Categorical Abstract Algebraic Logic:
Multi-Valued Referential Matrix System Semantics

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Abstract

Following work of Malinowski, the notion of a multi-valued referential gmatrix system is introduced to provide a semantics for logics formalized as π-institutions. A π-institution is said to be m-referential if it possesses an m-valued referential semantics. We show that it suffices to consider only semantics consisting of a single m-valued referential gmatrix system. Moreover, we identify conditions that characterize m-referential π-institutions.

1 Introduction

Consider a **language type** $\mathcal{L} = \langle \Lambda, \rho \rangle$, where Λ is a set of logical connectives/operation symbols and $\rho : \Lambda \to \omega$ is a function assigning to each operation symbol its arity. Let V be a countable set of variables. Denote by $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V), \mathcal{L} \rangle$ the free $\mathcal{L}$-algebra generated by V. A **logic** $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ consists of a language type together with a structural consequence relation on $\mathrm{Fm}_{\mathcal{L}}(V)$. As is well-known, structural consequence relations are in one-to-one correspondence with structural closure operators (see, e.g., page 33 of [2]). Thus, a logic may be equivalently represented as a pair $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where C is a structural closure operator on $\mathrm{Fm}_{\mathcal{L}}(V)$.

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A **generalized matrix**, or **gmatrix**, for $\mathcal{L}$ is a pair $\mathbb{A} = \langle \mathbf{A}, \mathcal{D} \rangle$, where $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ is an $\mathcal{L}$-algebra and $\mathcal{D}$ is a family of subsets of A.

A gmatrix $\mathbb{A} = \langle \mathbf{A}, \mathcal{D} \rangle$ **determines** a logic $\mathcal{S}^{\mathbb{A}} = \langle \mathcal{L}, C^{\mathbb{A}} \rangle$, defined, for all $\Phi \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\begin{array}{lll} \varphi \in C^{\mathbb{A}}(\Phi) & \text{iff} & \text{for all } h \in \mathrm{Hom}(\mathbf{Fm}_{\mathcal{L}}(V), \mathbf{A}) \text{ and all } D \in \mathcal{D}, \\ & & h(\Phi) \subseteq D \text{ implies } h(\varphi) \in D. \end{array}$$

Given a class K of gmatrices for $\mathcal{L}$, the logic **determined by** K is defined by $\mathcal{S}^{\mathsf{K}} = \langle \mathcal{L}, C^{\mathsf{K}} \rangle$, where $C^{\mathsf{K}} = \bigcap_{\mathbb{A} \in \mathsf{K}} C^{\mathbb{A}}$.

A class of gmatrices for $\mathcal{L}$ is said to form a **gmatrix semantics** for a logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ if $C^{\mathsf{K}} = C$.

A **referential algebra** for $\mathcal{L}$ is an $\mathcal{L}$-algebra $\mathbf{R} = \langle R, \mathcal{L}^{\mathbf{R}} \rangle$ such that R consists of a collection of subsets of a set U of **base** or **reference points**. For all $a \in U$, set $D_a = \{X \in R : a \in X\}$ and $\mathcal{D} = \{D_a : a \in U\}$. Then the gmatrix $\mathbb{R} = \langle \mathbf{R}, \mathcal{D} \rangle$ for $\mathcal{L}$ is called a **referential gmatrix** for $\mathcal{L}$ over U.

A logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is **self-extensional** if for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} C(\alpha) = C(\beta) & \text{implies} & C(\varphi(\alpha, \overline{z})) = C(\varphi(\beta, \overline{z})), \\ & & \text{for all } \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V). \end{array}$$

The relation $\Lambda(\mathcal{S})$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ defined, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$ by

$$\langle \alpha, \beta \rangle \in \Lambda(\mathcal{S}) \quad \text{iff} \quad C(\alpha) = C(\beta)$$

is called the **interderivability** or **Frege relation** of $\mathcal{S}$. The relation $\widetilde{\Omega}(\mathcal{S})$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ defined, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\begin{array}{lll} \langle \alpha, \beta \rangle \in \widetilde{\Omega}(\mathcal{S}) & \text{iff} & C(\varphi(\alpha, \overline{z})) = C(\varphi(\beta, \overline{z})), \\ & & \text{for all } \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V) \end{array}$$

is called the **Tarski relation** of $\mathcal{S}$. Thus, a logic $\mathcal{S}$ is self-extensional if and only if $\Lambda(\mathcal{S}) \subseteq \widetilde{\Omega}(\mathcal{S})$. Since the reverse inclusion always holds, a logic is self-extensional if and only if its Frege and its Tarski relations coincide. These relations have been studied extensively in the context of abstract algebraic logic (see, e.g., [3] and [2]).

A fundamental result due to Wójcicki [8] (see, also, [9]) asserts that a logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is self-extensional if and only if it has a referential semantics, i.e., if and only if $C = C^{\mathsf{K}}$, for a class K of referential gmatrices. In fact, Wójcicki shows that this holds if and only if $C = C^{\mathbb{R}}$, for a single referential gmatrix $\mathbb{R}$ (Proposition (A) on page 379 in [9]).

In [7], Malinowski, based on the aforementioned work of Wójcicki, as well as his own previous work on referential semantics (e.g., [5, 6]), defined the notion of multi-valued referential semantics for sentential logics. As is evident by the examples provided on pages 144-5 of [7], the motivation came from a desire to provide a referential-like semantics for sentential logics, like Łukasiewicz's multi-valued logics, which are built with the purpose of modeling multi-valued systems.

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a language type, m an integer, $m \geq 2$, $E_1, \ldots, E_{m-2}$ unary function symbols (fundamental or derived) in Λ and T a collection of **base** or **reference points**. A gmatrix $\mathbb{A} = \langle \mathbf{A}, \mathcal{D} \rangle$ is called an m-(**valued**) **referential gmatrix** (**based on**) T [7] if the following conditions are satisfied:

- The universe A of the algebra $\mathbf{A}$ is a subset of $\{e_0, e_1, \ldots, e_{m-1}\}^T$, i.e., consists of functions of the form $r : T \to \{e_0, e_1, \ldots, e_{m-1}\}$. The elements e_0 and e_{m-1} are denoted, respectively, by 0 and 1.
- $\mathcal{D} = \{D_t : t \in T\}$, where $D_t = \{r \in A : r(t) = 1\}$, for all $t \in T$.
- The function symbols $E_1, \ldots, E_{m-2}$ are interpreted in $\mathbf{A}$ as follows:

$$E_i^{\mathbf{A}}(r)(t) = \begin{cases} 1, & \text{if } r(t) = e_i, \\ 0, & \text{otherwise,} \end{cases} \qquad i \in \{1, \ldots, m-2\}.$$

In Corollary 1 of [7] Malinowski asserts that this is a genuine generalization of the notion of a referential gmatrix, since it reduces to that concept for $m = 2$ (for which the last condition becomes vacuous).

A logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is said to be m-**referential** if it possesses a semantics consisting of m-referential gmatrices. However, in Proposition 2 of [7], Malinowksi asserts that, as is the case with referential semantics, one only needs to consider semantics consisting of a single m-referential gmatrix in this context. This assertion is based on the construction given by Wójcicki on page 379 of [9] to prove the result for the case of self-extensional logics.

Malinowski then focuses on finding an intrinsic characterization of m-referential sentential logics, i.e., one that does not refer to the external m-referential gmatrix semantics of the logical system.

Let, again $\mathcal{L}$ be a language type, $m \geq 2$ an integer and $E_1, \ldots, E_{m-2}$ unary operations (fundamental or derived) of $\mathcal{L}$. A sentential logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is called m-**normal** [7] if the following axioms hold, for all $i, j \in \{1, \ldots, m-2\}$, $v \in V$ and $\alpha \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(N0) $C(E_i(v)) \neq \mathrm{Fm}_{\mathcal{L}}(V)$;

(N1) $C(\alpha, E_i(\alpha)) = \mathrm{Fm}_{\mathcal{L}}(V)$;

(N2) $C(E_i(E_j(\alpha))) = \mathrm{Fm}_{\mathcal{L}}(V)$;

(N3) $C(E_i(\alpha), E_j(\alpha)) = \mathrm{Fm}_{\mathcal{L}}(V)$, for $i \neq i$.

Further, for all $i \in \{1, \dots, m-2\}$, we define a relation $\sim^i$ on $\mathrm{Fm}_{\mathcal{L}}(V)$, by setting, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\alpha \sim^i \beta \quad \text{iff} \quad C(E_i(\alpha)) = C(E_i(\beta)).$$

In the main theorem of [7], Malinowski proves that a sentential logic $\mathcal{S}$ is m-referential if and only if it is m-normal and

$$\widetilde{\Omega}(\mathcal{S}) = \Lambda(\mathcal{S}) \cap \sim^1 \cap \cdots \cap \sim^{m-2}.$$

A corollary of this result (for $m = 2$) is that a logic is referential if and only if it is self-extensional, i.e., the result of Wójcicki [8].

In this work we introduce multi-valued gmatrix systems as a means of providing a referential-like semantics for π-institutions and provide analogs of the main results of Malinowski for logics formalized as π-institutions. Namely, after introducing the necessary notions and machinery, we prove that a π-institution that has a multi-valued referential semantics has necessarily one consisting of a single multi-valued gmatrix system and give a characterization of those π-institutions that possess a multi-valued gmatrix system semantics in terms of an analog of the notion of m-normality, appropriately abstracted to the categorical context.

2 Preliminaries

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a **Set**-valued functor. The **clone of all natural transformations on** SEN is the category U with collection of objects SEN^α, α an ordinal, and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition of $\langle \tau_i : i < \beta \rangle : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ with $\langle \sigma_j : j < \gamma \rangle : \mathrm{SEN}^\beta \to \mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory of this category with objets all objects of the form SEN^k, $k < \omega$, and such that:

- it contains all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$, $i < k$, $k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}$ given by

$$p^{k,i}_\Sigma(\overline{\phi}) = \phi_i, \text{ for all } \overline{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

- for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < \ell\}$ of natural transformations in N, $\langle\tau_i : i < \ell\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ is also in N,

is referred to as a **category of natural transformations on** SEN (see, e.g., Section 2 of [12]).

An **algebraic system** is a triple $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ consisting of:

- A category **Sign** of **signatures**;
- A functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ giving, for each signature $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{SEN}(\Sigma)$ of Σ-**sentences**;
- A category of natural transformations N on SEN.

Usually, in a specific context, a fixed underlying algebraic system is assumed, called the **base algebraic system** and denoted by $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$. Then, an $N^\flat$-**algebraic system** $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is one such that there exists a surjective functor $N^\flat \to N$ that preserves all projection natural transformations (and, consequently, all arities of natural transformations involved). In this situation, a typographical correspondence is used to denote the natural transformation in N that is the image of a specific natural transformation in $N^\flat$, such as, e.g., σ for the image of $\sigma^\flat$.

An **interpreted** $N^\flat$-**algebraic system** is a pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, such that $\mathbf{A}$ is an $N^\flat$-algebraic system and $\langle F, \alpha\rangle : \mathbf{A}^\flat \to \mathbf{A}$ is an algebraic system morphism. In other words:

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is a functor;
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ is a natural transformation, such that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \dots, \varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}^\flat(\Sigma)^k \arrow[r, "\alpha^k_\Sigma"] \arrow[d, "\sigma^\flat_\Sigma"'] & \mathrm{SEN}(F(\Sigma))^k \arrow[d, "\sigma_{F(\Sigma)}"] \\
\mathrm{SEN}^\flat(\Sigma) \arrow[r, "\alpha_\Sigma"'] & \mathrm{SEN}(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})) = \sigma_{F(\Sigma)}(\alpha_\Sigma(\varphi_0),\dots,\alpha_\Sigma(\varphi_{k-1})),$$

where, using the aforementioned convention, $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ denotes the image natural transformation in N of $\sigma^\flat$ in $N^\flat$.

A **gmatrix system (for $\mathbf{A}^\flat$)** is a pair $\mathbb{A} = \langle \mathcal{A}, \mathcal{D}\rangle$, where $\mathcal{A}$ is an interpreted $N^\flat$-algebraic system and $\mathcal{D} = \{D^i : i \in I\}$ is a collection of filter families on $\mathbf{A}$, i.e., $D^i = \{D^i_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that $D^i_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$ and all $i \in I$.

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. A **π-institution based on $\mathbf{A}^\flat$** (see [1] and, also, [4] for the closely related notion of an institution) is a pair $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$, where $C = \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a **closure system** on $\mathbf{A}^\flat$, i.e., a collection of closure operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$, $\Sigma \in |\mathbf{Sign}^\flat|$, which satisfies the **structurality condition**, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and let $\mathbb{A} = \langle \mathcal{A}, \mathcal{D}\rangle$ be a gmatrix system for $\mathbf{A}^\flat$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. The gmatrix system $\mathbb{A}$ **generates** a closure system $C^{\mathbb{A}}$ on $\mathbf{A}^\flat$ by the following rule: For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\varphi \in C^{\mathbb{A}}_\Sigma(\Phi) & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma') \text{ and all } i \in I,\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq D^i_{F(\Sigma')}\\ & & \qquad\qquad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in D^i_{F(\Sigma')}.\end{array}$$

If K is a class of gmatrix systems for $\mathbf{A}^\flat$, then we set

$$C^{\mathsf{K}} = \bigcap_{\mathbb{A}\in\mathsf{K}} C^{\mathbb{A}},$$

where the intersection is applied signature-wise. The corresponding π-institutions are denoted by $\mathcal{I}^{\mathbb{A}} = \langle \mathbf{A}^\flat, C^{\mathbb{A}}\rangle$ and $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{A}^\flat, C^{\mathsf{K}}\rangle$. Note that both are based on the base algebraic system $\mathbf{A}^\flat$.

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$ be a π-institution based on $\mathbf{A}^\flat$. We say that a class of gmatrix systems K for $\mathbf{A}^\flat$ is a **gmatrix system semantics for $\mathcal{I}$** in case $C^{\mathsf{K}} = C$.

3 Multi-Valued Referential Semantics

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system.

Let **Sign** be a category and $\mathrm{PTS} : |\mathbf{Sign}| \to \mathbf{Set}$ a functor, giving, for all $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{PTS}(\Sigma)$ of all Σ-**base** or Σ-**reference points**.

Let $m \geq 2$ be an integer and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}(\Sigma) \subseteq \{e_0, \dots, e_{m-1}\}^{\mathrm{PTS}(\Sigma)},$$

i.e., such that $\mathrm{SEN}(\Sigma)$ consists of functions $r : \mathrm{PTS}(\Sigma) \to \{e_0, \dots, e_{m-1}\}$. We set $0 \coloneqq e_0$ and $1 \coloneqq e_{m-1}$.

Let N be a category of natural transformations on SEN rendering $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ an $N^\flat$-algebraic system. We call an $N^\flat$-algebraic system of this form an m-**referential** $N^\flat$-**algebraic system**.

If $\langle F, \alpha \rangle : \mathbf{A}^\flat \to \mathbf{A}$ is an algebraic system morphism, then the pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is called an (**interpreted**) m-**referential** $N^\flat$-**algebraic system**. When the qualifier "interpreted" is omitted, we rely on context to clarify whether the system under consideration is interpreted (i.e., is accompanied by the morphism $\langle F, \alpha \rangle$) or not.

Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an m-referential $N^\flat$-algebraic system. Let $\Sigma \in |\mathbf{Sign}|$ and $p \in \mathrm{PTS}(\Sigma)$. Define the filter family $D^{\Sigma,p} = \{D^{\Sigma,p}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$ by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$D^{\Sigma,p}_{\Sigma'} = \begin{cases} \{r \in \mathrm{SEN}(\Sigma) : r(p) = 1\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma. \end{cases}.$$

Finally, let

$$\mathcal{D} = \{D^{\Sigma,p} : \Sigma \in |\mathbf{Sign}|, p \in \mathrm{PTS}(\Sigma)\}$$

and define $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$.

A gmatrix system for $\mathbf{A}^\flat$ of this form is called an m-**referential** $N^\flat$-**gmatrix system**.

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $m \geq 2$ and

$$E = \{E^{1\,\flat}, \dots, E^{(m-2)\,\flat}\}$$

be a set of $m-2$ unary natural transformations in $N^\flat$. An m-referential $N^\flat$-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$ based on PTS is said to be **normal** (**with respect to** E) if, for all $i = 1, \dots, m-2$, and all $r \in \mathrm{SEN}(\Sigma)$,

$$E^i_\Sigma(r) : \mathrm{PTS}(\Sigma) \to \{0, e_1, \dots, e_{m-2}, 1\}$$

is given, for all $p \in \mathrm{PTS}(\Sigma)$, by

$$E^i_\Sigma(r)(p) = \begin{cases} 1, & \text{if } r(p) = e_i, \\ 0, & \text{otherwise.} \end{cases}$$

Note that, for all i, since $E^i : \mathrm{SEN} \to \mathrm{SEN}$ is a natural transformation in N, we have that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $r \in \mathrm{SEN}(\Sigma)$,

$$\begin{tikzcd} \mathrm{SEN}(\Sigma) \arrow[r, "E^i_\Sigma"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\mathrm{SEN}(f)"] \\ \mathrm{SEN}(\Sigma') \arrow[r, "E^i_{\Sigma'}"'] & \mathrm{SEN}(\Sigma') \end{tikzcd}$$

$$\mathrm{SEN}(f)(E^i_\Sigma(r)) = E^i_{\Sigma'}(\mathrm{SEN}(f)(r)),$$

i.e., for all $p' \in \mathrm{PTS}(\Sigma')$,

$$\mathrm{SEN}(f)(E^i_\Sigma(r))(p') = E^i_{\Sigma'}(\mathrm{SEN}(f)(r))(p') = \begin{cases} 1, & \text{if } \mathrm{SEN}(f)(r)(p') = e_i, \\ 0, & \text{otherwise.} \end{cases}$$

For $m = 2$ (in which case $E = \varnothing$), we identify the notion of a normal 2-referential $N^\flat$-algebraic system with that of a 2-referential $N^\flat$-algebraic system (normality being vacuously satisfied).

Moreover, under the obvious identification of functions $r : \mathrm{PTS}(\Sigma) \to \{0, 1\}$ with subsets $X \subseteq \mathrm{PTS}(\Sigma)$, we have the following:

Proposition 1 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. A gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$ is a 2-referential gmatrix system if and only if it is a referential gmatrix system in the sense of* [13].

Now we prove a proposition characterizing the closure system $C^{\mathbb{A}}$ on a base algebraic system $\mathbf{A}^\flat$ generated by a given m-referential $N^\flat$-gmatrix system $\mathbb{A}$.

Proposition 2 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Let $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an m-referential gmatrix system for $\mathbf{A}^\flat$ based on* PTS. *Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,*

$$\begin{array}{lll} \varphi \in C^{\mathbb{A}}_\Sigma(\Phi) & \textit{iff} & \textit{for all } \Sigma' \in |\mathbf{Sign}^\flat|, \textit{ all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\ & & \textit{and all } p \in \mathrm{PTS}(F(\Sigma')), \\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))(p) = 1, \textit{ for all } \phi \in \Phi \\ & & \qquad\qquad \textit{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = 1. \end{array}$$

Proof: It suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\Sigma^* \in |\mathbf{Sign}|$, $p^* \in \mathrm{PTS}(\Sigma^*)$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in D^{\Sigma^*,p^*}_{F(\Sigma')} \quad \text{iff} \quad \Sigma^* = F(\Sigma') \text{ and } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p^*) = 1.$$

If $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in D^{\Sigma^*,p^*}_{F(\Sigma')}$, then $D^{\Sigma^*,p^*}_{F(\Sigma')} \neq \varnothing$. By the definition of D^{Σ^*,p^*}, this is possible only if $F(\Sigma') = \Sigma^*$. Moreover, since $D^{\Sigma^*,p^*}_{\Sigma^*} = \{r \in \mathrm{SEN}(\Sigma^*) : r(p^*) = 1\}$, we obtain that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p^*) = 1$.

Suppose, conversely, that $\Sigma^* = F(\Sigma')$ and $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p^*) = 1$. Then

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in \{r \in \mathrm{SEN}(\Sigma^*) : r(p^*) = 1\} = D^{\Sigma^*,p^*}_{\Sigma^*}.$$

This concludes the proof of the equivalence above. ∎

A π-institution of the form $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{A}^\flat, C^{\mathsf{K}}\rangle$, where K is a class of normal m-referential $\mathbf{N}^\flat$-gmatrix systems (with respect to E) will be called **m-referential (with respect to E)**.

In the following theorem, paralleling Proposition 2 of [7], it is shown that an m-referential π-institution may be seen as generated by a single normal m-referential $N^\flat$-gmatrix system, with respect to E. This result has a precursor in the categorical case in Corollary 5 of [12] which, in turn, originates from a corresponding result pertaining to the referential semantics of sentential logics, Proposition (A) on page 379 in [9].

Proposition 3 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $m \geq 2$ an integer and $E = \{E^{1\flat},\ldots,E^{(m-2)\flat}\}$ a set of $m-2$ unary natural transformations in $N^\flat$. A π-institution $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$ is m-referential with respect to E if and only if there exists a normal m-referential $\mathbf{N}^\flat$-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{D}\rangle$ with respect to E, such that $C = C^{\mathbb{A}}$.*

Proof: The sufficiency of the condition is obvious. For the necessity, suppose that $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$ is m-referential. Then $C = C^{\mathsf{K}}$, where

$$K = \{\mathbb{A}^i = \langle \mathcal{A}^i, \mathcal{D}^i\rangle : i \in I\}$$

is a collection of normal m-referential $\mathbf{A}^\flat$-gmatrix systems with respect to E. Assume that $\mathbb{A}^i = \langle \mathcal{A}^i, \mathcal{D}^i\rangle$, with $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$ and $\mathbf{A}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, N^i\rangle$, is based on $\mathrm{PTS}^i : |\mathbf{Sign}^i| \to \mathbf{Set}$, for all $i \in I$.

Let $\mathbf{Sign} = \prod_{i\in I} \mathbf{Sign}^i$. Define $\mathrm{PTS} : |\mathbf{Sign}| \to \mathbf{Set}$ by setting

$$\mathrm{PTS}(\langle \Sigma_i : i \in I\rangle) = \biguplus_{i\in I} \mathrm{PTS}^i(\Sigma_i),\ \Sigma_i \in |\mathbf{Sign}^i|, i \in I,$$

where $\uplus$ denotes disjoint union. For simplicity, we assume in the sequel that all sets of points are already disjoint and so $\uplus$ may be taken to be ordinary set-theoretic union.

Next define $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ based on PTS as follows:

- For all $i \in I$ and $\Sigma_i \in |\mathbf{Sign}^i|$,

$$\begin{aligned}\mathrm{SEN}(\langle \Sigma_i : i \in I\rangle) &= \{r \in \{0,1,\dots,m-1\}^{\mathrm{PTS}(\langle \Sigma_i : i\in I\rangle)} : \\ &\qquad r|_{\mathrm{PTS}^i(\Sigma_i)} \in \mathrm{SEN}^i(\Sigma_i) \text{ for all } i \in I\}.\end{aligned}$$

  Further, given $\Sigma_i, \Sigma_i' \in |\mathbf{Sign}^i|$ and $f_i \in \mathbf{Sign}^i(\Sigma_i, \Sigma_i')$, for all $i \in I$, define

$$\mathrm{SEN}(\langle f_i : i \in I\rangle) : \mathrm{SEN}(\langle \Sigma_i : i \in I\rangle) \to \mathrm{SEN}(\langle \Sigma_i' : i \in I\rangle)$$

  by setting, for all $r \in \mathrm{SEN}(\langle \Sigma_i : i \in I\rangle)$, all i and all $p' \in \mathrm{PTS}^i(\Sigma_i')$,

$$\mathrm{SEN}(\langle f_i : i \in I\rangle)(r)(p') = \mathrm{SEN}^i(f_i)(r|_{\mathrm{PTS}^i(\Sigma_i)})(p').$$

  With these definitions $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ becomes a functor. Indeed, for all $i \in I$, $\Sigma_i, \Sigma_i', \Sigma_i'' \in |\mathbf{Sign}^i|$ all $f_i \in \mathbf{Sign}^i(\Sigma_i, \Sigma_i')$, $g_i \in \mathbf{Sign}^i(\Sigma_i', \Sigma_i'')$ and all $r \in \mathrm{SEN}(\langle \Sigma_i : i \in I\rangle)$,

$$\langle \Sigma_i : i \in I\rangle \xrightarrow{\langle f_i : i \in I\rangle} \langle \Sigma_i' : i \in I\rangle \xrightarrow{\langle g_i : i \in I\rangle} \langle \Sigma_i'' : i \in I\rangle$$

$$\begin{aligned}&\mathrm{SEN}(\langle g_i : i \in I\rangle)(\mathrm{SEN}(\langle f_i : i \in I\rangle)(r))|_{\mathrm{PTS}^i(\Sigma_i)}\\ &\quad = \mathrm{SEN}^i(g_i)(\mathrm{SEN}(\langle f_i : i \in I\rangle)(r)|_{\mathrm{PTS}^i(\Sigma_i)})\\ &\quad = \mathrm{SEN}^i(g_i)(\mathrm{SEN}^i(f_i)(r|_{\mathrm{PTS}^i(\Sigma_i)}))\\ &\quad = \mathrm{SEN}^i(g_i \circ f_i)(r|_{\mathrm{PTS}^i(\Sigma_i)})\\ &\quad = \mathrm{SEN}(\langle g_i \circ f_i : i \in I\rangle)(r)\\ &\quad = \mathrm{SEN}(\langle g_i : i \in I\rangle \circ \langle f_i : i \in I\rangle)(r).\end{aligned}$$

- Next, suppose that $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural transformation in $N^\flat$, with image transformation $\sigma^i : (\mathrm{SEN}^i)^k \to \mathrm{SEN}^i$ in N^i, for all $i \in I$. Define $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ by letting, for all $\Sigma_i \in |\mathbf{Sign}^i|$, $i \in I$,

$$\sigma_{\langle \Sigma_i : i\in I\rangle} : \mathrm{SEN}(\langle \Sigma_i : i \in I\rangle)^k \to \mathrm{SEN}(\langle \Sigma_i : i \in I\rangle)$$

  be given, for all $r_0, \dots, r_{k-1} \in \mathrm{SEN}(\langle \Sigma_i : i \in I\rangle)$ and all $i \in I$, $p \in \mathrm{PTS}^i(\Sigma_i)$,

$$\sigma_{\langle \Sigma_i : i\in I\rangle}(r_0, \dots, r_{k-1})(p) = \sigma^i_{\Sigma_i}(r_0|_{\mathrm{PTS}^i(\Sigma_i)}, \dots, r_{k-1}|_{\mathrm{PTS}^i(\Sigma_i)})(p).$$

Thus defined, $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is a natural transformation. Indeed, for all $i \in I$, $\Sigma_i, \Sigma'_i \in |\mathbf{Sign}^i|$, all $f_i \in \mathbf{Sign}^i(\Sigma_i, \Sigma'_i)$ and all $r_0, \ldots, r_{k-1} \in \mathrm{SEN}(\langle \Sigma_i : i \in I \rangle)$,

$$\begin{tikzcd}
\mathrm{SEN}(\langle \Sigma_i : i \in I \rangle)^k \arrow[r, "\sigma_{\langle \Sigma_i : i \in I \rangle}"] \arrow[d, "\mathrm{SEN}(\langle f_i : i \in I \rangle)^k"'] & \mathrm{SEN}(\langle \Sigma_i : i \in I \rangle) \arrow[d, "\mathrm{SEN}(\langle f_i : i \in I \rangle)"] \\
\mathrm{SEN}(\langle \Sigma'_i : i \in I \rangle)^k \arrow[r, "\sigma_{\langle \Sigma'_i : i \in I \rangle}"'] & \mathrm{SEN}(\langle \Sigma'_i : i \in I \rangle)
\end{tikzcd}$$

$$\begin{aligned}
&\mathrm{SEN}(\langle f_i : i \in I \rangle)(\sigma_{\langle \Sigma_i : i \in I \rangle}(r_0, \ldots, r_{k-1}))|_{\mathrm{PTS}^i(\Sigma'_i)} \\
&= \mathrm{SEN}^i(f_i)(\sigma_{\langle \Sigma_i : i \in I \rangle}(r_0, \ldots, r_{k-1})|_{\mathrm{PTS}^i(\Sigma_i)}) \\
&= \mathrm{SEN}^i(f_i)(\sigma^i_{\Sigma_i}(r_0|_{\mathrm{PTS}^i(\Sigma_i)}, \ldots, r_{k-1}|_{\mathrm{PTS}^i(\Sigma_i)})) \\
&= \sigma^i_{\Sigma'_i}(\mathrm{SEN}^i(f_i)(r_0|_{\mathrm{PTS}^i(\Sigma_i)}), \ldots, \mathrm{SEN}^i(f_i)(r_{k-1}|_{\mathrm{PTS}^i(\Sigma_i)})) \\
&= \sigma^i_{\Sigma'_i}(\mathrm{SEN}(\langle f_i : i \in I \rangle)(r_0)|_{\mathrm{PTS}^i(\Sigma'_i)}, \ldots, \\
&\qquad\qquad \mathrm{SEN}(\langle f_i : i \in I \rangle)(r_{k-1})|_{\mathrm{PTS}^i(\Sigma'_i)}) \\
&= \sigma_{\langle \Sigma'_i : i \in I \rangle}(\mathrm{SEN}(\langle f_i : i \in I \rangle)(r_0), \ldots, \mathrm{SEN}(\langle f_i : i \in I \rangle)(r_{k-1}))|_{\mathrm{PTS}^i(\Sigma'_i)}.
\end{aligned}$$

Moreover, the collection N of all such σ, for $\sigma^\flat$ in $N^\flat$, forms a category of natural transformations on SEN.

We conclude that the triple $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an m-referential $\mathbf{A}^\flat$-algebraic system.

Now define an algebraic system morphism $\langle F, \alpha \rangle : \mathbf{A}^\flat \to \mathbf{A}$ as follows:

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is defined on objects, for all $\Sigma \in |\mathbf{Sign}^\flat|$, by
$$F(\Sigma) = \langle F^i(\Sigma) : i \in I \rangle,$$
and on morphisms, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, by
$$F(f) = \langle F^i(f) : i \in I \rangle.$$
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ is defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(F(\Sigma))$ be given, for all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, by
$$\alpha_\Sigma(\varphi) = r \in \{0, 1, \ldots, m-1\}^{\mathrm{PTS}(\langle F^i(\Sigma) : i \in I \rangle)}$$
specified by
$$r|_{\mathrm{PTS}^i(F^i(\Sigma))} = \alpha^i_\Sigma(\varphi), \text{ for all } i \in I.$$

Thus defined, $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ is a natural transformation. Indeed, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}^\flat(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}^\flat(f)"'] & \mathrm{SEN}(F(\Sigma)) \arrow[d, "\mathrm{SEN}(F(f))"] \\
\mathrm{SEN}^\flat(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}(F(\Sigma'))
\end{tikzcd}$$

$$\begin{array}{rcl}
\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))|_{\mathrm{PTS}^i(F^i(\Sigma))} & = & \alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \\
& = & \mathrm{SEN}^i(F^i(f))(\alpha^i_\Sigma(\varphi)) \\
& = & \mathrm{SEN}^i(F^i(f))(\alpha_\Sigma(\varphi)|_{\mathrm{PTS}^i(F^i(\Sigma))}) \\
& = & \mathrm{SEN}(F(f))(\alpha_\Sigma(\varphi))|_{\mathrm{PTS}^i(F^i(\Sigma))}.
\end{array}$$

Moreover, the pair $\langle F, \alpha \rangle : \mathbf{A}^\flat \to \mathbf{A}$ is an algebraic system morphism, since, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}^\flat(\Sigma)^k \arrow[r, "\sigma^\flat_\Sigma"] \arrow[d, "\alpha^k_\Sigma"'] & \mathrm{SEN}^\flat(\Sigma) \arrow[d, "\alpha_\Sigma"] \\
\mathrm{SEN}(F(\Sigma))^k \arrow[r, "\sigma_{F(\Sigma)}"'] & \mathrm{SEN}(F(\Sigma))
\end{tikzcd}$$

$$\begin{array}{l}
\alpha_\Sigma(\sigma^\flat_\Sigma(\varphi_0, \ldots, \varphi_{k-1}))|_{\mathrm{PTS}^i(F^i(\Sigma))} \\
= \alpha^i_\Sigma(\sigma^\flat_\Sigma(\varphi_0, \ldots, \varphi_{k-1})) \\
= \sigma^i_{F^i(\Sigma)}(\alpha^i_\Sigma(\varphi_0), \ldots, \alpha^i_\Sigma(\varphi_{k-1})) \\
= \sigma^i_{F^i(\Sigma)}(\alpha_\Sigma(\varphi_0)|_{\mathrm{PTS}^i(F^i(\Sigma))}, \ldots, \alpha_\Sigma(\varphi_{k-1})|_{\mathrm{PTS}^i(F^i(\Sigma))}) \\
= \sigma_{\langle F^i(\Sigma) : i \in I \rangle}(\alpha_\Sigma(\varphi_0), \ldots, \alpha_\Sigma(\varphi_{k-1}))|_{\mathrm{PTS}^i(F^i(\Sigma))} \\
= \sigma_{F(\Sigma)}(\alpha_\Sigma(\varphi_0), \ldots, \alpha_\Sigma(\varphi_{k-1}))|_{\mathrm{PTS}^i(F^i(\Sigma))}
\end{array}$$

We conclude that the pair $\langle F, \alpha \rangle : \mathbf{A}^\flat \to \mathbf{A}$ is a well-defined algebraic system morphism and, therefore, that the pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an interpreted m-referential $N^\flat$-algebraic system.

Finally, we declare that $\mathcal{D} = \{D^{\Sigma,p} : \Sigma \in |\mathbf{Sign}|, p \in \mathrm{PTS}(\Sigma)\}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $p \in \mathrm{PTS}(\Sigma)$, $D^{\Sigma,p} = \{D^{\Sigma,p}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$ is given by setting, for all

$\Sigma' \in |\mathbf{Sign}|$,

$$D^{\Sigma,p}_{\Sigma'} = \begin{cases} \{r \in \mathrm{SEN}(\Sigma) : r(p) = 1\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma. \end{cases}$$

Then $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$ is an m-referential gmatrix system and it only remains to show that it is normal with respect to E and that $C^{\mathbb{A}} = \bigcap_{i \in I} C^{\mathbb{A}^i} (=: C^{\mathsf{K}})$.

To show that $\mathbb{A}$ is normal, let $k \in \{1, \ldots, m-2\}$. Since each $\mathbb{A}^i$ is normal with respect to $E = \{E^1, \ldots, E^{m-2}\}$, we have that, for all $i \in I$, all $\Sigma_i \in |\mathbf{Sign}^i|$, all $r \in \mathrm{SEN}(\langle \Sigma_i : i \in I \rangle)$ and all $p_i \in \mathrm{PTS}^i(\Sigma_i)$,

$$E^{ki}_{\Sigma_i}(r|_{\mathrm{PTS}^i(\Sigma_i)})(p_i) = \begin{cases} 1, & \text{if } r|_{\mathrm{PTS}^i(\Sigma_i)}(p_i) = e_k, \\ 0, & \text{otherwise.} \end{cases}$$

But this is equivalent to, for all $\Sigma = \langle \Sigma_i : i \in I \rangle \in |\mathbf{Sign}|$, all $r \in \mathrm{SEN}(\Sigma)$ and all $p \in \mathrm{PTS}(\Sigma)$,

$$E^k_\Sigma(r)(p) = \begin{cases} 1, & \text{if } r(p) = e_k, \\ 0, & \text{otherwise.} \end{cases}$$

So $\mathbb{A}$ is indeed normal.

Finally, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have $\varphi \in C^{\mathsf{K}}_\Sigma(\Phi)$ iff, for all $i \in I$, $\varphi = C^{\mathbb{A}^i}_\Sigma(\Phi)$ iff, by Proposition 2, for all $i \in I$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p_i \in \mathrm{PTS}^i(F^i(\Sigma'))$,

$$\begin{aligned} &\alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))(p_i) = 1, \text{ for all } \phi \in \Phi, \\ &\qquad\qquad \text{implies} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p_i) = 1, \end{aligned}$$

iff, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p \in \mathrm{PTS}(F(\Sigma'))$,

$$\begin{aligned} &\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))(p) = 1, \text{ for all } \phi \in \Phi, \\ &\qquad\qquad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = 1, \end{aligned}$$

iff, again by Proposition 2, $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$. ■

4 m-Normal π-Institutions

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $m \geq 2$ an integer, and assume that $N^\flat$ contains a set of $m-2$ unary natural transformations $E = \{E^{1\flat}, \ldots, E^{(m-2)\flat}\}$, $m \geq 3$. We say that a π-institution $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ based on $\mathbf{A}^\flat$ is m-**normal** (**with respect to** E) if the following conditions hold, for all $i, j \in \{1, \ldots, m-2\}$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$:

(N1) $C_\Sigma(\varphi, E^{i\flat}_\Sigma(\varphi)) = \mathrm{SEN}^\flat(\Sigma)$;

(N2) $C_\Sigma(E^{i\flat}_\Sigma(E^{j\flat}_\Sigma(\varphi))) = \mathrm{SEN}^\flat(\Sigma)$;

(N3) $C_\Sigma(E^{i\flat}_\Sigma(\varphi), E^{j\flat}_\Sigma(\varphi)) = \mathrm{SEN}^\flat(\Sigma)$, for $i \neq j$.

We now recall the notions of the interderivability equivalence system and of the Tarski congruence system of a π-institution $\mathcal{I}$.

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and let $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ be a π-institution based on $\mathbf{A}^\flat$. Define the **interderivability**, or **Frege**, **relation family** $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\Lambda_\Sigma(\mathcal{I}) \subseteq \mathrm{SEN}^\flat(\Sigma)^2$ be given, for all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

Define, also, the **Tarski relation family** $\widetilde{\Omega}(\mathcal{I}) = \{\widetilde{\Omega}_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\widetilde{\Omega}_\Sigma(\mathcal{I}) \subseteq \mathrm{SEN}^\flat(\Sigma)^2$ be given, for all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, by $\langle \varphi, \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$ iff

$$\begin{array}{l} \text{for all } \sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \text{ in } N^\flat, \text{all } \Sigma' \in |\mathbf{Sign}^\flat|, \\ f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \text{ and all } \overline{\chi} \in \mathrm{SEN}^\flat(\Sigma')^{k-1}, \\ C_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi), \overline{\chi})) = C_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \overline{\chi})). \end{array}$$

Here, to simplify notation, we adopt the convention that the last condition means that $\mathrm{SEN}^\flat(f)(\varphi)$ and $\mathrm{SEN}^\flat(f)(\psi)$ may occupy any position - and not only the first - in $\sigma^\flat_{\Sigma'}$, as long as they occupy the same position in the two sides of the equation.

Finally, in case $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ is a normal π-institution with respect to E, we also define, for all $i \in \{1, \ldots, m-2\}$, $\sim^i = \{\sim^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\sim^i_\Sigma \subseteq \mathrm{SEN}^\flat(\Sigma)^2$ be given, for all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\varphi \sim^i_\Sigma \psi \quad \text{iff} \quad C_\Sigma(E^{i\flat}_\Sigma(\varphi)) = C_\Sigma(E^{i\flat}_\Sigma(\psi)).$$

We have the following proposition concerning the status of these relation systems:

Proposition 4 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ be a π-institution based on $\mathbf{A}^\flat$. Then $\Lambda(\mathcal{I})$ is an equivalence system on $\mathbf{A}^\flat$ and $\widetilde{\Omega}(\mathcal{I})$ is a congruence system on $\mathbf{A}^\flat$. Moreover, if $\mathcal{I}$ is m-normal with respect to E, then $\sim^i$ is also an equivalence system on $\mathbf{A}^\flat$, for all $i \in \{1, \ldots, m-2\}$.*

Proof: The results about $\Lambda(\mathcal{I})$ and $\widetilde{\Omega}(\mathcal{I})$ are well-known in categorical abstract algebraic logic (see Theorem 4 of [11] and Proposition 3.2 of [10]). Suppose that $\mathcal{I}$ is m-normal with respect to $E = \{E^{1\flat}, \dots, E^{(m-2)\flat}\}$ and let $i \in \{1, \dots, m-2\}$. Then it is clear from the definition that $\sim^i$ is an equivalence family on $\mathbf{A}^\flat$, i.e., that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\sim^i_\Sigma$ is an equivalence relation on $\mathrm{SEN}^\flat(\Sigma)$. To show that it also satisfies the system property, suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\varphi \sim^i_\Sigma \psi$. This means that $C_\Sigma(E^{i\flat}_\Sigma(\varphi)) = C_\Sigma(E^{i\flat}_\Sigma(\psi))$. By structurality, then, we get that

$$\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\varphi)) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\psi))).$$

Since $E^{i\flat} : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is a natural transformation, we get that

$$E^{i\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in C_{\Sigma'}(E^{i\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))).$$

Hence, by symmetry,

$$C_{\Sigma'}(E^{i\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))) = C_{\Sigma'}(E^{i\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))).$$

This shows that $\mathrm{SEN}^\flat(f)(\varphi) \sim^i_{\Sigma'} \mathrm{SEN}^\flat(f)(\psi)$, which verifies that $\sim^i$ is indeed an equivalence system on $\mathbf{A}^\flat$. ■

We are now ready to formulate the main theorem of the paper, an analog of the main theorem of [7]. It characterizes m-referential π-institutions in terms of normality and a certain relationship between the relation systems associated with a π-institution introduced above.

Theorem 5 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $m \geq 2$ an integer and suppose that $N^\flat$ contains a set of $m-2$ unary natural transformations $E = \{E^{1\flat}, \dots, E^{(m-2)\flat}\}$. Then a π-institution $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ is m-referential with respect to E if and only if the following conditions hold:*

(i) *$\mathcal{I}$ is m-normal with respect to $E^\flat$;*

(ii) $\widetilde{\Omega}(\mathcal{I}) = \Lambda(\mathcal{I}) \cap {\sim^1} \cap \cdots \cap {\sim^{m-2}}$.

Proof: Suppose, first, that $\mathcal{I}$ is m-referential. Thus, there exists, by Proposition 3, a normal m-referential $N^\flat$-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, such that $C = C^{\mathbb{A}}$. We must show that $\mathcal{I}$ satisfies Conditions (i) and (ii) of the statement.

To show that $\mathcal{I}$ is m-normal with respect to E, we verify conditions (N1)-(N3).

For (N1), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi \in \mathrm{SEN}^\flat(\Sigma)$. We show that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p \in \mathrm{PTS}(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = 1 \quad \text{and} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\varphi)))(p) = 1$$

is impossible. This will imply, using Proposition 2, that $C^{\mathbb{A}}_\Sigma(\varphi, E^{i\flat}_\Sigma(\varphi)) = \mathrm{SEN}^\flat(\Sigma)$ and, thus, that $C_\Sigma(\varphi, E^{i\flat}_\Sigma(\varphi)) = \mathrm{SEN}^\flat(\Sigma)$. In fact, if

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = 1,$$

then we have

$$\begin{aligned}
&\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\varphi)))(p)\\
&\quad = \alpha_{\Sigma'}(E^{i\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)))(p)\\
&\quad = E^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)))(p)\\
&\quad = 0 \quad (\text{since } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) \neq e_i).
\end{aligned}$$

For (N2), suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi \in \mathrm{SEN}^\flat(\Sigma)$. We show that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p \in \mathrm{PTS}(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(E^{j\flat}_\Sigma(\varphi))))(p) \neq 1$$

and use again Proposition 2. We have, indeed,

$$\begin{aligned}
&\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(E^{j\flat}_\Sigma(\varphi))))(p)\\
&\quad = \alpha_{\Sigma'}(E^{i\flat}_{\Sigma'}(E^{j\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))))\\
&\quad = E^i_{F(\Sigma')}(E^j_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))))(p)\\
&\quad = \begin{cases} 1, & \text{if } E^j_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)))(p) = e_i\\ 0, & \text{otherwise}\end{cases}\\
&\quad = 0 \quad (\text{since } E^j_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)))(p) \in \{0,1\}).
\end{aligned}$$

For (N3), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $p \in \mathrm{PTS}(F(\Sigma'))$, we have

$$\begin{aligned}
&\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\varphi)))(p)\\
&\quad = \alpha_{\Sigma'}(E^{i\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)))(p)\\
&\quad = E^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)))(p)\\
&\quad = \begin{cases} 1, & \text{if } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = e_i\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

A similar computation yields

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{j\flat}_\Sigma(\varphi)))(p) = \begin{cases} 1, & \text{if } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = e_j,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, it is not possible to have

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\varphi)))(p) = 1 = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{j\flat}_\Sigma(\varphi)))(p).$$

This and Proposition 2 show (N3).

We thus, conclude that $\mathcal{I}$ is an m-normal π-institution with respect to E. We now turn to proving Condition (ii).

First, it is obvious that $\widetilde{\Omega}(\mathcal{I}) \leq \Lambda(\mathcal{I})$ ($\leq$ denotes signature-wise inclusion). Moreover, since $\widetilde{\Omega}(\mathcal{I})$ is a congruence system on $\mathbf{A}^\flat$ and E is a subset of $N^\flat$, we get that $\widetilde{\Omega}(\mathcal{I}) \leq \sim^i$, for all $i = 1, \ldots, m-2$. Thus, we have that $\widetilde{\Omega}(\mathcal{I}) \leq \Lambda(\mathcal{I}) \cap \sim^1 \cap \cdots \cap \sim^{m-2}$. It suffices, now, to show the reverse inclusion.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I})$ and $\varphi \sim^i_\Sigma \psi$, for all $i = 1, \ldots, m-2$. Then we have $C_\Sigma(\varphi) = C_\Sigma(\psi)$ and $C_\Sigma(E^{i\flat}_\Sigma(\varphi)) = C_\Sigma(E^{i\flat}_\Sigma(\psi))$, for all $i = 1, \ldots, m-2$. Since, by hypothesis, $C = C^{\mathbb{A}}$, we get, by Proposition 2, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p \in \mathrm{PTS}(F(\Sigma'))$,

$$\begin{array}{rcl}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = 1 & \text{iff} & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))(p) = 1, \\ \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\varphi)))(p) = 1 & \text{iff} & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E^{i\flat}_\Sigma(\psi)))(p) = 1, \\ & & \qquad m = 1, \ldots, m-2.\end{array}$$

The latter family implies that, for all $i = 1, \ldots, m-2$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) = e_i \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)) = e_i.$$

Altogether, we conclude that, for all $p \in \mathrm{PTS}(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(p) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))(p),$$

i.e., that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. But now we get, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\overline{\chi} \in \mathrm{SEN}^\flat(\Sigma')^{k-1}$,

$$\begin{array}{l}\alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi), \overline{\chi})) \\ \qquad = \sigma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)), \alpha^{k-1}_{\Sigma'}(\overline{\chi})) \\ \qquad = \sigma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)), \alpha^{k-1}_{\Sigma'}(\overline{\chi})) \\ \qquad = \alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \overline{\chi})).\end{array}$$

This shows that $C^{\mathbb{A}}_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi), \overline{\chi})) = C^{\mathbb{A}}_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \overline{\chi}))$. By hypothesis, this is equivalent to

$$C_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi), \overline{\chi})) = C_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \overline{\chi})).$$

Hence, by definition, $\langle\varphi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$. Thus, $\Lambda(\mathcal{I}) \cap {\sim}^1 \cap \cdots \cap {\sim}^{m-2} \leq \widetilde{\Omega}(\mathcal{I})$ and this completes the "only if" direction of the proof.

Suppose, conversely, that $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$, with $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ is m-normal with respect to E and that $\widetilde{\Omega}(\mathcal{I}) = \Lambda(\mathcal{I}) \cap {\sim}^1 \cap \cdots \cap {\sim}^{m-2}$. We then show that $\mathcal{I}$ is m-referential with respect to E. To this end we construct step-by-step a normal m-referential $N^\flat$-gmatrix system $\mathbb{A}$, such that $C = C^{\mathbb{A}}$.

Let $\mathbf{Sign} = \mathbf{Sign}^\flat$. Define $\mathrm{PTS} : |\mathbf{Sign}| \to \mathbf{Set}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{PTS}(\Sigma) = \mathrm{Th}_\Sigma(\mathcal{I})\backslash\{\mathrm{SEN}^\flat(\Sigma)\} =: \mathrm{Th}^*_\Sigma(\mathcal{I}),$$

the collection of all Σ-theories of $\mathcal{I}$ other than $\mathrm{SEN}^\flat(\Sigma)$. This defines the functor PTS of base points on which $\mathbb{A}$ will be built.

Now we specify the functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ giving the sentences of $\mathbb{A}$. Recall that sentences must be functions from the set of reference points to $\{e_0,\ldots,e_{m-1}\}$.

First, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, let $\overline{\varphi} : \mathrm{PTS}(\Sigma) \to \{0, e_1, \ldots, e_{m-2}, 1\}$ be given, for all $T \in \mathrm{Th}^*_\Sigma(\mathcal{I})$ by

$$\overline{\varphi}(T) = \begin{cases} 1, & \text{if } \varphi \in T, \\ e_i, & \text{if } E^{i\flat}_\Sigma(\varphi) \in T, \quad i = 1,\ldots,m-2, \\ 0, & \text{otherwise.}\end{cases}$$

Claim: $\overline{\varphi} : \mathrm{PTS}(\Sigma) \to \{0, e_1, \ldots, e_{m-2}, 1\}$ is well-defined.
Proof: First, by (N1), if $\varphi \in T$, then, for all $i = 1,\ldots,m-2$, $E^{i\flat}_\Sigma(\varphi) \notin T$. Second, by (N3), if $E^{i\flat}(\varphi) \in T$, then $E^{j\flat}_\Sigma(\varphi) \notin T$, for all $j \neq i$. ■

Based on this definition, we define $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ as follows:
For all $\Sigma \in |\mathbf{Sign}|$, we set

$$\mathrm{SEN}(\Sigma) = \{\overline{\varphi} : \varphi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Furthermore, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}(\Sigma')$ is given, for all $\overline{\varphi} \in \mathrm{SEN}(\Sigma)$, by

$$\mathrm{SEN}(f)(\overline{\varphi}) = \overline{\mathrm{SEN}^\flat(f)(\varphi)}.$$

Claim: $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is well-defined and constitutes a functor.
Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\varphi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\overline{\varphi} = \overline{\psi}$. Then, for all $T \in \mathrm{Th}^*_\Sigma(\mathcal{I})$, we have $\overline{\varphi}(T) = \overline{\psi}(T)$. Therefore, by definition,

$$\begin{array}{rcll} \varphi \in T & \text{iff} & \psi \in T & \text{and} \\ E^{i\flat}_\Sigma(\varphi) \in T & \text{iff} & E^{i\flat}_\Sigma(\psi) \in T, & \text{for all } 1 \leq i \leq m-2. \end{array}$$

Thus, we get that $\langle\varphi,\psi\rangle \in \Lambda_\Sigma(\mathcal{I})$ and $\varphi \sim^i_\Sigma \psi$, for all $1 \le i \le m-2$. But, by Proposition 4, $\Lambda(\mathcal{I})$ and $\sim^i$, $i \in \{1,\dots,m-2\}$, are equivalence systems, whence we get, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,

$$\begin{gathered}\langle \mathrm{SEN}^\flat(f)(\varphi), \mathrm{SEN}^\flat(f)(\psi)\rangle \in \Lambda_{\Sigma'}(\mathcal{I}) \quad \text{and} \\ \mathrm{SEN}^\flat(f)(\varphi) \sim^i_{\Sigma'} \mathrm{SEN}^\flat(f)(\psi), \quad 1 \le i \le m-2.\end{gathered}$$

Thus, we get $\overline{\mathrm{SEN}^\flat(f)(\varphi)} = \overline{\mathrm{SEN}^\flat(f)(\psi)}$. This shows that $\mathrm{SEN}(f)(\varphi) = \mathrm{SEN}(f)(\psi)$ and, hence, that $\mathrm{SEN}(f)$ is well-defined, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$.

$\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, thus, defined, is a functor, since, for all $\Sigma,\Sigma',\Sigma'' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $g \in \mathbf{Sign}(\Sigma',\Sigma'')$,

$$\Sigma \xrightarrow{\quad f \quad} \Sigma' \xrightarrow{\quad g \quad} \Sigma''$$

$$\begin{aligned}\mathrm{SEN}(g\circ f)(\overline{\varphi}) &= \overline{\mathrm{SEN}^\flat(g\circ f)(\varphi)} \\ &= \overline{\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\varphi))} \\ &= \mathrm{SEN}(g)(\overline{\mathrm{SEN}^\flat(f)(\varphi)}) \\ &= \mathrm{SEN}(g)(\mathrm{SEN}(f)(\overline{\varphi})).\end{aligned}$$

This concludes the proof of the claim. ∎

Now, for $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, let $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ be defined by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ be given, for all $\varphi_0,\dots,\varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\sigma_\Sigma(\overline{\varphi}_0,\dots,\overline{\varphi}_{k-1}) = \overline{\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})}.$$

We show that $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ is well-defined. Suppose that $\varphi_0,\psi_0,\dots,\varphi_{k-1},\psi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\overline{\varphi}_0 = \overline{\psi}_0$, …, $\overline{\varphi}_{k-1} = \overline{\psi}_{k-1}$. By definition, we get that, for all $T \in \mathrm{Th}^*_\Sigma(\mathcal{I})$, all $j < k$ and all $i \in \{1,\dots,m-2\}$,

$$(\varphi_j \in T \text{ iff } \psi_j \in T) \quad \text{and} \quad (E^{i\flat}_\Sigma(\varphi_j) \in T \text{ iff } E^{i\flat}_\Sigma(\psi_j) \in T).$$

But these imply that, for all $j < k$,

$$\langle\varphi_j,\psi_j\rangle \in \Lambda_\Sigma(\mathcal{I}) \text{ and } \varphi_j \sim^i_\Sigma \psi_j,\ i \in \{1,\dots,m-2\}.$$

Thus, we get, by hypothesis, that, for all $j < k$, $\langle\varphi_i,\psi_i\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$. Thus, since $\widetilde{\Omega}(\mathcal{I})$ is a congruence system, we get $\langle\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1}),\sigma^\flat_\Sigma(\psi_0,\dots,\psi_{k-1})\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$. Therefore, by compatibility and the congruence property, we obtain $\overline{\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})} = \overline{\sigma^\flat_\Sigma(\psi_0,\dots,\psi_{k-1})}$, i.e., that

$$\sigma_\Sigma(\overline{\varphi}_0,\dots,\overline{\varphi}_{k-1}) = \sigma_\Sigma(\overline{\psi}_0,\dots,\overline{\psi}_{k-1}).$$

Next we show that $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is a natural transformation. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\varphi_0, \dots, \varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^k \arrow[r, "\sigma_\Sigma"] \arrow[d, "\mathrm{SEN}(f)^k"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\mathrm{SEN}(f)"] \\
\mathrm{SEN}(\Sigma')^k \arrow[r, "\sigma_{\Sigma'}"'] & \mathrm{SEN}(\Sigma')
\end{tikzcd}$$

$$\begin{array}{lll}
\mathrm{SEN}(f)(\sigma_\Sigma(\overline{\varphi}_0, \dots, \overline{\varphi}_{k-1})) & = & \mathrm{SEN}(f)(\overline{\sigma^\flat_\Sigma(\varphi_0, \dots, \varphi_{k-1})}) \\
 & = & \overline{\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\varphi_0, \dots, \varphi_{k-1}))} \\
 & = & \overline{\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)^k(\varphi_0, \dots, \varphi_{k-1}))} \\
 & = & \sigma_{\Sigma'}(\overline{\mathrm{SEN}^\flat(f)^k(\varphi_0, \dots, \varphi_{k-1})}) \\
 & = & \sigma_{\Sigma'}(\mathrm{SEN}(f)^k(\overline{\varphi}_0, \dots, \overline{\varphi}_{k-1})).
\end{array}$$

Let N be the collection of all natural transformations of the form σ, for $\sigma^\flat$ in $N^\flat$. Then N is a category of natural transformations on SEN and, thus, we have defined an m-referential $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ based on PTS.

Next specify the pair $\langle I, \alpha \rangle : \mathbf{A}^\flat \to \mathbf{A}$ as follows:

- $I : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is the identity functor (which makes sense, since $\mathbf{Sign} = \mathbf{Sign}^\flat$).
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN}$ is defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(\Sigma)$ be given, for all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, by

  $$\alpha_\Sigma(\varphi) = \overline{\varphi}.$$

  This is natural transformation, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}^\flat(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}^\flat(f)"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\mathrm{SEN}(f)"] \\
\mathrm{SEN}^\flat(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}(\Sigma')
\end{tikzcd}$$

$$\begin{array}{rcl}\mathrm{SEN}(f)(\alpha_\Sigma(\varphi)) & = & \mathrm{SEN}(f)(\overline{\varphi}) \\ & = & \overline{\mathrm{SEN}^\flat(f)(\varphi)} \\ & = & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)).\end{array}$$

Moreover, the pair $\langle I,\alpha\rangle : \mathbf{A}^\flat \to \mathbf{A}$ is an algebraic system morphism, since, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0,\dots,\varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd} \mathrm{SEN}^\flat(\Sigma)^k \arrow[r, "\sigma^\flat_\Sigma"] \arrow[d, "\alpha^k_\Sigma"'] & \mathrm{SEN}^\flat(\Sigma) \arrow[d, "\alpha_\Sigma"] \\ \mathrm{SEN}(\Sigma)^k \arrow[r, "\sigma_\Sigma"'] & \mathrm{SEN}(\Sigma) \end{tikzcd}$$

$$\begin{array}{rcl}\alpha_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})) & = & \overline{\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})} \\ & = & \sigma_\Sigma(\overline{\varphi}_0,\dots,\overline{\varphi}_{k-1}) \\ & = & \sigma_\Sigma(\alpha_\Sigma(\varphi_0),\dots,\alpha_\Sigma(\varphi_{k-1})).\end{array}$$

We conclude that the pair $\mathcal{A} = \langle \mathbf{A}, \langle I,\alpha\rangle\rangle$ is an interpreted m-referential $N^\flat$-algebraic system based on PTS.

To finish the definition of the gmatrix system $\mathbb{A}$, let $\Sigma \in |\mathbf{Sign}|$ and $T \in \mathrm{Th}^*_\Sigma(\mathcal{I})$. Define $D^{\Sigma,T} = \{D^{\Sigma,T}_{\Sigma'}\}_{\Sigma'\in|\mathbf{Sign}|}$ by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}D^{\Sigma,T}_{\Sigma'} & = & \left\{\begin{array}{ll}\{\overline{\varphi} \in \mathrm{SEN}(\Sigma) : \overline{\varphi}(T) = 1\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma\end{array}\right. \\ & = & \left\{\begin{array}{ll}\{\overline{\varphi} \in \mathrm{SEN}(\Sigma) : \varphi \in T\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma.\end{array}\right.\end{array}$$

and define

$$\mathcal{D} = \{D^{\Sigma,T} : \Sigma \in |\mathbf{Sign}|, T \in \mathrm{Th}^*_\Sigma(\mathcal{I})\}.$$

Clearly, the pair $\mathbb{A} = \langle \mathcal{A},\mathcal{D}\rangle$ is an m-referential $N^\flat$-gmatrix system. Thus, to conclude the proof it suffices to show that $\mathbb{A}$ is m-normal with respect to E and that $C = C^{\mathbb{A}}$.

Claim: $\mathbb{A} = \langle \mathcal{A},\mathcal{D}\rangle$ is m-normal.

Proof: Let $i \in \{1,\dots,m-2\}$. Then, for all $\Sigma \in |\mathbf{Sign}|$, all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \mathrm{Th}^*_\Sigma(\mathcal{I})$, we get

$$E^i_\Sigma(\overline{\varphi})(T) = \overline{E^{i\,\flat}_\Sigma(\varphi)}(T) = \left\{\begin{array}{ll} 1, & \text{if } E^{i\,\flat}_\Sigma(\varphi) \in T, \\ e_j, & \text{if } E^{j\,\flat}_\Sigma(E^{i\,\flat}_\Sigma(\varphi)) \in T,\ j = 1,\dots,m-2, \\ 0, & \text{otherwise.}\end{array}\right.$$

We distinguish the following cases:

(a) If $\overline{\varphi}(T) = e_i$, then $E^{i\flat}_\Sigma(\varphi) \in T$ and, hence, $E^i_\Sigma(\overline{\varphi})(T) = 1$. Moreover, for all $j \neq i$, by (N3), $E^{j\flat}_\Sigma(\varphi) \notin T$ and, by (N2) $E^{k\flat}_\Sigma(E^{j\flat}_\Sigma(\varphi)) \notin T$. Thus, we get $E^j_\Sigma(\overline{\varphi})(T) = 0$.

(b) If $\overline{\varphi}(T) = 1$, then $\varphi \in T$. So, by (N1), $E^{i\flat}_\Sigma(\varphi) \notin T$ and, by (N2), $E^{j\flat}_\Sigma(E^{i\flat}_\Sigma(\varphi)) \notin T$. So $E^i_\Sigma(\overline{\varphi})(T) = 0$.

(c) If $\overline{\varphi}(T) = 0$, then $\varphi \notin T$ and $E^{i\flat}_\Sigma(\varphi) \notin T$. But, also, $E^{j\flat}_\Sigma(E^{i\flat}_\Sigma(\varphi)) \notin T$, by (N2). Thus, we get $E^i_\Sigma(\overline{\varphi})(T) = 0$.

Therefore, we conclude that $\mathbb{A}$ is normal. ■

Finally, we turn to the last claim that concludes the proof:
Claim: $C = C^{\mathbb{A}}$.
Proof: Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$. Thus, by structurality, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\mathrm{SEN}^\flat(f)(\varphi) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

This implies that, for all $T' \in \mathrm{Th}_{\Sigma'}(\mathcal{I})$,

$$\mathrm{SEN}^\flat(f)(\Phi) \subseteq T' \quad \text{implies} \quad \mathrm{SEN}^\flat(f)(\varphi) \in T'.$$

This is equivalent to asserting that

$$\overline{\mathrm{SEN}^\flat(f)(\Phi)} \subseteq D^{\Sigma',T'}_{\Sigma'} \quad \text{implies} \quad \overline{\mathrm{SEN}^\flat(f)(\varphi)} \in D^{\Sigma',T'}_{\Sigma'}.$$

Thus, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq D^{\Sigma',T'}_{\Sigma'} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \subseteq D^{\Sigma',T'}_{\Sigma'}.$$

Thus, by definition, $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$.
For the converse, we reverse all the steps followed above. ■

This concludes the proof of the main theorem. ■

We note, in closing, that the case of $m = 2$ gives a theorem that was previously obtained by the author as Theorem 8 of [12].

Corollary 6 (Theorem 8 of [12]) *A π-institution $\mathcal{I}$ is 2-referential (i.e., referential in the sense of* [12]*) if and only if $\widetilde{\Omega}(\mathcal{I}) = \Lambda(\mathcal{I})$ (i.e., if it is self-extensional).*

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