Chu Spaces, Concept Lattices and Information Systems in n Dimensions

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February 1, 2007

Abstract

Three areas of computer science that were developed independently but have strong interconnections were brought together in recent work of G.-Q. Zhang and his collaborators: Chu spaces and concept lattices, on the one hand, and domains and information systems on the other. In a different direction, inspired by the work of Wille on formal concept analysis, the author developed a theory of n-adic concept analysis that led to the definitions of n-closure systems, n-closure operators and of n-information systems. The infinitary versions of the ordinary concepts, discussed in Zhang's work, are special cases of the n-dimensional ones when restricted to the 2 dimensions. In this work, some of the interconnections revealed in Zhang's work between the 2-dimensional concepts are lifted to the n-dimensional framework. The hope is that the present work may help further clarify these relationships and also provide some impetus for considering applications of some of the recent work in this multi-dimensional framework in fields, such as data-mining, knowledge discovery, ontology and ontological engineering.

1 Introduction

In [23], Guo-Qiang Zhang revisited three very well established areas of theoretical computer science and of applied mathematics and explored some interconnections between them. On the one hand were Chu spaces, introduced by Barr and Chu [2, 3] in a categorical setting, and used as models of linear logic in [3, 15] and, later, further developed by Pratt [9, 10, 11, 12, 13] and others at Stanford for other applications. On the other hand was domain theory [1, 5, 8, 21, 22], which was introduced by Scott for studying in a mathematically rigorous way the semantics of programming languages. Finally, in the applied mathematics side was the notion of a formal context, as introduced by Wille [19, 4] and later developed further and in various directions by other members of the Darmstadt universal algebra and

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[0] *Keywords:* Chu Spaces, Formal Contexts, Formal Concepts, Domains, Information Systems

2000 AMS Subject Classification: Primary: 03B70 Secondary: 06A15, 06B23, 08A70, 68P99, 68Q55

lattice theory group. Formal contexts have proven extremely useful in the order-theoretic analysis of scientific data.

In [23] Zhang summarized a wealth of results, known from the literature, concerning relationships between the three frameworks and, then, further built on this work in the collaborations [24, 7, 6], providing new insights and obtaining many interesting new results of category-theoretic character. Some of the key relationships that were described in [23] between Chu spaces, concept lattices and domains will be briefly reviewed in this paragraph, since they will be useful in better understanding the account presented here of similar interconnections in the n dimensions, for a fixed, but arbitrary, finite n.

Recall from formal concept analysis that a formal context $\mathbb{K} = \langle O, A, I\rangle$ consists of a collection O of objects, a collection A of attributes and a binary relation $I \subseteq O \times A$. Such a formal context gives rise to a Galois connection between sets of objects and sets of attributes, whose fixed points form closure systems on both the set of objects and the set of attributes. The resulting complete lattices of object concepts and of attribute concepts are order-anti-isomorphic to each other. A formal context is identical as a structure with a Chu space but Chu spaces, developed in a categorical setting, come together with morphisms. Zhang exploited this additional feature to endow formal contexts with these Chu space mappings. However, as he shows by means of an example (Example 3.6 in [23]), Chu mappings of formal contexts do not preserve formal concepts. In Propositions 3.7-3.10, Zhang provides a summary of various conditions between the components of a Chu mapping that are sufficient for the preservation of formal concepts. In Section 4 of his account, Zhang presents two of the fundamental results connecting the notions that he is considering. First, the Representation Theorem of Formal Concept Analysis, which assures that every complete lattice is isomorphic to the complete lattice of concepts of some formal context. Thus, studying concept lattices is tantamount to studying complete lattices. Second, after recalling the definition of an information system of Scott, he reviews the Basic Theorem of Domain Theory, stating that the collection of information states of an information system is a domain and that, conversely, every domain is order-isomorphic to the partial ordering of information states of some information system. Thus, studying domains is, in some sense, tantamount to studying information systems. Based on this theorem, given a formal context, a construction is provided of an information system and it is shown that, if the collection of attributes of the formal context is finite, then a subset of the set of attributes of the formal context is an attribute concept if and only if it is an information state of the derived information system. The possibility of a formal context being "infinitary", as opposed to the finitarity always present in information systems, creates a discrepancy between the two frameworks that is depicted by Example 4.8 in [23]. In the present setting, this problem is avoided by considering n-information systems with possibly infinitary deduction mechanisms. Finally, in [23] some results geared towards data-mining applications are provided along with some pointers to various possibilities that are opening for future theoretical and applied work in this field. Zhang himself continued theoretical work via collaborations with other scientists in [24, 7, 6].

In all three areas of Chu spaces, concept analysis and domain theory, a central role is played by a binary relation between two sets. Both in the context of Chu spaces and in that of formal concept analysis these are the set of objects and the set of attributes, whereas in the context of domain theory, more precisely, the context of information systems, which provide a logical framework in which one studies derivations in domain theory, these are the set of consistent finite subsets of the set of tokens and the set of tokens itself.

In [16], inspired by work of Wille [20] extending formal concept analysis to three dimensions, the author created a framework for analyzing n-dimensional formal concepts. Our goal in this note is to give an idea on how Zhang's account in [23] may be coupled with these higher dimensional results to provide a framework potentially helpful in dealing with higher dimensional applications, such as, for instance, multidimensional bits of information in the context of web data mining, an application that was in Zhang's mind when writing [23] (see Section 5 of [23] and also the concluding remarks and suggestions for future work therein).

2 Preliminaries

Some of the terminology and the basic elements of the theories of n-ordered sets, n-closure systems and n-closure operators and polyadic formal concept analysis are reviewed in this section.

2.1 Complete n-Ordered Sets

A relational structure $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$, where $\lesssim_1, \ldots, \lesssim_n$ are quasi-orders on P, is called an *ordinal structure*. Denote by $\sim_i := \lesssim_i \cap \gtrsim_i$, for all $i = 1, \ldots, n$. An ordinal structure $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ is called an *n-ordered set* if, for all $x, y \in P$ and all $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$,

1. $x \sim_1 y, \ldots, x \sim_n y$ imply that $x = y$ (*Uniqueness Condition*)
2. $x \lesssim_{i_1} y, \ldots, x \lesssim_{i_{n-1}} y$ imply $y \lesssim_{i_n} x$ (*Antiordinal Dependency*)

Each quasiorder $\lesssim_i$ induces in the standard way an order $\leq_i$ on the set of equivalence classes $P/{\sim_i} = \{[x]_i : x \in P\}, i = 1, 2, \ldots, n$, where $[x]_i = \{y \in P : x \sim_i y\}$.

Let $\mathbf{P} = \langle P, \lesssim_1, \lesssim_2, \ldots, \lesssim_n \rangle$ be an n-ordered set, $j_1, j_2, \ldots, j_{n-1} \in \{1, 2, \ldots, n\}$ be distinct and $X_1, X_2, \ldots, X_{n-1} \subseteq P$.

An element $b \in P$ is called a $(j_{n-1}, \ldots, j_1)$-*bound* of $(X_{n-1}, X_{n-2}, \ldots, X_1)$ if $x_i \lesssim_{j_i} b$, for all $x_i \in X_i$ and all $i = 1, \ldots, n-1$. The set of all $(j_{n-1}, \ldots, j_1)$-bounds of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$.

A $(j_{n-1}, \ldots, j_1)$-bound $l \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$ of $(X_{n-1}, \ldots, X_1)$ is called a $(j_{n-1}, \ldots, j_1)$-*limit* of $(X_{n-1}, \ldots, X_1)$ if $l \gtrsim_{j_n} b$, for all $(j_{n-1}, \ldots, j_1)$-bounds $b \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$. The set of all $(j_{n-1}, \ldots, j_1)$-limits of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}}$.

The following proposition was proved in [16].

Proposition 1 *Let* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *be an* n*-ordered set,* $X_1, \ldots, X_{n-1} \subseteq P$ *and* $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$*. Then, there exists at most one* $(j_{n-1}, \ldots, j_1)$*-limit* $\bar{l}$ *of* $(X_{n-1}, \ldots, X_1)$ *satisfying*

(C) $\bar{l}$ *is the largest in* $\lesssim_{j_2}$ *among the largest limits in* $\lesssim_{j_3}$ *among ... among the largest limits in* $\lesssim_{j_{n-1}}$ *among the largest limits in* $\lesssim_{j_n}$ *or, equivalently,*

(C') $\bar{l}$ *is the smallest in* $\lesssim_{j_1}$ *among the largest limits in* $\lesssim_{j_3}$ *among ... among the largest limits in* $\lesssim_{j_{n-1}}$ *among the largest limits in* $\lesssim_{j_n}$*.*

If it exists, a $(j_{n-1}, \ldots, j_1)$-limit satisfying the statement in Proposition 1 is called the $(j_{n-1}, \ldots, j_1)$-*join* of $(X_{n-1}, \ldots, X_1)$ and denoted by $\nabla_{j_{n-1},\ldots,j_1}(X_{n-1}, \ldots, X_1)$.

An n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ is said to be a *complete* n*-lattice* if all $(j_{n-1}, \ldots, j_1)$-joins exist in $\mathbf{P}$, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. It is said to be an n*-lattice* if all joins of the form $\nabla_{j_{n-1},\ldots,j_1}(\{x_{n-1,1}, x_{n-1,2}\}, \ldots, \{x_{1,1}, x_{1,2}\})$ exist, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. It was shown in the Reduction of Arity Theorem 12 of [17] that this condition is equivalent to the existence in $\mathbf{P}$ of all joins of the form $\nabla_{j_{n-1},\ldots,j_1}(X_{n-1}, \ldots, X_1)$, with X_i finite, $i = 1, \ldots, n-1$, for all $\{j_1, \ldots, jn\} = \{1, \ldots, n\}$. Finally, $\mathbf{P}$ is called a *complete* n*-semilattice* (see [18]) if $\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$ exists in $\mathbf{P}$, for all $X_1, \ldots, X_{n-1} \subseteq P$.

2.2 n-Closure Systems and n-Closure Operators

Given a set K, by $\mathcal{P}(K)$ will be denoted the powerset of the set K. The following definitions of an n-closure system and of an n-closure operator have been first formulated in [18].

Definition 2 *Let* $K_1, \ldots, K_n$ *be* n *sets. An* n-closure system $\mathcal{L}$ *on* $K_1, \ldots, K_n$ *is defined to be a collection of* n*-tuples of subsets* $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$*, such that,*

1. $(A_1, \ldots, A_n) \subseteq_i (B_1, \ldots, B_n), i \neq k$*, imply* $(B_1, \ldots, B_n) \subseteq_k (A_1, \ldots, A_n)$*, for all* $k = 1, \ldots, n$*,*
2. *for all* $X_i \subseteq K_i, i = 1, \ldots, n-1$*, there exists unique* $A = (A_1, \ldots, A_n) \in \mathcal{L}$*, such that* A *has the largest second component among all* n*-tuples in* $\mathcal{L}$ *with the largest third component among ... among all* n*-tuples with the largest* n*-th component among all* n*-tuples* $B = (B_1, \ldots, B_n)$ *in* $\mathcal{L}$ *such that* $X_i \subseteq B_i, i = 1, \ldots, n-1$*.*

Using a variant of the notation introduced in [20] and adopted in [16], we denote the element $A \in \mathcal{L}$ in Condition 2 of the definition of an n-closure system by $\beta_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$.

Next, the notion of an n-closure operator is introduced. n-closure systems and n-closure operators are in a relation similar to the one satisfied by ordinary (2-dimensional) closure systems and closure operators.

Definition 3 *Let* $K_1, K_2, \ldots, K_n$ *be arbitrary sets. An* n-closure operator *on* $K_1, \ldots, K_n$ *is a mapping from* $\mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1})$ *to* $\mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$*, such that the following conditions hold:*

1. *If* $C(X_1,\ldots,X_{n-1})=(A_1,\ldots,A_n)$ *and* $x\in X_i$*, then* $x\in A_i$*, for all* $i=1,\ldots,n-1$.
2. *If* $C(X_1,\ldots,X_{n-1})=(A_1,\ldots,A_n)$*,* $C(Y_1,\ldots,Y_{n-1})=(B_1,\ldots,B_n)$ *and*
$$X_i\subseteq Y_i, i=1,\ldots,n-1, \text{ then } B_n\subseteq A_n.$$
3. *If* $C(X_1,\ldots,X_{n-1})=(A_1,\ldots,A_n)$*,* $C(Y_1,\ldots,Y_{n-1})=(B_1,\ldots,B_n)$ *and*
$$X_i\subseteq Y_i,\ i\le k,\ \text{and } A_i=B_i,\ i>k,\ \text{for some } k=1,\ldots,n-1,\ \text{then } B_k\subseteq A_k.$$
4. *If* $C(X_1,\ldots,X_{n-1})=(A_1,\ldots,A_n)$*,* $C(Y_1,\ldots,Y_{n-1})=(B_1,\ldots,B_n)$ *and*
$$A_i\subseteq B_i,\ i\neq k,\ \text{for some } k=1,\ldots,n,\ \text{then } B_k\subseteq A_k.$$
5. *If* $C(X_1,\ldots,X_{n-1})=(A_1,\ldots,A_n)$*, then* $C(A_1,\ldots,A_{n-1})=(A_1,\ldots,A_n)$.

If $C(X_1,\ldots,X_{n-1})=(A_1,\ldots,A_n)$, then we adopt the notation
$$C_i(X_1,\ldots,X_{n-1}):=A_i,\quad i=1,\ldots,n,$$
with the *warning* that this is just a notational convention and it is *not meant* to imply that C_i is a closure operator in the traditional sense.

It was shown in Lemma 4 of [18] that the collection of all closed sets of an n-closure operator form an n-closure system.

Lemma 4 (Lemma 4 of [18]) *Suppose that* $C:\mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_{n-1})\to\mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_n)$ *is an* n*-closure operator. Then the collection*
$$\mathcal{L}=\{(A_1,\ldots,A_n)\in\mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_n):C(A_1,\ldots,A_{n-1})=(A_1,\ldots,A_n)\}$$
is an n*-closure system.*

It was furthermore shown in Proposition 11 of [18] that the collection of sets in an n-closure system forms a complete n-semilattice under the n component-wise inclusion relations.

Proposition 5 (Proposition 11 of [18]) *If* $\mathcal{L}\subseteq\mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_n)$ *is an* n*-closure system, then* $\langle\mathcal{L},\subseteq_1,\ldots,\subseteq_n\rangle$ *is a complete* n*-semilattice.*

2.3 n-ary Relations and n-Closure Operators

Let $K_1, \ldots, K_n$ be sets and $R \subseteq K_1 \times \cdots \times K_n$ an n-ary relation between $K_1, \ldots, K_n$. For all $j = 1, \ldots, n$, and all $X_i \subseteq K_i, i \neq j$, define

$$\begin{aligned} &R_j(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n) = \\ &\qquad \{a \in K_j : (x_1, \ldots, x_{j-1}, a, x_{j+1}, \ldots, x_n) \in R, \text{ for all } x_i \in X_i, i \neq j\}. \end{aligned}$$

Now consider $X_i \subseteq K_i, i = 1, \ldots, n-1$. Define, by downward induction on $k = n, \ldots, 1$, the n-tuple $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1) = \langle \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_1, \ldots, \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_n \rangle$ as follows:

$$\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_n = R_n(X_1, \ldots, X_{n-1}),$$

and, given $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_i$, for all $i = n, n-1, \ldots, k+1$,

$$\begin{aligned} &\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_k = \\ &\qquad R_k(X_1, \ldots, X_{k-1}, \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_{k+1}, \ldots, \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_n). \end{aligned}$$

It is not difficult to show that $(X_1, \ldots, X_{n-1}) \mapsto \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$ forms an n-closure operator $\mathfrak{b}_{n-1,\ldots,1} : \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$. This result goes back to Proposition 3 of [16].

Proposition 6 *Let $K_1, \ldots, K_n$ be sets and $R \subseteq K_1 \times \cdots \times K_n$ an n-ary relation between $K_1, \ldots, K_n$. Then, the mapping $C : \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$, defined by $C(X_1, \ldots, X_{n-1}) = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, for all $X_i \subseteq K_i, i = 1, \ldots, n-1$, is an n-closure operator.*

Proof:
All Properties 1-5 in the definition of an n-closure operator are routine to verify. So the details of the proof are left to the reader. ■

3 Polyadic Contexts and Complete n-lattices

The following definition first appeared in [16] and introduces polyadic formal contexts. The inspiration came from Wille's triadic formal contexts [20]. Wille generalized an earlier framework of his, based on a binary relation between objects and attributes, to one based on a ternary relation involving objects, attributes and situations. The author abstracted this application-oriented ideas of Wille to an arbitrary n-dimensional setting. The idea of introducing mappings between n-adic formal contexts of the kind defined in Definition 7 comes from Zhang's introduction of Chu morphisms in the dyadic formal context setting [23].

Definition 7 *An* n-adic formal context *is an $(n+1)$-tuple $\mathbb{K} = \langle K_1, \ldots, K_n, Y \rangle$, where $K_1, \ldots, K_n$ are sets and $Y \subseteq K_1 \times \cdots \times K_n$ is an n-ary relation between $K_1, \ldots, K_n$.*

Given two n-adic formal contexts $\mathbb{K} = \langle K_1, \ldots, K_n, Y\rangle$ *and* $\mathbb{L} = \langle L_1, \ldots, L_n, Z\rangle$*, a* mapping from $\mathbb{K}$ to $\mathbb{L}$ *is an n-tuple of functions* $\langle f_1, \ldots, f_n\rangle$*, with* $f_n : K_n \to L_n$ *and* $f_i : L_i \to K_i$*, for all* $i = 1, \ldots, n-1$*, such that the following condition is satisfied, for all* $y_n \in K_n$ *and all* $x_i \in L_i, i = 1, \ldots, n-1$,

$$(f_1(x_1), \ldots, f_{n-1}(x_{n-1}), y_n) \in Y \quad \textit{iff} \quad (x_1, \ldots, x_{n-1}, f_n(y_n)) \in Z.$$

Given an n-adic formal context $\mathbb{K} = \langle K_1, \ldots, K_n, Y\rangle$ and $i = 1, \ldots, n$, recall, from the relational framework of the previous section, that $Y_i : \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{i-1}) \times \mathcal{P}(K_{i+1}) \times \cdots \times \mathcal{P}(K_n) \to \mathcal{P}(K_i)$ is defined by

$$\begin{aligned} &Y_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) = \\ &\qquad \{a \in K_i : (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n), \text{ for all } x_j \in X_j, j \neq i\}. \end{aligned}$$

An n-tuple $\langle X_1, \ldots, X_n\rangle$, with $X_i \subseteq K_i, i = 1, \ldots, n$, is said to be *closed* or an (n*-adic formal*) *concept* if

$$Y_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) = X_i, \text{ for all } i = 1, \ldots, n.$$

The collection of all n-adic concepts of the n-adic formal context $\mathbb{K}$ is denoted by $\mathcal{C}(\mathbb{K})$. They are ordered by the the n component-wise quasi-orderings $\subseteq_i, i = 1, \ldots, n$. In this way an ordinal structure $\mathcal{C}(\mathbb{K}) = \langle \mathcal{C}(\mathbb{K}), \subseteq_1, \ldots, \subseteq_n\rangle$ is formed.

The following result was proven in [16]:

Theorem 8 (Part 1 of Theorem 6 of [16]) *Let* $\mathbb{K} = (K_1, \ldots, K_n, Y)$ *be an n-adic context. Then* $\mathcal{C}(\mathbb{K}) = \langle \mathcal{C}(\mathbb{K}), \subseteq_1, \ldots, \subseteq_n\rangle$ *is a complete n-lattice for which the* $(j_{n-1}, \ldots, j_1)$*-joins* $(\{j_1, \ldots, j_n\} = \{1, \ldots, n\})$ *are described by*

$$\nabla_{j_{n-1},\ldots,j_1} \mathfrak{X}_{j_n'} = \mathfrak{b}_{j_{n-1},\ldots,j_1}(\langle \bigcup \{A_i : (A_1, \ldots, A_n) \in \mathfrak{X}_i\} : i \neq j_n\rangle).$$

Here, if $i = j_n$, $\mathfrak{X}_{j_n'} = \langle \mathfrak{X}_1, \ldots, \mathfrak{X}_{i-1}, \mathfrak{X}_{i+1}, \ldots, \mathfrak{X}_n\rangle$*, and* $\mathfrak{X}_{j_i} \subseteq \mathcal{C}(\mathbb{K})$*, for all* $i \neq n$.

As a consequence of Theorem 8, Proposition 6 and Lemma 4 we obtain the following corollary:

Corollary 9 *Let* $\mathbb{K} = (K_1, \ldots, K_n, Y)$ *be an n-adic context. Then* $\mathcal{C}(\mathbb{K})$ *forms an n-closure system*

Chu mappings do not necessarily preserve n-adic concepts. This was shown in the dyadic case, i.e., the case of Chu mappings, in Example 3.6 of G.-Q. Zhang [23]. The dyadic case is a special case of our more general n-adic framework.

We close this section by studying a few analogs of dyadic results that clarify the interaction of n-adic formal context mappings with complete n-adic concept lattices.

Given a mapping $f : A \to B$, we adopt the notation of Definition 2.5 of [23]:

$$f^+ : \mathcal{P}(A) \to \mathcal{P}(B) \quad \text{with} \quad X \mapsto \{f(a) : a \in X\},$$

$$f^- : \mathcal{P}(B) \to \mathcal{P}(A) \quad \text{with} \quad Y \mapsto \{a : f(a) \in Y\}.$$

Then we obtain the following interesting analogs of Propositions 3.7-3.10 of [23] in the n dimensions.

Proposition 10 *Suppose that* $\mathbb{K} = \langle K_1, \ldots, K_n, R\rangle, \mathbb{L} = \langle L_1, \ldots, L_n, S\rangle$ *are two* n*-adic formal contexts and* $f = \langle f_1, \ldots, f_n\rangle$ *a morphism* $f : \mathbb{K} \to \mathbb{L}$*. Then we have, for all* $X_i \subseteq L_i, i = 1, \ldots, n-1,$

$$R_n(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1})) = f_n^-(S_n(X_1, \ldots, X_{n-1})).$$

$$\begin{tikzcd} K_1 & K_2 & & K_{n-1} & K_n \arrow[d, "f_n"] \\ L_1 \arrow[u, "f_1"] & L_2 \arrow[u, "f_2"] & \cdots & L_{n-1} \arrow[u, "f_{n-1}"] & L_n \end{tikzcd}$$

Furthermore, for all $i = 1, \ldots, n-1$*, all* $X_j \subseteq L_j, j = 1, \ldots, n-1, j \neq i$*, and all* $Y_n \subseteq K_n$*,*

$$f_i^-(R_i(f_1^+(X_1), \ldots, f_{i-1}^+(X_{i-1}), f_{i+1}^+(X_{i+1}), \ldots, f_{n-1}^+(X_{n-1}), Y_n)) = \\ S_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n-1}, f_n^+(Y_n)).$$

Proof:

Suppose, first, that $x \in K_n$ is such that $x \in R_n(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1}))$. Then, for all $x_i \in X_i, i = 1, \ldots, n-1$, we have that $(f_1(x_1), \ldots, f_{n-1}(x_{n-1}), x) \in R$. This holds if and only if $(x_1, \ldots, x_{n-1}, f_n(x)) \in S$, whence $f_n(x) \in S_n(X_1, \ldots, X_{n-1})$. Therefore $x \in f_n^-(S_n(X_1, \ldots, X_{n-1}))$. Since all steps in the above implications are reversible, we obtain the desired equality.

Suppose, next, that $x \in L_i$, such that

$$x \in f_i^-(R_i(f_1^+(X_1), \ldots, f_{i-1}^+(X_{i-1}), f_{i+1}^+(X_{i+1}), \ldots, f_{n-1}^+(X_{n-1}), Y_n)).$$

Then we have that $f_i(x) \in R_i(f_1^+(X_1), \ldots, f_{i-1}^+(X_{i-1}), f_{i+1}^+(X_{i+1}), \ldots, f_{n-1}^+(X_{n-1}), Y_n)$. Therefore, for all $x_j \in X_j, j = 1, \ldots, n-1, j \neq i$, and all $y \in Y_n$, we have that

$$(f_1(x_1), \ldots, f_{i-1}(x_{i-1}), f_i(x), f_{i+1}(x_{i+1}), \ldots, f_{n-1}(x_{n-1}), y) \in R.$$

Therefore, we obtain that $(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{n-1}, f_n(y)) \in S$, which yields $x \in S_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n-1}, f_n^+(Y))$, as was to be shown. Once more, the above steps are all reversible, whence the stated equality follows. ■

We are now ready to formulate Proposition 11, which reveals some connections between the n-adic concept operators and the morphisms of n-adic contexts. Proposition 11 is an n-dimensional analog of Proposition 3.9 of [23].

Proposition 11 *Suppose that* $\mathbb{K} = \langle K_1, \ldots, K_n, R\rangle, \mathbb{L} = \langle L_1, \ldots, L_n, S\rangle$ *are two* n*-adic formal contexts and* $f = \langle f_1, \ldots, f_n\rangle$ *a morphism* $f : \mathbb{K} \to \mathbb{L}$*, such that* f_i *is surjective, for all* $i = 1, \ldots, n$*. Then*

1. *if* $(X_1, \ldots, X_n) \in \mathcal{C}(\mathbb{L})$*, then* $(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n)) \in \mathcal{C}(\mathbb{K})$ *and*
2. *if* $(Y_1, \ldots, Y_n) \in \mathcal{C}(\mathbb{K})$*, then* $(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1}), f_n^+(Y_n)) \in \mathcal{C}(\mathbb{L})$*.*

Proof:

1. Assume that $(X_1, \ldots, X_n) \in \mathcal{C}(\mathbb{L})$. We show, first, that

$$f_n^-(X_n) = R_n(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1})).$$

We have $y \in f_n^-(X_n)$ if and only if $f_n(y) \in X_n$ if and only if, by hypothesis, $f_n(y) \in S_n(X_1, \ldots, X_{n-1})$ if and only if $y \in f_n^-(S_n(X_1, \ldots, X_{n-1}))$ if and only if, by Proposition 10, $y \in R_n(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1}))$, as was to be shown. Next, it must be shown that, for every $i = 1, \ldots, n-1$,

$$f_i^+(X_i) = R_i(f_1^+(X_1), \ldots, f_{i-1}^+(X_{i-1}), f_{i+1}^+(X_{i+1}), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n)).$$

We only show the case $i = 1$. The remaining cases follow then by symmetry. We have $y \in f_1^+(X_1)$ if and only if, there exists $x \in X_1$, such that $y = f_1(x)$, if and only if, by hypothesis, there exists $x \in S_1(X_2, \ldots, X_{n-1}, X_n)$, such that $y = f_1(x)$, iff, by surjectivity, there exists $x \in S_1(X_2, \ldots, X_{n-1}, f_n^+(f_n^-(X_n)))$, such that $y = f_1(x)$, if and only if, by Proposition 10, there exists $x \in f_1^-(R_1(f_2^+(X_2), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n)))$, such that $y = f_1(x)$, if and only if $y = f_1(x) \in R_1(f_2^+(X_2), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n))$, as was to be shown.

2. Assume that $(Y_1, \ldots, Y_n) \in \mathcal{C}(\mathbb{K})$. We first show that

$$f_n^+(Y_n) = S_n(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1})).$$

We have $x \in f_n^+(Y_n)$ if and only if, there exists $y \in Y_n$, such that $x = f_n(y)$, if and only if, by the hypothesis, there exists $y \in R_n(Y_1, \ldots, Y_{n-1})$, such that $x = f_n(y)$, if and only if, by surjectivity, there exists $y \in R_n(f_1^+(f_1^-(Y_1)), \ldots, f_{n-1}^+(f_{n-1}^-(Y_{n-1})))$, such that $x = f_n(y)$, if and only if, by Proposition 10, there exists $y \in f_n^-(S_n(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1})))$, such that $y = f_n(x)$, if and only if $x = f_n(y) \in S_n(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1}))$. Finally, it must be shown that, for all $i = 1, \ldots, n-1$,

$$f_i^-(Y_i) = S_i(f_1^-(Y_1), \ldots, f_{i-1}^-(Y_{i-1}), f_{i+1}^-(Y_{i+1}), \ldots, f_{n-1}^-(Y_{n-1}), f_n^+(Y_n)).$$

We only show the case $i = 1$. The remaining cases then follow by symmetry. We have $y \in f_1^-(Y_1)$ if and only if $f_1(x) \in Y_1$ if and only if, by the hypothesis, $f_1(x) \in R_1(Y_2, \ldots, Y_{n-1}, Y_n)$ if and only if $x \in f_1^-(R_1(Y_2, \ldots, Y_{n-1}, Y_n))$ if and only if, by surjectivity, $x \in f_1^-(R_1(f_2^+(f_2^-(Y_2)), \ldots, f_{n-1}^+(f_{n-1}^-(Y_{n-1})), Y_n))$ if and only if, by Proposition 10, $x \in S_1(f_2^-(Y_2), \ldots, f_{n-1}^-(Y_{n-1}), f_n^+(Y_n))$.

■

Finally, some conditions are given under which the converse implications of the ones presented in Proposition 11 hold.

Proposition 12 *Suppose that* $\mathbb{K} = \langle K_1, \ldots, K_n, R\rangle, \mathbb{L} = \langle L_1, \ldots, L_n, S\rangle$ *are two* n*-adic formal contexts and* $f = \langle f_1, \ldots, f_n\rangle$ *a morphism* $f : \mathbb{K} \to \mathbb{L}$*.*

1. *If* f_n *is surjective and* f_i *is injective, for all* $i = 1, \ldots, n-1$*, then,*

$$\text{if } (f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n)) \in \mathcal{C}(\mathbb{K}), \text{ then } (X_1, \ldots, X_n) \in \mathcal{C}(\mathbb{L}).$$

2. *If* f_i *is surjective, for all* $i = 1, \ldots, n-1$*, and* f_n *is injective, then,*

$$\text{if } (f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1}), f_n^+(Y_n)) \in \mathcal{C}(\mathbb{L}), \text{ then } (Y_1, \ldots, Y_n) \in \mathcal{C}(\mathbb{K}).$$

Proof:

1. Suppose that $(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n)) \in \mathcal{C}(\mathbb{K})$. To show that, under the given hypotheses, $(X_1, \ldots, X_n) \in \mathcal{C}(\mathbb{L})$, we show, first, that

$$X_n = S_n(X_1, \ldots, X_{n-1}).$$

We have $x \in X_n$ if and only if, by surjectivity, $x \in f_n^+(f_n^-(X_n))$ if and only if, there exists $y \in f_n^-(X_n)$, such that $x = f_n(y)$, if and only if, by the hypothesis, there exists $y \in R_n(f_1^+(X_1), \ldots, f_{n-1}^+(X_{n-1}))$, such that $x = f_n(y)$, if and only if, by Proposition 10, there exists $y \in f_n^-(S_n(X_1, \ldots, X_{n-1}))$, such that $x = f_n(y)$, if and only if $x = f_n(y) \in S_n(X_1, \ldots, X_{n-1})$. Next, it must be shown that, for all $i = 1, \ldots, n-1$,

$$X_i = S_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n-1}, X_n).$$

We show the case $i = 1$. The remaining cases then follow by symmetry. We have $x \in X_1$ if and only if, by injectivity, $f_1(x) \in f_1^+(X_1)$ if and only if, by the hypothesis, $f_1(x) \in R_1(f_2^+(X_2), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n))$ if and only if $x \in f_1^-(R_1(f_2^+(X_2), \ldots, f_{n-1}^+(X_{n-1}), f_n^-(X_n)))$ if and only if, once again by Proposition 10, $x \in S_1(X_2, \ldots, X_{n-1}, f_n^+(f_n^-(X_n)))$ if and only if, by surjectivity, $x \in S_1(X_2, \ldots, X_{n-1}, X_n)$.

2. Suppose that $(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1}), f_n^+(Y_n)) \in \mathcal{C}(\mathbb{L})$. To show that, under the given hypothesis, $(Y_1, \ldots, Y_n) \in \mathcal{C}(\mathbb{K})$, we show, first, that

$$Y_n = R_n(Y_1, \ldots, Y_{n-1}).$$

We have $y \in Y_n$ if and only if, by injectivity, $f_n(y) \in f_n^+(Y_n)$ if and only if, by the hypothesis, $f_n(y) \in S_n(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1}))$ if and only if $y \in f_n^-(S_n(f_1^-(Y_1), \ldots, f_{n-1}^-(Y_{n-1})))$ if and only if, by Proposition 10,

$$y \in R_n(f_1^+(f_1^-(Y_1)), \ldots, f_{n-1}^+(f_{n-1}^-(Y_{n-1})))$$

if and only if, by surjectivity, $y \in R_n(Y_1, \ldots, Y_{n-1})$. Next, it must be shown that, for all $i = 1, \ldots, n-1$, we have that

$$Y_i = R_i(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_{n-1}, Y_n).$$

We only show the case $i = 1$. The remaining cases then follow by symmetry. We have $y \in Y_1$ if and only if, by surjectivity, $y \in f_1^+(f_1^-(Y_1))$ if and only if, there exists $x \in f_1^-(Y_1)$, such that $y = f_1(x)$, if and only if, by the hypothesis, there exists $x \in S_1(f_2^-(Y_2), \ldots, f_{n-1}^-(Y_{n-1}), f_n^+(Y_n))$, such that $y = f_1(x)$, if and only if, by Proposition 10, there exists $x \in f_1^-(R_1(f_2^+(f_2^-(Y_2)), \ldots, f_{n-1}^+(f_{n-1}^-(Y_{n-1})), Y_n))$, such that $y = f_1(x)$, if and only if, by surjectivity, there exists $x \in f_1^-(R_1(Y_2, \ldots, Y_{n-1}, Y_n))$, such that $y = f_1(x)$, if and only if $y = f_1(x) \in R_1(Y_2, \ldots, Y_{n-1}, Y_n)$.

■

In Theorem 8, it was shown that the collection of closed sets of an n-adic formal context forms a complete n-lattice under the n component-wise quasi-orderings. Theorem 6 of [16] is actually a stronger result containing a converse of this statement that will be presented now. The origins of this result go back to Wille's work on the triadic case and his Fundamental Theorem of Triadic Concept analysis in [20].

Theorem 13 (Representation Theorem) *For every complete n-lattice $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$, there exists an n-adic formal context $\mathbb{K}$, such that $\mathbf{L} \cong \mathcal{C}(\mathbb{K})$.*

We note that, given $\mathbf{L}$ in the Representation Theorem 13, the n-adic formal context that serves the purpose of verifying the conclusion is the context $\mathbb{K} = \langle L, \ldots, L, Y_{\mathbf{L}} \rangle$, where $Y_{\mathbf{L}}$ is the n-ary relation on L, defined by

$$Y_{\mathbf{L}} = \{(x_1, \ldots, x_n) \in L^n : (x_1, \ldots, x_n) \text{ is joined}\},$$

where $(x_1, \ldots, x_n) \in L^n$ is *joined* in $\mathbf{L}$ if and only if, by definition, there exists $y \in L$, such that $x_i \lesssim_i y$, for all $i = 1, \ldots, n$.

We note that, actually, Theorem 6 of [16] is more general than Theorem 13 and refer the reader to the proof of Theorem 6 of [16] for clues on how to prove Theorem 13 rather than repeating the proof here.

4 Complete n-Semilattices and n-Information Systems

The final concept that is related to the ones dealt with here is that of an n-information system, that was inspired by the concept of an information system of Scott [14]. Note, however, that, for the purposes of the present exposition, no consideration will be given to consistent sets of tokens. This is due to our desire to allow in the present setting possibly infinitary deductions, so that the current notion may fit exactly the framework of n-adic formal contexts and the induced (possibly infinitary) n-closure operators. A treatment of the finitary case, at both the level of n-closure operators and of n-information systems, is postponed for future study.

Notice how the conditions given for an n-information system in Definition 14 reflect the conditions imposed on an n-closure operator, in a way analogous to the entailment conditions of an ordinary information system reflecting the conditions imposed on an ordinary closure operator.

Definition 14 *An* n-information system *is a* $2n$*-tuple* $\mathbf{A} = \langle A_1, \ldots, A_n, \vdash_1, \ldots, \vdash_n \rangle$ *consisting of*

1. *a set* A_i *of* i-tokens, $i = 1, \ldots, n$,
2. *a relation of* i-entailment $\vdash_i$ *between members of* $\mathcal{P}(A_1), \ldots, \mathcal{P}(A_{n-1})$ *and members of* A_i, $i = 1, \ldots, n$, *(formally* $\vdash_i \subseteq \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_{n-1}) \times A_i$*) satisfying*

   (a) *If* $a \in Y_j$ *for some* $j = 1, \ldots, n-1$, *then* $Y_1, \ldots, Y_{n-1} \vdash_j a$.

   (b) *For all* $Y_i, Z_i \subseteq A_i, i = 1, \ldots, n-1$, *such that* $Y_i \subseteq Z_i$,
   $$Z_1, \ldots, Z_{n-1} \vdash_n a \quad \textit{implies} \quad Y_1, \ldots, Y_{n-1} \vdash_n a.$$

   (c) *For all* $X_i, Y_i \subseteq A_i, i = 1, \ldots, n-1$, *if, for some* $k = 1, \ldots, n-1$,
   - $X_i \subseteq Y_i$, *for all* $i \leq k$, *and*
   - $X_1, \ldots, X_{n-1} \vdash_i a$ *iff* $Y_1, \ldots, Y_{n-1} \vdash_i a$, *for all* $a \in A_i$ *and all* $i > k$,

   then $Y_1, \ldots, Y_{n-1} \vdash_k a$ *implies* $X_1, \ldots, X_{n-1} \vdash_k a$, *for all* $a \in A_k$.

   (d) *For all* $X_i, Y_i \subseteq A_i, i = 1, \ldots, n-1$, *if, for some* $k = 1, \ldots, n-1$,
   $$X_1, \ldots, X_{n-1} \vdash_i a \quad \textit{implies} \quad Y_1, \ldots, Y_{n-1} \vdash_i a, \textit{ for all } a \in A_i \textit{ and all } i \neq k,$$
   then, for all $a \in A_k$, $Y_1, \ldots, Y_{n-1} \vdash_k a$ *implies* $X_1, \ldots, X_{n-1} \vdash_k a$.

   (e) *If* $Y_i \subseteq A_i, i = 1, \ldots, n-1$, *and* $Z_i = \{a \in A_i : Y_1, \ldots, Y_{n-1} \vdash_i a\}, i = 1, \ldots, n-1$, *then, for all* $k = 1, \ldots, n$, *and all* $a \in A_k$,
   $$Z_1, \ldots, Z_{n-1} \vdash_k a, \quad \textit{implies} \quad Y_1, \ldots, Y_{n-1} \vdash_k a.$$

Suppose that $\mathbf{A} = \langle A_1, \ldots, A_n, \vdash_1, \ldots, \vdash_n \rangle$ is an n-information system. Given $X_i \subseteq A_i$, $i = 1, \ldots, n-1$, define
$$\overline{(X_1, \ldots, X_{n-1})}^i := \{a \in A_i : X_1, \ldots, X_{n-1} \vdash_i a\}.$$
This is the set of tokens of the n-information system that are *i-deducible from* the $(n-1)$-tuple $(X_1, \ldots, X_{n-1})$.

An n-tuple $\langle E_1, \ldots, E_n \rangle$ of subsets $E_i \subseteq A_i, i = 1, \ldots, n$, is said to be an *element* or an (*information*) *state* of the n-information system $\mathbf{A}$ if

1. $E_1, \ldots, E_{n-1} \vdash_i a$ implies $a \in E_i$, for all $i = 1, \ldots, n-1, a \in A_i$, and
2. $E_1, \ldots, E_{n-1} \vdash_n a$ iff $a \in E_n$, for all $a \in A_n$.

By $|\mathbf{A}|$ is denoted the collection of all information states of the n-information system $\mathbf{A}$.

In the following proposition, it is shown that

$$\overline{(X_1, \ldots, X_{n-1})} = \langle \overline{(X_1, \ldots, X_{n-1})}^1, \ldots, \overline{(X_1, \ldots, X_{n-1})}^n \rangle$$

is an information state whenever $X_i \subseteq A_i$, for all $i = 1, \ldots, n-1$, and, conversely, that every information state of the n-information system $\mathbf{A}$ has this form.

Proposition 15 *Suppose that $\mathbf{A} = \langle A_1, \ldots, A_n, \vdash_1, \ldots, \vdash_n \rangle$ is an n-information system. For all $X_i \subseteq A_i$, $i = 1, \ldots, n-1$, the n-tuple*

$$\overline{(X_1, \ldots, X_{n-1})} = \langle \overline{(X_1, \ldots, X_{n-1})}^1, \ldots, \overline{(X_1, \ldots, X_{n-1})}^n \rangle$$

is an information state of $\mathbf{A}$. Conversely, every information state $\langle E_1, \ldots, E_n \rangle$ of $\mathbf{A}$ has the form

$$\overline{(X_1, \ldots, X_{n-1})} = \langle \overline{(X_1, \ldots, X_{n-1})}^1, \ldots, \overline{(X_1, \ldots, X_{n-1})}^n \rangle,$$

for some $X_i \subseteq A_i$, $i = 1, \ldots, n-1$.

Proof:

Let $X_i \subseteq A_i, i = 1, \ldots, n-1$, and denote by $Z_i = \overline{(X_1, \ldots, X_{n-1})}^i, i = 1, \ldots, n$. Then, we have that, for all $i = 1, \ldots, n$ and all $z_i \in Z_i$, $X_1, \ldots, X_{n-1} \vdash_i z_i$. Now suppose that

$$\overline{(X_1, \ldots, X_{n-1})}^1, \ldots, \overline{(X_1, \ldots, X_{n-1})}^{n-1} \vdash_k a,$$

for some $k = 1, \ldots, n$, and some $a \in A_k$. This is equivalent to $Z_1, \ldots, Z_{n-1} \vdash_k a$, whence by Condition 2(e) of an n-information system, we get that $X_1, \ldots, X_{n-1} \vdash_k a$, i.e., that $a \in \overline{(X_1, \ldots, X_{n-1})}^k$. Therefore, $\overline{(X_1, \ldots, X_{n-1})}$ is $\vdash_k$-closed. This shows that $\overline{(Z_1, \ldots, Z_{n-1})}^k = \overline{(X_1, \ldots, X_{n-1})}^k$, for all $k = 1, \ldots, n-1$. Finally, by Condition 2(d) of Definition 14, we get that $Z_n = \overline{(X_1, \ldots, X_{n-1})}^n = \overline{(Z_1, \ldots, Z_{n-1})}^n$.

Suppose, conversely, that $\langle E_1, \ldots, E_n \rangle$ is an information state of $\mathbf{A}$. It will be shown that $E_i = \overline{(E_1, \ldots, E_{n-1})}^i$, for all $i = 1, \ldots, n$. By Condition 2 in the definition of an information state, it suffices to do this for $k = 1, \ldots, n-1$. By Condition 2(a) of the definition of an n-information system, $E_k \subseteq \overline{(E_1, \ldots, E_{n-1})}^k$. On the other hand, by the hypothesis, since $\langle E_1, \ldots, E_n \rangle$ is an information state, we get that $\overline{(E_1, \ldots, E_n)}^k \subseteq E_k$. Therefore, for all $k = 1, \ldots, n-1$, $\overline{(E_1, \ldots, E_{n-1})}^k = E_k$. ■

Next, it is shown that the map $(X_1, \ldots, X_{n-1}) \mapsto \overline{(X_1, \ldots, X_{n-1})}$ is an n-dimensional closure operator on $A_1, \ldots, A_n$.

Proposition 16 *Suppose that* $\mathbf{A} = \langle A_1, \ldots, A_n, \vdash_1, \ldots, \vdash_n\rangle$ *is an* n*-information system. The mapping* $\langle X_1, \ldots, X_{n-1}\rangle \mapsto \overline{(X_1, \ldots, X_{n-1})}$ *is an* n*-closure operator* $^- : \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_{n-1}) \to \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$.

Proof:

It is shown that all five conditions of Definition 3 hold for for the mapping $\langle X_1, \ldots, X_{n-1}\rangle \mapsto \overline{(X_1, \ldots, X_{n-1})}$.

Condition 1 of Definition 3 follows from Condition 2(a) of Definition 14. Condition 2 of Definition 3 follows from Condition 2(b) of Definition 14. Condition 3 of Definition 3 follows from Condition 2(c) of Definition 14. Condition 4 of Definition 3 follows from Condition 2(d) of Definition 14. Finally, Condition 5 of Definition 3 follows from Condition 2(e) of Definition 14. ■

Propositions 15 and 16 show that every n-information system gives rise to an n-closure operator, whose associated n-closure system consists of the entire collection of all information states of the n-information system. Therefore, we obtain

Theorem 17 *The collection* $|\mathbf{A}|$ *of information states of an* n*-information system* $\mathbf{A}$ *is an* n*-closure system.*

It will now be shown that, conversely, every n-closure system gives rise to an n-information system.

Definition 18 *Suppose that* $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ *be an* n*-closure system. Define the* $2n$*-tuple* $\mathbf{IS}(\mathcal{L}) = \langle A_1, \ldots, A_n, \vdash_1, \ldots, \vdash_n\rangle$ *by setting*

- $A_i = K_i$, *for all* $i = 1, \ldots, n$,
- $X_1, \ldots, X_{n-1} \vdash_j a$ *if and only if* $a \in \beta_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_j$, *for all* $X_i \subseteq A_i$, $i = 1, \ldots, n-1$, *all* $j = 1, \ldots, n$ *and all* $a \in A_j$.

Proposition 19 *Given an* n*-closure system* $\mathcal{L}$, $\mathbf{IS}(\mathcal{L})$ *is an* n*-information system.*

Proof:

By Lemma 5 of [18], $(X_1, \ldots, X_{n-1}) \mapsto \beta_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$ is an n-closure operator on $A_1, \ldots, A_n$. Therefore all five conditions of Definition 3 are satisfied by $\beta_{n-1,\ldots,1}$. Now it is not difficult to see that $\mathbf{IS}(\mathcal{L})$, as defined by Definition 18, satisfies all five corresponding conditions of Definition 14. Therefore $\mathbf{IS}(\mathcal{L})$ is indeed an n-information system. ■

The passage from an n-information system $\mathbf{A}$ to the n-closure system $|\mathbf{A}|$ of its information states, as established in Theorem 17, and the passage from an n-closure system $\mathcal{L}$ to the corresponding n-information system $\mathbf{IS}(\mathcal{L})$, given in Proposition 19, are inverses of each other in a sense made precise by the following theorem.

Theorem 20 *The mappings* $\mathcal{L} \mapsto \mathbf{IS}(\mathcal{L})$ *and* $\mathbf{A} \mapsto |\mathbf{A}|$ *are mutually inverse and set up a bijective correspondence between the class of all* n*-closure systems and the class of all* n*-information systems.*

Proof:
Suppose given an n-closure system $\mathcal{L}$ and an n-information system $\mathbf{A}$. Then, by following the relevant definitions, we obtain

$$\begin{array}{rll}(X_1,\ldots,X_n)\in|\mathbf{IS}(\mathcal{L})| & \text{iff} & \overline{(X_1,\ldots,X_n)}=(X_1,\ldots,X_n) \text{ in } \mathbf{IS}(\mathcal{L}) \\ & \text{iff} & \beta_{n-1,\ldots,1}(A_{n-1},\ldots,A_1)=(A_1,\ldots,A_n) \text{ in } \mathcal{L} \\ & \text{iff} & (A_1,\ldots,A_n)\in\mathcal{L},\end{array}$$

and, similarly,

$$\begin{array}{rll}(X_1,\ldots,X_n)\vdash_i a \text{ in } \mathbf{IS}(|\mathbf{A}|) & \text{iff} & a\in\beta_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_i \text{ in } |\mathbf{A}| \\ & \text{iff} & a\in\overline{(X_1,\ldots,X_{n-1})}^i \text{ in } \mathbf{A} \\ & \text{iff} & X_1,\ldots,X_{n-1}\vdash_i a \text{ in } \mathbf{A}.\end{array}$$

■

Finally, a connection is revealed between n-adic formal contexts and n-information systems and between n-adic formal concepts and information states. Similarly with the 2-dimensional case, in the n-dimensional framework, an n-adic formal context determines an n-information system, whose information states coincide with the n-adic formal concepts of the n-adic context.

Definition 21 *Suppose that* $\mathbb{K}=\langle K_1,\ldots,K_n,R\rangle$ *is an n-adic formal context. Consider the system* $\mathbf{IS}(\mathbb{K})=\langle K_1,\ldots,K_n,\vdash_1,\ldots,\vdash_n\rangle$*, where, for all* $i=1,\ldots,n$*,*

$$X_1,\ldots,X_{n-1}\vdash_i a \quad \textit{if and only if} \quad a\in\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_i,$$

for all $X_j\subseteq K_j, j=1,\ldots,n-1,$ *all* $a\in K_i$*.*

Given an n-adic formal context $\mathbb{K}$, the $2n$-tuple $\mathbf{IS}(\mathbb{K})$ is an n-information system.

Proposition 22 *Given an n-adic formal context* $\mathbb{K}=\langle K_1,\ldots,K_n,R\rangle$*, the system*

$$\mathbf{IS}(\mathbb{K})=\langle K_1,\ldots,K_n,\vdash_1,\ldots,\vdash_n\rangle$$

is an n-information system.

Proof:
Recall that, given an n-adic formal context $\mathbb{K}=\langle K_1,\ldots,K_n,R\rangle$, the mapping $(X_1,\ldots,X_{n-1})\mapsto\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)$, defined, by downward induction on $k=n,\ldots,1$, by

$$\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_n=R_n(X_1,\ldots,X_{n-1}),$$

and, given $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_i$, for all $i=n,n-1,\ldots,k+1$,

$$\begin{aligned}&\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_k=\\&\qquad R_k(X_1,\ldots,X_{k-1},\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_{k+1},\ldots,\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)_n),\end{aligned}$$

was shown in Proposition 6 to be an n-closure operator. Denote it by $C : \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$. By Lemma 4, C gives rise to an n-closure system, denoted here by $\mathcal{L}(C)$. This n-closure system gives, in turn, rise to $\mathbf{IS}(\mathcal{L}(C))$, which, by Proposition 19, is an n-information system. It suffices now to put the relevant definitions together to see that $\mathbf{IS}(\mathbb{K}) = \mathbf{IS}(\mathcal{L}(C))$. Thus $\mathbf{IS}(\mathbb{K})$ is in fact an n-information system. ■

Moreover, it may be shown that an n-tuple of sets is a closed set of the n-adic context $\mathbb{K}$ if and only if it is an information state of the n-information system $\mathbf{IS}(\mathbb{K})$.

Theorem 23 *Given an n-adic formal context $\mathbb{K} = \langle K_1, \ldots, K_n, R\rangle$, an n-tuple $\langle E_1, \ldots, E_n\rangle$, with $E_i \subseteq K_i, i = 1, \ldots, n$, is an information state of the n-information system $\mathbf{IS}(\mathbb{K}) = \langle K_1, \ldots, K_n, \vdash_1, \ldots, \vdash_n\rangle$ if and only it is an n-adic concept of the n-adic formal context $\mathbb{K}$.*

Proof:

Using the same notation as in the proof of Proposition 22, we have $(E_1, \ldots, E_n)$ is an information state of $\mathbf{IS}(\mathbb{K})$ if and only if, by Theorem 20, $(E_1, \ldots, E_n)$ belongs to $\mathcal{L}(C)$ if and only if $C(E_1, \ldots, E_{n-1}) = (E_1, \ldots, E_n)$ if and only if $\mathfrak{b}_{n-1,\ldots,1}(E_{n-1}, \ldots, E_1) = (E_1, \ldots, E_n)$, i.e., if and only if $(E_1, \ldots, E_{n-1})$ is an n-adic concept of the n-adic formal context $\mathbb{K}$. ■

Before closing, we remark that, in work currently in progress, we address some of the relationships that can be established when the notions that are considered in this paper are all taken to be finitary in a specific technical sense. The n-dimensional theory of finitary closure operators, finitary concept lattices and finitary information systems will contain as special cases the finitary notions considered by Zhang [23] in the computer science framework.

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