Categorical Abstract Algebraic Logic:
Pragmatic Matrix System Semantics

George Voutsadakis*

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Abstract

Taking after work of Tokarz, pragmatic matrix systems are introduced to provide a semantics for logics formalized as π-institutions. Unlike referential semantics which can only be associated with self-extensional π-institutions, but similarly with the case of pseudo-referential semantics, it is shown that every π-institution can be endowed with a pragmatic matrix system semantics. Self-extensional π-institutions are characterized as exactly those that possess a pragmatic matrix system semantics with respect to which the underlying algebraic system of the π-institution satisfies a specific property called extensionality.

1 Introduction

The area of *referential semantics* deals with the general problem of defining the primitive notion of **truth** of a sentence in some formal language at a "world". To assign a meaning to a given sentence one may use one of the following two methods, among others:

(a) A map from the set of possible worlds W to $\{0, 1\}$ which takes the value 1 for exactly those worlds in W in which the sentence is true.

*School of Mathematics and Computer Science, Lake Superior State University, Sault Sainte Marie, MI 49783, USA, gvoutsad@lssu.edu

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(b) A map from the set of possible worlds W into itself that maps a certain world a to a world b exactly in case the sentence describes in world a the world b.

The first approach is the one adopted in the study of referential semantics for sentential logics by Wójcicki in [7, 8] and of pseudo-referential semantics for sentential logics by Malinowski [4] and by Marek [5]. Similarly, it was the approach taken in studying referential and pseudo-referential semantics for π-institutions in [10] and [12], respectively. In the present work, we follow the second approach, adopted by Tokarz [6] that led to the introduction of the so called *pragmatic matrix semantics* for sentential logics.

Consider a **language type** $\mathcal{L} = \langle \Lambda, \rho \rangle$, where Λ is a set of logical connectives/operation symbols and $\rho : \Lambda \to \omega$ is a function assigning to each operation symbol its arity. Let V be a countable set of variables. Denote by $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V), \mathcal{L} \rangle$ the free $\mathcal{L}$-algebra generated by V. A **logic** $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ consists of a language type together with a structural consequence relation on $\mathrm{Fm}_{\mathcal{L}}(V)$. As is well-known, structural consequence relations are in one-to-one correspondence with structural closure operators (see, e.g., page 33 of [2]). Thus, a logic may be equivalently represented as a pair $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where C is a structural closure operator on $\mathrm{Fm}_{\mathcal{L}}(V)$.

A **generalized matrix**, or **gmatrix**, for $\mathcal{L}$ is a pair $\mathbb{A} = \langle \mathbf{A}, \mathcal{D} \rangle$, where $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ is an $\mathcal{L}$-algebra and $\mathcal{D}$ is a family of subsets of A.

A gmatrix $\mathbb{A} = \langle \mathbf{A}, \mathcal{D} \rangle$ **determines** a logic $\mathcal{S}^{\mathbb{A}} = \langle \mathcal{L}, C^{\mathbb{A}} \rangle$, defined, for all $\Phi \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\begin{array}{lll} \varphi \in C^{\mathbb{A}}(\Phi) & \text{iff} & \text{for all } h \in \mathrm{Hom}(\mathbf{Fm}_{\mathcal{L}}(V), \mathbf{A}) \text{ and all } D \in \mathcal{D}, \\ & & h(\Phi) \subseteq D \text{ implies } h(\varphi) \in D. \end{array}$$

Given a class K of gmatrices for $\mathcal{L}$, the logic **determined by** K is defined by $\mathcal{S}^{\mathsf{K}} = \langle \mathcal{L}, C^{\mathsf{K}} \rangle$, where $C^{\mathsf{K}} = \bigcap_{\mathbb{A} \in \mathsf{K}} C^{\mathbb{A}}$.

A class of gmatrices for $\mathcal{L}$ is said to form a **gmatrix semantics** for a logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ if $C^{\mathsf{K}} = C$.

A **referential algebra** for $\mathcal{L}$ is an $\mathcal{L}$-algebra $\mathbf{R} = \langle R, \mathcal{L}^{\mathbf{R}} \rangle$ such that R consists of a collection of subsets of a set U of **base** or **reference points**. For all $a \in U$, set $D_a = \{X \in R : a \in X\}$ and $\mathcal{D} = \{D_a : a \in U\}$. Then the gmatrix $\mathbb{R} = \langle \mathbf{R}, \mathcal{D} \rangle$ for $\mathcal{L}$ is called a **referential gmatrix** for $\mathcal{L}$ over U.

A logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is **self-extensional** if for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} C(\alpha) = C(\beta) & \text{implies} & C(\varphi(\alpha, \overline{z})) = C(\varphi(\beta, \overline{z})), \\ & & \text{for all } \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V). \end{array}$$

A fundamental result due to Wójcicki [7] (see, also, [9]) asserts that a logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is self-extensional if and only if it has a referential semantics, i.e., if and only if $C = C^{\mathbb{R}}$, for some referential gmatrix $\mathbb{R}$. Thus, non-self-extensional logics do not possess a referential gmatrix semantics.

Malinowski defined in [4] pseudo-referential gmatrices. The concept is a generalization of referential gmatrices and it is obtained by considering, in addition to the set U of reference points, a distinguished collection $U^* \subseteq \mathcal{P}(U)$ of subsets of the set of reference points. We define, for all $V \in U^*$, the set

$$D_V = \{X \in R : X \cap V \neq \varnothing\}$$

and set $\mathcal{D} = \{D_V : V \in U^*\}$. A matrix of the form $\mathbb{R} = \langle \mathbf{R}, \mathcal{D} \rangle$ is called a **pseudo-referential gmatrix** for $\mathcal{L}$. Note that, by taking $U^* = \{\{u\} : u \in U\}$ one obtains referential gmatrices as a special case. Malinowski shows in the Theorem of [4] that every logic - not just self-extensional ones - has a pseudo-referential gmatrix semantics. This work of Malinowski initiated an effort to provide a semantics along the lines of referential semantics to a class of sentential logics wider than the class of self-extensional ones.

In [6] Tokarz, switching from the first to the second approach outlined at the beginning of this Introduction, devised pragmatic gmatrices as an alternative to Malinowksi's pseudo-referential gmatrices. Besides following a different philosophical paradigm, pragmatic gmatrices have the advantage of being more intuitive than pseudo-referential ones.

The main idea is to replace the requirement that the underlying universe of the algebra be a subset of $\mathcal{P}(U)$, i.e., of 2^U, as in the case of referential algebras, by that of being a subset of U^U, i.e., a collection P of functions from the set of base points to itself. Now to define the filter family of the gmatrix that serves as the model of the sentential logic one needs a distinguished set of base points $T \subseteq U$ whose elements are termed **facts**. We define, for all $u \in U$

$$D_u = \{p \in P : p(u) \in T\}$$

and we set $\mathcal{D} = \{D_u : u \in U\}$. A **pragmatic gmatrix system** for $\mathcal{L}$ is one of the form $\mathbb{P} = \langle \mathbf{P}, \mathcal{D} \rangle$, where $\mathbf{P} = \langle P, \mathcal{L}^{\mathbf{P}} \rangle$ is an $\mathcal{L}$-algebra. Tokarz shows in Theorem 1 of [6] that every logic has a pragmatic gmatrix semantics. Tokarz then defines the notions of extensional and strongly extensional languages with respect to a pragmatic gmatrix $\mathbb{P}$ and shows that they characterize self-extensional logics. Thus, if one restricts to languages that are extensional or strongly extensional in this sense, one captures exactly the logics for which the referential gmatrices of Wójcicki form a suitable semantics.

The author has studied in a series of papers referentiality and self-extensionality and has established results paralleling those of Wójcicki for logical systems formalized as π-institutions (see, e.g., [10, 11]). Moreover, in [12], paralleling the work of Malinowksi, pseudo-referential gmatrix system semantics was introduced for π-institutions and it was shown that every π-institution possesses a pseudo-referential gmatrix semantics.

In this work, pragmatic gmatrix system semantics is introduced for π-institutions motivated by the same goal that led Tokarz to the introduction of the pragmatic gmatrix semantics for sentential logics, i.e., to provide an alternative, based on a different paradigm, to pseudo-referential matrix system semantics that would be applicable to a wider class of π-institutions than just the class of self-extensional ones. We, in fact, show in Theorem 1 that every π-institution has a pragmatic gmatrix system semantics. Moreover, by introducing the notion of a base algebraic system that is extensional with respect to a pragmatic gmatrix system, we show in Corollary 4 that self-extensional π-institutions, which are π-institutions having a referential gmatrix system semantics, are captured exactly by those that are based on extensional algebraic systems with respect to some pragmatic gmatrix system.

2 Preliminaries

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a **Set**-valued functor. The **clone of all natural transformations on** SEN is the category U with collection of objects SEN^α, α an ordinal, and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition of $\langle \tau_i : i < \beta \rangle : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ with $\langle \sigma_j : j < \gamma \rangle : \mathrm{SEN}^\beta \to \mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory of this category with all objects of the form SEN^k, $k < \omega$, and such that:

- it contains all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$, $i < k$, $k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}$ given by

$$p^{k,i}_\Sigma(\overline{\phi}) = \phi_i, \text{ for all } \overline{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

- for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < \ell\}$ of natural transformations in N, $\langle \tau_i : i < \ell \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ is also in N,

is referred to as a **category of natural transformations on** SEN (see, e.g., Section 2 of [10]).

An **algebraic system** is a triple $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ consisting of:

- A category **Sign** of **signatures**;
- A functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ giving for each signature $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{SEN}(\Sigma)$ of Σ-**sentences**;
- A category of natural transformations N on SEN.

Usually, in a specific context, a fixed underlying algebraic system is assumed, called the **base algebraic system** and denoted by $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$. Then, an $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is one such that there exists a surjective functor $N^\flat \to N$ that preserves all projection natural transformations (and, consequently, all arities of natural transformations involved).

An **interpreted** $N^\flat$-algebraic system is a pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, such that $\mathbf{A}$ is an $N^\flat$-algebraic system and $\langle F, \alpha \rangle : \mathbf{A}^\flat \to \mathbf{A}$ is an algebraic system morphism. In other words:

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is a functor;
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ is a natural transformation, such that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}^\flat(\Sigma)^k \arrow[r, "\alpha_\Sigma^k"] \arrow[d, "\sigma_\Sigma^\flat"'] & \mathrm{SEN}(F(\Sigma))^k \arrow[d, "\sigma_{F(\Sigma)}"] \\
\mathrm{SEN}^\flat(\Sigma) \arrow[r, "\alpha_\Sigma"'] & \mathrm{SEN}(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\sigma_\Sigma^\flat(\varphi_0, \ldots, \varphi_{k-1})) = \sigma_{F(\Sigma)}(\alpha_\Sigma(\varphi_0), \ldots, \alpha_\Sigma(\varphi_{k-1})),$$

where $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is the image natural transformation on SEN of $\sigma^\flat$ in $N^\flat$.

A **gmatrix system** (**for $\mathbf{A}^\flat$**) is a pair $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$, where $\mathcal{A}$ is an interpreted $N^\flat$-algebraic system and $\mathcal{D} = \{D^i : i \in I\}$ is a collection of filter families on $\mathbf{A}$, i.e., $D^i = \{D^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that $D^i_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$ and all $i \in I$.

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. A **π-institution based on $\mathbf{A}^\flat$** (see [1] and, also, [3] for the closely related notion of an institution) is a pair $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$, where $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is a closure system on $\mathbf{A}^\flat$, i.e., a collection of closure operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$, $\Sigma \in |\mathbf{Sign}^\flat|$, which satisfies the **structurality condition**, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and let $\mathbb{A} = \langle \mathcal{A}, \mathcal{D} \rangle$ be a gmatrix system for $\mathbf{A}^\flat$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$. The gmatrix system $\mathbb{A}$ **generates** a closure system $C^{\mathbb{A}}$ on $\mathbf{A}^\flat$ by the following rule: For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \varphi \in C^{\mathbb{A}}_\Sigma(\Phi) & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \text{ and all } i \in I, \\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq D^i_{F(\Sigma')} \\ & & \qquad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in D^i_{F(\Sigma')}. \end{array}$$

If K is a class of gmatrix systems for $\mathbf{A}^\flat$, then we set

$$C^{\mathsf{K}} = \bigcap_{\mathbb{A} \in \mathsf{K}} C^{\mathbb{A}},$$

where the intersection is applied signature-wise. The corresponding π-institutions are denoted by $\mathcal{I}^{\mathbb{A}} = \langle \mathbf{A}^\flat, C^{\mathbb{A}} \rangle$ and $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{A}^\flat, C^{\mathsf{K}} \rangle$. Note that both are based on the base algebraic system $\mathbf{A}^\flat$.

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ be a π-institution based on $\mathbf{A}^\flat$. We say that a class of gmatrix systems K for $\mathbf{A}^\flat$ is a **gmatrix system semantics for $\mathcal{I}$** in case $C^{\mathsf{K}} = C$.

The remainder of this work will focus on a special kind of gmatrix system semantics for π-institutions, the so-called *pragmatic gmatrix system semantics*, introduced in Section 3.

3 Pragmatic Gmatrix Systems

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.

Consider a category **Sign** and a functor $\mathrm{PTS} : |\mathbf{Sign}| \to \mathbf{Set}$ giving, for all $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{PTS}(\Sigma)$ of **Σ-base** or **Σ-reference points**.

Let, also, $\mathrm{FCT} : |\mathbf{Sign}| \to \mathbf{Set}$ be such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{FCT}(\Sigma) \subseteq \mathrm{PTS}(\Sigma).$$

This functor gives, for every $\Sigma \in |\mathbf{Sign}|$, the set of **$\Sigma$-facts**. Thus, Σ-facts are also Σ-base points.

A **pragmatic $N^\flat$-algebraic system** (**based on** PTS) is an $N^\flat$-algebraic system $\mathbf{P} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}(\Sigma) \subseteq \mathrm{PTS}(\Sigma)^{\mathrm{PTS}(\Sigma)},$$

i.e., a subset of the set of all functions from $\mathrm{PTS}(\Sigma)$ to itself.

A **pragmatic gmatrix system** (**based on** PTS **over** FCT) is a gmatrix system $\mathbb{P} = \langle \mathcal{P}, \mathcal{D}\rangle$, where $\mathcal{P} = \langle \mathbf{P}, \langle F, \alpha\rangle\rangle$ is an interpreted pragmatic $N^\flat$-algebraic system based on PTS and

$$\mathcal{D} = \{D^{\Sigma,p} : \Sigma \in |\mathbf{Sign}|, p \in \mathrm{PTS}(\Sigma)\},$$

where

$$D^{\Sigma,p}_{\Sigma'} = \begin{cases} \{k \in \mathrm{SEN}(\Sigma) : k(p) \in \mathrm{FCT}(\Sigma)\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma. \end{cases}$$

The same argument presented by Tokarz in Section II of [6] shows that it is not the case that $C^{\mathbb{P}}$ is self-extensional, for every pragmatic gmatrix system $\mathbb{P}$.

Theorem 1 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$ a π-institution based on $\mathbf{A}^\flat$. Then, there exists a pragmatic gmatrix system $\mathbb{P} = \langle \mathcal{P}, \mathcal{D}\rangle$, based on* PTS *over* FCT*, with $\mathcal{P} = \langle \mathbf{P}, \langle F, \alpha\rangle\rangle$ and $\mathbf{P} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, such that $C = C^{\mathbb{P}}$.*

Proof: Set $\mathbf{Sign} = \mathbf{Sign}^\flat$. Define, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{PTS}(\Sigma) = \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)),$$

where, as usual, $\mathcal{P}$ here denotes the powerset operator. Now, set, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{FCT}(\Sigma) = \{\mathrm{SEN}^\flat(\Sigma) - \{\varphi\} : \varphi \in \mathrm{SEN}^\flat(\Sigma)\} \subseteq \mathrm{PTS}(\Sigma).$$

Next, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, define a function $k^{\Sigma,\varphi} : \mathrm{PTS}(\Sigma) \to \mathrm{PTS}(\Sigma)$ by setting, for all $X \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$k^{\Sigma,\varphi}(X) = \begin{cases} \mathrm{SEN}^\flat(\Sigma) - \{\varphi\}, & \text{if } \varphi \in C_\Sigma(X), \\ \mathrm{SEN}(\Sigma), & \text{otherwise.} \end{cases}$$

Claim: For all $\Sigma \in |\mathbf{Sign}|$, and all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if $\varphi \neq \psi$, then $k^{\Sigma,\varphi} \neq k^{\Sigma,\psi}$.
Proof: If $\varphi \neq \psi$, then

$$\begin{array}{rcl} k^{\Sigma,\varphi}(\mathrm{SEN}^\flat(\Sigma)) & = & \mathrm{SEN}^\flat(\Sigma) - \{\varphi\}, \\ k^{\Sigma,\psi}(\mathrm{SEN}^\flat(\Sigma)) & = & \mathrm{SEN}^\flat(\Sigma) - \{\psi\}. \end{array}$$

Since $\varphi \neq \psi$, we clearly have

$$k^{\Sigma,\varphi}(\mathrm{SEN}^\flat(\Sigma)) \neq k^{\Sigma,\psi}(\mathrm{SEN}^\flat(\Sigma)),$$

whence $k^{\Sigma,\varphi} \neq k^{\Sigma,\psi}$. ■

Now we define the pragmatic $N^\flat$-algebraic system $\mathbf{P} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ as follows:

- For all $\Sigma \in |\mathbf{Sign}|$,

  $$\mathrm{SEN}(\Sigma) = \{k^{\Sigma,\varphi} : \varphi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

  For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}(\Sigma')$ is given by setting, for all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$,

  $$\mathrm{SEN}(f)(k^{\Sigma,\varphi}) = k^{\Sigma',\mathrm{SEN}^\flat(f)(\varphi)}.$$

  It is clear that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, thus defined, is a functor, since, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, we have, for all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$,

  $$\mathrm{SEN}(\Sigma) \xrightarrow{\mathrm{SEN}(f)} \mathrm{SEN}(\Sigma') \xrightarrow{\mathrm{SEN}(g)} \mathrm{SEN}(\Sigma'')$$

  $$\begin{array}{rcl} \mathrm{SEN}(g)(\mathrm{SEN}(f)(k^{\Sigma,\varphi})) & = & \mathrm{SEN}(g)(k^{\Sigma',\mathrm{SEN}^\flat(f)(\varphi)}) \\ & = & k^{\Sigma'',\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\varphi))} \\ & = & k^{\Sigma'',\mathrm{SEN}^\flat(gf)(\varphi)} \\ & = & \mathrm{SEN}(gf)(k^{\Sigma,\varphi}). \end{array}$$

- For all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, we define $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ be given, for all $\varphi_0,\dots,\varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma_\Sigma(k^{\Sigma,\varphi_0},\dots,k^{\Sigma,\varphi_{k-1}}) = k^{\Sigma,\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})}.$$

$\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is a natural transformation, since, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\varphi_0,\dots,\varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^k \arrow[r, "\sigma_\Sigma"] \arrow[d, "\mathrm{SEN}(f)^k"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\mathrm{SEN}(f)"] \\
\mathrm{SEN}(\Sigma')^k \arrow[r, "\sigma_{\Sigma'}"'] & \mathrm{SEN}(\Sigma')
\end{tikzcd}$$

$$\begin{aligned}
&\sigma_{\Sigma'}(\mathrm{SEN}(f)(k^{\Sigma,\varphi_0}),\dots,\mathrm{SEN}(f)(k^{\Sigma,\varphi_{k-1}})) \\
&\quad = \sigma_{\Sigma'}(k^{\Sigma',\mathrm{SEN}^\flat(f)(\varphi_0)},\dots,k^{\Sigma',\mathrm{SEN}^\flat(f)(\varphi_{k-1})}) \\
&\quad = k^{\Sigma',\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi_0),\dots,\mathrm{SEN}^\flat(f)(\varphi_{k-1}))} \\
&\quad = k^{\Sigma',\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1}))} \\
&\quad = \mathrm{SEN}(f)(k^{\Sigma,\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})}) \\
&\quad = \mathrm{SEN}(f)(\sigma_\Sigma(k^{\Sigma,\varphi_0},\dots,k^{\Sigma,\varphi_{k-1}})).
\end{aligned}$$

Finally, we set N be the category of natural transformations on SEN consisting of all natural transformations of the form σ, for $\sigma^\flat$ in $N^\flat$.

Now define the algebraic system morphism $\langle I,\alpha\rangle : \mathbf{A}^\flat \to \mathbf{P}$ as follows:

- $I : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is the identity functor (recall $\mathbf{Sign} = \mathbf{Sign}^\flat$);
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN}$ is defined by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(\Sigma)$$

be given, for all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\alpha_\Sigma(\varphi) = k^{\Sigma,\varphi}.$$

Again this is a bona fide natural transformation since, for all $\Sigma,\Sigma' \in$

$|\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}^\flat(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}^\flat(f)"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\mathrm{SEN}(f)"] \\
\mathrm{SEN}^\flat(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}(\Sigma')
\end{tikzcd}$$

$$\begin{array}{lcl}
\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) & = & k^{\Sigma',\mathrm{SEN}^\flat(f)(\varphi)} \\
& = & \mathrm{SEN}(f)(k^{\Sigma,\varphi}) \\
& = & \mathrm{SEN}(f)(\alpha_\Sigma(\varphi)).
\end{array}$$

Thus, the pair $\mathcal{P} = \langle \mathbf{P}, \langle I, \alpha\rangle\rangle$ is an interpreted pragmatic $N^\flat$-algebraic system based on PTS.

Now, following the standard procedure for a pragmatic gmatrix system, we define, for all $\Sigma \in |\mathbf{Sign}|$ and all $X \in \mathrm{PTS}(\Sigma)$, i.e., $X \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\Sigma,X} = \{D^{\Sigma,X}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$$

by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$D^{\Sigma,X}_{\Sigma'} = \begin{cases} \{k^{\Sigma,\varphi} \in \mathrm{SEN}(\Sigma) : k^{\Sigma,\varphi}(X) \in \mathrm{FCT}(\Sigma)\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma. \end{cases}$$

and let

$$\mathcal{D} = \{D^{\Sigma,X} : \Sigma \in |\mathbf{Sign}|, X \subseteq \mathrm{SEN}^\flat(\Sigma)\}.$$

Then the structure $\mathbb{P} = \langle \mathcal{P}, \mathcal{D}\rangle$ is a pragmatic gmatrix system based on PTS over FCT.

Before continuing with the claim that will conclude the proof of the theorem, we note that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, because of the definition of $k^{\Sigma,\varphi}$ and of $\mathrm{FCT}(\Sigma)$, we get

$$D^{\Sigma,X}_{\Sigma'} = \begin{cases} \{k^{\Sigma,\varphi} \in \mathrm{SEN}(\Sigma) : \varphi \in C_\Sigma(X)\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma. \end{cases}$$

Claim: $C = C^{\mathbb{P}}$.

Proof: Suppose, first, that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\varphi \notin C_\Sigma(\Phi)$. Then, for all $\phi \in \Phi$,

$$\alpha_\Sigma(\phi)(\Phi) = k^{\Sigma,\phi}(\Phi) = \mathrm{SEN}^\flat(\Sigma) - \{\phi\} \in \mathrm{FCT}(\Sigma).$$

Thus, by the definition of $\mathcal{D}$, $\alpha_\Sigma(\phi) \in D_\Sigma^{\Sigma,\Phi}$, for all $\phi \in \Phi$. On the other hand,

$$\alpha_\Sigma(\varphi)(\Phi) = k^{\Sigma,\varphi}(\Phi) = \mathrm{SEN}^\flat(\Sigma) \notin \mathrm{FCT}(\Sigma).$$

So $\alpha_\Sigma(\varphi) \notin D_\Sigma^{\Sigma,\Phi}$. It follows that $\varphi \notin C_\Sigma^{\mathbb{P}}(\Phi)$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$. Consider $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, such that, for all $\phi \in \Phi$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in D_{\Sigma'}^{\Sigma',X'},$$

for some $X' \subseteq \mathrm{SEN}^\flat(\Sigma')$. Then, for all $\phi \in \Phi$, $k^{\Sigma',\mathrm{SEN}^\flat(f)(\phi)} \in D_{\Sigma'}^{\Sigma',X'}$. This implies that $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(X')$, for all $\phi \in \Phi$. Since, by hypothesis, $\varphi \in C_\Sigma(\Phi)$, we get, by structurality,

$$\mathrm{SEN}^\flat(f)(\varphi) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq C_{\Sigma'}(X').$$

This shows that $k^{\Sigma',\mathrm{SEN}^\flat(f)(\varphi)}(X') \in D_{\Sigma'}^{\Sigma',X'}$, or, equivalently, that

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in D_{\Sigma'}^{\Sigma',X'}.$$

Hence, we conclude that $\varphi \in C_\Sigma^{\mathbb{P}}(\Phi)$. ■

By the claim, we conclude that every π-institution $\mathcal{I}$ has a pragmatic gmatrix system semantics, namely one consisting of the single pragmatic gmatrix system constructed in this proof, which may be called the **canonical pragmatic gmatrix system associated with** the π-institution $\mathcal{I}$. ■

In Section 4, we characterize those π-institutions that are self-extensional based on the type of the available pragmatic gmatrix system semantics for them.

4 Extensionality

Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.

Let $\mathbb{P} = \langle \mathcal{P}, \mathcal{D} \rangle$ be a pragmatic gmatrix system based on PTS over FCT, with $\mathcal{P} = \langle \mathbf{P}, \langle F, \alpha \rangle \rangle$ and $\mathbf{P} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$.

Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then φ, ψ are called **coreferential with respect to** $\mathbb{P}$, or $\mathbb{P}$-**coreferential**, in symbols $\varphi \sim_\Sigma^{\mathbb{P}} \psi$, if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $a \in \mathrm{PTS}(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(a) \in \mathrm{FCT}(F(\Sigma')) \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))(a) \in \mathrm{FCT}(F(\Sigma')).$$

We say that $\mathbf{A}^\flat$ is **extensional with respect to** $\mathbb{P}$, or $\mathbb{P}$-**extensional**, written $\mathbb{P} \in \mathsf{Ext}(\mathbf{A}^\flat)$, if, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0, \psi_0, \ldots, \varphi_{k-1}, \psi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\varphi_0 \sim^{\mathbb{P}}_\Sigma \psi_0, \ldots, \varphi_{k-1} \sim^{\mathbb{P}}_\Sigma \psi_{k-1} \quad \text{imply} \quad \sigma^\flat_\Sigma(\varphi_0, \ldots, \varphi_{k-1}) \sim^{\mathbb{P}}_\Sigma \sigma^\flat_\Sigma(\psi_0, \ldots, \psi_{k-1}).$$

Let again $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then φ and ψ are called **synonymous with respect to** $\mathbb{P}$, or $\mathbb{P}$-**synonymous**, written $\varphi \approx^{\mathbb{P}}_\Sigma \psi$, if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).$$

Moreover, $\mathbf{A}^\flat$ is said to be **strongly extensional with respect to** $\mathbb{P}$, or **strongly** $\mathbb{P}$-**extensional**, in symbols $\mathbb{P} \in \mathbf{Ext}(\mathbf{A}^\flat)$, if $\sim^{\mathbb{P}} \leq \approx^{\mathbb{P}}$, where, here, $\leq$ denotes signature-wise inclusion.

Since, obviously, $\approx^{\mathbb{P}} \leq \sim^{\mathbb{P}}$, we have that

$$\mathbb{P} \in \mathbf{Ext}(\mathbf{A}^\flat) \quad \text{iff} \quad \sim^{\mathbb{P}} = \approx^{\mathbb{P}} \quad \text{implies} \quad \mathbb{P} \in \mathsf{Ext}(\mathbf{A}^\flat).$$

Thus, $\mathbf{Ext}(\mathbf{A}^\flat) \subseteq \mathsf{Ext}(\mathbf{A}^\flat)$.

Given a base algebraic system $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ based on $\mathbf{A}^\flat$, recall that the **intederivabilty relation system** of $\mathcal{I}$ is the equivalence system $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ by setting, for all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

In general, $\Lambda(\mathcal{I})$ is an equivalence system but not an $N^\flat$-congruence system. The π-institution $\mathcal{I}$ is called **self-extensional** if $\Lambda(\mathcal{I})$ is a congruence system, i.e., if, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0, \psi_0, \ldots, \varphi_{k-1}, \psi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\varphi_i \Lambda_\Sigma(\mathcal{I}) \psi_i, \text{ for all } i < k, \text{ imply } \sigma^\flat_\Sigma(\varphi_0, \ldots, \varphi_{k-1}) \Lambda_\Sigma(\mathcal{I}) \sigma^\flat_\Sigma(\psi_0, \ldots, \psi_{k-1}).$$

Theorem 2 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathbb{P} = \langle \mathcal{P}, \mathcal{D} \rangle$ a pragmatic gmatrix system for $\mathbf{A}^\flat$. If $\mathbb{P} \in \mathsf{Ext}(\mathbf{A}^\flat)$, then $\mathcal{I}^{\mathbb{P}} = \langle \mathbf{A}^\flat, C^{\mathbb{P}} \rangle$ is a self-extensional π-institution.*

Proof: Suppose $\mathbb{P} \in \mathsf{Ext}(\mathbf{A}^\flat)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$ and $\varphi_0, \psi_0, \ldots, \varphi_{k-1}, \psi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\varphi_i \Lambda_\Sigma(\mathcal{I}^{\mathbb{P}}) \psi_i$, for all $i < k$. This means that $C^{\mathbb{P}}_\Sigma(\varphi_i) = C^{\mathbb{P}}_\Sigma(\psi_i)$, for all $i < k$. By the definition of $C^{\mathbb{P}}$, this implies that $\varphi_i \sim^{\mathbb{P}}_\Sigma \psi_i$, for all $i < k$. Therefore, by the $\mathbb{P}$-extensionality of $\mathbf{A}^\flat$,

$$\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1}) \sim^{\mathbb{P}}_\Sigma \sigma^\flat_\Sigma(\psi_0,\dots,\psi_{k-1}).$$

Thus, reversing the steps, we now obtain

$$C^{\mathbb{P}}_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})) = C^{\mathbb{P}}_\Sigma(\sigma^\flat_\Sigma(\psi_0,\dots,\psi_{k-1})).$$

This gives that $\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})\Lambda_\Sigma(\mathcal{I}^{\mathbb{P}})\sigma^\flat_\Sigma(\psi_0,\dots,\psi_{k-1})$, which shows that $\mathcal{I}^{\mathbb{P}}$ is self-extensional. ■

Consider again a base algebraic system $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$. Let $\mathbf{Sign}$ be a category and $\mathrm{PTS}^R : |\mathbf{Sign}| \to \mathbf{Set}$. A gmatrix system $\mathbb{R} = \langle \mathcal{R}, \mathcal{C}\rangle$ for $\mathbf{A}^\flat$, with $\mathcal{R} = \langle \mathbf{R}, \langle F, \alpha^R\rangle\rangle$ and $\mathbf{R} = \langle \mathbf{Sign}, \mathrm{SEN}^R, N^R\rangle$, is called a **referential gmatrix system** (**based on** PTS^R) if:

- $\mathcal{R}$ is an interpreted $N^\flat$-algebraic system, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}^R(\Sigma) \subseteq \mathcal{P}(\mathrm{PTS}^R(\Sigma))$;
- The collection of filter families $\mathcal{C}$ is given by

$$\mathcal{C} = \{C^{\Sigma,p} : \Sigma \in |\mathbf{Sign}|, p \in \mathrm{PTS}^R(\Sigma)\},$$

where, for all $\Sigma \in |\mathbf{Sign}|$ and all $p \in \mathrm{PTS}^R(\Sigma)$, $C^{\Sigma,p} = \{C^{\Sigma,p}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$ is defined by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$C^{\Sigma,p}_{\Sigma'} = \begin{cases} \{X \in \mathrm{SEN}^R(\Sigma) : p \in X\}, & \text{if } \Sigma' = \Sigma,\\ \varnothing, & \text{if } \Sigma' \neq \Sigma.\end{cases}$$

Note that, if $\mathbb{R}$ is a referential gmatrix system for $\mathbf{A}^\flat$, then we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \varphi \in C^{\mathbb{R}}_\Sigma(\Phi) & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|,\ f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\ & & \bigcap_{\phi\in\Phi} \alpha^R_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq \alpha^R_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)).\end{array}$$

It was shown in Theorem 8 of [10], based on a fundamental result of Wójcicki, that a π-institution $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$ based on $\mathbf{A}^\flat$ is self-extensional if and only if it is of the form $\mathcal{I}^{\mathbb{R}} = \langle \mathbf{A}^\flat, C^{\mathbb{R}}\rangle$ for some referential gmatrix system $\mathbb{R}$.

Theorem 3 *Let $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. Suppose that $\mathbb{R} = \langle \mathcal{R}, \mathcal{C}\rangle$, with $\mathcal{R} = \langle \mathbf{R}, \langle F, \alpha^R\rangle\rangle$ and $\mathbf{R} = \langle \mathbf{Sign}, \mathrm{SEN}^R, N^R\rangle$, is a referential gmatrix system for $\mathbf{A}^\flat$. Then there exists a pragmatic gmatrix system $\mathbb{P} = \langle \mathcal{P}, \mathcal{D}\rangle \in \mathbf{Ext}(\mathbf{A}^\flat)$, such that $C^{\mathbb{P}} = C^{\mathbb{R}}$.*

Proof: Let $\mathbb{R} = \langle \mathcal{R}, \mathcal{C} \rangle$, with $\mathcal{R} = \langle \mathbf{R}, \langle F, \alpha^R \rangle \rangle$ and $\mathbf{R} = \langle \mathbf{Sign}, \mathrm{SEN}^R, N^R \rangle$, be a referential gmatrix system for $\mathbf{A}^\flat$ based on $\mathrm{PTS}^R : |\mathbf{Sign}| \to \mathbf{Set}$, i.e., such that $\mathrm{SEN}^R(\Sigma) \subseteq \mathcal{P}(\mathrm{PTS}^R(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}|$.

For every $\Sigma \in |\mathbf{Sign}|$, let $p_\Sigma \in \mathrm{PTS}^R(\Sigma)$ and $q_\Sigma \notin \mathrm{PTS}^R(\Sigma)$. We define $\mathrm{PTS}^P : |\mathbf{Sign}| \to \mathbf{Set}$ by setting

$$\mathrm{PTS}^P(\Sigma) = \mathrm{PTS}^R(\Sigma) \cup \{q_\Sigma\}, \quad \text{for all } \Sigma \in |\mathbf{Sign}|.$$

Define, also $\mathrm{FCT} : |\mathbf{Sign}| \to \mathbf{Set}$ by setting

$$\mathrm{FCT}(\Sigma) = \{q_\Sigma\}, \quad \text{for all } \Sigma \in |\mathbf{Sign}|.$$

Note that $\mathrm{FCT} \leq \mathrm{PTS}^P$. Next, for all $\Sigma \in |\mathbf{Sign}|$, $X \in \mathrm{SEN}^R(\Sigma)$, define $p^{\Sigma,X} : \mathrm{PTS}^P(\Sigma) \to \mathrm{PTS}^P(\Sigma)$ by setting, for all $a \in \mathrm{PTS}^P(\Sigma)$,

$$p^{\Sigma,X}(a) = \begin{cases} q_\Sigma, & \text{if } (a = q_\Sigma \text{ or } a \in X), \\ p_\Sigma, & \text{otherwise.} \end{cases}$$

Claim: If $X, Y \in \mathrm{SEN}^R(\Sigma)$, with $X \neq Y$, then $p^{\Sigma,X} \neq p^{\Sigma,Y}$.
Proof: If $X \neq Y$, then there exists $x \in X \backslash Y$ or $y \in Y \backslash X$. Assume without loss of generality that the first occurs. Then, we have $p^{\Sigma,X}(x) = q_\Sigma$, whereas $p^{\Sigma,Y}(x) = p_\Sigma \neq q_\Sigma$. Thus, $p^{\Sigma,X} \neq p^{\Sigma,Y}$. ■

We now proceed to define the pragmatic gmatrix system $\mathbb{P}$ based on PTS^P over FCT.

First, define the algebraic system $\mathbf{P} = \langle \mathbf{Sign}, \mathrm{SEN}^P, N^P \rangle$ as follows:

- Foe every $\Sigma \in |\mathbf{Sign}|$, define

  $$\mathrm{SEN}^P(\Sigma) = \{p^{\Sigma,X} : X \in \mathrm{SEN}^R(\Sigma)\}.$$

  Moreover, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we define $\mathrm{SEN}^P(f) : \mathrm{SEN}^P(\Sigma) \to \mathrm{SEN}^P(\Sigma')$ by setting, for all $X \in \mathrm{SEN}^R(\Sigma)$,

  $$\mathrm{SEN}^P(f)(p^{\Sigma,X}) = p^{\Sigma',\mathrm{SEN}^R(f)(X)}.$$

  This makes $\mathrm{SEN}^P : \mathbf{Sign} \to \mathbf{Set}$ a functor.

- Now consider $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$. We define $\sigma^P : (\mathrm{SEN}^P)^k \to \mathrm{SEN}^P$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\sigma^P_\Sigma : \mathrm{SEN}^P(\Sigma)^k \to \mathrm{SEN}^P(\Sigma)$ be given, for all $X_0, \ldots, X_{k-1} \in \mathrm{SEN}^R(\Sigma)$, by

  $$\sigma^P_\Sigma(p^{\Sigma,X_0}, \ldots, p^{\Sigma,X_{k-1}}) = p^{\Sigma,\sigma^R_\Sigma(X_0,\ldots,X_{k-1})}.$$

  This is a well defined natural transformation. We set N^P to be the collection of all natural transformations of the form σ^P as $\sigma^\flat$ ranges in $N^\flat$. Then N^P is a category of natural transformations on SEN^P.

The triple $\mathbf{P} = \langle \mathbf{Sign}, \mathrm{SEN}^P, N^P \rangle$ is the algebraic system reduct of $\mathbb{P}$. It is clearly a pragmatic $N^\flat$-algebraic system based on PTS^P.

Next, define the pair $\langle F, \alpha^P \rangle : \mathbf{A}^\flat \to \mathbf{P}$ as follows:

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is the functor inherited by the referential gmatrix system $\mathbb{R}$.
- $\alpha^P : \mathrm{SEN}^\flat \to \mathrm{SEN}^P \circ F$ is defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\alpha^P_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^P(F(\Sigma))$ be given, for all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, by
$$\alpha^P_\Sigma(\varphi) = p^{F(\Sigma), \alpha^R_\Sigma(\varphi)}.$$
Then $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN}^P \circ F$ is a well defined natural transformation.

With these definitions the pair $\mathcal{P} = \langle \mathbf{P}, \langle F, \alpha^P \rangle \rangle$ becomes an interpreted pragmatic $N^\flat$-algebraic system based on PTS^P.

Finally, for all $a \in \mathrm{PTS}^P(\Sigma)$, set $D^{\Sigma,a} = \{D^{\Sigma,a}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$, where, for all $\Sigma' \in |\mathbf{Sign}|$,

$$D^{\Sigma,a}_{\Sigma'} = \begin{cases} \{p^{\Sigma,X} \in \mathrm{SEN}^P(\Sigma) : p^{\Sigma,X}(a) = q_\Sigma\}, & \text{if } \Sigma' = \Sigma, \\ \varnothing, & \text{if } \Sigma' \neq \Sigma. \end{cases}$$

and let

$$\mathcal{D} = \{D^{\Sigma,a} : \Sigma \in |\mathbf{Sign}|, a \in \mathrm{PTS}^P(\Sigma)\}.$$

This completes the definition of the pragmatic gmatrix system $\mathbb{P}$. We note that, because of the definition of $\mathbb{P}$, if one defines $\alpha : \mathrm{SEN}^R \to \mathrm{SEN}^P$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma : \mathrm{SEN}^R(\Sigma) \to \mathrm{SEN}^P(\Sigma)$ be given, for all $X \in \mathrm{SEN}^R(\Sigma)$, by

$$\alpha_\Sigma(X) = p^{\Sigma,X},$$

then the following diagrams commute:

$$\begin{tikzcd} & \mathbf{Sign}^\flat \arrow[dl, "F"'] \arrow[dr, "F"] & \\ \mathbf{Sign} \arrow[rr, "\equiv"] & & \mathbf{Sign} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{SEN}^\flat(\Sigma) \arrow[dl, "\alpha^R_\Sigma"'] \arrow[dr, "\alpha^P_\Sigma"] & \\ \mathrm{SEN}^R(F(\Sigma)) \arrow[rr, "\alpha_{F(\Sigma)}"'] & & \mathrm{SEN}^P(F(\Sigma)) \end{tikzcd}$$

It only remains to show that $\mathbb{P} \in \mathbf{Ext}(\mathbf{A}^\flat)$ and that $C^{\mathbb{P}} = C^{\mathbb{R}}$.

Claim: $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ is strongly extensional with respect to $\mathbb{P} = \langle \mathcal{P}, \mathcal{D} \rangle$.

Proof: It suffices to show that $\sim\,\leq\,\approx$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\varphi \sim^{\mathbb{P}}_\Sigma \psi$. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and $a \in \mathrm{PTS}^P(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(a) \in \mathrm{FCT}(F(\Sigma')) \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))(a) \in \mathrm{FCT}(F(\Sigma')),$$

i.e., for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and $a \in \mathrm{PTS}^P(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))(a) = q_{F(\Sigma')} \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))(a) = q_{F(\Sigma')}.$$

Equivalently, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)),$$

i.e., $\varphi \approx^{\mathbb{P}}_\Sigma \psi$. ∎

And the final claim:
Claim: $C^{\mathbb{P}} = C^{\mathbb{R}}$.
Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, that $\varphi \in C^{\mathbb{P}}_\Sigma(\Phi)$ if and only if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $a \in \mathrm{PTS}^P(F(\Sigma'))$,

$$\begin{aligned} &\alpha^P_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{p^{F(\Sigma'),X} \in \mathrm{SEN}^P(F(\Sigma')) : p^{F(\Sigma'),X}(a) = q_{F(\Sigma')}\} \\ &\text{implies} \\ &\alpha^P_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in \{p^{F(\Sigma'),X} \in \mathrm{SEN}^P(F(\Sigma')) : p^{F(\Sigma'),X}(a) = q_{F(\Sigma')}\}. \end{aligned}$$

iff, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $a \in \mathrm{PTS}^P(F(\Sigma'))$,

$$p^{F(\Sigma'),\alpha^R_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))}(a) = q_{F(\Sigma')}, \text{ all } \phi \in \Phi, \quad \text{implies} \quad p^{F(\Sigma'),\alpha^R_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))}(a) = q_{F(\Sigma')},$$

iff, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p \in \mathrm{PTS}^R(F(\Sigma'))$,

$$p \in \bigcap_{\phi \in \Phi} \alpha^R_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \quad \text{implies} \quad p \in \alpha^R_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)),$$

if and only if $\varphi \in C^{\mathbb{R}}_\Sigma(\Phi)$. ∎

This completes the proof of the theorem. ∎

We close this work by providing a corollary that summarizes what was accomplished:

Corollary 4 *Let* $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathcal{I} = \langle \mathbf{A}^\flat, C \rangle$ *a* π*-institution based on* $\mathbf{A}^\flat$*. Then, the following statements are equivalent:*

(i) $C = C^{\mathbb{R}}$ *for some referential gmatrix system* $\mathbb{R}$*;*

(ii) $C = C^{\mathbb{P}}$ *for some pragmatic gmatrix system* $\mathbb{P}$ *with respect to which* $\mathbf{A}^\flat$ *is strongly extensional;*

(iii) $C = C^{\mathbb{P}}$ *for some pragmatic gmatrix system* $\mathbb{P}$ *with respect to which* $\mathbf{A}^\flat$ *is extensional;*

(iv) $\mathcal{I}$ *is self-extensional.*

Proof:

(i)⇔(iv) This is essentially Theorem 8 of [10].

(i)⇒(ii) By Theorem 3.

(ii)⇒(iii) Follows from $\mathbf{Ext}(\mathbf{A}^\flat) \subseteq \mathsf{Ext}(\mathbf{A}^\flat)$.

(iii)⇒(iv) By Theorem 2.

■

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