An Equational Theory of n-Lattices

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March 11, 2003

Abstract

Biedermann introduced the algebra of trilattices as the algebraic counterpart of the order-theoretic notion of a trilattice, which arose in Wille's triadic concept analysis. The author introduced polyadic concept analysis as a generalization of triadic concept analysis. There, n-lattices, for arbitrary n, appeared as a generalization of Wille's trilattices. In this paper the algebra of n-lattices is presented as the algebraic counterpart of n-lattices. The theory is a generalization of the theory of Biedermann to n dimensions.

1 Introduction

Ganter and Wille [4] introduced formal concept analysis in order to provide the theoretical foundations of a rigorous lattice-theoretic data analysis. A *formal context* consists of two sets, that of *objects* and that of *attributes*, together with a binary relation between objects and attributes. This relation induces in the standard way a Galois connection between sets of objects and sets of attributes whose closed sets, called *formal concepts*, form a complete lattice, the *lattice of formal concepts*. Because it employs a binary relation, a formal context is, roughly speaking, two dimensional. In [5], *triadic concept analysis* was introduced as a generalization of formal concept analysis to three dimensions. Apart from the sets of objects and attributes, a third

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[0]*Keywords:* formal concept analysis, formal contexts, triadic concept analysis, triordered sets, complete trilattices, trilattices, equational theory of trilattices, polyadic concept analysis, n-ordered sets, n-lattices
1991 AMS Subject Classification: Primary: 06A23 Secondary: 62-07

set, called the set of *conditions*, was introduced and a ternary relation between objects, attributes and conditions took the place of the binary relation of formal contexts. *Complete trilattices* were the lattice-theoretic structures that arose in place of complete lattices out of this generalization. They are triordered sets in which six operations of arbitrary arity, called the *ik-joins* exist. Biedermann [1] studies trilattices in which finitary ik-joins exist from the algebraic rather than the order-theoretic viewpoint. He shows that six operations of small arities are enough to algebraically represent trilattices and provides an equational basis for their theory. Inspired by the work of Wille, the author introduced in [6] *polyadic concept analysis*, which generalizes triadic contexts and concepts to n dimensions for arbitrary n. A generalization of Wille's Basic Theorem of Triadic Concept Analysis to arbitrary dimensions was also provided. The question naturally arises of whether there exists an algebraic theory for n-lattices, arising in polyadic concept analysis, generalizing the theory of Biedermann for trilattices. This question is the one that the present paper answers. It is shown that, in the general n-adic case, there are $n!$ operations of small arity that suffice to represent n-lattices. An equational basis for their theory, generalizing the theory of Biedermann is also provided.

[2] and [3] provide the basic notions and notation pertaining to ordered sets. Formal Concept Analysis and related notions are introduced in [4]. Triadic Concept Analysis is presented in [5], where one finds the definitions of triordered sets, ik-joins and complete trilattices. The equational theory of trilattices is introduced in [1]. Finally, [6] proposes the generalizations of triordered sets, ik-joins and complete trilattices to n-ordered sets, $(j_{n-1},\dots,j_1)$-joins and complete n-lattices, respectively.

2 n-Ordered Sets and Operations

n-ordered sets arise in n-adic concept analysis, for $n \geq 2$. They are generalizations of triordered sets, which, in turn, are generalizations of ordered sets.

Definition 1 *An* ordinal structure $\langle P, \lesssim_1, \lesssim_2, \dots, \lesssim_n\rangle$ *is a relational structure whose n relations are quasiorders. Let* $\sim_i = \lesssim_i \cap \gtrsim_i$, *for* $i = 1, 2, \dots, n$. *An* n-ordered set $\langle P, \lesssim_1, \dots, \lesssim_n\rangle$ *is an ordinal structure, such that, for all* $x, y \in P$ *and all* $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$,

1. $x \sim_{i_1} y, \dots, x \sim_{i_{n-1}} y$ *imply* $x = y$ *(Uniqueness Condition)*
2. $x \lesssim_{i_1} y, \dots, x \lesssim_{i_{n-1}} y$ *imply* $x \gtrsim_{i_n} y$ *(Antiordinal Dependency)*

Each quasiorder $\lesssim_i$ induces in the standard way an order $\leq_i$ on the set of equivalence classes $P/{\sim_i} = \{[x]_i : x \in P\}, i = 1, 2, \dots, n$, where $[x]_i = \{y \in P : x \sim_i y\}$.

Triadic diagrams have been employed (see, for instance, [5] and [1]) in representing 3-ordered (triordered) structures and a direct generalization of these has also been used in [6] for 4-ordered structures. However, the visual representations become increasingly cumbersome and less and less illuminating as the dimension increases.

In trilattices, one takes ik-bounds, ik-limits and ik-joins, for $\{i, j, k\} = \{1, 2, 3\}$, to define the triadic operations [5]. These were generalized in [6] to $(j_1, \ldots, j_{n-1})$-bounds, $(j_1, \ldots, j_{n-1})$-limits and $(j_1, \ldots, j_{n-1})$-joins, respectively, for $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. This process is, roughly speaking, the analog of the two-step process of taking least elements of sets of upper bounds and greatest elements of sets of lower bounds to compute suprema and infima, respectively, in lattices, but it requires three instead of two steps.

We follow [1] in establishing the following terminology and notation:

Definition 2 *Let $\langle P, \lesssim_1, \lesssim_2, \ldots, \lesssim_n \rangle$ be a triordered set, $j_1, j_2, \ldots, j_{n-1} \in \{1, 2, \ldots, n\}$ be distinct and $X_1, X_2, \ldots, X_{n-1}$ subsets of P.*

1. *An element $b \in P$ is called a $(j_{n-1}, \ldots, j_1)$-*bound *of $(X_{n-1}, X_{n-2}, \ldots, X_1)$ if $x_i \lesssim_{j_i} b$, for all $x_i \in X_i$ and all $i = 1, \ldots, n-1$. The set of all $(j_{n-1}, \ldots, j_1)$-bounds of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$.*

2. *A $(j_{n-1}, \ldots, j_1)$-bound $l \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$ of $(X_{n-1}, \ldots, X_1)$ is called a $(j_{n-1}, \ldots, j_1)$-*limit *of $(X_{n-1}, \ldots, X_1)$ if $l \gtrsim_{j_n} b$, for all $(j_{n-1}, \ldots, j_1)$-bounds $b \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$. The set of all $(j_{n-1}, \ldots, j_1)$-limits of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}}$.*

Obviously,

$$(X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)} = (X_{\sigma(n-1)}, \ldots, X_{\sigma(1)})^{(j_{\sigma(n-1)}, \ldots, j_{\sigma(1)})}$$

and

$$(X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}} = (X_{\sigma(n-1)}, \ldots, X_{\sigma(1)})^{\overline{(j_{\sigma(n-1)}, \ldots, j_{\sigma(1)})}},$$

for $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$ and every permutation σ of $\{1, \ldots, n-1\}$. Note also that if $(X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}} \neq \emptyset$ and $l_1, l_2 \in (X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}}$ then $l_1 \sim_{j_n} l_2$ since $(j_{n-1}, \ldots, j_1)$-limits are also $(j_{n-1}, \ldots, j_1)$-bounds.

Proposition 3 is generalizing Proposition 2.4 of [1] in n dimensions.

Proposition 3 *Let $\langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ be an n-ordered set, $X_1, \ldots, X_{n-1} \subseteq P$ and $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. Then, there exists at most one $(j_{n-1}, \ldots, j_1)$-limit $\bar{l}$ of $(X_{n-1}, \ldots, X_1)$ satisfying*

(C) $\bar{l}$ *is the largest in* $\lesssim_{j_2}$ *among the largest limits in* $\lesssim_{j_3}$ *among* $\ldots$ *among the largest limits in* $\lesssim_{j_{n-1}}$ *among the largest limits in* $\lesssim_{j_n}$ *or, equivalently,*

(C') $\bar{l}$ *is the smallest in* $\lesssim_{j_1}$ *among the largest limits in* $\lesssim_{j_3}$ *among* $\ldots$ *among the largest limits in* $\lesssim_{j_{n-1}}$ *among the largest limits in* $\lesssim_{j_n}$.

Proof:

Suppose $\bar{l}, \bar{l}'$ both satisfy (C). Since they are $(j_{n-1}, \ldots, j_1)$-limits, in particular $(j_{n-1}, \ldots, j_1)$-bounds, of $(X_{n-1}, \ldots, X_1)$ we must have $\bar{l} \lesssim_{j_n} \bar{l}'$ and $\bar{l}' \lesssim_{j_n} \bar{l}$, i.e., $\bar{l} \sim_{j_n} \bar{l}'$. But then, by (C), $\bar{l} \sim_{j_{n-1}} \bar{l}'$, whence, again by (C), $\bar{l} \sim_{j_{n-2}} \bar{l}'$, and so on, down to $\bar{l} \sim_{j_2} \bar{l}'$. Thus $\bar{l} \sim_{j_i} \bar{l}'$, for $i = 2, \ldots, n$, which, together with $\bigcap_{i=2}^{n} \sim_{j_i} = \Delta_P$, gives $\bar{l} = \bar{l}'$. ■

Definition 4 *If it exists, a* $(j_{n-1}, \ldots, j_1)$*-limit satisfying the statement in Proposition 3 is called the* $(j_{n-1}, \ldots, j_1)$-join *of* $(X_{n-1}, \ldots, X_1)$ *and denoted by* $\nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1}, \ldots, X_1)$.

Definition 5 *A* complete n-lattice $\langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ *is an* n*-ordered set in which all* $(j_{n-1}, \ldots, j_1)$*-joins* $\nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1}, \ldots, X_1)$ *exist, for all* $X_1, \ldots, X_{n-1} \subseteq L$ *and all* $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. *A complete trilattice is* bounded *by* $0_{j_n} := \nabla_{(j_{n-1},\ldots,j_1)}(L, L, \ldots, L)$, *where* $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$.

Proposition 6 *In any* n*-ordered set* $\langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *the idempotent laws*

$$\nabla_{(j_{n-1},\ldots,j_1)}(\{x\}, \ldots, \{x\}) = x \tag{1}$$

hold for all $x \in P$ *and all* $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. *Conversely, if the* $(j_{n-1}, \ldots, j_1)$*-join of* $(\{x\}, \ldots, \{x\})$ *exists in the ordinal structure* $\langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *and satisfies* (1), *for all* $x \in P$ *and all* $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$, *then* $\langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *is an* n*-ordered set.*

Proof:

Clearly, x is a $(j_{n-1}, \ldots, j_1)$-bound of $(\{x\}, \ldots, \{x\})$. Suppose that l is another such bound, i.e., $x \lesssim_{j_i} l$, for all $i = 1, \ldots, n-1$. Thus, by the antiordinal property, we have $l \lesssim_{j_n} x$, whence x is a $(j_{n-1}, \ldots, j_1)$-limit of $(\{x\}, \ldots, \{x\})$. Finally, given such a limit $\bar{l}$, i.e., such that $x \lesssim_{j_n} \bar{l}$, we get $x \lesssim_{j_i} \bar{l}$, for all $i \neq n-1$, whence $\bar{l} \lesssim_{j_{n-1}} x$. And, in a similar fashion, assuming that $\bar{l}$ is among the largest limits in $\lesssim_{k+1}, \ldots, \lesssim_{j_n}$, we may similarly conclude that $\bar{l} \lesssim_{j_k} x$, for all $k = n-2, \ldots, 2$. This proves that $x = \nabla_{(j_{n-1},\ldots,j_1)}(\{x\}, \ldots, \{x\})$.

Conversely, to show the antiordinal property, suppose $x \lesssim_{j_1} y, \ldots, x \lesssim_{j_{n-1}} y$, whence, since $x = \nabla_{(j_{n-1},\ldots,j_1)}(\{x\}, \ldots, \{x\})$, we get $y \lesssim_{j_n} x$, as desired. Similarly, for the uniqueness property, suppose $x \sim_{j_1} y, \ldots, x \sim_{j_{n-1}} y$. Then, as above, we get $y \lesssim_{j_n} x$, and, reasoning symmetrically, using $y = \nabla_{(j_{n-1},\ldots,j_1)}(\{y\}, \ldots, \{y\})$, we get $x \lesssim_{j_n} y$, whence $x \sim_{j_n} y$. Thus $x = \nabla_{(j_{n-1},\ldots,j_1)}(\{x\}, \ldots, \{x\}) = \nabla_{(j_{n-1},\ldots,j_1)}(\{y\}, \ldots, \{y\}) = y$. ∎

Definition 7 *For a $(j_{n-1}, \ldots, j_1)$-join*

$$\nabla_{(j_{n-1},\ldots,j_1)}(\{x_{n-1,1}, \ldots, x_{n-1,i_{n-1}}\}, \ldots, \{x_{1,1}, \ldots, x_{1,i_1}\})$$

which exists in an n-ordered set $\langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ with $P \neq \emptyset$, we define the operation

$$\nabla^{i_{n-1}\ldots i_1}_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_1) := \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_1) :=$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\{x_{n-1,1}, \ldots, x_{n-1,i_{n-1}}\}, \ldots, \{x_{1,1}, \ldots, x_{1,i_1}\}),$$

*which will also be called the $(j_{n-1}, \ldots, j_1)$-*join *of $(\{x_{n-1,1}, \ldots, x_{n-1,i_{n-1}}\}, \ldots, \{x_{1,1}, \ldots, x_{1,i_1}\})$. This $(j_{n-1}, \ldots, j_1)$-joins will be called $(i_{n-1}, \ldots, i_1)$*-ary operations*, in the sense that they are functions*

$$\nabla^{i_{n-1}\ldots i_1}_{(j_{n-1},\ldots,j_1)} : P^{i_{n-1}} \times \ldots \times P^{i_1} \to P.$$

It is not difficult to see that the following commutative laws hold:

$$\nabla^{i_{n-1}\ldots i_1}_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_1) = \nabla^{i_{n-1}\ldots i_1}_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_{k+1}, \sigma(\vec{x}_k), \vec{x}_{k-1}, \ldots, \vec{x}_1),$$

for all $1 \leq k \leq n-1$ and all permutations σ of $\{1, \ldots, i_k\}$, where $\sigma(\vec{x}_k) = (x_{k,\sigma(1)}, \ldots, x_{k,\sigma(i_k)})$.

3 Reduction Theorem and n-Lattices

From this point on in the paper we make the following **global notational convention** to simplify cumbersome notation:

- Given nonnegative integers $k < l$ we write $x_{k,\ldots,l} := (x_k, x_{k+1}, \ldots, x_l)$ and $x_{l,\ldots,k} := (x_l, x_{l-1}, \ldots, x_k)$.
- Given nonnegative integers $k < l$ we write $X_{k,\ldots,l} := (X_k, X_{k+1}, \ldots, X_l)$ and $X_{l,\ldots,k} := (X_l, X_{l-1}, \ldots, X_k)$.

We also use freely the natural isomorphisms between different direct products without explicit mention, for instance, if $k < m < l$, $x_{k,\ldots,l} = (x_{k,\ldots,m}, x_{m+1,\ldots,l})$ has the obvious meaning.

Existence of joins of arity $(2, \dots, 2)$ clearly implies existence of all joins of arities $(\epsilon_1, \dots, \epsilon_{n-1})$, for all $\epsilon_i \in \{1, 2\}, i = 1, \dots, n-1$.

To prove the Reduction of Arity Theorem, the analog of Theorem 2.9 of [1], we need to be able to construct arbitrary joins from joins of arity $(2, \dots, 2)$. Thus, we need a way to go "upwards" in arities. The following two lemmas will serve this purpose.

Lemma 8 *Suppose that the $(j_{n-1}, \dots, j_1)$-joins of arities $(i_{n-1,\dots,k+1}, i_k - 1, i_{k-1,\dots,1})$, $(i_{n-1,\dots,k+1}, 1, i_{k-1,\dots,1})$ and $(i_{n-1,\dots,k+1}, 2, i_{k-1,\dots,1})$ exist, for $2 \le k \le n-1$. Then*

$$t := \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, (\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, \{x_{k,1}, \dots, x_{k,i_k-1}\}, \vec{x}_{k-1,\dots,1}),$$

$$\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, x_{k,i_k}, \vec{x}_{k-1,\dots,1})), \vec{x}_{k-1,\dots,1}) \in (\vec{x}_{n-1}, \dots, \vec{x}_1)^{(j_{n-1},\dots,j_1)}.$$

Proof:

We have $x_{p,q} \lesssim_{j_p} t$, for all $1 \le p \le n-1, p \ne k, 1 \le q \le i_p$, and

$$x_{k,q} \lesssim_{j_k} \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, \{x_{k,1}, \dots, x_{k,i_k-1}\}, \vec{x}_{k-1,\dots,1}) \lesssim_{j_k} t,$$

$1 \le q \le i_k - 1$, and

$$x_{k,i_k} \lesssim_{j_k} \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, x_{k,i_k}, \vec{x}_{k-1,\dots,1}) \lesssim_{j_k} t,$$

whence $t \in (\vec{x}_{n-1}, \dots, \vec{x}_1)^{(j_{n-1},\dots,j_1)}$. ∎

Lemma 9 *Suppose that the $(j_{n-1}, \dots, j_1)$-joins of arities $(i_{n-1,\dots,k+1}, i_k - 1, i_{k-1,\dots,1})$, $(i_{n-1,\dots,k+1}, 1, i_{k-1,\dots,1})$ and $(i_{n-1,\dots,k+1}, 2, i_{k-1,\dots,1})$ exist, for $2 \le k \le n-1$. Then*

$$t := \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, (\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, \{x_{k,1}, \dots, x_{k,i_k-1}\}, \vec{x}_{k-1,\dots,1}),$$

$$\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, x_{k,i_k}, \vec{x}_{k-1,\dots,1})), \vec{x}_{k-1,\dots,1}) \in (\vec{x}_{n-1}, \dots, \vec{x}_1)^{\overline{(j_{n-1},\dots,j_1)}}.$$

Proof:

By Lemma 8, $t \in (\vec{x}_{n-1}, \dots, \vec{x}_1)^{(j_{n-1},\dots,j_1)}$. So it suffices to show that, if $b \in (\vec{x}_{n-1}, \dots, \vec{x}_1)^{(j_{n-1},\dots,j_1)}$, then $b \lesssim_{j_n} t$. Let $b \in (\vec{x}_{n-1}, \dots, \vec{x}_1)^{(j_{n-1},\dots,j_1)} \ne \emptyset$, by Lemma 8. Then $b \in (\vec{x}_{n-1,\dots,k+1}, (x_{k,1}, \dots, x_{k,i_k-1}), \vec{x}_{k-1,\dots,1})^{(j_{n-1},\dots,j_1)}$ and $b \in (\vec{x}_{n-1,\dots,k+1}, x_{k,i_k}, \vec{x}_{k-1,\dots,1})^{(j_{n-1},\dots,j_1)}$. Thus

$$b \lesssim_{j_n} \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, (x_{k,1}, \dots, x_{k,i_k-1}), \vec{x}_{k-1,\dots,1}) \quad \text{and}$$

$$b \lesssim_{j_n} \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, x_{k,i_k}, \vec{x}_{k-1,\dots,1}). \tag{2}$$

Now consider the term

$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},b,\vec{x}_{k-1,\ldots,1}),$$

which exists by assumption. We have by (2), that both $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},(x_{k,1},\ldots,x_{k,i_k-1}),\vec{x}_{k-1,\ldots,1})$, and $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},x_{k,i_k},\vec{x}_{k-1,\ldots,1})$ are in $(\vec{x}_{n-1,\ldots,k+1},\{b\},\vec{x}_{k-1,\ldots,1})^{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}$, whence

$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},(x_{k,1},\ldots,x_{k,i_k-1}),\vec{x}_{k-1,\ldots,1})\lesssim_{j_k} \nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},b,\vec{x}_{k-1,\ldots,1})$$

and

$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},x_{k,i_k},\vec{x}_{k-1,\ldots,1})\lesssim_{j_k}\nabla_{(j_{n-1},\ldots,k+1,j_n,j_{k-1},\ldots,1)}(\vec{x}_{n-1,\ldots,k+1},b,\vec{x}_{k-1,\ldots,1}).$$

Therefore

$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},b,\vec{x}_{k-1,\ldots,1})\in(\vec{x}_{n-1,\ldots,k+1},\{\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},(x_{k,1},\ldots,x_{k,i_k-1}),\vec{x}_{k-1,\ldots,1}),\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},x_{k,i_k},\vec{x}_{k-1,\ldots,1})\},\vec{x}_{k-1,\ldots,1})^{(j_{n-1},\ldots,j_1)}. \tag{3}$$

Thus $b\lesssim_{j_n}\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},b,\vec{x}_{k-1,\ldots,1})\lesssim_{j_n} t$, whence $t\in(\vec{x}_{n-1},\ldots,\vec{x}_1)^{\overline{(j_{n-1},\ldots,j_1)}}$. ■

Lemma 10 *Suppose that the* $(j_{n-1},\ldots,j_1)$*-joins of arities* $(i_{n-1,\ldots,k+1},i_k-1,i_{k-1,\ldots,1})$, $(i_{n-1,\ldots,k+1},1,i_{k-1,\ldots,1})$ *and* $(i_{n-1,\ldots,k+1},2,i_{k-1,\ldots,1})$ *exist, for* $2\le k\le n-1$. *Then*

$$t:=\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},\{x_{k,1},\ldots,x_{k,i_k-1}\},\vec{x}_{k-1,\ldots,1}),\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},x_{k,i_k},\vec{x}_{k-1,\ldots,1})),\vec{x}_{k-1,\ldots,1})$$

is the $(j_{n-1},\ldots,j_1)$*-join of* $(\vec{x}_{n-1},\ldots,\vec{x}_1)$.

Proof:

By Lemma 9, $(\vec{x}_{n-1},\ldots,\vec{x}_1)^{\overline{(j_{n-1},\ldots,j_1)}}\neq\emptyset$. Assume $l\in(\vec{x}_{n-1},\ldots,\vec{x}_1)^{\overline{(j_{n-1},\ldots,j_1)}}$. The difficult inequality is the one that assures that $l\le_{j_k} t$, where $l\sim_{j_n} t,\ldots,l\sim_{j_{k+1}} t$. Since $l\in(\vec{x}_{n-1},\ldots,\vec{x}_1)^{(j_{n-1},\ldots,j_1)}$, we have, exactly as in the derivation of (3) in Lemma 9,

$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},l,\vec{x}_{k-1,\ldots,1})\in(\vec{x}_{n-1,\ldots,k+1},\{\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},(x_{k,1},\ldots,x_{k,i_k-1}),\vec{x}_{k-1,\ldots,1}),\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{k-1,\ldots,1},x_{k,i_k},\vec{x}_{k-1,\ldots,1})\},\vec{x}_{k-1,\ldots,1})^{(j_{n-1},\ldots,j_1)}. \tag{4}$$

But we also have $l \lesssim_{j_n} \nabla_{(j_{n-1},\dots,j_{k+1},j_n,j_{k-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, l, \vec{x}_{k-1,\dots,1})$, whence

$$\nabla_{(j_{n-1},\dots,j_{k+1},j_n,j_{k-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, l, \vec{x}_{k-1,\dots,1}) \in (\vec{x}_{n-1,\dots,k+1}, \{\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1},$$

$$(x_{k,1},\dots,x_{k,i_k-1}), \vec{x}_{k-1,\dots,1}), \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{k-1,\dots,1}, x_{k,i_k}, \vec{x}_{k-1,\dots,1})\}, \vec{x}_{k-1,\dots,1})^{\overline{(j_{n-1},\dots,j_1)}}.$$

Since also $l \in (\vec{x}_{n-1,\dots,k+1}, \{l\}, \vec{x}_{k-1,\dots,1})^{(j_{n-1},\dots,j_{k+1},j_n,j_{k-1},\dots,j_1)}$, i.e.,

$$l \lesssim_{j_k} \nabla_{(j_{n-1},\dots,j_{k+1},j_n,j_{k-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, l, \vec{x}_{k-1,\dots,1}),$$

we have that

$$l \lesssim_{j_k} \nabla_{(j_{n-1},\dots,j_{k+1},j_n,j_{k-1},\dots,j_1)}(\vec{x}_{n-1,\dots,k+1}, l, \vec{x}_{k-1,\dots,1}) \lesssim_{j_k} t.$$

That among all these l, we also have $l \lesssim_{j_{k-1}} t$, and so on, all the way down to j_2, i.e., that t is the $(j_{n-1},\dots,j_1)$-join of $(\vec{x}_{n-1},\dots,\vec{x}_1)$, now follows by the assumption on l and the join properties of t. ■

Lemma 11 *Suppose the joins of arity $(i_{n-2,\dots,1}, i_{n-1})$ and $(i_{n-2,\dots,1}, 1)$ exist. Then the joins of arity $(i_{n-1,\dots,1})$ also exist and*

$$\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,1}) = \nabla_{(j_{n-2},\dots,j_1,j_n)}(\vec{x}_{n-2,\dots,1}, \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1})).$$

Proof:
Let $t = \nabla_{(j_{n-2},\dots,j_1,j_n)}(\vec{x}_{n-2,\dots,1}, \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1}))$. We have

$$\nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1}) \in (\vec{x}_{n-2,\dots,1}, \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1}))^{(j_{n-2},\dots,j_1,j_n)},$$

whence $\nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1}) \lesssim_{j_{n-1}} t$, i.e., $t \in (\vec{x}_{n-1},\dots,\vec{x}_1)^{(j_{n-1},\dots,j_1)}$. This also yields $t \lesssim_{j_n} \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1})$. But we also have, by t's definition, that $t \gtrsim_{j_n} \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1})$, whence $t \sim_{j_n} \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1})$ and, therefore, $t \in (\vec{x}_{n-1},\dots,\vec{x}_1)^{\overline{(j_{n-1},\dots,j_1)}}$.

Now suppose that $l \in (\vec{x}_{n-1},\dots,\vec{x}_1)^{\overline{(j_{n-1},\dots,j_1)}}$. Then $l \sim_{j_n} \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1})$, whence $l \in (\vec{x}_{n-2,\dots,1}, \nabla_{(j_{n-2},\dots,j_1,j_{n-1})}(\vec{x}_{n-2,\dots,1}, \vec{x}_{n-1}))^{(j_{n-2},\dots,j_1,j_n)}$. Thus $l \lesssim_{j_{n-1}} t$. Similarly, if $l \in (\vec{x}_{n-1,\dots,1})^{\overline{(j_{n-1},\dots,j_1)}}$, such that for all other limit $l', l' \le_{j_{n-1}} l$, we have that $l \sim_{j_{n-1}} t$, whence $l \lesssim_{j_{n-2}} t$, by the join property of t. The same applies all the way down to j_2 and, hence, $t = \nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1\dots,3}, \vec{x}_2, \vec{x}_1)$. ■

Theorem 12 (Reduction of Arity Theorem) *If in an n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \dots, \lesssim_n \rangle$ all $(j_{n-1},\dots,j_1)$-joins of arity $(2,2,\dots,2)$ exist, then all finitary $(j_{n-1},\dots,j_1)$-joins $\nabla^{i_{n-1}\dots i_1}_{(j_{n-1},\dots,j_1)}$ exist.*

Proof:

We use the technique of [1] by first reducing the rank of the first $n-2$ arguments, using Lemma 10, then reducing the rank of the $n-1$-st argument by possibly increasing the rank of the first $n-2$ arguments again, using Lemma 11, and, finally, reducing the rank of the first $n-2$ arguments once more, using again Lemma 10.

If all arguments of the join are at most binary, then the join exists by hypothesis. If there exists at least one argument with arity at least 3, then we follow the following steps:

Step 1: If any of the first $n-2$ arguments has arity at least 3, then use Lemma 10 to write the join as a join of joins of lower arities until the arities of the first $n-2$ arguments are at most 2.

Step 2: Now, if the $(n-1)$-st argument has arity at least 3, use Lemma 11 with $i_{n-1} \leq 2$, to reduce the arity of the $(n-1)$-st argument to less than or equal to 2 by increasing the arity of the $(n-2)$-nd argument.

Step 3: Finally, if step 2 was performed, use Lemma 10 once more to reduce once more the arity of the $(n-2)$-nd argument that was increased in step 2.

■

In a complete n-lattice $\langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ the following identities hold:

$$\nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1,\ldots,k+1}, X_k \cup Y_k, X_{k-1,\ldots,1}) =$$

$$\nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1,\ldots,k+1}, \{\nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1,\ldots,k+1}, X_k, X_{k-1,\ldots,1}), \\ \nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1,\ldots,k+1}, Y_k, X_{k-1,\ldots,1})\}, X_{k-1,\ldots,1})$$

and

$$\nabla_{(j_{n-1},\ldots,j_1)}(X_{n-1,\ldots,1}) = \nabla_{(j_{n-2},\ldots,j_1,j_{n-1})}(X_{n-2,\ldots,1}, \nabla_{(j_{n-2},\ldots,j_1,j_{n-1}}(X_{n-2,\ldots,1}, X_{n-1})),$$

for all $X_1, \ldots, X_n, Y_k \subseteq L$, all $1 \leq k \leq n-1$, and all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$.

For n-ordered sets $\langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ in which all $(2, 2, \ldots, 2)$-ary $(j_{n-1}, \ldots, j_1)$-joins exist, we may use the first identity above to write

$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, \vec{x}_k, \vec{x}_{k-1,\ldots,1}) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},$$

$$(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, \vec{u}, \vec{x}_{k-1,\ldots,1}), \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, \vec{v}, \vec{x}_{k-1,\ldots,1})), \vec{x}_{k-1}, \vec{x}_1),$$

where $\vec{x}_l \in L^{i_l}, \vec{u} = (u_1, \ldots, u_p) \in L^p$ and $\vec{v} = (v_1, \ldots, v_q) \in L^q$, with $\{x_{k,1}, \ldots, x_{k,i_k}\} = \{u_1, \ldots, u_p, v_1, \ldots, v_q\}$.

Definition 13 *An* n-lattice *is an* n*-ordered set* $\langle L, \lesssim_1, \ldots, \lesssim_n\rangle$ *in which all the* $(2, \ldots, 2)$*-ary* $(j_{n-1}, \ldots, j_1)$*-joins exist. The following notation will be used for these joins:*

$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_1) := \nabla_{(j_{n-1},\ldots,j_1)}(x_{n-1,1}, x_{n-1,2}, \ldots, x_{1,1}, x_{1,2}) :=$$

$$\nabla_{(j_{n-1},\ldots,j_1)}((x_{n-1,1}, x_{n-1,2}), \ldots, (x_{1,1}, x_{1,2})),$$

where $\vec{x}_i = (x_{i,1}, x_{i,2}), x_{i,1}, x_{i,2} \in L, i = 1, \ldots, n-1$. *The* derived operations *of arities* $(\epsilon_{n-1}, \ldots, \epsilon_1)$*, where* $\epsilon_i \in \{1, 2\}$*,* $i = 1, \ldots, n-1$*, are all the operations defined by the joins above by taking the argument corresponding to* $\vec{x}_i$ *to be* $\vec{x}_i = (x_{i,1}, x_{i,2})$*, if* $\epsilon_i = 2$*, and* $\vec{x}_i = (x_i, x_i)$*, if* $\epsilon_i = 1$*,* $i = 1, \ldots, n-1$.

4 Equational Base for n-Lattices

The following theorem generalizes Theorem 3.1 of Biedermann [1] to n-lattices. It is very interesting to point out that the equational theory consists of the nine axioms (T1)-(T9) of Biedermann, appropriately generalized to the context of n-lattices.

Theorem 14 *In an* n*-lattice* $\langle L, \lesssim_1, \ldots, \lesssim_n\rangle$ *the following equations hold for all* $\{j_1, \ldots, j_n\} = \{j'_1, \ldots, j'_n\} = \{1, \ldots, n\}$ *and for all* $\vec{x}_i = (x_{i,1}, x_{i,2}), i = 1, \ldots, n$*,*

1. (Idempotent Law) $\nabla_{(j_{n-1},\ldots,j_1)}(x, x, \ldots, x) = x$

2. (k-th Component Commutativity) $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, \vec{x}_k, \vec{x}_{k-1,\ldots,1}) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, (x_{k,2}, x_{k,1}), \vec{x}_{k-1,\ldots,1}), k = 2, \ldots, n-1,$

3. (Bound Laws) $\nabla_{(j'_{n-1},\ldots,j'_1)}(\vec{y}_{n-1,\ldots,k+1}, (x_{l,1}, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})), \vec{y}_{k-1,\ldots,1}) = \nabla_{(j'_{n-1},\ldots,j'_1)}(\vec{y}_{n-1,\ldots,k+1}, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \vec{y}_{k-1,\ldots,1})$*, where* $j'_k = j_l, 2 \leq k \leq n-1$.

4. (Limit Laws) $\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \ldots, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),$ $(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \nabla_{(j_{n-1},\ldots,j_{k+1},j_1,j_{k-1},\ldots,j_2,j_k)}(\vec{x}_{n-1,\ldots,k+1}, \vec{x}_1, \vec{x}_{k-1,\ldots,2}, \vec{x}_k)),$ $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \ldots, \nabla_{(j_{n-1},\ldots,1)}(\vec{x}_{n-1,\ldots,1})) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),\ 2 \leq k \leq n-1.$

5. (Antiordinal Laws) $\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), \ldots,$ $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), (\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}),$ $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})), \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), \ldots, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1},$ $x_{k,1}, \vec{x}_{k-1,\ldots,1}) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1})),\ 2 \leq k \leq n-1.$

6. (Commutative Laws) $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_1) = \nabla_{(j_{n-2},\ldots,j_1,j_n)}(\vec{x}_{n-2}, \ldots, \vec{x}_1, \nabla_{(j_{n-2},\ldots,j_1,j_{n-1})}(\vec{x}_{n-2}, \ldots, \vec{x}_1, \vec{x}_{n-1}))$

7. (Separation Laws) $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, (\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,2}, \vec{x}_{k-1,\ldots,1})), \vec{x}_{k-1,\ldots,1})), \vec{x}_{k-1,\ldots,1})$, $2 \leq k \leq n-1$

8. (Absorption Laws) $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1},\ldots,\vec{x}_1) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1},\ldots,\vec{x}_{k+1}, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1},\ldots,\vec{x}_1), \vec{x}_{k-1},\ldots,\vec{x}_1)$, $2 \leq k \leq n-1$,

9. (Associative Laws) $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, (\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, (x_{k,1}, x_{k,2}), \vec{x}_{k-1,\ldots,1}), \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n,\ldots,k+1}, x_{k,3}, \vec{x}_{k-1,\ldots,1})), \vec{x}_{k-1,\ldots,1}) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, (\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n,\ldots,k+1}, (x_{k,2}, x_{k,3}), \vec{x}_{k-1,\ldots,1})), \vec{x}_{k-1,\ldots,1})$ $2 \leq k \leq n-1$.

Proof:

Proposition 6 takes care of the idempotent law. The commutative law was the content of Lemma 11. The absorption, the associative laws, and the separation laws follow from Lemma 10. The k-th component commutative laws are reflecting the set theoretic origin of the arguments. So the laws whose validity remains to be demonstrated are the bound, the limit and the antiordinal laws.

For the bound laws, observe that $x_{l,1} \lesssim_{j_l} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$, whence

$$(\vec{y}_{n-1,\ldots,k+1}, \{x_{l,1}, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})\}, \vec{y}_{k-1,\ldots,1})^{(j'_{n-1},\ldots,j'_1)} =$$

$$(\vec{y}_{n-1,\ldots,k+1}, \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \vec{y}_{k-1,\ldots,1})^{(j'_{n-1},\ldots,j'_1)}$$

and the laws follow.

For the limit laws, set

$$\begin{aligned} t &= \nabla_{(j_{n-1},\ldots,k+1,j_n,j_{k-1},\ldots,1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \\ &\quad (\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \nabla_{(j_{n-1},\ldots,j_{k+1},j_1,j_{k-1},\ldots,j_2,j_k)}(\vec{x}_{n-1,\ldots,k+1}, \vec{x}_1, \vec{x}_{k-1,\ldots,2}, \vec{x}_k)), \\ &\quad \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})). \end{aligned}$$

Observe, first, that $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}) \sim_{j_n} \nabla_{(j_{n-1},\ldots,j_{k+1},j_1,j_{k-1},\ldots,j_2,j_k)}(\vec{x}_{n-1,\ldots,k+1}, \vec{x}_1, \vec{x}_{k-1,\ldots,2}, \vec{x}_k)$. The bound property of t yields $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}) \lesssim_{j_i} t, i \neq k$. This yields, by the antiordinal property, $t \lesssim_{j_k} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$. But we also have $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}) \lesssim_{j_k} t$, by the limit property of t. Hence $t \sim_{j_k} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$. The equality now follows.

For the antiordinal laws, let

$$\begin{aligned} t &= \nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}),\ldots, \\ &\quad \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), (\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}), \\ &\quad \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})), \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}),\ldots, \\ &\quad \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1})). \end{aligned}$$

By the bound property for t we get $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}) \lesssim_{j_i} t$, for $i \neq k$. Thus, by the antiordinal property, $t \lesssim_{j_k} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1})$. But the limit property of t yields $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1}) \lesssim_{j_k} t$, whence $t \sim_{j_k} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,k+1}, x_{k,1}, \vec{x}_{k-1,\ldots,1})$. Now the bound properties, combined with this equivalence yield the required equality. ■

Lemma 15 *In an n-lattice $\langle L, \lesssim_1, \ldots, \lesssim_n\rangle$, the following equivalences hold, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\} = \{j'_1, \ldots, j'_n\}$, where $j_k = j'_l$, and all $x_1, x_2 \in L$,*

$$\begin{aligned} x_1 \lesssim_{j_k} x_2 &\iff x_2 = \nabla_{(j_{n-1},\ldots,j_{k-1},j_k,j_{k-1},\ldots,j_1)}(x_2, \ldots, x_2, (x_1, x_2), x_2, \ldots, x_2) \\ &\iff x_2 = \nabla_{(j'_{n-1},\ldots,j'_{l-1},j'_l,j'_{l-1},\ldots,j'_1)}(x_2, \ldots, x_2, (x_1, x_2), x_2, \ldots, x_2) \end{aligned}$$

where in the above equations, (x_1, x_2) appears in the $j_k = j'_l$ position.

Proof:

Suppose, first that $x_1 \lesssim_{j_k} x_2$. Then

$$(\{x_2\}, \ldots, \{x_2\}, \{x_1, x_2\}, \{x_2\}, \ldots, \{x_2\})^{(j_{n-1},\ldots,j_1)} = (\{x_2\}, \ldots, \{x_2\})^{(j_{n-1},\ldots,j_1)},$$

whence $\nabla_{(j_{n-1},\ldots,j_1)}(x_2, \ldots, x_2, (x_1, x_2), x_2, \ldots, x_2) = \nabla_{(j_{n-1},\ldots,j_1)}(x_2, \ldots, x_2) = x_2$, by Proposition 6.

Conversely, if $x_2 = \nabla_{(j_{n-1},\ldots,j_1)}(x_2, \ldots, x_2, (x_1, x_2), x_2, \ldots, x_2)$, then $x_1 \lesssim_{j_k} x_2$ follows from the fact that joins are bounds. ■

The following lemmata will be used in the proof of the main theorem, showing that the equations in Theorem 14 form in fact an equational basis for the theory of n-lattices.

The *following are common assumptions for Lemmata* 16-20:

$\langle L, \{\nabla_{(j_{n-1},\ldots,j_1)}\}_{\{j_1,\ldots,j_n\}=\{1,\ldots,n\}}\rangle$ is a nonempty set equipped with $n!$ $(2, 2, \ldots, 2)$-ary operations, which satisfy the equations in Theorem 14. $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$ $= \{j'_1, \ldots, j'_n\}$. All variables denote elements or vectors of elements of L.

Lemma 16 (Last Component Commutativity) $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_1) = \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, \ldots, \vec{x}_2, (x_{1,2}, x_{2,1}))$.

Proof:

$$\begin{aligned} &\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}) = \\ &= \nabla_{(j_{n-2},\ldots,j_1,j_n)}(\vec{x}_{n-2,\ldots,1}, \nabla_{(j_{n-2},\ldots,j_1,j_{n-1})}(\vec{x}_{n-2,\ldots,1}, \vec{x}_{n-1})) \\ &= \nabla_{(j_{n-2},\ldots,j_1,j_n)}(\vec{x}_{n-2,\ldots,2}, (x_{1,2}, x_{1,1}), \nabla_{(j_{n-2},\ldots,j_1,j_{n-1})}(\vec{x}_{n-2,\ldots,2}, (x_{1,2}, x_{1,1}), \vec{x}_{n-1})) \\ &= \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2}, (x_{1,2}, x_{1,1})) \end{aligned}$$

the first and third equalities follow by the commutative laws and the second follows by $(n-2)$-component commutativity. ■

Lemma 17 *If* $\nabla_{(j_{n-1},\ldots,j_1)}(x_2,\ldots,x_2,(x_1,x_2),x_2,\ldots,x_2)=x_2$, *where* (x_1,x_2) *appears in the* j_k *position,* $2\le k\le n-1$, *then*

$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},(x_1,x_2),\vec{x}_{l-1,\ldots,1}) =$$

$$= \nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},x_2,\vec{x}_{l-1,\ldots,1}),$$

where $j_k=j'_l, l=1,\ldots,n-1$.

Proof:
For $2\le l\le n-1$, we have

$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},(x_1,x_2),\vec{x}_{l-1,\ldots,1}) =$$

$$\begin{aligned}
&= \nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},(x_1,\nabla_{(j_{n-1},\ldots,j_1)}(x_2,\ldots,x_2,(x_1,x_2),\\
&\quad x_2,\ldots,x_2)),\vec{x}_{l-1,\ldots,1})\\
&= \nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},\nabla_{(j_{n-1},\ldots,j_1)}(x_2,\ldots,x_2,(x_1,x_2),\\
&\quad x_2,\ldots,x_2),\vec{x}_{l-1,\ldots,1})\\
&= \nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},x_2,\vec{x}_{l-1,\ldots,1}),
\end{aligned}$$

where the first equality holds by hypothesis, the second by the bound laws and the last again by hypothesis. Now for $l=1$,

$$\nabla_{(j'_{n-1},\ldots,j'_2,j'_1)}(\vec{x}_{n-1,\ldots,2},(x_1,x_2)) =$$

$$\begin{aligned}
&= \nabla_{(j'_{n-2},\ldots,j'_1,j'_n)}(\vec{x}_{n-2,\ldots,2},(x_1,x_2),\nabla_{(j'_{n-2},\ldots,j'_1,j'_{n-1})}(\vec{x}_{n-2,\ldots,2},(x_1,x_2),\vec{x}_{n-1}))\\
&= \nabla_{(j'_{n-2},\ldots,j'_1,j'_n)}(\vec{x}_{n-2,\ldots,2},x_2,\nabla_{(j'_{n-2},\ldots,j'_1,j'_{n-1})}(\vec{x}_{n-2,\ldots,2},x_2,\vec{x}_{n-1}))\\
&= \nabla_{(j'_{n-1},\ldots,j'_2,j'_1)}(\vec{x}_{n-1,\ldots,2},x_2),
\end{aligned}$$

where, the first equality holds by commutativity, the second by the first case presented above and the last one again by commutativity. ∎

Lemma 18 *Suppose* $\nabla_{(j_{n-1},\ldots,j_1)}(x_2,\ldots,x_2,(x_1,x_2),x_2,\ldots,x_2)=x_2$ *and*

$$\nabla_{(j_{n-1},\ldots,j_1)}(x_1,\ldots,x_1,(x_2,x_1),x_1,\ldots,x_1)=x_1,$$

where (x_1,x_2) *and* (x_2,x_1) *appear in the* j_k *position. Then*

$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},(x_1,x),\vec{x}_{l-1,\ldots,1}) =$$

$$= \nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(\vec{x}_{n-1,\ldots,l+1},(x_2,x),\vec{x}_{l-1,\ldots,1}),$$

where $j_k=j'_l$, *for* $1\le k\le n-1$.

Proof:

We have for $2 \leq l \leq n-1$,

$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(x_1,x),\vec{x}_{l-1,\dots,1}) =$$

$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},x_1,\vec{x}_{l-1,\dots,1}),$$
$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},x,\vec{x}_{l-1,\dots,1})),\vec{x}_{l-1,\dots,1}) =$$
$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(x_2,x_1),$$
$$\vec{x}_{l-1,\dots,1}),\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},x,\vec{x}_{l-1,\dots,1})),\vec{x}_{l-1,\dots,1}) =$$
$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(x_1,x_2),$$
$$\vec{x}_{l-1,\dots,1}),\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},x,\vec{x}_{l-1,\dots,1})),\vec{x}_{l-1,\dots,1}) =$$
$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},x_2,$$
$$\vec{x}_{l-1,\dots,1}),\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},x,\vec{x}_{l-1,\dots,1})),\vec{x}_{l-1,\dots,1}) =$$
$$\nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(\vec{x}_{n-1,\dots,l+1},(x_2,x),\vec{x}_{l-1,\dots,1}),$$

where the first equality holds by separation, the second by Lemma 17, the third by k-th component commutativity, the fourth by Lemma 17 and the last by separation.

For $l = 1$,

$$\nabla_{(j'_{n-1},\dots,j'_2,j'_1)}(\vec{x}_{n-1,\dots,2},(x_1,x)) =$$
$$\nabla_{(j'_{n-2},\dots,j'_1,j'_n)}(\vec{x}_{n-2,\dots,2},(x_1,x),\nabla_{(j'_{n-2},\dots,j'_1,j'_{n-1})}(\vec{x}_{n-2,\dots,2},(x_1,x),\vec{x}_{n-1})) =$$
$$\nabla_{(j'_{n-2},\dots,j'_1,j'_n)}(\vec{x}_{n-2,\dots,2},(x_2,x),\nabla_{(j'_{n-2},\dots,j'_1,j'_{n-1})}(\vec{x}_{n-2,\dots,2},(x_2,x),\vec{x}_{n-1})) =$$
$$\nabla_{(j'_{n-1},\dots,j'_2,j'_1)}(\vec{x}_{n-1,\dots,2},(x_2,x)),$$

where the first equality holds by commutativity, the second by the first part above and the last one again by commutativity. ■

Lemma 19 (Last Component Antiordinal Laws)

$$\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1}) =$$

$$\nabla_{(j_{n-1},\dots,j_{k+1},j_n,j_{k-1},\dots,j_1)}(\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1}),\dots,\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1}),$$
$$(\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},\vec{x}_1),\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1})),\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1}),\dots,$$
$$\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1}),\nabla_{(j_{n-1},\dots,j_1)}(\vec{x}_{n-1,\dots,2},x_{1,1}))$$

Proof:

$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),$$
$$(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1})),\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}))=$$
$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),$$
$$(\nabla_{(j_{n-1},\ldots,j_{k+1},j_1,j_{k-1},\ldots,j_2,j_k)}(\vec{x}_{n-1,\ldots,k+1},\vec{x}_1,\vec{x}_{k-1,\ldots,2},\vec{x}_k),$$
$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_1,j_{k-1},\ldots,j_2,j_k)}(\vec{x}_{n-1,\ldots,k+1},x_{1,1},\vec{x}_{k-1,\ldots,2},\vec{x}_k)),$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}))=$$
$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),$$
$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_1,j_{k-1},\ldots,j_2,j_k)}(\vec{x}_{n-1,\ldots,k+1},x_{1,1},\vec{x}_{k-1,\ldots,2},\vec{x}_k),$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}))=$$
$$\nabla_{(j_{n-1},\ldots,j_{k+1},j_n,j_{k-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),\ldots,\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}))=$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2},x_{1,1}),$$

where the first equality holds by the limit laws and Lemma 18, the second by the antiordinal laws and Lemma 17, the third by the limit laws and Lemma 18 once more and the last by the idempotent laws. ■

Lemma 20 (Additional Bound Laws)

$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{y}_{n-1,\ldots,l+1},\vec{y}_{l-1,\ldots,1},(x_{k,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})))=$$
$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{y}_{n-1,\ldots,l+1},\vec{y}_{l-1,\ldots,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}))$$

and

$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{y}_{n-1,\ldots,k+1},(x_{l,1},\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{x}_{n-1,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l)),\vec{y}_{k-1,\ldots,1})=$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{y}_{n-1,\ldots,k+1},\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{x}_{n-1,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l),\vec{y}_{k-1,\ldots,1}),$$

where $j_k=j'_l, 2\leq k\leq n-1$.

Proof:
For the first equation we have

$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{y}_{n-1,\ldots,l+1},\vec{y}_{l-1,\ldots,1},(x_{k,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}))) =$$
$$\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l,j'_n)}(\vec{y}_{n-2,\ldots,l+1},\vec{y}_{l-1,\ldots,1},(x_{k,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})),$$
$$\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1 j'_l,j'_{n-1})}(\vec{y}_{n-2,\ldots,l+1},\vec{y}_{l-1,\ldots,1},(x_{k,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})),\vec{y}_{n-1}))) =$$
$$\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l,j'_n)}(\vec{y}_{n-2,\ldots,l+1},\vec{y}_{l-1,\ldots,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),$$
$$\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1 j'_l,j'_{n-1})}(\vec{y}_{n-2,\ldots,l+1},\vec{y}_{l-1,\ldots,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),\vec{y}_{n-1}))) =$$
$$\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{y}_{n-1,\ldots,l+1},\vec{y}_{l-1,\ldots,1},\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})),$$

where the first equality holds by commutativity, the second from the bound laws and the third again by commutativity.

For the second equation:

$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{y}_{n-1,\ldots,k+1},(x_{l,1},\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{x}_{n-1,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l)),\vec{y}_{k-1,\ldots,1}) =$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{y}_{n-1,\ldots,k+1},(x_{l,1},\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l,j'_n)}(\vec{x}_{n-2,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l,$$
$$\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l,j'_{n-1})}(\vec{x}_{n-2,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l,\vec{x}_{n-1}))),\vec{y}_{k-1,\ldots,1}) =$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{y}_{n-1,\ldots,k+1},\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l,j'_n)}(\vec{x}_{n-2,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l,$$
$$\nabla_{(j'_{n-2},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l,j'_{n-1})}(\vec{x}_{n-2,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l,\vec{x}_{n-1})),\vec{y}_{k-1,\ldots,1}) =$$
$$\nabla_{(j_{n-1},\ldots,j_1)}(\vec{y}_{n-1,\ldots,k+1},\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{x}_{n-1,\ldots,l+1},\vec{x}_{l-1,\ldots,1},\vec{x}_l),\vec{y}_{k-1,\ldots,1}),$$

where the first equality holds by commutativity, the second by the bound laws and the last also by commutativity. ■

Theorem 21 *Let $\langle L,\{\nabla_{(j_{n-1},\ldots,j_1)}\}_{\{j_1,\ldots,j_n\}=\{1,\ldots,n\}}\rangle$ be a nonempty set equipped with $n!$ $(2,2,\ldots,2)$-ary operations, which satisfy the equations in Theorem 14. Then, for all $\{j_1,\ldots,j_n\}=\{1,\ldots,n\}=\{j'_1,\ldots,j'_n\}$, with $j_k=j'_l$, $x_1,x_2\in L$, we have*

$$\begin{array}{l} x_2=\nabla_{(j_{n-1},\ldots,j_{k+1},j_k,j_{k-1},\ldots,j_1)}(x_2,\ldots,x_2,(x_1,x_2),x_2,\ldots,x_2) \iff \\ x_2=\nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_l,j'_{l-1},\ldots,j'_1)}(x_2,\ldots,x_2,(x_1,x_2),x_2,\ldots,x_2) \end{array}$$

Moreover, $\langle L,\lesssim_1,\ldots,\lesssim_n\rangle$ is an n-lattice, where

$$x_1\lesssim_{j_k} x_2 \Leftrightarrow x_2=\nabla_{(j_{n-1},\ldots,j_{k+1},j_k,j_{k-1},\ldots,j_1)}(x_2,\ldots,x_2,(x_1,x_2),x_2,\ldots,x_2),$$

where (x_1,x_2) appears in the j_k position. In particular, the $n!$ operations are the $(j_{n-1},\ldots,j_1)$-joins in $\langle L,\lesssim_1,\ldots,\lesssim_n\rangle$ as defined previously order-theoretically.

Proof:
We first show that

$$x_2 = \nabla_{(j_{n-1},\dots,j_{k+1},j_k,j_{k-1},\dots,j_1)}(x_2,\dots,x_2,(x_1,x_2),x_2,\dots,x_2) \iff$$
$$x_2 = \nabla_{(j'_{n-1},\dots,j'_{l+1},j'_l,j'_{l-1},\dots,j'_1)}(x_2,\dots,x_2,(x_1,x_2),x_2,\dots,x_2),$$

where $2 \le k, l \le n-1$. So suppose that $x_2 = \nabla_{(j_{n-1},\dots,j_1)}(x_2,\dots,x_2,(x_1,x_2),x_2,\dots,x_2)$, where (x_1,x_2) appears in the $j_k = j'_l$ position. Then we have

$$\nabla_{(j'_{n-1},\dots,j'_1)}(x_2,\dots,x_2,(x_1,x_2),x_2,\dots,x_2) = \nabla_{(j'_{n-1},\dots,j'_1)}(x_2,\dots,x_2) = x_2,$$

where the first equality follows from Lemma 17 and the second from the idempotent law. The reverse implication holds by symmetry. Next, show the same equivalence for the case where exactly one of $k = 1$ or $l = 1$ holds. Assume, without loss of generality due to symmetry, that $l = 1$. Then the implication left to right follows in the same way as before. For the reverse implication suppose that $x_2 = \nabla_{(j'_{n-1},\dots,j'_1)}(x_2,\dots,x_2,(x_1,x_2))$, where (x_1,x_2) appears in the $j'_l = j'_1$ position. Then we have

$$\nabla_{(j_{n-1},\dots,j_1)}(x_2,\dots,x_2,(x_1,x_2),x_2,\dots,x_2) =$$
$$\nabla_{(j_{n-1},\dots,j_1)}(x_2,\dots,x_2,(x_1,\nabla_{(j'_{n-1},\dots,j'_1)}(x_2,\dots,x_2,(x_1,x_2))),x_2,\dots,x_2) =$$
$$\nabla_{(j_{n-1},\dots,j_1)}(x_2,\dots,x_2,\nabla_{(j'_{n-1},\dots,j'_1)}(x_2,\dots,x_2,(x_1,x_2)),x_2,\dots,x_2) =$$
$$\nabla_{(j_{n-1},\dots,j_1)}(x_2,\dots,x_2,x_2,x_2,\dots,x_2) = x_2,$$

where the first equality follows by hypothesis, the second by Lemma 20, the third by the hypothesis and the last by the idempotent law.

Having established these equivalences, we may now define $x \lesssim_{j_k} y$ if and only if $y = \nabla_{(j_{n-1},\dots,j_1)}(y,\dots,y,(x,y),y,\dots,y)$, where (x,y) appears in the j_k-th position. This is independent of the choice of $j_i, i = 1,\dots,n-1$, and of their order because of the equivalences proved above.

The idempotent law yields immediately the reflexivity of $\lesssim_{j_k}$.

For the transitivity, let $x \lesssim_{j_k} y$ and $y \lesssim_{j_k} z$. Then, by definition,

$$y = \nabla_{(j_{n-1},\dots,j_1)}(y,\dots,y,(x,y),y,\dots,y) \text{ and } z = \nabla_{(j_{n-1},\dots,j_1)}(z,\dots,z,(y,z),z,\dots,z).$$

Thus, we get

$$\nabla_{(j_{n-1},\dots,j_1)}(z,\dots,z,(x,z),z,\dots,z) =$$

$$\begin{aligned}
&= \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(\nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,x,z,\ldots,z),\\
&\quad \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,z,z,\ldots,z)),z,\ldots,z)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(\nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,x,z,\ldots,z),z),z,\ldots,z)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(\nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,x,z,\ldots,z)\\
&\quad \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(y,z),z,\ldots,z)),z,\ldots,z)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(\nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(x,y),z,\ldots,z),\\
&\quad \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,z,z,\ldots,z)),z,\ldots,z)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(\nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,y,z,\ldots,z),\\
&\quad \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,z,z,\ldots,z)),z,\ldots,z)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(z,\ldots,z,(y,z),z,\ldots,z)\\
&= z,
\end{aligned}$$

where the first equality holds by the separation laws, the second by the idempotent laws, the third by the hypothesis, the fourth by associativity, the fifth by Lemma 17, the sixth by separation once more and the last by the hypothesis.

We have thus shown that $\lesssim_{j_k}$, as defined above, is a quasiordering on L.

To complete the first part of the theorem, it now remains to show the uniqueness condition and the antiordinal dependency.

For the uniqueness, suppose that $x \sim_{j_k} y$, for all $1 \le k \le n-1$. Then

$$\begin{aligned}
x &= \nabla_{(j_{n-1},\ldots,j_1)}(x,\ldots,x)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(y,x,\ldots,x)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(y,y,x\ldots,x)\\
&= \ldots\\
&= \nabla_{(j_{n-1},\ldots,j_1)}(y,\ldots,y)\\
&= y,
\end{aligned}$$

where the first and the last equalities hold by the idempotent law and all intermediate ones by Lemma 17. This proves uniqueness. For the antiordinal dependency suppose $x \lesssim_{j_1} y,\ldots,x \lesssim_{j_{n-1}} y$. Then

$$\begin{aligned}
y &= \nabla_{(j_{n-1},\ldots,j_1)}((x,y),y,\ldots,y)\\
&= \nabla_{(j_{n-1},\ldots,j_1)}((x,y),(x,y),y,\ldots,y)\\
&= \ldots\\
&= \nabla_{(j_{n-1},\ldots,j_1)}((x,y),(x,y),\ldots,(x,y))\\
&\lesssim_{j_n} \nabla_{(j_{n-1},\ldots,j_1)}(x,(x,y),\ldots,(x,y))\\
&\lesssim_{j_n} \nabla_{(j_{n-1},\ldots,j_1)}(x,x,(x,y),\ldots,(x,y))\\
&\lesssim_{j_n} \ldots\\
&\lesssim_{j_n} \nabla_{(j_{n-1},\ldots,j_1)}(x,\ldots,x,(x,y))\\
&\lesssim_{j_n} \nabla_{(j_{n-1},\ldots,j_1)}(x,\ldots,x)\\
&= x,
\end{aligned}$$

where the first equality follows by the hypothesis, the series of subsequent equalities follow by Lemma 17, the series of the inequalities by the antiordinal laws, the last inequality by Lemma 19 and the last equality by the idempotent laws. This completes the part of the theorem asserting that $\langle L, \lesssim_1, \lesssim_2, \ldots, \lesssim_n\rangle$ is an n-ordered set.

Finally, we need to show that it is an n-lattice, where $\nabla_{(j_{n-1},\ldots,j_1)}, \{j_1, \ldots, j_n\} = \{1, \ldots, n\}$ are the order-theoretically defined $(j_{n-1}, \ldots, j_1)$-joins in $\langle L, \lesssim_1, \ldots, \lesssim_n\rangle$.

By Lemma 20, setting all y's equal to

$$t = \nabla_{(j'_{n-1},\ldots,j'_{l+1},j'_{l-1},\ldots,j'_1,j'_l)}(\vec{x}_{n-1,\ldots,l+1}, \vec{x}_{l-1,\ldots,1}, \vec{x}_l),$$

we get

$$\nabla_{(j_{n-1},\ldots,j_1)}(t, \ldots, t, (x_{l,1}, t), t, \ldots, t) = \nabla_{(j_{n-1},\ldots,j_1)}(t, \ldots, t, t, t, \ldots, t),$$

whence $x_{l,1} \lesssim_{j_l} t$. This establishes the bound property of the joins for the last component. The other components may be treated similarly using the bound laws instead of Lemma 20. Thus we have that $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$ is the $(j_{n-1}, \ldots, j_1)$-bound of $(\vec{x}_{n-1}, \ldots, \vec{x}_1)$.

Next, we show that $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$ is a $(j_{n-1}, \ldots, j_1)$-limit of $(\vec{x}_{n-1}, \ldots, \vec{x}_1)$. So suppose that $x_{i,1}, x_{i,2} \lesssim_{j_i} b$, for all $1 \leq i \leq n-1$. Then we have

$$\begin{array}{rcl}
b & = & \nabla_{(j_{n-1},\ldots,j_1)}(b, \ldots, b) \\
 & = & \nabla_{(j_{n-1},\ldots,j_1)}((x_{n-1,1}, b), b, \ldots, b) \\
 & = & \nabla_{(j_{n-1},\ldots,j_1)}((\nabla_{(j_{n-1},\ldots,j_1)}(x_{n-1,1}, b, \ldots, b), \nabla_{(j_{n-1},\ldots,j_1)}(b, b, \ldots, b)), b, \ldots, b) \\
 & = & \nabla_{(j_{n-1},\ldots,j_1)}((\nabla_{(j_{n-1},\ldots,j_1)}(x_{n-1,1}, b, \ldots, b), \\
 & & \nabla_{(j_{n-1},\ldots,j_1)}((x_{n-1,2}, b), b, \ldots, b)), b, \ldots, b) \\
 & = & \nabla_{(j_{n-1},\ldots,j_1)}((\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, b, \ldots, b), \nabla_{(j_{n-1},\ldots,j_1)}(b, b, \ldots, b)), b, \ldots, b) \\
 & \lesssim_{j_n} & \nabla_{(j_{n-1},\ldots,j_1)}(\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, b, \ldots, b), b, \ldots, b) \\
 & = & \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1}, b, \ldots, b) \\
 & = & \ldots \lesssim_{j_n} \ldots \\
 & = & \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,2}, b) \\
 & \sim_{j_n} & \nabla_{(j_1,j_{n-1},\ldots,j_2)}(b, \vec{x}_{n-1,\ldots,2}) \\
 & = & \ldots \lesssim_{j_n} \ldots \\
 & = & \nabla_{(j_1,j_{n-1},\ldots,j_2)}(\vec{x}_1, \vec{x}_{n-1,\ldots,2}) \\
 & \sim_{j_n} & \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}),
\end{array}$$

where the first equality holds by the idempotent law, the second by Lemma 17, the third by separation, the fourth again by Lemma 17, the fifth by associativity, the inequality by the antiordinal law, the sixth equality by the absorption law and the $= \ldots \lesssim_{j_n} \ldots$ signify repetition of the initial series of equalities and the inequality for another of the argument positions. This completes the proof that $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$ is the $(j_{n-1}, \ldots, j_1)$-limit of $(\vec{x}_{n-1}, \ldots, \vec{x}_1)$.

Finally, it remains to show that $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$ is the $(j_{n-1},\ldots,j_1)$-join of $(\vec{x}_{n-1},\ldots,\vec{x}_1)$. To this end, suppose that l is a $(j_{n-1},\ldots,j_1)$-limit of $(\vec{x}_{n-1},\ldots,\vec{x}_1)$. By the limit property of both l and $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$, we get that

$$l \sim_{j_n} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}).$$

Hence, by the limit laws,

$$l \sim_{j_n} \nabla_{(j_{n-2},\ldots,j_1,j_{n-1})}(\vec{x}_{n-2,\ldots,1},\vec{x}_{n-1}). \tag{5}$$

Now we have

$$\begin{array}{rcl} l & = & \nabla_{(j_{n-2},\ldots,j_1,j_n)}(l,l,\ldots,l) \\ & = & \ldots \lesssim_{j_{n-1}} \ldots \\ & = & \nabla_{(j_{n-2},\ldots,j_1,j_n)}(\vec{x}_{n-2,\ldots,1},l) \\ & = & \nabla_{(j_{n-2},\ldots,j_1,j_n)}(\vec{x}_{n-2,\ldots,1},\nabla_{(j_{n-2},\ldots,j_1,j_{n-1})}(\vec{x}_{n-2,\ldots,1},\vec{x}_{n-1})) \\ & = & \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1}), \end{array}$$

where the first equality follows by the idempotent law, the $= \ldots \lesssim_{j_{n-1}} \ldots$ signify the same work that was done in the preceding part of the proof based on the fact that l is a $(j_{n-2},\ldots,j_1,j_n)$-bound of $(\vec{x}_{n-2},\ldots,\vec{x}_1,l)$, the next equality holds by (5) and the last equality by the commutative laws. One works similarly to get that $l \lesssim_{j_{n-2}} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$ and so on down to $l \lesssim_{j_2} \nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$, and this finishes the proof that $\nabla_{(j_{n-1},\ldots,j_1)}(\vec{x}_{n-1,\ldots,1})$ is the $(j_{n-1},\ldots,j_1)$-join of $(\vec{x}_{n-1},\ldots,\vec{x}_1)$. ■

5 Discussion and Directions

The remarks on trilattices made by Biedermann in [1] are valid in this generalized context as well. We mention the appropriately generalized versions here and suggest them as topics for future research.

- Are there any structures lying between n-ordered sets (where $(1,1,\ldots,1)$-joins exist) and n-lattices (where $(2,2,\ldots,2)$-joins exist)? I.e., does postulating the existence of some but not all of the $(\alpha_{n-1},\ldots,\alpha_1)$-joins, $\alpha_i \in \{1,2\}, i = 1,\ldots,n-1$, lead to structures strictly between n-ordered sets and n-lattices?
- Is our equational basis for n-lattices an irredundant basis?
- n-lattices, as presented here, form a universal algebraic variety. Thus, the exploration of this variety's universal algebraic properties and of its subvarieties is an interesting open direction of investigation.

Acknowledgements

Were it not for the equations of Biedermann for trilattices, the discovery of equations for n-lattices would have been an almost intractable task. The author is indebted to him for his discovery.

As always, I thank once more Don Pigozzi and Charles Wells for having been there when I needed them the most.

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