Categorical Abstract Algebraic Logic:
The Subdirect Product Theorem

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Abstract

In their celebrated monograph "A General Algebraic Semantics for Sentential Logics", Font and Jansana introduced full models of a sentential logic $\mathcal{S}$ and proved the "Subdirect Product Theorem": For any sentential logic $\mathcal{S}$, the class of all algebraic reducts of (Tarski) reduced full models of $\mathcal{S}$ coincides with the closure under subdirect products of the class of all algebraic reducts of (Leibniz) reduced matrix models of $\mathcal{S}$. In this note the required machinery is developed that leads to the formulation of an analog of this theorem for logics formalized as π-institutions.

1 Introduction

Let $\mathcal{L}$ be an algebraic language and let V be a fixed countably infinite set of propositional variables. The set of terms or formulas over $\mathcal{L}$ is denoted by $\mathrm{Fm}_{\mathcal{L}}(V)$. It is well-known that the set of terms forms a free $\mathcal{L}$-algebra over V, which is denoted by $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V), \mathcal{L} \rangle$.

A **sentential logic** $\mathcal{S} = \langle \mathcal{L}, C \rangle$ consists of an algebraic language $\mathcal{L}$ together with a structural closure operator $C : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$. Structurality means that, for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$h(C(\Phi)) \subseteq C(h(\Phi)), \quad \text{for all } \Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V).$$

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Sentential logics constitute the underlying structures of the general theory of abstract algebraic logic. (See the monographs [2, 6, 7] and the books [4, 5] for overviews of this theory.)

An $\mathcal{L}$-**matrix** is a pair $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A} = \langle A, \mathcal{L}^{\boldsymbol{A}}\rangle$ is an $\mathcal{L}$-algebra and $F \subseteq A$. Given a sentential logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$, an $\mathcal{L}$-matrix $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ is called an $\mathcal{S}$-**matrix** and F is called an $\mathcal{S}$-**filter** if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} \phi \in C(\Phi) & \text{implies} & h(\Phi) \subseteq F \Rightarrow h(\phi) \in F, \\ & & \text{for every } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A}. \end{array}$$

The collection of all $\mathcal{S}$-matrices is denoted by $\mathrm{Mat}(\mathcal{S})$ and the collection of all $\mathcal{S}$-filters on an algebra $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$. The collection of all $\mathcal{S}$-filters on $\boldsymbol{A}$ forms a complete lattice under inclusion, denoted by $\mathbf{Fi}_{\mathcal{S}}(\boldsymbol{A}) = \langle \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}), \subseteq\rangle$.

Given an $\mathcal{S}$-matrix $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$, the **Leibniz congruence of** $\mathfrak{A}$, or **of** F **on** $\boldsymbol{A}$ (see [2], Theorem 1.5, and, also, [4], Theorem 0.5.3), denoted $\Omega(\mathfrak{A}) = \Omega^{\boldsymbol{A}}(F)$, is the largest congruence θ on $\boldsymbol{A}$ that is compatible with F in the sense that, for all $a, a' \in A$,

$$\langle a, a'\rangle \in \theta \quad \text{and} \quad a \in F \quad \text{imply} \quad a' \in F.$$

The $\mathcal{S}$-matrix $\mathfrak{A}$ is called (**Leibniz**) **reduced** ([4], Definition 0.5.6) if $\Omega(\mathfrak{A}) = \mathrm{id}_A$, the identity congruence on $\boldsymbol{A}$. The collection of all reduced $\mathcal{S}$-matrices is denoted by $\mathrm{Mat}^*(\mathcal{S})$. In the early days of abstract algebraic logic, when the focus of the investigations was almost exclusively on the so-called protoalgebraic logics [1], the widest class of logics thought to be amenable to algebraic investigations (see, e.g., Introduction of [2]), the class of algebras thought to constitute the "right" algebraic counterpart of a sentential logic $\mathcal{S}$ was the class of algebraic reducts of reduced $\mathcal{S}$-matrices, denoted by $\mathrm{Alg}^*(\mathcal{S})$:

$$\begin{array}{rcl} \mathrm{Alg}^*(\mathcal{S}) & = & \{\boldsymbol{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}))(\langle \boldsymbol{A}, F\rangle \in \mathrm{Mat}^*(\mathcal{S}))\} \\ & = & \{\boldsymbol{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}))(\Omega^{\boldsymbol{A}}(F) = \mathrm{id}_A)\}. \end{array}$$

An $\mathcal{L}$-**generalized matrix**, or $\mathcal{L}$-**gmatrix** for short, is a pair $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{F}\rangle$, where $\boldsymbol{A} = \langle A, \mathcal{L}^{\boldsymbol{A}}\rangle$ is an $\mathcal{L}$-algebra and $\mathcal{F} \subseteq \mathcal{P}(A)$ is a collection of subsets of A. When $\mathcal{F}$ is a closure system, $\mathbb{A}$ is also known as an **abstract logic** (see [6], p. 15, and, also, [4], p. 410). Given a sentential logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$, an $\mathcal{L}$-gmatrix $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{F}\rangle$ is called an $\mathcal{S}$-**gmatrix** if, for all $F \in \mathcal{F}$, $F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$. The collection of all $\mathcal{S}$-gmatrices is denoted by $\mathrm{GMat}(\mathcal{S})$. Given an $\mathcal{S}$-gmatrix $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{F}\rangle$, the **Tarski congruence of** $\mathbb{A}$, or **of** $\mathcal{F}$ **on** $\boldsymbol{A}$, denoted

$\widetilde{\Omega}(\mathbb{A}) = \widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{F})$, is the largest congruence θ on $\boldsymbol{A}$ that is compatible with all filters $F \in \mathcal{F}$. It is not very difficult to see that

$$\widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{F}) = \bigcap_{F \in \mathcal{F}} \Omega^{\boldsymbol{A}}(F). \tag{1}$$

The $\mathcal{S}$-gmatrix $\mathbb{A}$ is called (**Tarski**) **reduced** if $\widetilde{\Omega}(\mathbb{A}) = \mathrm{id}_A$ (see Definition 1.12 of [6]). The collection of all reduced $\mathcal{S}$-gmatrices is denoted by $\mathrm{GMat}^*(\mathcal{S})$. During their investigations in [6], Font and Jansana realized, based on the study of a variety of examples, that in the most general case of (not necessarily protoalgebraic) sentential logics, the class of algebras that constitutes the "right" algebraic counterpart for a sentential logic $\mathcal{S}$ may be the class $\mathrm{Alg}(\mathcal{S})$ of algebraic reducts of reduced $\mathcal{S}$-gmatrices, which are called **$\mathcal{S}$-algebras** (see Definition 2.16 of [6]):

$$\begin{array}{rcl}\mathrm{Alg}(\mathcal{S}) & = & \{\boldsymbol{A} : (\exists \mathcal{F} \subseteq \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}))(\langle \boldsymbol{A}, \mathcal{F}\rangle \in \mathrm{GMat}^*(\mathcal{S}))\} \\ & = & \{\boldsymbol{A} : (\exists \mathcal{F} \subseteq \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}))(\widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{F}) = \mathrm{id}_A)\}.\end{array}$$

In Theorem 2.23 of [6], it is shown that, for any sentential logic $\mathcal{S}$, the class $\mathrm{Alg}\mathcal{S}$ is the class of all subdirect products of algebras in the class $\mathrm{Alg}^*\mathcal{S}$, i.e., that $\mathrm{Alg}\mathcal{S} = \mathbf{P}_{\mathrm{S}}\mathrm{Alg}^*\mathcal{S}$.

The structure of the present work follows the development of the relevant machinery that leads to the formulation of an analog of Theorem 2.23 of [6] to logics formalized as π-institutions.

In Section 2 we introduce some of the basics of the theory of π-institutions as related to categorical abstract algebraic logic, with the goal of defining formally the classes of algebraic systems and of algebraic system reducts of reduced matrix systems associated with a given π-institution. These two classes correspond in the categorical framework to the classes $\mathrm{Alg}\mathcal{S}$ and $\mathrm{Alg}^*\mathcal{S}$, described above, in the framework of sentential logics. Section 3 introduces the notion of subdirect product for algebraic systems. The notion aims at capturing as closely as possible the ordinary notion used in universal algebra (see, e.g., Definition 8.1 of [3]). The main difference, apart from the added complexity of the notion of an algebraic system as compared with that of a universal algebra, is that we consider, so-called, interpreted algebraic systems, i.e., algebraic systems with a fixed interpretation from a base algebraic system to their elements. The base algebraic system is taken to be the underlying algebraic system over which the π-institution modeling the logical system is based. These broadens the notion of the free algebra of formulas (or terms) used in (universal) abstract algebraic logic. Finally, in Section 4, the main result of the paper, the Subdirect

Product Theorem, which is the analog of Theorem 2.23 of [6], is formulated, based on the notions introduced in Sections 2 and 3, and its proof is given.

2 Preliminaries

An **algebraic system** $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ consists of a category **Sign**, a **Set**-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a category N of natural transformations on SEN (see, e.g., Section 2 of [9]).

Example 1 *Consider a language $\mathcal{L}$, the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ and an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \mathcal{L}^{\boldsymbol{A}}\rangle$. Both can be considered as algebraic systems in a rather trivial way. For the first, take $\mathbf{Sign}_{\mathcal{L}}$ to be the one element category $\bigstar_{\mathcal{L}}$, with collection of morphisms all assignments from variables to formulas, i.e., maps $a : V \to \mathrm{Fm}_{\mathcal{L}}(V)$. The identity of this category is the "insertion of variables" map and composition is the Kleisli composition $b \circ a : V \to \mathrm{Fm}_{\mathcal{L}}(V)$ given by $b \circ a = b^* a$, where $b^* : \mathrm{Fm}_{\mathcal{L}}(V) \to \mathrm{Fm}_{\mathcal{L}}(V)$ is the extension of the assignment $b : V \to \mathrm{Fm}_{\mathcal{L}}(V)$ to formulas, effected by the freeness of $\mathbf{Fm}_{\mathcal{L}}(V)$ on V. The functor $\mathrm{SEN}_{\mathcal{L}} : \bigstar_{\mathcal{L}} \to \mathbf{Set}$ sends the only object $*$ of $\bigstar_{\mathcal{L}}$ to $\mathrm{Fm}_{\mathcal{L}}(V)$ and an assignment a to the endomorphism a^*. Finally, the category $N_{\mathcal{L}}$ of natural transformations is the clone of finitary operations of $\mathbf{Fm}_{\mathcal{L}}(V)$ generated by the basic operations in $\mathcal{L}$. So the free algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ is formalized as the algebraic system $\boldsymbol{F} = \langle \bigstar_{\mathcal{L}}, \mathrm{SEN}_{\mathcal{L}}, N_{\mathcal{L}}\rangle$.*

In some contexts it is preferable to drop the morphisms altogether and consider $\bigstar_{\mathcal{L}}$ to be the trivial category. In that case, we write $\boldsymbol{F}^t$, where the superscript refers to the trivialization of the signature category.

In a similar way we formalize the $\mathcal{L}$-algebra $\boldsymbol{A}$ as an algebraic system $\boldsymbol{A} = \langle \bigstar_{\boldsymbol{A}}, \mathrm{SEN}_{\boldsymbol{A}}, N_{\boldsymbol{A}}\rangle$ (using the same letter and relying on context to avoid confusion). The category $\bigstar_{\boldsymbol{A}}$ has a unique object $*$ and its morphisms are all the endomorphisms of $\boldsymbol{A}$ with ordinary composition. The functor $\mathrm{SEN}_{\boldsymbol{A}} : \bigstar_{\boldsymbol{A}} \to \mathbf{Set}$ maps $*$ to A and an endomorphism $h : \boldsymbol{A} \to \boldsymbol{A}$ to its underlying mapping $h : A \to A$. Finally, $N_{\boldsymbol{A}}$ is the clone of operations on A generated by the fundamental operations $\mathcal{L}^{\boldsymbol{A}}$ of the $\mathcal{L}$-algebra $\boldsymbol{A}$.

In analogy with the preceding case, we may want to drop the morphisms, in which case we write $\boldsymbol{A}^t$ for the resulting algebraic system.

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a functor. A **closure system** $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ **on** SEN is a collection of closure operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, $\Sigma \in |\mathbf{Sign}|$, that satisfy the **structurality condition**

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\Phi)),$$

for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$.

A **closure system on** an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is a closure system on SEN.

A π-**institution** $\mathcal{I} = \langle \boldsymbol{A}, C \rangle$ is a pair consisting of an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and a closure system $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on $\boldsymbol{A}$.

Example 2 *Consider a sentential logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$. The sentential logic $\mathcal{S}$ may be formalized as a π-institution in a rather trivial way. In fact, we let $\mathcal{I}_{\mathcal{S}} = \langle \boldsymbol{F}, C \rangle$, where $C_* = C$. Note that structurality of C gives the required structurality of C_*, expressed in this case by the inclusion $a^*(C_*(\Phi)) \subseteq C_*(a^*(\Phi))$, for all assignments $a : V \to \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$.*

Alternatively, we can also formalize $\mathcal{S}$ as $\mathcal{I}^t_{\mathcal{S}} = \langle \boldsymbol{F}^t, C \rangle$. In this case structurality of C_* is trivial.

Suppose, for the remainder of this discussion that $\mathcal{I} = \langle \boldsymbol{A}, C \rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, is a fixed π-institution.

An (N-)**algebraic system** $\boldsymbol{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ is an algebraic system, such that there exists a surjective functor $O' : N \to N'$ that preserves all projection natural transformations. This condition implies that O' preserves the arities of all objects and, therefore, also the arities of all natural transformations in N. We usually denote $\sigma' = O'(\sigma)$, the natural transformation in N' that is the image of σ in N.

Example 3 *Note that $\mathbf{A}$ as an algebraic system is an $N_{\mathcal{L}}$-algebraic system. The surjective functor from $N_{\mathcal{L}}$ onto $N_{\boldsymbol{A}}$ preserves all projections and maps the fundamental term operations in $\mathcal{L}$ as applied to $\mathrm{Fm}_{\mathcal{L}}(V)$ to the corresponding fundamental operations in $\mathcal{L}^{\boldsymbol{A}}$. This map has a unique extension to the clone of operations on $\mathbf{Fm}_{\mathcal{L}}(V)$, since this clone is generated by the operations in $\mathcal{L}$. Moreover, it is onto since the clone of operations $N_{\boldsymbol{A}}$ is generated by the the operations in $\mathcal{L}^{\boldsymbol{A}}$.*

An (**interpreted**) **algebraic system** (**for** $\mathcal{I}$) is a pair $\mathcal{A}' = \langle \boldsymbol{A}', \langle F, \alpha \rangle \rangle$, where

- $\boldsymbol{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ is an algebraic system;
- $\langle F, \alpha \rangle : \boldsymbol{A} \to \boldsymbol{A}'$ is an algebraic system morphism, i.e., a pair consisting of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$, such that

$$\alpha_{\Sigma'}(\sigma_\Sigma(\phi_0, \dots, \phi_{k-1})) = \sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \dots, \alpha_\Sigma(\phi_{k-1})),$$

for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{k-1} \in \mathrm{SEN}(\Phi)$.

Example 4 *Consider the π-institution $\mathcal{I}^t_{\mathcal{S}} = \langle \boldsymbol{F}^t, C \rangle$.*

- *The pair $\mathcal{F} = \langle \boldsymbol{F}, \langle J, \zeta \rangle \rangle$, where $J : \boldsymbol{F}^t \to \boldsymbol{F}$ is the only available functor and $\zeta_* = \mathrm{id}_{\mathrm{Fm}_{\mathcal{L}}(V)}$, is an interpreted algebraic system for $\mathcal{I}^t_{\mathcal{S}}$.*
- *The pair $\mathcal{A} = \langle \boldsymbol{A}. \langle F, \alpha \rangle \rangle$, where $F : \boldsymbol{F}^t \to \boldsymbol{A}$ is the only available functor and $\alpha_* : \mathrm{Fm}_{\mathcal{L}}(V) \to A$ is the underlying set map of an $\mathcal{L}$-homomorphism $\alpha_* : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A}$, is also an interpreted algebraic system for $\mathcal{I}^t_{\mathcal{S}}$.*

A **matrix system** (**for** $\mathcal{I}$) is a pair $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$, where $\mathcal{A}'$ is an algebraic system and $T' = \{T'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ is a **sentence family** of $\mathcal{A}'$, i.e., a collection of subsets $T'_{\Sigma'} \subseteq \mathrm{SEN}'(\Sigma')$, for all $\Sigma' \in |\mathbf{Sign}'|$. We denote the collection of all sentence families of $\mathcal{A}'$ by $\mathrm{SenFam}(\mathcal{A}')$.[1]

A matrix system $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ **defines** (or **generates** or **induces**) a closure system $C^{\mathfrak{A}'}$ on SEN in the following way: For all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{lll} \phi \in C^{\mathfrak{A}'}_\Sigma(\Phi) & \text{iff} & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \Rightarrow \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T'_{F(\Sigma')}, \\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'). \end{array}$$

We denote the defining condition more succinctly by $\Phi \vDash^{\mathfrak{A}'}_\Sigma \phi$. So we have, by definition, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C^{\mathfrak{A}'}_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vDash^{\mathfrak{A}'}_\Sigma \phi.$$

Example 5 *Consider again the algebraic system $\boldsymbol{F}^t$ and the interpreted algebraic system $\mathcal{F} = \langle \boldsymbol{F}, \langle J, \zeta \rangle \rangle$. Let $T \in \mathrm{Th}(\mathcal{S})$ be a $*$-theory of the π-institution $\mathcal{I}^t_{\mathcal{S}}$, i.e., a theory of the sentential logic $\mathcal{S}$. Consider the matrix system for $\mathcal{I}^t_{\mathcal{S}}$ given by*

$$\mathfrak{F} = \langle \mathcal{F}, \{T\} \rangle.$$

Then, we have, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} \phi \in C^{\mathfrak{F}}_*(\Phi) & \textit{iff} & \Phi \subseteq T \;\Rightarrow\; \phi \in T \\ & \textit{iff} & \phi \in C^{\{T, \mathrm{Fm}_{\mathcal{L}}(V)\}}(\Phi), \end{array}$$

where $C^{\{T, \mathrm{Fm}_{\mathcal{L}}(V)\}}$ is the closure system on $\mathrm{Fm}_{\mathcal{L}}(V)$, with T and $\mathrm{Fm}_{\mathcal{L}}(V)$ as its only closed sets.

[1] In preceding papers by the author sentence families have also been called "axiom families" and their collection was denoted by $\mathrm{AxFam}(\mathcal{A}')$.

A matrix system $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ is called an $\mathcal{I}$-**matrix system** if $C \leq C^{\mathfrak{A}'}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{implies} \quad \phi \in C^{\mathfrak{A}'}_\Sigma(\Phi).$$

In this case, we say that T' is an $\mathcal{I}$-**filter family** of $\mathcal{A}'$. The collection of all $\mathcal{I}$-filter families of $\mathcal{A}'$ is denoted by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ and the collection of all $\mathcal{I}$-matrix systems is denoted by $\mathrm{MatSys}(\mathcal{I})$.

Example 6 *Consider the algebraic system* $\boldsymbol{F}^t$ *and the interpreted algebraic system* $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$. *Let* $T' \subseteq A$ *be a subset of* A. *We have:*

$T' \in \mathrm{FiFam}^{\mathcal{I}^t_{\mathcal{S}}}(\mathcal{A})$
iff *for all* $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V), \phi \in C_*(\Phi) \Rightarrow \phi \in C^{\langle \mathcal{A}, T' \rangle}_*(\Phi)$
iff *for all* $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V), \phi \in C_*(\Phi) \Rightarrow (\alpha_*(\Phi) \subseteq T' \Rightarrow \alpha_*(\phi) \in T')$.

A few comments concerning the current context as contrasted to that encountered in the theory of sentential logics are in order. Under the formalization in Example 6, a matrix system does not capture the notion of the logical matrix from the theory of sentential logics. Whereas in the theory of sentential logics one requires the condition

$$\phi \in C(\Phi) \Rightarrow (\forall \alpha_* : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A})(\alpha_*(\Phi) \subseteq T' \Rightarrow \alpha_*(\phi) \in T'),$$

for T' to be an $\mathcal{S}$-filter on $\boldsymbol{A}$, in the present case, one works with a fixed algebraic morphism $\langle F, \alpha \rangle$ interpreting $\mathcal{I}^t_{\mathcal{S}}$ into $\boldsymbol{A}$. In the context of π-institutions, this is justified by the fact that, in considering interpretations across institutions, one is having in mind mapping signatures of one logical system to signatures of another, not merely homomorphisms between structures over the same signature in the same logical system (which, after all, should be morphisms of a single institution representing the logical system under consideration and its models). Against this radically different background, it seems reasonable to admit as a model of one logical system another system into which the symbols of the first can be mapped under a specific interpretation, rather than across all possible interpretations.

Continuing with the general theory, a **generalized matrix system** (**for** $\mathcal{I}$) or **gmatrix system** (**for** $\mathcal{I}$) is a pair $\mathbb{A}' = \langle \mathcal{A}', \mathcal{T}' \rangle$, where $\mathcal{A}'$ is an algebraic system and $\mathcal{T}' \subseteq \mathrm{SenFam}(\mathcal{A}')$ is a collection of sentence families of $\mathcal{A}'$.

A gmatrix system $\mathbb{A}' = \langle \mathcal{A}', \mathcal{T}' \rangle$ **defines** (or **generates** or **induces**) a closure system $C^{\mathbb{A}'}$ on SEN in the following way:

$$C^{\mathbb{A}'} = \bigcap_{T' \in \mathcal{T}'} C^{\langle \mathcal{A}', T' \rangle}.$$

So we have, by definition, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma^{\mathbb{A}'}(\Phi) \quad \text{iff} \quad \Phi \vDash_\Sigma^{\langle \mathcal{A}', T' \rangle} \phi, \text{ for all } T' \in \mathcal{T}'.$$

A gmatrix system $\mathbb{A}' = \langle \mathcal{A}', \mathcal{T}' \rangle$ is called an $\mathcal{I}$-**gmatrix system** if $C \leq C^{\mathbb{A}'}$. The collection of all $\mathcal{I}$-gmatrix systems is denoted by $\mathrm{GMatSys}(\mathcal{I})$.

Let $\mathcal{A}'$ be an algebraic system and T' a sentence family of $\mathcal{A}'$. Recall that a **congruence system** θ on $\mathcal{A}'$ is a collection $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$, such that, for all $\Sigma \in |\mathbf{Sign}'|$, θ_Σ is an equivalence relation on $\mathrm{SEN}'(\Sigma)$ and such that the family θ is invariant under both natural transformations in N' (making each θ_Σ a *congruence*) and signature morphisms in $\mathbf{Sign}'$ (making the family θ a *system*), i.e.,

- for all $\Sigma \in |\mathbf{Sign}'|$, all $\sigma_\Sigma : \mathrm{SEN}'(\Sigma)^k \to \mathrm{SEN}'(\Sigma)$, and all $\varphi_0, \psi_0, \ldots, \varphi_{k-1}, \psi_{k-1} \in \mathrm{SEN}'(\Sigma)$,

  $$\langle \varphi_i, \psi_i \rangle \in \theta_\Sigma, i < k, \text{ implies } \langle \sigma'_\Sigma(\varphi_0, \ldots, \varphi_{k-1}), \sigma'_\Sigma(\psi_0, \ldots, \psi_{k-1}) \rangle \in \theta_\Sigma;$$

- for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, all $f : \mathbf{Sign}'(\Sigma, \Sigma')$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$,

  $$\langle \varphi, \psi \rangle \in \theta_\Sigma \text{ implies } \langle \mathrm{SEN}'(f)(\varphi), \mathrm{SEN}'(f)(\psi) \rangle \in \theta_{\Sigma'}.$$

The **Leibniz congruence system** $\Omega^{\mathcal{A}'}(T')$ **of** T' is the largest congruence system on $\mathcal{A}'$ that is compatible with T' (see Proposition 2.2 of [8]) in the sense that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi, \psi \in \mathrm{SEN}'(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\mathcal{A}'}(T') \quad \text{and} \quad \phi \in T'_\Sigma \quad \text{imply} \quad \psi \in T'_\Sigma.$$

Equivalently, T' is a signature-wise union of congruence classes modulo $\Omega^{\mathcal{A}'}(T')$.

Let $\mathcal{A}'$ be an algebraic system and $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. The **Tarski congruence system** $\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}')$ of $\mathcal{T}'$ is the largest congruence system on $\mathcal{A}'$ that is compatible with all $T' \in \mathcal{T}'$ (see Theorem 3 of [9]). It is not difficult to see that

$$\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') = \bigcap_{T' \in \mathcal{T}'} \Omega^{\mathcal{A}'}(T').$$

An $\mathcal{I}$-matrix system $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ is said to be (**Leibniz**) **reduced** if $\Omega^{\mathcal{A}'}(T') = \Delta^{\mathcal{A}'}$, the (signature-wise) identity congruence system on $\mathcal{A}'$. The collection of all reduced $\mathcal{I}$-matrix systems is denoted by $\mathrm{MatSys}^*(\mathcal{I})$.

Similarly, an $\mathcal{I}$-gmatrix system $\mathbb{A}' = \langle \mathcal{A}', \mathcal{T}' \rangle$ is said to be (**Tarski**) **reduced** if $\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') = \Delta^{\mathcal{A}'}$. The collection of all reduced $\mathcal{I}$-gmatrix systems is denoted by $\mathrm{GMatSys}^*(\mathcal{I})$.

An algebraic system $\mathcal{A}'$ is called an $\mathcal{I}^*$**-algebraic system** if $\mathcal{A}'$ is the algebraic system reduct of a reduced $\mathcal{I}$-matrix system, i.e., if and only if there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, such that $\langle \mathcal{A}', T' \rangle \in \mathrm{MatSys}^*(\mathcal{I})$. The collection of all $\mathcal{I}^*$-algebraic systems is denoted by $\mathrm{AlgSys}^*(\mathcal{I})$:

$$\mathrm{AlgSys}^*(\mathcal{I}) = \{\mathcal{A}' : (\exists T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'))(\Omega^{\mathcal{A}'}(T') = \Delta^{\mathcal{A}'})\}.$$

On the other hand, an algebraic system $\mathcal{A}'$ is called an $\mathcal{I}$**-algebraic system** if $\mathcal{A}'$ is the algebraic system reduct of a reduced $\mathcal{I}$-gmatrix system, i.e., if and only if there exists $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, such that $\langle \mathcal{A}', \mathcal{T}' \rangle \in \mathrm{GMatSys}^*(\mathcal{I})$. The collection of all $\mathcal{I}$-algebraic systems is denoted by $\mathrm{AlgSys}(\mathcal{I})$:

$$\mathrm{AlgSys}(\mathcal{I}) = \{\mathcal{A}' : (\exists \mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'))(\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') = \Delta^{\mathcal{A}'})\}.$$

3 Subdirect Products

Consider an algebraic system $\boldsymbol{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$. An algebraic system $\boldsymbol{A}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', N'' \rangle$ is called an **algebraic subsystem** of $\boldsymbol{A}'$ if the following hold:

- $\mathbf{Sign}'' = \mathbf{Sign}'$;
- $\mathrm{SEN}''(\Sigma) \subseteq \mathrm{SEN}'(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'|$, and
  $\mathrm{SEN}''(f) = \mathrm{SEN}'(f) \upharpoonright_{\mathrm{SEN}''(\Sigma)}$, for all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$;
- $\sigma'' = \sigma' \upharpoonright_{\mathrm{SEN}''(\Sigma)}$, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N.

Moreover, if $\mathcal{A}' = \langle \boldsymbol{A}', \langle F', \alpha' \rangle \rangle$ is an interpreted algebraic system, then $\mathcal{A}'' = \langle \boldsymbol{A}'', \langle F'', \alpha'' \rangle \rangle$ is an (**interpreted**) **algebraic subsystem** of $\mathcal{A}'$ if $\boldsymbol{A}''$ is an algebraic subsystem of $\boldsymbol{A}'$ and, in addition, the following diagram commutes

$$\begin{tikzcd}
 & \boldsymbol{A} \arrow[dl, "\langle F'', \alpha'' \rangle"'] \arrow[dr, "\langle F', \alpha' \rangle"] & \\
\boldsymbol{A}'' \arrow[rr, "\langle I_{\mathbf{Sign}}, \iota \rangle"'] & & \boldsymbol{A}'
\end{tikzcd}$$

where $\langle I_{\mathbf{Sign}}, \iota \rangle : \boldsymbol{A}'' \to \boldsymbol{A}'$ denotes the signature-wise inclusion morphism.

Consider, next, a collection $\{\boldsymbol{A}^i : i \in I\}$ of algebraic systems $\boldsymbol{A}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, N^i \rangle$. The **product algebraic system**

$$\prod_{i \in I} \boldsymbol{A}^i = \langle \prod_{i \in I} \mathbf{Sign}^i, \prod_{i \in I} \mathrm{SEN}^i, \prod_{i \in I} N^i \rangle$$

is defined as follows:

- $\prod_{i\in I}\mathbf{Sign}^i$ is the category with objects $\prod_{i\in I}|\mathbf{Sign}^i|$. For all $\Sigma^i,\Sigma'^i \in |\mathbf{Sign}^i|$, $i\in I$,

$$\prod_{i\in I}\mathbf{Sign}^i(\langle\Sigma^i : i\in I\rangle,\langle\Sigma'^i : i\in I\rangle) = \prod_{i\in I}\mathbf{Sign}^i(\Sigma^i,\Sigma'^i).$$

- $\prod_{i\in I}\mathrm{SEN}^i : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ is defined by setting

$$\prod_{i\in I}\mathrm{SEN}^i(\langle\Sigma^i : i\in I\rangle) = \prod_{i\in I}\mathrm{SEN}^i(\Sigma^i),\ \text{for all } \Sigma^i\in|\mathbf{Sign}^i|,\ i\in I,$$

  and

$$\prod_{i\in I}\mathrm{SEN}^i(\langle f^i : i\in I\rangle)(\langle\phi^i : i\in I\rangle) = \langle\mathrm{SEN}^i(f^i)(\phi^i) : i\in I\rangle,$$

  for all $\Sigma^i,\Sigma'^i\in|\mathbf{Sign}^i|$, $f^i\in\mathbf{Sign}^i(\Sigma^i,\Sigma'^i)$ and all $\phi^i\in\mathrm{SEN}^i(\Sigma^i)$, $i\in I$.

- $\sigma^{\prod_{i\in I}N^i} : (\prod_{i\in I}\mathrm{SEN}^i)^k \to \prod_{i\in I}\mathrm{SEN}^i$ is defined, for all $\Sigma^i\in|\mathbf{Sign}^i|$ and all $\phi_0^i,\dots,\phi_{k-1}^i\in\mathrm{SEN}^i(\Sigma^i)$, $i\in I$, by

$$\sigma^{\prod N^i}_{\langle\Sigma^i : i\in I\rangle}(\langle\phi_0^i : i\in I\rangle,\dots,\langle\phi_{k-1}^i : i\in I\rangle) = \langle\sigma^i_{\Sigma^i}(\phi_0^i,\dots,\phi_{k-1}^i) : i\in I\rangle.$$

Furthermore, if $\mathcal{A}^i = \langle\boldsymbol{A}^i,\langle F^i,\alpha^i\rangle\rangle$, $i\in I$, are interpreted algebraic systems, then their (**interpreted**) **product algebraic system** is $\prod_{i\in I}\mathcal{A}^i = \langle\prod_{i\in I}\boldsymbol{A}^i,\prod_{i\in I}\langle F^i,\alpha^i\rangle\rangle$, where $\prod_{i\in I}\boldsymbol{A}^i$ is the product algebraic system of the $\boldsymbol{A}^i$, $i\in I$, and $\prod_{i\in I}\langle F^i,\alpha^i\rangle : \boldsymbol{A}\to\prod_{i\in I}\boldsymbol{A}^i$ is the unique algebraic system morphism that makes the following diagram commute

$$\begin{tikzcd}
& \boldsymbol{A} \arrow[dl, "\prod_{i\in I}\langle F^i,\alpha^i\rangle"'] \arrow[dr, "\langle F^i,\alpha^i\rangle"] & \\
\prod\boldsymbol{A}^i \arrow[rr, "\langle P^i,p^i\rangle"'] & & \boldsymbol{A}^i
\end{tikzcd}$$

where $\langle P^i,p^i\rangle : \prod_{i\in I}\boldsymbol{A}^i\to\boldsymbol{A}^i$ is the i-th projection morphism.

Consider again a collection $\{\boldsymbol{A}^i : i\in I\}$ of algebraic systems. An algebraic system $\boldsymbol{A}'$ is a **subdirect product algebraic system** of the $\boldsymbol{A}^i$, $i\in I$, if it is an algebraic subsystem of the direct product algebraic system $\prod_{i\in I}\boldsymbol{A}^i$, such that, for all $i \in I$, the composition

$$\begin{tikzcd} \boldsymbol{A}' \arrow[r, hook, "\langle I_{\mathbf{Sign}'}, \iota\rangle"] & \prod_{i\in I}\boldsymbol{A}^i \arrow[r, "\langle P^i, p^i\rangle"] & \boldsymbol{A}^i \end{tikzcd}$$

is a surjective algebraic system morphism.

On the other hand, an interpreted algebraic system $\mathcal{A}' = \langle \boldsymbol{A}', \langle F', \alpha'\rangle\rangle$ is an (**interpreted**) **subdirect product algebraic system** of the interpreted algebraic systems $\mathcal{A}^i = \langle \boldsymbol{A}^i, \langle F^i, \alpha^i\rangle\rangle$, $i \in I$, if $\boldsymbol{A}'$ is a subdirect product algebraic system of the $\boldsymbol{A}^i$, $i \in I$, and the following diagram commutes

$$\begin{tikzcd} & \boldsymbol{A} \arrow[dl, "\langle F', \alpha'\rangle"'] \arrow[d, "\prod_{i\in I}\langle F^i, \alpha^i\rangle"] \arrow[dr, "\langle F^i, \alpha^i\rangle"] & \\ \boldsymbol{A}' \arrow[r, hook, "\langle I_{\mathbf{Sign}'}, \iota\rangle"'] & \prod_{i\in I}\boldsymbol{A}^i \arrow[r, "\langle P^i, p^i\rangle"'] & \boldsymbol{A}^i \end{tikzcd} \tag{2}$$

We will denote by $\mathbf{P}_{\mathrm{S}}(\mathrm{K})$ the class of all subdirect product algebraic systems of families of algebraic systems belonging to the class K.

4 The Subdirect Product Theorem

Having built the necessary machinery, we are now in a position to formulate the promised analog of Theorem 2.23 of [6], which was the main goal of our work.

Theorem 7 *Let $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$ be a π-institution based on the algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Then*

$$\mathrm{AlgSys}(\mathcal{I}) = \mathbf{P}_{\mathrm{S}}(\mathrm{AlgSys}^*(\mathcal{I})).$$

Proof: Suppose that $\mathcal{A}^i = \langle \boldsymbol{A}^i, \langle F^i, \alpha^i\rangle\rangle \in \mathrm{AlgSys}^*(\mathcal{I})$, for all $i \in I$. Then, there exist $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, such that $\langle \mathcal{A}^i, T^i\rangle \in \mathrm{MatSys}^*(\mathcal{I})$. Consider a subdirect product algebraic system $\mathcal{A}' = \langle \boldsymbol{A}', \langle F', \alpha'\rangle\rangle$ of the collection $\mathcal{A}^i = \langle \boldsymbol{A}^i, \langle F^i, \alpha^i\rangle\rangle$, as in Diagram (2). We must show $\mathcal{A}' \in \mathrm{AlgSys}(\mathcal{I})$, i.e., we must exhibit a collection $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, such that $\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') = \Delta^{\mathcal{A}'}$.

For all $i \in I$, we define $S^i = \iota^{-1}(p^{i^{-1}}(T^i))$. Then, since $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, we get that $S^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. We define $\mathcal{T}' = \{S^i : i \in I\}$. Then, we have

$$\begin{array}{lcl}
\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') & = & \widetilde{\Omega}^{\mathcal{A}'}(\{S^i : i \in I\}) \\
 & = & \widetilde{\Omega}^{\mathcal{A}'}(\{(p^i\iota)^{-1}(T^i) : i \in I\}) \\
 & = & \bigcap_{i\in I} \Omega^{\mathcal{A}'}((p^i\iota)^{-1}(T^i)) \quad \text{(by Equation (1))} \\
 & = & \bigcap_{i\in I} (p^i\iota)^{-1}(\Omega^{\mathcal{A}^i}(T^i)) \quad (p^i\iota \text{ surjective}) \\
 & = & \bigcap_{i\in I} (p^i\iota)^{-1}(\Delta^{\mathcal{A}^i}) \quad (\langle \mathcal{A}^i, T^i\rangle \text{ reduced}) \\
 & = & \Delta^{\mathcal{A}'}.
\end{array}$$

Since $\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') = \Delta^{\mathcal{A}'}$, we get $\mathcal{A}' \in \mathrm{AlgSys}(\mathcal{I})$, as desired.

Suppose, conversely, that $\mathcal{A}' = \langle \boldsymbol{A}', \langle F', \alpha'\rangle\rangle \in \mathrm{AlgSys}(\mathcal{I})$. Then, there exists $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, such that $\langle \mathcal{A}', \mathcal{T}'\rangle \in \mathrm{GMatSys}^*(\mathcal{I})$, i.e., such that $\widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') = \Delta^{\mathcal{A}'}$. We consider the algebraic systems

$$\mathcal{A}'/\Omega^{\mathcal{A}'}(T') = \langle \boldsymbol{A}'/\Omega^{\mathcal{A}'}(T'), \langle F', \pi^{\Omega^{\mathcal{A}'}(T')}\alpha\rangle\rangle, \quad T' \in \mathcal{T}'.$$

$$\begin{tikzcd}
 & \boldsymbol{A} \arrow[dl, "{\langle F', \alpha'\rangle}"'] \arrow[dr, "{\langle F', \pi^{\Omega^{\mathcal{A}'}(T')}\alpha'\rangle}"] & \\
\boldsymbol{A}' \arrow[rr, "{\langle I_{\mathbf{Sign}'}, \pi^{\Omega^{\mathcal{A}'}(T')}\rangle}"'] & & \boldsymbol{A}'/\Omega^{\mathcal{A}'}(T')
\end{tikzcd}$$

Clearly, since $\Omega^{\mathcal{A}'/\Omega^{\mathcal{A}'}(T')}(T'/\Omega^{\mathcal{A}'}(T')) = \Delta^{\mathcal{A}'/\Omega^{\mathcal{A}'}(T')}$, for all $T' \in \mathcal{T}'$, we get that $\mathcal{A}'/\Omega^{\mathcal{A}'}(T') \in \mathrm{AlgSys}^*(\mathcal{I})$, for all $T' \in \mathcal{T}'$. We, next form the direct product algebraic system $\prod_{T'\in\mathcal{T}'} \mathcal{A}'/\Omega^{\mathcal{A}'}(T')$ (right triangle in the diagram below). We claim that the following diagram gives a (interpreted) subdirect product algebraic system.

$$\begin{tikzcd}
 & \boldsymbol{A} \arrow[dl, "{\langle F', \alpha'\rangle}"'] \arrow[d, "{\prod_{T'}\langle F', \pi^{\Omega^{\mathcal{A}'}(T')}\alpha'\rangle}"] \arrow[dr, "{\langle F', \pi^{\Omega^{\mathcal{A}'}(T')}\alpha'\rangle}"] & \\
\boldsymbol{A}' \arrow[r, "{\prod_{T'}\langle I_{\mathbf{Sign}'}, \pi^{\Omega^{\mathcal{A}'}(T')}\rangle}"'] & \prod_{T'} \boldsymbol{A}'/\Omega^{\mathcal{A}'}(T') \arrow[r, "{\langle P^{T'}, p^{T'}\rangle}"'] & \boldsymbol{A}'/\Omega^{\mathcal{A}'}(T')
\end{tikzcd}$$

This will conclude the proof, since then $\mathcal{A}' \in \mathbf{P}_{\mathrm{S}}(\mathrm{AlgSys}^*(\mathcal{I}))$.

First, we show that $\prod_{T'}\langle I_{\mathbf{Sign}'},\pi^{\Omega^{\mathcal{A}'}(T')}\rangle$ is injective. Let $\Sigma \in |\mathbf{Sign}'|$ and $\phi,\psi \in \mathrm{SEN}'(\Sigma)$, such that

$$\langle \pi_\Sigma^{\Omega^{\mathcal{A}'}(T')}(\phi) : T' \in \mathcal{T}'\rangle = \langle \pi_\Sigma^{\Omega^{\mathcal{A}'}(T')}(\phi) : T' \in \mathcal{T}'\rangle.$$

This means, by the definitions involved, that $\phi/\Omega^{\mathcal{A}'}_\Sigma(T') = \psi/\Omega^{\mathcal{A}'}_\Sigma(T')$, for all $T' \in \mathcal{T}'$. Hence

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \bigcap_{T'\in\mathcal{T}'}\Omega^{\mathcal{A}'}_\Sigma(T')\\ & = & \widetilde{\Omega}^{\mathcal{A}'}(\mathcal{T}') \quad \text{(by Equation (1))}\\ & = & \Delta^{\mathcal{A}'} \quad (\langle\mathcal{A}',\mathcal{T}'\rangle \text{ reduced}).\end{array}$$

Therefore, $\phi = \psi$ and $\prod_{T'}\langle I_{\mathbf{Sign}'},\pi^{\Omega^{\mathcal{A}'}(T')}\rangle$ is injective.

Next, we show that, for all $T' \in \mathcal{T}'$, the composition

$$\boldsymbol{A}' \xrightarrow{\prod_{T'}\langle I_{\mathbf{Sign}'},\pi^{\Omega^{\mathcal{A}'}(T')}\rangle} \prod_{T'}\boldsymbol{A}'/\Omega^{\mathcal{A}'}(T') \xrightarrow{\langle P^{T'},p^{T'}\rangle} \boldsymbol{A}'/\Omega^{\mathcal{A}'}(T')$$

is a surjective morphism. To this end, let $\Sigma \in |\mathbf{Sign}'|$ and $\phi/\Omega^{\mathcal{A}'}_\Sigma(T') \in \mathrm{SEN}'(\Sigma)/\Omega^{\mathcal{A}'}_\Sigma(T')$. Then, we have

$$\begin{array}{lll}p^{T'}_\Sigma(\langle\pi_\Sigma^{\Omega^{\mathcal{A}'}(T')}(\phi) : T' \in \mathcal{T}'\rangle) & = & p^{T'}_\Sigma(\langle\phi/\Omega^{\mathcal{A}'}_\Sigma(T') : T' \in \mathcal{T}'\rangle)\\ & = & \phi/\Omega^{\mathcal{A}'}_\Sigma(T').\end{array}$$

Therefore, $\langle P^{T'},p^{T'}\rangle\circ\prod_{T'}\langle I_{\mathbf{Sign}'},\pi^{\Omega^{\mathcal{A}'}(T')}\rangle$ is surjective, for all $T' \in \mathcal{T}'$, showing that $\mathcal{A}'$ is a subdirect product algebraic system. ∎

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