Proposition 88, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is a G-model of $\mathbb{S}$ and, by Proposition 87, it is clearly the weakest one. ∎

Models of the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ are called **basic full G-models**.

We show, next, in an analog of Proposition 2.11 of [28], that the class of full G-models of a sentential G-logic is closed under bilogical G-morphisms.

Proposition 90 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The class* FMod($\mathbb{S}$) *is closed under bilogical G-morphisms. That is, if $\mathbb{L} = \langle \mathcal{A}, D\rangle$ and $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$ are G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ is a bilogical G-morphism, then $\mathbb{L}$ is a full G-model of $\mathbb{S}$ if and only if $\mathbb{L}'$ is a full G-model of $\mathbb{S}$. In particular, a G-logic $\mathbb{L}$ is a full G-model of $\mathbb{S}$ if and only if its reduction $\mathbb{L}^*$ is a full G-model of $\mathbb{S}$.*

Proof: Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then, by Proposition 74, $\mathbb{L}^* \cong \mathbb{L}'^*$. Therefore, we have

$$\begin{array}{lll}\mathbb{L} \text{ full } G\text{-model} & \text{iff} & \mathrm{Cl}(\mathbb{L}^*) = \mathrm{Fi}_{\mathbb{S}}(\mathbf{A}^*)\\ & \text{iff} & \mathrm{Cl}(\mathbb{L}'^*) = \mathrm{Fi}_{\mathbb{S}}(\mathbf{A}'^*)\\ & \text{iff} & \mathbb{L}' \text{ full } G\text{-model.}\end{array}$$

The second statement follows immediately, since, by Lemma 69, the quotient G-morphism $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism. ∎

Recall that in Proposition 89, it was shown that basic full G-models of $\mathbb{S}$ are special cases of full G-models of $\mathbb{S}$. It follows from Proposition 90 that any G-model that can be mapped onto a basic full G-model of $\mathbb{S}$ is a full G-model of $\mathbb{S}$. The converse of this also holds.

Corollary 91 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. $\mathbb{L}$ is a full G-model of $\mathbb{S}$ if and only if there is a bilogical G-morphism from $\mathbb{L}$ onto a basic full G-model, i.e., a G-logic of the form $\langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$.*

Proof: Suppose, first, that $\mathbb{L}$ is a full G-model of $\mathbb{S}$. Then, by Lemma 69, the projection $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism and, by the definition of a full G-model, $\mathbb{L}^*$ is a basic full G-model.

Assume, conversely, that there exists a bilogical G-morphism $h : \mathbb{L} \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$. By Proposition 89, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ is a full G-model of $\mathbb{S}$ and, by Proposition 90, $\mathbb{L}$ is a full G-model of $\mathbb{S}$. ∎

The preceding results allow us to characterize the class of full G-models as the smallest class containing all basic full G-models and closed under bilogical G-morphisms (in both the forward and backward direction). This forms an analog of Corollary 2.13 of [28].

Corollary 92 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic. The class* $\mathrm{FMod}(\mathbb{S})$ *is the smallest class of* G*-logics containing all basic full* G*-models, i.e.,* G*-logics of the form* $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$*, and closed under (both images and pre-images of) bilogical* G*-morphisms.*

Proof: Denote by L the smallest class of G-logics containing all basic full G-models and closed under bilogical G-morphisms. We must show that $\mathsf{L} = \mathrm{FMod}(\mathbb{S})$.

First, by Proposition 89, every basic full G-model is in $\mathrm{FMod}(\mathbb{S})$. Moreover, by Proposition 90, $\mathrm{FMod}(\mathbb{S})$ is closed under bilogical G-morphisms. Therefore, by the definition of L, $\mathsf{L} \subseteq \mathrm{FMod}(\mathbb{S})$.

Conversely, suppose $\mathbb{L} \in \mathrm{FMod}(\mathbb{S})$. Then, by Lemma 69, the quotient G-morphism $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism and, by the definition of full G-models, $\mathbb{L}^*$ is a basic full G-model. Therefore, $\mathbb{L} \in \mathsf{L}$. ∎

Full G-models assume their name from the fact that the closed sets of their reductions include all $\mathbb{S}$-filters on the reduced G-algebra. The next theorem gives another sense in which full G-models are "full". The collection of closed sets consists of all $\mathbb{S}$-filters with which the Tarski G-congruence of the G-model is compatible. These are the reasons given by Font and Jansana in [28] for choosing the name for these models in the sentential logic framework. The second justification they provide is the content of Theorem 2.14 of [28] whose analog for G-logics is the next result.

Theorem 93 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic and* $\mathbb{L} = \langle \mathcal{A}, D\rangle$*, with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*, a* G*-logic.* $\mathbb{L}$ *is a full* G*-model of* $\mathbb{S}$ *if and only if*

$$\mathrm{Cl}(\mathbb{L}) = \{F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) : \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) \le \Omega_{\mathcal{A}}(F)\}.$$

Proof: Suppose that $\mathbb{L}$ is a full G-model of $\mathbb{S}$. We must show that the displayed equality holds.

Assume, first, that $F \in \mathrm{Cl}(\mathbb{L})$. By Propositions 89 and 87, $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Moreover, by the discussion preceding Proposition 62, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) \le \Omega_{\mathcal{A}}(F)$.

Assume, next, that $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) \le \Omega_{\mathcal{A}}(F)$. Then, by Proposition 80, There exists $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$, such that $F = H \circ \pi$, where $\pi : \mathcal{A} \to \mathcal{A}^*$ is the quotient G-morphism. But, by Lemma 69, $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism. Moreover, since, by hypothesis, $\mathbb{L}$ is a full G-model, $\mathrm{Cl}(\mathbb{L}^*) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. Thus, $F \in \mathrm{Cl}(\mathbb{L})$.

Suppose, conversely, that the displayed equality holds. Using Proposition 80, we can see that $\pi : \mathbb{L} \to_b \langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$ is a bilogical G-morphism. Hence, $\mathrm{Cl}(\mathbb{L})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. So, by definition, $\mathbb{L}$ is a full G-model of $\mathbb{S}$. ∎

3.11 $\mathbb{S}$-Algebras

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. A G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is called an $\mathbb{S}$-**algebra** if the G-logic $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is reduced. Equivalently, $\mathcal{A}$ is an $\mathbb{S}$-algebra if and only if it is the G-algebraic reduct of a reduced full G-model of $\mathbb{S}$. We denote the class of all $\mathbb{S}$-algebras by $\mathrm{Alg}(\mathbb{S})$.

Note that the quotient G-algebra $\mathcal{F}m^*_{\mathcal{L}}(V) = \mathcal{F}m_{\mathcal{L}}(V)/\widetilde{\Omega}(\mathbb{S})$ is an $\mathbb{S}$-algebra. This follows from Proposition 82 and Corollary 73. The following is an analog of Proposition 2.17 of [28].

Proposition 94 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. Then the following are equivalent:*

(i) $\mathbb{L}$ *is a reduced full G-model of* $\mathbb{S}$*;*

(ii) $\mathbb{L}$ *is reduced and* $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$*;*

(iii) $\mathcal{A}$ *is an $\mathbb{S}$-algebra and* $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose that $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a reduced full G-model of $\mathbb{S}$. The fact that it is reduced shows that its reduction is $\mathbb{L}$ and the fact that it is full shows that $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(ii)⇒(iii) Suppose that $\mathbb{L}$ is reduced and $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Then $\mathcal{A}$ is an $\mathbb{S}$-algebra by definition. Moreover, by hypothesis, $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(iii)⇒(i) Since $\mathcal{A}$ is an $\mathbb{S}$-algebra, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is reduced. Since $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, $\mathbb{L}$ is a reduced full G-model of $\mathbb{S}$.

■

Tarski G-congruences of full G-models of a G-logic $\mathbb{S}$ are $\mathrm{Alg}(\mathbb{S})$-congruences, as is the case with sentential logics (see Proposition 2.18 of [28]).

Proposition 95 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a full G-model of $\mathbb{S}$. Then $\mathcal{A}^*$ is an $\mathbb{S}$-algebra and, hence, $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$.*

Proof: Assume $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full G-model of $\mathbb{S}$. Then, by definition, its reduction is $\langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$. Thus, again by definition, $\mathcal{A}^*$ is an $\mathbb{S}$-algebra. This shows that $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. ■

The class $\mathrm{Alg}(\mathbb{S})$ may be characterized as the class of all G-algebraic reducts of all reduced G-models of $\mathbb{S}$. Notice that this characterization, which forms an analog of Proposition 2.19 of [28], does not employ the notion of fullness of models.

Proposition 96 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic. The class of* $\mathbb{S}$*-algebras is the class of* G*-algebraic reducts of all reduced* G*-models of* $\mathbb{S}$*.*

Proof: Let K be the class of algebraic reducts of all reduced G-models of $\mathbb{S}$. We need to show that $\mathsf{K} = \mathrm{Alg}(\mathbb{S})$. If $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$, then, by definition, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is a reduced full G-model of $\mathbb{S}$. Therefore, taking into account Proposition 88, $\mathcal{A} \in \mathsf{K}$. On the other hand, assume $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}$. Then, there exists a reduced G-model $\mathbb{L} = \langle \mathcal{A}, D\rangle$ of $\mathbb{S}$. Taking into account Proposition 87, consider the G-model $\mathbb{L}' = \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ of $\mathbb{S}$. We have $\mathbb{L}' \leq \mathbb{L}$, whence, by Proposition 63 and the hypothesis,

$$E \leq \widetilde{\Omega}(\mathbb{L}') \leq \widetilde{\Omega}(\mathbb{L}) = E.$$

Thus, $\mathbb{L}'$ is a reduced full G-model of $\mathbb{S}$, showing that $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$. ■

Further, the class $\mathrm{Alg}(\mathbb{S})$ is an abstract class, that is, it is closed under isomorphisms.

Proposition 97 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic. The class of* $\mathbb{S}$*-algebras is closed under isomorphisms.*

Proof: Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic G-algebras. Using Proposition 79, we may show that $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ are also isomorphic as lattices. Now, taking into account Propositions 81, 64 and 68, we get that $\mathbb{L} = \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ and $\mathbb{L}' = \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ are G-isomorphic G-logics. Therefore, taking into account Propositions 67 and 68, $\mathbb{L}$ is reduced if and only if $\mathbb{L}'$ is reduced. Thus, $\mathcal{A}$ is an $\mathbb{S}$-algebra if and only if $\mathcal{A}'$ is an $\mathbb{S}$-algebra. ■

Full G-models of a G-logic $\mathbb{S}$ may be characterized in terms of $\mathbb{S}$-algebras. This is an analog of Proposition 2.21 of [28].

Proposition 98 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic. Then the following conditions are equivalent:*

(i) $\mathbb{L} = \langle \mathcal{A}, D\rangle$*, with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*, is a full* G*-model of* $\mathbb{S}$*;*

(ii) $\mathcal{A}^*$ *is an* $\mathbb{S}$*-algebra and* $\mathrm{Cl}(\mathbb{L})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$*;*

(iii) There is a bilogical G*-morphism between* $\mathbb{L}$ *and a* G*-logic* $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$*, such that* $\mathcal{A}'$ *is an* $\mathbb{S}$*-algebra and* $\mathrm{Cl}(\mathbb{L}') = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$*.*

Proof:

(i)⇒(ii) Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full G-model of $\mathbb{S}$. By definition, $\mathrm{Cl}(\mathbb{L})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. Thus, also by definition, $\mathcal{A}^* \in \mathrm{Alg}(\mathbb{S})$.

(ii)⇒(iii) We take $\mathbb{L}' = \mathbb{L}^*$ and consider the quotient G-morphism.

(iii)⇒(i) By Proposition 94, $\mathbb{L}'$ is a reduced full G-model of $\mathbb{S}$. By Corollary 91, $\mathbb{L}$ is a full G-model of $\mathbb{S}$.

■

We have seen in Proposition 86 that a sentential G-logic $\mathbb{S}$ is complete with respect to any class L of its G-models that includes $\mathbb{S}$ or $\mathbb{S}^*$, and also with respect to the corresponding reduced class L^*. We inferred that $\mathbb{S}$ is complete with respect to the class of all its G-models and with respect to the class of all its reduced G-models. We prove next an analog of Theorem 2.22 of [28] which is a "better" or "more advanced" completeness result, in the sense that it asserts completeness with respect to smaller, more targeted, classes of models, which, however, are still large enough to serve the purpose.

Theorem 99 (Completeness Theorem) *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is complete with respect to the following classes of G-logics:*

1. *The class of all full G-models of $\mathbb{S}$;*
2. *The class of all basic full G-models of $\mathbb{S}$, i.e., those of the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$, for any G-algebra $\mathcal{A}$;*
3. *The class of all reduced full G-models of $\mathbb{S}$, that is, the class of all G-logics of the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$, $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$.*

Proof: Observe that all three classes described consist of G-models of $\mathbb{S}$. Moreover, $\mathbb{S}^*$ belongs to all three. Therefore, by Proposition 86, $\mathbb{S}$ is complete with respect to each of these classes. ■

Recall the class $\mathrm{Alg}^*(\mathbb{S})$ of all G-algebraic reducts of reduced $\mathbb{S}$-matrices. There is a strong connection between this class and the class of $\mathbb{S}$-algebras. The main thread tying them together is the relation between the Tarski G-congruence of an $\mathbb{S}$-model and the Leibniz G-congruences of the $\mathbb{S}$-matrices associated with its closed functions. Theorem 100 constitutes an analog of Theorem 2.23 of [28] for G-logics and their associated classes of G-algebras.

Theorem 100 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Then $\mathrm{Alg}(\mathbb{S})$ is the class of all subdirect products of G-algebras in the class $\mathrm{Alg}^*(\mathbb{S})$.*

Proof: Suppose, first, that $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathrm{Alg}(\mathbb{S})$. Then

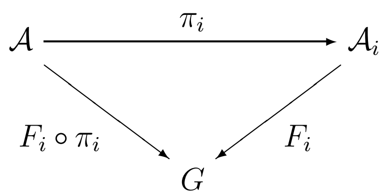
$$E = \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})) = \bigwedge\{\Omega_{\mathcal{A}}(F) : F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\}.$$

By Proposition 56, $\mathcal{A}$ is a subdirect product of $\{\mathcal{A}/\Omega_{\mathcal{A}}(F) : F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\} \subseteq \mathrm{Alg}^*(\mathbb{S})$.

Assume, conversely, that $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is a subdirect product of

$$\{\mathcal{A}_i = \langle \mathbf{A}_i, E_i\rangle : i \in I\} \subseteq \mathrm{Alg}^*(\mathbb{S}).$$

By definition of $\mathrm{Alg}^*(\mathbb{S})$, for every $i \in I$, there exists $F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}_i)$, such that $\Omega_{\mathcal{A}_i}(F_i) = E_i$.



Let D be the G-operator on $\mathcal{A}$ generated by the family of $\mathbb{S}$-filters $F_i \circ \pi_i$, $i \in I$, where $\pi_i : \mathcal{A} \to \mathcal{A}_i$ denotes the i-th projection morphism of the product restricted to $\mathcal{A}$. Then, by Propositions 79 and 87, $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a G-model of $\mathbb{S}$. We show it is reduced. We have, for all $a, b \in A$,

$$\begin{array}{rcll}
\widetilde{\Omega}(\mathbb{L}) & = & \bigwedge_{H \in \mathrm{Cl}(\mathbb{L})} \Omega_{\mathcal{A}}(H) & \text{(Tarski \& Leibniz operators)} \\
& \leq & \bigwedge_{i \in I} \Omega_{\mathcal{A}}(F_i \circ \pi_i) & (F_i \circ \pi_i \in \mathrm{Cl}(\mathbb{L})) \\
& = & \bigwedge_{i \in I} \Omega_{\mathcal{A}_i}(F_i) \circ \pi_i^2 & \text{(Theorem 58)} \\
& = & \bigwedge_{i \in I} E_i \circ \pi_i^2 & (\mathcal{A}_i \in \mathrm{Alg}^*(\mathbb{S})) \\
& = & E. & (\mathcal{A} \subseteq_{\mathsf{sd}} \prod_{i \in I} \mathcal{A}_i)
\end{array}$$

This shows that $\mathbb{L}$ is reduced. Therefore, by Proposition 96, $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$. ■

Corollary 101 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic. Then*

$$\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$$

and the two classes coincide iff $\mathrm{Alg}^*(\mathbb{S})$ is closed under subdirect products.

Proof: Directly from Theorem 100. ■

Finally, we can state a relationship between the corresponding associated classes of G-algebras of two sentential G-logics one of which is an extension of the other. This forms an analog of Proposition 2.27 of [28].

Proposition 102 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ and $\mathbb{S}' = \langle \mathcal{F}m_{\mathcal{L}}(V), C' \rangle$ be sentential G-logics, such that $\mathbb{S} \leq \mathbb{S}'$. Then $\mathrm{Alg}(\mathbb{S}') \subseteq \mathrm{Alg}(\mathbb{S})$ and $\mathrm{Alg}^*(\mathbb{S}') \subseteq \mathrm{Alg}^*(\mathbb{S})$.*

Proof: Assume $\mathbb{S} \leq \mathbb{S}'$. Then, by Lemma 77, for every G-algebra $\mathcal{A}$, $\mathrm{Fi}_{\mathbb{S}'}(\mathcal{A}) \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Therefore, directly by the respective definitions, $\mathrm{Alg}^*(\mathbb{S}') \subseteq \mathrm{Alg}^*(\mathbb{S})$. By Theorem 100, we obtain, also, that $\mathrm{Alg}(\mathbb{S}') \subseteq \mathrm{Alg}(\mathbb{S})$. ■

3.12 The Lattice of Full G-Models

In this section, we present an analog for G-logics, G-models and G-congruences of the celebrated Isomorphism Theorem 2.30 of [28], which asserts that $\widetilde{\Omega}_{\mathbf{A}}$ is an order-isomorphism between the ordered set of full models of a sentential logic on an algebra $\mathbf{A}$ and that of $\mathrm{Alg}(\mathcal{S})$-congruences on $\mathbf{A}$.

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. We denote by

$$\widetilde{H}_{\mathcal{A}}(\Theta) = \langle \mathcal{A}, C_\Theta\rangle$$

the G-logic on $\mathcal{A}$ projectively generated from $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ by the quotient G-morphism $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$.

The definition implies that

$$\pi_{\hat{\Theta}} : \widetilde{H}_{\mathcal{A}}(\Theta) \to_b \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$$

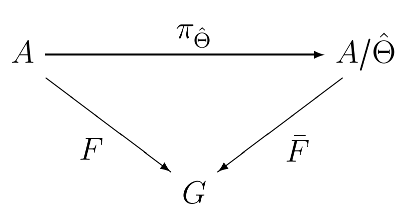
is a bilogical G-morphism.

Lemma 103 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. Then:*

- $\Theta \in \mathrm{Gon}(\widetilde{H}_{\mathcal{A}}(\Theta))$;
- $\widetilde{H}_{\mathcal{A}}(\Theta)/\Theta = \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$;
- $\widetilde{H}_{\mathcal{A}}(\Theta) \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$.

Moreover, the mapping $\Theta \mapsto \widetilde{H}_{\mathcal{A}}(\Theta)$ is order preserving, that is, if $\Theta, \Theta' \in \mathrm{Gon}(\mathcal{A})$, such that $\Theta \le \Theta'$, then $\widetilde{H}_{\mathcal{A}}(\Theta) \le \widetilde{H}_{\mathcal{A}}(\Theta')$.

Proof: Fist, we show that $\Theta \in \mathrm{Gon}(\widetilde{H}_{\mathcal{A}}(\Theta))$. Let $a, b \in A$ and consider $F \in \mathrm{Fi}_{\mathbb{S}}(\widetilde{H}_{\mathcal{A}}(\Theta))$. Then, since $\widetilde{H}_{\mathcal{A}}(\Theta)$ is projectively generated from $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$, there exists $\bar{F} \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)$, such that $F = \bar{F} \circ \pi_{\hat{\Theta}}$.

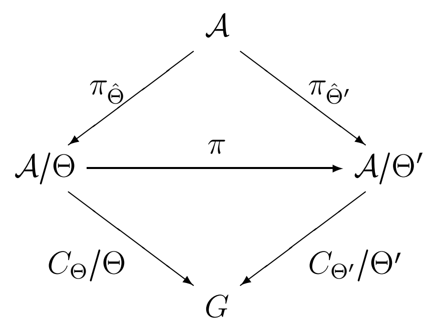


Now we get

$$\begin{array}{rcll}\Theta(a,b) \wedge F(a) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{F}(a/\hat{\Theta}) & (\text{Definitions of } \bar{\Theta} \text{ and } \bar{F})\\ & \le & \bar{F}(b/\hat{\Theta}) \quad (\bar{F} \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)) & \\ & = & F(b). \quad (\text{Definition of } \bar{F}) & \end{array}$$

Since $\Theta \in \mathrm{Gon}(\widetilde{H}_{\mathcal{A}}(\Theta))$, it makes sense to consider the quotient $\widetilde{H}_{\mathcal{A}}(\Theta)/\Theta$ and, by the definition of $\widetilde{H}_{\mathcal{A}}(\Theta)$, $\widetilde{H}_{\mathcal{A}}(\Theta)/\Theta = \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$. By Proposition 89, $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ is a full G-model of $\mathbb{S}$, whence, by Corollary 91, $\widetilde{H}_{\mathcal{A}}(\Theta) \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$.

Finally, let $\Theta, \Theta' \in \mathrm{Gon}(\mathcal{A})$, with associated quotient morphisms $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$ and $\pi_{\hat{\Theta}'} : \mathcal{A} \to \mathcal{A}/\Theta'$. Assume that $\Theta \le \Theta'$ and let $\pi : \mathcal{A}/\Theta \to \mathcal{A}/\Theta'$ be the homomorphism $a/\hat{\Theta} \mapsto a/\hat{\Theta}'$, which is well-defined because of the inequality $\Theta \le \Theta'$ and, moreover, makes the following diagram commute.



Now we have

$$\begin{array}{rcll}
\mathrm{Cl}(\widetilde{H}_{\mathcal{A}}(\Theta')) & = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta') \circ \pi_{\hat{\Theta}'} & (\pi_{\hat{\Theta}'} : \widetilde{H}_{\mathcal{A}}(\Theta') \to_b \langle \mathcal{A}/\Theta', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta')\rangle) \\
& = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta') \circ \pi \circ \pi_{\hat{\Theta}} & (\pi_{\hat{\Theta}'} = \pi \circ \pi_{\hat{\Theta}}) \\
& \subseteq & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta) \circ \pi_{\hat{\Theta}} & (\text{Proposition 79}) \\
& = & \mathrm{Cl}(\widetilde{H}_{\mathcal{A}}(\Theta)). & (\pi_{\hat{\Theta}} : \widetilde{H}_{\mathcal{A}}(\Theta) \to_b \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle)
\end{array}$$

Therefore, by Proposition 61, we conclude that $\widetilde{H}_{\mathcal{A}}(\Theta) \le \widetilde{H}_{\mathcal{A}}(\Theta')$. ■

The following Isomorphism Theorem is a version in the current setting of the well known Isomorphism Theorem (Theorem 2.30 of [28]) devised by Font and Jansana in the framework of sentential logics.

Theorem 104 (The Isomorphism Theorem) *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is an order isomorphism between $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \le\rangle$ and $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \le\rangle$, with $\widetilde{H}_{\mathcal{A}}$ as its inverse,*

$$\widetilde{\Omega}_{\mathcal{A}} : \ \langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \le\rangle \rightleftarrows \langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \le\rangle \ : \widetilde{H}_{\mathcal{A}}$$

Proof: By Proposition 95, if $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$, then we have $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. Conversely, by Lemma 103, if $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$, then $\widetilde{H}_{\mathcal{A}}(\Theta) \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$. Thus, both mappings are well defined. We show that they are inverse bijections.

Suppose, first, that $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$. By Proposition 95, $\mathcal{A}^* \in \mathrm{Alg}(\mathbb{S})$ and $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. Moreover, $\mathbb{L}$ is projectively generated from $\langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$ by the quotient G-morphism. Thus, by definition, $\mathbb{L} = \widetilde{H}_{\mathcal{A}}(\widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}))$.

Suppose, on the other hand, that $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. By definition, $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ is reduced, i.e.,

$$\widetilde{\Omega}_{\mathcal{A}/\Theta}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)) = \bar{\Theta}.$$

Then, we have

$$\begin{aligned}
\widetilde{\Omega}_{\mathcal{A}}(\widetilde{H}_{\mathcal{A}}(\Theta)) &= \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta) \circ \pi_{\hat{\Theta}}) \quad (\text{Definition of } \widetilde{H}_{\mathcal{A}}(\Theta))\\
&= \widetilde{\Omega}_{\mathcal{A}/\Theta}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)) \circ \pi^2_{\hat{\Theta}} \quad (\text{Proposition 67})\\
&= \bar{\Theta} \circ \pi^2_{\hat{\Theta}} \quad (\text{Display above})\\
&= \Theta. \quad (\text{Definition of } \bar{\Theta})
\end{aligned}$$

We conclude that $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse bijections. Finally, note that, by Proposition 63, $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving and, by Lemma 103, $\widetilde{H}_{\mathcal{A}}$ is also order preserving, whence they are order isomorphisms between $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \leq\rangle$ and $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \leq\rangle$. ■

It can be shown that one of the two ordered sets proven isomorphic in the Isomorphism Theorem has the structure of a complete lattice and this allows us, via the Isomorphism Theorem, to conclude that the other does also. See Theorem 2.31 and Corollary 2.32 of [28] for the corresponding statements in the context of sentential logics.

Theorem 105 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. The ordered set $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \leq\rangle$ is a complete lattice, with the meet operation $\wedge$ in* $\mathbf{Gon}(\mathcal{A})$ *as its meet.*

Proof: Let $\{\Theta_i : i \in I\} \subseteq \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ be nonempty and consider $\Theta := \bigwedge_{i\in I}\Theta_i$. We must show that $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. Define, for every $i \in I$, $h_i : \mathcal{A}/\Theta \to \mathcal{A}/\Theta_i$ by

$$h_i(a/\hat{\Theta}) = a/\hat{\Theta}_i, \quad a \in A.$$

This is well defined. To see this, note that, for all $a, b \in A$,

$$\begin{aligned}
\langle a,b\rangle \in \hat{\Theta} \quad &\text{iff} \quad \Theta(a,b) = \top \quad (\text{Definition of } \hat{\Theta})\\
&\text{iff} \quad \textstyle\bigwedge_{i\in I}\Theta_i(a,b) = \top \quad (\text{Definition of } \Theta)\\
&\text{iff} \quad \Theta_i(a,b) = \top,\ i \in I, \quad (\text{Property of } \textstyle\bigwedge)\\
&\text{iff} \quad \langle a,b\rangle \in \hat{\Theta}_i,\ i \in I. \quad (\text{Definition of } \hat{\Theta}_i)
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) &= \Theta(a,b) \quad (\text{Definition of } \bar{\Theta})\\
&= \textstyle\bigwedge_{i\in I}\Theta_i(a,b) \quad (\text{Definition of } \Theta)\\
&= \textstyle\bigwedge_{i\in I}\bar{\Theta}_i(a/\hat{\Theta}_i, b/\hat{\Theta}_i) \quad (\text{Definition of } \bar{\Theta}_i)\\
&= \textstyle\bigwedge_{i\in I}\bar{\Theta}_i(h_i(a/\hat{\Theta}), h_i(b/\hat{\Theta})). \quad (\text{Definition of } h_i)
\end{aligned}$$

Since, for every $i \in I$, $\Theta_i \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$, the G-logic $\mathbb{L}_i = \langle \mathcal{A}/\Theta_i, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i)\rangle$ is reduced, that is, we have, for all $i \in I$,

$$\widetilde{\Omega}_{\mathcal{A}/\Theta_i}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i)) = \bar{\Theta}_i.$$

Our goal is to show that $\mathbb{L} = \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ is reduced. We calculate

$$\begin{array}{rcl}\widetilde{\Omega}(\mathbb{L}) & \leq & \bigwedge_{i\in I}\widetilde{\Omega}_{\mathcal{A}/\Theta}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i)\circ h_i) \quad \text{(Proposition 79)}\\ & = & \bigwedge_{i\in I}\widetilde{\Omega}_{\mathcal{A}/\Theta_i}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i))\circ h_i^2 \quad \text{(Theorem 58)}\\ & = & \bigwedge_{i\in I}\bar{\Theta}_i\circ h_i^2 \quad \text{(Displayed above)}\\ & = & \bar{\Theta}. \quad \text{(Shown above)}\end{array}$$

We conclude that $\mathbb{L}$ is reduced and $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$.

Finally, observe that any one-element algebra belongs to $\mathrm{Alg}(\mathbb{S})$, whence $\nabla^{\mathcal{A}} \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ and, hence, it is the largest element in this lattice. It corresponds to the G-logic whose only closed G-set is the function mapping every sentence to $\top$ in G, which is clearly a full G-model of $\mathbb{S}$. ■

Corollary 106 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. Then $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \leq\rangle$ is a complete lattice and the Tarski operator is a lattice isomorphism from $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \leq\rangle$ to $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \leq\rangle$.*

Proof: This follows from Theorems 104 and 105. ■

$\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ is a subset of the complete lattice of all G-logics over $\mathcal{A}$. However, it is not a sublattice. As a consequence of the preceding results, we can see that, given a collection $\{\mathbb{L}_i : i \in I\} \subseteq \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$, its meet in $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ can be obtained as the G-logic projectively generated from $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ by the projection homomorphism $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$, where $\Theta = \bigwedge_{i\in I}\widetilde{\Omega}(\mathbb{L}_i)$.

Some extensions of the isomorphisms detailed above may be established via the use of bilogical morphisms. The following proposition and corollary are analogs in the G-logic framework of Proposition 2.33 and of Corollary 2.34, respectively, of [28].

Proposition 107 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$ two full G-models of $\mathbb{S}$ and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism, with h' a section of h. Then the mapping*

$$\mathcal{X} \mapsto \{X \circ h' : X \in \mathcal{X}\}$$

establishes an isomorphism between the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}$ extending $\mathbb{L}$ and the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}'$ extending $\mathbb{L}'$. Moreover, the principal ideals of $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ and of $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}')$ determined by $\widetilde{\Omega}(\mathbb{L})$ and $\widetilde{\Omega}(\mathbb{L}')$, respectively, are isomorphic.

Proof: By Corollary 66, the given mapping is an isomorphism between the lattice of all G-logics extending $\mathbb{L}$ and the lattice of all G-logics extending $\mathbb{L}'$. By Proposition 90, this restricts to an isomorphism between the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}$ extending $\mathbb{L}$ and the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}'$ extending $\mathbb{L}'$. The last statement is a direct consequence of the Isomorphism Theorem 104. ■

Corollary 108 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic,* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ G*-algebras and* $h : \mathcal{A} \to \mathcal{A}'$ *an epimorphism, such that* $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ *is a bilogical* G*-morphism. Then* h *induces an isomorphism between the complete lattices* $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ *and* $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A}')$*. In addition, the lattices* $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ *and of* $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}')$ *are isomorphic.*

Proof: By Proposition 89, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is the weakest full G-model of $\mathbb{S}$ on $\mathcal{A}$ and $\langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ is the weakest full G-model of $\mathbb{S}$ on $\mathcal{A}'$. Therefore, by Proposition 107, h induces an isomorphism between the complete lattices $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ and $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A}')$. ■

3.13 Protoalgebraic G-Logics

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. We say that $\mathbb{S}$ is **protoalgebraic** if, for all $X \in \mathrm{Th}(\mathbb{S})$, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$ is compatible with every $X' \in \mathrm{Th}(\mathbb{S})^X$. Assuming that G has an implication $\to$, this can be expressed by the condition that, for all $X \in \mathrm{Th}(\mathbb{S})$ and all all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)(\varphi,\psi) \le \bigwedge_{X \le X' \in \mathrm{Th}(\mathbb{S})} (X'(\varphi) \leftrightarrow X'(\psi)).$$

This definition, which in the traditional framework is encapsulated in the motto "indistinguishability implies interderivability", turns out to be equivalent to the, more semantic in flavor, property of the Leibniz operator being monotone on $\mathrm{Th}(\mathbb{S})$.

Proposition 109 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic.* $\mathbb{S}$ *is protoalgebraic if and only if* $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ *is monotone on* $\mathrm{Th}(\mathbb{S})$*.*

Proof: Suppose, first, that $\mathbb{S}$ is protoalgebraic. To show monotonicity, let $X, X' \in \mathrm{Th}(\mathbb{S})$, such that $X \le X'$. By definition of protoalgebraicity, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$ is compatible with X'. Hence, by the maximality property of $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X')$, we get $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X) \le \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X')$. So $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$.

Suppose, conversely, that $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. Let $X, X' \in \mathrm{Th}(\mathbb{S})$, with $X \le X'$. Then, by monotonicity, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X') \le \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X')$.

This shows that $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$ is compatible with X'. Thus, $\mathbb{S}$ is protoalgebraic. ■

Further, monotonicity of the Leibniz operator on $\mathrm{Th}(\mathbb{S})$ is equivalent to the monotonicity of $\Omega_{\mathcal{A}}$ on the $\mathbb{S}$-filters of any G-algebra $\mathcal{A}$. The fact that the property of monotonicity of the Leibniz operator "transfers" from $\mathrm{Th}(\mathbb{S})$ to $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, for every G-algebra $\mathcal{A}$, is an example of a class of theorems of a similar type which are collectively termed **transfer theorems**.

Proposition 110 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is protoalgebraic if and only if, for every G-algebra $\mathcal{A}$, admitting a surjective G-morphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.*

Proof: First, suppose the given condition holds. Then take $\mathcal{A} = \mathcal{F}m_{\mathcal{L}}(V)$ and recall that, by Lemma 78, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V)) = \mathrm{Th}(\mathbb{S})$. Hence, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. Therefore, by Proposition 109, $\mathbb{S}$ is protoalgebraic.

For the converse, suppose $\mathbb{S}$ is protoalgebraic. Thus, by Proposition 109, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. Now assume $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F \le F'$. Let $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$ be a surjective G-morphism. Recall, by Lemma 77, that if $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, then $F \circ h \in \mathrm{Th}(\mathbb{S})$. So we have

$$\begin{array}{rcll}\Omega_{\mathcal{A}}(F) \circ h^2 & = & \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(F \circ h) & \text{(Lemma 58)}\\ & \le & \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(F' \circ h) & \text{(Hypothesis)}\\ & = & \Omega_{\mathcal{A}}(F') \circ h^2. & \text{(Lemma 58)}\end{array}$$

Hence, by the surjectivity of h, $\Omega_{\mathcal{A}}(F) \le \Omega_{\mathcal{A}}(F')$. Thus, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. ■

Another characterization involves the commutativity of the Leibniz operator with arbitrary meets.

Corollary 111 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is protoalgebraic if and only if, for every G-algebra $\mathcal{A}$, admitting a surjective homomorphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$, $\Omega_{\mathcal{A}}$ commutes with arbitrary meets, i.e., for all $F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$,*

$$\Omega_{\mathcal{A}}\left(\bigwedge_{i\in I} F_i\right) = \bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i).$$

Proof: First, assume that $\mathbb{S}$ is protoalgebraic. Since, by definition, $\bigwedge_{i\in I} F_i \le F_i$, for all $i \in I$, we get, by Proposition 109, $\Omega_{\mathcal{A}}(\bigwedge_{i\in I} F_i) \le \Omega_{\mathcal{A}}(F_i)$. Since this holds for all $i \in I$, $\Omega_{\mathcal{A}}(\bigwedge_{i\in I} F_i) \le \bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i)$. Conversely, suppose $a, b \in A$. Then

$$\begin{array}{rcl}\bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i)(a,b) \wedge \bigwedge_{i\in I} F_i(a) & \le & \Omega_{\mathcal{A}}(F_i)(a,b) \wedge F_i(a)\\ & \le & F_i(b).\end{array}$$

So we get

$$\bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i)(a) \wedge \bigwedge_{i\in I} F_i(a) \le \bigwedge_{i\in I} F_i(b).$$

This shows that $\bigwedge_{i\in I}\Omega_{\mathcal{A}}(F_i)$ is compatible with $\bigwedge_{i\in I}F_i$. Thus, by the maximality property of the Leibniz G-congruence, $\bigwedge_{i\in I}\Omega_{\mathcal{A}}(F_i) \le \Omega_{\mathcal{A}}(\bigwedge_{i\in I}F_i)$.

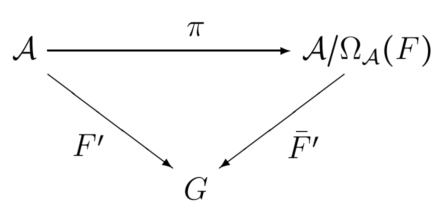
Assume, conversely, that $\Omega_{\mathcal{A}}$ is meet continuous and let $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F \le F'$. Then, we have

$$\Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F \wedge F') = \Omega_{\mathcal{A}}(F) \wedge \Omega_{\mathcal{A}}(F').$$

Thus, $\Omega_{\mathcal{A}}(F) \le \Omega_{\mathcal{A}}(F')$. We conclude that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and, hence, by Proposition 109, $\mathbb{S}$ is protoalgebraic. ∎

Note that the preceding results hold for all G-algebras such that they hold for all their subalgebras generated by sets of generators equipotent with the cardinality of the free algebra. This ensures the existence of surjective homomorphisms from the formula algebra onto those subalgebras. *In the sequel we assume that such homomorphisms exist.*

We dwell briefly on the *compatibility property*, since it gives protoalgebraic logics their distinctive flavor and, as Blok and Pigozzi noted, makes them amenable to the methods of Universal Algebra. Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra. Let $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F \le F'$. Then protoalgebraicity implies that $\Omega_{\mathcal{A}}(F) \le \Omega_{\mathcal{A}}(F')$. Hence, the G-congruence $\Omega_{\mathcal{A}}(F)$ is compatible with the $\mathbb{S}$-filter F'. That is, F' is constant on the equivalence classes of $\widehat{\Omega_{\mathcal{A}}(F)}$. Another way to express this is that, for the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$, we have that, for all $F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, with $F \le F'$, there exists an $\bar{F}' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, such that the following triangle commutes.



The correspondence $F' \mapsto \bar{F}'$ establishes a lattice isomorphism between the lattice of all $\mathbb{S}$-filters on $\mathcal{A}$ greater than F and the lattice of all $\mathbb{S}$-filters on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$ greater than $\bar{F}$.

The next proposition, an analog of Proposition 3.1 of [28], characterizes protoalgebraicity via the behavior of the Tarski operator. It shows that protoalgebraicity of the G-logic is tantamount to the Tarski G-congruence of any G-model of the logic having the same value as the Leibniz G-congruence of the least closed G-set of the G-model on the underlying G-algebra.

Proposition 112 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The following conditions are equivalent.*

(i) $\mathbb{S}$ is protoalgebraic;

(ii) For any G-model $\mathbb{L} = \langle \mathcal{A}, D\rangle$ of $\mathbb{S}$, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) = \Omega_{\mathcal{A}}(D(\bot))$;

(iii) *For any* $\mathbb{S}$-*matrix* $\mathbb{L} = \langle \mathcal{A}, F\rangle$, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$;

(iv) *For any* $X \in \mathrm{Th}(\mathbb{S})$, $\widetilde{\Omega}_{\mathcal{F}m_{\mathcal{L}}(V)}(\mathrm{Th}(\mathbb{S})^X) = \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$.

Proof:

(i)⇒(ii) Suppose $\mathbb{S}$ is protoalgebraic and let $\mathbb{L} = \langle \mathcal{A}, D\rangle$ be a G-model of $\mathbb{S}$. Then, by Proposition 87, $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Therefore, by Proposition 109, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Cl}(\mathbb{L})$. Now we get

$$\begin{array}{rcl} \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) & = & \bigwedge\{\Omega_{\mathcal{A}}(F) : F \in \mathrm{Cl}(\mathbb{L})\} \quad \text{(Before Proposition 62)} \\ & = & \Omega_{\mathcal{A}}(D(\bot)). \quad \text{(Monotonicity of } \Omega_{\mathcal{A}}) \end{array}$$

(ii)⇒(iii) Take $\mathbb{L} = \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ in Part (ii).

(iii)⇒(iv) Take $\mathcal{A} = \mathcal{F}m_{\mathcal{L}}(V)$ and $F = X$ in Part (iii).

(iv)⇒(i) Let $X, X' \in \mathrm{Th}(\mathbb{S})$, such that $X \leq X'$. Then $X' \in \mathrm{Th}(\mathbb{S})^X$. Hence,

$$\begin{array}{rcl} \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X) & = & \widetilde{\Omega}_{\mathcal{F}m_{\mathcal{L}}(V)}(\mathrm{Th}(\mathbb{S})^X) \quad \text{(Hypothesis)} \\ & \leq & \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X'). \quad (X' \in \mathrm{Th}(\mathbb{S})^X) \end{array}$$

So $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. By Proposition 109, $\mathbb{S}$ is protoalgebraic.

■

Proposition 101 asserted that, for any G-logic $\mathbb{S}$, the class $\mathrm{Alg}^*(\mathbb{S})$ is, in general, included in the class $\mathrm{Alg}(\mathbb{S})$ of $\mathbb{S}$-algebras. It is now shown that for protoalgebraic G-logics the two classes coincide. This is an analog of Proposition 3.2 of [28].

Proposition 113 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a protoalgebraic sentential G-logic. Then* $\mathrm{Alg}(\mathbb{S}) = \mathrm{Alg}^*(\mathbb{S})$.

Proof: By Proposition 101, $\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$. Suppose, conversely, that $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathrm{Alg}(\mathbb{S})$. Then, there exists $\mathbb{L} = \langle \mathcal{A}, D\rangle$, such that $\widetilde{\Omega}(\mathbb{L}) = E$. But then, if F is the least function in $\mathrm{Cl}(\mathbb{L})$, we get, by Proposition 112, $\Omega_{\mathcal{A}}(F) = \widetilde{\Omega}(\mathbb{L}) = E$ and, hence, $\mathcal{A} \in \mathrm{Alg}^*(\mathbb{S})$. ■

Further, protoalgebraicity of a G-logic $\mathbb{S}$ entails that a full G-model of the logic on a given G-algebra is fully determined by its least closed function, in the sense that any two full G-models on the same G-algebra which happen to have the same least closed functions are identical. See Lemma 3.3 of [28] for the corresponding result in the context of sentential logics.

Lemma 114 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a protoalgebraic sentential G-logic and* $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\mathbb{L}' = \langle \mathcal{A}, D'\rangle$ *be full G-models of* $\mathbb{S}$ *over the same G-algebra* $\mathcal{A} = \langle \mathbf{A}, E\rangle$. *If the least elements* F *and* F' *of* $\mathrm{Cl}(\mathbb{L})$ *and* $\mathrm{Cl}(\mathbb{L}')$, *respectively, coincide, then* $\mathbb{L} = \mathbb{L}'$.

Proof: By Proposition 112 and the hypothesis,

$$\widetilde{\Omega}(\mathbb{L}) = \Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F') = \widetilde{\Omega}(\mathbb{L}').$$

Hence, by Theorem 104, $\mathbb{L} = \mathbb{L}'$. ∎

We turn, now, to the question of when all full G-models of a G-logic $\mathbb{S}$ over a G-algebra $\mathcal{A}$ have the form $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$, for some $\mathbb{S}$-filter F on $\mathcal{A}$. The answer, expressed in an analog of Theorem 3.4 of [28], provides an additional view and characterization of protoalgebraicity in terms of the structure of full G-models.

Theorem 115 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is protoalgebraic iff, for any G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, all full G-models of $\mathbb{S}$ over $\mathcal{A}$ have the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$, for some $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.*

Proof: Suppose, first, that $\mathbb{S}$ is protoalgebraic. Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a full G-model of $\mathbb{S}$. Consider the least closed set $F \in \mathrm{Cl}(\mathbb{L})$. Clearly, $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$. We must show the reverse inclusion. By protoalgebraicity and Proposition 112, $\widetilde{\Omega}(\mathbb{L}) = \Omega_{\mathcal{A}}(F)$. Thus, the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \mathbb{L} \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), \mathrm{Cl}(\mathbb{L})/\Omega_{\mathcal{A}}(F)\rangle.$$

Since $\mathbb{L}$ is a full model, we get $\mathrm{Cl}(\mathbb{L})/\Omega_{\mathcal{A}}(F) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Now let $F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$. Since $\mathbb{S}$ is protoalgebraic, $\Omega_{\mathcal{A}}(F)$ is compatible with F'. Therefore, if $\pi' : \mathcal{A}/\Omega_{\mathcal{A}}(F) \to \mathcal{A}$ is a section of π, we get $F' = (F' \circ \pi') \circ \pi$. By the surjectivity of π and Proposition 79, $F' \circ \pi' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Thus, by the bilogical G-morphism property, $F' = (F' \circ \pi') \circ \pi \in \mathrm{Cl}(\mathbb{L})$. Therefore, $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$.

Assume, conversely, that every full G-model of $\mathbb{S}$ over $\mathcal{A}$ is of the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H\rangle$, for some $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Let $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F \leq F'$. By Corollary 101, $\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$. Therefore, $\Omega_{\mathcal{A}}(F) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. By the Isomorphism Theorem 104, there exists a full G-model $\mathbb{L}$ of $\mathbb{S}$, such that $\Omega_{\mathcal{A}}(F) = \widetilde{\Omega}(\mathbb{L})$. By fullness, the quotient morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical G-morphism

$$\pi : \mathbb{L} \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))\rangle.$$

Since $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, we get $F \circ \pi' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, whence, by the bilogical G-morphism property, $F = (F \circ \pi') \circ \pi \in \mathrm{Cl}(\mathbb{L})$. Now assume, according to the hypothesis, that $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H$. Then we get $H \leq F \leq F'$. So $F' \in \mathrm{Cl}(\mathbb{L})$, whence

$$\Omega_{\mathcal{A}}(F) = \widetilde{\Omega}(\mathbb{L}) \leq \Omega_{\mathcal{A}}(F').$$

Thus, $\Omega_{\mathcal{A}}$ is monotone and, hence, by Proposition 110, $\mathbb{S}$ is protoalgebraic. ∎

3.14 Leibniz G-Filters

The characterization of protoalgebraicity in terms of the form of full G-models in Theorem 115 raises another natural question. To single out, if possible, among all $\mathbb{S}$-filters F of a protoalgebraic G-logic $\mathbb{S}$ those for which $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$ is the set of closed functions of a full G-model.

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Given an algebra $\mathcal{A}$, define

$$\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A}) = \left\{F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) : \left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right\rangle \in \mathrm{FMod}(\mathbb{S})\right\}.$$

The $\mathbb{S}$-filters in $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$ are called **Leibniz $\mathbb{S}$-filters**.

Based on the Isomorphism Theorem 104 and the form of full G-models of a protoalgebraic G-logic, given by Theorem 115, we may establish another isomorphism, akin to the one established in Proposition 3.5 of [28], between Leibniz $\mathbb{S}$-filters on a G-algebra and $\mathrm{Alg}^*(\mathbb{S})$-congruences on the same G-algebra.

Theorem 116 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a protoalgebraic sentential G-logic. Then, for every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$ and $\mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A}) = \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$.*

Proof: For the proof we compose two isomorphisms.

$$\begin{array}{ccccc}
\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A}) & \longrightarrow & \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}) & \longrightarrow & \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}) \\
F & \longmapsto & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F & \longmapsto & \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right)
\end{array}$$

By the definition of $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$, the first one is well defined. It is one-to-one, and it is both order preserving and order reflecting. Finally, since $\mathbb{S}$ is protoalgebraic, by Theorem 115, it is surjective. Therefore, it is an order isomorphism.

By Theorem 104, $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ is isomorphic to $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ via the Tarski operator. Thus, $F \mapsto \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F)$ is an isomorphism from $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$ to $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. However, by Proposition 112, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$. Moreover, by Proposition 113, $\mathrm{Alg}(\mathbb{S}) = \mathrm{Alg}^*(\mathbb{S})$ and, therefore, $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}) = \mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A})$. Combining, we get the result. ■

The Leibniz $\mathbb{S}$-filters on a G-algebra $\mathcal{A}$ have a characterization that does not rely on the notion of full G-model. To formalize this characterization, we define, for every G-algebra $\mathcal{A}$, a binary relation $\sim_\Omega$ on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ by setting, for all $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$,

$$F \sim_\Omega F' \quad \text{iff} \quad \Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F').$$

That is, $\sim_\Omega$ is the kernel of the Leibniz operator acting on the $\mathbb{S}$-filters on the G-algebra $\mathcal{A}$. By Theorem 116, when $\mathbb{S}$ is protoalgebraic, at most one

$\mathbb{S}$-filter in each $\sim_\Omega$-equivalence class can belong to $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$. We obtain a characterization of this filter.

Suppose $\mathbb{S}$ is protoalgebraic. Let $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and denote by $[F]_\Omega$ its $\sim_\Omega$-equivalence class. Then $\bigwedge[F]_\Omega \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and, moreover,

$$\begin{aligned} \Omega_{\mathcal{A}}(\bigwedge[F]_\Omega) &= \bigwedge_{H\in[F]_\Omega}\Omega_{\mathcal{A}}(H) \quad \text{(Corollary 111)} \\ &= \Omega_{\mathcal{A}}(F). \quad \text{(Intersection over } [F]_\Omega) \end{aligned}$$

Thus, $\bigwedge[F]_\Omega \in [F]_\Omega$. This shows that $[F]_\Omega$ has a minimum element.

The following result characterizes Leibniz $\mathbb{S}$-filters on any G-algebra $\mathcal{A}$ as being the minimum elements in their own $\sim_\Omega$-equivalence class and, also, as those that induce the least $\mathbb{S}$-filter on the quotient G-algebra formed by their Leibniz G-congruence. It constitutes an analog of Proposition 3.6 of [28] for protoalgebraic G-logics,

Proposition 117 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a protoalgebraic sentential G-logic. Then, for all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, the following conditions are equivalent:*

(i) $F \in \mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$, *i.e.,* $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle \in \mathrm{FMod}(\mathbb{S})$;

(ii) *F is the minimum element in* $[F]_\Omega$;

(iii) *$F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$, where π' is a section of the quotient morphism* $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Proof:

(ii)⇒(iii) Suppose F is the minimum element in $[F]_\Omega$. Consider a filter $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Construct $F' = (H \circ \pi) \wedge F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. We have

$$\begin{aligned} F' &= (H\circ\pi) \wedge ((F\circ\pi')\circ\pi) \quad (\Omega_{\mathcal{A}}(F) \text{ compatible with } F) \\ &= (H \wedge (F\circ\pi'))\circ\pi. \end{aligned}$$

Thus, for all $a, b \in A$,

$$\begin{aligned} &\Omega_{\mathcal{A}}(F)(a,b) \wedge F'(a) \\ &\quad = \overline{\Omega_{\mathcal{A}}(F)}(a/\widehat{\Omega_{\mathcal{A}}(F)}, a/\widehat{\Omega_{\mathcal{A}}(F)}) \wedge (H \wedge (F\circ\pi'))(a/\widehat{\Omega_{\mathcal{A}}(F)}) \\ &\quad\quad \text{(Definition of } \overline{\Omega_{\mathcal{A}}(F)} \text{ and } F' = (H\wedge(F\circ\pi'))\circ\pi) \\ &\quad \leq (H\wedge(F\circ\pi'))(b/\widehat{\Omega_{\mathcal{A}}(F)}) \\ &\quad\quad (H \wedge (F\circ\pi') \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))) \\ &\quad = F'(b). \quad (F' = (H\wedge(F\circ\pi'))\circ\pi) \end{aligned}$$

Thus, $\Omega_{\mathcal{A}}(F)$ is compatible with F'. By the maximality property of the Leibniz G-congruence, $\Omega_{\mathcal{A}}(F) \leq \Omega_{\mathcal{A}}(F')$. However, since, by definition, $F' \leq F$ and $\mathbb{S}$ is protoalgebraic, we get $\Omega_{\mathcal{A}}(F') \leq \Omega_{\mathcal{A}}(F)$. Hence,

$\Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F')$. So $F \sim_\Omega F'$ and, by hypothesis, $F \leq F'$. We conclude that $F = F'$ and, hence, $F \leq H \circ \pi$. This yields

$$F \circ \pi' \leq (H \circ \pi) \circ \pi' = H.$$

Thus, $F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

(iii)⇒(i) Suppose $F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$. Taking into account that, by protoalgebraicity, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$, our goal is to prove that

$$\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \circ \pi' \cong \mathrm{Fi}_{\mathbb{S}}\left(\mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right)\right).$$

We have

$$\begin{array}{rcll} \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \circ \pi' & \cong & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{F \circ \pi'} & \text{(Protoalgebraicity)} \\ & = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F)) & \text{(Hypothesis)} \\ & = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F)). & \text{(Protoalgebraicity)} \end{array}$$

(i)⇒(ii) Suppose $F \in \mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$. By protoalgebraicity, its class $[F]_\Omega$ has a minimum element, say H. Applying the implications (ii)⇒(iii)⇒(i) for H, we get that $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H \rangle$ is a full G-model of $\mathbb{S}$. Thus, we have

$$\widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H\right) = \Omega_{\mathcal{A}}(H) = \Omega_{\mathcal{A}}(F) = \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right).$$

By the Isomorphism Theorem 104, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$, whence $F = H$. So F is the minimum element in its $\sim_\Omega$-class $[F]_\Omega$.

■

An important consequence for what follows is the fact that injectivity of the Leibniz operator on all $\mathbb{S}$-filters on a G-algebra is equivalent to the condition that every $\mathbb{S}$-filter on the G-algebra is actually a Leibniz $\mathbb{S}$-filter. This forms an analog of Proposition 3.7 of [28].

Proposition 118 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ *be a protoalgebraic sentential G-logic. Then, for all G-algebras* $\mathcal{A} = \langle \mathbf{A}, E \rangle$, $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ *iff the Leibniz operator* $\Omega_{\mathcal{A}}$ *is injective on* $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

Proof: We have $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ if and only if, by Proposition 117, for all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, F is the least element in $[F]_\Omega$ if and only if, for all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, $[F]_\Omega = \{F\}$ if and only if $\Omega_{\mathcal{A}}$ is injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. ■

3.15 Weakly Algebraizable G-Logics

We have seen that protoalgebraicity of a G-logic $\mathbb{S}$ is characterized by the monotonicity of the Leibniz operator on the theories of the G-logic and, also, by the monotonicity of the Leibniz operator on the $\mathbb{S}$-filters of the G-logic on any G-algebra. Another important class of logics in the traditional algebraic hierarchy is that of *weakly algebraizable logics*. It is obtained from the class of protoalgebraic logics if one insists that, apart from being monotone, the Leibniz operator also be injective. So we add, and study the effects of, injectivity of the Leibniz operator in order to introduce the class of weakly algebraizable G-logics.

Theorem 119, an analog of Theorem 3.8 of [28], characterizes those sentential G-logics $\mathbb{S}$ for which the Leibniz operator is both monotone and injective on the $\mathbb{S}$-filters of any G-algebra.

Theorem 119 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The following conditions are equivalent:*

(i) $\mathbb{S}$ is protoalgebraic and, for all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, $F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$, where π' is a section of the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$;

(ii) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ is monotone and injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$;

(iii) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the mapping $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ is a bijection (hence, a lattice isomorphism) from $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ onto $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$;

(iv) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\Omega_{\mathcal{A}} : \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \to \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ is a lattice isomorphism;

(v) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\Omega_{\mathcal{A}} : \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \to \mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A})$ is a lattice isomorphism.

Proof:

(i)⇔(ii) Since $\mathbb{S}$ is protoalgebraic, by Proposition 110, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Moreover, by Proposition 117, for every G-algebra $\mathcal{A}$, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) = \mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$. Hence, by Proposition 118, $\Omega_{\mathcal{A}}$ is injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(i)⇒(iii) The mapping $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ is injective. By Proposition 117, it is well defined and, by Theorem 115, it is surjective.

(iii)⇒(iv) Since $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ is surjective, by Theorem 115, $\mathbb{S}$ is protoalgebraic. Composing the postulated isomorphism with the one given by the Isomorphism Theorem 104, we get an isomorphism from $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ onto $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$, given by

$$F \mapsto \widetilde{\Omega}\left(\left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \right\rangle\right) \stackrel{\text{Prop. 112}}{=} \Omega_{\mathcal{A}}(F),$$

i.e., it is the Leibniz operator on $\mathcal{A}$.

(iv)⇒(v) By Corollary 101, the inclusion $\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$ holds in general and gives

$$\mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A}) \subseteq \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}).$$

On the other hand, by hypothesis, every congruence in $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ is of the form $\Omega_{\mathcal{A}}(F)$, for some $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. This yields the reverse of the displayed inclusion, whence we get the isomorphism of Part (v).

(v)⇒(ii) This is straightforward.

■

A sentential G-logic $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ is called **weakly algebraizable** if, for every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ is monotone and injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

Given a G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, we say that the Leibniz operator on $\mathcal{A}$ is **continuous** if, for every directed family $\{F_i : i \in I\}$ of $\mathbb{S}$-filters on $\mathcal{A}$, we have $\bigvee_{i\in I} F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and

$$\Omega_{\mathcal{A}}\left(\bigvee_{i\in I} F_i\right) = \bigvee_{i\in I} \Omega_{\mathcal{A}}(F_i).$$

If, in addition to $\Omega_{\mathcal{A}}$ being monotone and injective on every G-algebra $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is also continuous, then we call the sentential G-logic $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ **algebraizable**.

Given a G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, we say that the Tarski operator on $\mathcal{A}$ is **continuous** if, for every directed family $\{\mathbb{L}_i : i \in I\}$ of full G-models of $\mathbb{S}$ on $\mathcal{A}$, $\bigvee_{i\in I} \mathbb{L}_i$ is also a full G-model of $\mathbb{S}$ and

$$\widetilde{\Omega}_{\mathcal{A}}\left(\bigvee_{i\in I} \mathbb{L}_i\right) = \bigvee_{i\in I} \widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}_i).$$

Then we have the following partial analog of Theorem 3.10 of [28] in the context of weakly algebraizable sentential G-logics.

Theorem 120 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a weakly algebraizable sentential G-logic. Then the following conditions are equivalent:*

(i) $\mathbb{S}$ *is algebraizable;*

(ii) For every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\Omega_{\mathcal{A}}$ is continuous on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$*;*

(iii) For every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$*.*

Proof: By the definition of algebraizability and Theorem 119, Statements (i) and (ii) are equivalent. So it suffices to prove the equivalence of (ii) and (iii).

Suppose, first, that $F \in \mathrm{Fi}_{\$}(\mathcal{A})$. Define

$$\Phi_{\mathcal{A}}(F) = \left\langle \mathcal{A}, \mathrm{Fi}_{\$}(\mathcal{A})^F \right\rangle.$$

By protoalgebraicity and Proposition 112, $\Omega_{\mathcal{A}} = \widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}}$. By Theorem 119, $\Phi_{\mathcal{A}}$ is a bijection. Consequently, $\widetilde{\Omega}_{\mathcal{A}} = \Omega_{\mathcal{A}} \circ \Phi_{\mathcal{A}}^{-1}$. Suppose $\Omega_{\mathcal{A}}$ is continuous on $\mathrm{Fi}_{\$}(\mathcal{A})$. Let $\{\mathbb{L}_i : i \in I\} \subseteq \mathrm{FMod}_{\$}(\mathcal{A})$ be directed. Consider $F_i = \Phi_{\mathcal{A}}^{-1}(\mathbb{L}_i)$. Then $\{F_i : i \in I\} \subseteq \mathrm{Fi}_{\$}(\mathcal{A})$ is also directed, whence, by hypothesis, $\bigvee_{i \in I} F_i \in \mathrm{Fi}_{\$}(\mathcal{A})$. Thus,

$$\Phi_{\mathcal{A}}\left(\bigvee_{i \in I} F_i\right) = \left\langle \mathcal{A}, \mathrm{Fi}_{\$}(\mathcal{A})^{\bigvee F_i} \right\rangle \in \mathrm{FMod}_{\$}(\mathcal{A}).$$

Moreover,

$$\mathrm{Fi}_{\$}(\mathcal{A})^{\bigvee F_i} = \bigcap_{i \in I} \mathrm{Fi}_{\$}(\mathcal{A})^{F_i}.$$

So $\Phi_{\mathcal{A}}(\bigvee_{i \in I} F_i) = \bigvee_{i \in I} \mathbb{L}_i$. Now we get

$$\begin{array}{rcl}
\widetilde{\Omega}_{\mathcal{A}}(\bigvee_{i \in I} \mathbb{L}_i) & = & (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}})(\Phi_{\mathcal{A}}^{-1}(\bigvee_{i \in I} \mathbb{L}_i)) \quad (\Phi_{\mathcal{A}} \circ \Phi_{\mathcal{A}}^{-1} = Id) \\
& = & \Omega_{\mathcal{A}}(\bigvee_{i \in I} F_i) \quad (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} = \Omega_{\mathcal{A}}) \\
& = & \bigvee_{i \in I} \Omega_{\mathcal{A}}(F_i) \quad (\text{Hypothesis}) \\
& = & \bigvee_{i \in I} \widetilde{\Omega}_{\mathcal{A}}(\Phi_{\mathcal{A}}(F_i)) \quad (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} = \Omega_{\mathcal{A}}) \\
& = & \bigvee_{i \in I} \widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}_i). \quad (\Phi_{\mathcal{A}}(F_i) = \mathbb{L}_i)
\end{array}$$

This proves that $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\$}(\mathcal{A})$.

Suppose, conversely, that $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\$}(\mathcal{A})$. Consider a directed $\{F_i : i \in I\} \subseteq \mathrm{Fi}_{\$}(\mathcal{A})$. Clearly, $\{\Phi_{\mathcal{A}}(F_i) : i \in I\}$ is also directed. Moreover,

$$\begin{array}{rcl}
\Omega_{\mathcal{A}}(\bigvee_{i \in I} F_i) & = & \widetilde{\Omega}_{\mathcal{A}}(\Phi_{\mathcal{A}}(\bigvee_{i \in I} F_i)) \quad (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} = \Omega_{\mathcal{A}}) \\
& = & \widetilde{\Omega}_{\mathcal{A}}(\bigvee_{i \in I} \mathbb{L}_i) \quad (\Phi_{\mathcal{A}}(\bigvee_{i \in I} F_i) = \bigvee_{i \in I} \mathbb{L}_i) \\
& = & \bigvee_{i \in I} \widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}_i) \quad (\text{Hypothesis}) \\
& = & \bigvee_{i \in I} \Omega_{\mathcal{A}}(\Phi_{\mathcal{A}}^{-1}(\mathbb{L}_i)) \quad (\widetilde{\Omega}_{\mathcal{A}} = \Omega_{\mathcal{A}} \circ \Phi_{\mathcal{A}}^{-1}) \\
& = & \bigvee_{i \in I} \Omega_{\mathcal{A}}(F_i). \quad (\Phi_{\mathcal{A}}^{-1}(\mathbb{L}_i) = F_i)
\end{array}$$

Therefore, $\Omega_{\mathcal{A}}$ is continuous on $\mathrm{Fi}_{\$}(\mathcal{A})$. ∎

Corollary 121 *Let $\$ = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic. Then $\$$ is algebraizable if and only if, for every G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$, the mapping*

$$F \mapsto \left\langle \mathcal{A}, \mathrm{Fi}_{\$}(\mathcal{A})^F \right\rangle$$

is a bijection between $\mathrm{Fi}_{\$}(\mathcal{A})$ and $\mathrm{FMod}_{\$}(\mathcal{A})$ and the Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\$}(\mathcal{A})$.

Proof: By definition, $\mathcal{S}$ is algebraizable if and only if the Leibniz operator is monotone, injective and continuous on $\mathrm{Fi}_{\mathcal{S}}(\mathcal{A})$ if and only if, by Theorems 119 and 120, the mapping $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathcal{S}}(\mathcal{A})^F \rangle$ is a bijection between $\mathrm{Fi}_{\mathcal{S}}(\mathcal{A})$ and $\mathrm{FMod}_{\mathcal{S}}(\mathcal{A})$ and the Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\mathcal{S}}(\mathcal{A})$. ■

Chapter 4

Graded Classes of Models

4.1 Introduction

In work done during the last quarter of the twentieth century by, among others, Czelakowski [13, 14], Blok and Pigozzi [5, 6] and Font and Jansana [28] (see, also, the monograph [15], the survey [29] and the textbook [27]) a truly abstract framework was developed for the algebraization of arbitrary sentential logics. Depending on the strength of the ties this process establishes between a logic and its associated class of algebras, logics are classified in the steps of a hierarchy, known as the algebraic or Leibniz hierarchy.

In particular, in [6] (see, also, [8] and [7]) the authors pointed out the special role that first order logic without equality plays in formalizing a deductive system. Following this line of thought in earnest, two doctoral dissertations in Barcelona in the mid '90s went quite a long way in pursuing this point of view and in clarifying the scope of this approach. The first was Elgueta's Dissertation [22] and accompanying work [23, 24, 25] and [26] and the second was Dellunde's Dissertation [17] and accompanying work [18, 19] and [20].

Here, we are concerned especially with part of the work of Elgueta (which partially overlaps with other works in the preceding paragraph, e.g., [20]) presented in [23], culminating in the characterization of classes of models of first order structures defined without equality. In Section 5, titled "Main Theorems", Elgueta presents characterizations of classes of first order models without equality, including elementary classes (Subsection 5.1), universal classes (Subsection 5.2), universal Horn classes (Subsection 5.3) and universal atomic classes (Subsection 5.4). In each case several characterizations are provided based on operators on classes of models that preserve those classes. Moreover, Elgueta characterizes the corresponding reduced classes, i.e., those resulting by applying to models the operation of reduction modulo Leibniz equality (which is first order definable without equality and plays the role of "equality" in this equality free setting).

Following Elgueta [23], we make an attempt at developing the fundamentals of an extended framework in which interpretations of first order formulas (without equality) are multi-valued. However, we are faced with rather limited success. Many wished-for analogs of the more difficult results of [23] remain elusive. It is not clear yet whether this is due to inherent difficulties, or whether modifications in the framework are needed, or the failures are because different techniques are needed. So the work in this chapter cannot but be viewed as preliminary and exploratory in nature.

Let us give, nevertheless, a brief overview of what is accomplished and point out some key elements that are lacking and need to be modified and/or addressed differently.

In Section 4.2, basic notation and terminology are introduced and the values of a Boolean algebra (or, perhaps, more generally, a structure with the appropriate operations) are used to interpret formulas of a first order language without equality. Section 4.3 deals with *substructures*, *filter ex-*

tensions, *elementary substructures* and *elementarily equivalent structures*. Section 4.4 deals with morphisms of structures. We present morphisms, epimorphisms, embeddings and the corresponding strict versions, which are the ones that, roughly speaking, preserve and reflect the values of the interpreted symbols. We also define image structures and pre-image structures and what it means for a morphism to be elementary.

Section 4.5 is the one that introduces products.

In Section 4.6, we introduce graded congruences, or G-congruences, of structures. Graded congruences are G-congruences on the underlying G-algebra which are compatible with all relations of the structure. We use compatibility to define the Leibniz G-congruence of a structure. We prove some properties essential for applying the theory of Leibniz G-congruences. Key among those are the way they interact with morphisms and, in particular, the fact that they commute with inverse strict epimorphisms. Section 4.7 is dedicated into proving an analog of the well known result of Blok and Pigozzi characterizing Leibniz G-congruences.

Section 4.8 is dedicated entirely to studying quotients of structures and special morphisms relating them. Given a G-congruence Θ of a structure $\mathfrak{A}$, we define the quotient structure $\mathfrak{A}/\Theta$ of $\mathfrak{A}$ by Θ. The definition makes the canonical projection $\pi_\Theta : A \to A/\Theta$ a reductive morphism, i.e., a strict epimorphism, between the corresponding structures. Working along the lines of the classical universal algebraic results, we are able to obtain a sequence of analogs of the Homomorphism Theorems. More precisely, we prove analogs of the Homomorphism, the Second and Third Isomorphism and of the Correspondence Theorems.

A natural quotient to consider, as it is the one that reduces the structures "to the largest extent possible", is the quotient by the largest G-congruence of the structure, i.e., its Leibniz G-congruence. For this reason this quotient is termed the Leibniz quotient. It is considered in Section 4.9. There it is shown that every reductive morphism between two structures induces an isomorphism between the corresponding Leibniz quotients.

Section 4.10 quickly readjusts the notions of models and of semantic consequence to the current framework. A novelty here, already encompassed in [23], is that, apart from ordinary model classes, one may use reduced model classes, which are classes of models obtained by applying the Leibniz reduction operation to each model of a certain class.

Class operators on structures are introduced and studied in Section 4.11. Let us summarize these operators. We consider the operator $\mathbb{S}$ of taking substructures, $\mathbb{F}$ of taking filter extensions, $\mathbb{H}$ of taking morphic images, $\mathbb{R}$ of taking reductions and $\mathbb{E}$ of taking extensions (preimages under reductive morphisms), and $\mathbb{P}$ of taking products. The first result of the section asserts that all these operators are idempotent. Subsequent results examine the way any two different of these operators compose with one another.

Section 4.12 adapts the Diagram Lemma of Model Theory to the multi-valued framework. The result is an extension of the version established by Elgueta (Theorem 4.5 of [23]). Section 4.13 revisits and establishes an analog of the Reduction Operator Lemma (Theorem 4.7 of [23]). The success in extending those two key lemmas gave (false?) hope that many of the results of the main section, Section 5 on "The Main Theorems" of [23], would be adaptable to the multi-valued context. The main theorems include characterizations of (from narrower to wider) universal atomic classes (Subsection 5.4 of [23]), universal Horn classes (Subsection 5.3 ibid.), universal classes (Subsection 5.2 ibid.) and elementary classes (Subsection 5.1 ibid.). However, we were only able to characterize universal atomic classes in Section 4.14. It is not clear yet whether the problems encountered in characterizing the other classes in a similar way are inherent, whether they need some tweaking of the framework or whether they require new methodology. For now these characterizations are left open for future work. So this work (and this report) should be considered as preliminary and leaves many questions open for further investigation.

4.2 Notation and Terminology

Let $\mathbf{G} = \langle G, \leq \rangle$ be a fixed complete Boolean algebra, which may be assumed to have additional structure, as needed. Recall that, given a set X, the set of all functions G^X is ordered pointwise, i.e., by setting, for all $f, g : X \to G$,

$$f \leq g \quad \text{iff, for all } x \in X,\ f(x) \leq g(x).$$

We consider a first-order language $\mathcal{L} = \langle F, R, \rho \rangle$, where F is a set of function symbols, R is a set of relation symbols and ρ is the arity function from $F \cup R$ into the natural numbers. The formulas of $\mathcal{L}$ are the usual first-order formulas. That is, the set of **$\mathcal{L}$-terms** $\mathrm{Tm}_{\mathcal{L}}(V)$ is constructed using the function symbols in F starting from a countably infinite collection V of individual variables. Moreover, the set of **$\mathcal{L}$-formulas** $\mathrm{Fm}_{\mathcal{L}}(V)$ is constructed using the usual logical connectives (say, $\neg$, $\wedge$ and $\vee$ and the quantifiers $\forall$ and $\exists$) starting from atomic formulas of the form $r(t_1, \ldots, t_n)$, where $r \in R$, with $\rho(r) = n$, and $t_1, \ldots, t_n$ are arbitrary $\mathcal{L}$-terms.

To introduce the notion of $\mathcal{L}$-structure, let us revisit, first, the notions of G-equivalence and G-congruence and, also, of reduced G-congruence.

Let A be a set. A **G-equivalence Θ on** A is a function $\Theta : A^2 \to G$, such that:

(**Reflexivity**) $\Theta(a, a) = \top$, for all $a \in A$;

(**Symmetry**) $\Theta(b, a) = \Theta(a, b)$, for all $a, b \in A$;

(**Transitivity**) $\Theta(a, b) \wedge \Theta(b, c) \leq \Theta(a, c)$, for all $a, b, c \in A$.

Θ is said to be **reduced** if, for all $a, b \in A$,

$$\Theta(a,b) = \top \quad \text{iff} \quad a = b.$$

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be an F-algebra. An operation $f^{\mathbf{A}} \in F^{\mathbf{A}}$, with arity $\rho(f) = n$, is said to be **compatible with** Θ if, for all $a_1, b_1, \dots, a_n, b_n \in A$,

$$\bigwedge_{i=1}^{n} \Theta(a_i, b_i) \le \Theta(\lambda^{\mathbf{A}}(a_1, \dots, a_n), \lambda^{\mathbf{A}}(b_1, \dots, b_n)).$$

If all operations in $F^{\mathbf{A}}$ are compatible with Θ, then Θ is called a G-**congruence of A**.

Let R be a set of relation symbols with arity function $\rho : R \to \omega$. A G-**relation** $r^A \in R^A$, with $\rho(r) = n$, is a function $r^A : A^n \to G$. It is said to be **compatible with** Θ if, for all $a_1, b_1, \dots, a_n, b_n \in A$,

$$\bigwedge_{i=1}^{n} \Theta(a_i, b_i) \wedge r^A(a_1, \dots, a_n) \le r^A(b_1, \dots, b_n).$$

If all relations in $R^{\mathbf{A}}$ are compatible with Θ, then Θ is called a G-**congruence of** $\langle A, R^A \rangle$.

A **structure over** $\mathcal{L}$ or $\mathcal{L}$-**structure** $\mathfrak{A} = \langle A, E, F^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ consists of:

- A set A, the **universe** of $\mathfrak{A}$;
- A reduced G-equivalence E on A;
- The **interpretations** $F^{\mathfrak{A}}$ of the function symbols as functions $f^{\mathfrak{A}} : A^n \to A$, where $f \in F$, with $\rho(f) = n$, all of which are compatible with E, i.e., such that E is a reduced G-congruence on $\mathbf{A} = \langle A, F^{\mathfrak{A}} \rangle$;
- The **interpretations** $R^{\mathfrak{A}}$ of the relation symbols as G-relations $r^{\mathfrak{A}} : A^n \to G$, where $r \in R$, with $\rho(r) = n$, all of which are compatible with E, i.e., such that E is a reduced G-congruence of $\langle A, R^{\mathfrak{A}} \rangle$.

In general, capital Gothic letters $\mathfrak{A}, \mathfrak{B}, \dots$ represent structures over $\mathcal{L}$. The corresponding boldface letter $\mathbf{A}$ is used to denote the underlying algebra $\mathbf{A} = \langle A, F^{\mathfrak{A}} \rangle$ of $\mathfrak{A}$, and we sometimes write $f^{\mathbf{A}}$ instead of $f^{\mathfrak{A}}$. The corresponding calligraphic letter $\mathcal{A}$ is used to denote the underlying G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$ of $\mathfrak{A}$. Lowercase boldface letters $\mathbf{a}, \mathbf{b}, \dots$ are used to indicate members of the cartesian product of some family of sets. So, if $\mathfrak{A}$ is an $\mathcal{L}$-structure, $\mathbf{a} = \langle a_1, \dots, a_n \rangle$ belongs to A^n, $f \in F$, with $\rho(f) = n$, $r \in R$, with $\rho(r) = n$, and h is any mapping with domain A, then $f^{\mathbf{A}}(\mathbf{a})$, $r^{\mathfrak{A}}(\mathbf{a})$ and $h(\mathbf{a})$ are short-hand for $f^{\mathbf{A}}(a_1, \dots, a_n)$, $r^{\mathfrak{A}}(a_1, \dots, a_n)$ and $\langle h(a_1), \dots, h(a_n) \rangle$, respectively.

By an $\mathcal{L}$-**algebra** we mean the underlying algebra of any $\mathcal{L}$-structure. By a G-**algebra** we mean the underlying G-algebra of any $\mathcal{L}$-structure. If the set of function symbols is empty, an $\mathcal{L}$-algebra simply means an arbitrary set and a G-algebra means a set with a reduced G-equivalence on it. The absolutely free $\mathcal{L}$-algebra over the set V of variables is the algebra of all $\mathcal{L}$-terms over V. It is denoted by $\mathbf{Tm}_{\mathcal{L}}(V)$. We also write $\mathcal{T}m_{\mathcal{L}}(V) = \langle \mathbf{Tm}_{\mathcal{L}}(V), \Delta_{\mathrm{Tm}_{\mathcal{L}}(V)}\rangle$ for the G-algebra of terms, where the reduced G-congruence is simply the identity.

Formulas are represented by lowercase Greek letters $\varphi, \psi, \ldots$, and uppercase ones are used to denote sets of formulas. We write $\varphi(x_1, \ldots, x_n)$ to indicate that the free variables that occur in φ are among $x_1, \ldots, x_n$.

Now we define the truth value $\varphi^{\mathfrak{A}}[h]$ of an $\mathcal{L}$-formula φ in a structure $\mathfrak{A}$ under an assignment $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$.

For atomic $\varphi = r(t_1, \ldots, t_n)$, we get the value

$$\varphi^{\mathfrak{A}}[h] \coloneqq r^{\mathfrak{A}}(h(t_1), \ldots, h(t_n)).$$

Assume, inductively, that the values $\varphi^{\mathfrak{A}}[h]$, $\varphi_1^{\mathfrak{A}}[g]$ and $\varphi_2^{\mathfrak{A}}[h]$ have been computed. Then

$$\begin{array}{rcl}(\neg\varphi)^{\mathfrak{A}}[h] & = & \neg\varphi^{\mathfrak{A}}[h];\\ (\varphi_1 \wedge \varphi_2)^{\mathfrak{A}}[h] & = & \varphi_1^{\mathfrak{A}}[h] \wedge \varphi_2^{\mathfrak{A}}[h];\\ (\varphi_1 \vee \varphi_2)^{\mathfrak{A}}[h] & = & \varphi_1^{\mathfrak{A}}[h] \vee \varphi_2^{\mathfrak{A}}[h].\end{array}$$

Finally, suppose, inductively, that the values $\varphi(x)^{\mathfrak{A}}[h]$, for all h, have been computed. Denote, as usual, by $h[a/x]$ or $h(a/x)$ the assignment that maps x to a and acts as h on the remaining variables. Then

$$\begin{array}{rcl}(\forall x\varphi)^{\mathfrak{A}}[h] & = & \bigwedge_{a\in A}\varphi^{\mathfrak{A}}[h(a/x)];\\ (\exists x\varphi)^{\mathfrak{A}}[h] & = & \bigvee_{a\in A}\varphi^{\mathfrak{A}}[h(a/x)].\end{array}$$

It is clear that, in this context, the De Morgan Laws hold, that is, for a structure $\mathfrak{A}$ under an assignment $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$, we have

$$\begin{array}{rcl}[\neg(\varphi_1 \wedge \varphi_2)]^{\mathfrak{A}}[h] & = & (\neg\varphi_1 \vee \neg\varphi_2)^{\mathfrak{A}}[h],\\ [\neg(\varphi_1 \vee \varphi_2)]^{\mathfrak{A}}[h] & = & (\neg\varphi_1 \wedge \neg\varphi_2)^{\mathfrak{A}}[h],\\ (\neg\forall x\varphi)^{\mathfrak{A}}[h] & = & (\exists x\neg\varphi)^{\mathfrak{A}}[h],\\ (\neg\exists x\varphi)^{\mathfrak{A}}[h] & = & (\forall x\neg\varphi)^{\mathfrak{A}}[h].\end{array}$$

This implies that, in a context where a proof by structural induction on a formula is to be carried out, once the case of negation has been dealt with, then only one of the cases of conjunction or disjunction and of universal or existential quantifications have to be undertaken.

We show that the compatibility of the functions and of the G-relations of a structure with its reduced G-congruence relation extends to the compatibility of all its term functions and all its formula defined G-relations with the reduced G-congruence.

Proposition 122 *Let $\mathfrak{A} = \langle A, E, F^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-structure. Then, for all $\mathcal{L}$-terms $t(\bar{x})$ and all $\mathcal{L}$-formulas $\varphi(\bar{x})$, we have, for all $a_1, b_1, \ldots, a_n, b_n \in A$,* *(a)* $\bigwedge_{i=1}^n E(a_i,b_i) \le E(t^{\mathbf{A}}(\mathbf{a}),t^{\mathbf{A}}(\mathbf{b}))$;

(b) $\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \le \varphi^{\mathfrak{A}}(\mathbf{b})$.

Proof:

(a) We work by induction on the structure of an $\mathcal{L}$-term. Suppose, for the base case, that $t = x_j$ is a variable. Then, we get

$$\begin{array}{lll} \bigwedge_{i=1}^n E(a_i,b_i) & \le & E(a_j,b_j) \quad \text{(Property of } \wedge) \\ & = & E(x_j^{\mathbf{A}}(\mathbf{a}),x_j^{\mathbf{A}}(\mathbf{b})). \quad \text{(Definition of } x_j^{\mathbf{A}}) \end{array}$$

Assume, for the induction hypothesis, that, for the $\mathcal{L}$-terms $t_1(\bar{x}),\ldots,$ $t_m(\bar{x})$, we have

$$\bigwedge_{i=1}^n E(a_i,b_i) \le E(t_j^{\mathbf{A}}(\mathbf{a}),t_j^{\mathbf{A}}(\mathbf{b})).$$

Consider the term $t(\bar{x}) = f(t_1,\ldots,t_m)$, where $f \in F$, with $\rho(f) = m$. Then, we get

$$\begin{array}{lll} \bigwedge_{i=1}^n E(a_i,b_i) & \le & \bigwedge_{j=1}^m E(t_j^{\mathbf{A}}(\mathbf{a}),t_j^{\mathbf{A}}(\mathbf{b})) \\ & & \quad \text{(Induction Hypothesis and Property of } \wedge) \\ & \le & E(f^{\mathbf{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\ldots,t_m^{\mathbf{A}}(\mathbf{a})),f^{\mathbf{A}}(t_1^{\mathbf{A}}(\mathbf{b}),\ldots,t_m^{\mathbf{A}}(\mathbf{b}))) \\ & & \quad \text{(Compatibility of } f^{\mathbf{A}} \text{ with } E) \\ & = & E(t^{\mathbf{A}}(\mathbf{a}),t^{\mathbf{A}}(\mathbf{b})). \quad \text{(Definition of } t) \end{array}$$

So, for all terms $t(\bar{x})$, $\bigwedge_{i=1}^n E(a_i,b_i) \le E(t^{\mathbf{A}}(\mathbf{a}),t^{\mathbf{A}}(\mathbf{b}))$.

(b) We work by induction on the structure of $\mathcal{L}$-formulas. For the base case, let $r(t_1,\ldots,t_m)$ be atomic, with $t_j = t_j(\bar{x})$, for $j = 1,\ldots,m$. Then, we get

$$\begin{array}{l} \bigwedge_{i=1}^n E(a_i,b_i) \wedge r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\ldots,t_m^{\mathbf{A}}(\mathbf{a})) \\ \quad \le \bigwedge_{j=1}^m E(t_j^{\mathbf{A}}(\mathbf{a}),t_j^{\mathbf{A}}(\mathbf{b})) \wedge r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\ldots,t_m^{\mathbf{A}}(\mathbf{a})) \\ \qquad \text{(Part (a) and Property of } \wedge) \\ \quad \le r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{b}),\ldots,t_m^{\mathbf{A}}(\mathbf{b})). \\ \qquad \text{(Compatibility of } r^{\mathfrak{A}} \text{ with } E) \end{array}$$

For negation, consider the formula $\varphi(\bar{x}) = \neg\psi(\bar{x})$. Assume inductively, that, for all $\mathbf{a} = \langle a_1,\ldots,a_n\rangle$ and $\mathbf{b} = \langle b_1,\ldots,b_n\rangle$,

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge \psi^{\mathfrak{A}}(\mathbf{a}) \le \psi^{\mathfrak{A}}(\mathbf{b}).$$

This is equivalent to $\bigwedge_{i=1}^n E(a_i,b_i) \le \psi^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow \psi^{\mathfrak{A}}(\mathbf{b})$. Thus, we get

$$\bigwedge_{i=1}^n E(a_i,b_i) \le \neg\psi^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow \neg\psi^{\mathfrak{A}}(\mathbf{b}),$$

i.e., $\bigwedge_{i=1}^n E(a_i,b_i) \leq \varphi^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow \varphi^{\mathfrak{A}}(\mathbf{b})$. In particular, we obtain

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \leq \varphi^{\mathfrak{A}}(\mathbf{b}).$$

We deal next with conjunction. Suppose $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$ and, inductively, that

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi_1^{\mathfrak{A}}(\mathbf{a}) \leq \varphi_1^{\mathfrak{A}}(\mathbf{b}) \quad \text{and} \quad \bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi_2^{\mathfrak{A}}(\mathbf{a}) \leq \varphi_2^{\mathfrak{A}}(\mathbf{b}).$$

Then we get

$$\begin{array}{l} \bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \\ \qquad = \bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi_1^{\mathfrak{A}}(\mathbf{a}) \wedge \varphi_2^{\mathfrak{A}}(\mathbf{a}) \\ \qquad\quad \text{(Definition of } \varphi) \\ \qquad \leq \varphi_1^{\mathfrak{A}}(\mathbf{b}) \wedge \varphi_2^{\mathfrak{A}}(\mathbf{b}) \\ \qquad\quad \text{(Induction Hypothesis and Property of } \wedge) \\ \qquad \leq \varphi^{\mathfrak{A}}(\mathbf{b}). \quad \text{(Definition of } \varphi) \end{array}$$

Finally, for universal quantification, assume $\varphi(\bar{x}) = \forall y \psi(\bar{x},y)$ and that, inductively,

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge E(a,b) \wedge \psi^{\mathfrak{A}}(\mathbf{a},a) \leq \psi^{\mathfrak{A}}(\mathbf{b},b).$$

Then we get

$$\begin{array}{rcl} \bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) & = & \bigwedge_{i=1}^n E(a_i,b_i) \wedge \bigwedge_{a\in A} \psi^{\mathfrak{A}}(\mathbf{a},a) \\ & & \quad \text{(Definition of } \varphi^{\mathfrak{A}}(\mathbf{a})) \\ & = & \bigwedge_{a\in A}(\bigwedge_{i=1}^n E(a_i,b_i) \wedge \psi^{\mathfrak{A}}(\mathbf{a},a)) \\ & & \quad \text{(Property of } \wedge) \\ & \leq & \bigwedge_{a\in A} \psi^{\mathfrak{A}}(\mathbf{b},a) \\ & & \quad \text{(Induction Hypothesis)} \\ & = & \varphi^{\mathfrak{A}}(\mathbf{b}). \quad \text{(Definition of } \varphi^{\mathfrak{A}}(\mathbf{b})) \end{array}$$

So, for every $\mathcal{L}$-formula φ, $\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \leq \varphi^{\mathfrak{A}}(\mathbf{b})$.

■

Let $X \subseteq G$. Then we write $\mathfrak{A} \vDash_X \varphi[h]$ to signify that $\varphi^{\mathfrak{A}}[h] \in X$. Further, $\mathfrak{A} \vDash_X \varphi$ expresses $\mathfrak{A} \vDash_X \forall\varphi$, where $\forall\varphi$ denotes the universal closure of φ. When we write

$$\mathfrak{A} \vDash_X \varphi(x_1,\ldots,x_k)[a_1,\ldots,a_k],$$

we mean $\mathfrak{A} \vDash \varphi[h]$, with respect to any assignment h, such that $h(x_i) = a_i$, for all $1 \leq i \leq k$.

We finish the section with a remark concerning **languages with** and **languages without G-equality**. We say $\mathcal{L}$ is a **language with G-equality** if R contains a binary relation symbol e, such that, for all structures $\mathfrak{A}$ and all $a, b \in A$,

$$e^{\mathfrak{A}}(a,b) = E(a,b).$$

$\mathcal{L}$ is **without G-equality** if such a relation symbol (forced to be interpreted in such a way in all structures) is not present in R.

4.3 Substructures and Elementarity

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be two $\mathcal{L}$-structures. We say that $\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if $\mathcal{A} = \langle \mathbf{A}, E^{\mathfrak{A}} \rangle$ is a subalgebra of $\mathcal{B} = \langle \mathbf{B}, E^{\mathfrak{B}} \rangle$ ($\mathcal{A} \leq \mathcal{B}$) and, for all $r \in R$,

$$r^{\mathfrak{A}} = r^{\mathfrak{B}} \restriction_{A^{\rho(r)}}.$$

The $\mathcal{L}$-structure $\mathfrak{B}$ is called a **filter extension of** $\mathfrak{A}$, written $\mathfrak{A} \precsim \mathfrak{B}$, if $\mathcal{A} = \mathcal{B}$ and, for all $r \in R$,

$$r^{\mathfrak{A}} \leq r^{\mathfrak{B}},$$

that is, for all $\mathbf{a} \in A^{\rho(r)}$, $r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(\mathbf{a})$.

Suppose that $\mathfrak{A} = \langle \mathbf{A}, R^{\mathfrak{A}} \rangle$ is an $\mathcal{L}$-structure. Let X be a subset of A and denote by $[X]$ the universe of the subalgebra of $\mathbf{A}$ generated by X. Then the **substructure of $\mathfrak{A}$ generated by** X, denoted $\mathfrak{A}\restriction_X$, is the substructure of $\mathfrak{A}$, with universe $[X]$, i.e.,

$$\mathfrak{A}\restriction_X = \left\langle [X], E^{\mathfrak{A}} \restriction_{[X]}, \left\{ f^{\mathfrak{A}}\restriction_{[X]} : f \in F \right\}, \left\{ r^{\mathfrak{A}}\restriction_{[X]} : r \in R \right\} \right\rangle.$$

The $\mathcal{L}$-structure $\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$, denoted $\mathfrak{A} \subseteq_e \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and, in addition, for every formula φ and any assignment $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\varphi^{\mathfrak{A}}[h] = \varphi^{\mathfrak{B}}[h].$$

More generally, we call two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ **elementarily equivalent** and write $\mathfrak{A} \equiv \mathfrak{B}$ if, for every sentence φ over $\mathcal{L}$,

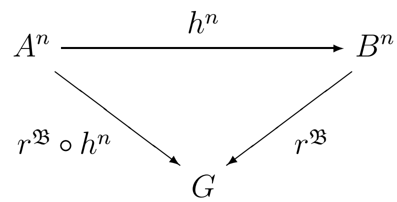
$$\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}.$$

Obviously, if $\mathfrak{A} \subseteq_e \mathfrak{B}$, we also have $\mathfrak{A} \equiv \mathfrak{B}$.

4.4 Morphisms

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be two $\mathcal{L}$-structures. A mapping $h : A \to B$ is called a **morphism** from $\mathfrak{A}$ to $\mathfrak{B}$, written $h : \mathfrak{A} \to \mathfrak{B}$, if it is a

G-algebra homomorphism (G-morphism) $h : \mathcal{A} \to \mathcal{B}$ and, in addition, for all $r \in R$, with $\rho(r) = n$, and all $\mathbf{a} \in A^n$,



$$r^{\mathfrak{A}}(\mathbf{a}) \le r^{\mathfrak{B}}(h(\mathbf{a})).$$

A G-morphism $h : \langle \mathbf{A}, E^{\mathfrak{A}} \rangle \to \langle \mathbf{B}, E^{\mathfrak{B}} \rangle$ is a G-**embedding**, written $h : \mathcal{A} \rightarrowtail \mathcal{B}$, if, for all $a_1, a_2 \in A$, $E^{\mathfrak{A}}(a_1, a_2) = E^{\mathfrak{B}}(h(a), h(b))$. Note that, since both $E^{\mathfrak{A}}$ and $E^{\mathfrak{B}}$ are reduced, this implies that $h : \mathbf{A} \rightarrowtail \mathbf{B}$ is an $\mathcal{L}$-algebra embedding. A morphism $h : \mathfrak{A} \to \mathfrak{B}$ is an **embedding** (of $\mathcal{L}$-structures) if it is a G-embedding. In this case we write $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$. It is an **epimorphism** if it is an epimorphism between the corresponding $\mathcal{L}$-algebras. This is written $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ and we say that $\mathfrak{B}$ is a **morphic image** of $\mathfrak{A}$. A morphism $h : \mathfrak{A} \to \mathfrak{B}$ is an **isomorphism**, written $h : \mathfrak{A} \cong \mathfrak{B}$, if it is bijective and $h^{-1} : \mathfrak{B} \to \mathfrak{A}$ is also a morphism.

A mapping $h : A \to B$ is called a **strong** or **strict morphism** from $\mathfrak{A}$ to $\mathfrak{B}$, written $h : \mathfrak{A} \to_s \mathfrak{B}$, if $h : \mathfrak{A} \to \mathfrak{B}$ and, moreover, the reverse inequality of the one displayed above, for all $r \in R$, with $\rho(r) = n$, and all $\mathbf{a} \in A^n$, also holds, that is, we have

$$r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(h(\mathbf{a})).$$

A **strict embedding**, written $h : \mathfrak{A} \rightarrowtail_s \mathfrak{B}$ is a strict morphism that is also an embedding. A **strict epimorphism**, more often called a **reductive morphism**, written $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, is a strict morphism that is an epimorphism. In case there exists a reductive morphism $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, we say that $\mathfrak{B}$ is a **reduction** of $\mathfrak{A}$ and $\mathfrak{A}$ is an **expansion** of $\mathfrak{B}$.

Based on the following lemma, one can show that a reductive morphism, which is also an embedding, is an isomorphism and, moreover, if the language has G-equality, then a reductive morphism is an isomorphism.

Lemma 123 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathcal{A} \to \mathcal{B}$ a G-morphism.*

(i) $h : \mathfrak{A} \to \mathfrak{B}$ iff, for all $r \in R$, and all $\mathbf{b} \in B^{\rho(r)}$,

$$\bigvee_{\mathbf{a} \in h^{-1}(\mathbf{b})} r^{\mathfrak{A}}(\mathbf{a}) \le r^{\mathfrak{B}}(\mathbf{b}).$$

(ii) $h : \mathfrak{A} \to_s \mathfrak{B}$ iff, for all $r \in R$, and all $\mathbf{b} \in h(A)^{\rho(r)}$,

$$r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(\mathbf{b}), \text{ for all } \mathbf{a} \in h^{-1}(\mathbf{b}).$$

(iii) $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *implies, for all* $r \in R$,

- $r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(h(\mathbf{a}))$, *for all* $\mathbf{a} \in A^{\rho(r)}$;
- $r^{\mathfrak{B}}(\mathbf{b}) = r^{\mathfrak{A}}(\mathbf{a})$, *for all* $\mathbf{b} \in B^{\rho(r)}$ *and* $\mathbf{a} \in h^{-1}(\mathbf{b})$.

Proof:

(i) For the implication left to right, consider $\mathbf{b} \in B^{\rho(r)}$. Let $\mathbf{a} \in h^{-1}(\mathbf{b})$. By definition, $h(\mathbf{a}) = \mathbf{b}$. By hypothesis, $r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(h(\mathbf{a})) = r^{\mathfrak{B}}(\mathbf{b})$. Hence, since $\mathbf{a} \in h^{-1}(\mathbf{b})$ was arbitrary, $\bigvee_{\mathbf{a}\in h^{-1}(\mathbf{b})} r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(\mathbf{b})$.

For the converse, assume $\mathbf{a} \in A^{\rho(r)}$ and let $\mathbf{b} = h(\mathbf{a})$. Then we have, using the hypothesis,

$$r^{\mathfrak{A}}(\mathbf{a}) \leq \bigvee_{\mathbf{a}\in h^{-1}(\mathbf{b})} r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(\mathbf{b}) = r^{\mathfrak{B}}(h(\mathbf{a})).$$

(ii) We have, for all $\mathbf{a} \in A^{\rho(r)}$ and all $\mathbf{b} \in h(A)^{\rho(r)}$,

$$\mathbf{b} = h(\mathbf{a}) \quad \text{iff} \quad \mathbf{a} \in h^{-1}(\mathbf{b}).$$

Thus, the condition $r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(h(\mathbf{a}))$, for all $\mathbf{a} \in A^{\rho(r)}$, is equivalent to the condition $r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(\mathbf{b})$, for all $\mathbf{b} \in h(A)^{\rho(r)}$ and all $\mathbf{a} \in h^{-1}(\mathbf{b})$.

(iii) By Part (ii), taking into account the fact that $h(A) = B$.

■

Corollary 124 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathcal{A} \to \mathcal{B}$.

(i) If $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *is an embedding, then* $h : \mathfrak{A} \cong \mathfrak{B}$.

(ii) If the language has G-equality and $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, *then* $h : \mathfrak{A} \cong \mathfrak{B}$.

Proof: Part (i) follows from the definition of an embedding and from Part (iii) of Lemma 123. Part (ii) is a consequence of Part (i) and the fact that the presence of G-equality, together with strictness, implies that h is an embedding. ■

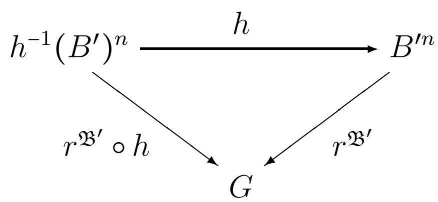
Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to \mathfrak{B}$. Let $\mathfrak{A}' \subseteq \mathfrak{A}$. The **image of $\mathfrak{A}'$ under** h is the substructure of $\mathfrak{B}$ generated by $h(A')$,

$$h(\mathfrak{A}') = \langle h(A'), E^{\mathfrak{B}}\restriction_{h(A')}, F^{\mathfrak{B}}\restriction_{h(A')}, R^{\mathfrak{B}}\restriction_{h(A')} \rangle.$$

On the other hand, consider a substructure $\mathfrak{B}' \subseteq \mathfrak{B}$. The **preimage of $\mathfrak{B}'$ under** h is the structure

$$h^{-1}(\mathfrak{B}') = \langle h^{-1}(B'), E^{\mathfrak{A}}\restriction_{h^{-1}(B')}, F^{\mathfrak{A}}\restriction_{h^{-1}(B')}, R^{h^{-1}(\mathfrak{B}')} \rangle,$$

where $R^{h^{-1}(\mathfrak{B}')} = \{r^{h^{-1}(\mathfrak{B}')} : r \in R\}$ and, for all $r \in R$ and all $\mathbf{a} \in h^{-1}(B')^{\rho(r)}$,



$$r^{h^{-1}(\mathfrak{B}')}(\mathbf{a}) = r^{\mathfrak{B}'}(h(\mathbf{a})).$$

Both $h(\mathfrak{A}')$ and $h^{-1}(\mathfrak{B}')$ are $\mathcal{L}$-structures. For $h(\mathfrak{A}')$ this holds by definition. For $h^{-1}(\mathfrak{B}')$, on the other hand, one must show that all operations in $F^{\mathfrak{A}}\restriction_{h^{-1}(B')}$ and all G-relations in $R^{h^{-1}(\mathfrak{B}')}$ are compatible with $E^{\mathfrak{A}}\restriction_{h^{-1}(B')}$. The proof for operations is a direct consequence of the compatibility of $F^{\mathfrak{A}}$ with $E^{\mathfrak{A}}$. For relations, let us assume that $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in h^{-1}(B')$. Then we have

$$\begin{array}{l} \bigwedge_{i=1}^n E^{\mathfrak{A}}\restriction_{h^{-1}(B')}(a_i, b_i) \wedge r^{h^{-1}(\mathfrak{B}')}(\mathbf{a}) \\ \quad = \bigwedge_{i=1}^n E^{\mathfrak{A}}(a_i, b_i) \wedge r^{\mathfrak{B}'}(h(\mathbf{a})) \\ \qquad (\text{Definitions of } E^{\mathfrak{A}}\restriction_{h^{-1}(B')} \text{ and of } r^{h^{-1}(\mathfrak{B}')}) \\ \quad \leq \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \wedge r^{\mathfrak{B}'}(h(\mathbf{a})) \\ \qquad (h : \mathfrak{A} \to \mathfrak{B}) \\ \quad = \bigwedge_{i=1}^n E^{\mathfrak{B}'}(h(a_i), h(b_i)) \wedge r^{\mathfrak{B}'}(h(\mathbf{a})) \\ \qquad (a_1, b_1, \ldots, a_n, b_n \in h^{-1}(B')) \\ \quad \leq r^{\mathfrak{B}'}(h(\mathbf{b})) \\ \qquad (r^{\mathfrak{B}'} \text{ compatible with } E^{\mathfrak{B}'}) \\ \quad = r^{h^{-1}(\mathfrak{B}')}(\mathbf{b}). \quad (\text{Definition of } r^{h^{-1}(\mathfrak{B}')}(\mathbf{b})) \end{array}$$

Furthermore, by definition, $h(\mathfrak{A}') \subseteq \mathfrak{B}$. However, in general, $h^{-1}(\mathfrak{B}')$ may not be a substructure of $\mathfrak{A}$. This is the case, however, if $h : \mathfrak{A} \to_s \mathfrak{B}$ is a strict morphism.

Lemma 125 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to_s \mathfrak{B}$. If $\mathfrak{B}' \subseteq \mathfrak{B}$, then $h^{-1}(\mathfrak{B}') \subseteq \mathfrak{A}$.*

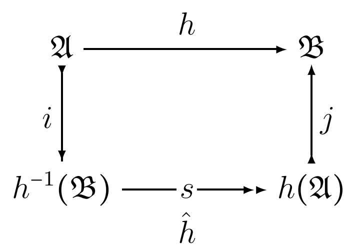
Proof: Let $\mathbf{a} \in h^{-1}(B')$. Then

$$\begin{array}{rcll} r^{h^{-1}(\mathfrak{B}')}(\mathbf{a}) & = & r^{\mathfrak{B}'}(h(\mathbf{a})) & (\text{Definition of } r^{h^{-1}(\mathfrak{B}')}(\mathbf{a})) \\ & = & r^{\mathfrak{B}}(h(\mathbf{a})) & (\mathfrak{B}' \subseteq \mathfrak{B} \text{ and } h(\mathbf{a}) \in B') \\ & = & r^{\mathfrak{A}}(\mathbf{a}). & (h : \mathfrak{A} \to_s \mathfrak{B}) \end{array}$$

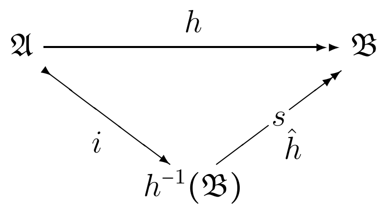
So $h^{-1}(\mathfrak{B}') \subseteq \mathfrak{A}$. ■

We detail, next, how any given surjective morphism from a structure $\mathfrak{A}$ onto a structure $\mathfrak{B}$ can be canonically decomposed through a reductive morphism. Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, , R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to \mathfrak{B}$. The map $h : A \to B$ gives rise to a reductive morphism $\hat{h}$:

$h^{-1}(\mathfrak{B}) \twoheadrightarrow_s h(\mathfrak{A})$. Denoting by $i : \mathfrak{A} \rightarrowtail h^{-1}(\mathfrak{B})$ and by $j : h(\mathfrak{A}) \rightarrowtail \mathfrak{B}$ the identity mappings, we get the commutative diagram



Thus, in case $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, we obtain the decomposition



This is quite important for what follows because, as Elgueta points out [23], strict morphisms are the appropriate ones by which to replace homomorphisms when attempting to lift universal algebraic results to languages that may contain relation symbols in addition to function symbols.

Let, again, $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to \mathfrak{B}$ a morphism. h is called **elementary**, written $h : \mathfrak{A} \to_e \mathfrak{B}$, if, for all $\mathcal{L}$-formulas φ and all assignments $g : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\varphi^{\mathfrak{A}}[g] = \varphi^{\mathfrak{B}}[h \circ g].$$

We close the section by showing that every reductive morphism is, in fact, elementary. A weak converse will be proven later on.

Proposition 126 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures. If $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, then $h : \mathfrak{A} \to_e \mathfrak{B}$.*

Proof: Suppose $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. We use structural induction on φ to show that, for every φ and every $g : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\varphi^{\mathfrak{A}}[g] = \varphi^{\mathfrak{B}}[h \circ g].$$

For $\varphi = r(t_1, \ldots, t_n)$ atomic, we have

$$\begin{array}{rcll} \varphi^{\mathfrak{A}}[g] & = & r^{\mathfrak{A}}(g(t_1), \ldots, g(t_n)) & \text{(Definition)} \\ & = & r^{\mathfrak{B}}(h(g(t_1)), \ldots, h(g(t_n))) & (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & = & \varphi^{\mathfrak{B}}[h \circ g]. & \text{(Definition)} \end{array}$$

For negation, we have

$$\begin{array}{rcll} (\neg\varphi)^{\mathfrak{A}}[g] & = & \neg\varphi^{\mathfrak{A}}[g] & \text{(Definition)} \\ & = & \neg\varphi^{\mathfrak{B}}[h \circ g] & \text{(Induction Hypothesis)} \\ & = & (\neg\varphi)^{\mathfrak{B}}[h \circ g]. & \text{(Definition)} \end{array}$$

The cases of conjunction and disjunction can be handled similarly.

For existential quantification, we obtain

$$\begin{array}{rcll}(\exists x\varphi)^{\mathfrak{A}}[g] & = & \bigvee_{a\in A}\varphi^{\mathfrak{A}}[g(a/x)] & \text{(Definition)}\\ & = & \bigvee_{a\in A}\varphi^{\mathfrak{B}}[h\circ g(a/x)] & \text{(Induction Hypothesis)}\\ & = & \bigvee_{a\in A}\varphi^{\mathfrak{B}}[(h\circ g)(h(a)/x)] & \\ & = & \bigvee_{b\in B}\varphi^{\mathfrak{B}}[(h\circ g)(b/x)] & (h\text{ Surjective})\\ & = & (\exists x\varphi)^{\mathfrak{B}}[h\circ g]. & \text{(Definition)}\end{array}$$

Universal quantification can be handled analogously. ■

4.5 Products

Let $\mathfrak{A}_i = \langle \mathbf{A}_i, E^{\mathfrak{A}_i}, R^{\mathfrak{A}_i}\rangle$, $i \in I$, be a family of $\mathcal{L}$-structures. The **direct product** of the $\mathfrak{A}_i$, $i \in I$, is the $\mathcal{L}$-structure defined by

$$\prod_{i\in I}\mathfrak{A}_i = \left\langle \prod_{i\in I}\mathbf{A}_i, E^{\prod\mathfrak{A}_i}, R^{\prod\mathfrak{A}_i}\right\rangle,$$

where:

- $\prod_{i\in I}\mathbf{A}_i$ is the ordinary direct product of the $\mathcal{L}$-algebras $\mathbf{A}_i$, $i \in I$;
- $E^{\prod\mathfrak{A}_i}$ is defined, for all $\mathbf{a},\mathbf{b} \in \prod_{i\in I}A_i$,
$$E^{\prod_i\mathfrak{A}_i}(\mathbf{a},\mathbf{b}) = \bigwedge_{i\in I}E^{\mathfrak{A}_i}(a_i,b_i);$$
- $R^{\prod\mathfrak{A}_i}$ is defined by $R^{\prod\mathfrak{A}_i} = \{r^{\prod\mathfrak{A}_i} : r \in R\}$, where, for each $r \in R$, with $\rho(r) = n$, and all $\mathbf{a}_1,\ldots,\mathbf{a}_n \in \prod_{i\in I}A_i$,
$$r^{\prod_i\mathfrak{A}_i}(\mathbf{a}_1,\ldots,\mathbf{a}_n) = \bigwedge_{i\in I}r^{\mathfrak{A}_i}(a_{1i},\ldots,a_{ni}).$$

In this construction, I is allowed to be empty. In this case, we assume that $\prod_{i\in\varnothing}\mathbf{A}_i$ is the trivial one-element $\mathcal{L}$-algebra and all relations (including the reduced G-congruence) interpret the unique tuple in the domain as $\top$.

We show that this construction does indeed give a bona fide $\mathcal{L}$-structure. To verify this, we must show that all operations in $F^{\prod\mathfrak{A}_i}$ and all G-relations in $R^{\prod\mathfrak{A}_i}$ are compatible with $E^{\prod\mathfrak{A}_i}$. Let $f \in F$, with $\rho(f) = n$, and $\mathbf{a}_1,\mathbf{b}_1,\ldots,\mathbf{a}_n,\mathbf{b}_n \in \prod_{i\in I}A_i$. Then, by definition, for all $i \in I$,

$$\bigwedge_{j=1}^{n}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \le E^{\mathfrak{A}_i}(f^{\mathfrak{A}_i}(a_{1i},\ldots,a_{ni}), f^{\mathfrak{A}_i}(b_{1i},\ldots,b_{ni})).$$

Thus,

$$\bigwedge_{i\in I}\bigwedge_{j=1}^{n}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \le \bigwedge_{i\in I}E^{\mathfrak{A}_i}(f^{\mathfrak{A}_i}(a_{1i},\ldots,a_{ni}), f^{\mathfrak{A}_i}(b_{1i},\ldots,b_{ni})).$$

This yields

$$\bigwedge_{j=1}^{n}\bigwedge_{i\in I}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \leq \bigwedge_{i\in I}E^{\mathfrak{A}_i}(f^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni}),f^{\mathfrak{A}_i}(b_{1i},\dots,b_{ni})),$$

i.e.,

$$\bigwedge_{j=1}^{n}E^{\prod\mathfrak{A}_i}(\mathbf{a}_j,\mathbf{b}_j) \leq E^{\prod\mathfrak{A}_i}(f^{\prod\mathfrak{A}_i}(\mathbf{a}_1,\dots,\mathbf{a}_n),f^{\prod\mathfrak{A}_i}(\mathbf{b}_1,\dots,\mathbf{b}_n)).$$

This proves that $F^{\prod\mathfrak{A}_i}$ is compatible with $E^{\prod\mathfrak{A}_i}$. Suppose, next, that $r\in R$, with $\rho(r)=n$, and $\mathbf{a}_1,\mathbf{b}_1,\dots,\mathbf{a}_n,\mathbf{b}_n\in\prod_{i\in I}A_i$. We get

$$\begin{aligned}
&\textstyle\bigwedge_{j=1}^{n}E^{\prod\mathfrak{A}_i}(\mathbf{a}_j,\mathbf{b}_j)\wedge r^{\prod\mathfrak{A}_i}(\mathbf{a}_1,\dots,\mathbf{a}_n)\\
&\quad=\textstyle\bigwedge_{j=1}^{n}\bigwedge_{i\in I}E^{\mathfrak{A}_i}(a_{ji},b_{ji})\wedge\bigwedge_{i\in I}r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni})\\
&\quad=\textstyle\bigwedge_{i\in I}\bigwedge_{j=1}^{n}E^{\mathfrak{A}_i}(a_{ji},b_{ji})\wedge\bigwedge_{i\in I}r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni})\\
&\quad=\textstyle\bigwedge_{i\in I}\left[\bigwedge_{j=1}^{n}E^{\mathfrak{A}_i}(a_{ji},b_{ji})\wedge r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni})\right]\\
&\quad\leq\textstyle\bigwedge_{i\in I}r^{\mathfrak{A}_i}(b_{1i},\dots,b_{ni})\\
&\quad=r^{\prod\mathfrak{A}_i}(\mathbf{b}_1,\dots,\mathbf{b}_n).
\end{aligned}$$

This proves that $R^{\prod\mathfrak{A}_i}$ is also compatible with $E^{\prod\mathfrak{A}_i}$. Thus $\prod_{i\in I}\mathfrak{A}_i$ is an $\mathcal{L}$-structure.

4.6 G-Congruences

Consider a set A. Recall that a mapping $\Theta : A^2\to G$ is a G-equivalence on A if, for all $a,b,c\in A$,

(**Reflexivity**) $\Theta(a,a)=\top$;

(**Symmetry**) $\Theta(a,b)=\Theta(b,a)$;

(**Transitivity**) $\Theta(a,b)\wedge\Theta(b,c)\leq\Theta(a,c)$.

Suppose that $\mathbf{A}=\langle A,F^{\mathbf{A}}\rangle$ is an F-algebra. Recall that $\Theta : A^2\to G$ is G-congruence on $\mathbf{A}$ if Θ is a G-equivalence on A and, in addition, it satisfies, for all f in F of arity n and all $a_1,\dots,a_n$ and $b_1,\dots,b_n$ in A,

$$\bigwedge_{i=1}^{n}\Theta(a_i,b_i)\leq\Theta(f^{\mathbf{A}}(a_1,\dots,a_n),f^{\mathbf{A}}(b_1,\dots,b_n)).$$

Let $\mathrm{Gon}(\mathbf{A})$ denote the collection of all G-congruences on $\mathbf{A}$. This set is naturally ordered by

$$\Theta\leq\Theta'\quad\text{iff}\quad\Theta(a,b)\leq\Theta'(a,b),\ \text{for all } a,b\in A.$$

Lemma 127 *Let* $\mathbf{A}=\langle A,F^{\mathbf{A}}\rangle$ *be an* F*-algebra. Then* $\mathbf{Gon(A)}=\langle\mathrm{Gon}(\mathbf{A}),$ $\leq\rangle$ *is a complete lattice.*

Proof: Clearly, the function $\top : A^2 \to G$, with $\top(a,b) = \top$, for all $a,b \in A$, is a G-congruence on $\mathbf{A}$. Next, let Θ_i, $i \in I$, be a nonempty collection of G-congruences on $\mathbf{A}$. Then, for all $a,b,c \in A$,

- $\bigwedge_i \Theta_i(a,a) = \bigwedge_i \top = \top$;
- $\bigwedge_i \Theta_i(a,b) = \bigwedge_i \Theta_i(b,a)$;
- $\bigwedge_i \Theta_i(a,b) \wedge \bigwedge_i \Theta_i(b,c) = \bigwedge_i(\Theta_i(a,b) \wedge \Theta_i(b,c)) \leq \bigwedge_i \Theta_i(a,c)$.

Moreover, for all n-ary $f \in F$ and all $a_1,b_1,\ldots,a_n,b_n \in A$,

$$\begin{array}{rcl}\bigwedge_{j=1}^n \bigwedge_i \Theta_i(a_j,b_j) & = & \bigwedge_i \bigwedge_{j=1}^n \Theta_i(a_j,b_j)\\ & \leq & \bigwedge_i \Theta_i(f^{\mathbf{A}}(a_1,\ldots,a_n), f^{\mathbf{A}}(b_1,\ldots,b_n)).\end{array}$$

Thus, $\mathbf{Gon}(\mathbf{A})$ forms a complete lattice under $\leq$. ■

Let $\mathcal{A} = \langle \mathbf{A}, E^{\mathcal{A}}\rangle$ be a G-algebra. A G-congruence $\Theta \in \mathrm{Gon}(\mathbf{A})$ is a **G-congruence on $\mathcal{A}$** if

$$E^{\mathcal{A}} \leq \Theta.$$

We denote by $\mathrm{Gon}(\mathcal{A})$ the collection of all G-congruences on $\mathcal{A}$. Clearly, $\mathrm{Gon}(\mathcal{A}) \subseteq \mathrm{Gon}(\mathbf{A})$ and, hence, $\mathrm{Gon}(\mathcal{A})$ inherits the order $\leq$ from $\mathbf{Gon}(\mathbf{A})$.

Lemma 128 *Let $\mathcal{A} = \langle \mathbf{A}, E^{\mathcal{A}}\rangle$ be a G-algebra. Then $\mathbf{Gon}(\mathcal{A}) = \langle \mathrm{Gon}(\mathcal{A}), \leq\rangle$ is a principal filter of $\mathbf{Gon}(\mathbf{A})$, and, hence, a complete lattice.*

Proof: This is clear from definition, since $\Theta \in \mathrm{Gon}(\mathcal{A})$ if and only if $\Theta \in \mathrm{Gon}(\mathbf{A})$ and $E^{\mathcal{A}} \leq \Theta$. ■

Now, given a set A, a mapping $\Theta : A^2 \to G$ and a G-relation $r : A^n \to G$, we say that Θ is **compatible with** r if, for all $a_1,b_1,\ldots,a_n,b_n \in A$, we have

$$\bigwedge_{i=1}^n \Theta(a_i,b_i) \leq r(a_1,\ldots,a_n) \leftrightarrow r(b_1,\ldots,b_n).$$

Note that, if Θ is symmetric, this is equivalent to saying that the G-relation r is compatible with Θ, according to previously introduced terminology.

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-structure. We say that $\Theta : A^2 \to G$ is a **G-congruence on $\mathfrak{A}$** if:

- Θ is a G-congruence on $\mathcal{A} = \langle \mathbf{A}, E^{\mathfrak{A}}\rangle$;
- Θ is compatible with $r^{\mathfrak{A}}$, for all $r \in R$.

Denote by $\mathrm{Gon}(\mathfrak{A})$ the collection of all G-congruences on $\mathfrak{A}$. Clearly, we have $\mathrm{Gon}(\mathfrak{A}) \subseteq \mathrm{Gon}(\mathcal{A})$ and, furthermore, $\mathrm{Gon}(\mathfrak{A})$ inherits from $\mathbf{Gon}(\mathcal{A})$ the ordering $\leq$.

Since, for a given structure $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$, $E^{\mathfrak{A}} \in \mathrm{Gon}(\mathfrak{A})$ by definition, we get that, for every structure $\mathfrak{A}$, the largest G-congruence on $\mathfrak{A}$ exists (under sufficient conditions on the complete lattice $\mathbf{G}$, of course). It will be called the **Leibniz G-congruence** of $\mathfrak{A}$ and denoted by $\Omega(\mathfrak{A})$. So we have

$$\mathrm{Gon}(\mathfrak{A}) = \{\Theta \in \mathrm{Gon}(\mathcal{A}) : \Theta \leq \Omega(\mathfrak{A})\}.$$

We have the following observations that follow directly from the definitions.

Lemma 129 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. Suppose there exists an n-ary relation symbol $r \in R$, such that, for some $a_1, b_1, \dots, a_n, b_n \in A$,*

$$r^{\mathfrak{A}}(a_1, \dots, a_n) \not\leftrightarrow r^{\mathfrak{A}}(b_1, \dots, b_n).$$

Then $\Omega(\mathfrak{A}) \neq \top$.

Proof: Suppose, towards a contradiction, that $\Omega(\mathfrak{A}) = \top$ and let $r \in R$ and $a_1, b_1, \dots, a_n, b_n \in A$. Then, we have

$$\top = \bigwedge_{i=1}^{n} \Omega(\mathfrak{A})(a_i, b_i) \leq r^{\mathfrak{A}}(a_1, \dots, a_n) \leftrightarrow r^{\mathfrak{A}}(b_1, \dots, b_n).$$

This contradicts the hypothesis. ■

If the signature $\mathcal{L}$ includes a binary relation symbol r whose interpretation $r^{\mathfrak{A}}$ in a structure $\mathfrak{A}$ happens to be a G-congruence on $\mathfrak{A}$, then $r^{\mathfrak{A}}$ is necessarily the Leibniz G-congruence on $\mathfrak{A}$.

Lemma 130 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. If $\mathcal{L}$ has a binary relation symbol r, such that $r^{\mathfrak{A}} \in \mathrm{Gon}(\mathfrak{A})$, then $r^{\mathfrak{A}} = \Omega(\mathfrak{A})$.*

Proof: By hypothesis and the maximality property of $\Omega(\mathfrak{A})$, we have $r^{\mathfrak{A}} \leq \Omega(\mathfrak{A})$. On the other hand, for all $a, b \in A$,

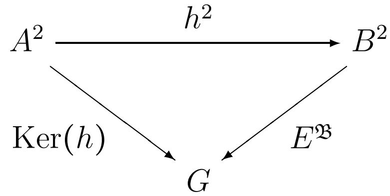
$$\begin{array}{rcl}\Omega(\mathfrak{A})(a,b) & = & \Omega(\mathfrak{A})(a,b) \wedge \Omega(\mathfrak{A})(b,b) \quad (\Omega(\mathfrak{A})(b,b) = \top) \\ & \leq & r^{\mathfrak{A}}(a,b) \leftrightarrow r^{\mathfrak{A}}(b,b) \quad (\Omega(\mathfrak{A}) \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{A}}(a,b). \quad (r^{\mathfrak{A}}(b,b) = \top)\end{array}$$

Since a, b were arbitrary, $\Omega(\mathfrak{A}) \leq r^{\mathfrak{A}}$. Combining, we get the conclusion. ■

We provide, next, a series of lemmas relating G-congruences with strong homomorphisms, substructures and strong homomorphic images.

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathbf{A} \to \mathbf{B}$. Define the **$G$-kernel** $\mathrm{Ker}(h) : A^2 \to G$ **of** h by

$$\mathrm{Ker}(h) = E^{\mathfrak{B}} \circ h,$$



that is, for all $a, b \in A$,

$$\mathrm{Ker}(h)(a,b) = E^{\mathfrak{B}}(h(a), h(b)).$$

Lemma 131 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathbf{A} \to \mathbf{B}$*. If* $h : \mathfrak{A} \to_s \mathfrak{B}$*, then* $\mathrm{Ker}(h) \in \mathrm{Gon}(\mathfrak{A})$*.*

Proof: First, we show that $\mathrm{Ker}(h)$ is a G-equivalence on A. Let $a, b, c \in A$.

- For Reflexivity,

$$\begin{array}{rcl} \mathrm{Ker}(h)(a,a) & = & E^{\mathfrak{B}}(h(a), h(a)) \quad \text{(Definition of } \mathrm{Ker}(h)) \\ & = & \top; \quad (E^{\mathfrak{B}}\ G\text{-Congruence}) \end{array}$$

- For Symmetry,

$$\begin{array}{rcl} \mathrm{Ker}(h)(b,a) & = & E^{\mathfrak{B}}(h(b), h(a)) \quad \text{(Definition of } \mathrm{Ker}(h)) \\ & = & E^{\mathfrak{B}}(h(a), h(b)) \quad (E^{\mathfrak{B}}\ G\text{-Congruence}) \\ & = & \mathrm{Ker}(h)(a,b); \quad \text{(Definition of } \mathrm{Ker}(h)) \end{array}$$

- For Transitivity,

$$\begin{array}{rcl} \mathrm{Ker}(h)(a,b) \wedge \mathrm{Ker}(h)(b,c) & = & E^{\mathfrak{B}}(h(a), h(b)) \wedge E^{\mathfrak{B}}(h(b), h(c)) \\ & & \quad \text{(Definition of } \mathrm{Ker}(h)) \\ & \leq & E^{\mathfrak{B}}(h(a), h(c)) \\ & & \quad (E^{\mathfrak{B}}\ G\text{-Congruence}) \\ & = & \mathrm{Ker}(h)(a,c). \\ & & \quad \text{(Definition of } \mathrm{Ker}(h)) \end{array}$$

Next, we show that $\mathrm{Ker}(h)$ is compatible with all operations in $F^{\mathfrak{A}}$. Let $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$.

$$\begin{array}{rcl} \bigwedge_{i=1}^n \mathrm{Ker}(h)(a_i, b_i) & = & \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \\ & & \quad \text{(Definition of } \mathrm{Ker}(h)) \\ & \leq & E^{\mathfrak{B}}(f^{\mathfrak{B}}(h(\mathbf{a})), f^{\mathfrak{B}}(h(\mathbf{b}))) \\ & & \quad (E^{\mathfrak{B}}\ G\text{-Congruence}) \\ & = & E^{\mathfrak{B}}(h(f^{\mathfrak{A}}(\mathbf{a})), h(f^{\mathfrak{A}}(\mathbf{b}))) \\ & & \quad (h : \mathbf{A} \to \mathbf{B}) \\ & = & \mathrm{Ker}(h)(f^{\mathfrak{A}}(\mathbf{a}), f^{\mathfrak{A}}(\mathbf{b})). \\ & & \quad \text{(Definition of } \mathrm{Ker}(h)) \end{array}$$

By definition of a morphism of structures, $E^{\mathfrak{A}} \leq E^{\mathfrak{B}} \circ h = \mathrm{Ker}(h)$.

Finally, we show that $\mathrm{Ker}(h)$ is compatible with all G-relations in $R^{\mathfrak{A}}$. Note that this is the only part where strictness is needed. We have, for all $r \in R$, with $\rho(r) = n$, and all $a_1, b_1, \dots, a_n, b_n \in A$,

$$\begin{array}{rcl} \bigwedge_{i=1}^n \mathrm{Ker}(h)(a_i, b_i) \wedge r^{\mathfrak{A}}(\mathbf{a}) & = & \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \wedge r^{\mathfrak{A}}(\mathbf{a}) \\ & & \quad (\text{Definition of } \mathrm{Ker}(h)) \\ & = & \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \wedge r^{\mathfrak{B}}(h(\mathbf{a})) \\ & & \quad (h : \mathfrak{A} \to_s \mathfrak{B}) \\ & \leq & r^{\mathfrak{B}}(h(\mathbf{b})) \quad (E^{\mathfrak{B}}\ G\text{-Congruence}) \\ & = & r^{\mathfrak{A}}(\mathbf{b}). \quad (h : \mathfrak{A} \to_s \mathfrak{B}) \end{array}$$

We conclude that $\mathrm{Ker}(h) \in \mathrm{Gon}(\mathfrak{A})$. ■

Lemma 132 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures, such that $\mathfrak{B} \subseteq \mathfrak{A}$. Given $\Theta : A^2 \to G$, define $\Theta_B = \Theta{\restriction}_{B^2}$. If $\Theta \in \mathrm{Gon}(\mathfrak{A})$, then $\Theta_B \in \mathrm{Gon}(\mathfrak{B})$.*

Proof: It follows almost immediately from the definition of Θ_B and the fact that Θ is a G-congruence on $\mathbf{A}$ that Θ_B is a G-congruence on $\mathbf{B}$. Further, for all $b_1, b_2 \in B$, we have

$$\begin{array}{rcl} E^{\mathfrak{B}}(b_1, b_2) & = & E^{\mathfrak{A}}(b_1, b_2) \quad (\mathfrak{B} \subseteq \mathfrak{A}) \\ & \leq & \Theta(b_1, b_2) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \Theta_B(b_1, b_2). \quad (\text{Definition of } \Theta_B) \end{array}$$

Finally, for all $r \in R$, with $\rho(r) = n$, and all $b_1, b_1', \dots, b_n, b_n' \in B$, we have

$$\begin{array}{rcl} \bigwedge_{i=1}^n \Theta_B(b_i, b_i') \wedge r^{\mathfrak{B}}(\mathbf{b}) & = & \bigwedge_{i=1}^n \Theta(b_i, b_i') \wedge r^{\mathfrak{B}}(\mathbf{b}) \quad (\Theta_B = \Theta{\restriction}_{B^2}) \\ & = & \bigwedge_{i=1}^n \Theta(b_i, b_i') \wedge r^{\mathfrak{A}}(\mathbf{b}) \quad (\mathfrak{B} \subseteq \mathfrak{A}) \\ & \leq & r^{\mathfrak{A}}(\mathbf{b}') \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{B}}(\mathbf{b}'). \quad (\mathfrak{B} \subseteq \mathfrak{A}) \end{array}$$

Thus, $\Theta_B \in \mathrm{Gon}(\mathfrak{B})$. ■

Lemma 133 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to_s \mathfrak{B}$.*

(a) If $\Theta \in \mathrm{Gon}(\mathfrak{B})$, then $\Theta \circ h \in \mathrm{Gon}(\mathfrak{A})$.

(b) If $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ and $\Theta \in \mathrm{Gon}(\mathfrak{A})$, such that $\mathrm{Ker}(h)$ is compatible with Θ, then, there exists $\Theta' \in \mathrm{Gon}(\mathfrak{B})$, such that $\Theta = \Theta' \circ h$.

Proof:

(a) First, we must show that, if Θ is a G-congruence on $\mathbf{B}$, then $\Theta \circ h$ is a G-congruence on $\mathbf{A}$. Let us show Transitivity and Congruence. Suppose $a_1, a_2, a_3 \in A$. We have

$$\begin{aligned}&(\Theta \circ h)(a_1, a_2) \wedge (\Theta \circ h)(a_2, a_3)\\ &\qquad = \Theta(h(a_1), h(a_2)) \wedge \Theta(h(a_2), h(a_3)) \quad \text{(Composition)}\\ &\qquad \leq \Theta(h(a_1), h(a_3)) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\ &\qquad = (\Theta \circ h)(a_1, a_3). \quad \text{(Composition)}\end{aligned}$$

For Congruence, suppose $f \in F$, with $\rho(f) = n$, and let $a_1, \ldots, a_n$, $a'_1, \ldots, a'_n \in A$. We have

$$\begin{aligned}&\textstyle\bigwedge_{i=1}^n (\Theta \circ h)(a_i, a'_i)\\ &\qquad = \textstyle\bigwedge_{i=1}^n \Theta(h(a_i), h(a'_i)) \quad \text{(Composition)}\\ &\qquad \leq \Theta(f^{\mathbf{B}}(h(\mathbf{a})), f^{\mathbf{B}}(h(\mathbf{a}'))) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\ &\qquad = \Theta(h(f^{\mathbf{A}}(\mathbf{a})), h(f^{\mathbf{A}}(\mathbf{a}'))) \quad (h : \mathbf{A} \to \mathbf{B})\\ &\qquad = (\Theta \circ h)(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{a}')). \quad \text{(Composition)}\end{aligned}$$

Further, for all $a_1, a_2 \in A$,

$$\begin{aligned}E^{\mathfrak{A}}(a_1, a_2) &\leq E^{\mathfrak{B}}(h(a_1), h(a_2)) \quad (h : \mathfrak{A} \to \mathfrak{B})\\ &\leq \Theta(h(a_1), h(a_2)) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\ &= (\Theta \circ h)(a_1, a_2). \quad \text{(Composition)}\end{aligned}$$

It remains to show that, if Θ is compatible with $R^{\mathfrak{B}}$, then $\Theta \circ h$ is compatible with $R^{\mathfrak{A}}$. Suppose $r \in R$, with $\rho(r) = n$, and let $a_1, a'_1, \ldots, a_n$, $a'_n \in A$. We have

$$\begin{aligned}&\textstyle\bigwedge_{i=1}^n (\Theta \circ h)(a_i, a'_i) \wedge r^{\mathfrak{A}}(\mathbf{a})\\ &\qquad = \textstyle\bigwedge_{i=1}^n \Theta(h(a_i), h(a'_i)) \wedge r^{\mathfrak{A}}(\mathbf{a}) \quad \text{(Composition)}\\ &\qquad = \textstyle\bigwedge_{i=1}^n \Theta(h(a_i), h(a'_i)) \wedge r^{\mathfrak{B}}(h(\mathbf{a})) \quad (h : \mathfrak{A} \to_s \mathfrak{B})\\ &\qquad \leq r^{\mathfrak{B}}(h(\mathbf{a}')) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\ &\qquad = r^{\mathfrak{A}}(\mathbf{a}') \quad (h : \mathfrak{A} \to_s \mathfrak{B})\end{aligned}$$

Thus, if $\Theta \in \mathrm{Gon}(\mathfrak{B})$, then $\Theta \circ h \in \mathrm{Gon}(\mathfrak{A})$.

(b) The condition that Θ' needs to satisfy compels the definition of Θ'. Suppose $b_1, b_2 \in B$. Since h is reductive, there exist $a_1, a_2 \in A$, such that $b_1 = h(a_1)$ and $b_2 = h(a_2)$. We define

$$\Theta'(b_1, b_2) = \Theta(a_1, a_2).$$

Of course, one has to show that Θ' is well defined. So suppose, a_1, a'_1, $a_2, a'_2 \in A$, such that $h(a_1) = h(a'_1) = b_1$ and $h(a_2) = h(a'_2) = b_2$. Then $E^{\mathfrak{B}}(h(a_1), h(a'_1)) = \top$ and $E^{\mathfrak{B}}(h(a_2), h(a'_2)) = \top$. By the definition of $\mathrm{Ker}(h)$, $\mathrm{Ker}(h)(a_1, a'_1) = \top$ and $\mathrm{Ker}(h)(a_2, a'_2) = \top$. Thus, since, by hypothesis, $\mathrm{Ker}(h) \leq \Theta$, $\Theta(a_1, a_1') = \Theta(a_2, a_2') = \top$. Thus, using Transitivity,

$$\begin{aligned} \Theta(a_1, a_2) &= \top \wedge \Theta(a_1, a_2) \wedge \top \\ &= \Theta(a_1', a_1) \wedge \Theta(a_1, a_2) \wedge \Theta(a_2, a_2') \\ &\leq \Theta(a_1', a_2') \end{aligned}$$

and, by symmetry, $\Theta(a_1, a_2) = \Theta(a_1', a_2')$. Hence, Θ' is well-defined.

Since the definition yields $\Theta = \Theta' \circ h$, it suffices to show that Θ' is a congruence on $\mathfrak{B}$. We show Transitivity, Congruence, Inclusion of $E^{\mathfrak{B}}$ and Compatibility. Reflexivity and Symmetry can be proven similarly.

Let $b_1, b_2, b_3 \in B$. Then there exist $a_1, a_2, a_3 \in A$, such that $b_i = h(a_i)$, $i = 1, 2, 3$. So we have

$$\begin{aligned} \Theta'(b_1, b_2) \wedge \Theta'(b_2, b_3) &= \Theta(a_1, a_2) \wedge \Theta(a_2, a_3) \quad (\text{Definition of } \Theta') \\ &\leq \Theta(a_1, a_3) \quad (\Theta \in \mathrm{Gon}(\mathbf{A})) \\ &= \Theta'(b_1, b_3). \quad (\text{Definition of } \Theta') \end{aligned}$$

Next, suppose $f \in F$, with $\rho(f) = n$, and let $b_1, b_1', \ldots, b_n, b_n' \in B$. Consider $a_1, a_1', \ldots, a_n, a_n' \in A$, such that $\mathbf{b} = h(\mathbf{a})$ and $\mathbf{b}' = h(\mathbf{a}')$. Note that $f^{\mathbf{B}}(\mathbf{b}) = f^{\mathbf{B}}(h(\mathbf{a})) = h(f^{\mathbf{A}}(\mathbf{a}))$ and, similarly, $f^{\mathbf{B}}(\mathbf{b}') = h(f^{\mathbf{A}}(\mathbf{a}'))$. Then we have

$$\begin{aligned} \textstyle\bigwedge_{i=1}^n \Theta'(b_i, b_i') &= \textstyle\bigwedge_{i=1}^n \Theta(a_i, a_i') \quad (\text{Definition of } \Theta') \\ &\leq \Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{a}')) \quad (\Theta \in \mathrm{Gon}(\mathbf{A})) \\ &= \Theta'(f^{\mathbf{B}}(\mathbf{b}), f^{\mathbf{B}}(\mathbf{b}')). \quad (\text{Definition of } \Theta') \end{aligned}$$

Next, for $b_1, b_2 \in B$, such that $b_1 = h(a_1)$ and $b_2 = h(a_2)$, for some $a_1, a_2 \in A$, we get

$$\begin{aligned} E^{\mathfrak{B}}(b_1, b_2) &= E^{\mathfrak{B}}(h(a_1), h(a_2)) \quad (b_i = h(a_i)) \\ &\leq \Theta(a_1, a_2) \quad (\mathrm{Ker}(h) \leq \Theta) \\ &= \Theta'(b_1, b_2). \quad (\text{Definition of } \Theta') \end{aligned}$$

Finally, suppose $r \in R$, with $\rho(r) = n$, and let $b_1, b_1', \ldots, b_n, b_n' \in B$. Consider $a_1, a_1', \ldots, a_n, a_n' \in A$, such that $\mathbf{b} = h(\mathbf{a})$ and $\mathbf{b}' = h(\mathbf{a}')$. Note that, since $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, $r^{\mathfrak{B}}(\mathbf{b}) = r^{\mathfrak{B}}(h(\mathbf{a})) = r^{\mathfrak{A}}(\mathbf{a})$ and, similarly, $r^{\mathfrak{B}}(\mathbf{b}') = r^{\mathfrak{A}}(\mathbf{a}')$. Thus, we have

$$\begin{aligned} \textstyle\bigwedge_{i=1}^n \Theta'(b_i, b_i') \wedge r^{\mathfrak{B}}(\mathbf{b}) &= \textstyle\bigwedge_{i=1}^n \Theta(a_i, a_i') \wedge r^{\mathfrak{A}}(\mathbf{a}) \\ &\qquad (\text{Definition of } \Theta' \text{ and } h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ &\leq r^{\mathfrak{A}}(\mathbf{a}') \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ &= r^{\mathfrak{B}}(\mathbf{b}'). \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \end{aligned}$$

Hence, $\Theta' \in \mathrm{Gon}(\mathfrak{B})$.

■

We can also show that Leibniz G-congruences commute with inverse reductive morphisms. This property abstracts a similar property in the setting of ordinary first order structures that was presented in Theorem 2.5 of [23].

Corollary 134 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Then*

$$\Omega(\mathfrak{A}) = \Omega(\mathfrak{B}) \circ h.$$

Proof: Let us set $\Theta := \Omega(\mathfrak{B}) \circ h$. By Part (a) of Lemma 133, $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Hence, by the maximality property of $\Omega(\mathfrak{A})$, we get $\Omega(\mathfrak{B}) \circ h = \Theta \leq \Omega(\mathfrak{A})$.

Assume, conversely, taking into account Lemma 131, that $\Theta' \in \mathrm{Gon}(\mathfrak{B})$ is the G-congruence obtained by applying Part (b) of Lemma 133 to the G-congruence $\Omega(\mathfrak{A}) \in \mathrm{Gon}(\mathfrak{A})$. Then, by the lemma and the maximality of $\Omega(\mathfrak{B})$, we have

$$\Theta' \circ h = \Omega(\mathfrak{A}) \quad \text{and} \quad \Theta' \leq \Omega(\mathfrak{B}).$$

Hence, $\Omega(\mathfrak{A}) = \Theta' \circ h \leq \Omega(\mathfrak{B}) \circ h$. We conclude that $\Omega(\mathfrak{B}) \circ h = \Omega(\mathfrak{A})$. ■

4.7 Leibniz Equality

A **Leibniz formula over $\mathcal{L}$** is a formula $\psi(x, y)$, with two free variables, that has the form

$$\psi(x, y) := \forall \bar{z}(\varphi(x, \bar{z}) \leftrightarrow \varphi(y, \bar{z})),$$

for some atomic $\mathcal{L}$-formula $\varphi(x, \bar{z})$, with at least one free variable x.

Using Leibniz formulas, we may obtain characterizations of the Leibniz G-congruence on a given $\mathcal{L}$-structure.

Theorem 135 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. Then, for all $a, b \in A$, we have*

$$\Omega(\mathfrak{A})(a, b) = \bigwedge \{\psi^{\mathfrak{A}}(a, b) : \psi \text{ Leibniz}\}.$$

Proof: Let us define

$$\Theta := \bigwedge \{\psi^{\mathfrak{A}}(a, b) : \psi \text{ Leibniz}\}.$$

Suppose, first, that $a, b \in A$ and let $\varphi(x, \bar{z})$ be an atomic formula and $\mathbf{c}$ in A. Then, using compatibility of $\Omega(\mathfrak{A})$ with $R^{\mathfrak{A}}$, we get

$$\Omega(\mathfrak{A})(a, b) \leq \varphi^{\mathfrak{A}}(a, \mathbf{c}) \leftrightarrow \varphi^{\mathfrak{A}}(b, \mathbf{c}).$$

Thus, for every Leibniz formula $\psi(x, y)$,

$$\Omega(\mathfrak{A})(a, b) \leq \psi^{\mathfrak{A}}(a, b).$$

This yields

$$\Omega(\mathfrak{A})(a,b) \leq \bigwedge\{\psi^{\mathfrak{A}}(a,b) : \psi \text{ Leibniz}\},$$

i.e., $\Omega(\mathfrak{A}) \leq \Theta$.

For the reverse inequality, by the maximality property of $\Omega(\mathfrak{A})$, it suffices to show that Θ is a G-congruence on $\mathfrak{A}$. As done previously, we show Transitivity, Congruence, Inclusion of $E^{\mathfrak{A}}$ and Compatibility.

Let $a, b, c \in A$. Then

$$\begin{array}{rcl} \Theta(a,b) \wedge \Theta(b,c) & = & \bigwedge_\psi \psi^{\mathfrak{A}}(a,b) \wedge \bigwedge_\psi \psi^{\mathfrak{A}}(b,c) \quad \text{(Definition of } \Theta) \\ & = & \bigwedge_\psi (\psi^{\mathfrak{A}}(a,b) \wedge \psi^{\mathfrak{A}}(b,c)) \quad \text{(Property of meet)} \\ & \leq & \bigwedge_\psi \psi^{\mathfrak{A}}(a,c) \quad \text{(Property of } \leftrightarrow) \\ & = & \Theta(a,c). \quad \text{(Definition of } \Theta) \end{array}$$

Next, suppose $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then we get

$$\begin{array}{rcl} \bigwedge_{i=1}^n \Theta(a_i, b_i) & = & \bigwedge_{i=1}^n \bigwedge_\psi \psi^{\mathfrak{A}}(a_i, b_i) \quad \text{(Definition of } \Theta) \\ & \leq & \bigwedge_\psi \bigwedge_{i=1}^n \psi^{\mathfrak{A}}(f^{\mathbf{A}}(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n), \\ & & \qquad f^{\mathbf{A}}(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n)) \\ & & \quad \text{(Meets over special formulas)} \\ & \leq & \bigwedge_\psi \psi^{\mathfrak{A}}(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \quad \text{(Property of } \leftrightarrow) \\ & = & \Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})). \quad \text{(Definition of } \Theta) \end{array}$$

Next, recall that all relations of $\mathfrak{A}$ are compatible with the reduced G-congruence $E^{\mathfrak{A}}$. Thus, for every atomic $\varphi(x, \bar{z})$ and all $a, b, \mathbf{c} \in A$,

$$E^{\mathfrak{A}}(a,b) \leq \varphi^{\mathfrak{A}}(a, \mathbf{c}) \leftrightarrow \varphi^{\mathfrak{A}}(b, \mathbf{c}).$$

This yields, that, for every Leibniz $\mathcal{L}$-formula ψ, $E^{\mathfrak{A}}(a,b) \leq \psi^{\mathfrak{A}}(a,b)$ and, therefore, $E^{\mathfrak{A}} \leq \Theta$.

Finally, suppose $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then we get

$$\begin{array}{rcl} \bigwedge_{i=1}^n \Theta(a_i, b_i) & = & \bigwedge_{i=1}^n \bigwedge_\psi \psi^{\mathfrak{A}}(a_i, b_i) \quad \text{(Definition of } \Theta) \\ & \leq & \bigwedge_{i=1}^n r^{\mathfrak{A}}(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n) \\ & & \qquad \leftrightarrow r^{\mathfrak{A}}(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n) \\ & & \quad \text{(Meets over special formulas)} \\ & \leq & r^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow r^{\mathfrak{A}}(\mathbf{b}). \quad \text{(Property of } \leftrightarrow) \end{array}$$

We conclude that $\Theta \leq \Omega(\mathfrak{A})$. Now the characterization of the statement follows. ∎

Applying induction on the structure of formulas, we obtain the following consequence.

Corollary 136 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure. Then, for all* $a, b \in A$*, we have*

$$\Omega(\mathfrak{A})(a,b) = \bigwedge\{\varphi^{\mathfrak{A}}(a, \mathbf{c}) \leftrightarrow \varphi^{\mathfrak{A}}(b, \mathbf{c}) : \varphi(x, \bar{z}) \text{ an } \mathcal{L}\text{-formula, } \mathbf{c} \text{ in } A\}.$$

4.8 Quotient Structures and Morphisms

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure and suppose $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Define $\hat{\Theta} = \{\hat{\Theta}_g\}_{g \in G}$, where, for all $g \in G$, $\hat{\Theta}_g \subseteq A^2$ is given by

$$\hat{\Theta}_g = \{\langle a, b \rangle \in A : \Theta(a, b) \geq g\}.$$

Lemma 137 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Then, for all $g \in G$, $\hat{\Theta}_g$ is a congruence of the algebra $\mathbf{A}$.*

Proof: Since, for all $a \in A$, $\Theta(a, a) = \top \geq g$, we get $\langle a, a \rangle \in \hat{\Theta}_g$, whence $\hat{\Theta}_g$ is reflexive.

If $\langle a, b \rangle \in \hat{\Theta}_g$, then $\Theta(b, a) = \Theta(a, b) \geq g$. Hence, $\langle b, a \rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is symmetric.

If $\langle a, b \rangle, \langle b, c \rangle \in \hat{\Theta}_g$, then

$$\Theta(a, c) \geq \Theta(a, b) \wedge \Theta(b, c) \geq g,$$

whence $\langle a, c \rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is also transitive.

Finally, if $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$, such that $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \hat{\Theta}_g$, then

$$\Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \geq \bigwedge_{i=1}^{n} \Theta(a_i, b_i) \geq g.$$

Hence, $\langle f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{a}}(\mathbf{b}) \rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is a congruence on $\mathbf{A}$. ∎

$\hat{\Theta}$ is called the **stratified congruence associated with** the G-congruence Θ. More generally, a **stratified congruence on A** is any G-indexed family of congruences $\{\theta_g\}_{g \in G}$, such that, for all $g, g' \in G$,

$$g \leq g' \quad \text{implies} \quad \theta_{g'} \subseteq \theta_g.$$

We call θ_g the g-**stratum** of the stratified congruence. By slightly overloading notation, we shall write $\hat{\Theta}$ to denote $\hat{\Theta}_\top$ as well.

Consider the quotient algebra $\mathbf{A}/\hat{\Theta}$. Define the tuple

$$\mathfrak{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta}, R^{\mathfrak{A}/\Theta} \rangle$$

by setting:

- For all $a, b \in A$,
$$\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta(a, b);$$
- $R^{\mathfrak{A}/\Theta} = \{r^{\mathfrak{A}/\Theta} : r \in R\}$, where, for all $r \in R$, with $\rho(r) = n$, and all $a_1, \ldots, a_n \in A$,
$$r^{\mathfrak{A}/\Theta}(a_1/\hat{\Theta}, \ldots, a_n/\hat{\Theta}) = r^{\mathfrak{A}}(a_1, \ldots, a_n).$$

These definitions do not depend on the chosen representatives. Assume, first, that $a, a', b, b' \in A$, such that $\langle a, a'\rangle, \langle b, b'\rangle \in \hat{\Theta}$. Then

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a,b) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \Theta(a',a) \wedge \Theta(a,b) \wedge \Theta(b,b') \quad (\Theta(a',a) = \Theta(b,b') = \top) \\ & \leq & \Theta(a',b') \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a'/\hat{\Theta}, b'/\hat{\Theta}). \quad (\text{Definition of } \bar{\Theta})\end{array}$$

Thus, $\bar{\Theta}$ is well defined. Further, if $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$, such that $\langle a_i, b_i\rangle \in \hat{\Theta}$, for all $i = 1, \ldots, n$, then

$$\begin{array}{rcl}r^{\mathfrak{A}/\Theta}(a_1/\hat{\Theta}, \ldots, a_n/\hat{\Theta}) & = & r^{\mathfrak{A}}(a_1, \ldots, a_n) \quad (\text{Definition of } r^{\mathfrak{A}/\Theta}) \\ & = & \bigwedge_{i=1}^n \Theta(a_i, b_i) \wedge r^{\mathfrak{A}}(a_1, \ldots, a_n) \quad (\Theta(a_i, b_i) = \top) \\ & \leq & r^{\mathfrak{A}}(b_1, \ldots, b_n) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{A}/\Theta}(b_1/\hat{\Theta}, \ldots, b_n/\hat{\Theta}). \quad (\text{Definition of } r^{\mathfrak{A}/\Theta})\end{array}$$

Hence $R^{\mathfrak{A}/\Theta}$ is also well defined.

Lemma 138 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *be an* $\mathcal{L}$*-structure and* $\Theta \in \mathrm{Gon}(\mathfrak{A})$*. Then* $\mathfrak{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta}, R^{\mathfrak{A}/\Theta}\rangle$ *is an* $\mathcal{L}$*-structure.*

Proof: We must show that $\bar{\Theta}$ is a reduced G-congruence and that $\bar{\Theta}$ is compatible with all operations in $F^{\mathbf{A}/\Theta}$ and all G-relations in $R^{\mathfrak{A}/\Theta}$.

Let $a, a \in A$. Then

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, a/\hat{\Theta}) & = & \Theta(a,a) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \top. \quad (\Theta \in \mathrm{Gon}(\mathfrak{A}))\end{array}$$

Let $a, b \in A$. Then

$$\begin{array}{rcl}\bar{\Theta}(b/\hat{\Theta}, a/\hat{\Theta}) & = & \Theta(b,a) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \Theta(a,b) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}). \quad (\text{Definition of } \bar{\Theta})\end{array}$$

Let $a, b, c \in A$. Then

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{\Theta}(b/\hat{\Theta}, c/\hat{\Theta}) & = & \Theta(a,b) \wedge \Theta(b,c) \quad (\text{Definition of } \bar{\Theta}) \\ & \leq & \Theta(a,c) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, c/\hat{\Theta}). \quad (\text{Definition of } \bar{\Theta})\end{array}$$

Let $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcl}\bigwedge_{i=1}^n \bar{\Theta}(a_i/\hat{\Theta}, b_i/\hat{\Theta}) & = & \bigwedge_{i=1}^n \Theta(a_i, b_i) \quad (\text{Definition of } \bar{\Theta}) \\ & \leq & \Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta}, f^{\mathbf{A}}(\mathbf{b}/\hat{\Theta})) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \bar{\Theta}(f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}), f^{\mathbf{A}/\Theta}(\mathbf{b}/\hat{\Theta})). \\ & & \quad (\text{Definition of } \mathbf{A}/\Theta)\end{array}$$

To see that $\bar{\Theta}$ is reduced, suppose $a, b \in A$, such that $\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \top$. Then $\Theta(a, b) = \top$. But this gives $\langle a, b\rangle \in \hat{\Theta}$. Thus, $a/\hat{\Theta} = b/\hat{\Theta}$. Therefore, $\bar{\Theta}$ is a reduced G-congruence on $\mathbf{A}/\Theta$. To show compatibility with the G-relations, suppose $r \in R$, with $\rho(r) = n$, and let $a_1, b_1, \dots, a_n, b_n \in A$. Then

$$\begin{array}{rcl} \bigwedge_{i=1}^n \bar{\Theta}(\mathbf{a}/\hat{\Theta}) \wedge r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & = & \bigwedge_{i=1}^n \Theta(\mathbf{a}) \wedge r^{\mathfrak{A}}(\mathbf{a}) \\ & & \quad \text{(Definitions of } \bar{\Theta} \text{ and } r^{\mathfrak{A}/\Theta}) \\ & \le & r^{\mathfrak{A}}(\mathbf{b}) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta}). \quad \text{(Definition of } r^{\mathfrak{A}/\Theta}) \end{array}$$

■

We call $\mathfrak{A}/\Theta$ the **quotient structure of $\mathfrak{A}$ by Θ**.

We define the **natural** or **canonical projection morphism**, or **quotient morphism**, $\pi_\Theta : A \to A/\hat{\Theta}$ by setting, for all $a \in A$,

$$\pi_\Theta(a) = a/\hat{\Theta}.$$

Then the following proposition applies, which forms a kind of converse to Lemma 131.

Proposition 139 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-structure and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Then $\pi_\Theta : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\Theta$ is a reductive morphism, with $\mathrm{Ker}(\pi_\Theta) = \Theta$.*

Proof: First, let us show that π_Θ is an algebra homomorphism. Let $f \in F$, with $\rho(f) = n$, and $a_1, \dots, a_n \in A$. We have

$$\begin{array}{rcl} \pi_\Theta(f^{\mathbf{A}}(\mathbf{a})) & = & f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta} \quad \text{(Definition of } \pi_\Theta) \\ & = & f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad \text{(Definition of } f^{\mathbf{A}/\Theta}) \\ & = & f^{\mathbf{A}/\Theta}(\pi_\Theta(\mathbf{a})). \quad \text{(Definition of } \pi_\Theta) \end{array}$$

Next, we show that it is, in fact, a G-algebra morphism. Let $a, b \in A$. We have

$$\begin{array}{rcl} E^{\mathfrak{A}}(a, b) & \le & \Theta(a, b) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad \text{(Definition of } \bar{\Theta}) \\ & = & \bar{\Theta}(\pi_\Theta(a), \pi_\Theta(b)). \quad \text{(Definition of } \pi_\Theta) \end{array}$$

To see that it is a reductive morphism of $\mathcal{L}$-structures, suppose $r \in R$, with $\rho(r) = n$, and $a_1, \dots, a_n \in A$. Then

$$\begin{array}{rcl} r^{\mathfrak{A}}(\mathbf{a}) & = & r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad \text{(Definition of } r^{\mathfrak{A}/\Theta}) \\ & = & r^{\mathfrak{A}/\Theta}(\pi_\Theta(\mathbf{a})). \quad \text{(Definition of } \pi_\Theta) \end{array}$$

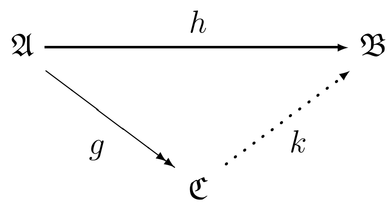
Finally, for the last claim in the statement, we have, for all $a, b \in A$,

$$\begin{array}{rcl} \mathrm{Ker}(\pi_\Theta)(a, b) & = & \bar{\Theta}(\pi_\Theta(a), \pi_\Theta(b)) \quad \text{(Definition of } \mathrm{Ker}(\pi_\Theta)) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad \text{(Definition of } \pi_\Theta) \\ & = & \Theta(a, b). \quad \text{(Definition of } \bar{\Theta}) \end{array}$$

So $\mathrm{Ker}(\pi_\Theta) = \Theta$ and all claims in the statement are proven. ■

Our next goal is to prove a series of Homomorphism Theorems paralleling the ones in Universal Algebra. On our way we establish a lemma.

Lemma 140 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *and* $\mathfrak{C} = \langle \mathbf{C}, E^{\mathfrak{C}}, R^{\mathfrak{C}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \to \mathfrak{B}$ *and* $g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$*, such that* $\mathrm{Ker}(g) \le \mathrm{Ker}(h)$*. Then, there exists unique* $k : \mathfrak{C} \to \mathfrak{B}$*, such that* $h = k \circ g$*.*



Moreover, the morphism h *is strict if and only if* k *is strict.*

Proof: To define k, let $c \in C$. Then, since g is surjective, there exist $a \in A$, such that $g(a) = c$. We let

$$k(c) = h(a).$$

To see this is well defined, note that, if $a_1, a_2 \in A$ are such that $g(a_1) = g(a_2) = c$, then, since $\mathrm{Ker}(g) \le \mathrm{Ker}(h)$,

$$\top = E^{\mathfrak{C}}(g(a_1), g(a_2)) \le E^{\mathfrak{B}}(h(a_1), h(a_2)).$$

Hence, $E^{\mathfrak{B}}(h(a_1), h(a_2)) = \top$ and, since $E^{\mathfrak{B}}$ is reduced, $h(a_1) = h(a_2)$. So k is well defined.

Next, we show that k is an algebra homomorphism $k : \mathbf{C} \to \mathbf{B}$. Indeed, if $f \in F$, with $\rho(f) = n$, and $c_1, \ldots, c_n \in C$, such that $c_i = g(a_i)$, $i = 1, \ldots, n$, for some $a_1, \ldots, a_n \in A$, then we have $g(f^{\mathbf{A}}(\mathbf{a})) = f^{\mathbf{C}}(g(\mathbf{a})) = f^{\mathbf{C}}(\mathbf{c})$, whence

$$\begin{array}{rcll} f^{\mathbf{B}}(k(\mathbf{c})) & = & f^{\mathbf{B}}(h(\mathbf{a})) & (\text{Definition of } k \text{ and } \mathbf{c} = g(\mathbf{a})) \\ & = & h(f^{\mathbf{A}}(\mathbf{a})) & (h : \mathbf{A} \to \mathbf{B}) \\ & = & k(f^{\mathbf{C}}(\mathbf{c})). & (\text{Definition of } k \text{ and } f^{\mathbf{C}}(\mathbf{c}) = g(f^{\mathbf{A}}(\mathbf{a}))) \end{array}$$

Further, for all $c_1, c_2 \in C$, such that $g(a_1) = c_1$ and $g(a_2) = c_2$, for some $a_1, a_2 \in A$,

$$\begin{array}{rcll} E^{\mathfrak{C}}(c_1, c_2) & = & E^{\mathfrak{C}}(g(a_1), g(a_2)) & (\text{Hypothesis}) \\ & \le & E^{\mathfrak{B}}(h(a_1), h(a_2)) & (\mathrm{Ker}(g) \le \mathrm{Ker}(h)) \\ & = & E^{\mathfrak{B}}(k(c_1), k(c_2)). & (\text{Definition of } k) \end{array}$$

Hence k is a G-algebra morphism.

Next, we show that $k : \mathfrak{C} \to \mathfrak{A}$. So suppose $r \in R$, with $\rho(r) = n$, and let $c_1, \ldots, c_n \in C$, such that $c_i = g(a_i)$, $i = 1, \ldots, n$, for some $a_1 \ldots, a_n \in A$. Then we have

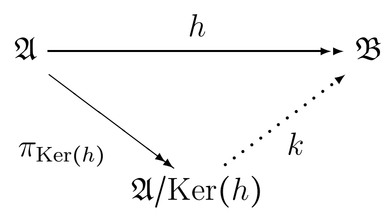
$$\begin{array}{rcll} r^{\mathfrak{C}}(\mathbf{c}) & = & r^{\mathfrak{C}}(g(\mathbf{a})) & (\mathbf{c} = g(\mathbf{a})) \\ & = & r^{\mathfrak{A}}(\mathbf{a}) & (g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}) \\ & \le & r^{\mathfrak{B}}(h(\mathbf{a})) & (h : \mathfrak{A} \to \mathfrak{B}) \\ & = & r^{\mathfrak{B}}(k(\mathbf{c})). & (\text{Definition of } k) \end{array}$$

So $k : \mathfrak{C} \to \mathfrak{A}$. From this sequence, we can deduce that, if h is strict, then so is k. And the converse of this is straightforward by the fact that $h = k \circ g$. So h is strict if and only if k is strict. ∎

We are now ready for the sequence of Homomorphism Theorems.

Theorem 141 (Homomorphism) *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*. Then* $\mathfrak{A}/\mathrm{Ker}(h) \cong \mathfrak{B}$*.*

Proof: By Lemma 131, $\mathrm{Ker}(h) \in \mathrm{Gon}(\mathfrak{A})$. By Lemma 138, $\mathfrak{A}/\mathrm{Ker}(h)$ is an $\mathcal{L}$-structure and, by Lemma 139, $\pi_{\mathrm{Ker}(h)} : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\mathrm{Ker}(h)$ is a reductive morphism. Consider now the following diagram, where $k : \mathfrak{A}/\mathrm{Ker}(h) \to_s \mathfrak{B}$ is the strict morphism guaranteed by Lemma 140.



It now suffices to show that k is a bijection, and that it preserves the reduced congruences. That k is surjective follows from the fact that h is surjective. For injectivity, suppose $a, a' \in A$, such that $k(a/\widehat{\mathrm{Ker}(h)}) = k(a'/\widehat{\mathrm{Ker}(h)})$. Then, by definition, $h(a) = h(a')$. Hence, $E^{\mathfrak{B}}(h(a), h(a')) = \top$. By definition of $\mathrm{Ker}(h)$, $\mathrm{Ker}(h)(a, a') = \top$. Therefore, $a/\widehat{\mathrm{Ker}(h)} = a'/\widehat{\mathrm{Ker}(h)}$. So k is indeed one-to-one.

Finally, for $a, a' \in A$, we have

$$
\begin{aligned}
E^{\mathfrak{A}/\mathrm{Ker}(h)}(a/\widehat{\mathrm{Ker}(h)}, a'/\widehat{\mathrm{Ker}(h)}) &= \overline{\mathrm{Ker}(h)}(a/\widehat{\mathrm{Ker}(h)}, a'/\widehat{\mathrm{Ker}(h)}) \\
&\quad \text{(Definition of } E^{\mathfrak{A}/\mathrm{Ker}(h)}) \\
&= \mathrm{Ker}(h)(a, a') \\
&\quad \text{(Definition of } \overline{\mathrm{Ker}(h)}) \\
&= E^{\mathfrak{B}}(h(a), h(a')) \\
&\quad \text{(Definition of } \mathrm{Ker}(h)) \\
&= E^{\mathfrak{B}}(k(a/\widehat{\mathrm{Ker}(h)}), k(a'/\widehat{\mathrm{Ker}(h)})). \\
&\quad \text{(Definition of } k)
\end{aligned}
$$

Therefore, $k : \mathfrak{A}/\mathrm{Ker}(h) \cong \mathfrak{B}$. ∎

The next theorem in the series is the analog of the well known Second Isomorphism Theorem of Universal Algebra.

Theorem 142 (Second Isomorphism) *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure and* $\Theta, \Theta' \in \mathrm{Gon}(\mathfrak{A})$*, such that* $\Theta \leq \Theta'$*. Then the function*

$$\pi : (A/\hat{\Theta})/(\hat{\Theta}'/\hat{\Theta}) \to A/\hat{\Theta}',$$

defined, for all $a \in A$, *by*

$$\pi((a/\hat{\Theta})/(\hat{\Theta}'/\hat{\Theta})) = a/\hat{\Theta}',$$

establishes an isomorphism $\pi : (\mathfrak{A}/\Theta)/(\Theta'/\Theta) \cong \mathfrak{A}/\Theta'$.

Proof: Since $\Theta \in \mathrm{Gon}(\mathfrak{A})$, by Lemma 138, $\mathfrak{A}/\Theta$ is an $\mathcal{L}$-structure. On $A/\hat{\Theta}$, define the G-relation $\Theta'/\Theta : (A/\hat{\Theta})^2 \to G$ by setting, for all $a, b \in A$,

$$(\Theta'/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta'(a, b).$$

To see that Θ'/Θ is well defined, let $a, a', b, b' \in A$, such that $\langle a, a'\rangle \in \hat{\Theta}$ and $\langle b, b'\rangle \in \hat{\Theta}$. This means that $\Theta(a', a) = \top$ and $\Theta(b, b') = \top$. Thus, we get

$$\begin{array}{rcl}\Theta'(a,b) & = & \Theta(a',a) \wedge \Theta'(a,b) \wedge \Theta(b,b') \\ & & \quad (\Theta(a',a) = \top \text{ and } \Theta(b,b') = \top) \\ & \le & \Theta'(a',a) \wedge \Theta'(a,b) \wedge \Theta'(b,b') \quad (\Theta \le \Theta') \\ & \le & \Theta'(a',b'). \quad (\Theta' \in \mathrm{Gon}(\mathfrak{A}))\end{array}$$

By symmetry, $\Theta'(a, b) = \Theta'(a', b')$, whence Θ'/Θ is well defined. Since $\Theta' \in \mathrm{Gon}(\mathfrak{A})$, it is not difficult to see that $\Theta'/\Theta \in \mathrm{Gon}(\mathfrak{A}/\Theta)$. Let us show Congruence, Inclusion of $E^{\mathfrak{A}/\Theta}$ and Compatibility.

Suppose $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \dots, a_n, b_n \in A$. Then

$$\begin{array}{rcl}\bigwedge_{i=1}^n (\Theta'/\Theta)(a_i/\hat{\Theta}, b_i/\hat{\Theta}) & = & \bigwedge_{i=1}^n \Theta'(a_i, b_i) \quad (\text{Definition of } \Theta'/\Theta) \\ & \le & \Theta'(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \quad (\Theta' \in \mathrm{Gon}(\mathfrak{A})) \\ & = & (\Theta'/\Theta)(f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta}, f^{\mathbf{A}}(\mathbf{b})/\hat{\Theta}) \\ & & \quad (\text{Definition of } \Theta'/\Theta) \\ & = & (\Theta'/\Theta)(f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}), f^{\mathbf{A}/\Theta}(\mathbf{b}/\hat{\Theta})). \\ & & \quad (\text{Definition of } f^{\mathbf{A}/\Theta})\end{array}$$

Next, let $a, b \in A$. We obtain

$$\begin{array}{rcl}E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad (\text{Definition of } E^{\mathfrak{A}/\Theta}) \\ & = & \Theta(a,b) \quad (\text{Definition of } \bar{\Theta}) \\ & \le & \Theta'(a,b) \quad (\Theta \le \Theta') \\ & = & (\Theta'/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}). \quad (\text{Definition of } \Theta'/\Theta)\end{array}$$

Finally, consider $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \dots, a_n, b_n \in A$. Then

$$\begin{array}{rcl}\bigwedge_{i=1}^n (\Theta'/\Theta)(a_i/\hat{\Theta}, b_i/\hat{\Theta}) \wedge r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & = & \bigwedge_{i=1}^n \Theta'(a_i, b_i) \wedge r^{\mathfrak{A}}(\mathbf{a}) \\ & & (\text{Definitions of } \Theta'/\Theta \text{ and } r^{\mathfrak{A}/\Theta}) \\ & \le & r^{\mathfrak{A}}(\mathbf{b}) \quad (\Theta' \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta}). \quad (\text{Definition of } r^{\mathfrak{A}/\Theta})\end{array}$$

Now, by Proposition 139, we get that $\pi_{\Theta'/\Theta} : \mathfrak{A}/\Theta \twoheadrightarrow_s (\mathfrak{A}/\Theta)/(\Theta'/\Theta)$ is a reductive morphism.

Define, next, $h : A/\hat{\Theta} \to A/\hat{\Theta}'$ by setting, for all $a \in A$,

$$h(a/\hat{\Theta}) = a/\hat{\Theta}'.$$

We show that this is a reductive morphism $h : \mathfrak{A}/\Theta \twoheadrightarrow_s \mathfrak{A}/\Theta'$. It is definitely well defined, since $\langle a, b\rangle \in \hat{\Theta}$ implies $\top = \Theta(a,b) \le \Theta'(a,b)$ and, hence, $\langle a, b\rangle \in \hat{\Theta}'$. By Universal Algebra, it is an algebra homomorphism. Further, for all $a, b \in A$,

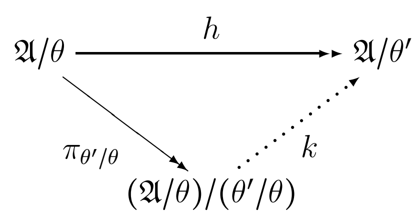
$$\begin{array}{rcll} E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } E^{\mathfrak{A}/\Theta}) \\ & = & \Theta(a,b) & (\text{Definition of } \bar{\Theta}) \\ & \le & \Theta'(a,b) & (\Theta \le \Theta') \\ & = & \bar{\Theta}'(a/\hat{\Theta}', b/\hat{\Theta}') & (\text{Definition of } \bar{\Theta}') \\ & = & E^{\mathfrak{A}/\Theta'}(h(a/\hat{\Theta}), h(b/\hat{\Theta})). & (\text{Definition of } E^{\mathfrak{A}/\Theta'}) \end{array}$$

Hence, h is a G-algebra morphism. Finally, for all $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in A$,

$$\begin{array}{rcll} r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & = & r^{\mathfrak{A}}(\mathbf{a}) & (\text{Definition of } r^{\mathfrak{A}/\Theta}) \\ & = & r^{\mathfrak{A}/\Theta'}(\mathbf{a}/\hat{\Theta}') & (\text{Definition of } r^{\mathfrak{A}/\Theta'}) \\ & = & r^{\mathfrak{A}/\Theta'}(h(\mathbf{a}/\hat{\Theta})). & (\text{Definition of } h) \end{array}$$

Since h is surjective and preserves the value of the G-relations, it is indeed a reductive morphism $h : \mathfrak{A}/\Theta \twoheadrightarrow_s \mathfrak{A}/\Theta'$.

We have now the following diagram with given the two solid reductive morphisms.



By Theorem 141, we get a unique isomorphism $k : (\mathfrak{A}/\Theta)/(\Theta'/\Theta) \cong \mathfrak{A}/\Theta'$, such that $k \circ \pi_{\Theta'/\Theta} = h$, i.e., such that, for all $a \in A$,

$$k((a/\hat{\Theta})/(\hat{\Theta}'/\hat{\Theta})) = a/\hat{\Theta}'.$$

This concludes the proof. ∎

In order to formalize the Third Isomorphism Theorem, one has to introduce a couple of additional notions, which we have partially encountered previously. We start with an $\mathcal{L}$-structure $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ and consider $B \subseteq A$ and a G-congruence Θ on $\mathfrak{A}$. We augment B so as to be "closed under $\hat{\Theta}$-classes", obtaining

$$B^{\Theta} = \{a \in A : B \cap a/\hat{\Theta} \ne \varnothing\}.$$

Then take $\mathfrak{B}^{\Theta}$ to be the substructure of $\mathfrak{A}$ generated by B^{Θ}. As before, write $\Theta_B = \Theta{\restriction}_B$.

Lemma 143 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures, such that $\mathfrak{B} \subseteq \mathfrak{A}$ and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. The universe of $\mathfrak{B}^\Theta$ is B^Θ.*

Proof: Suppose $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in B^\Theta$. By definition, there exist $b_1, \ldots, b_n \in A$, such that $b_i \in B \cap a_i/\hat{\Theta}$, for all $i = 1, \ldots, n$. Now we have

$$\begin{array}{rcl} B \cap f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta} & = & B \cap f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad \text{(Definition of } f^{\mathbf{A}/\Theta}\text{)} \\ & = & B \cap f^{\mathbf{A}/\Theta}(\mathbf{b}/\hat{\Theta}) \quad (\mathbf{a}/\hat{\Theta} = \mathbf{b}/\hat{\Theta}) \\ & = & B \cap f^{\mathbf{A}}(\mathbf{b})/\hat{\Theta} \quad \text{(Definition of } f^{\mathbf{A}/\Theta}\text{)} \\ & \neq & \varnothing. \quad (\mathfrak{B} \subseteq \mathfrak{A}) \end{array}$$

Thus, $f^{\mathbf{A}/\Theta}(\mathbf{a}) \in B^\Theta$ and, hence, B^Θ is a subuniverse of $\mathfrak{A}$. ∎

Now we have available the machinery and notation needed to formulate the analog of the Third Isomorphism Theorem.

Theorem 144 (Third Isomorphism) *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures, such that $\mathfrak{B} \subseteq \mathfrak{A}$, and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Then*

$$\mathfrak{B}/\Theta_B \cong \mathfrak{B}^\Theta/\Theta_{B^\Theta}.$$

Proof: We define a mapping $\pi : B/\hat{\Theta}_B \to B^\Theta/\hat{\Theta}_{B^\Theta}$ by setting, for all $b \in B$,

$$\pi(b/\hat{\Theta}_B) = b/\hat{\Theta}_{B^\Theta}.$$

We must show that $\pi : \mathfrak{B}/\Theta_B \cong \mathfrak{B}^\Theta/\Theta_{B^\Theta}$. We know that π gives an isomorphism of the underlying algebras. Moreover, it is an isomorphism of G-algebras, since, for all $b_1, b_2 \in B$,

$$\begin{array}{rcl} E^{\mathfrak{B}/\Theta_B}(b_1/\hat{\Theta}_B, b_2/\hat{\Theta}_B) & = & \bar{\Theta}_B(b_1/\hat{\Theta}_B, b_2/\hat{\Theta}_B) \quad \text{(Definition of } E^{\mathfrak{B}/\Theta_B}\text{)} \\ & = & \Theta_B(b_1, b_2) \quad \text{(Definition of } \bar{\Theta}_B\text{)} \\ & = & \Theta(b_1, b_2) \quad \text{(Definition of } \Theta_B\text{)} \\ & = & \Theta_{B^\Theta}(b_1, b_2) \quad \text{(Definition of } \Theta_{B^\Theta}\text{)} \\ & = & \bar{\Theta}_{B^\Theta}(b_1/\hat{\Theta}_{B^\Theta}, b_2/\hat{\Theta}_{B^\Theta}) \quad \text{(Definition of } \bar{\Theta}_{B^\Theta}\text{)} \\ & = & E^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}(\pi(b_1/\hat{\Theta}_B), \pi(b_2/\hat{\Theta}_B)). \\ & & \quad \text{(Definition of } E^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}\text{)} \end{array}$$

To see that it is a strict homomorphism of $\mathcal{L}$-structures, note that, for all $r \in R$, with $\rho(r) = n$, and all $b_1, \ldots, b_n \in B$,

$$\begin{array}{rcl} r^{\mathfrak{B}/\Theta_B}(\mathbf{b}/\Theta_B) & = & r^{\mathfrak{B}}(\mathbf{b}) \quad \text{(Definition of } r^{\mathfrak{B}/\Theta_B}\text{)} \\ & = & r^{\mathfrak{A}}(\mathbf{b}) \quad (\mathfrak{B} \subseteq \mathfrak{A}) \\ & = & r^{\mathfrak{B}^\Theta}(\mathbf{b}) \quad (\mathfrak{B}^\Theta \subseteq \mathfrak{A}) \\ & = & r^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}(\mathbf{b}/\Theta_{B^\Theta}). \quad \text{(Definition of } r^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}\text{)} \end{array}$$

Thus, π is a strict homomorphism. ∎

For the Correspondence Theorem, we need the interval notation for lattices. For $[a, b]$ a closed interval of a lattice $\mathbf{L}$, where $a \leq b$, we define $\mathbf{[}a, b\mathbf{]}$ to be the corresponding sublattice of $\mathbf{L}$.

Theorem 145 (Correspondence) *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Then the mapping α defined on the interval $[\Theta, \Omega(\mathfrak{A})]$ of $\mathbf{Gon}(\mathfrak{A})$ by*

$$\alpha(\Theta') = \Theta'/\Theta,$$

is a lattice isomorphism from $[\Theta, \Omega(\mathfrak{A})]$ to $\mathbf{Gon}(\mathfrak{A}/\Theta)$, where $[\Theta, \Omega(\mathfrak{A})]$ is the interval sublattice of $\mathbf{Gon}(\mathfrak{A})$.

Proof: By Theorem 142, the mapping α is well defined. If $\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)$, then we define $\beta(\Theta'') : A^2 \to G$ by setting, for all $a, b \in A$,

$$\beta(\Theta'')(a, b) = \Theta''(a/\hat{\Theta}, b/\hat{\Theta}).$$

We show that $\beta(\Theta'') \in \mathrm{Gon}(\mathfrak{A})$ and that $\beta(\Theta'')/\Theta = \Theta''$. We skip Reflexivity, Symmetry and Transitivity and show Congruence, Inclusion of $E^{\mathfrak{A}}$ and Compatibility with $R^{\mathfrak{A}}$. Let $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcll} \bigwedge_{i=1}^n \beta(\Theta'')(a_i, b_i) & = & \bigwedge_{i=1}^n \Theta''(a_i/\hat{\Theta}, b_i/\hat{\Theta}) & (\text{Definition of } \beta(\Theta'')) \\ & \leq & \Theta''(f^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}), f^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta})) & (\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)) \\ & = & \Theta''(f^{\mathfrak{A}}(\mathbf{a})/\hat{\Theta}, f^{\mathfrak{A}}(\mathbf{b})/\hat{\Theta}) & (\text{Definition of } f^{\mathfrak{A}/\Theta}) \\ & = & \beta(\Theta'')(f^{\mathfrak{A}}(\mathbf{a}), f^{\mathfrak{A}}(\mathbf{b})). & (\text{Definition of } \beta(\Theta'')) \end{array}$$

Next, let $a, b \in A$. We have

$$\begin{array}{rcll} E^{\mathfrak{A}}(a, b) & \leq & \Theta(a, b) & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } \bar{\Theta}) \\ & = & E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } E^{\mathfrak{A}/\Theta}) \\ & \leq & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}). & (\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)) \end{array}$$

Now consider $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcl} \bigwedge_{i=1}^n \beta(\Theta'')(a_i, b_i) \wedge r^{\mathfrak{A}}(\mathbf{a}) & = & \bigwedge_{i=1}^n \Theta''(a_i/\hat{\Theta}, b_i/\hat{\Theta}) \wedge r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) \\ & & \quad (\text{Definitions of } \beta(\Theta'') \text{ and } r^{\mathfrak{A}/\Theta}) \\ & \leq & r^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)) \\ & = & r^{\mathfrak{A}}(\mathbf{b}). \quad (\text{Definition of } r^{\mathfrak{A}/\Theta}) \end{array}$$

Finally, for all $a, b \in A$,

$$\begin{array}{rcll} (\beta(\Theta'')/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}) & = & \beta(\Theta'')(a, b) & (\text{Definition of } \beta(\Theta'')/\Theta) \\ & = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}). & (\text{Definition of } \beta(\Theta'')) \end{array}$$

We conclude that α and β are inverse maps.

Finally, notice that, on the one hand, $\Theta/\Theta = \bar{\Theta} = E^{\mathfrak{A}/\Theta}$. Indeed, for all $a, b \in A$,

$$\begin{array}{rcll} (\Theta/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a, b) & (\text{Definition of } \Theta/\Theta) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } \bar{\Theta}) \\ & = & E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}). & (\text{Definition of } E^{\mathfrak{A}/\Theta}) \end{array}$$

On the other hand, by Corollary 134, taking $\pi_\Theta : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\Theta$, $\Omega(\mathfrak{A}/\Theta) = \Omega(\mathfrak{A})/\Theta$. ■

4.9 Leibniz Quotient

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. Its **Leibniz quotient** is the $\mathcal{L}$-structure

$$\mathfrak{A}^* = \mathfrak{A}/\Omega(\mathfrak{A}) := \langle \mathbf{A}/\widehat{\Omega(\mathfrak{A})}, \overline{\Omega(\mathfrak{A})}, R^{\mathfrak{A}/\Omega(\mathfrak{A})} \rangle.$$

We write $\mathbf{A}^*$ for the underlying algebra $\mathbf{A}/\Omega(\mathfrak{A}) := \mathbf{A}/\widehat{\Omega(\mathfrak{A})}$ of $\mathfrak{A}^*$, $\mathcal{A}^*$ for the underlying G-algebra $\mathcal{A}/\Omega(\mathfrak{A})$ of $\mathfrak{A}^*$, and we use a^* to denote an element $a/\widehat{\Omega(\mathfrak{A})}$ of the quotient, for $a \in A$.

Given $\mathcal{L}$-structures $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ and a mapping $h : A \to B$, we write $h^* : A^* \to B^*$ for the correspondence

$$a^* \mapsto (h(a))^*$$

induced by h on the quotients. However, this is not, in general, a well defined mapping.

By Proposition 139, $\mathfrak{A}^*$ is a reduction of $\mathfrak{A}$. By the Correspondence Theorem, $\mathfrak{A}^*$ is a reduced structure. So we have $\mathfrak{A}^{**} \cong \mathfrak{A}^*$. We can show that $\mathfrak{A}^*$ is minimal in the sense that it is a reduction of any other reduction of the $\mathcal{L}$-structure $\mathfrak{A}$.

Theorem 146 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and let $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Then $h^* : \mathfrak{A}^* \cong \mathfrak{B}^*$. More generally, if $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $h^* : h^{-1}(\mathfrak{B})^* \cong \mathfrak{B}^*$.*

Proof: Let $a, a' \in A$. Then, we have

$$\begin{array}{lll} a^* = a'^* & \text{iff} & \langle a, a' \rangle \in \widehat{\Omega(\mathfrak{A})} \quad (\text{Definition of } {}^*) \\ & \text{iff} & \langle h(a), h(a') \rangle \in \widehat{\Omega(\mathfrak{B})} \quad (\text{Corollary 134}) \\ & \text{iff} & h(a)^* = h(a')^*. \quad (\text{Definition of } {}^*) \end{array}$$

So h^* is both well defined and one-to-one. It is clearly onto, since h is onto. To see that it is an algebra homomorphism, let $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$. Then

$$\begin{array}{rcll} h^*(f^{\mathbf{A}^*}(\mathbf{a}^*)) & = & h^*(f^{\mathbf{A}}(\mathbf{a})^*) & (\text{Definition of } \mathbf{A}^*) \\ & = & h(f^{\mathbf{A}}(\mathbf{a}))^* & (\text{Definition of } h^*) \\ & = & f^{\mathbf{B}}(h(\mathbf{a}))^* & (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & = & f^{\mathbf{B}^*}(h(\mathbf{a})^*) & (\text{Definition of } \mathbf{B}^*) \\ & = & f^{\mathbf{B}^*}(h^*(\mathbf{a}^*)). & (\text{Definition of } h^*) \end{array}$$

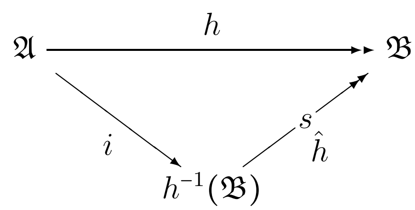
In fact, it is a G-algebra morphism, since, for all $a_1, a_2 \in A$,

$$\begin{array}{rcll} E^{\mathfrak{A}^*}(a_1^*, a_2^*) & = & \overline{\Omega(\mathfrak{A})}(a_1^*, a_2^*) & (\text{Definition of } E^{\mathfrak{A}^*}) \\ & = & \Omega(\mathfrak{A})(a_1, a_2) & (\text{Definition of } \overline{\Omega(\mathfrak{A})}) \\ & = & \Omega(\mathfrak{B})(h(a_1), h(a_2)) & (\text{Corollary 134}) \\ & = & \overline{\Omega(\mathfrak{B})}(h(a_1)^*, h(a_2)^*) & (\text{Definition of } \overline{\Omega(\mathfrak{B})}) \\ & = & E^{\mathfrak{B}^*}(h^*(a_1^*), h^*(a_2^*)). & (\text{Definitions of } E^{\mathfrak{B}^*} \text{ and } h^*) \end{array}$$

Finally, to see that it is a strict homomorphism of structures, consider $r \in R$, with $\rho(r) = n$, and $a_1, \dots, a_n \in A$. We have

$$\begin{aligned} r^{\mathfrak{A}^*}(\mathbf{a}^*) &= r^{\mathfrak{A}}(\mathbf{a}) \quad \text{(Definition of } r^{\mathfrak{A}^*}) \\ &= r^{\mathfrak{B}}(h(\mathbf{a})) \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ &= r^{\mathfrak{B}^*}(h(\mathbf{a})^*) \quad \text{(Definition of } r^{\mathfrak{B}^*}) \\ &= r^{\mathfrak{B}^*}(h^*(\mathbf{a}^*)). \quad \text{(Definition of } h^*) \end{aligned}$$

So $h^* : \mathfrak{A}^* \cong \mathfrak{B}^*$. For the more general statement, assume that $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$. Consider the diagram following Lemma 125.



The conclusion follows by using this together with the case just proven. ∎

Corollary 147 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures, such that $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Then $\mathfrak{A}^*$ is a reduction of $\mathfrak{B}$. Moreover, if $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A}^* \cong \mathfrak{B}^*$.*

Proof: Suppose $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Then, by Theorem 146, $\mathfrak{A}^* \cong \mathfrak{B}^*$. Thus, since $\mathfrak{B}^*$ is a reduction of $\mathfrak{B}$, $\mathfrak{A}^*$ is a reduction of $\mathfrak{B}$. If $\mathfrak{A} \cong \mathfrak{B}$, then, in particular, $\mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Thus, by Theorem 146, $\mathfrak{A}^* \cong \mathfrak{B}^*$. ∎

4.10 Models and Semantic Consequence

Consider a **graded collection**

$$\Phi = \{\Phi_g : g \in G\}$$

of pairwise disjoint sets of $\mathcal{L}$-formulas. Define,

$$\mathsf{Mod}(\Phi) = \{\mathfrak{A} : \varphi^{\mathfrak{A}} = g, \text{ for all } \varphi \in \Phi_g, g \in G\}$$

and

$$\mathsf{Mod}^*(\Phi) = \{\mathfrak{A} \in \mathsf{Mod}(\Phi) : \mathfrak{A} \text{ reduced}\}.$$

$\mathsf{Mod}(\Phi)$ and $\mathsf{Mod}^*(\Phi)$ are called, respectively, the **full model class** and the **reduced model class** of Φ. The two classes $\mathsf{Mod}(\Phi)$ and $\mathsf{Mod}^*(\Phi)$ are closely related. Let K be a class of $\mathcal{L}$-structures. Define

$$\mathbb{L}(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{B}^*, \text{ for some } \mathfrak{B} \in \mathsf{K}\}.$$

Sometimes we write K^* in lieu of $\mathbb{L}(\mathsf{K})$. By Proposition 126, $\mathfrak{A}^*$ is elementarily equivalent to $\mathfrak{A}$. Thus, we have

$$\mathsf{Mod}^*(\Phi) = \mathbb{L}(\mathsf{Mod}(\Phi)).$$

The operator $\mathbb{L}$ is called the **reduction operator**. If K is an arbitrary class of $\mathcal{L}$-structures, we say K is a **full class** whenever it is closed under expansions and reductions. We say that K is a **reduced class** if it is obtained by applying the reduction operator to some class. In particular, the whole class of reduced $\mathcal{L}$-structures is called **reduced semantics** to differentiate it from the class of all $\mathcal{L}$-structures, named **full semantics**.

Observe that, if, in every member $\mathfrak{A}$ of a class K, $r^{\mathfrak{A}}$ is constant, for all $r \in R$, then K^* consists of one-element algebras in which all relations are constant (the same constant value as in the parent structure in K).

Let $\Gamma = \{\Gamma_g : g \in G\}$ and $\Phi = \{\Phi_g : g \in G\}$ be graded collections of $\mathcal{L}$-formulas. We write $\Gamma \vDash \Phi$ to signify that, for all $\mathcal{L}$-structures $\mathfrak{A}$ and all assignments h,

$$\mathfrak{A} \vDash \Gamma[h] \quad \text{implies} \quad \mathfrak{A} \vDash \Phi[h].$$

Additionally, we write $\Gamma \vDash^* \Phi$ to signify that, for all reduced $\mathcal{L}$-structures $\mathfrak{A}$ and all assignments h,

$$\mathfrak{A} \vDash \Gamma[h] \quad \text{implies} \quad \mathfrak{A} \vDash \Phi[h].$$

4.11 Class Operators and Properties

We introduced operators on classes of $\mathcal{L}$-structures corresponding to the constructions introduced thus far in the text. Then we formulate a few technical lemmas that investigate some of the properties of those operators.

$$\begin{array}{rcl}
\mathbb{S}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{C} \subseteq \mathfrak{B}, \text{ for some } \mathfrak{B} \in \mathsf{K}\};\\
\mathbb{F}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{B} \lesssim \mathfrak{C}, \text{ for some } \mathfrak{B} \in \mathsf{K}\};\\
\mathbb{H}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow \mathfrak{C}, \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\};\\
\mathbb{R}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}, \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\};\\
\mathbb{E}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}, \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\};\\
\mathbb{P}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i \text{ and } \mathfrak{A}_i \in \mathsf{K}, \text{ for all } \in I\}.
\end{array}$$

Suppose $\mathbb{O}$ and $\mathbb{O}'$ are two of these operators. Then $\mathbb{O}\mathbb{O}'$ denotes their composition and $\mathbb{O} \leq \mathbb{O}'$ means that, for every class K of $\mathcal{L}$-structures, $\mathbb{O}(\mathsf{K}) \subseteq \mathbb{O}'(\mathsf{K})$. Moreover, $\mathbb{O}^*$ denotes the operator $\mathbb{L}\mathbb{O}$. By definition, the operator $\mathbb{P}$ applied to any class (even the empty class) yields a nonempty class of structures, since, for an empty index set, one gets the trivial structure, with the trivial underlying algebra, in which the reduced G-congruence and all G-relations are interpreted as $\top$. We denote by $\overline{\mathbb{P}}$ the operator $\mathbb{P}$ applied only to constructions with nonempty index sets.

Lemma 148 *For $\mathbb{O}$ any of the operators defined above, $\mathbb{O}^2 = \mathbb{O}$.*

Proof: Let $\mathbb{O} = \mathbb{S}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{S}^2(\mathsf{K})$. Assume, for simplicity, that there exists $\mathfrak{B} \in \mathbb{S}(\mathsf{K})$, such that $\mathfrak{A} \subseteq \mathfrak{B}$. Hence, there exists $\mathfrak{C} \in \mathsf{K}$, such that $\mathfrak{B} \subseteq \mathfrak{C}$. Thus, $\mathfrak{A} \subseteq \mathfrak{C} \in \mathsf{K}$, yielding $\mathfrak{A} \in \mathbb{S}(\mathsf{K})$.

Let $\mathbb{O} = \mathbb{F}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{F}^2(\mathsf{K})$. Assume, for simplicity, that there exists $\mathfrak{B} \in \mathbb{F}(\mathsf{K})$, such that $\mathfrak{B} \lesssim \mathfrak{A}$. Hence, there exists $\mathfrak{C} \in \mathsf{K}$, such that $\mathfrak{C} \lesssim \mathfrak{B}$. Thus, we get $\mathsf{K} \ni \mathfrak{C} \lesssim \mathfrak{A}$, yielding $\mathfrak{A} \in \mathbb{F}(\mathsf{K})$.

Let $\mathbb{O} = \mathbb{H}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{H}^2(\mathsf{K})$. Assume, for simplicity, that there exists $\mathfrak{B} \in \mathbb{H}(\mathsf{K})$, such that $\mathfrak{B} \twoheadrightarrow \mathfrak{A}$. Hence, there exists $\mathfrak{C} \in \mathsf{K}$, such that $\mathfrak{C} \twoheadrightarrow \mathfrak{B}$. Thus, we get $\mathsf{K} \ni \mathfrak{C} \twoheadrightarrow \mathfrak{A}$, yielding $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$.

The same reasoning applies for $\mathbb{O} = \mathbb{R}$ and for $\mathbb{O} = \mathbb{E}$, since the composite of two reductive homomorphisms is a reductive homomorphism.

Let $\mathbb{O} = \mathbb{P}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{P}^2(\mathsf{K})$. Assume, for simplicity, that there exist $\mathfrak{A}_i \in \mathbb{P}(\mathsf{K})$, such that $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$. Hence, there exist $\mathfrak{A}_{ij} \in \mathsf{K}$, $j \in J_i$, such that $\mathfrak{A}_i = \prod_{j\in J_i} \mathfrak{A}_{ij}$. Thus, we get $\mathfrak{A} = \prod_{i\in I} \prod_{j\in J_i} \mathfrak{A}_{ij}$, yielding $\mathfrak{A} \in \mathbb{P}(\mathsf{K})$.

■

Next we look at how the reduction and expansion operators interact with some of the remaining operators.

Lemma 149 *(a) $\mathbb{SE} \leq \mathbb{ES}$ and $\mathbb{PE} \leq \mathbb{EP}$;*

(b) $\mathbb{SR} \leq \mathbb{RS}$ and $\mathbb{PR} \leq \mathbb{RP}$.

Proof:

(a) Suppose $\mathfrak{A} \in \mathbb{SE}(\mathsf{K})$. Then $\mathfrak{A} \subseteq \mathfrak{C}$ and there exists $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. We consider the structure $h(\mathfrak{A})$, with $h(\mathfrak{A}) \subseteq \mathfrak{B}$. Moreover, denoting by $h{\restriction_{\mathfrak{A}}} : \mathfrak{A} \to h(\mathfrak{A})$, the restriction of h on A, we get that $h{\restriction_{\mathfrak{A}}} : \mathfrak{A} \twoheadrightarrow_s h(\mathfrak{A})$, since, for every $r \in R$, with $\rho(r) = n$, and all $a_1, \ldots, a_n \in A$,

$$\begin{array}{rcll} r^{\mathfrak{A}}(\mathbf{a}) & = & r^{\mathfrak{C}}(\mathbf{a}) & (\mathfrak{A} \subseteq \mathfrak{C}) \\ & = & r^{\mathfrak{B}}(h(\mathbf{a})) & (h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}) \\ & = & r^{h(\mathfrak{A})}(h(\mathbf{a})). & (h(\mathfrak{A}) \subseteq \mathfrak{B}) \end{array}$$

This shows that $\mathfrak{A} \in \mathbb{ES}(\mathsf{K})$.

Suppose $\mathfrak{A} \in \mathbb{PE}(\mathsf{K})$. Then $\mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i$, such that, there exist $h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i$, for some $\mathfrak{B}_i \in \mathsf{K}$, for all $i \in I$. Consider the mapping $h : \prod_{i\in I} \mathfrak{A}_i \to \prod_{i\in I} \mathfrak{B}_i$, defined, for all $\mathbf{a} = \langle a_i : i \in I\rangle \in \prod_{i\in I} A_i$, by

$$h(\mathbf{a}) = \langle h_i(a_i) : i \in I\rangle.$$

This is an $\mathcal{L}$-algebra homomorphism. Further, it is a G-algebra morphism, since, for all $\mathbf{a}, \mathbf{b} \in \prod_{i\in I} A_i$,

$$\begin{array}{rcll} E^{\prod \mathfrak{A}_i}(\mathbf{a}, \mathbf{b}) & = & \bigwedge_{i\in I} E^{\mathfrak{A}_i}(a_i, b_i) & (\text{Definition of } E^{\prod \mathfrak{A}_i}) \\ & \leq & \bigwedge_{i\in I} E^{\mathfrak{B}_i}(h_i(a_i), h_i(b_i)) & (h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i) \\ & = & E^{\prod \mathfrak{B}_i}(h(\mathbf{a}), h(\mathbf{b})). & (\text{Definition of } E^{\prod \mathfrak{B}_i}) \end{array}$$

Moreover, for all $r \in R$, with $\rho(r) = n$, and all $\mathbf{a}^1, \dots, \mathbf{a}^n \in (\prod_{i\in I} A_i)^n$,

$$\begin{array}{rcl} r^{\prod \mathfrak{B}_i}(h(\mathbf{a}_1), \dots, h(\mathbf{a}_n)) & = & \bigwedge_{i\in I} r^{\mathfrak{B}_i}(h_i(a_{1i}), \dots, h_i(a_{n_i})) \\ & & \quad \text{(Definition of } r^{\prod \mathfrak{B}_i}) \\ & = & \bigwedge_{i\in I} r^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni}) \\ & & \quad (h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i) \\ & = & r^{\prod \mathfrak{A}_i}(\mathbf{a}_1, \dots, \mathbf{a}_n). \\ & & \quad \text{(Definition of } r^{\prod \mathfrak{A}_i}) \end{array}$$

So $h : \prod_{i\in I} \mathfrak{A}_i \twoheadrightarrow_s \prod_{i\in I} \mathfrak{B}_i$, which shows that $\mathfrak{A} \in \mathbb{EP}(\mathsf{K})$.

(b) Suppose $\mathfrak{A} \in \mathbb{SR}(\mathsf{K})$. Then $\mathfrak{A} \subseteq \mathfrak{C}$ and there exists $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. By Lemma 125, $\mathfrak{A}' = h^{-1}(\mathfrak{A}) \subseteq \mathfrak{B}$. Moreover, $h{\restriction_{\mathfrak{A}'}} : \mathfrak{A}' \twoheadrightarrow_s \mathfrak{A}$ is a reductive morphism. Thus, $\mathfrak{A} \in \mathbb{RS}(\mathsf{K})$.

Suppose $\mathfrak{A} = \mathbb{PR}(\mathsf{K})$. Then $\mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i$ and, for all $i \in I$, there exists $h_i : \mathfrak{B}_i \twoheadrightarrow_s \mathfrak{A}_i$, for some $\mathfrak{B}_i \in \mathsf{K}$. Then, the mapping $h : \prod_{i\in I} B_i \to \prod_{i\in I} A_i$ given, for all $\mathbf{b} \in \prod_{i\in I} B_i$, by

$$h(\mathbf{b}) = \langle h_i(b_i) : i \in I \rangle$$

is such that $h : \prod_{i\in I} \mathfrak{B}_i \twoheadrightarrow_s \prod_{i\in I} \mathfrak{A}_i$ (see Part (a)). Hence, we obtain $\mathfrak{A} \in \mathbb{RP}(\mathsf{K})$.

■

Next we turn to properties involving filter extensions.

Lemma 150 *(a)* $\mathbb{EF} \leq \mathbb{FE}$*;*

(b) $\mathbb{FR} \leq \mathbb{RF} = \mathbb{H}$*;*

(c) $\mathbb{FS} \leq \mathbb{SF}$*.*

Proof:

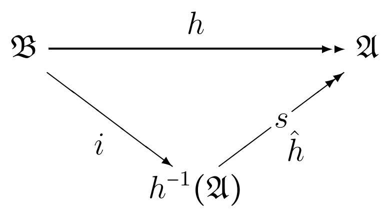
(a) Suppose $\mathfrak{A} \in \mathbb{EF}(\mathsf{K})$. Let $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \lesssim \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. Consider the structure $h^{-1}(\mathfrak{B})$. For every $r \in R$, with $\rho(r) = n$, and all $a_1, \dots, a_n \in A$,

$$\begin{array}{rcll} r^{h^{-1}(\mathfrak{B})}(\mathbf{a}) & = & r^{\mathfrak{B}}(h(\mathbf{a})) & \text{(Definition of } r^{h^{-1}(\mathfrak{B})}) \\ & \leq & r^{\mathfrak{C}}(h(\mathbf{a})) & (\mathfrak{B} \lesssim \mathfrak{C}) \\ & = & r^{\mathfrak{A}}(\mathbf{a}). & (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}) \end{array}$$

Thus, $h^{-1}(\mathfrak{B}) \lesssim \mathfrak{A}$. Moreover, by definition, $h^{-1}(\mathfrak{B}) \twoheadrightarrow_s \mathfrak{B}$. So we get $\mathfrak{A} \in \mathbb{FE}(\mathsf{K})$.

(b) Suppose that $\mathfrak{A} \in \mathbb{FR}(\mathsf{K})$. Then $\mathfrak{C} \lesssim \mathfrak{A}$ and there exists $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. In this case $\mathfrak{B} \lesssim h^{-1}(\mathfrak{A})$ and $h : h^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}$. So we get $\mathfrak{A} \in \mathbb{RF}(\mathsf{K})$.

To conclude this part, we must show that $\mathbb{RF} = \mathbb{H}$. One direction is straightforward. Suppose $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and that $\mathfrak{B} \lesssim \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. Then $h : \mathfrak{B} \twoheadrightarrow \mathfrak{A}$ is an epimorphism and, hence $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$. Assume, conversely, that $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$. Hence, there exists an epimorphism $h : \mathfrak{B} \twoheadrightarrow \mathfrak{A}$, with $\mathfrak{B} \in \mathsf{K}$. We have seen that such an epimorphism factors.



In the diagram $\mathfrak{B} \lesssim h^{-1}(\mathfrak{A})$ and $\hat{h} : h^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}$. Hence, $\mathfrak{A} \in \mathbb{RF}(\mathsf{K})$.

(c) Suppose that $\mathfrak{A} \in \mathbb{FS}(\mathsf{K})$. Hence, we have $\mathfrak{C} \lesssim \mathfrak{A}$ and $\mathfrak{C} \subseteq \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. We define the $\mathcal{L}$-structure $\mathfrak{D} = \langle \mathcal{B}, R^{\mathfrak{D}} \rangle$, such that $R^{\mathfrak{D}} = \{r^{\mathfrak{D}} : r \in R\}$ is given by setting, for all $r \in R$, with $\rho(r) = n$, and all $b_1, \ldots, b_n \in B$,

$$r^{\mathfrak{D}}(\mathbf{b}) = \begin{cases} r^{\mathfrak{A}}(\mathbf{b}), & \text{if } \mathbf{b} \in C, \\ r^{\mathfrak{B}}(\mathbf{b}), & \text{if } \mathbf{b} \notin C. \end{cases}$$

We have $\mathfrak{B} \lesssim \mathfrak{D}$, since, by definition, the underlying algebras are identical and, for all $r \in R$, $r^{\mathfrak{B}} \leq r^{\mathfrak{D}}$. Next, we show that $\mathfrak{A} \subseteq \mathfrak{D}$. First, note that $E^{\mathfrak{A}} = E^{\mathfrak{D}} \restriction_C$, since, for all $a_1, a_2 \in A$,

$$\begin{array}{rcll} E^{\mathfrak{A}}(a_1, a_2) & = & E^{\mathfrak{C}}(a_1, a_2) & (\mathfrak{C} \lesssim \mathfrak{A}) \\ & = & E^{\mathfrak{B}}(a_1, a_2) & (\mathfrak{C} \subseteq \mathfrak{B}) \\ & = & E^{\mathfrak{D}}(a_1, a_2). & (\mathfrak{D} = \langle \mathcal{B}, R^{\mathfrak{D}} \rangle) \end{array}$$

Moreover, by definition of $r^{\mathfrak{D}}$, we have, for all $r \in R$, with $\rho(r) = n$, and all $a_1, \ldots, a_n \in A$,

$$r^{\mathfrak{D}}(\mathbf{a}) = r^{\mathfrak{A}}(\mathbf{a}).$$

We conclude that $\mathfrak{A} \subseteq \mathfrak{D}$ and, hence, $\mathfrak{A} \in \mathbb{SF}(\mathsf{K})$.

■

Next, we explore how the image and preimage operators under reductive morphisms $\mathbb{R}$ and $\mathbb{E}$, respectively, interact with the model reduction operator $\mathbb{L}$ and with each other.

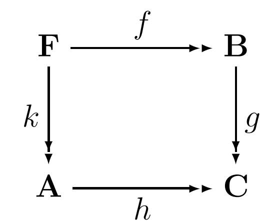
Lemma 151 *(a)* $\mathbb{EL} = \mathbb{ER} = \mathbb{RE}$*;*

(b) $\mathbb{LE} = \mathbb{LR} = \mathbb{RL} = \mathbb{L} \leq \mathbb{EL}$*.*

Proof:

(a) Suppose that $\mathfrak{A} \in \mathbb{RE}(\mathsf{K})$. Then there exist $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and $g : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, with $\mathfrak{B} \in \mathsf{K}$. Then, by Theorem 146, $\mathfrak{C}^* \cong \mathfrak{A}^*$ and $\mathfrak{C}^* \cong \mathfrak{B}^*$. Hence, there exist $\mathfrak{A} \twoheadrightarrow_s \mathfrak{C}^*$ and $\mathfrak{B} \twoheadrightarrow_s \mathfrak{C}^*$. These show that $\mathfrak{A} \in \mathbb{ER}(\mathsf{K})$.

Suppose, conversely, that $\mathfrak{A} \in \mathbb{ER}(\mathsf{K})$. Then, there exist $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$ and $g : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathsf{K}$. Let $\mathbf{F}$ be an algebra of $\mathcal{L}$-terms, such that there exists algebra homomorphisms $k : \mathbf{F} \twoheadrightarrow \mathbf{A}$ and $f : \mathbf{F} \twoheadrightarrow \mathbf{B}$, such that the rectangle commutes.



Then, clearly, $\langle \mathcal{F}, \{\bot\}_{r \in R}\rangle$, with $\mathcal{F} = \langle \mathbf{F}, \Delta^{\mathbf{F}}\rangle$, is an $\mathcal{L}$-structure and

$$h \circ k = g \circ f : \langle \mathcal{F}, \{\bot\}_{r \in R}\rangle \to \mathfrak{C}$$

is a morphism. Let $\mathfrak{F} = (h \circ k)^{-1}(\mathfrak{C})$. Then, for all $r \in R$, with $\rho(r) = n$, and all $t_1, \ldots, t_n \in F$,

$$\begin{array}{rcl} r^{(h\circ k)^{-1}(\mathfrak{C})}(t_1, \ldots, t_n) & = & r^{\mathfrak{C}}(h(k(t_1)), \ldots, h(k(t_n))) \\ & & \text{(Definition of } r^{(h\circ k)^{-1}(\mathfrak{C})}) \\ & = & r^{\mathfrak{A}}(k(t_1), \ldots, k(t_n)) \\ & & (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}) \end{array}$$

and, similarly,

$$\begin{array}{rcl} r^{(h\circ k)^{-1}(\mathfrak{C})}(t_1, \ldots, t_n) & = & r^{(g\circ f)^{-1}(\mathfrak{C})}(t_1, \ldots, t_n) \\ & = & r^{\mathfrak{C}}(g(f(t_1)), \ldots, g(f(t_n))) \\ & = & r^{\mathfrak{B}}(f(t_1), \ldots, f(t_n)). \end{array}$$

Hence, $k : \mathfrak{F} \twoheadrightarrow_s \mathfrak{A}$ and $f : \mathfrak{F} \twoheadrightarrow_s \mathfrak{B}$. This shows that $\mathfrak{A} \in \mathbb{RE}(\mathsf{K})$.

Finally, notice that $\mathbb{L} \le \mathbb{R}$. Hence $\mathbb{EL} \le \mathbb{ER}$. So it suffices to show the reverse inclusion. Assume $\mathfrak{A} \in \mathbb{ER}(\mathsf{K})$. Then, there exist $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$ and $g : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathsf{K}$. Thus, there exist $\mathfrak{A} \twoheadrightarrow_s \mathfrak{C}^*$ and $\mathfrak{B} \twoheadrightarrow_s \mathfrak{C}^*$. This shows that $\mathfrak{A} \in \mathbb{EL}(\mathsf{K})$.

(b) Clearly, $\mathbb{L} \le \mathbb{LE}$, $\mathbb{L} \le \mathbb{LR}$, $\mathbb{L} \le \mathbb{RL}$ and $\mathbb{L} \le \mathbb{EL}$. So we must show that the reverse inclusions of the first three also hold.

Suppose $\mathfrak{A} \in \mathbb{LE}(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$ and there exists $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. Then, by Theorem 146, $\mathfrak{A} \cong \mathfrak{C}^* \cong \mathfrak{B}^*$. Hence, $\mathfrak{A} \in \mathbb{L}(\mathsf{K})$.

Let $\mathfrak{A} \in \mathbb{LR}(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$ and there exists $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. Using again Theorem 146, we get $\mathfrak{A} \cong \mathfrak{C}^* \cong \mathfrak{B}^*$. Thus, $\mathfrak{A} \in \mathbb{L}(\mathsf{K})$.

Finally, suppose $\mathfrak{A} \in \mathbb{RL}(\mathsf{K})$. Then there exist $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and $\mathfrak{B} \in \mathsf{K}$, such that $\mathfrak{C} \cong \mathfrak{B}^*$. Note that $\mathfrak{C}^* \cong (\mathfrak{B}^*)^* \cong \mathfrak{B}^* \cong \mathfrak{C}$. Thus, by Lemma 131, $\mathrm{Ker}(h) = E^{\mathfrak{C}}$. Thus $\mathfrak{A} \overset{h}{\cong} \mathfrak{C} \cong \mathfrak{B}^*$, showing that $\mathfrak{A} \in \mathbb{L}(\mathsf{K})$.

■

4.12 The Diagram Lemma

This section revisits some standard constructions from model theory. However, they are viewed under the light of languages without equality and in view of the interpretations in G adopted for the relation symbols.

Consider an $\mathcal{L}$-structure $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$. We let $\mathcal{L}_A$ be the expansion of $\mathcal{L}$ resulting by adding to $\mathcal{L}$ new individual constants c_a, for $a \in A$. One uses $\bar{a}$ to denote the sequence of all elements of A in some predetermined order and $\bar{c}$ for the sequence of the corresponding constants. Further, $\mathcal{L}_A$-structures are denoted by $(\mathfrak{B}, b_a)_{a \in A}$, where $\mathfrak{B}$ is an $\mathcal{L}$-structure and $b_a \in B$, for every $a \in A$.

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. The **diagram** $D(\mathfrak{A})$ of $\mathfrak{A}$ is the collection

$$D(\mathfrak{A}) = \{D^g(\mathfrak{A}) : g \in G\},$$

where $D^g(\mathfrak{A})$ is the set of all atomic sentences φ over $\mathcal{L}_A$ satisfying

$$\varphi^{(\mathfrak{A}, a)_{a \in A}} = g.$$

Sometimes we denote this by $(\mathfrak{A}, a)_{a \in A} \vDash_g \varphi$. Moreover, we abbreviate by $(\mathfrak{A}, a)_{a \in A} \vDash D(\mathfrak{A})$ the statement $(\mathfrak{A}, a)_{a \in A} \vDash_g D^g(\mathfrak{A})$, for all $g \in G$.

Define

$$L(\mathfrak{A}) = \{L^g(\mathfrak{A}) : g \in G\},$$

where $L^g(\mathfrak{A})$ is the set of all infinitary conjunctions of Leibniz sentences over $\mathcal{L}_A$

$$\bigwedge_{\psi \text{ Leibniz}} \psi(t(\bar{c}), t'(\bar{c})),$$

such that

$$\Omega(\mathfrak{A})(t^{\mathbf{A}}(\bar{a}), t'^{\mathbf{A}}(\bar{a})) = g.$$

The **Leibniz diagram** $D_\ell(\mathfrak{A})$ of $\mathfrak{A}$, is the collection

$$D_\ell(\mathfrak{A}) = \{D_\ell^g(\mathfrak{A}) : g \in G\},$$

resulting by adding $L(\mathfrak{A})$ to $D(\mathfrak{A})$, that is,

$$D_\ell^g(\mathfrak{A}) = D^g(\mathfrak{A}) \cup L^g(\mathfrak{A}).$$

Lemma 152 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and let $h : A \to B$. If $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$, then $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$.*

Proof: Suppose $(\mathfrak{B}, h(a))_{a \in A} \vDash_g D_\ell^g(\mathfrak{A})$, for all $g \in G$.

We start by showing that $h^* : A^* \to B^*$ is well-defined. Suppose that, for some $a, a' \in A$, $a^* = a'^*$, i.e., $\Omega(\mathfrak{A})(a, a') = \top$. By Theorem 135, we have

$$\bigwedge_{\psi \text{ Leibniz}} \psi(c_a, c_{a'}) \in D_\ell^\top(\mathfrak{A}).$$

By hypothesis, $\bigwedge_\psi \psi^{\mathfrak{B}}(h(a), h(a')) = \top$. By Theorem 135,

$$\Omega(\mathfrak{B})(h(a), h(a')) = \top.$$

Hence, $h(a)^* = h(a')^*$. Thus, by definition of h^*, $h^*(a^*) = h^*(a'^*)$.

Next, we show that $h^* : \mathbf{A}^* \to \mathbf{B}^*$ is a homomorphism. Let $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$. Because of the interpretation of the constants in $(\mathfrak{A}, a)_{a \in A}$, we have

$$\bigwedge_{\psi \text{ Leibniz}} \psi(c_{f^{\mathbf{A}}(\mathbf{a})}, f(c_{a_1}, \ldots, c_{a_n})) \in D_\ell^\top(\mathfrak{A}).$$

By hypothesis, $\bigwedge_\psi \psi^{\mathfrak{B}}(h(f^{\mathbf{A}}(\mathbf{a})), f^{\mathbf{B}}(h(\mathbf{a}))) = \top$. Thus, by Theorem 135,

$$\Omega(\mathfrak{B})(h(f^{\mathbf{A}}(\mathbf{a})), f^{\mathbf{B}}(h(\mathbf{a}))) = \top.$$

So $h^* : \mathbf{A}^* \to \mathbf{B}^*$ is a homomorphism.

To see that it is a G-morphism, suppose that $a, a' \in A$. Notice that, since $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$, we have, $\Omega(\mathfrak{A})(a, a') = \Omega(\mathfrak{B})(h(a), h(a'))$. Thus, we obtain

$$\begin{array}{rcll} E^{\mathfrak{A}^*}(a^*, a'^*) & = & \overline{\Omega(\mathfrak{A})}(a^*, a'^*) & \text{(Definition of } E^{\mathfrak{A}^*}) \\ & = & \Omega(\mathfrak{A})(a, a') & \text{(Definition of } \overline{\Omega(\mathfrak{A})}) \\ & = & \Omega(\mathfrak{B})(h(a), h(a')) & \text{(Comment above)} \\ & = & \overline{\Omega(\mathfrak{B})}(h(a)^*, h(a')^*) & \text{(Definition of } \overline{\Omega(\mathfrak{B})}) \\ & = & E^{\mathfrak{B}^*}(h^*(a^*), h^*(a'^*)). & \text{(Definition of } E^{\mathfrak{B}^*} \text{ and } h^*) \end{array}$$

Let us see now that h^* is a strict morphism from $\mathfrak{A}^*$ to $\mathfrak{B}^*$. Suppose $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in A$, such that $r^{\mathfrak{A}}(a_1, \ldots, a_n) = g$. Then we have $r(c_{a_1}, \ldots, c_{a_n}) \in D_\ell^g(\mathfrak{A})$. Thus, by hypothesis, $r^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) = g$. This proves that, for all $r \in R$, with $\rho(r) = n$, and all $a_1, \ldots, a_n \in A$,

$$\begin{array}{rcll} r^{\mathfrak{A}^*}(\mathbf{a}^*) & = & r^{\mathfrak{A}}(\mathbf{a}) & \text{(Definition of } r^{\mathfrak{A}^*}) \\ & = & r^{\mathfrak{B}}(h(\mathbf{a})) & \text{(Shown above)} \\ & = & r^{\mathfrak{B}^*}(h(\mathbf{a})^*) & \text{(Definition of } r^{\mathfrak{B}^*}) \\ & = & r^{\mathfrak{B}^*}(h^*(\mathbf{a}^*)), & \text{(Definition of } h^*) \end{array}$$

that is, $h^* : \mathfrak{A}^* \to_s \mathfrak{B}^*$.

From the fact that, for all $a, a' \in A$,

$$E^{\mathfrak{A}^*}(a^*, a'^*) = E^{\mathfrak{B}^*}(h^*(a^*), h^*(a'^*)),$$

which was shown above, it follows that $h^* : A^* \rightarrowtail B^*$ is an embedding. ■

In the reverse direction, we obtain

Lemma 153 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and let $h : \mathfrak{A} \to \mathfrak{B}$, such that $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$. Then $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$.*

Proof: Assume that $h : \mathfrak{A} \to \mathfrak{B}$ and that $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$.

Suppose, first, that $r \in R$, with $\rho(r) = n$, and $t_1(c_\mathbf{a}), \ldots, t_n(c_\mathbf{a})$ are closed terms in $\mathcal{L}_A$, such that $r(t_1, \ldots, t_n) \in D^g(\mathfrak{A})$. Thus, $r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) = g$. Now we compute

$$\begin{array}{rcl}
r^{\mathfrak{B}}(t_1^{\mathbf{B}}(h(\mathbf{a})), \ldots, t_n^{\mathbf{B}}(h(\mathbf{a}))) & = & r^{\mathfrak{B}^*}(t_1^{\mathbf{B}^*}(h(\mathbf{a})^*), \ldots, t_n^{\mathbf{B}^*}(h(\mathbf{a})^*)) \\
& & \quad \text{(Definition of } r^{\mathfrak{B}^*}) \\
& = & r^{\mathfrak{B}^*}(t_1^{\mathbf{B}^*}(h^*(\mathbf{a}^*)), \ldots, t_n^{\mathbf{B}^*}(h^*(\mathbf{a}^*))) \\
& & \quad \text{(Definition of } h^*) \\
& = & r^{\mathfrak{B}^*}(h^*(t_1^{\mathbf{A}^*}(\mathbf{a}^*)), \ldots, h^*(t_n^{\mathbf{A}^*}(\mathbf{a}^*))) \\
& & \quad (h^* : \mathbf{A}^* \rightarrowtail \mathbf{B}^*) \\
& = & r^{\mathfrak{A}^*}(t_1^{\mathbf{A}^*}(\mathbf{a}^*), \ldots, t_n^{\mathbf{A}^*}(\mathbf{a}^*)) \\
& & \quad (h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*) \\
& = & r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) \\
& & \quad \text{(Definition of } r^{\mathfrak{A}^*}) \\
& = & g. \quad \text{(Hypothesis)}
\end{array}$$

Hence, $(\mathfrak{B}, h(a))_{a \in A} \vDash D(\mathfrak{A})$

Finally, suppose $\bigwedge_{\psi \text{ Leibniz}} \psi(t, t') \in L^g(\mathfrak{A})$. Then, by the definition of $L^g(\mathfrak{A})$,

$$\Omega(\mathfrak{A})(t^{\mathbf{A}}(\mathbf{a}), t'^{\mathbf{A}}(\mathbf{a})) = g.$$

Since $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$, we get

$$\Omega(\mathfrak{B})(h(t^{\mathbf{A}}(\mathbf{a})), h(t'^{\mathbf{A}}(\mathbf{a}))) = g.$$

Since $h : \mathfrak{A} \to \mathfrak{B}$,

$$\Omega(\mathfrak{B})(t^{\mathbf{B}}(h(\mathbf{a})), t'^{\mathbf{B}}(h(\mathbf{a}))) = g.$$

Hence, by Theorem 135,

$$\bigwedge_{\psi \text{ Leibniz}} \psi^{\mathfrak{B}}(t^{\mathbf{B}}(h(\mathbf{a})), t'^{\mathbf{B}}(h(\mathbf{a}))) = g.$$

This shows that $(\mathfrak{B}, h(a))_{a \in A} \vDash L(\mathfrak{A})$. We conclude that $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$. ■

4.13 The Reduction Operator Lemma

In this section we focus on the commutativity properties of the reduction operator $\mathbb{L}$ with respect to other class operators. These are crucial in deriving characterizations of reduced classes of models from corresponding characterizations of full classes (see treatment in [23]).

Lemma 154 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}}\rangle$ *be* $\mathcal{L}$*-structures, such that* $\mathfrak{A} \subseteq \mathfrak{B}$*, and* $\Theta \in \mathrm{Gon}(\mathfrak{B})$*. Let* $\Theta_A = \Theta{\restriction}_A$ *(which, by Lemma 132, is a* G*-congruence on* $\mathfrak{A}$*) and define* $h : A/\hat{\Theta}_A \to B/\hat{\Theta}$ *by setting, for all* $a \in A$*,*

$$h(a/\hat{\Theta}_A) = a/\hat{\Theta}.$$

Then $h : \mathfrak{A}/\Theta_A \rightarrowtail_s \mathfrak{B}/\Theta$*.*

Proof: First, h is well-defined. If $a, a' \in A$, such that $\langle a, a'\rangle \in \hat{\Theta}_A$, then, by definition, $\langle a, a'\rangle \in \hat{\Theta}$, whence $h(a/\hat{\Theta}_A) = h(a'/\hat{\Theta}_A)$.

Further, $h : \mathbf{A}/\Theta_A \to \mathbf{B}/\Theta$ is an algebra homomorphism. For $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned}
h(f^{\mathbf{A}/\Theta_A}(\mathbf{a}/\hat{\Theta}_A)) &= h(f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta}_A) \quad (\text{Definition of } f^{\mathbf{A}/\Theta_A})\\
&= f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta} \quad (\text{Definition of } h)\\
&= f^{\mathbf{B}}(\mathbf{a})/\hat{\Theta} \quad (\mathfrak{A} \subseteq \mathfrak{B})\\
&= f^{\mathbf{B}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad (\text{Definition of } f^{\mathbf{B}/\Theta})\\
&= f^{\mathbf{B}/\Theta}(h(\mathbf{a}/\hat{\Theta}_A)). \quad (\text{Definition of } h)
\end{aligned}$$

Next, let $a, a' \in A$. Then we have

$$\begin{aligned}
E^{\mathfrak{A}/\Theta_A}(a/\hat{\Theta}_A, a'/\hat{\Theta}_A) &= \bar{\Theta}_A(a/\hat{\Theta}_A, a'/\hat{\Theta}_A) \quad (\text{Definition of } E^{\mathfrak{A}/\Theta_A})\\
&= \Theta_A(a, a') \quad (\text{Definition of } \bar{\Theta}_A)\\
&= \Theta(a, a') \quad (\text{Definition of } \Theta_A)\\
&= \bar{\Theta}(a/\hat{\Theta}, a'/\hat{\Theta}) \quad (\text{Definition of } \bar{\Theta})\\
&= E^{\mathfrak{B}/\Theta}(a/\hat{\Theta}, a'/\hat{\Theta}) \quad (\text{Definition of } E^{\mathfrak{B}/\Theta})\\
&= E^{\mathfrak{B}/\Theta}(h(a/\hat{\Theta}_A), h(a'/\hat{\Theta}_A)). \quad (\text{Definition of } h)
\end{aligned}$$

Hence, h is a morphism of G-algebras. Next, we show that h is a strict homomorphism of $\mathcal{L}$-structures. Let $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in n$. Then

$$\begin{aligned}
r^{\mathfrak{A}/\Theta_A}(\mathbf{a}/\hat{\Theta}_A) &= r^{\mathfrak{A}}(\mathbf{a}) \quad (\text{Definition of } r^{\mathfrak{A}/\Theta_A})\\
&= r^{\mathfrak{B}}(\mathbf{a}) \quad (\mathfrak{A} \subseteq \mathfrak{B})\\
&= r^{\mathfrak{B}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad (\text{Definition of } r^{\mathfrak{B}/\Theta})\\
&= r^{\mathfrak{B}/\Theta}(h(\mathbf{a}/\hat{\Theta}_A)). \quad (\text{Definition of } h)
\end{aligned}$$

Since, as was shown above, for all $a, a' \in A$,

$$E^{\mathfrak{A}/\Theta_A}(a/\hat{\Theta}_A, a'/\hat{\Theta}_A) = E^{\mathfrak{B}/\Theta}(h(a/\hat{\Theta}_A), h(a'/\hat{\Theta}_A)),$$

we get that $h : \mathfrak{A}/\theta_A \rightarrowtail_s \mathfrak{B}/\theta$. ■

Theorem 155 (Reduction Operator Lemma) *We have* $\mathbb{LS} = \mathbb{LSL}$ *and* $\mathbb{LP} = \mathbb{LPL}$, *that is,* $\mathbb{S}^* = \mathbb{S}^*\mathbb{L}$ *and* $\mathbb{P}^* = \mathbb{P}^*\mathbb{L}$.

Proof: Suppose $\mathfrak{A} \in \mathbb{S}^*(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$, for some $\mathfrak{C}$, such that $\mathfrak{C} \subseteq \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. Consider $\Omega(\mathfrak{B})$ and define $\Omega(\mathfrak{B})_C = \Omega(\mathfrak{B})\restriction_C$. By Lemma 154, $h : \mathfrak{C}/\Omega(\mathfrak{B})_C \rightarrowtail_s \mathfrak{B}^*$. By Theorem 141, $h : \mathfrak{C}/\Omega(\mathfrak{B})_C \cong h(\mathfrak{C}/\Omega(\mathfrak{B})_C)$. Moreover, by the definition of $h(\mathfrak{C}/\Omega(\mathfrak{B})_C)$, $h(\mathfrak{C}/\Omega(\mathfrak{B})_C) \subseteq \mathfrak{B}^*$. By Theorem 146, $(\mathfrak{C}/\Omega(\mathfrak{B})_C)^* \cong \mathfrak{C}^*$. So we have

$$\mathfrak{A} \cong \mathfrak{C}^* \cong (\mathfrak{C}/\Omega(\mathfrak{B})_C)^* \quad \text{and} \quad \mathfrak{C}/\Omega(\mathfrak{B})_C \cong h(\mathfrak{C}/\Omega(\mathfrak{B})_C) \subseteq \mathfrak{B}^*.$$

Therefore, we obtain $\mathfrak{A} \in \mathbb{S}^*\mathbb{L}(\mathsf{K})$.

Suppose, conversely, that $\mathfrak{A} \in \mathbb{S}^*\mathbb{L}(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$, with $\mathfrak{C} \subseteq \mathfrak{B}^*$, for some $\mathfrak{B} \in \mathsf{K}$. Consider the canonical projection $\pi : \mathfrak{B} \twoheadrightarrow_s \mathfrak{B}^*$. Then, by Lemma 125, $\pi^{-1}(\mathfrak{C}) \subseteq \mathfrak{B}$ and, moreover, $\pi\restriction_{\pi^{-1}(\mathfrak{C})} : \pi^{-1}(\mathfrak{C}) \twoheadrightarrow_s \mathfrak{C}$. Hence, by Theorem 146, $\mathfrak{A} \cong \mathfrak{C}^* \cong \pi^{-1}(\mathfrak{C})^* \in \mathbb{S}^*(\mathsf{K})$.

For the second equality, the key property to be proven is that, for every collection $\mathfrak{A}_i$, $i \in I$,

$$\left(\prod_{i\in I}\mathfrak{A}_i\right)^* \cong \left(\prod_{i\in I}\mathfrak{A}_i^*\right)^*.$$

Suppose this isomorphism has been shown. Then we have

$$\begin{array}{lll}\mathfrak{A} \in \mathbb{P}^*(\mathsf{K}) & \text{iff} & \mathfrak{A} \cong (\prod_{i\in I}\mathfrak{A}_i)^*, \text{ for some } \mathfrak{A}_i \in \mathsf{K}, i \in I,\\ & \text{iff} & \mathfrak{A} \cong (\prod_{i\in I}\mathfrak{A}_i^*)^*, \text{ for some } \mathfrak{A}_i \in \mathsf{K}, i \in I,\\ & \text{iff} & \mathfrak{A} \in \mathbb{P}^*\mathbb{L}(\mathsf{K}).\end{array}$$

Thus, the displayed isomorphism suffices to show $\mathbb{P}^* = \mathbb{P}^*\mathbb{L}$. So we turn to proving the displayed isomorphism. We first define

$$h : \prod_{i\in I}\mathfrak{A}_i^* \twoheadrightarrow_s \left(\prod_{i\in I}\mathfrak{A}_i\right)^*.$$

For all $\mathbf{a} = \langle a_i : i \in I\rangle \in \prod_{I\in I}\mathfrak{A}_i$, let

$$h(\langle a_i^* : i \in I\rangle) = \langle a_i : i \in I\rangle^*.$$

We abbreviate this as $h(\mathbf{a}^*) = (\mathbf{a})^*$. To see that this is well defined, suppose $\mathbf{a}^* = \mathbf{b}^*$, i.e., $a_i^* = b_i^*$, $i \in I$. Consider an arbitrary atomic $\mathcal{L}$-formula $\varphi(x, z_1, \ldots, z_k)$ and elements $\mathbf{c}_1, \ldots, \mathbf{c}_k \in \prod_{i\in I} A_i$. We obtain

$$\begin{array}{ll}\multicolumn{2}{l}{\varphi^{\prod\mathfrak{A}_i}(\mathbf{a}, \mathbf{c}_1, \ldots, \mathbf{c}_k) = g}\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i}(a_i, c_{1i}, \ldots, c_{ki}) = g \quad (\text{Definition of } \varphi^{\prod\mathfrak{A}_i})\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i^*}(a_i^*, c_{1i}^*, \ldots, c_{ki}^*) = g \quad (\text{Definition of } \varphi^{\mathfrak{A}_i^*})\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i^*}(b_i^*, c_{1i}^*, \ldots, c_{ki}^*) = g \quad (a_i^* = b_i^*, i \in I)\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i}(b_i, c_{1i}, \ldots, c_{ki}) = g \quad (\text{Definition of } \varphi^{\mathfrak{A}_i^*})\\ \text{iff} & \varphi^{\prod\mathfrak{A}_i}(\mathbf{b}, \mathbf{c}_1, \ldots, \mathbf{c}_k) = g. \quad (\text{Definition of } \varphi^{\prod\mathfrak{A}_i})\end{array}$$

By Theorem 135, $(\mathbf{a})^* = (\mathbf{b})^*$, showing that h is well defined.

Next, we show that h is an algebra homomorphism. We have, for all $f \in F$, with $\rho(f) = n$, and all $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{i\in I} A_i$,

$$\begin{aligned}
&h(f^{\prod \mathfrak{A}_i^*}(\mathbf{a}_1^*, \dots, \mathbf{a}_n^*))\\
&\quad = h(\langle f^{\mathfrak{A}_i^*}(a_{1i}^*, \dots, a_{ni}^*) : i \in I\rangle) \quad \text{(Definition of } f^{\prod \mathfrak{A}_i^*})\\
&\quad = h(\langle f^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni})^* : i \in I\rangle) \quad \text{(Definition of } f^{\mathfrak{A}_i^*})\\
&\quad = \langle f^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni}) : i \in I\rangle^* \quad \text{(Definition of } h)\\
&\quad = f^{\prod \mathfrak{A}_i}(\mathbf{a}_1, \dots, \mathbf{a}_n)^* \quad \text{(Definition of } f^{\prod \mathfrak{A}_i})\\
&\quad = f^{(\prod \mathfrak{A}_i)^*}((\mathbf{a}_1)^*, \dots, (\mathbf{a}_n)^*) \quad \text{(Definition of } f^{(\prod \mathfrak{A}_i)^*})\\
&\quad = f^{(\prod \mathfrak{A}_i)^*}(h(\mathbf{a}_1^*), \dots, h(\mathbf{a}_n^*)). \quad \text{(Definition of } h)
\end{aligned}$$

To see that h is a G-algebra morphism, let $\mathbf{a}, \mathbf{b} \in \prod_{i\in I} A_i$. Then

$$\begin{aligned}
E^{\prod \mathfrak{A}_i^*}(\mathbf{a}^*, \mathbf{b}^*) &= \textstyle\bigwedge_{i\in I} E^{\mathfrak{A}_i^*}(a_i^*, b_i^*) \quad \text{(Definition of } E^{\prod \mathfrak{A}_i^*})\\
&= \textstyle\bigwedge_{i\in I} E^{\mathfrak{A}_i}(a_i, b_i) \quad \text{(Definition of } E^{\mathfrak{A}_i^*})\\
&= E^{\prod \mathfrak{A}_i}(\mathbf{a}, \mathbf{b}) \quad \text{(Definition of } E^{\prod \mathfrak{A}_i})\\
&= E^{(\prod \mathfrak{A}_i)^*}((\mathbf{a})^*, (\mathbf{b})^*) \quad \text{(Definition of } E^{(\prod \mathfrak{A}_i)^*})\\
&= E^{(\prod \mathfrak{A}_i)^*}(h(\mathbf{a}^*), h(\mathbf{b}^*)). \quad \text{(Definition of } h)
\end{aligned}$$

Let us, finally, show that h is a strict morphism. Consider $r \in R$, with $\rho(r) = n$, and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{i\in I} A_i$. Then

$$\begin{aligned}
&r^{(\prod \mathfrak{A}_i)^*}(h(\mathbf{a}_1^*), \dots, h(\mathbf{a}_n^*))\\
&\quad = r^{(\prod \mathfrak{A}_i)^*}((\mathbf{a}_1)^*, \dots, (\mathbf{a}_n)^*) \quad \text{(Definition of } h)\\
&\quad = r^{\prod \mathfrak{A}_i}(\mathbf{a}_1, \dots, \mathbf{a}_n) \quad \text{(Definition of } r^{(\prod \mathfrak{A}_i)^*})\\
&\quad = \textstyle\bigwedge_{i\in I} r^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni}) \quad \text{(Definition of } r^{\prod \mathfrak{A}_i})\\
&\quad = \textstyle\bigwedge_{i\in I} r^{\mathfrak{A}_i^*}(a_{1i}^*, \dots, a_{ni}^*) \quad \text{(Definition of } r^{\mathfrak{A}_i^*})\\
&\quad = r^{\prod \mathfrak{A}_i^*}(\mathbf{a}_1^*, \dots, \mathbf{a}_n^*) \quad \text{(Definition of } r^{\prod \mathfrak{A}_i^*})
\end{aligned}$$

By Theorem 146, $h^* : (\prod_{i\in I} \mathfrak{A}_i^*/\mathcal{F})^* \cong (\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})^*$.

■

4.14 Universal Atomic Classes

Let K be a class of $\mathcal{L}$-structures. K is a **universal atomic class** if $\mathsf{K} = \mathrm{Mod}(\Gamma)$, for some set $\Gamma = \{\Gamma_g : g \in G\}$ of atomic $\mathcal{L}$-formulas, where Γ_g represents formulas that should be evaluated to $\geq g$. Let $\mathrm{Atm}(\mathsf{K}) = \{\mathrm{Atm}_g(\mathsf{K}) : g \in G\}$ be the set of all atomic $\mathcal{L}$-formulas evaluated to $\geq g$ in all members of K. Then K is a universal atomic class if and only if $\mathsf{K} = \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}))$.

Lemma 156 *Let K be a class of $\mathcal{L}$-structures. If K is universal atomic, then it is closed under $\mathbb{H}$, $\mathbb{E}$, $\mathbb{S}$ and $\mathbb{P}$.*

Proof: Since $\mathsf{K} = \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}))$, to show that a structure is in K, it suffices to show that it "satisfies" an arbitrary atomic formula $r(t_1(\bar{x}), \dots, t_n(\bar{x}))$ in $\mathrm{Atm}_g(\mathsf{K})$, for all $g \in G$.

Suppose, first, that $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$. Then, there exists $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{B} \overset{h}{\twoheadrightarrow} \mathfrak{A}$. Let $r(t_1(\bar{x}), \ldots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a}$ in A. Since h is surjective, there exists $\mathbf{b}$ in B, such that $h(\mathbf{b}) = \mathbf{a}$. We obtain

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) & = & r^{\mathfrak{A}}(t_1^{\mathbf{A}}(h(\mathbf{b})), \ldots, t_n^{\mathbf{A}}(h(\mathbf{b}))) \\ & = & r^{\mathfrak{A}}(h(t_1^{\mathbf{B}}(\mathbf{b})), \ldots, h(t_n^{\mathbf{A}}(\mathbf{b}))) \quad (h : \mathbf{B} \to \mathbf{A}) \\ & \geq & r^{\mathfrak{B}}(t_1^{\mathbf{B}}(\mathbf{b}), \ldots, t_n^{\mathbf{B}}(\mathbf{b})). \quad (h : \mathfrak{B} \to \mathfrak{A}) \\ & \geq & g. \quad (\mathfrak{B} \in \mathsf{K}) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under morphic images.

Suppose, next, that $\mathfrak{A} \in \mathbb{E}(\mathsf{K})$. Then, there exists $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \overset{h}{\twoheadrightarrow}_s \mathfrak{B}$. Let $r(t_1(\bar{x}), \ldots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a}$ in A. We get

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) & = & r^{\mathfrak{B}}(h(t_1^{\mathbf{A}}(\mathbf{a})), \ldots, h(t_n^{\mathbf{A}}(\mathbf{a}))) \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & = & r^{\mathfrak{B}}(t_1^{\mathbf{B}}(h(\mathbf{a})), \ldots, t_n^{\mathbf{B}}(h(\mathbf{a}))) \quad (h : \mathbf{A} \to \mathbf{B}) \\ & \geq & g. \quad (\mathfrak{B} \in \mathsf{K}) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under expansions.

Suppose, now, that $\mathfrak{A} \in \mathbb{S}(\mathsf{K})$. Then, there exists $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \subseteq \mathfrak{B}$. We have, for all $r(t_1(\bar{x}), \ldots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a}$ in A,

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) & = & r^{\mathfrak{B}}(t_1^{\mathbf{B}}(\mathbf{a}), \ldots, t_n^{\mathbf{B}}(\mathbf{a})) \quad (\mathfrak{A} \subseteq \mathfrak{B}) \\ & \geq & g. \quad (\mathfrak{B} \in \mathsf{K}) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under substructures.

Suppose, finally, that $\mathfrak{A} \in \mathbb{P}(\mathsf{K})$. Then, there exist $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$, such that $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i$. We have, for all $r(t_1(\bar{x}), \ldots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a} = \langle \mathbf{a}_i : i \in I \rangle$ in A,

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) & = & \bigwedge_{i \in I} r^{\mathfrak{A}_i}(t_1^{\mathbf{A}_i}(\mathbf{a}_i), \ldots, t_n^{\mathbf{A}_i}(\mathbf{a}_i)) \quad (\text{Definition of } r^{\mathfrak{A}}) \\ & \geq & g. \quad (\mathfrak{A}_i \in \mathsf{K},\ i \in I) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under products. ∎

Theorem 157 *Let K be a class of $\mathcal{L}$-structures. The following statements are equivalent:*

(i) K *is a universal atomic class;*

(ii) K *is closed under* $\mathbb{H}$, $\mathbb{E}$, $\mathbb{S}$ *and* $\mathbb{P}$*;*

(iii) $\mathsf{K} = \mathbb{HESP}(\mathsf{K}')$, *for some class* K' *of $\mathcal{L}$-structures.*

Proof:

(i)⇒(ii) By Lemma 156.

(ii)⇒(iii) This is clear by taking $\mathsf{K}' = \mathsf{K}$.

(iii)⇒(i) We have that

$$\mathsf{K} \subseteq \mathrm{Mod}(\mathrm{Atm}(\mathsf{K})) = \mathrm{Mod}(\mathrm{Atm}(\mathbb{HESP}(\mathsf{K}'))) \subseteq \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}')).$$

For the reverse inclusion, suppose $\mathfrak{A} \in \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}'))$. To show that $\mathfrak{A} \in \mathsf{K}$, it suffices, by hypothesis, to show that $\mathfrak{A} \in \mathbb{HESP}(\mathsf{K}')$. Let $\Delta = \{\Delta_g : g \in G\}$, where Δ_g is the set of atomic $\mathcal{L}_A$-sentences $\varphi(c_{\mathbf{a}})$, such that $\varphi^{(\mathfrak{A},a)_{a\in A}} \ngeq g$.

Claim: If $\varphi(c_{\mathbf{a}}) \in \Delta_g$, there exist $\mathfrak{B}_\varphi \in \mathsf{K}'$ and $\{b_{a,\varphi} : a \in A\} \subseteq B_\varphi$, such that

$$\varphi^{(\mathfrak{B}_\varphi, b_{a,\varphi})_{a\in A}} \ngeq g.$$

Suppose to the contrary. Then, for all $\mathfrak{B} \in \mathsf{K}'$ and all $\mathbf{b}$ in B, $\varphi^{\mathfrak{B}}(\mathbf{b}) \geq g$. Hence, $\varphi(\bar{x}) \in \mathrm{Atm}_g(\mathsf{K}')$. Thus, since $\mathfrak{A} \in \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}'))$, $\varphi^{(\mathfrak{A},a)_{a\in A}} \geq g$. This contradicts the hypothesis $\varphi \in \Delta_g$.

Define

$$\begin{array}{rcl} \mathfrak{B} & = & \prod_{\varphi\in\Delta} \mathfrak{B}_\varphi; \\ b_a & = & \langle b_{a,\varphi} : \varphi \in \Delta\rangle, \quad a \in A; \\ \mathfrak{C} & = & \mathfrak{B}{\restriction}_{\{b_a : a\in A\}}. \end{array}$$

By construction, $\mathfrak{C} \in \mathbb{SP}(\mathsf{K}')$. Also by construction, for all $g \in G$ and all $\varphi \in \Delta_g$, $\varphi^{(\mathfrak{B},b_a)_{a\in A}} \ngeq g$. Thus, for all $g \in G$ and all $\varphi \in \Delta_g$, $\varphi^{(\mathfrak{C},b_a)_{a\in A}} \ngeq g$. Let $V = \{x_a : a \in A\}$ and consider $\mathbf{T} := \mathbf{Tm}_{\mathcal{L}}(V)$, the absolutely free $\mathcal{L}$-algebra generated by V and let $\mathcal{T} = \langle \mathbf{T}, \Delta_{\mathbf{T}}\rangle$. Let $h : \mathcal{T} \to \mathcal{C}$ be specified by

$$h(x_a) = b_a, \quad a \in A.$$

Consider the preimage $h^{-1}(\mathfrak{C})$ of $\mathfrak{C}$ under h. Then $h : h^{-1}(\mathfrak{C}) \twoheadrightarrow_s \mathfrak{C}$.

Claim: $g : \mathcal{T} \to \mathcal{A}$, specified by $x_a \mapsto a$, defines a surjective homomorphism $g : h^{-1}(\mathfrak{C}) \twoheadrightarrow \mathfrak{A}$.

Clearly, $g : \mathcal{T} \to \mathcal{A}$ is a surjective morphism of G-algebras. To see that it is a morphism of structures, consider $r \in R$, with $\rho(r) = n$, and $t_1(x_{a_1}, \ldots, x_{a_k}), \ldots, t_n(x_{a_1}, \ldots, x_{a_k})$ $\mathcal{L}$-terms in k variables. Then

$$\begin{array}{l} r^{h^{-1}(\mathfrak{C})}(t_1(x_{a_1}, \ldots, x_{a_k}), \ldots, t_n(x_{a_1}, \ldots, x_{a_k})) \\ \qquad = r^{\mathfrak{C}}(t_1^{\mathbf{C}}(b_{a_1}, \ldots, b_{a_k}), \ldots, t_n^{\mathbf{C}}(b_{a_1}, \ldots, b_{a_k})) \\ \qquad\quad \text{(Definition of } r^{h^{-1}(\mathfrak{C})}) \\ \qquad \leq r^{\mathfrak{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_k), \ldots, t_n^{\mathbf{A}}(a_1, \ldots, a_k)) \\ \qquad\quad (\varphi \in \Delta_g \text{ implies } \varphi^{(\mathfrak{C},b_a)_{a\in A}} \ngeq g) \\ \qquad = r^{\mathfrak{A}}(g(t_1(x_{a_1}, \ldots, x_{a_k})), \ldots, g(t(x_{a_1}, \ldots, x_{a_k})). \\ \qquad\quad \text{(Definition of } g) \end{array}$$

This shows that $g : h^{-1}(\mathfrak{C}) \twoheadrightarrow \mathfrak{A}$. We have the following formation.

$$\begin{array}{ccc} h^{-1}(\mathfrak{C}) & \xrightarrow{\quad s \quad} & \mathfrak{C} \\ \Big\downarrow & & \Big\downarrow \subseteq \\ \mathfrak{A} & & \prod_{\varphi \in \Delta} \mathfrak{B}_\varphi \end{array}$$

Since $\mathfrak{A}$ is a homomorphic image of $h^{-1}(\mathfrak{C})$, $h^{-1}(\mathfrak{C})$ is an extension of $\mathfrak{C}$ and $\mathfrak{C} \in \mathbb{SP}(\mathsf{K}')$, we get that $\mathfrak{A} \in \mathbb{HESP}(\mathsf{K}')$. Therefore, by the hypothesis, $\mathfrak{A} \in \mathsf{K}$.

■

Remark: The preceding result yields an analog of Birkhoff's Variety Theorem when one specializes to languages having G-equality. In that case reductive homomorphisms are exactly isomorphisms.

Let K be a class of $\mathcal{L}$-algebras. The **(full) universal atomic class generated by** K, denoted K^V, is the class

$$\mathsf{K}^V = \mathrm{Mod}(\mathrm{Atm}(\mathsf{K})).$$

The **reduced universal atomic class generated by** K is the class

$$\mathsf{K}^{V*} = \mathbb{L}(\mathrm{Mod}(\mathrm{Atm}(\mathsf{K}))).$$

Lemma 158 *Let* K *be a class of* $\mathcal{L}$*-structures.*

(a) $\mathsf{K}^V = \mathbb{HESP}(\mathsf{K})$*;*

(b) $\mathsf{K}^{V*} = \mathbb{F}^*\mathbb{ESP}(\mathsf{K})$*.*

Proof:

(a) By the proof of Theorem 157, we get $\mathrm{Mod}(\mathrm{Atm}(\mathsf{K})) = \mathbb{HESP}(\mathsf{K})$. So, by definition, $\mathsf{K}^V = \mathbb{HESP}(\mathsf{K})$.

(b) It suffices to show that $\mathbb{LHESP} = \mathbb{F}^*\mathbb{ESP}$. We have

$$\begin{array}{rcll} \mathbb{LHESP} & = & \mathbb{LRFESP} & \text{(by Lemma 150)} \\ & = & \mathbb{LFESP} & \text{(by Lemma 151)} \\ & = & \mathbb{F}^*\mathbb{ESP}. & \end{array}$$

This proves Part (b).

■

Open Questions

There are many problems that we have not addresed concerning the framework developed here (or, possibly, some modified version of it, if more appropriate and/or convenient). To mention the few very obvious ones, inspired by classical model theory and the adaptation in [23], how could one work to obtain analogs of the characterization of classes axiomatized (in some way) by arbitrary first-order sentences, by universal sentences or by universal Horn sentences?

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