Categorical Abstract Algebraic Logic:
Selfextensional π-Institutions with Conjunction

George Voutsadakis*

February 1, 2007

Abstract

The work of Jansana on selfextensional logics with conjunction, that was partially based on the well-known work of Font and Jansana on providing a general algebraic semantics for sentential logics, is abstracted to cover selfextensional logics with conjunction, formalized as π-institutions. Analogs are provided in this more general context of the main results of Jansana: 1. The class of algebras $\mathbf{Alg}\mathcal{S}$ naturally associated with a selfextensional logic $\mathcal{S}$ with a conjunction is a variety. 2. Every selfextensional logic with a conjunction is fully selfextensional. 3. For every algebraic signature with a binary term $\wedge$, there is a dual isomorphism between the set of selfextensional logics with conjunction $\wedge$, ordered by extension, and the set of all subvarieties of the variety axiomatized by the semilattice equations with respect to $\wedge$, ordered by inclusion. In order to prove analogs of these results at the categorical level, we use the powerful machinery developed in the last few years in this area, including results from the theory of varieties and quasi-varieties of algebraic systems.

1 Introduction

Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, and a set $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, the *Frege relation* $\Lambda_{\mathcal{S}}(\Gamma)$ *of* $\mathcal{S}$ *relative to* Γ is the binary relation on $\mathrm{Fm}_{\mathcal{L}}(V)$ defined, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\langle \phi, \psi \rangle \in \Lambda_{\mathcal{S}}(\Gamma) \quad \text{iff} \quad \Gamma, \phi \vdash_{\mathcal{S}} \psi \quad \text{and} \quad \Gamma, \psi \vdash_{\mathcal{S}} \phi.$$

If this relation is a congruence on the formula algebra, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, then $\mathcal{S}$ is said to be *Fregean*. The name comes from the fact that this property may be viewed as the formal counterpart of Frege's compositionality principle for truth and is originally due to Suszko

*School of Mathematics and Computer Science, Lake Superior State University, Sault Sainte Marie, MI 49783, USA, `gvoutsad@lssu.edu`

[0] *Keywords:* Abstract Algebraic Logic, Selfextensional Logics, Fregean Logics, Protoalgebraic Logics, N-Protoalgebraic π-Institutions, Full Models, Property of Conjunction

2000 AMS Subject Classification: Primary: 03B22, 03C05, 03G25 Secondary: 08B05, 08C05

(see, e.g., [20, 21]). Fregean deductive systems were extensively studied by Czelakowski and Pigozzi in [8, 9].

Given, on the other hand, a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and an axiom family $F = \{F_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, the *Frege equivalence system* $\Lambda^{\mathcal{I}}(F) = \{\Lambda^{\mathcal{I}}_\Sigma(F)\}_{\Sigma\in|\mathbf{Sign}|}$ *of* $\mathcal{I}$ *relative to* F is the equivalence system on SEN, defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\begin{array}{lll}\langle \phi,\psi\rangle \in \Lambda^{\mathcal{I}}_\Sigma(F) & \text{iff} & C_{\Sigma'}(F_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = C_{\Sigma'}(F_{\Sigma'}, \mathrm{SEN}(f)(\psi)),\\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma').\end{array}$$

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be N-*Fregean* if, for every axiom family F, the Frege relation $\Lambda^{\mathcal{I}}(F)$ is an N-congruence system on SEN. This property directly generalizes the previously defined Fregean property for deductive systems.

There are many logics in the literature that are not Fregean. Many of these, however, do satisfy a weaker property called selfextensionality. Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$, the *interderivability relation* $\Lambda(\mathcal{S})$ *of* $\mathcal{S}$ is the binary relation on $\mathrm{Fm}_{\mathcal{L}}(V)$ defined, for all $\phi,\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ by

$$\langle \phi,\psi\rangle \in \Lambda(\mathcal{S}) \quad \text{iff} \quad \phi \vdash_{\mathcal{S}} \psi \quad \text{and} \quad \psi \vdash_{\mathcal{S}} \phi.$$

The deductive system $\mathcal{S}$ is called *selfextensional* if $\Lambda(\mathcal{S})$ is a congruence relation on the formula algebra. The name is due to Wójcicki (see [35]). The Tarski congruence $\widetilde{\Omega}(\mathcal{S})$ is the largest congruence on the formula algebra that is included in the interderivability relation of $\mathcal{S}$, whence an equivalent condition to selfextensionality is the condition $\Lambda(\mathcal{S}) = \widetilde{\Omega}(\mathcal{S})$.

As one would imagine, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the *interderivability* or *Frege equivalence system of* $\mathcal{I}$ is the equivalence system $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN, that is defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\langle \phi,\psi\rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

A π-institution $\mathcal{I}$, as above, with N a category of natural transformations on SEN, is said to be N-*selfextensional* if $\Lambda(\mathcal{I})$ is an N-congruence system on SEN. Since, in the π-institution framework, it is also true that the Tarski N-congruence system $\widetilde{\Omega}^N(\mathcal{I})$ is the largest N-congruence system that is included in the Frege equivalence system $\Lambda(\mathcal{I})$, we obtain that $\mathcal{I}$ is N-selfextensional if and only if $\Lambda(\mathcal{I}) = \widetilde{\Omega}^N(\mathcal{I})$.

One of the main achievements of Abstract Algebraic Logic has been the classification of sentential logics into different steps of an algebraic hierarchy of logics that roughly reflect the strength of the ties between the logic and a naturally associated algebraic counterpart. The stronger these ties are, the closer the connection between metalogical properties that the logic possesses and corresponding algebraic properties of the algebraic counterpart. The sentential logics that form the widest class in this hierarchy are the protoalgebraic logics, that were introduced and studied by Czelakowski [6] and Blok and Pigozzi [3]. A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ is *protoalgebraic* if, for every theory T of $\mathcal{S}$ and all $\phi,\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\langle \phi,\psi\rangle \in \Omega(T) \quad \text{implies} \quad T,\phi \vdash_{\mathcal{S}} \psi \quad \text{and} \quad T,\psi \vdash_{\mathcal{S}} \phi,$$

in other words, $\mathcal{S}$ is protoalgebraic if, whenever two formulas are congruent modulo the Leibniz congruence of the theory T, they must also be interderivable modulo T, for every theory T of $\mathcal{S}$. Two other characterizations of protoalgebraicity turn out to be very useful in various contexts. The first states that $\mathcal{S}$ is protoalgebraic if and only if the Leibniz operator on the theories of the logic is monotone, i.e., for every $T_1, T_2 \in \mathrm{Th}(\mathcal{S})$, if $T_1 \subseteq T_2$, then $\Omega(T_1) \subseteq \Omega(T_2)$. The second, syntactic in nature, asserts that $\mathcal{S}$ is protoalgebraic if and only if, there exists a set Δ of formulas in two variables, that satisfies

1. $\vdash_{\mathcal{S}} \Delta(\phi, \phi)$, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$, and
2. $\phi, \Delta(\phi, \psi) \vdash_{\mathcal{S}} \psi$, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$.

Other classes that one encounters in the algebraic hierarchy of sentential logics are the classes of equivalential logics [18, 5], of weakly algebraizable logics [7], of algebraizable logics [14, 15, 16] and of finitely algebraizable logics [4] among others.

One of the main reasons that selfextensional deductive systems have been recently at the focus of many studies in Abstract Algebraic Logic is that they can be found in every class of the Leibniz hierarchy. Therefore, as Jansana points out in [17], their study "can bring to the surface phenomena that do not come to it naturally when studying the classes of the Leibniz hierarchy".

Recently, analogs of the Tarski operator and of the Leibniz operator have also been introduced for π-institutions (see [22] for the former and [27] for the latter). This has led to the introduction of a categorical abstract hierarchy of π-institutions that consists of classes similar to the classes in which the standard hierarchy classifies deductive systems. Protoalgebraic π-institutions were introduced in [27], equivalential π-institutions in [29], weakly algebraizable π-institutions in [30] and syntactically N-algebraizable π-institutions in [34]. The introduction of these classes and their study leads naturally to the attempt, initiated in [33] and continued in the present work, of studying selfextensional π-institutions and their properties with the hope that they will provide in the categorical theory insights as valuable as those provided by the study of selfextensional deductive systems.

In the remainder of this introduction, we summarize a few facts concerning protoalgebraic π-institutions, that constitute the widest of the main classes in the categorical abstract hierarchy of π-institutions, and revisit a few concepts from the theory of models of π-institutions [23] that will be needed to define full N-selfextensionality and full N-Fregeanity.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, is said to be N-*protoalgebraic*, if, for every theory family T of $\mathcal{I}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega^N_\Sigma(T) \quad \text{implies} \quad C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi).$$

A characterization of N-protoalgebraicity using the N-Leibniz operator states that $\mathcal{I}$ is N-protoalgebraic if and only if the N-Leibniz operator is monotone on the collection of all theory families of $\mathcal{I}$, i.e., if and only if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, if $T \leq T'$, then

$\Omega^N(T) \leq \Omega^N(T')$. The syntactic characterization of protoalgebraicity for deductive systems, given above, is not valid in the context of π-institutions. The existence of a collection Δ of binary natural transformations on SEN in N, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

1. $\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset)$ and
2. $\psi \in C_\Sigma(\phi, \Delta_\Sigma(\phi, \psi))$

provides only a sufficient condition for $\mathcal{I}$ to be N-protoalgebraic rather than characterizing N-protoalgebraicity (see proposition 3.2 of [28]).

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, an (N, N')-*model of* $\mathcal{I}$ is a π-institution $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on SEN', together with an (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. The π-institution $\mathcal{I}'$ is the $\langle F, \alpha\rangle$-*min* (N, N')-*model of* $\mathcal{I}$ *on* SEN' if, for every C'', such that $\mathcal{I}'' = \langle \mathbf{Sign}', \mathrm{SEN}', C''\rangle$ is also a model of $\mathcal{I}$ via $\langle F, \alpha\rangle$, we have that $C' \leq C''$. An (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ via an (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ will be said to be a *full* (N, N')-*model of* $\mathcal{I}$ if its Tarski-reduction $\mathcal{I}'^{N'} = \langle \mathbf{Sign}', \mathrm{SEN}'^{N'}, C'^{N'}\rangle$ is the $\langle F, \pi_F^{N'}\alpha\rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$.

$$\mathcal{I} \xrightarrow{\langle F, \alpha\rangle} \mathcal{I}' \xrightarrow{\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{N'}\rangle} \mathcal{I}'^{N'}$$

A strengthening of the notion of an N-selfextensional π-institution is that of a fully N-selfextensional π-institution and a strengthening of the notion of an N-Fregean π-institution is that of a fully N-Fregean π-institution. Both terms are analogs of corresponding notions pertaining to deductive systems [11]. A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be *fully N-selfextensional* if and only if, for every full (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism, the π-institution $\mathcal{I}'$ is N'-selfextensional. Similarly, $\mathcal{I}$ is *fully N-Fregean*, if, for every full (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism, $\mathcal{I}'$ is N'-Fregean. Babyonyshev [1] has shown that, for deductive systems, the corresponding notions are such that fully selfextensional deductive systems form a proper subclass of all selfextensional deductive systems and that fully Fregean deductive systems form a proper subclass of Fregean deductive systems. Since deductive systems may be easily recast as π-institutions, his results also show that the notion of full N-selfextensionality is different from that of N-selfextensionality in general and the same applies to the notion of full N-Fregeanity versus N-Fregeanity.

Many of the selfextensional deductive systems $\mathcal{S}$ that have been studied in the literature have a conjunction, i.e., a binary term $\wedge$, such that the following rules

$$\phi, \psi \vdash_{\mathcal{S}} \phi \wedge \psi, \quad \phi \wedge \psi \vdash_{\mathcal{S}} \phi, \quad \phi \wedge \psi \vdash_{\mathcal{S}} \psi$$

are either primitive or derived rules of $\mathcal{S}$. This has led, first, Font and Jansana in [11] and, later, Jansana in [17] to study selfextensional logics with conjunction in the context of Abstract Algebraic Logic. The main methodological difference between the two treatments is that [11] uses the tool of Gentzen systems for its studies, whereas [17] revisits many of the results already proven in [11] but provides alternative proofs avoiding the use of Gentzen systems.

Our main goal in this work is to follow the work of Jansana [17] and provide generalizations of many of his results for N-selfextensional π-institutions with an N-conjunction without using machinery from the theory of Gentzen systems, despite the fact that, by now, the categorical theory has been expanded enough to also contain analogs of many of the results of [11] as applying to Gentzen systems (see, e.g., [25]). Some of the main results that will be proven in the present paper are summarized next. All relevant definitions will be given in detail in subsequent sections. The following list is only meant as a quick preview.

1. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a symmetrically N-rule based π-institution that is surjectively N-semilattice based and has theorems. Then the class of N-algebraic systems $\mathrm{Alg}^N(\mathcal{I})$ constitutes the N-core $\mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}})$ of the intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ of the π-institution $\mathcal{I}$.

2. If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a symmetrically N-rule based π-institution that is N-selfextensional and has an N-conjunction, then $\mathcal{I}$ is fully N-selfextensional.

3. Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, be a symmetrically N-rule based functor and $\wedge$ a binary natural transformation in N. Then there exists a dual isomorphism between the set of all non-pseudo-axiomatic, surjectively N-semilattice based π-institutions on SEN that have $\wedge$ as an N-conjunction, ordered by extension, and the set of all (SEN, N)-surjective subvarieties of the variety of N-algebraic systems axiomatized by the semilattice N-equations,
$$x \wedge x \approx x, \quad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z, \quad x \wedge y \approx y \wedge x,$$
ordered by inclusion.

Besides these three main results, a characterization will be provided of the symmetrically N-rule based and N-selfextensional π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ with an N-conjunction $\wedge$ as being exactly those π-institutions for which there exists a class $\mathtt{K}$ of N-algebraic systems that satisfies the semilattice equations relative to $\wedge$ and, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$, $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$ if and only, for every N-algebraic system $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and every surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$,
$$\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi).$$

2 Preliminaries

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor and N a category of natural transformations on SEN. When such a distinguished functor is under consideration, all varieties or quasi-varieties that will be discussed will be varieties or quasi-varieties of N-algebraic systems defined by collections of N-equations or N-quasi-equations in the sense of [32]. As a consequence, if $\mathtt{K}$ is a class of N-algebraic systems and $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle, \mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F'' \rangle \rangle$ are two N-algebraic systems in $\mathtt{K}$, by an N-**morphism** $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{B}$ we will always mean an (N', N'')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN}' \to \mathrm{SEN}''$, such that the following triangle commutes

$$\begin{tikzcd} & N \arrow[dl, "F'"'] \arrow[dr, "F''"] & \\ N' \arrow[rr, dashed, no head] & & N'' \end{tikzcd}$$

where the dotted line denotes the correspondence established between N' and N'' by the (N', N'')-epimorphic property of $\langle F, \alpha \rangle$, i.e., given any $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, it will always be assumed that $\sigma' := F'(\sigma)$ and $\sigma'' := F''(\sigma)$ correspond under the (N', N'')-epimorphic property of $\langle F, \alpha \rangle$.

The basic logical structures that will serve as the underlying structure of our investigations are π-institutions. They were introduced in [10] as a modification of institutions [12, 13] with the intention of keeping the category-theoretic institutional framework that successfully handles the syntactic aspects of multi-signature logical systems while, at the same time, stripping the institution formalism from its model-theoretic content. Recall from [10] that a π-**institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a triple consisting of

(i) a category **Sign**, whose objects are called **signatures**;

(ii) a set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ from the category **Sign** of signatures, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences** over that signature Σ or Σ-**sentences**;

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called Σ-**closure**, such that

   (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,
   (b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,
   (c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,
   (d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is said to be **finitary**, if, for all $\Sigma \in |\mathbf{Sign}|$, $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a finitary closure operator in the usual sense, i.e., if, for every $\Sigma \in |\mathbf{Sign}|$ and every $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Phi') : \Phi' \subseteq_\omega \Phi\},$$

where by $\subseteq_\omega$ is denoted the finite subset relation.

Given a functor SEN : **Sign** $\rightarrow$ **Set**, a collection $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that θ_Σ is an equivalence relation on SEN(Σ), for all $\Sigma \in |\mathbf{Sign}|$, is called an **equivalence family on** SEN. If, in addition, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, θ satisfies

$$\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2},$$

then θ is said to be an **equivalence system on** SEN. If N is a category of natural transformations on SEN and an equivalence system θ on SEN satisfies, for all $\sigma : \mathrm{SEN}^n \rightarrow \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi_i, \psi_i \rangle \in \theta_\Sigma,\ i < n, \quad \text{imply} \quad \langle \sigma_\Sigma(\phi_0, \ldots, \phi_{n-1}), \sigma_\Sigma(\psi_0, \ldots, \psi_{n-1}\rangle \in \theta_\Sigma,$$

then θ is a said to be an N-**congruence system on** SEN.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, one may associate with $\mathcal{I}$ an N-congruence system and an equivalence system that have played very significant roles in the Abstract Algebraic Logic literature in classifying sentential logics and π-institutions. Given $\Sigma \in |\mathbf{Sign}|$, a Σ-**theory** of $\mathcal{I}$ is a subset $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, such that $C_\Sigma(T_\Sigma) = T_\Sigma$. A **theory family** $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$ is a collection of Σ-theories of $\mathcal{I}$, $\Sigma \in |\mathbf{Sign}|$. A theory family T is called a **theory system** if, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}$. Notice that this terminology conforms with the one introduced for equivalence families/systems on SEN, given previously. The collection of all theory families of $\mathcal{I}$ is denoted ThFam($\mathcal{I}$). Ordered by signature-wise inclusion, which is denoted by $\leq$, it forms a complete lattice, which is denoted by $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq\rangle$. The collection of all theory systems of $\mathcal{I}$ is denoted by ThSys($\mathcal{I}$) and forms a complete sublattice of $\mathbf{ThFam}(\mathcal{I})$, denoted by $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq\rangle$.

The **Tarski** N-**congruence system** $\widetilde{\Omega}^N(\mathcal{I})$ of $\mathcal{I}$ is the largest N-congruence system on SEN that is compatible with every theory family of $\mathcal{I}$ in the sense that, for every theory family $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}) \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Such an N-congruence system is called a **logical** N-**congruence system** of $\mathcal{I}$. The **Frege equivalence system** of $\mathcal{I}$ is the equivalence system $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, by

$$\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

Note that the Tarski N-congruence system of $\mathcal{I}$ is the largest N-congruence system of $\mathcal{I}$ that is included in the Frege equivalence system of $\mathcal{I}$. In [26] a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, was called N-**selfextensional** if its Frege equivalence system is an N-congruence system. Since the Tarski N-congruence system of $\mathcal{I}$ is the largest N-congruence system of $\mathcal{I}$ that is included in the Frege equivalence system of $\mathcal{I}$, $\mathcal{I}$ being N-selfextensional is equivalent to the condition that $\widetilde{\Omega}^N(\mathcal{I}) = \Lambda(\mathcal{I})$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called **fully N-selfextensional** if, for every full (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, $\mathcal{I}'$ is N'-selfextensional.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the **Frege operator** $\Lambda^{\mathcal{I}}$ maps an axiom family $F = \{F_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$ to the equivalence system $\Lambda^{\mathcal{I}}(F) = \{\Lambda^{\mathcal{I}}_\Sigma(F)\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$ that is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$ by

$$\begin{array}{ll}\langle \phi, \psi\rangle \in \Lambda^{\mathcal{I}}_\Sigma(F) \quad \text{iff} & C_{\Sigma'}(F_{\Sigma'} \cup \{\mathrm{SEN}(f)(\phi)\}) = C_{\Sigma'}(F_{\Sigma'} \cup \{\mathrm{SEN}(f)(\psi)\}), \\ & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma').\end{array}$$

If F happens to be an axiom system (rather than simply an axiom family), we have $\Lambda^{\mathcal{I}}_\Sigma(F) = \Lambda_\Sigma(\mathcal{I}^F)$, where $\mathcal{I}^F = \langle \mathbf{Sign}, \mathrm{SEN}, C^F\rangle$ is given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi\cup\{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\phi \in C^F_\Sigma(\Phi) \quad \text{iff} \quad \phi \in C_\Sigma(F_\Sigma \cup \Phi).$$

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called **N-Fregean** if, for every theory family T of $\mathcal{I}$, $\Lambda^{\mathcal{I}}(T)$ is an N-congruence system on SEN. Of course, by considering the theorem system $\mathrm{Thm} = \{\mathrm{Thm}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|} := \{C_\Sigma(\emptyset)\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$ it is easy to see that, if $\mathcal{I}$ is N-Fregean, then it is also N-selfextensional.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. A closure system C on SEN and the corresponding π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ are said to be **N-rule based** if, for all $\Sigma \in |\mathbf{Sign}|, \Phi\cup\{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists an N-rule $\langle X, \sigma\rangle$ of C of length at most $|\Phi|^+$, and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\psi}) \subseteq \Phi$ and $\sigma_\Sigma(\vec{\psi}) = \phi$, i.e., such that ϕ follows from Φ by an application of $\langle X, \sigma\rangle$. This definition of an N-rule based π-institution was borrowed from [31], where it was used as a platform to discuss a generalized version of Bloom's Theorem for π-institutions. The reader may consult that paper for the definition of an N-rule and for many more details on these two concepts.

A finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ will be said to be **symmetrically N-rule based** if it is N-rule based and, in addition, if, for some $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$, then, there exist natural transformations $\sigma^{\langle\Sigma,\phi\rangle}, \sigma^{\langle\Sigma,\psi\rangle} : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{\chi}) = \phi, \sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{\chi}) = \psi$ and $\langle\{\sigma^{\langle\Sigma,\phi\rangle}\}, \sigma^{\langle\Sigma,\psi\rangle}\rangle$, $\langle\{\sigma^{\langle\Sigma,\psi\rangle}\}, \sigma^{\langle\Sigma,\phi\rangle}\rangle$ are both N-rules of $\mathcal{I}$.

A set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, is said to be **symmetrically N-rule based** if, for every finitary closure system C on SEN, the π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is symmetrically N-rule based.

In the last part of this section, we will briefly revisit some of the definitions and results concerning the two classes $\mathtt{K}^N_{\mathcal{I}}$ and $\mathrm{Alg}^N(\mathcal{I})$ of N-algebraic systems associated with a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN. All the results mentioned here without proof are proven in detail in Section 2 of [33].

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Consider the triple $\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle$, where $\mathrm{SEN}^N : \mathbf{Sign} \to \mathbf{Set}$ is the quotient functor $\mathrm{SEN}/\widetilde{\Omega}^N(\mathcal{I})$, $\overline{N}$ is the quotient category of N by $\widetilde{\Omega}^N(\mathcal{I})$ and $\overline{F} : N \to \overline{N}$ maps a natural transformation $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N to its quotient $\overline{\sigma} : (\mathrm{SEN}^N)^n \to \mathrm{SEN}^N$. All these concepts were defined in [22], where they were shown to be well-defined. The triple $\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle$ is an N-algebraic system. The variety that it generates in the sense of [32] will be denoted by $\mathtt{K}^N_{\mathcal{I}}$ and will be called, by analogy with the intrinsic variety $\mathtt{K}_{\mathcal{S}}$ associated with a deductive system $\mathcal{S}$, the **intrinsic** N**-variety** of the π-institution $\mathcal{I}$. An N-equation $\sigma \approx \tau$, with $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, is an N-identity of the intrinsic variety $\mathtt{K}^N_{\mathcal{I}}$ of a π-institution $\mathcal{I}$ if and only if, for every $\lambda : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n, \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})), \vec{\chi})) = C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})), \vec{\chi})). \tag{1}$$

Note that Equation (1) abbreviates the following sets of equations, for all $i < k$:

$$\begin{aligned} &C_{\Sigma'}(\lambda_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})), \chi_{i+1}, \ldots, \chi_{k-1}) = \\ &\qquad\qquad C_{\Sigma'}(\lambda_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})), \chi_{i+1}, \ldots, \chi_{k-1})). \end{aligned}$$

The abbreviating convention in Equation (1) will be followed throughout the paper when it is convenient to shorten the longer expressions that it represents.

Proposition 1 (Proposition 1 of [33]) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*. Then, for every $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\mathtt{K}^N_{\mathcal{I}} \models \sigma \approx \tau$ if and only if, for every $\lambda : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n, \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,*

$$C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})), \vec{\chi})) = C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})), \vec{\chi})).$$

From the proof of Proposition 1, we also infer that, if $\mathcal{I}$ happens to be N-selfextensional, then

$$\begin{array}{lll} \mathtt{K}^N_{\mathcal{I}} \models \sigma \approx \tau & \text{iff} & \langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}), \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n, \\ & \text{iff} & \langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \Lambda_\Sigma(\mathcal{I}), \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n, \\ & \text{iff} & C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi})) \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n. \end{array} \tag{2}$$

The class $\mathrm{Alg}^N(\mathcal{I})$ was also defined in [33]. A class with the same name had been defined in [24], but the definition of [33] was modified so as to require model morphisms to be surjective, a requirement not imposed in [24].

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. The N-algebraic system $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ is said to be an $(\mathcal{I}, N)$-**algebraic system** if and only if there exists a surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-reduced, i.e., iff $\mathcal{I}'$ is a reduced (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha\rangle$. Let $\mathrm{Alg}^N(\mathcal{I})$ denote the class of all $(\mathcal{I}, N)$-algebraic systems.

The next proposition relates the two classes $\mathtt{K}^N_{\mathcal{I}}$ and $\mathrm{Alg}^N(\mathcal{I})$. More specifically, it states that $\mathrm{Alg}^N(\mathcal{I})$ is a subclass of $\mathtt{K}^N_{\mathcal{I}}$ and that, moreover, the class $\mathtt{K}^N_{\mathcal{I}}$ is the variety of N-algebraic systems that is generated by the class $\mathrm{Alg}^N(\mathcal{I})$.

Proposition 2 (Proposition 2 of [33]) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_{\mathcal{I}}$ and, moreover, $\mathtt{K}^N_{\mathcal{I}} = \mathbf{V}^N(\mathrm{Alg}^N(\mathcal{I}))$, where $\mathbf{V}^N$ denotes the variety operator (which was shown in Theorem 4 of [32], an analog of Birkhoff's Theorem, to be equal to the operator* **HSP**).

Observe, now, that Proposition 2 yields the following interesting corollary:

Corollary 3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then, if the class $\mathrm{Alg}^N(\mathcal{I})$ of all $(\mathcal{I}, N)$-algebraic systems is a variety, it is necessarily equal to the intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ of $\mathcal{I}$.*

3 Semilattice-Based π-Institutions

In this section, following the work of Jansana [17], we introduce and study the notion of a semilattice-based π-institution. It is shown that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is N-semilattice-based if and only if it is N-selfextensional and has an N-conjunction.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. A natural transformation $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N is said to be an N-**conjunction** of $\mathcal{I}$ if,

1. for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \wedge_\Sigma \psi \in C_\Sigma(\phi, \psi)$ and
2. for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi, \psi \in C_\Sigma(\phi \wedge_\Sigma \psi)$.

It is not difficult to see that if $\mathcal{I}$ has two N-conjunctions $\wedge$ and $\wedge'$, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, the two Σ-sentences $\phi \wedge_\Sigma \psi$ and $\phi \wedge'_\Sigma \psi$ are Σ-interderivable in $\mathcal{I}$.

Lemma 4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN, *and that $\wedge, \wedge' : \mathrm{SEN}^2 \to \mathrm{SEN}$ are two N-conjunctions of $\mathcal{I}$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi \wedge_\Sigma \psi) = C_\Sigma(\phi \wedge'_\Sigma \psi)$.*

Proof:
Since both $\wedge$ and $\wedge'$ are N-conjunctions of $\mathcal{I}$, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi \wedge_\Sigma \psi) = C_\Sigma(\phi, \psi) = C_\Sigma(\phi \wedge'_\Sigma \psi)$. ■

$\mathcal{I}$ is said to be N-**conjunctive** if it has an N-conjunction.

Lemma 4 together with the well-known characterization of the Tarski N-congruence system $\widetilde{\Omega}^N(\mathcal{I})$ of a π-institution $\mathcal{I}$ (see [22]) give the following

Proposition 5 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is N-selfextensional with two N-conjunctions $\wedge, \wedge' : \mathrm{SEN}^2 \to \mathrm{SEN}$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$ and all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$, we have that*

$$C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi) \wedge_{\Sigma'} \mathrm{SEN}(f)(\psi), \vec{\chi})) = C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi) \wedge'_{\Sigma'} \mathrm{SEN}(f)(\psi), \vec{\chi})).$$

Proof:

Since $\mathcal{I}$ is N-selfextensional, we have that $\Lambda(\mathcal{I}) = \widetilde{\Omega}^N(\mathcal{I})$, i.e., that the Frege relation system $\Lambda(\mathcal{I})$ of $\mathcal{I}$ is an N-congruence system. The result now follows by Lemma 4 and the N-congruence system property of $\Lambda(\mathcal{I})$. ■

Some of the critical properties of an N-conjunction are "transferred" from a π-institution $\mathcal{I}$ to all its (N, N')-models via surjective (N, N')-logical morphisms. One of them is the defining property of the N-conjunction. The fact that it transfers asserts that all the (N, N')-models of $\mathcal{I}$ via surjective (N, N')-logical morphisms are N'-conjunctive.

Proposition 6 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*, that has an N-conjunction $\wedge$. If $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN$'$*, is a model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, then, for all $\Sigma' \in |\mathbf{Sign}'|$ and all $\phi', \psi' \in$* $\mathrm{SEN}'(\Sigma')$*, we have*

1. $C'_{\Sigma'}(\phi', \psi') = C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi')$ *and*
2. $\phi' \in C'_{\Sigma'}(\psi')$ *if and only if* $C'_{\Sigma'}(\psi') = C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi')$.

Proof:

1. Suppose that $\Sigma' \in |\mathbf{Sign}'|$ and $\phi', \psi' \in \mathrm{SEN}'(\Sigma')$. Then, by the surjectivity of $\langle F, \alpha\rangle$, we have that, there exist $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\phi) = \phi', \alpha_\Sigma(\psi) = \psi'$. Since $\wedge$ is an N-conjunction of $\mathcal{I}$, we have that $C_\Sigma(\phi, \psi) = C_\Sigma(\phi \wedge_\Sigma \psi)$. Therefore, since $\langle F, \alpha\rangle$ is an (N, N')-logical morphism, we get that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi \wedge_\Sigma \psi))$ and $\alpha_\Sigma(\phi \wedge_\Sigma \psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi))$. These are equivalent, respectively, to $\phi', \psi' \in C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi')$ and $\phi' \wedge'_{\Sigma'} \psi' \in C'_{\Sigma'}(\phi', \psi')$. Therefore, we obtain that $C'_{\Sigma'}(\phi', \psi') = C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi')$.
2. Part 2 follows easily from Part 1.

■

Suppose, now, that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, with N a category of natural transformations on SEN, and $\mathtt{K}$ is a class of N-algebraic systems. $\mathtt{K}$ is said to be N-**semilattice-based** if there exists a $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, such that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, with $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi, \psi, \chi \in \mathrm{SEN}'(\Sigma)$, we have that

1. $\phi \wedge'_\Sigma \phi = \phi$
2. $\phi \wedge'_\Sigma (\psi \wedge'_\Sigma \chi) = (\phi \wedge'_\Sigma \psi) \wedge'_\Sigma \chi$
3. $\phi \wedge'_\Sigma \psi = \psi \wedge'_\Sigma \phi$,

where, of course, by $\wedge' : \mathrm{SEN}'^2 \to \mathrm{SEN}'$ is denoted, as usual, the natural transformation in N', with $\wedge' := F'(\wedge)$. In this case, we will say that $\mathtt{K}$ is **N-semilattice-based relative to $\wedge$** or that $\mathtt{K}$ is an **N-semilattice class relative to $\wedge$**.

For every $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, with $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, a posystem $\leq' = \{\leq'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ may be defined on SEN', by letting, for all $\Sigma \in |\mathbf{Sign}'|$, $\leq'_\Sigma$ be given by

$$\phi \leq'_\Sigma \psi \quad \text{iff} \quad \phi \wedge'_\Sigma \psi = \phi, \quad \text{for all } \phi, \psi \in \mathrm{SEN}'(\Sigma).$$

Given an N-semilattice-based class $\mathtt{K}$ relative to $\wedge$ and a $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, an axiom family or axiom system $F = \{F_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ on SEN' is said to be an **N-semilattice filter family** or an **N-semilattice filter system**, respectively, if

1. $F_\Sigma \neq \emptyset$, for all $\Sigma \in |\mathbf{Sign}'|$,
2. $\phi, \psi \in F_\Sigma$ imply that $\phi \wedge'_\Sigma \psi \in F_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi, \psi \in \mathrm{SEN}'(\Sigma)$,
3. if $\phi \leq'_\Sigma \psi$ and $\phi \in F_\Sigma$, then $\psi \in F_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi, \psi \in \mathrm{SEN}'(\Sigma)$.

Given $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, as before, and, for all $\Sigma \in |\mathbf{Sign}'|$, $\phi_\Sigma \in \mathrm{SEN}'(\Sigma)$, the collection $[\phi_\Sigma)_{\Sigma \in |\mathbf{Sign}'|}$, where $[\phi_\Sigma) = \{\psi \in \mathrm{SEN}'(\Sigma) : \phi_\Sigma \leq'_\Sigma \psi\}$, for all $\Sigma \in |\mathbf{Sign}'|$, is an N-semilattice filter family of $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, called the **$N$-semilattice filter family generated by** the $\{\phi_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$.

A finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is **N-semilattice-based** if there exists a natural transformation $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N and a class $\mathtt{K}$ of N-algebraic systems, such that $\mathtt{K}$ is N-semilattice-based relative to $\wedge$ and, for all $\Sigma \in |\mathbf{Sign}|$, all $n > 0$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \text{iff} & \alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi), \\ & & \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K} \\ & & \text{and all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'. \end{array} \tag{3}$$

If this is the case $\mathcal{I}$ is said to be **N-semilattice-based relative to $\wedge$ and $\mathtt{K}$**. If $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}$ in such a way that, for every $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, there exists at least one surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, then $\mathcal{I}$ will be said to be **surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{K}$**. It is simply called **surjectively N-semilattice-based** if it is surjectively N-semilattice-based relative to some $\wedge$ and $\mathtt{K}$.

Condition (3) implies that, if $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}$, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} C_\Sigma(\phi) = C_\Sigma(\psi) & \text{iff} & \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi), \text{ for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K} \\ & & \text{and all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'. \end{array}$$

This remark immediately yields:

Proposition 7 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a surjectively N-semilattice-based π-institution relative to $\wedge$ and* K. *Then, for all $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, we have that*

$$\mathtt{K} \models \sigma \approx \tau \quad \textit{iff} \quad C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi})), \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n.$$

Proof:
Let $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ be in N. We have that $\mathtt{K} \models \sigma \approx \tau$ if and only if, by definition, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\vec{\phi'} \in \mathrm{SEN}'(\Sigma')^n$, $\sigma'_{\Sigma'}(\vec{\phi'}) = \tau'_{\Sigma'}(\vec{\phi'})$, which holds iff, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, $\sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))$, which is equivalent to $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau_\Sigma(\vec{\phi}))$, which, since $\mathcal{I}$ is N-semilattice based relative to $\wedge$ and K, holds, by the preceding remark, if and only if $C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi}))$. ■

Recall now that a finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be symmetrically N-rule based if it is N-rule based and, in addition, if, for some $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$, then, there exist natural transformations $\sigma^{\langle\Sigma,\phi\rangle}, \sigma^{\langle\Sigma,\psi\rangle} : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{\chi}) = \phi, \sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{\chi}) = \psi$ and $\langle\{\sigma^{\langle\Sigma,\phi\rangle}\}, \sigma^{\langle\Sigma,\psi\rangle}\rangle$, $\langle\{\sigma^{\langle\Sigma,\psi\rangle}\}, \sigma^{\langle\Sigma,\phi\rangle}\rangle$ are both N-rules of $\mathcal{I}$.

Proposition 8 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a symmetrically N-rule based π-institution that is surjectively N-semilattice-based relative to $\wedge$ and* K. *Then, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \mathtt{K} \models \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}.$$

Proof:
Suppose, first, that $C_\Sigma(\phi) = C_\Sigma(\psi)$. Then, since both $\langle\{\sigma^{\langle\Sigma,\phi\rangle}\}, \sigma^{\langle\Sigma,\psi\rangle}\rangle$ and $\langle\{\sigma^{\langle\Sigma,\psi\rangle}\}, \sigma^{\langle\Sigma,\phi\rangle}\rangle$ are N-rules of $\mathcal{I}$, we get that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{\chi})) = C_\Sigma(\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{\chi}))$. Therefore, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, we get that $\alpha_\Sigma(\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{\chi})) = \alpha_\Sigma(\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{\chi}))$. Thus, we get that $\sigma^{\langle\Sigma,\phi\rangle'}_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) = \sigma^{\langle\Sigma,\psi\rangle'}_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))$. Since $\langle F, \alpha\rangle$ is surjective, we obtain $\mathtt{K} \models \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}$.

Suppose, conversely, that $\mathtt{K} \models \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}$. Then, for all $\Sigma' \in |\mathbf{Sign}|$, all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$, all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, we get that $\sigma^{\langle\Sigma,\phi\rangle'}_{F(\Sigma')}(\alpha_{\Sigma'}(\vec{\chi})) = \sigma^{\langle\Sigma,\psi\rangle'}_{F(\Sigma')}(\alpha_{\Sigma'}(\vec{\chi}))$, which implies that $\alpha_{\Sigma'}(\sigma^{\langle\Sigma,\phi\rangle}_{\Sigma'}(\vec{\chi})) = \alpha_{\Sigma'}(\sigma^{\langle\Sigma,\psi\rangle}_{\Sigma'}(\vec{\chi}))$. In particular, we get that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, whence, since $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and K, we get that $C_\Sigma(\phi) = C_\Sigma(\psi)$. ■

Given a class K of N-algebraic systems, recall from [32] the variety of N-algebraic systems generated by K. This variety will be denoted by $\mathbf{V}^N(\mathtt{K})$ in the sequel. By one of the main results of [32], forming an analog of the well-known Birkhoff's Theorem of Universal Algebra for N-algebraic systems, $\mathbf{V}^N(\mathtt{K})$ is generated by taking homomorphic images of subsystems of direct products of the N-algebraic systems in $\mathtt{K}$. Using this terminology, Proposition 8 has the following corollary:

Corollary 9 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a symmetrically N-rule based π-institution that is surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{K}$. Then $\mathcal{I}$ is also N-semilattice-based relative to $\wedge$ and $\mathbf{V}^N(\mathtt{K})$.*

Proof:

Since $\mathbf{V}^N(\mathtt{K})$ satisfies exactly the same N-equations as $\mathtt{K}$, $\mathtt{K}$ is N-semilattice based relative to $\wedge$ and the N-semilattice property is defined by N-equations, we conclude that $\mathbf{V}^N(\mathtt{K})$ is also N-semilattice based relative to $\wedge$.

Clearly, since $\mathtt{K} \subseteq \mathbf{V}^N(\mathtt{K})$, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$, that, if, for every $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathbf{V}^N(\mathtt{K})$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi)$, then $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$ are such that $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$. Then we have that $C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})$. Then, by Proposition 8, $\mathtt{K} \models \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle} \approx \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}$. Thus, we get that $\mathbf{V}^N(\mathtt{K}) \models \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle} \approx \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}$. This shows that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathbf{V}^N(\mathtt{K})$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, we have that $\sigma_{F(\Sigma)}^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle'}(\alpha_\Sigma(\vec{\chi})) = \sigma_{F(\Sigma)}^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle'}(\alpha_\Sigma(\vec{\chi}))$, which yields that

$$\alpha_\Sigma(\sigma_\Sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle}(\vec{\chi})) = \alpha_\Sigma(\sigma_\Sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}(\vec{\chi}))$$

and, therefore, $\alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = \alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})$. Since $\mathbf{V}^N(\mathtt{K})$ is N-semilattice based relative to $\wedge$, we get that $\alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi)$. ■

Next, it will be shown that if $\mathcal{I}$ is a symmetrically N-rule based π-institution that is surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{K}$ and, also, surjectively N-semilattice-based relative to $\wedge'$ and $\mathtt{K}'$, then we must have that $\mathbf{V}^N(\mathtt{K}) = \mathbf{V}^N(\mathtt{K}')$.

Proposition 10 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a symmetrically N-rule based π-institution that is surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{K}$ and, also, surjectively N-semilattice-based relative to $\wedge'$ and $\mathtt{K}'$. Then $\mathbf{V}^N(\mathtt{K}) = \mathbf{V}^N(\mathtt{K}')$.*

Proof:

It suffices to show that, for all $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, we have that $\mathbf{V}^N(\mathtt{K}) \models \sigma \approx \tau$ if and only if $\mathbf{V}^N(\mathtt{K}') \models \sigma \approx \tau$. We indeed have

$$\begin{array}{lll}
\mathbf{V}^N(\mathtt{K}) \models \sigma \approx \tau & \text{iff} & \mathtt{K} \models \sigma \approx \tau \quad \text{(by definition)} \\
& \text{iff} & C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi})) \\
& & \text{for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n \quad \text{(by Proposition 7)} \\
& \text{iff} & \mathtt{K}' \models \sigma \approx \tau \quad \text{(by Proposition 7)} \\
& \text{iff} & \mathbf{V}^N(\mathtt{K}') \models \sigma \approx \tau. \quad \text{(by definition)}
\end{array}$$

$\blacksquare$

The unique variety of N-algebraic systems generated by any class $\mathtt{K}$ relative to which $\mathcal{I}$ is surjectively N-semilattice based, as specified by Proposition 10, will be denoted by $\mathbf{V}^N(\mathcal{I})$.

It will be shown in the next proposition that every N-semilattice-based π-institution relative to $\wedge$ is N-selfextensional and N-conjunctive relative to $\wedge$. This result will help in Theorem 12 to show that if $\mathcal{I}$ is a symmetrically N-rule based π-institution that is surjectively N-semilattice-based, then the class $\mathbf{V}^N(\mathcal{I})$ is the intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ of $\mathcal{I}$.

Proposition 11 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-semilattice-based π-institution relative to $\wedge$. Then*

1. *$\mathcal{I}$ is N-selfextensional;*
2. *$\wedge$ is an N-conjunction of $\mathcal{I}$.*

Proof:

Suppose that $\mathcal{I}$ is N-semilattice-based relative to $\wedge$. Thus, there exists a class $\mathtt{K}$ of N-algebraic systems, such that $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}$. This means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we have that

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \text{iff} \quad \begin{array}{l} \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi), \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K} \\ \text{and all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'. \end{array}$$

This implies that, if $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $C_\Sigma(\phi) = C_\Sigma(\psi)$, then, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$\sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) = \sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})).$$

Therefore, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, $\alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}))$. Since this is true for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, we get that

$$C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})).$$

Therefore, by Theorem 4 of [22], $\langle \phi, \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$ and $\mathcal{I}$ is N-selfextensional.

For the second statement, by the assumption that $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}$, we get that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi \wedge_\Sigma \psi) = \alpha_\Sigma(\phi) \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi), \alpha_\Sigma(\psi).$$

Therefore, we have that $C_\Sigma(\phi \wedge_\Sigma \psi) = C_\Sigma(\phi, \psi)$, i.e., that $\wedge$ is an N-conjunction of $\mathcal{I}$. $\blacksquare$

Proposition 11 will be used now to show that, if $\mathcal{I}$ is a symmetrically N-rule based π-institution that is surjectively N-semilattice-based, then the class $\mathbf{V}^N(\mathcal{I})$ is the intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ of $\mathcal{I}$.

Theorem 12 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a symmetrically N-rule based π-institution that is surjectively N-semilattice-based. Then, $\mathbf{V}^N(\mathcal{I}) = \mathtt{K}^N_{\mathcal{I}}$.*

Proof:
Suppose that $\mathcal{I}$ is surjectively N-semilattice based relative to $\wedge$ and $\mathtt{K}$. Since both $\mathbf{V}^N(\mathcal{I})$ and $\mathtt{K}^N_{\mathcal{I}}$ are, by definition, varieties, it suffices to show that, for every $\sigma : \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, we have that $\mathtt{K}^N_{\mathcal{I}} \models \sigma \approx \tau$ if and only if $\mathbf{V}^N(\mathcal{I}) \models \sigma \approx \tau$. We have that $\mathtt{K}^N_{\mathcal{I}} \models \sigma \approx \tau$ if and only if, by Proposition 1 (Proposition 1 of [33]), for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Since, by Proposition 11, $\mathcal{I}$ is N-selfextensional, this holds if and only if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \Lambda_\Sigma(\mathcal{I})$, i.e., $C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi}))$. But, by Proposition 7, this is equivalent to $\mathtt{K} \models \sigma \approx \tau$, which holds if and only if $\mathbf{V}^N(\mathcal{I}) \models \sigma \approx \tau$. ■

In the following theorem, Theorem 15, one of the main theorems of the paper, it is shown that a symmetrically N-rule based π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where N is a category of natural transformations on SEN, is N-semilattice-based if and only if it is N-selfextensional and N-conjunctive. This is an analog of Theorem 3.2 of [17] for π-institutions. Lemma 13 (along with its Corollary 14), that precedes the main theorem, provides a technical result that will be used in the proof of Theorem 15

Lemma 13 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a symmetrically N-rule based, N-selfextensional and N-conjunctive π-institution relative to $\wedge$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) \quad \textit{iff} \quad \langle \mathrm{SEN}^N, \langle \overline{N}, -^N\rangle\rangle \models \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle} \approx \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}.$$

Proof:
We have that $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$ if and only if, since $\mathcal{I}$ is N-conjunctive relative to $\wedge$, $\phi \in C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})$ if and only if, for the same reason, $C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})$ if and only if, since $\mathcal{I}$ is symmetrically N-rule based, for all $\Sigma' \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^n$, $C_{\Sigma'}(\sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle}_{\Sigma'}(\vec{\chi})) = C_{\Sigma'}(\sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}_{\Sigma'}(\vec{\chi}))$ if and only if, since $\mathcal{I}$ is N-selfextensional, for all $\Sigma' \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^n$, $\langle \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle}_{\Sigma'}(\vec{\chi}), \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}_{\Sigma'}(\vec{\chi})\rangle \in \widetilde{\Omega}^N_{\Sigma'}(\mathcal{I})$, if and only if, for all $\Sigma' \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^n$, $\sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle^N}_{\Sigma'}(\vec{\chi}^N) = \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle^N}_{\Sigma'}(\vec{\chi}^N)$, i.e., if and only if, by definition, $\langle \mathrm{SEN}^N, \langle \overline{N}, -^N\rangle\rangle \models \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle} \approx \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}$. ■

Lemma 13 gives immediately the following corollary that will provide the exact form in which its content will be used in the proof of Theorem 15.

Corollary 14 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a symmetrically N-rule based, N-selfextensional and N-conjunctive*

π-institution relative to $\wedge$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \textit{iff} & \alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \leq^N_{F(\Sigma)} \alpha_\Sigma(\phi) \\ & & \textit{for all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^N.\end{array}$$

Proof:

The right-to-left implication is clear if one considers the natural $(N, \overline{N})$-epimorphic projection $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^N\rangle : \mathrm{SEN} \rightarrow \mathrm{SEN}^N$. In fact, if the right-hand side of the equivalence holds, then $\phi_0^N \wedge^N_\Sigma \cdots \wedge^N_\Sigma \phi^N_{n-1} \leq^N_\Sigma \phi^N$, whence $(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})^N \leq^N_\Sigma \phi^N$, and, therefore, $\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \leq_\Sigma \phi$, which shows, since $\mathcal{I}$ is N-conjunctive relative to $\wedge$, that $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$.

For the reverse implication, assume that $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$. Then, by Lemma 13, we have that $\langle \mathrm{SEN}^N, \langle \overline{N}, -^N\rangle\rangle \models \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle} \approx \sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}$. Considering the instance of this N-equation for Σ and $\vec{\chi}$, such that $\sigma_\Sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle}(\vec{\chi}) = \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi$ and $\sigma_\Sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}(\vec{\chi}) = \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}$, we obtain that, for all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^N$, $\sigma_{F(\Sigma)}^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle^N}(\alpha_\Sigma(\vec{\chi})) = \sigma_{F(\Sigma)}^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle^N}(\alpha_\Sigma(\vec{\chi}))$. Hence, for all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^N$,

$$\alpha_\Sigma(\sigma_\Sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi\rangle}(\vec{\chi})) = \alpha_\Sigma(\sigma_\Sigma^{\langle \Sigma, \phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}\rangle}(\vec{\chi})),$$

which gives, by the choice of $\vec{\chi}$, that $\alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = \alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})$. Thus, $\alpha_\Sigma(\phi_0) \wedge^N_{F(\Sigma)} \cdots \wedge^N_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \wedge^N_{F(\Sigma)} \alpha_\Sigma(\phi) = \alpha_\Sigma(\phi_0) \wedge^N_{F(\Sigma)} \cdots \wedge^N_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1})$, which gives that $\alpha_\Sigma(\phi_0) \wedge^N_\Sigma \cdots \wedge^N_\Sigma \alpha_\Sigma(\phi_{n-1}) \leq^N_{F(\Sigma)} \alpha_\Sigma(\phi)$, i.e., that $\alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \leq^N_{F(\Sigma)} \alpha_\Sigma(\phi)$. This concludes the proof of the corollary. ■

Theorem 15 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ with N a category of natural transformations on* SEN*, is a symmetrically N-rule based π-institution. $\mathcal{I}$ is N-selfextensional and N-conjunctive if and only if it is N-semilattice-based.*

Proof:

The implication from right to left is the content of Proposition 11. For the reverse implication, suppose that $\mathcal{I}$ is a symmetrically N-rule based π-institution, that is N-selfextensional and N-conjunctive relative to $\wedge$. Then, the singleton class

$$\mathtt{K} = \{\langle \mathrm{SEN}^N, \langle \overline{N}, -^N\rangle\rangle\}$$

is an N-semilattice-based class relative to $\wedge$. By Corollary 14, we have that for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \text{iff} & \alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \leq^N_{F(\Sigma)} \alpha_\Sigma(\phi) \\ & & \text{for all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^N,\end{array}$$

which shows that $\mathcal{I}$ is N-semilattice based relative to $\wedge$ and $\mathtt{K}$. ■

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. $\mathcal{I}$ is said to **have theorems** if, for every $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Thm}_\Sigma := C_\Sigma(\emptyset) \neq \emptyset$. It is said to be **non-pseudo-axiomatic** if, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma) \neq \emptyset$ and the set of all its Σ-theorems is the set of all Σ-sentences that are derivable from every Σ-formula, i.e., if $C_\Sigma(\emptyset) = \bigcap_{\phi \in \mathrm{SEN}(\Sigma)} C_\Sigma(\phi)$, for all $\Sigma \in |\mathbf{Sign}|$. This is equivalent to saying that the set of all its Σ-theorems is the intersection of all its nonempty Σ-theories, i.e., that $C_\Sigma(\emptyset) = \bigcap_{\emptyset \neq T \in \mathrm{Th}_\Sigma(\mathcal{I})}(T)$. These definitions are direct generalizations of the corresponding definitions of [17] and will be used to formulate an analog of Lemma 3.3 of [17] showing that a π-institution with theorems is non-pseudo-axiomatic and, moreover, that, if $\mathcal{I}$ is surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{K}$, then, for every N-algebraic system $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ in $\mathtt{K}$ and any of its signatures Σ', the semilattice $\langle \mathrm{SEN}'(\Sigma'), \wedge'_{\Sigma'}\rangle$ has a greatest element which is determined as the image of an appropriately chosen theorem of $\mathcal{I}$ under any surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ (at least one such exists since $\mathcal{I}$ is surjectively N-semilattice based relative to $\mathtt{K}$).

Lemma 16 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a π-institution with theorems. Then $\mathcal{I}$ is non-pseudo-axiomatic. Moreover, if $\mathcal{I}$ is surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{K}$, then, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, with $\mathrm{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$, and all $\Sigma' \in |\mathbf{Sign}'|$, the semilattice $\langle \mathrm{SEN}'(\Sigma'), \wedge'_{\Sigma'}\rangle$ has a greatest element, which is $\alpha_\Sigma(\phi)$, for every surjective $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, every $\Sigma \in |\mathbf{Sign}|$, with $F(\Sigma) = \Sigma'$ and every Σ-theorem ϕ of $\mathcal{I}$.*

Proof:

Obviously, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Thm}_\Sigma \subseteq \bigcap\{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : T \neq \emptyset\}$. For the reverse inclusion, suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \bigcap\{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : T \neq \emptyset\}$. Then, since $\mathcal{I}$ has theorems, we have that $\phi \in \bigcap_{T \in \mathrm{Th}_\Sigma(\mathcal{I})} T = \mathrm{Thm}_\Sigma$.

Finally, let $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, with $\mathrm{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$, $\Sigma' \in |\mathbf{Sign}'|$, $\chi \in \mathrm{SEN}'(\Sigma')$ and suppose that $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ is surjective, that $\Sigma \in |\mathbf{Sign}|$ is such that $F(\Sigma) = \Sigma'$, that $\psi \in \mathrm{SEN}(\Sigma)$ is such that $\alpha_\Sigma(\psi) = \chi$ and that $\phi \in \mathrm{Thm}_\Sigma$. Then we have $\phi \in C_\Sigma(\psi)$, whence, since $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}$ and $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, we get that $\alpha_\Sigma(\psi) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi)$, i.e., $\chi \leq'_{\Sigma'} \alpha_\Sigma(\phi)$. Since $\Sigma' \in |\mathbf{Sign}'|$ and $\chi \in \mathrm{SEN}'(\Sigma')$ are arbitrary, this concludes the proof of the lemma. ■

Suppose now that $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ is a set-valued functor, with N a category of natural transformations on SEN, and $\mathtt{K}$ is an N-semilattice-based class of N-algebraic systems relative to $\wedge$. We define on SEN a closure system $C^{\mathtt{K}} = \{C^{\mathtt{K}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ as follows: For all

$\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in C^{\mathtt{K}}_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \text{iff} & \alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi), \\ & & \text{for all } \langle \mathrm{SEN}', \langle N', F' \rangle\rangle \in \mathtt{K} \\ & & \text{and all surjective } \langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'. \end{array}$$

$$\begin{array}{lll} \phi \in C^{\mathtt{K}}_\Sigma(\emptyset) & \text{iff} & \psi \leq'_{F(\Sigma)} \alpha_\Sigma(\phi), \\ & & \text{for all } \langle \mathrm{SEN}', \langle N', F' \rangle\rangle \in \mathtt{K}, \text{ all } \psi \in \mathrm{SEN}'(F(\Sigma)) \\ & & \text{and all surjective } \langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'. \end{array}$$

This definition is extended so that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\phi \in C^{\mathtt{K}}_\Sigma(\Phi) \quad \text{iff} \quad \text{there exists finite } \Phi' \subseteq \Phi, \text{ such that } \phi \in C^{\mathtt{K}}_\Sigma(\Phi').$$

It is shown in the following proposition that, thus defined, $\mathcal{I}^{\mathtt{K}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathtt{K}} \rangle$ is a finitary non-pseudo-axiomatic π-institution.

Proposition 17 *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is a set-valued functor, with N a category of natural transformations on* SEN, *and* $\mathtt{K}$ *is an N-semilattice-based class of N-algebraic systems relative to* $\wedge$. *Then* $\mathcal{I}^{\mathtt{K}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathtt{K}} \rangle$ *is a finitary, non-pseudo-axiomatic π-institution.*

Proof:

It is easy to show that, for every $\Sigma \in |\mathbf{Sign}|$, $C^{\mathtt{K}}_\Sigma$ is reflexive and monotone. So, to show that it is a closure operator on $\mathrm{SEN}(\Sigma)$, we just show idempotency. Suppose that $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C^{\mathtt{K}}_\Sigma(C^{\mathtt{K}}_\Sigma(\Phi))$. Thus, there exists $\Psi = \{\psi_0, \ldots, \psi_{n-1}\} \subseteq_\omega C^{\mathtt{K}}_\Sigma(\Phi)$, such that $\phi \in C^{\mathtt{K}}_\Sigma(\Psi)$. Thus, for every $i < n$, there exists $\Psi^i = \{\psi^i_0, \ldots, \psi^i_{n_i-1}\} \subseteq_\omega \Phi$, such that $\psi_i \in C^{\mathtt{K}}_\Sigma(\Psi^i)$. Hence, by the definition of $C^{\mathtt{K}}$, we get that, for all $\langle \mathrm{SEN}', \langle N', F' \rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$,

$$\alpha_\Sigma(\psi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi)$$

and, also, for all $i < n$,

$$\alpha_\Sigma(\psi^i_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi^i_{n_i-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\psi_i).$$

Therefore, we obtain

$$\begin{array}{lll} \bigwedge_{i<n}(\alpha_\Sigma(\psi^i_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi^i_{n_i-1})) & \leq'_{F(\Sigma)} & \alpha_\Sigma(\psi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi_{n-1}) \\ & \leq'_{F(\Sigma)} & \alpha_\Sigma(\phi). \end{array}$$

This proves that $\phi \in C^{\mathtt{K}}_\Sigma(\bigcup_{i<n} \Psi^i)$, which, since $\Psi^i \subseteq_\omega \Phi$, for all $i \in I$, establishes that $\phi \in C^{\mathtt{K}}_\Sigma(\Phi)$, showing that $C^{\mathtt{K}}_\Sigma$ is also idempotent, i.e., it is a closure operator on $\mathrm{SEN}(\Sigma)$.

To finish the proof that $C^{\mathtt{K}}$ is a closure system on SEN, it suffices to show that it is structural. To this end, suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and $\phi_0, \ldots, \phi_{n-1}, \phi \in$

$\mathrm{SEN}(\Sigma)$, such that $\phi \in C^{\mathtt{K}}_\Sigma(\phi_0,\ldots,\phi_{n-1})$. Thus, by the definition of $C^{\mathtt{K}}$, we get that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, we have that

$$\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi).$$

Since $\mathtt{K}$ is N-semilattice-based relative to $\wedge$, $\leq'$ is a posystem on SEN', which yields that

$$\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1})) \leq'_{F(\Sigma')} \mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)).$$

Since $\langle F,\alpha\rangle$ is (N,N')-epimorphic, this holds if and only if

$$\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})) \leq'_{F(\Sigma')} \mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)).$$

This, in turn, is equivalent, since α is a natural transformation, with

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1})) \leq'_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)).$$

Now, the fact that $\wedge$ is a natural transformation yields that

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi_0) \wedge_{\Sigma'} \cdots \wedge_{\Sigma'} \mathrm{SEN}(f)(\phi_{n-1})) \leq'_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)).$$

Finally, once more using the fact that $\langle F,\alpha\rangle$ is (N,N')-epimorphic, we get that

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi_0)) \wedge'_{F(\Sigma')} \cdots \wedge'_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi_{n-1})) \leq'_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)),$$

which proves that $\mathrm{SEN}(f)(\phi) \in C^{\mathtt{K}}_{\Sigma'}(\mathrm{SEN}(f)(\phi_0),\ldots,\mathrm{SEN}(f)(\phi_{n-1}))$, and establishes structurality.

That $C^{\mathtt{K}}$ is a finitary closure system on SEN is straightforward from its definition. So to conclude the proof of the proposition, it suffices now to show that $\mathcal{I}^{\mathtt{K}}$ is non-pseudo-axiomatic. To this end, it suffices to show that $\bigcap\{T \in \mathrm{Th}_\Sigma(\mathcal{I}^{\mathtt{K}}) : T \neq \emptyset\} \subseteq \mathrm{Thm}^{\mathtt{K}}_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$. To do this, we show that $\bigcap_{\psi\in\mathrm{SEN}(\Sigma)} C^{\mathtt{K}}_\Sigma(\psi) \subseteq \mathrm{Thm}^{\mathtt{K}}_\Sigma$. Suppose, in fact, that $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in C^{\mathtt{K}}_\Sigma(\psi)$, for all $\psi \in \mathrm{SEN}(\Sigma)$. This is equivalent, by the definition of $C^{\mathtt{K}}$, to the statement that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, we have $\alpha_\Sigma(\psi) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi)$. Thus, since $\langle F,\alpha\rangle$ is surjective, we get that, for all $\chi \in \mathrm{SEN}'(F(\Sigma))$, $\chi \leq'_{F(\Sigma)} \alpha_\Sigma(\phi)$. Therefore, by the definition of $C^{\mathtt{K}}$, we obtain that $\phi \in \mathrm{Thm}^{\mathtt{K}}_\Sigma$, as required. ■

It is now shown that, if $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a set-valued functor, with N a category of natural transformations on SEN, and $\mathtt{K}$ is N-semilattice-based relative to $\wedge$, then the π-institution $\mathcal{I}^{\mathtt{K}}$ is also N-semilattice based relative to $\wedge$ and $\mathtt{K}$. This forms an analog of Proposition 3.4 of [17] in the context of π-institutions.

Proposition 18 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a set-valued functor, with* N *a category of natural transformations on* SEN, *and* $\mathtt{K}$ *an* N*-semilattice-based class of* N*-algebraic systems relative to* $\wedge$*. Then* $\mathcal{I}^{\mathtt{K}}$ *is* N*-semilattice based relative to* $\wedge$ *and* $\mathtt{K}$*. If, moreover,* $\mathcal{I}^{\mathtt{K}}$ *is symmetrically* N*-rule based and surjectively* N*-semilattice-based relative to* $\wedge$ *and a subclass* $\mathtt{L}$ *of* $\mathtt{K}$*, such that* $\mathtt{K} = \mathbf{V}^N(\mathtt{L})$*, then* $\mathbf{V}^N(\mathcal{I}^{\mathtt{K}}) = \mathtt{K}$.

Proof:

The proof of the first statement is straightforward if one takes into account the definitions of $\mathcal{I}^{\mathtt{K}}$ and that of an N-semilattice-based π-institution relative to $\wedge$ and $\mathtt{K}$.

For the second statement, suppose that $\mathcal{I}^{\mathtt{K}}$ is symmetrically N-rule based and surjectively N-semilattice-based relative to $\wedge$ and $\mathtt{L}$, such that $\mathtt{K} = \mathbf{V}^N(\mathtt{L})$. Then, we have, by the definition of $\mathbf{V}^N(\mathcal{I}^{\mathtt{L}})$,

$$\begin{array}{rcll} \mathtt{K} & = & \mathbf{V}^N(\mathtt{L}) & \text{(by the hypothesis)} \\ & = & \mathbf{V}^N(\mathcal{I}^{\mathtt{L}}) & \text{(by Proposition 10)} \\ & = & \mathbf{V}^N(\mathcal{I}^{\mathtt{K}}). & \text{(since } \mathcal{I}^{\mathtt{K}} = \mathcal{I}^{\mathtt{L}}) \end{array}$$

■

Moreover, it may now be shown that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a symmetrically N-rule-based and surjectively N-semilattice based π-institution, that is non-pseudo-axiomatic, then the π-institution $\mathcal{I}^{\mathbf{V}^N(\mathcal{I})}$ that is generated by its canonical class of N-algebraic systems, coincides with $\mathcal{I}$. This forms, in the present context, an analog of Proposition 3.5 of [17].

Proposition 19 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a non-pseudo-axiomatic, symmetrically N-rule-based and surjectively N-semilattice based π-institution. Then $\mathcal{I}^{\mathbf{V}^N(\mathcal{I})} = \mathcal{I}$.*

Proof:

It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$, we have that $\phi \in C_\Sigma^{\mathbf{V}^N(\mathcal{I})}(\phi_0, \ldots, \phi_{n-1})$ if and only if $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$. We indeed have

$$\begin{array}{ll} \multicolumn{2}{l}{\phi \in C_\Sigma^{\mathbf{V}^N(\mathcal{I})}(\phi_0, \ldots, \phi_{n-1})} \\ \text{iff} & \alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \leq'_{F(\Sigma)} \alpha_\Sigma(\phi), \\ & \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathbf{V}^N(\mathcal{I}) \text{ and all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}', \\ \text{iff} & \alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi) = \alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}), \\ & \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathbf{V}^N(\mathcal{I}) \text{ and all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}', \\ \text{iff} & \alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = \alpha_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}), \\ & \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathbf{V}^N(\mathcal{I}) \text{ and all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}', \\ \text{iff} & C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \\ \text{iff} & \phi \in C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}). \end{array}$$

■

An N-semilattice-based and non-psedo-axiomatic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ relative to $\wedge$, where N is a category of natural transformations on SEN, is determined by the pairs of all Σ-sentences which are interderivable, for every $\Sigma \in |\mathbf{Sign}|$, i.e., by its Frege relation $\Lambda(\mathcal{I})$. Moreover the relation of extension between π-institutions over the same sentence functor that are N-semilattice-based relative to $\wedge$ and non-pseudo-axiomatic corresponds to the signature-wise inclusion relation between their Frege relations.

Notice that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ are any two π-institutions, such that $C \leq C'$, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I})$, then $C_\Sigma(\phi) = C_\Sigma(\psi)$, which immediately implies that $C'_\Sigma(\phi) = C'_\Sigma(\psi)$, i.e., that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}')$. Therefore $\Lambda(\mathcal{I}) \leq \Lambda(\mathcal{I}')$. For the special case where $\mathcal{I}$ and $\mathcal{I}'$ are N-semilattice-based relative to $\wedge$ and non-pseudo-axiomatic the converse implication also holds. This results in the following analog of Proposition 3.6 of [17].

Proposition 20 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$, with N a category of natural transformations on* SEN*, be two non-pseudo-axiomatic, N-semilattice-based π-institutions relative to $\wedge$. Then*

$$\Lambda(\mathcal{I}) \leq \Lambda(\mathcal{I}') \quad \textit{if and only if} \quad \mathcal{I} \leq \mathcal{I}'.$$

Therefore, if $\Lambda(\mathcal{I}) = \Lambda(\mathcal{I}')$, then $\mathcal{I} = \mathcal{I}'$.

Proof:

We already noticed that the right-to-left implication holds. For the reverse implication, assume that $\mathcal{I}, \mathcal{I}'$ are non-pseudo-axiomatic and N-semilattice-based relative to $\wedge$, such that $\Lambda(\mathcal{I}) \leq \Lambda(\mathcal{I}')$. Then we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lcl}
\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \text{iff} & \phi \in C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \\
 & \text{iff} & C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = C_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \\
 & \text{implies} & C'_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1} \wedge_\Sigma \phi) = C'_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \\
 & \text{iff} & \phi \in C'_\Sigma(\phi_0 \wedge_\Sigma \cdots \wedge_\Sigma \phi_{n-1}) \\
 & \text{iff} & \phi \in C'_\Sigma(\phi_0, \ldots, \phi_{n-1}).
\end{array}$$

If, on the other hand, $\phi \in C_\Sigma(\emptyset)$, then, since $\mathcal{I}$ is non-pseudo-axiomatic, we get that $\phi \in C_\Sigma(\psi)$, for all $\psi \in \mathrm{SEN}(\Sigma)$, whence $C_\Sigma(\phi \wedge_\Sigma \psi) = C_\Sigma(\psi)$, for all $\psi \in \mathrm{SEN}(\Sigma)$, i.e., $\langle \phi \wedge_\Sigma \psi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I})$, for all $\psi \in \mathrm{SEN}(\Sigma)$. Therefore, $\langle \phi \wedge_\Sigma \psi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}')$, for all $\psi \in \mathrm{SEN}(\Sigma)$, showing that $\phi \in C'_\Sigma(\psi)$, for all $\psi \in \mathrm{SEN}(\Sigma)$ and, thus, since $\mathcal{I}'$ is also non-pseudo-axiomatic, $\phi \in C'_\Sigma(\emptyset)$. Hence $\mathcal{I} \leq \mathcal{I}'$. ■

Let SEN : $\mathbf{Sign} \rightarrow \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN. A variety $\mathtt{K}$ of N-algebraic systems will be said to be (SEN, N)-**surjective** if it is generated by a subclass $\mathtt{L} \subseteq \mathtt{K}$, such that, for every $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{L}$, there exists at least one surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$.

If Propositions 18, 19 are combined with Proposition 20, we obtain an analog of an isomorphism obtained in [19] in the case of sentential logics using Gentzen systems. It is a bijection between the non-pseudo-axiomatic, surjectively semilattice-based π-institutions relative to $\wedge$ on a symmetrically N-rule based sentence functor SEN and the (SEN, N)-surjective semilattice-based varieties relative to $\wedge$, that becomes a dual isomorphism when the orderings of the π-institutions by extension and of the varieties by inclusion are taken into account. The isomorphism result of [19] was revisited in [17], where a proof that is not based on Gentzen systems is provided.

Theorem 21 *Let* SEN : **Sign** $\to$ **Set**, *with* N *a category of natural transformations on* SEN, *be a symmetrically* N*-rule based functor, and* $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N. *Then, there exists a dual isomorphism between the non-pseudo-axiomatic, surjectively* N*-semilattice based* π*-institutions on* SEN *relative to* $\wedge$, *ordered under extension, and the collection of all* (SEN, N)*-surjective subvarieties of the variety of all* N*-algebraic systems satisfying the equations*

$$x \wedge x \approx x;$$
$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z;$$
$$x \wedge y \approx y \wedge x,$$

ordered by inclusion. The isomorphism is given by $\mathcal{I} \mapsto \mathbf{V}^N(\mathcal{I})$.

Proof:
If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ are non-pseudo-axiomatic, symmetrically N-rule-based and surjectively N-semilattice based π-institutions on SEN relative to $\wedge$, such that $\mathbf{V}^N(\mathcal{I}) = \mathbf{V}^N(\mathcal{I}')$, then , by Proposition 19, we get that $\mathcal{I} = \mathcal{I}'$. Therefore $\mathcal{I} \mapsto \mathbf{V}^N(\mathcal{I})$ is injective. That it is onto is given by Proposition 18, since we have that every N-semilattice-based variety $\mathtt{K}$ relative to $\wedge$ defines an N-semilattice-based π-institution $\mathcal{I}^{\mathtt{K}}$, such that $\mathbf{V}^N(\mathcal{I}^{\mathtt{K}}) = \mathtt{K}$.

If $\mathcal{I}$ is a symmetrically N-rule-based and N-selfextensional π-institution, then its Frege relation system determines the equations that hold in the variety $\mathbf{V}^N(\mathcal{I})$ and, hence, determines the variety itself. Now, by Proposition 20, given two non-pseudo-axiomatic, symmetrically N-rule-based and N-semilattice based π-institutions on SEN relative to $\wedge$, $\mathcal{I} \le \mathcal{I}'$ if and only if $\mathbf{V}^N(\mathcal{I}') \subseteq \mathbf{V}^N(\mathcal{I})$. Thus, $\mathcal{I} \mapsto \mathbf{V}^N(\mathcal{I})$ is a dual isomorphism. ■

4 Full N-Selfextensionality

It has been shown in Theorem 12 that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a symmetrically N-rule based π-institution that is surjectively N-semilattice-based, then, $\mathbf{V}^N(\mathcal{I}) = \mathtt{K}^N_{\mathcal{I}}$. In this section it is shown that, for a symmetrically N-rule-based and surjectively N-semilattice based π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ relative to $\wedge$, the two classes $\mathbf{V}^N(\mathcal{I})$ and $\mathrm{Alg}^N(\mathcal{I})$ coincide. A proof is also provided of the fact that every N-conjunctive and N-selfextensional, symmetrically N-rule-based π-institution is fully N-selfextensional. These two results parallel in the context of π-institutions two of the main theorems, Theorems 3.12 and 3.13, respectively, of [17].

We start with an analog of Lemma 3.8 of [17], showing that, given a symmetrically N-rule based π-institution $\mathcal{I}$ that is N-semilattice-based and has theorems, then, for every N-algebraic system $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}^N_{\mathcal{I}}$ and all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, the theory families of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN' coincide with the N-semilattice filter families of $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$.

Lemma 22 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, *with* N *a category of natural transformations on* SEN, *be a symmetrically* N*-rule based* π*-institution, that is surjectively* N*-semilattice-based* *and has theorems. For every* $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathtt{K}^N_{\mathcal{I}}$ *and all surjective* (N, N')*-epimorphic translations* $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$*, the theory families of the* $\langle F, \alpha \rangle$*-min* (N, N')*-models of* $\mathcal{I}$ *on* SEN *are exactly the* N*-semilattice filter families of* $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$*.*

Proof:

Let $T = \{T_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ be a theory family of the $\langle F, \alpha \rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ of $\mathcal{I}$ on SEN'. Since $\mathcal{I}$ has theorems and $\langle F, \alpha \rangle$ is surjective, we have that $T_{\Sigma'} \neq \emptyset$, for all $\Sigma' \in |\mathbf{Sign}'|$. Moreover, since, by Proposition 11, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we have that $C_\Sigma(\phi \wedge_\Sigma \psi) = C_\Sigma(\phi, \psi)$, and $\langle F, \alpha \rangle$ is surjective, we must have that, for all $\Sigma' \in |\mathbf{Sign}'|$ and $\phi', \psi' \in \mathrm{SEN}'(\Sigma')$, $C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi') = C'_{\Sigma'}(\phi', \psi')$. Therefore, for every theory family T of $\mathcal{I}'$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\phi', \psi' \in \mathrm{SEN}'(\Sigma')$, we have that $\phi' \wedge'_{\Sigma'} \psi' \in T_{\Sigma'}$ if and only if $\phi', \psi' \in T'_{\Sigma'}$. Therefore T is indeed an N-semilattice filter family of $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$.

Suppose, conversely, that $T = \{T_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ is an N-semilattice filter family of $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathtt{K}^N_{\mathcal{I}}$. To see that T is a theory family of the $\langle F, \alpha \rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ of $\mathcal{I}$ on SEN', it suffices to show, by surjectivity of $\langle F, \alpha \rangle$, that, for all $\Sigma \in |\mathbf{Sign}'|$, we have $C'_{F(\Sigma)}(T_{F(\Sigma)}) \subseteq T_{F(\Sigma)}$.

Suppose, to this end, that $\phi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(T_{F(\Sigma)})$. By Lemma 2.1 of [26], $\mathcal{I}'$ is also finitary, which supplies $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$, with $\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}) \in T_{F(\Sigma)}$, such that $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$. Following the notation used in the proof of Lemma 2.1 of [26] and the proof technique of Lemma 13 of [33], it suffices to show, by induction on n, that, for all $n \geq 0$, $X^n_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) \subseteq T_{F(\Sigma)}$.

If $n = 0$, then $X^0_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) = \{\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})\} \subseteq T_{F(\Sigma)}$. Suppose, as the induction hypothesis, that $X^k_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) \subseteq T_{F(\Sigma)}$. For the induction step, let $\chi' \in X^{k+1}_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$. Then, there exist, by definition, $\chi_0, \ldots, \chi_{m-1}, \chi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\chi) = \chi'$, $\alpha_\Sigma(\chi_i) \in X^k_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$, for all $i < m$, and $\chi \in C_\Sigma(\chi_0, \ldots, \chi_{m-1})$. Now, since $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}^N_{\mathcal{I}}$, we obtain that $\alpha_\Sigma(\chi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\chi_{m-1}) \wedge'_{F(\Sigma)} \alpha_\Sigma(\chi) = \alpha_\Sigma(\chi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\chi_{m-1})$. Thus, since, by the inductive hypothesis, $\alpha_\Sigma(\chi_0), \ldots, \alpha_\Sigma(\chi_{m-1}) \in X^k_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) \subseteq T_{F(\Sigma)}$, we get, taking into account that $T_{F(\Sigma)}$ is the Σ'-component of an N-semilattice filter family of $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, that

$$\begin{array}{rcl} \alpha_\Sigma(\chi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\chi_{m-1}) \wedge'_{F(\Sigma)} \alpha_\Sigma(\chi) & = & \alpha_\Sigma(\chi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\chi_{m-1}) \\ & \in & T_{F(\Sigma)}, \end{array}$$

whence, again using the fact that $T_{F(\Sigma)}$ is the Σ'-component of an N-semilattice filter family of $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, we obtain that $\chi' = \alpha_\Sigma(\chi) \in T_{F(\Sigma)}$ as well. ■

Next, the task of showing that, given a symmetrically N-rule based and surjectively N-semilattice based π-institution, any $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathtt{K}^N_{\mathcal{I}}$, and any surjective $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, the Frege relation system of the $\langle F, \alpha \rangle$-min (N, N')-model of $\mathcal{I}$ on SEN' is N'-reduced is undertaken. This will help us establish in the sequel one of the main results of the paper, namely, that $\mathrm{Alg}^N(\mathcal{I}) = \mathtt{K}^N_{\mathcal{I}}$, and, as a consequence, that $\mathrm{Alg}^N(\mathcal{I})$, in this specific case, is a variety of N-algebraic systems.

Lemma 23 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a symmetrically N-rule based and surjectively N-semilattice-based π-institution. Then, for every $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}^N_{\mathcal{I}}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, the Frege relation $\Lambda(\mathcal{I}')$ of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on* SEN′ *is the identity relation system on* SEN′*. Therefore $\mathcal{I}'$ is N'-reduced.*

Proof:

By the surjectivity of $\langle F, \alpha\rangle$, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) \neq \alpha_\Sigma(\psi)$, we have that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \notin \Lambda_{F(\Sigma)}(\mathcal{I}')$. To this end consider the sets

$$T^{\alpha_\Sigma(\phi)} = \{\chi \in \mathrm{SEN}'(F(\Sigma)) : \alpha_\Sigma(\phi) \leq'_{F(\Sigma)} \chi\}$$

and

$$T^{\alpha_\Sigma(\psi)} = \{\chi \in \mathrm{SEN}'(F(\Sigma)) : \alpha_\Sigma(\psi) \leq'_{F(\Sigma)} \chi\}.$$

Since $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}^N_{\mathcal{I}}$ and $\mathcal{I}$ is surjectively N-semilattice-based, both $T^{\alpha_\Sigma(\phi)}$ and $T^{\alpha_\Sigma(\psi)}$ are $F(\Sigma)$-components of N-semilattice filter families of $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. Therefore, by Lemma 22, they are both $F(\Sigma)$-theories of $\mathcal{I}'$. Since $\alpha_\Sigma(\phi) \neq \alpha_\Sigma(\psi)$, we must have $\alpha_\Sigma(\phi) \notin T^{\alpha_\Sigma(\psi)}$ or $\alpha_\Sigma(\psi) \notin T^{\alpha_\Sigma(\phi)}$. This shows that $C'_{F(\Sigma)}(\alpha_\Sigma(\phi)) \neq C'_{F(\Sigma)}(\alpha_\Sigma(\psi))$ and, hence, that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \notin \Lambda_{F(\Sigma)}(\mathcal{I}')$, as required.

That $\mathcal{I}'$ is N'-reduced follows now from the fact that $\widetilde{\Omega}^N(\mathcal{I}') \leq \Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$. This also verifies that $\mathcal{I}'$ is N'-selfextensional in this case. ■

Lemma 24 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a symmetrically N-rule based π-institution, that is surjectively N-semilattice-based and has theorems. Then, for every finitary (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, such that $\Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$, we have that*

1. *$\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}^N_{\mathcal{I}}$ and*
2. *C' coincides with the closure system $C'^{\min}$ of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ of $\mathcal{I}$ on* SEN′.

Proof:

1. We have, by the hypothesis, $\widetilde{\Omega}^{N'}(\mathcal{I}') \leq \Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$, whence $\widetilde{\Omega}^{N'}(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$, showing that $\mathcal{I}'$ is N'-reduced, and, therefore, that $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_{\mathcal{I}}$; the last inclusion provided by Proposition 2.
2. Let $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ denote the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN′. To show that $C' = C'^{\min}$, it suffices to show that, given $T \in \mathrm{ThFam}(\mathcal{I}'^{\min})$, we have, for all $\Sigma \in |\mathbf{Sign}|$, $C'_{F(\Sigma)}(T_{F(\Sigma)}) \subseteq T_{F(\Sigma)}$. Suppose, to this end, that $\phi \in$

$\mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(T_{F(\Sigma)})$. Thus, since $\mathcal{I}'$ is finitary, there exist $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$, with $\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}) \in T_{F(\Sigma)}$, such that $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$. Therefore, since $\mathcal{I}$ is N-semilattice-based relative to $\wedge$ and $\mathtt{K}^N_{\mathcal{I}}$ and $\mathcal{I}'$ is a model of $\mathcal{I}$, we get that

$$\begin{aligned} C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi)) & \\ = C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}), \alpha_\Sigma(\phi)) & \\ = C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) & \\ = C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1})). & \end{aligned}$$

But, then, since $\Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$, we get that $\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) = \alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi)$, whence, since $T_{F(\Sigma)}$ is, by Lemma 22, the $F(\Sigma)$-component of an N-semilattice filter family of $\mathcal{I}'^{\min}$ and $\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}) \in T_{F(\Sigma)}$, we get that $\alpha_\Sigma(\phi_0) \wedge'_{F(\Sigma)} \cdots \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi_{n-1}) \wedge'_{F(\Sigma)} \alpha_\Sigma(\phi) \in T_{F(\Sigma)}$, and, therefore, by the same property, $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Thus $T_{F(\Sigma)} = C'_{F(\Sigma)}(T_{F(\Sigma)})$, i.e., T is indeed a theory family of $\mathcal{I}'$.

■

We are now ready to show that, given a symmetrically N-rule based π-institution $\mathcal{I}$, that is surjectively N-semilattice-based and has theorems, the class $\mathrm{Alg}^N(\mathcal{I})$ consists exactly of those members $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ of its intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ that are such there exists a surjective (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$. This class was called the N-**core** of $\mathtt{K}^N_{\mathcal{I}}$ and denoted by $\mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}})$ in previous work. Theorem 25 is an analog in the context of π-institutions of another of the main theorems, Theorem 3.12, of [17].

Theorem 25 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, *with* N *a category of natural transformations on* SEN, *be a symmetrically* N*-rule based* π*-institution, that is surjectively* N*-semilattice-based and has theorems. Then*

1. $\mathrm{Alg}^N(\mathcal{I}) = \mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}})$;
2. $\mathcal{I}$ *is* N*-semilattice-based relative to* $\mathrm{Alg}^N(\mathcal{I})$.

Proof:
By Proposition 2, we have that $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_{\mathcal{I}}$. Moreover, by the definition of the class $\mathrm{Alg}^N(\mathcal{I})$, for every $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathrm{Alg}^N(\mathcal{I})$, there exists at least one surjective (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$. Therefore $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}})$. By Lemma 23, we also get that $\mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}}) \subseteq \mathrm{Alg}^N(\mathcal{I})$. Therefore we obtain that $\mathrm{Alg}^N(\mathcal{I}) = \mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}})$. Finally, since, by Corollary 9 and Theorem 12, $\mathcal{I}$ is N-semilattice-based relative to $\mathtt{K}^N_{\mathcal{I}}$, we get that $\mathcal{I}$ is N-semilattice-based relative to $\mathrm{Alg}^N(\mathcal{I})$. ■

Theorem 25 together with Theorem 15 imply the following

Corollary 26 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a symmetrically N-rule based π-institution with theorems, that is N-selfextensional and has an N-conjunction. Then $\mathrm{Alg}^N(\mathcal{I}) = \mathrm{cor}^N(\mathtt{K}^N_{\mathcal{I}})$.*

Finally, the section concludes with the analog of a theorem first proved in [11] via the use of Gentzen systems and subsequently revisited in [17], where it is given as Theorem 3.13 with a proof avoiding the use of Gentzen systems. Roughly speaking, it states that every symmetrically N-rule based π-institution that is N-conjunctive and N-selfextensional is also fully N-selfextensional.

Recall, before stating Theorem 27 formally, that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be fully N-selfextensional if, for every (N, N')-full model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-epimorphic translation, $\mathcal{I}'$ is N'-selfextensional.

Theorem 27 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a symmetrically N-rule based π-institution, that is N-conjunctive and N-selfextensional. Then $\mathcal{I}$ is fully N-selfextensional.*

Proof:

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a symmetrically N-rule based π-institution, that is N-selfextensional and has an N-conjunction $\wedge$. Then, by Theorem 15 and its proof, $\mathcal{I}$ is surjectively N-semilattice based and it is N-semilattice based relative to $\wedge$ and $\mathtt{K}^N_{\mathcal{I}}$. By Proposition 2, $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_{\mathcal{I}}$, whence, by Lemma 23, if $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I})$ and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation, then the Frege relation system of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ on SEN' is the identity equivalence system on SEN', i.e., $\Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$ and, therefore, $\mathcal{I}'$ has the N'-congruence property.

Now suppose that $\mathcal{I}'$ is a full (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. Then, by definition, $\mathcal{I}'^{N'}$ is the $\langle F, \pi^{N'}_F\alpha\rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$. Therefore, by what was said in the previous paragraph, $\mathcal{I}'^{N'}$ has the $\overline{N'}$-congruence property. But, by Proposition 3.7 of [26], the congruence property is preserved by bilogical morphisms and $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{N'}\rangle : \mathcal{I}' \vdash^{se} \mathcal{I}'^{N'}$ is an $(N', \overline{N'})$-bilogical morphism, whence $\mathcal{I}'$ also has the N'-congruence property. This shows that $\mathcal{I}$ is fully N-selfextensional. ■

Acknowledgements

The author wishes to thank Josep Maria Font and Ramon Jansana for their work on sentential logics. Ramon's beautiful work on selfextensionality has been one of my main motivations for studying selfextensionality at the categorical level. Another one has been the study of Fregeanity and of its consequences by Janusz Czelakowski and Don Pigozzi, whom I also would like to thank. Warm thanks go also to Charles Wells and to Giora Slutzki for providing encouragement and support over many years.

References

[1] Babyonyshev, S.V., *Fully Fregean Logics*, Reports on Mathematical Logic, Vol. 37 (2003)

[2] Barr, M., and Wells, C., *Category Theory for Computing Science*, Third Edition, Les Publications CRM, Montréal 1999

[3] Blok, W.J., and Pigozzi, D., *Protoalgebraic Logics,* Studia Logica, Vol. 45 (1986), pp. 337-369

[4] Blok, W.J., and Pigozzi, D., *Algebraizable Logics,* Memoirs of the American Mathematical Society, Vol. 77, No. 396 (1989)

[5] Czelakowski, J., *Equivalential Logics I,II,* Studia Logica, Vol. 40 (1981), pp. 227-236, 355-372

[6] Czelakowski, J., *Filter Distributive Logics*, Studia Logica, Vol. 43 (1984), pp. 353-377

[7] Czelakowski, J., and Jansana, R., *Weakly Algebraizable Logics,* Journal of Symbolic Logic, Vol. 65, No. 2 (2000), pp. 641-668

[8] Czelakowski, J., and Pigozzi, D., *Fregean Logics*, Annals of Pure and Applied Logic, Vol. 127 (2004), pp. 17-76

[9] Czelakowski, J., and Pigozzi, D., *Fregean Logics with the Multiterm Deduction Theorem and Their Algebraization*, Studia Logica, Vol. 78 (2004), pp. 171-212

[10] Fiadeiro, J., and Sernadas, A., *Structuring Theories on Consequence,* in Recent Trends in Data Type Specification, Donald Sannella and Andrzei Tarlecki, Eds., Lecture Notes in Computer Science, Vol. 332, Springer-Verlag, New York 1988, pp. 44-72

[11] Font, J.M., and Jansana, R., *A General Algebraic Semantics for Sentential Logics,* Lecture Notes in Logic, Vol. 7 (1996), Springer-Verlag, Berlin Heidelberg 1996

[12] Goguen, J.A., and Burstall, R.M., *Introducing Institutions,* in Proceedings of the Logic of Programming Workshop, E. Clarke and D. Kozen, Eds., Lecture Notes in Computer Science, Vol. 164, Springer-Verlag, New York 1984, pp.221-256

[13] Goguen, J.A., and Burstall, R.M., *Institutions: Abstract Model Theory for Specification and Programming,* Journal of the Association for Computing Machinery, Vol. 39, No. 1 (1992), pp. 95-146

[14] Herrmann, B., *Equivalential Logics and Definability of Truth,* Dissertation, Freie Universitat Berlin, Berlin 1993

[15] Herrmann, B., *Equivalential and Algebraizable Logics,* Studia Logica, Vol. 57 (1996), pp. 419-436

[16] Herrmann, B., *Characterizing Equivalential and Algebraizable Logics by the Leibniz Operator,* Studia Logica, Vol. 58 (1997), pp. 305-323

[17] Jansana, R., *Selfextensional Logics with Conjunction*, Studia Logica,

[18] Prucnal, T., and Wroński, A., *An algebraic characterization of the notion of structural completeness*, Bulletin of the Section of Logic, Vol. 3 (1974), pp. 30-33

[19] Pynko, A., *Definitional Equivalence and Algebraizability of Generalized Logical Systems*, Annals of Pure and Applied Logic, Vol. 98 (1999), pp. 1-68

[20] Suszko, R., *Ontology in the Tractatus of L. Wittgenstein*, Notre Dame Journal of Formal Logic, Vol. 9 (1968), pp. 7-33

[21] Suszko, R., *Non-Frege logic and the theories that are based on it*, in Nonclassical Logic, Izdat. "Nauka", 1970, pp. 349-383

[22] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Tarski Congruence Systems, Logical Morphisms and Logical Quotients*, Submitted to the Annals of Pure and Applied Logic, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[23] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Models of π-Institutions*, Notre Dame Journal of Formal Logic, Vol. 46, No. 4 (2005), pp. 439-460

[24] Voutsadakis, G., *Categorical Abstract Algebraic Logic: $(\mathcal{I}, N)$-Algebraic Systems*, Applied Categorical Structures, Vol. 13, No. 3 (2005), pp. 265-280

[25] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Gentzen π-Institutions*, To be submitted to the Far East Journal of Mathematical Sciences, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[26] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Full Models, Frege Systems and Metalogical properties*, Reports on Mathematical Logic, Vol. 41 (2006), pp. 31-62

[27] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Prealgebraicity and Protoalgebraicity*, To appear in Studia Logica, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[28] Voutsadakis, G., *Categorical Abstract Algebraic Logic: More on Protoalgebraicity*, Notre Dame Journal of Formal Logic, Vol. 47, No. 4 (2006), pp. 487-514

[29] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Equivalential π-Institutions*, To appear in the Australasian Journal of Logic, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[30] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Weakly Algebraizable π-Institutions*, Submitted to Studia Logica, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[31] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Bloom's Theorem for Rule-Based π-Institutions*, Submitted to the Logic Journal of the IGPL, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[32] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Ordered Equational Logic and Algebraizable PoVarieties*, To appear in Order, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[33] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Selfextensional π-Institutions with Implication*, Submitted to Applied Categorical Structures, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[34] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Syntactically Algebraizable π-Institutions*, Submitted to the Reports on Mathematical Logic, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`

[35] Wójcicki, R., *Theory of Logical Calculi, Basic Theory of Consequence Operations*, Vol. 199, Synthese Library, Reidel, Dordrecht, 1988