

# Categorical Abstract Algebraic Logic: Pseudo-Referential Matrix System Semantics

George Voutsadakis\*

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To the memory of A.I. Mal'cev,  
50 years since his passing.

## Abstract

This work adapts techniques and results first developed by Malinowski and by Marek in the context of referential semantics of sentential logics to the context of logics formalized as  $\pi$ -institutions. More precisely, the notion of a pseudo-referential matrix system is introduced and it is shown how this construct generalizes that of a referential matrix system. It is then shown that every  $\pi$ -institution has a pseudo-referential matrix system semantics. This contrasts with referential matrix system semantics which is only available for self-extensional  $\pi$ -institutions by a previous result of the author obtained as an extension of a classical result of Wójcicki. Finally, it is shown that it is possible to replace an arbitrary pseudo-referential matrix system semantics by a *discrete* pseudo-referential matrix system semantics.

## 1 Introduction

Let  $\mathcal{L} = \langle \Lambda, \rho \rangle$  be a logical signature/algebraic type, i.e., a set of logical connectives/operation symbols  $\Lambda$  with attached finite arities given by the function  $\rho : \Lambda \rightarrow \omega$ . Let also  $V$  be a countably infinite set of propositional

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\*School of Mathematics and Computer Science, Lake Superior State University, Sault Sainte Marie, MI 49783, USA, [gvoutsad@lssu.edu](mailto:gvoutsad@lssu.edu)

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variables and  $T$  a set of **reference/base points**. Wójcicki [10] defines a **referential algebra  $\mathbf{A}$  over  $T$**  (or **based on  $T$** ) to be an  $\mathcal{L}$ -algebra with universe  $A \subseteq \{0, 1\}^T$ , or, equivalently,  $A \subseteq \mathcal{P}(T)$ .

Let  $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathbf{Fm}_{\mathcal{L}}(V), \mathcal{L} \rangle$  be the free  $\mathcal{L}$ -algebra generated by the set  $V$  of variables. A homomorphism from  $\mathbf{Fm}_{\mathcal{L}}(V)$  into a referential algebra  $\mathbf{A}$  over  $T$  may be viewed as an interpretation of the formulas of  $\mathbf{Fm}_{\mathcal{L}}(V)$  in  $\mathbf{A}$ . We conceive of a formula  $\alpha \in \mathbf{Fm}_{\mathcal{L}}(V)$  as **being true at point  $t \in T$  under  $h$**  if and only if  $t \in h(\alpha)$ . This notion of truth gives rise to a consequence operation on  $\mathbf{Fm}_{\mathcal{L}}(V)$ . Namely, a referential algebra  $\mathbf{A}$  determines the consequence operator  $C^{\mathbf{A}}$  on  $\mathbf{Fm}_{\mathcal{L}}(V)$  by setting, for all  $X \cup \{\alpha\} \subseteq \mathbf{Fm}_{\mathcal{L}}(V)$ ,  $\alpha \in C^{\mathbf{A}}(X)$  iff, for all  $h : \mathbf{Fm}_{\mathcal{L}}(V) \rightarrow \mathbf{A}$  and all  $t \in T$ ,

$$h(\beta)(t) = 1, \text{ for all } \beta \in X, \text{ implies } h(\alpha)(t) = 1,$$

or equivalently, iff, for all  $h : \mathbf{Fm}_{\mathcal{L}}(V) \rightarrow \mathbf{A}$ ,

$$\bigcap_{\beta \in X} h(\beta) \subseteq h(\alpha).$$

Wójcicki calls a propositional logic  $\mathcal{S} = \langle \mathcal{L}, C \rangle$ , where  $C = C^{\mathbf{A}}$ , for a referential algebra  $\mathbf{A}$ , a **referential** (or **referentially truth-functional**) **propositional logic**.

Wójcicki shows in [10] that, given a class  $\mathbf{K}$  of referential algebras, there exists a single referential algebra  $\mathbf{A}$ , such that  $C^{\mathbf{K}} := \bigcap_{\mathbf{K} \in \mathbf{K}} C^{\mathbf{K}} = C^{\mathbf{A}}$ . Thence follows that a propositional logic is referential if and only if it is defined by a class of referential algebras.

Given a propositional logic  $\mathcal{S} = \langle \mathcal{L}, C \rangle$ , the **Frege or interderivability relation** of  $\mathcal{S}$ , denoted  $\Lambda(\mathcal{S})$ , is the equivalence relation on  $\mathbf{Fm}_{\mathcal{L}}(V)$ , defined, for all  $\alpha, \beta \in \mathbf{Fm}_{\mathcal{L}}(V)$ , by

$$\langle \alpha, \beta \rangle \in \Lambda(\mathcal{S}) \quad \text{iff} \quad C(\alpha) = C(\beta).$$

The **Tarski congruence**  $\tilde{\Omega}(\mathcal{S})$  of  $\mathcal{S}$  (see [5]) is the largest congruence relation on  $\mathbf{Fm}_{\mathcal{L}}(V)$  that is compatible with all theories of  $\mathcal{S}$ . The Tarski congruence is a special case of the **Suszko congruence**  $\tilde{\Omega}^{\mathcal{S}}(T)$  associated with a given theory  $T$  of  $\mathcal{S}$ , which is defined as the largest congruence on  $\mathbf{Fm}_{\mathcal{L}}(V)$  that is compatible with all theories of  $\mathcal{S}$  that contain the given theory  $T$  (see [3]). In fact, by definition,  $\tilde{\Omega}(\mathcal{S}) = \tilde{\Omega}^{\mathcal{S}}(C(\emptyset))$ , i.e., the Tarski congruence of  $\mathcal{S}$  is the Suszko congruence associated with the set of theorems of the logic  $\mathcal{S}$ . Font and Jansana [5], extending Czelakowski's [2] (see also [1]) well-known characterization of the *Leibniz congruence*  $\Omega(T)$  associated

with a theory  $T$  of a sentential logic, have shown that, for all  $\alpha, \beta \in \text{Fm}_{\mathcal{L}}(V)$ ,

$$\langle \alpha, \beta \rangle \in \tilde{\Omega}(\mathcal{S}) \quad \text{iff} \quad \begin{array}{l} \text{for all } \varphi(p, \bar{q}) \in \text{Fm}_{\mathcal{L}}(V), \\ C(\varphi(\alpha, \bar{q})) = C(\varphi(\beta, \bar{q})). \end{array}$$

Whereas  $\tilde{\Omega}(\mathcal{S}) \subseteq \Lambda(\mathcal{S})$ , for every propositional logic  $\mathcal{S}$ , the reverse inclusion does not hold in general. A propositional logic is called **selfextensional** in [10] if  $\Lambda(\mathcal{S}) \subseteq \tilde{\Omega}(\mathcal{S})$ . In fact, Wójcicki shows in what has become a fundamental theorem in the theory of referential semantics, Theorem 2 of [10], that a propositional logic is referential if and only if it is self-extensional. This result shows that, unless a propositional logic  $\mathcal{S}$  is self-extensional,  $\mathcal{S}$  cannot possess a referential algebraic semantics.

Let  $\mathcal{L}$  be a logical signature. An  $\mathcal{L}$ -**g-matrix**  $\mathbb{A} = \langle \mathbf{A}, \mathcal{C} \rangle$  consists of an  $\mathcal{L}$ -algebra  $\mathbf{A}$  together with a collection  $\mathcal{C} \subseteq \mathcal{P}(A)$ . A g-matrix  $\mathbb{A}$  generates a consequence operator  $C^{\mathbb{A}}$  on  $\text{Fm}_{\mathcal{L}}(V)$  as follows: For all  $X \cup \{\alpha\} \subseteq \text{Fm}_{\mathcal{L}}(V)$ ,

$$\alpha \in C^{\mathbb{A}}(X) \quad \text{iff} \quad \begin{array}{l} \text{for all } h : \mathbf{Fm}_{\mathcal{L}}(V) \text{ and all } F \in \mathcal{C}, \\ h(X) \subseteq F \text{ implies } h(\alpha) \in F. \end{array}$$

A g-matrix  $\mathbb{A}$  is said to constitute a **g-matrix semantics** for a propositional logic  $\mathcal{S} = \langle \mathcal{L}, C \rangle$  in case  $C^{\mathbb{A}} = C$ .

Consider now a referential algebra  $\mathbf{A}$  over a set  $T$  of reference points. Let, for all  $t \in T$ ,

$$D_t = \{a \in A : t \in a\}.$$

Define the collection  $\mathcal{D} = \{D_t : t \in T\}$ . We call  $\langle \mathbf{A}, \mathcal{D} \rangle$  the **referential g-matrix** associated with the referential algebra  $\mathbf{A}$ .

It can be shown that the consequence operator  $C^{\langle \mathbf{A}, \mathcal{D} \rangle}$  generated by the g-matrix system  $\langle \mathbf{A}, \mathcal{D} \rangle$  is identical to  $C^{\mathbf{A}}$ . Thus, it follows that, unless  $\mathcal{S}$  is self-extensional it does not possess a referential g-matrix semantics.

To address this shortcoming of referential g-matrices in providing a semantics for arbitrary propositional logics, Malinowski introduced in [8] pseudo-referential g-matrices, as a generalization of referential g-matrices, and showed that every propositional logic possesses a pseudo-referential g-matrix semantics.

Let, once more,  $T$  be a set of reference points and consider, also, a collection  $T^* \subseteq \mathcal{P}(T)$  of subsets of  $T$ . According to [8] a **pseudo-referential g-matrix**  $\mathbb{A} = \langle \mathbf{A}, \mathcal{D} \rangle$  **relative to**  $(T, T^*)$  is a g-matrix, such that  $\mathbf{A}$  is a referential algebra based on  $T$  and

$$\begin{aligned} \mathcal{D} &= \{ \{a \in A : (\exists t \in t^*)(t \in a)\} : t^* \in T^* \} \\ &= \{ \{a \in A : a \cap t^* \neq \emptyset\} : t^* \in T^* \}. \end{aligned}$$

Note that this concept generalizes referentiality, since a referential g-matrix associated with a referential algebra  $\mathbf{A}$  based on  $T$  is obtained as a special case of a pseudo-referential g-matrix relative to  $(T, T^*)$ , with  $T^* = \{\{t\} : t \in T\}$ .

In the Theorem of [8] it is shown that every propositional logic  $\mathcal{S}$  has a strongly adequate pseudo-referential g-matrix  $\mathbb{A}$ , which may be termed the **canonical pseudo-referential g-matrix associated with  $\mathcal{S}$** .

Malinowski's work was followed by Marek [9]. Marek defines a **discrete pseudo-referential g-matrix** as a pseudo-referential g-matrix relative to a pair  $(T, T^*)$ , such that  $T^* \subseteq \{\{t\} : t \in T\}$ . She then shows that every g-matrix is isomorphic to, and, hence, generates the same sentential logic as, a *discrete* pseudo-referential g-matrix. Thus, since, as is well-known, every propositional logic has a strongly adequate g-matrix semantics, it follows that it also has a strongly adequate discrete pseudo-referential g-matrix semantics (see Corollary of [9]).

The author, taking after the work of Wójcicki, showed in previous work [11, 12] that a logic formalized as a  $\pi$ -institution (see Section 2) is referential, i.e., has a referential g-matrix system semantics, if and only if it is self-extensional. Thus, it turns out that, similarly to the case of propositional logics, for these logics, unless the condition of self-extensionality is fulfilled, no referential g-matrix system semantics is available. The present work, inspired by the previously mentioned work of Malinowski [8] and Marek [9], addresses this constraint on the availability of a referential g-matrix system semantics by introducing a pseudo-referential g-matrix system semantics (see Section 4). It is shown in Theorem 5 that every  $\pi$ -institution possesses a pseudo-referential g-matrix system semantics. Finally, improving on this result, we show in Section 7, in a parallel to the Theorem of Marek [9], that, for every g-matrix system, there exists a discrete pseudo-referential g-matrix system that generates the same closure system (see Theorem 6). It then follows that every logic formalized as a  $\pi$ -institution has a discrete pseudo-referential g-matrix semantics.

## 2 $\pi$ -Institutions and Closure Systems

We describe  $\pi$ -institutions [4] (see, also [6] for the closely related notion of an institution) on which our logical systems will be based.

Let  $\mathbf{Sign}^b$  be a category, called the **category of signatures**. Let  $\mathbf{SEN}^b : \mathbf{Sign}^b \rightarrow \mathbf{Set}$  be a set-valued functor, called the **sentence functor**. Let  $N^b$  be a category of natural transformations on  $\mathbf{SEN}^b$  (see Section 2 of [12]).

We call the triple  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$  the **base algebraic system**.

A collection  $T^b = \{T_\Sigma^b\}_{\Sigma \in |\mathbf{Sign}^b|}$ , such that  $T_\Sigma^b \subseteq \mathbf{SEN}^b(\Sigma)$ , for all  $\Sigma \in |\mathbf{Sign}^b|$ , is called a **sentence family** of  $\mathbf{A}^b$ .

A  $\pi$ -**institution based on  $\mathbf{A}^b$**  is a pair  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$ , where

$$C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^b|}$$

is a **closure (operator) system**, i.e., a  $|\mathbf{Sign}^b|$ -indexed collection of closure operators  $C_\Sigma : \mathcal{P}(\mathbf{SEN}^b(\Sigma)) \rightarrow \mathcal{P}(\mathbf{SEN}^b(\Sigma))$  that satisfy the **structurality condition**:

For all  $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma_1, \Sigma_2)$  and  $\Phi \subseteq \mathbf{SEN}^b(\Sigma_1)$ ,

$$\mathbf{SEN}^b(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\mathbf{SEN}^b(f)(\Phi)).$$

For  $\Sigma \in |\mathbf{Sign}^b|$ , a set  $T_\Sigma^b \subseteq \mathbf{SEN}^b(\Sigma)$  is called a  $\Sigma$ -**theory** of  $\mathcal{I}$  if it is closed under consequence, i.e., if  $C_\Sigma(T_\Sigma^b) = T_\Sigma^b$ . The collection of all  $\Sigma$ -theories of  $\mathcal{I}$  is denoted by  $\text{Th}_\Sigma(\mathcal{I})$ . A collection  $T^b = \{T_\Sigma^b\}_{\Sigma \in |\mathbf{Sign}^b|}$ , such that  $T_\Sigma^b \in \text{Th}_\Sigma(\mathcal{I})$ , for all  $\Sigma \in |\mathbf{Sign}^b|$ , is called a **theory family** of  $\mathcal{I}$ . The collection of all theory families of  $\mathcal{I}$  is denoted by  $\text{ThFam}(\mathcal{I})$ . It is well-known that they form a complete lattice under signature-wise inclusion  $\leq$ , whose meet coincides with signature-wise intersection.

Note that closure systems on  $\mathbf{A}^b$  are ordered as follows:

$$C^1 \leq C^2 \quad \text{iff} \quad \text{for all } \Sigma \in |\mathbf{Sign}^b|, \Phi \subseteq \mathbf{SEN}^b(\Sigma), \\ C_\Sigma^1(\Phi) \subseteq C_\Sigma^2(\Phi).$$

Under this ordering the collection of all closure systems on  $\mathbf{A}^b$  also forms a complete lattice whose meet is given by signature-wise intersection.

Given a base algebraic system  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ , an  $N^b$ -**algebraic system**  $\mathbf{A} = \langle \mathbf{Sign}, \mathbf{SEN}, N \rangle$  is an algebraic system, such that there exists a surjective functor  $N^b \rightarrow N$  preserving all projection natural transformations and, as a consequence, also all the arities of the natural transformations involved. We denote by  $\sigma : \mathbf{SEN}^k \rightarrow \mathbf{SEN}$  the natural transformation that is the image of  $\sigma^b : (\mathbf{SEN}^b)^k \rightarrow \mathbf{SEN}^b$  in  $N^b$ , and, in general use similar typing conventions to keep track of mappings of natural transformations in  $N^b$  to those on  $N^b$ -algebraic systems.

An **interpreted  $N^b$ -algebraic system** is a pair  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ , where

- $\mathbf{A}$  is an  $N^b$ -algebraic system and
- $\langle F, \alpha \rangle : \mathbf{A}^b \rightarrow \mathbf{A}$  is an algebraic system morphism.

We will use the term *algebraic system* to refer to both an  $N^b$ -algebraic system and an interpreted  $N^b$ -algebraic system relying on the context to clear the ambiguity.

Let  $\mathbf{A}^b$  be an algebraic system and  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  a  $\pi$ -institution based on  $\mathbf{A}^b$ . We define, next, the notion of a matrix system and of a g-matrix system for  $\mathbf{A}^b$  and of a matrix system model and g-matrix system model for  $\mathcal{I}$ .

A **matrix system** for  $\mathbf{A}^b$  is a pair  $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ , where  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  is an interpreted algebraic system and  $T$  is a sentence family of  $\mathcal{A}$ .

A matrix system  $\mathfrak{A}$  defines a closure system  $C^{\mathfrak{A}}$  (and hence a  $\pi$ -institution  $\mathcal{I}^{\mathfrak{A}} = \langle \mathbf{A}^b, C^{\mathfrak{A}} \rangle$ ) on  $\mathbf{A}^b$  as follows: For all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^b(\Sigma)$ ,

$$\varphi \in C_{\Sigma}^{\mathfrak{A}}(\Phi) \quad \text{iff} \quad \Phi \models_{\Sigma}^{\mathfrak{A}} \varphi,$$

where the relation on the right means that, for all  $\Sigma' \in |\mathbf{Sign}^b|$  and all  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$ ,

$$\alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}^b(\varphi)) \in T_{F(\Sigma')}.$$

A **generalized matrix system** for  $\mathbf{A}^b$  (or **g-matrix system**, for short) is a pair  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ , where  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  is an interpreted algebraic system and  $\mathcal{T}$  is a collection of sentence families of  $\mathcal{A}$ .

A g-matrix systems  $\mathbb{A}$  defines a closure system  $C^{\mathbb{A}}$  (and hence a  $\pi$ -institution  $\mathcal{I}^{\mathbb{A}} = \langle \mathbf{A}^b, C^{\mathbb{A}} \rangle$ ) on  $\mathbf{A}^b$  by setting  $C^{\mathbb{A}} = \bigcap_{\mathfrak{A} \in \mathbb{A}} C^{\mathfrak{A}}$ , where  $\mathfrak{A} = \langle \mathbf{A}, T \rangle \in \mathbb{A}$  means that  $T \in \mathcal{T}$ . Thus, equivalently, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^b(\Sigma)$ ,

$$\varphi \in C_{\Sigma}^{\mathbb{A}}(\Phi) \quad \text{iff} \quad (\forall \mathfrak{A} \in \mathbb{A})(\Phi \models_{\Sigma}^{\mathfrak{A}} \varphi).$$

A **matrix system model** for  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  or an  **$\mathcal{I}$ -matrix system** is a matrix system  $\mathfrak{A} = \langle \mathcal{A}, T \rangle$  for  $\mathbf{A}^b$ , such that  $C \leq C^{\mathfrak{A}}$ .

Similarly, a **g-matrix system model** for  $\mathcal{I}$  or an  **$\mathcal{I}$ -g-matrix system** is a g-matrix system  $\mathbb{A}$ , such that  $C \leq C^{\mathbb{A}}$ .

### 3 Referential $\pi$ -Institutions

In this work we focus on a special kind of (interpreted)  $N^b$ -algebraic system  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ ,  $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$ . We require that, for all  $\Sigma \in |\mathbf{Sign}|$ , there is a set  $\text{PTS}(\Sigma)$ , called the set of  $\Sigma$ -**reference** or  $\Sigma$ -**base points**, and that, for all  $\Sigma \in |\mathbf{Sign}|$ ,  $\text{SEN}(\Sigma) \subseteq \mathcal{P}(\text{PTS}(\Sigma))$ , i.e., each  $\Sigma$ -sentence is a set of  $\Sigma$ -points.

In this context, an interpretation  $\langle F, \alpha \rangle : \mathbf{A}^b \rightarrow \mathbf{A}$  will be viewed as a valuation of sentences of  $\mathbf{A}^b$  in the following way: For all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi \in \text{SEN}^b(\Sigma)$ ,  $\varphi$  is **true at**  $p \in \text{PTS}(F(\Sigma))$  **under**  $\langle F, \alpha \rangle$  iff  $p \in \alpha_{\Sigma}(\varphi)$ .

An algebraic system of this special form is called a **referential algebraic system** and said to **be based on** PTS.

Note that this definition is a generalized version of the one given in Section 3 of [12]. The generalization stems from the fact that, in the present context, we no longer insist that the sentence functor  $\text{SEN}$  be a simple subfunctor (having the same domain) of the inverse powerset of a contravariant functor  $\mathbf{Sign} \rightarrow \mathbf{Set}^{\text{op}}$ .

Let  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  be an interpreted referential  $N^b$ -algebraic system. Then  $\mathcal{A}$  determines a closure system  $C^{\mathcal{A}}$  on  $\mathbf{A}^b$  according to the following definition:

For all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^b(\Sigma)$ ,  $\varphi \in C_{\Sigma}^{\mathcal{A}}(\Phi)$  iff, for all  $\Sigma' \in |\mathbf{Sign}^b|$  and all  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$ ,

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) \subseteq \alpha_{\Sigma'}(\text{SEN}(f)^b(\varphi)).$$

Essentially the same proof as that of Proposition 1 of [12] yields the following

**Proposition 1 (Proposition 1 of [12])** *Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$  be a base algebraic system and  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  an interpreted referential  $N^b$ -algebraic system. Then  $C^{\mathcal{A}}$  is a closure system on  $\mathbf{A}^b$ .*

Since  $C^{\mathcal{A}}$  is a closure system on  $\mathbf{A}^b$ , the pair  $\mathcal{I}^{\mathcal{A}} = \langle \mathbf{A}^b, C^{\mathcal{A}} \rangle$  is a  $\pi$ -institution. We call an institution having this form a **referential  $\pi$ -institution**. Such  $\pi$ -institutions correspond in the theory of categorical abstract algebraic logic to the referential propositional logics of Wójcicki [10].

Let  $\mathbf{A} = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$  be a base algebraic system and  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  a  $\pi$ -institution based on  $\mathbf{A}^b$ . We define the **Frege equivalence system**  $\Lambda(\mathcal{I})$  of  $\mathcal{I}$ , also known as the **interderivability equivalence system**, by setting, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi, \psi \in \text{SEN}^b(\Sigma)$ ,

$$\langle \varphi, \psi \rangle \in \Lambda_{\Sigma}(\mathcal{I}) \quad \text{if and only if} \quad C_{\Sigma}(\varphi) = C_{\Sigma}(\psi).$$

The **Tarski congruence system**  $\widetilde{\Omega}(\mathcal{I})$  of  $\mathcal{I}$  ([5] for the universal algebraic notion and [13] for its categorical extension) is the largest congruence system on  $\mathbf{A}^b$  that is compatible with every theory family  $T \in \text{ThFam}(\mathcal{I})$ .

Clearly, it is always the case that  $\widetilde{\Omega}(\mathcal{I}) \leq \Lambda(\mathcal{I})$ . We call the  $\pi$ -institution  $\mathcal{I}$  **self-extensional** if  $\Lambda(\mathcal{I}) \leq \widetilde{\Omega}(\mathcal{I})$ . In view of the preceding remark,  $\mathcal{I}$  is self-extensional if and only if  $\Lambda(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$ .

A generalization to  $\pi$ -institutions of Wójcicki's Theorem (see Theorem 2 of [10], but, also, Theorem 2.2 of [7] for a complete proof), provides a

characterization of referential sentential logics. This is essentially Theorem 8 of [12], with the aforementioned generalization pertaining to the signature category not affecting the proof.

**Theorem 2 (Theorem 8 of [12])** *A  $\pi$ -institution  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  is referential if and only if it is self-extensional.*

We recall here a version of the construction of the *canonical referential algebraic system* associated with a given selfextensional  $\pi$ -institution that witnesses one implication of Theorem 2.

Let  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$ , with  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ , be a self-extensional  $\pi$ -institution. For each  $\Sigma \in |\mathbf{Sign}^b|$ , we take as the set of  $\Sigma$ -points the set  $\text{Th}_\Sigma(\mathcal{I})$  of  $\Sigma$ -theories of  $\mathcal{I}$ .

Define the functor  $\mathbf{SEN} : \mathbf{Sign}^b \rightarrow \mathbf{Set}$  as follows:

For every  $\Sigma \in |\mathbf{Sign}^b|$ ,

$$\mathbf{SEN}(\Sigma) = \{\text{Th}_\Sigma(\varphi) : \varphi \in \mathbf{SEN}^b(\Sigma)\},$$

where  $\text{Th}_\Sigma(\varphi) = \{T \in \text{Th}_\Sigma(\mathcal{I}) : \varphi \in T\}$ , for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi \in \mathbf{SEN}^b(\Sigma)$ .

Moreover, for all  $\Sigma, \Sigma' \in |\mathbf{Sign}^b|$ , and all  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$ , we define  $\mathbf{SEN}(f) : \mathbf{SEN}(\Sigma) \rightarrow \mathbf{SEN}(\Sigma')$  by setting

$$\mathbf{SEN}(f)(\text{Th}_\Sigma(\varphi)) = \text{Th}_{\Sigma'}(\mathbf{SEN}^b(f)(\varphi)),$$

for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi \in \mathbf{SEN}^b(\Sigma)$ .

Define the category of natural transformations  $N$  on  $\mathbf{SEN}$  as follows: For every  $\sigma^b : (\mathbf{SEN}^b)^k \rightarrow \mathbf{SEN}^b$  in  $N^b$ , let  $\sigma : \mathbf{SEN}^k \rightarrow \mathbf{SEN}$  be defined by letting, for all  $\Sigma \in |\mathbf{Sign}^b|$ ,  $\sigma_\Sigma : \mathbf{SEN}(\Sigma)^k \rightarrow \mathbf{SEN}(\Sigma)$  be given by

$$\sigma_\Sigma(\text{Th}_\Sigma(\varphi_0), \dots, \text{Th}_\Sigma(\varphi_{k-1})) = \text{Th}_\Sigma(\sigma_\Sigma^b(\varphi_0, \dots, \varphi_{k-1})),$$

for all  $\varphi_0, \dots, \varphi_{k-1} \in \mathbf{SEN}^b(\Sigma)$ . Using self-extensionality one may show that this is well-defined. Moreover,  $\sigma$  is a natural transformation and the collection of natural transformations, thus defined, forms a category of natural transformations on  $\mathbf{SEN}$ . So the triple  $\mathbf{A} = \langle \mathbf{Sign}^b, \mathbf{SEN}, N \rangle$  constitutes an  $N^b$ -algebraic system.

Finally, the **canonical referential algebraic system** associated with  $\mathcal{I}$  is defined by  $\mathcal{A} = \langle \mathbf{A}, \langle I, \alpha \rangle \rangle$ , where:

- $I : \mathbf{Sign}^b \rightarrow \mathbf{Sign}^b$  is the identity functor;



- $\alpha : \text{SEN}^b \rightarrow \text{SEN}$  is the natural transformation defined by letting, for all  $\Sigma \in |\mathbf{Sign}^b|$ ,  $\alpha_\Sigma : \text{SEN}^b(\Sigma) \rightarrow \text{SEN}(\Sigma)$  be given by

$$\alpha_\Sigma(\varphi) = \text{Th}_\Sigma(\varphi), \text{ for all } \varphi \in \text{SEN}^b(\Sigma).$$

Note, now, that, for all  $\Sigma, \Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and  $\varphi \in \text{SEN}^b(\Sigma)$ ,

$$\begin{array}{ccc} \text{SEN}^b(\Sigma) & \xrightarrow{\alpha_\Sigma} & \text{SEN}(\Sigma) \\ \text{SEN}^b(f) \downarrow & & \downarrow \text{SEN}(f) \\ \text{SEN}^b(\Sigma') & \xrightarrow{\alpha_{\Sigma'}} & \text{SEN}(\Sigma') \end{array}$$

$$\begin{aligned} \text{SEN}(f)(\alpha_\Sigma(\varphi)) &= \text{SEN}(f)(\text{Th}_\Sigma(\varphi)) \\ &= \text{Th}_{\Sigma'}(\text{SEN}^b(f)(\varphi)) \\ &= \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)). \end{aligned}$$

It can be shown that, if  $\mathcal{I}$  is self-extensional, then  $\mathcal{A}$  is well-defined and, moreover,  $\mathcal{I} = \mathcal{I}^{\mathcal{A}}$ . Thus,  $\mathcal{I}$  is referential.

## 4 Pseudo-Referential Matrix Systems

Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$  be a base algebraic system and  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  a referential  $N^b$ -algebraic system based on PTS. The algebraic system  $\mathcal{A}$  will be said to be **supported** if it is endowed with a collection  $\mathcal{S} = \{S^i : i \in I\}$  of base point families

$$S^i = \{S_\Sigma^i\}_{\Sigma \in |\mathbf{Sign}|},$$

where  $S_\Sigma^i \subseteq \text{PTS}(\Sigma)$ , for all  $i \in I$  and all  $\Sigma \in |\mathbf{Sign}|$ . We refer to  $\mathcal{S}$  as the **support** of  $\mathcal{A}$  in this case.

Given a supported algebraic system  $\mathcal{A}$ , with support  $\mathcal{S}$ , a **pseudo-referential g-matrix system relative to**  $(\text{PTS}, \mathcal{S})$  is a pair

$$\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle,$$

where  $\mathcal{T} = \{T^i : i \in I\}$  is a collection of sentence families  $T^i = \{T_\Sigma^i\}_{\Sigma \in |\mathbf{Sign}|}$ , such that, for all  $i \in I$  and all  $\Sigma \in |\mathbf{Sign}|$ ,

$$T_\Sigma^i = \{X \in \text{SEN}(\Sigma) : X \cap S_\Sigma^i \neq \emptyset\}.$$

We close this section with two properties of pseudo-referential g-matrix systems. The first states that, in a precise model-theoretic sense, pseudo-referential g-matrix systems encompass referential algebraic systems. The second characterizes the closure system  $C^{\mathbb{A}}$  induced by a pseudo-referential g-matrix system on the base algebraic system  $\mathbf{A}^b$ .

Let  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  be a referential algebraic system, based on PTS. Consider the set  $\mathcal{P}$  of all  $|\mathbf{Sign}|$ -indexed tuples  $P$ , such that, for some  $\Sigma \in |\mathbf{Sign}|$ ,

$$P_{\Sigma'} \begin{cases} \in \{\{p\} : p \in \text{PTS}(\Sigma)\}, & \text{if } \Sigma' = \Sigma \\ = \emptyset, & \text{if } \Sigma' \neq \Sigma \end{cases}$$

Consider the pseudo-referential g-matrix system  $\mathbb{A}(\mathcal{A}) = \langle \mathcal{A}, \mathcal{T} \rangle$  relative to  $(\text{PTS}, \mathcal{P})$ . This is called the **pseudo-referential g-matrix system associated with  $\mathcal{A}$** . Then we have the following:

**Lemma 3** *Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$  be a base algebraic system,  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  a referential  $N^b$ -algebraic system and  $\mathbb{A}(\mathcal{A}) = \langle \mathcal{A}, \mathcal{T} \rangle$  the pseudo-referential g-matrix system associated with  $\mathcal{A}$ . Then  $C^{\mathcal{A}} = C^{\mathbb{A}(\mathcal{A})}$ .*

**Proof:** This follows easily from the fact that, according to the definitions involved, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi \in \text{SEN}^b(\Sigma)$ , we have

$$p \in \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)) \quad \text{iff} \quad \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)) \cap \{p\} \neq \emptyset,$$

for all  $\Sigma' \in |\mathbf{Sign}^b|$ , all  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and all  $p \in \text{PTS}(\Sigma')$ . ■

Thus, by identifying  $\mathcal{A}$  with  $\mathbb{A}(\mathcal{A})$  we may view referential algebraic semantics in the sense of [12] as a special case of pseudo-referential g-matrix system semantics.

We now obtain the following characterization of  $C^{\mathbb{A}}$  for an arbitrary pseudo-referential g-matrix system  $\mathbb{A}$ .

**Proposition 4** *Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$  be a base algebraic system,  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  an  $N^b$ -referential algebraic system based on PTS, and  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  a pseudo-referential g-matrix system relative to a pair  $(\text{PTS}, \mathcal{S})$ , with  $\mathcal{S} = \{S^i : i \in I\}$ . Then, for all  $\Sigma \in |\mathbf{Sign}^b|$ ,  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^b(\Sigma)$ ,  $\varphi \in C_{\Sigma}^{\mathbb{A}}(\Phi)$  iff, for all  $\Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and all  $i \in I$ ,*

$$\begin{aligned} \alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) \cap S_{F(\Sigma')}^i &\neq \emptyset, \text{ for all } \phi \in \Phi, \\ \text{implies } \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)) \cap S_{F(\Sigma')}^i &\neq \emptyset. \end{aligned}$$

**Proof:** Let  $\Sigma \in |\mathbf{Sign}^b|$  and  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^b(\Sigma)$ . We have  $\varphi \in C_{\Sigma}^{\mathbb{A}}(\Phi)$  iff, by definition, for all  $\Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and all  $i \in I$ ,

$$\alpha_{\Sigma'}(\text{SEN}(f)(\Phi)) \subseteq T_{F(\Sigma')}^i \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}(f)(\varphi)) \in T_{F(\Sigma')}^i.$$

By the definition of  $T^i$ , this is equivalent to having, for all  $\Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and all  $i \in I$ ,

$$\begin{aligned} \alpha_{\Sigma'}(\text{SEN}^b(f)(\Phi)) &\subseteq \{X \in \text{SEN}(F(\Sigma')) : X \cap S_{F(\Sigma')}^i \neq \emptyset\} \\ \text{implies } \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)) &\in \{X \in \text{SEN}(F(\Sigma')) : X \cap T_{F(\Sigma')}^i \neq \emptyset\}. \end{aligned}$$

Equivalently, for all  $\Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and all  $i \in I$ ,

$$\begin{aligned} \alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) \cap S_{F(\Sigma')}^i &\neq \emptyset, \text{ for all } \phi \in \Phi, \\ \text{implies } \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)) \cap S_{F(\Sigma')}^i &\neq \emptyset. \end{aligned}$$

■

## 5 Universality of the Semantics

In this section we show that every  $\pi$ -institution has a pseudo-referential matrix semantics. This contrasts with Theorem 2, which implies that not every  $\pi$ -institution has a referential algebraic semantics.

**Theorem 5** *Let  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  be a  $\pi$ -institution based on an algebraic system  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$ . Then, there exists a pseudo-referential g-matrix system  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  relative to a pair  $(\text{PTS}, \mathcal{S})$ , such that  $\mathcal{I} = \mathcal{I}^{\mathbb{A}}$ , i.e.,  $C = C^{\mathbb{A}}$ .*

**Proof:** Let  $\mathbf{Sign} = \mathbf{Sign}^b$ . For all  $\Sigma \in |\mathbf{Sign}^b|$ , let  $\text{PTS}(\Sigma) = \text{SEN}^b(\Sigma)$ . Now we define  $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$  based on PTS as follows:

- $\text{SEN}(\Sigma) = \{\{\varphi\} : \varphi \in \text{SEN}^b(\Sigma)\}$ , for all  $\Sigma \in |\mathbf{Sign}^b|$ . And, given,  $\Sigma, \Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$ ,

$$\text{SEN}(f)(\{\varphi\}) = \{\text{SEN}^b(f)(\varphi)\}, \text{ for all } \varphi \in \text{SEN}^b(\Sigma).$$

- For all  $\sigma^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$  in  $N^b$ , all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi_0, \dots, \varphi_{k-1} \in \text{SEN}^b(\Sigma)^k$ ,

$$\sigma_{\Sigma}(\{\varphi_0\}, \dots, \{\varphi_{k-1}\}) = \{\sigma_{\Sigma}^b(\varphi_0, \dots, \varphi_{k-1})\}.$$

We let  $N$  consist of all natural transformations of this form.

It is not difficult to see that, with these definitions, the triple  $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$  becomes a referential  $N^b$ -algebraic system based on PTS.

Next, define  $\langle I, \alpha \rangle : \mathbf{A}^b \rightarrow \mathbf{A}$  by setting

- $I : \mathbf{Sign}^b \rightarrow \mathbf{Sign}$  the identity functor;
- For all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi \in \text{SEN}^b(\Sigma)$ ,  $\alpha_\Sigma(\varphi) = \{\varphi\}$ .

Now  $\mathcal{A} = \langle \mathbf{A}, \langle I, \alpha \rangle \rangle$  is an interpreted referential  $N^b$ -algebraic system.

Let  $\mathcal{S} = \text{ThFam}(\mathcal{I}) = \{S^i : i \in I\}$ . This determines the pseudo-referential g-matrix system  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  relative to  $(\text{PTS}, \mathcal{S})$ . We have that  $\mathcal{T} = \{T^i : i \in I\}$ , with  $T^i = \{T_\Sigma^i\}_{\Sigma \in |\mathbf{Sign}|}$  given, for all  $i \in I$  and all  $\Sigma \in |\mathbf{Sign}|$ , by

$$\begin{aligned} T_\Sigma^i &= \{\{\varphi\} \in \text{SEN}(\Sigma) : \{\varphi\} \cap S_\Sigma^i \neq \emptyset\} \\ &= \{\{\varphi\} \in \text{SEN}(\Sigma) : \varphi \in S_\Sigma^i\}, \end{aligned}$$

We prove that  $C = C^\mathbb{A}$ , i.e., that, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^b(\Sigma)$ ,

$$\varphi \in C_\Sigma(\Phi) \quad \text{iff} \quad \varphi \in C_\Sigma^\mathbb{A}(\Phi).$$

$\Rightarrow$ : Suppose that  $\varphi \in C_\Sigma(\Phi)$ . Let  $\Sigma' \in |\mathbf{Sign}^b|$ ,  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and  $i \in I$ , such that  $\alpha_{\Sigma'}(\text{SEN}^b(f)(\phi)) \subseteq T_{\Sigma'}^i$ , for all  $\phi \in \Phi$ . By the definition of  $\alpha$ , this holds iff  $\{\text{SEN}^b(f)(\phi)\} \in T_{\Sigma'}^i$ , for all  $\phi \in \Phi$ . By the expression given above for  $T^i$ , this holds iff  $\text{SEN}^b(f)(\phi) \in S_{\Sigma'}^i$ , for all  $\phi \in \Phi$ , i.e., iff  $\text{SEN}^b(f)(\Phi) \subseteq S_{\Sigma'}^i$ . Then, since by hypothesis  $\varphi \in C_\Sigma(\Phi)$ , we get  $\text{SEN}^b(f)(\varphi) \in S_{\Sigma'}^i$ . This shows that  $\{\text{SEN}^b(f)(\varphi)\} \in T_{\Sigma'}^i$ , or, equivalently,  $\alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)) \in T_{\Sigma'}^i$ . Therefore,  $\varphi \in C_\Sigma^\mathbb{A}(\Phi)$ .

$\Leftarrow$ : Suppose that  $\varphi \in C_\Sigma^\mathbb{A}(\Phi)$ . Let  $i \in I$ , such that  $\Phi \subseteq S_\Sigma^i$ . This is equivalent to  $\{\phi\} \in T_\Sigma^i$ , for all  $\phi \in \Phi$ . Since, by hypothesis  $\varphi \in C_\Sigma^\mathbb{A}(\Phi)$ , we get that  $\{\varphi\} \in T_\Sigma^i$ . Equivalently,  $\varphi \in S_\Sigma^i$ . Since  $i \in I$  was arbitrary, we get that  $\varphi \in C_\Sigma(\Phi)$ . ■

We call the pseudo-referential g-matrix system  $\mathbb{A}$ , constructed in the proof of Theorem 5, such that  $\mathcal{I}^\mathbb{A} = \mathcal{I}$ , the **canonical pseudo-referential g-matrix system associated with  $\mathcal{I}$** .

## 6 Selfextensional $\pi$ -Institutions

In this section, we start with a selfextensional  $\pi$ -institution  $\mathcal{I}$  and show how, starting from the canonical pseudo-referential g-matrix system associated

with  $\mathcal{I}$ , a process of dividing out by the Frege equivalence system of  $\mathcal{I}$  (which is a congruence system due to selfextensionality), leads to the canonical referential g-matrix system for  $\mathcal{I}$  constructed in [12]. We present an outline, omitting some of the details that are easy to check.

Let  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  be a selfextensional  $\pi$ -institution based on the algebraic system  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$ . Consider the canonical pseudo-referential g-matrix system  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  associated with  $\mathcal{I}$ , based on  $(\mathbf{PTS}, \mathcal{S})$ , with  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  and  $\mathbf{A} = \langle \mathbf{Sign}^b, \mathbf{SEN}, N \rangle$ , as constructed in the proof of Theorem 5.

Recall that the Frege equivalence system  $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}^b|}$  of  $\mathcal{I}$  is defined, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi, \psi \in \mathbf{SEN}^b(\Sigma)$ , by

$$\langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

By selfextensionality,  $\Lambda(\mathcal{I})$  is a congruence system on  $\mathbf{A}^b$  and, in fact, coincides with the Tarski congruence system  $\tilde{\Omega}(\mathcal{I})$ .

We define on the underlying algebraic system  $\mathbf{A} = \langle \mathbf{Sign}^b, \mathbf{SEN}, N \rangle$  of the canonical pseudo-referential g-matrix system  $\mathbb{A}$  associated with  $\mathcal{I}$  the relation family  $\equiv^{\mathcal{I}} = \{\equiv_\Sigma^{\mathcal{I}}\}_{\Sigma \in |\mathbf{Sign}^b|}$ , by setting, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi, \psi \in \mathbf{SEN}^b(\Sigma)$ ,

$$\{\varphi\} \equiv_\Sigma^{\mathcal{I}} \{\psi\} \quad \text{iff} \quad \langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}).$$

Clearly,  $\equiv^{\mathcal{I}}$  is an equivalence family on  $\mathbf{A}$ . Moreover, it is an equivalence system because of structurality. This establishes that the quotient functor  $\mathbf{SEN}^{\equiv^{\mathcal{I}}} := \mathbf{SEN} / \equiv^{\mathcal{I}} : \mathbf{Sign}^b \rightarrow \mathbf{Set}$  is well-defined (see [13]).

Note that  $\mathbf{SEN}^{\equiv^{\mathcal{I}}}$  may be considered as a point-based functor, based on  $\mathbf{Th}(\mathcal{I}) = \{\mathbf{Th}_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}^b|}$  under the identification

$$\{\varphi\} / \equiv_\Sigma^{\mathcal{I}} \longleftrightarrow \mathbf{Th}_\Sigma(\varphi),$$

for all  $\varphi \in \mathbf{SEN}^b(\Sigma)$ ,  $\Sigma \in |\mathbf{Sign}^b|$  (which is well-defined by the definition of  $\equiv^{\mathcal{I}}$ ).

Next, observe that, by the self-extensionality of  $\mathcal{I}$ , the equivalence system  $\equiv^{\mathcal{I}}$  is actually a congruence system on  $\mathbf{A}$ . In fact, for all  $\sigma^b : (\mathbf{SEN}^b)^k \rightarrow \mathbf{SEN}^b$  in  $N^b$ , for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi_0, \psi_0, \dots, \varphi_{k-1}, \psi_{k-1} \in \mathbf{SEN}^b(\Sigma)$ , such that  $\{\varphi_i\} \equiv_\Sigma^{\mathcal{I}} \{\psi_i\}$ , for all  $i < k$ , we get that  $C_\Sigma(\varphi_i) = C_\Sigma(\psi_i)$ , for all  $i \in I$ , whence by self-extensionality,  $C_\Sigma(\sigma_\Sigma^b(\varphi_0, \dots, \varphi_{k-1})) = C_\Sigma(\sigma_\Sigma^b(\psi_0, \dots, \psi_{k-1}))$ , giving that  $\{\sigma_\Sigma^b(\varphi_0, \dots, \varphi_{k-1})\} \equiv_\Sigma^{\mathcal{I}} \{\sigma_\Sigma^b(\psi_0, \dots, \psi_{k-1})\}$ . But, by the definition of  $\sigma : \mathbf{SEN}^k \rightarrow \mathbf{SEN}$ , the latter is equivalent to  $\sigma_\Sigma(\{\varphi_0\}, \dots, \{\varphi_{k-1}\}) \equiv_\Sigma^{\mathcal{I}} \sigma_\Sigma(\{\psi_0\}, \dots, \{\psi_{k-1}\})$ .

Now we conclude that the quotient  $\mathbf{A}^{\equiv \mathcal{I}} := \mathbf{A}/\equiv^{\mathcal{I}} = \langle \mathbf{Sign}^b, \text{SEN}^{\equiv \mathcal{I}}, N^{\equiv \mathcal{I}} \rangle$  is a well-defined  $N^b$ -algebraic system.

Finally, recall that  $\mathcal{T} = \{T^i : i \in I\}$ , with  $T^i = \{T_{\Sigma}^i\}_{\Sigma \in |\mathbf{Sign}|}$  given, for all  $i \in I$  and all  $\Sigma \in |\mathbf{Sign}|$ , by

$$T_{\Sigma}^i = \{\{\varphi\} \in \text{SEN}(\Sigma) : \varphi \in S_{\Sigma}^i\}.$$

We note that  $\equiv^{\mathcal{I}}$  is compatible with  $T^i$ , for all  $i$ , and, therefore, it is a (g-matrix) congruence system of  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ . In fact, for all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi, \psi \in \text{SEN}^b(\Sigma)$ , such that  $\{\varphi\} \equiv_{\Sigma}^{\mathcal{I}} \{\psi\}$  and  $\{\varphi\} \in T_{\Sigma}^i$ , we get that  $C_{\Sigma}(\varphi) = C_{\Sigma}(\psi)$  and  $\varphi \in S_{\Sigma}^i \in \text{Th}_{\Sigma}(\mathcal{I})$ . Hence, we obtain  $\psi \in S_{\Sigma}^i$ , which shows that  $\{\psi\} \in T_{\Sigma}^i$ .

It follows that the quotient g-matrix system  $\mathbb{A}^{\equiv \mathcal{I}} = \langle \mathcal{A}^{\equiv \mathcal{I}}, \mathcal{T}^{\equiv \mathcal{I}} \rangle$  is well-defined.

To establish the equivalence of the canonical referential g-matrix system associated with  $\mathcal{I}$  with the quotient  $\mathbb{A}^{\equiv \mathcal{I}}$  of the canonical pseudo-referential g-matrix system  $\mathbb{A}$  associated with  $\mathcal{I}$  it suffices to note that the mapping

$$\text{Th}_{\Sigma}(\varphi) \mapsto \{\varphi\}/\equiv_{\Sigma}^{\mathcal{I}},$$

for all  $\Sigma \in |\mathbf{Sign}^b|$ ,  $\varphi \in \text{SEN}^b(\Sigma)$ , determines an isomorphism between these two g-matrix systems.

## 7 Discrete Pseudo-Referential Matrix Systems

Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \text{SEN}^b, N^b \rangle$  be an algebraic system and  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  a pseudo-referential g-matrix system relative to some  $(\text{PTS}, \mathcal{S})$ , with  $\mathcal{S} = \{S^i : i \in I\}$ , i.e., such that  $\mathcal{T} = \{T^i : i \in I\}$ , with

$$T_{\Sigma}^i = \{X \in \text{SEN}(\Sigma) : X \cap S_{\Sigma}^i \neq \emptyset\},$$

for all  $\Sigma \in |\mathbf{Sign}|$  and all  $i \in I$ .

The pseudo-referential g-matrix system  $\mathbb{A}$  will be called **discrete** if, for all  $i \in I$ , there exists  $\Sigma_i \in |\mathbf{Sign}|$ , such that, for all  $\Sigma \in |\mathbf{Sign}|$ ,

$$S_{\Sigma}^i \begin{cases} \in \{\{p\} : p \in \text{PTS}(\Sigma_i)\}, & \text{if } \Sigma = \Sigma_i, \\ = \emptyset, & \text{otherwise.} \end{cases}$$

In this section, taking after the work of Marek [9], we show that every  $\pi$ -institution  $\mathcal{I} = \langle \mathbf{A}^b, C \rangle$  has a strongly adequate discrete pseudo-referential matrix system semantics. This is done by exhibiting, for every g-matrix system, an equivalent discrete pseudo-referential g-matrix system.

**Theorem 6** *Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$  be an algebraic system. For every  $N^b$ -g-matrix system  $\mathbb{A}^\# = \langle \mathcal{A}^\#, \mathcal{T}^\# \rangle$ , with  $\mathcal{A}^\# = \langle \mathbf{A}^\#, \langle F^\#, \alpha^\# \rangle \rangle$ ,  $\mathbf{A}^\# = \langle \mathbf{Sign}^\#, \mathbf{SEN}^\#, N^\# \rangle$ , there exists a discrete pseudo referential g-matrix system  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  relative to some  $(\mathbf{PTS}, \mathcal{S})$ , such that  $\mathcal{I}^\mathbb{A} = \mathcal{I}^{\mathbb{A}^\#}$ .*

**Proof:** Let  $\mathbf{A}^b = \langle \mathbf{Sign}^b, \mathbf{SEN}^b, N^b \rangle$  be an algebraic system. Consider an  $N^b$ -g-matrix system  $\mathbb{A}^\# = \langle \mathcal{A}^\#, \mathcal{T}^\# \rangle$ , with  $\mathcal{A}^\# = \langle \mathbf{A}^\#, \langle F^\#, \alpha^\# \rangle \rangle$ ,  $\mathbf{A}^\# = \langle \mathbf{Sign}^\#, \mathbf{SEN}^\#, N^\# \rangle$  and  $\mathcal{T}^\# = \{T^{\#i} : i \in I\}$ .

For all  $\Sigma \in |\mathbf{Sign}^\#|$ , consider a collection  $\{x_\Sigma^i : i \in I\}$ , where, for all  $i \in I$ ,  $x_\Sigma^i \notin \mathbf{SEN}^\#(\Sigma)$  and, for all  $i, j \in I$ , with  $i \neq j$ ,  $x_\Sigma^i \neq x_\Sigma^j$ .

Now define

$$\mathbf{PTS}(\Sigma) = \mathbf{SEN}^\#(\Sigma) \cup \{x_\Sigma^i : i \in I\}, \text{ for all } \Sigma \in |\mathbf{Sign}^\#|.$$

Moreover, let  $\mathcal{S} = \{S^{\Sigma, i} : \Sigma \in |\mathbf{Sign}^\#|, i \in I\}$ , where, for all  $\Sigma \in |\mathbf{Sign}^\#|$  and all  $i \in I$ ,  $S^{\Sigma, i} = \{S_{\Sigma'}^{\Sigma, i}\}_{\Sigma' \in |\mathbf{Sign}^\#|}$  is defined by setting

$$S_{\Sigma'}^{\Sigma, i} = \begin{cases} \{x_\Sigma^i\}, & \text{if } \Sigma' = \Sigma \\ \emptyset, & \text{if } \Sigma' \neq \Sigma \end{cases}, \text{ for all } \Sigma' \in |\mathbf{Sign}^\#|.$$

Next, define, for all  $\Sigma \in |\mathbf{Sign}^\#|$  and all  $\varphi \in \mathbf{SEN}^\#(\Sigma)$ ,  $X_\varphi \subseteq \mathbf{PTS}(\Sigma)$ , by

$$p \in X_\varphi \iff p = \varphi \text{ or } (\exists i \in I)(p = x_\Sigma^i \text{ and } \varphi \in T_\Sigma^{\#, i}).$$

**Claim:** For all  $\Sigma \in |\mathbf{Sign}^\#|$  and all  $\varphi, \psi \in \mathbf{SEN}^\#(\Sigma)$ ,  $X_\varphi = X_\psi$  if and only if  $\varphi = \psi$ .

**Proof of the Claim:** The “if” direction is obvious. For the “only if”, reasoning by contraposition, we note that if  $\varphi \neq \psi$ , then  $\varphi \in X_\varphi$ , whereas  $\varphi \notin X_\psi$ . Therefore  $X_\varphi \neq X_\psi$ .  $\square$

Now define, for all  $\Sigma \in |\mathbf{Sign}^\#|$ ,

$$\mathbf{SEN}(\Sigma) = \{X_\varphi : \varphi \in \mathbf{SEN}^\#(\Sigma)\}$$

and, moreover, for all  $\Sigma, \Sigma' \in |\mathbf{Sign}^\#|$  and all  $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$ , let  $\mathbf{SEN}(f) : \mathbf{SEN}(\Sigma) \rightarrow \mathbf{SEN}(\Sigma')$  be given, for all  $\varphi \in \mathbf{SEN}^\#(\Sigma)$ , by

$$\mathbf{SEN}(f)(X_\varphi) = X_{\mathbf{SEN}^\#(f)(\varphi)}.$$

The fact that  $\mathbf{SEN} : \mathbf{Sign}^\# \rightarrow \mathbf{Set}$ , thus defined, is a functor follows from the fact that  $\mathbf{SEN}^\#$  is a functor.

Next, for all  $\sigma : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$  in  $N^b$ , we define  $\sigma : \text{SEN}^k \rightarrow \text{SEN}$  by letting, for all  $\Sigma \in |\mathbf{Sign}^\#|$ ,  $\sigma_\Sigma : \text{SEN}(\Sigma)^k \rightarrow \text{SEN}(\Sigma)$  be given by

$$\sigma_\Sigma(X_{\varphi_0}, \dots, X_{\varphi_{k-1}}) = X_{\sigma_\Sigma^\#(\varphi_0, \dots, \varphi_{k-1})},$$

for all  $\varphi_0, \dots, \varphi_{k-1} \in \text{SEN}^\#(\Sigma)$ .

This is well-defined by the preceding claim and, moreover, it is a bona fide natural transformation, since, for all  $\Sigma, \Sigma' \in |\mathbf{Sign}^\#|$ ,  $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$  and all  $\varphi_0, \dots, \varphi_{k-1} \in \text{SEN}^\#(\Sigma)$ , we have according to the preceding definitions,

$$\begin{array}{ccc} \text{SEN}^k(\Sigma) & \xrightarrow{\sigma_\Sigma} & \text{SEN}(\Sigma) \\ \text{SEN}^k(f) \downarrow & & \downarrow \text{SEN}(f) \\ \text{SEN}^k(\Sigma') & \xrightarrow{\sigma_{\Sigma'}} & \text{SEN}(\Sigma') \end{array}$$

$$\begin{aligned} & \text{SEN}(f)(\sigma_\Sigma(X_{\varphi_0}, \dots, X_{\varphi_{k-1}})) \\ &= \text{SEN}(f)(X_{\sigma_\Sigma^\#(\varphi_0, \dots, \varphi_{k-1})}) \\ &= X_{\text{SEN}^\#(f)(\sigma_\Sigma^\#(\varphi_0, \dots, \varphi_{k-1}))} \\ &= X_{\sigma_{\Sigma'}^\#(\text{SEN}^\#(f)(\varphi_0), \dots, \text{SEN}^\#(f)(\varphi_{k-1}))} \\ &= \sigma_{\Sigma'}(X_{\text{SEN}^\#(f)(\varphi_0)}, \dots, X_{\text{SEN}^\#(f)(\varphi_{k-1})}) \\ &= \sigma_{\Sigma'}(\text{SEN}(f)(X_{\varphi_0}, \dots, X_{\varphi_{k-1}})). \end{aligned}$$

Let  $N$  be the category consisting of all natural transformations  $\sigma$ , for  $\sigma^\#$  in  $N^\#$ . Then the triple  $\mathbf{A} = \langle \mathbf{Sign}^\#, \text{SEN}, N \rangle$  is a referential  $N^b$ -algebraic system.

Define  $\langle F, \alpha \rangle : \mathbf{A}^b \rightarrow \mathbf{A}$  as follows:

- $F : \mathbf{Sign}^b \rightarrow \mathbf{Sign}^\#$  is equal to  $F^\# : \mathbf{Sign}^b \rightarrow \mathbf{Sign}^\#$ ;
- $\alpha : \text{SEN}^b \rightarrow \text{SEN} \circ F$  is defined by letting, for all  $\Sigma \in |\mathbf{Sign}^b|$ ,  $\alpha_\Sigma : \text{SEN}^b(\Sigma) \rightarrow \text{SEN}(F(\Sigma))$  be given by

$$\alpha_\Sigma(\varphi) = X_{\alpha_\Sigma^\#(\varphi)}, \quad \text{for all } \varphi \in \text{SEN}^\#(\Sigma).$$

Again this definition makes  $\alpha : \text{SEN}^b \rightarrow \text{SEN} \circ F$  a bona fide natural transformation, since, for all  $\Sigma, \Sigma' \in |\mathbf{Sign}^b|$ , all  $f \in \mathbf{Sign}^b(\Sigma, \Sigma')$  and all  $\varphi \in$



$\text{SEN}^b(\Sigma)$ , we have

$$\begin{array}{ccc}
 \text{SEN}^b(\Sigma) & \xrightarrow{\alpha_\Sigma} & \text{SEN}(F(\Sigma)) \\
 \text{SEN}^b(f) \downarrow & & \downarrow \text{SEN}(F(f)) \\
 \text{SEN}^b(\Sigma') & \xrightarrow{\alpha_{\Sigma'}} & \text{SEN}(F(\Sigma'))
 \end{array}$$

$$\begin{aligned}
 \text{SEN}(F(f))(\alpha_\Sigma(\varphi)) &= \text{SEN}(F(f))(X_{\alpha_\Sigma^\#(\varphi)}) \\
 &= X_{\text{SEN}^\#(F^\#(f))(\alpha_\Sigma^\#(\varphi))} \\
 &= X_{\alpha_{\Sigma'}^\#(\text{SEN}^b(f)(\varphi))} \\
 &= \alpha_{\Sigma'}(\text{SEN}^b(f)(\varphi)).
 \end{aligned}$$

Moreover,  $\langle F, \alpha \rangle : \mathbf{A}^b \rightarrow \mathbf{A}$  is an algebraic system morphism. Indeed, for all  $\sigma^b : (\text{SEN}^b)^k \rightarrow \text{SEN}^b$  in  $N^b$ , all  $\Sigma \in |\mathbf{Sign}^b|$  and all  $\varphi_0, \dots, \varphi_{k-1} \in \text{SEN}^b(\Sigma)$ , we have

$$\begin{array}{ccc}
 \text{SEN}^b(\Sigma)^k & \xrightarrow{\sigma_\Sigma^b} & \text{SEN}^b(\Sigma) \\
 \alpha_\Sigma^k \downarrow & & \downarrow \alpha_\Sigma \\
 \text{SEN}(F(\Sigma))^k & \xrightarrow{\sigma_{F(\Sigma)}} & \text{SEN}(F(\Sigma))
 \end{array}$$

$$\begin{aligned}
 \alpha_\Sigma(\sigma_\Sigma^b(\varphi_0, \dots, \varphi_{k-1})) &= X_{\alpha_\Sigma^\#(\sigma_\Sigma^b(\varphi_0, \dots, \varphi_{k-1}))} \\
 &= X_{\sigma_{F^\#(\Sigma)}^\#(\alpha_\Sigma^\#(\varphi_0), \dots, \alpha_\Sigma^\#(\varphi_{k-1}))} \\
 &= \sigma_{F(\Sigma)}(X_{\alpha_\Sigma^\#(\varphi_0)}, \dots, X_{\alpha_\Sigma^\#(\varphi_{k-1})}) \\
 &= \sigma_{F(\Sigma)}(\alpha_\Sigma(\varphi_0), \dots, \alpha_\Sigma(\varphi_{k-1})).
 \end{aligned}$$

Thus, the pair  $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$  is an interpreted referential  $N^b$ -algebraic system.

Let  $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$  be the discrete pseudo-referential  $N^b$ -g-matrix system relative to  $(\text{PTS}, \mathcal{S})$ , where  $\mathcal{S} = \{S^{\Sigma, i} : \Sigma \in |\mathbf{Sign}^\#|, i \in I\}$ , as before, with  $\mathcal{T}^\# = \{T^{\#i} : i \in I\}$  being the collection of filter families of the g-matrix system  $\mathbb{A}^\#$ .

Then, for all  $i \in I$  and for all  $\Sigma \in |\mathbf{Sign}^\#|$ , we have  $T^{\Sigma,i} = \{T_{\Sigma'}^{\Sigma,i}\}_{\Sigma' \in |\mathbf{Sign}^\#|}$ , where, for all  $\Sigma' \in |\mathbf{Sign}^\#|$ ,

$$\begin{aligned} T_{\Sigma'}^{\Sigma,i} &= \{X \in \text{SEN}(\Sigma') : X \cap S_{\Sigma'}^{\Sigma,i} \neq \emptyset\} \\ &= \begin{cases} \emptyset, & \text{if } \Sigma' \neq \Sigma, \\ \{X_\varphi : x_\Sigma^i \in X_\varphi, \varphi \in \text{SEN}^\#(\Sigma)\}, & \text{if } \Sigma' = \Sigma \end{cases} \\ &= \begin{cases} \emptyset, & \text{if } \Sigma' \neq \Sigma \\ \{X_\varphi : \varphi \in T_{\Sigma}^{\#i}\}, & \text{if } \Sigma' = \Sigma \end{cases} \end{aligned}$$

Now notice that, for all  $\Sigma \in |\mathbf{Sign}^\#|$  and all  $\varphi \in \text{SEN}^\#(\Sigma)$ , we have that, for all  $\Sigma' \in |\mathbf{Sign}^\#|$ , all  $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$  and all  $i \in I$ ,

$$\alpha_{\Sigma'}(\text{SEN}^\#(f)(\varphi)) \in T_{F(\Sigma')}^{F(\Sigma'),i} \quad \text{iff} \quad \alpha_{\Sigma'}^\#(\text{SEN}^\#(f)(\varphi)) \in T_{F(\Sigma')}^{\#i}. \quad (1)$$

Equation (1) is true because, from the expression obtained from  $T^{\Sigma,i}$  above, we obtain

$$\begin{aligned} \alpha_{\Sigma'}(\text{SEN}^\#(f)(\varphi)) \in T_{F(\Sigma')}^{F(\Sigma'),i} &\quad \text{iff} \quad X_{\alpha_{\Sigma'}^\#(\text{SEN}^\#(f)(\varphi))} \in \{X_\varphi : \varphi \in T_{F(\Sigma')}^{\#i}\} \\ &\quad \text{iff} \quad \alpha_{\Sigma'}^\#(\text{SEN}^\#(f)(\varphi)) \in T_{F(\Sigma')}^{\#i}. \end{aligned}$$

Finally, we get the desired conclusion expressed in the following

**Claim:**  $\mathcal{I}^\mathbb{A} = \mathcal{I}^{\mathbb{A}^\#}$ .

Let  $\Sigma \in |\mathbf{Sign}^\#|$  and  $\Phi \cup \{\varphi\} \subseteq \text{SEN}^\#(\Sigma)$ . Then we have  $\varphi \in C_\Sigma^\mathbb{A}(\Phi)$  iff, for all  $\Sigma' \in |\mathbf{Sign}^\#|$ , all  $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$  and all  $i \in I$ ,

$$\alpha_{\Sigma'}(\text{SEN}^\#(f)(\Phi)) \subseteq T_{F(\Sigma')}^{F(\Sigma'),i} \quad \text{implies} \quad \alpha_{\Sigma'}(\text{SEN}^\#(f)(\varphi)) \in T_{F(\Sigma')}^{F(\Sigma'),i},$$

iff, by Equivalence (1), for all  $\Sigma' \in |\mathbf{Sign}^\#|$ , all  $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$  and all  $i \in I$ ,

$$\alpha_{\Sigma'}^\#(\text{SEN}^\#(f)(\Phi)) \subseteq T_{F(\Sigma')}^{\#i} \quad \text{implies} \quad \alpha_{\Sigma'}^\#(\text{SEN}^\#(f)(\varphi)) \in T_{F(\Sigma')}^{\#i},$$

iff  $\varphi \in C_\Sigma^{\mathbb{A}^\#}(\Phi)$ . Since  $C^\mathbb{A} = C^{\mathbb{A}^\#}$ , we conclude that  $\mathcal{I}^\mathbb{A} = \mathcal{I}^{\mathbb{A}^\#}$ , as required.  $\blacksquare$

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