Universal Dialgebra:
Unifying Universal Algebra and Coalgebra

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Abstract

The concept of dialgebra provides a platform under which universal algebra and coalgebra are unified in one theory. Other examples of dialgebras include universal multialgebras and partial algebras. In the dialgebraic setting, several relationships between common features of these various theories are clarified and, in many cases, rather similar proofs of closely related results are combined to a single proof. Moreover, as has been recently conjectured in the context of behavioral certification of evolving software requirements, this more general setting increases the potential of application of the various constituent theories by combining many of the desirable features of algebras and coalgebras that have already been widely applied in theoretical computer science. Several of the elementary results in universal algebra and coalgebra up to the isomorphism theorems are proven here for dialgebras and hints are given as to how these more general results reduce to the two special cases.

Contents

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1 Inroduction

Universal algebra came into being in the mid 1930's, when Garrett Birkhoff [4] introduced the concept of a *general algebra,* i.e., a set with an arbitrary collection of finitary operations on it, in an attempt to unify many similar results that had appeared in different branches of algebra and to continue their study under a common umbrella. At around the same time in the east (and later in the west) Alfred Tarski was laying the foundations of what is now known as first-order model theory of structures. His *structures* or *models* were sets with an arbitrary collection of finitary relations on them. Included in this framework as special instances are, together with universal algebras, *partial algebras* and *multialgebras,* which were also studied separately later (see, e.g., [9] and [11], Chapter 2, respectively). Also in the east, in the former Soviet Union, Anatoliĭ Mal'cev started a tradition very similar to the one founded by Birkhoff and Tarski in the west, by advocating the study of logical properties and advancing the application of techniques from logic in the domain of arbitrary algebras and structures. The field of universal algebra has enjoyed a tremendous development since and several books have appeared that are dedicated to laying the foundations and exploring elementary and more intricate aspects of universal algebras. A representative list includes [6, 7, 11, 17, 20, 22, 25]. There is a rather sizeable intersection of the material that will be presented in this paper with every one of these books, but in the sequel most references will be made to [6] and [22] because these are the most recent among them and, probably, the most accessible with the most modern notation (although notation has not changed much in the field).

More recently, many of the key ideas in universal algebra have found a very fruitful application in theoretical computer science in the domain of *denotational* or *initial semantics* of data structures and programming languages. The free or initial universal algebra of a class is used to model the allowable operations or *constructors* on the structure and its initiality is then exploited to obtain a unique morphism into any other algebra in the class, which is perceived as a semantical interpretation of these operations. This process also allows for the formulation of an *induction principle* that provides the tool for the proof of many properties that are relevant and desirable for the specified structures (see, e.g., [28, 16, 32] for further details and examples).

Although a lot of interesting structures may be specified by the use of this universal algebraic framework, there are many that require a different setting. Their specification has to be given via their "observable" properties. Two structures should be considered identical under this approach not when they are the same but rather when their observable properties are the same. This approach leads to the *operational* or *final semantics* of data structures and languages. It turned out that the appropriate framework in which this kind of semantics may be studied is the framework of *coalgebras,* which are, in some sense, dual to algebras. Their general theory, as parallels the theory of universal algebra, has been developed and studied relatively recently in [29, 30, 16, 12] and many other papers have also appeared in the literature on some more specialized aspects and results relating coalgebras to logic and universal algebra. In the operational semantics, the *cofree* or *final coalgebra* is used to model the observational properties or *destructors* of a given structure. Its finality is then put into action to give a unique morphism from any other coalgebra into it. This morphism is also interpreted as appropriately describing properties of the specified structure. The process provides now a *coinduction principle* that is used as a proof technique on properties of the behaviour of the structure.

Since the techniques of category theory play a rather major role in the development of universal coalgebra, many categorical definitions and elementary results will be considered known. However, the reader should not be discouraged since, apart from the well-known sources [2, 3, 5, 18, 21], sufficient short and excellent introductions for our purposes are given in [30, 12]. Both expositions deal exclusively with the category **Set** of sets and functions and the same will be done here, although many of the results (in fact most of them, if adequately translated) will be easily seen to hold in arbitrary categories.

Roughly speaking, given a functor $F : \mathbf{Set} \to \mathbf{Set}$, an F-*algebra* consists of a set A together with a mapping $\alpha : F(A) \to A$. An F-*coalgebra,* on the other hand, consists of a set A together with a mapping $\alpha : A \to F(A)$. A *homomorphism* from an algebra $\langle A, \alpha\rangle$ to an algebra $\langle B, \beta\rangle$ is a mapping $f : A \to B$ that makes the following diagram commute:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Similarly, a homomorphism from a coalgebra $\langle A, \alpha\rangle$ to a coalgebra $\langle B, \beta\rangle$ is a mapping $g : A \to B$, such that the following rectangle commutes:

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ F(A) & \xrightarrow[F(g)]{} & F(B) \end{array}$$

With those morphisms between them, the classes of F-algebras and F-coalgebras become categories. Although they are not dual to each other, the two categories have a lot in common. Many meaningful and interesting definitions may be formulated for both and a great deal of the elementary propositions in the field of universal coalgebra parallel their prototypes in the algebraic domain. It is only natural then to explore a common underlying formalism that may serve as the framework in which both theories may be developed and more general results, that specialize to the known ones in each domain, proven. Such a framework, centered around the notion of a *dialgebra*, was introduced over the category of sets (in a purely algebraic context) in [33, 34, 1], in theoretical computer science in [14, 15] and [27] and, more recently, from a more abstract point of view in [26]. A similar framework was introduced in [35] under the name of *generalized algebraic category,* but the focus there was different. Namely, the main interest was to explore conditions that would guarantee the existence of products or the lack of products in such categories. Given two functors $F, G : \mathbf{Set} \to \mathbf{Set}$ an $\langle F, G\rangle$-*dialgebra* is a set A together with a mapping $\alpha : F(A) \to G(A)$ and a *dialgebra homomorphism* from a dialgebra $\langle A, \alpha\rangle$ to a dialgebra $\langle B, \beta\rangle$ is a mapping $f : A \to B$, such that the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(B) \arrow[d, "\beta"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

These basic notions together with many examples of special dialgebras for specific functors that have appeared in the literature will be presented in Section 2. In Section 3, the notion of *homomorphism* that parallels its homonyms from algebra and coalgebra is introduced. Also the notion of *bisimulation* is generalized from coalgebra to this context. Bisimulations correspond exactly to substitutive relations in universal algebra. Thus, bisimulations that are equivalences correspond to congruence relations. In Section 4, the existence of limits and colimits in the category of $\langle F, G\rangle$-coalgebras is investigated and its relation to the preservation of limits and colimits and weak limits and colimits by the two functors F and G is explored in some detail. More precisely, it is proven in Theorem 10 that the underlying set forgetful functor from dialgebras to sets creates and preserves all types of limits that are preserved by G and, in Theorem 14, that it creates and preserves all types of colimits that are preserved by F. Section 5 gives some properties of monos and epis in the category of dialgebras. Preservation of weak pushouts by F amounts to the coincidence of the notions of surjective and epi, whereas preservation of weak pullbacks by G amounts to the coincidence of the notions of injective and mono. Several properties of bisimulations are provided in Section 6 together with a description of the lattice of bisimulations $\mathbf{Bis}(\mathbf{A}, \mathbf{B})$ from a dialgebra $\mathbf{A}$ to a dialgebra $\mathbf{B}$. Section 7 deals with subdialgebras, which generalize the concepts of subalgebras and subcoalgebras. The lattice of subdialgebras of a given dialgebra is also described. Section 8 takes up the formulation of the classical isomorphism theorems from universal algebra and their counterparts in coalgebra and reformulates them in the dialgebraic context. Section 9 briefly mentions the obvious analogs of the concepts of a simple algebra or coalgebra, initial algebra and final coalgebra in the dialgebraic framework, but we rush to remark that they do not seem to give anything interesting in general in the current context. It is desirable that further research be done towards a useful generalization of these concepts that would allow the extraction of some principle that would possibly abstract and generalize induction and coinduction. Finally, in Section 10, it is shown how a pair of natural transformations $\mu : M \to F$ and $\nu : G \to N$, where M, N, F, G are all endofunctors on $\mathbf{Set}$, gives a nicely behaved functor from the category of $\langle F, G\rangle$-dialgebras to the category of $\langle M, N\rangle$-dialgebras.

We would like to mention here the debt that this work has to the work of Rutten [30]. It is easy to see that the order of our presentation has been greatly influenced by his and, hopefully, the same has happened with the clarity and firmness of his exposition. Besides this work, his other expository papers [29, 16, 31] together with the excellent paper [12] of Gumm have also been of the utmost help in introducing the author to the main concepts of coalgebra and providing the motivation for its study.

Further, it is hoped that dialgebras will shed some light on the similarities and the differences between algebras and coalgebras and that they will provide the framework needed by other applications for which neither universal algebra nor universal coalgebra are readily suitable (see, e.g., [19]). Many results, then, might be available from this exposition. Finally, there are many more problems that are arising from this work but have not been addressed here. For example, we do not know if and how the concepts of initial algebra and final coalgebra may be unified under a common dialgebraic concept that has some significance and usefulness, similar to the merit each enjoys in its respective domain. The same holds for the induction and coinduction principles. Further work is definitely needed towards this direction, but better understanding of these relationships is bound to accompany the development of universal coalgebra along the lines of universal algebra.

2 Basic Notions and Examples

Let $F, G : \mathbf{Set} \to \mathbf{Set}$ be two endomorphisms on the category **Set** of small sets. An $\langle F, G\rangle$-**dialgebra** $\mathbf{A} = \langle A, \alpha\rangle$ is a pair consisting of a set A together with a mapping $\alpha : F(A) \to G(A)$.

$$\begin{tikzcd} F(A) \arrow[d, "\alpha"'] \\ G(A) \end{tikzcd}$$

Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be two $\langle F, G\rangle$-dialgebras. An $\langle F, G\rangle$-**dialgebra homomorphism** $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$, such that the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(h)"] \arrow[d, "\alpha"'] & F(B) \arrow[d, "\beta"] \\ G(A) \arrow[r, "G(h)"'] & G(B) \end{tikzcd}$$

Identity morphisms are $\langle F, G\rangle$-dialgebra homomorphisms and, given two dialgebra homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{B} \to \mathbf{C}$, the composition $g \circ f : A \to C$ is also a dialgebra homomorphism $h : \mathbf{A} \to \mathbf{C}$.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(B) \arrow[r, "F(g)"] \arrow[d, "\beta"] & F(C) \arrow[d, "\gamma"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \arrow[r, "G(g)"'] & G(C) \end{tikzcd}$$

Thus, $\langle F, G\rangle$-dialgebras together with $\langle F, G\rangle$-dialgebra homomorphisms between them form a category, called the **category of** $\langle F, G\rangle$**-dialgebras** and denoted by $\mathbf{Set}^F_G$. To simplify notation for special cases, define $\mathbf{Set}^F \overset{\text{def}}{=} \mathbf{Set}^F_{I_{\mathbf{Set}}}$ and $\mathbf{Set}_G \overset{\text{def}}{=} \mathbf{Set}^{I_{\mathbf{Set}}}_G$, where by $I_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is denoted the identity functor on $\mathbf{Set}$.

Next, some of the main examples that motivate the introduction of dialgebras are summarised, together with some pointers to the literature where the interested reader will find more detailed accounts and many more examples.

1 (a) (See also [28], Section 2, [29], Section 11, [30] Section 13, and [16], Section 5.) If $G = I_{\mathbf{Set}}$, then the category $\mathbf{Set}^F$ is the category of F-*algebras* and F-*homomorphisms.* Its objects are pairs $\mathbf{A} = \langle A, \alpha \rangle$, where A is a set and $\alpha : F(A) \to A$ is a mapping.

$$\begin{tikzcd} F(A) \arrow[d, "\alpha"'] \\ A \end{tikzcd}$$

Given two F-algebras $\mathbf{A} = \langle A, \alpha \rangle$ and $\mathbf{B} = \langle B, \beta \rangle$, an F-algebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$, such that the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(h)"] \arrow[d, "\alpha"'] & F(B) \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

(b) (See also [18], Chapter VI, [21], Chapter 1, [3], Section 14.3, and [2], Section 5.4.) Let $\mathbf{T} = \langle T, \eta, \mu \rangle$ be an algebraic theory in monoid form. I.e., $T : \mathbf{Set} \to \mathbf{Set}$ is a functor and $\eta : I_{\mathbf{Set}} \to T$ and $\mu : TT \to T$ are natural transformations, such that the following diagrams commute, for every set X,

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "i_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, "i_{T(X)}"] \\ & T(X) & \end{tikzcd} \qquad \begin{tikzcd} T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\ T(T(X)) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

A $\mathbf{T}$-*algebra* is a pair $\mathbf{A} = \langle A, \alpha \rangle$, where A is a set and $\alpha : T(A) \to A$ is a mapping that makes the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "i_A"'] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T(T(A)) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"'] & A \end{tikzcd}$$

Given two **T**-algebras $\mathbf{A} = \langle A, \alpha \rangle$ and $\mathbf{B} = \langle B, \beta \rangle$, a **T**-*algebra homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$, such that the following rectangle commutes

$$\begin{tikzcd} T(A) \arrow[r, "T(h)"] \arrow[d, "\alpha"'] & T(B) \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

T-algebras together with **T**-algebra homomorphisms between them form a category, called the *Eilenberg-Moore category* of the algebraic theory **T** in **Set** and denoted by $\mathbf{Set}^{\mathbf{T}}$. This is a subcategory of the category $\mathbf{Set}^T$ of $\langle T, I_{\mathbf{Set}} \rangle$-dialgebras.

(c) It is well-known ([21], Section 1.4) that all varieties of universal algebras viewed as categories are Eilenberg-Moore categories: Let $\mathcal{L}$ be an algebraic signature and $\mathcal{V}$ a variety of $\mathcal{L}$-algebras. By $\vec{\mathcal{V}}$ will be denoted the category corresponding to $\mathcal{V}$, i.e., it is the category with objects all algebras in $\mathcal{V}$ and with morphisms all $\mathcal{L}$-algebra homomorphisms between them. Then, there exists an algebraic theory $\mathbf{T}_{\mathcal{V}} = \langle T_{\mathcal{V}}, \eta_{\mathcal{V}}, \mu_{\mathcal{V}} \rangle$ in monoid form in **Set** such that $\mathbf{Set}^{\mathbf{T}_{\mathcal{V}}} = \vec{\mathcal{V}}$. Roughly speaking, the functor $T_{\mathcal{V}} : \mathbf{Set} \to \mathbf{Set}$ sends a set X to the carrier $F_{\mathcal{V}}(X)$ of the free algebra $\mathbf{F}_{\mathcal{V}}(X)$ in $\mathcal{V}$ generated by X, $\eta_{\mathcal{V}_X} : X \to F_{\mathcal{V}}(X)$ is the insertion-of-variables map and $\mu_{\mathcal{V}_X} : F_{\mathcal{V}}(F_{\mathcal{V}}(X)) \to F_{\mathcal{V}}(X)$ is the mapping that combines terms over terms over X, by "performing" the free algebra operations, to simple terms over X.

(d) (See also [12], Section 2.3, [30], Section 2, and [16], Section 5.) An alternative way to view the class of all $\mathcal{L}$-algebras as the class $\mathbf{Set}^{F_{\mathcal{L}}}$ for some functor $F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ is as follows:

Let $\mathcal{L} = \langle \Lambda, r \rangle$ be a *type of algebras,* i.e., Λ a set of *operation symbols* and $r : \Lambda \to \omega$ a *rank function* on Λ. Define the functor $F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ by

$$F_{\mathcal{L}}(A) = \sum_{\lambda \in \Lambda} A^{r(\lambda)}, \quad \text{for every set } A,$$

and, given $f : A \to B$ in **Set**, let $F_{\mathcal{L}}(f) : \sum_{\lambda \in \Lambda} A^{r(\lambda)} \to \sum_{\lambda \in \Lambda} B^{r(\lambda)}$ be defined by

$$F_{\mathcal{L}}(f)(x) = f^{r(\lambda)}(x), \quad \text{if} \quad x \in A^{r(\lambda)}, \quad \text{for all} \quad x \in \sum_{\lambda \in \Lambda} A^{r(\lambda)}.$$

For this functor $F_{\mathcal{L}}$, the category $\mathbf{Set}^{F_{\mathcal{L}}}$ consists of all $\mathcal{L}$-algebras with $\mathcal{L}$-algebra homomorphisms between them.

2 (a) (See, e.g., [12] and [30].) Let $F = I_{\mathbf{Set}}$. Then the category $\mathbf{Set}_G$ is the category of *G-coalgebras* and *G-coalgebra homomorphisms.* Its objects are pairs $\mathbf{A} = \langle A, \alpha \rangle$ where A is a set and $\alpha : A \to G(A)$ is a mapping.

$$\begin{tikzcd} A \arrow[d, "\alpha"'] \\ G(A) \end{tikzcd}$$

Given two G-coalgebras $\mathbf{A} = \langle A, \alpha\rangle$ and $\mathbf{B} = \langle B, \beta\rangle$, a G-coalgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$ such that the following rectangle commutes

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ G(A) \arrow[r, "G(h)"'] & G(B) \end{tikzcd}$$

Coalgebras have been used in computer science as a formalism in which, among other things, a theory of transition systems may be developed [29] and well-known data structures may be precisely specified via final coalgebra constructions ([28], Section 4, [29], [30], Section 11, [32] and [12], Section 2).

(b) (See, e.g., [29].) The special case of 2(a), where $G = \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ is the powerset functor, gives the category $\mathbf{Set}_{\mathcal{P}}$ of (nondeterministic, unlabeled) transition systems. A *transition system* $\mathbf{A} = \langle A, \alpha\rangle$ is a pair consisting of a set A of *states* and a *transition function* $\alpha : A \to \mathcal{P}(A)$ which gives, for each state $a \in A$, the set $\alpha(a)$ of possible states which $\mathbf{A}$ may enter in a single transition step from state a. For instance, the transition system given by $A = \{0, 1, 2, 3\}$ and

a	0	1	2	3
$\alpha(a)$	$\{0, 1, 2\}$	$\{1, 2\}$	$\{1, 3\}$	$\emptyset$

may go from state 0 in a single transition step to any of the states $0, 1, 2$, from state 2 to any of the states 1 or 3 and cannot make any legal transition if found in state 3.

3 (a) (See also [25], Section 1.1, [17], Section 4.5, [20], Sections I 2.2 and III 6.1, [11], Chapter 2 and [7], Section II 2.) If $F = F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ and $G = I_{\mathbf{Set}} + 1$, where $1 = \{\emptyset\}$ is the terminal object in $\mathbf{Set}$, then the category $\mathbf{Set}^{F_{\mathcal{L}}}_{I_{\mathbf{Set}}+1}$ is the category of *partial* $\mathcal{L}$*-algebras* with *partial homomorphisms* between them. A partial algebra is a pair $\mathbf{A} = \langle A, \alpha\rangle$, where A is a set and $\alpha : \sum_{\lambda\in\Lambda} A^{r(\lambda)} \to A+1$ a map that determines the partial fundamental operations of $\mathbf{A}$. Intuitively, $\mathbf{A}$ is a partial algebra, such that the fundamental partial operation $\lambda^{\mathbf{A}} : A^{r(\lambda)} \to A$ is defined on the $r(\lambda)$-tuple $\langle a_0, \ldots, a_{r(\lambda)-1}\rangle \in A^{r(\lambda)}$ if and only if $\alpha(a_0, \ldots, a_{r(\lambda)-1}) \in A$ and is undefined if and only if $\alpha(a_0, \ldots, a_{r(\lambda)-1}) = \emptyset$. Given two partial algebras $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$, a homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a function $h : A \to B$, such that, if $\lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1})$ is defined, then $\lambda^{\mathbf{B}}(h(a_0), \ldots, h(a_{r(\lambda)-1}))$ is also defined and

$$h(\lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1})) = \lambda^{\mathbf{B}}(h(a_0), \ldots, h(a_{r(\lambda)-1})).$$

(b) If $F = F_{\mathcal{L}}$ and $G = \mathcal{P}$, then $\mathbf{Set}^{F_{\mathcal{L}}}_{\mathcal{P}}$ is the category of *partial* $\mathcal{L}$*-multialgebras* with *partial multialgebra homomorphisms* between them. Here a fundamental operation $\lambda^{\mathbf{A}}$ of a partial multialgebra $\mathbf{A} = \langle A, \alpha\rangle$ maps a $r(\lambda)$-tuple $\vec{a} \in A^{r(\lambda)}$ to a set of values in A if $\alpha(\vec{a}) \neq \emptyset$, and it is undefined if $\alpha(\vec{a}) = \emptyset$.

(c) (See also [9].) If (b) is altered so that $G = \mathcal{P}^*$, where $\mathcal{P}^*$ sends a set to the set of all its nonempty subsets, one obtains the category $\mathbf{Set}_{\mathcal{P}^*}^{F_\mathcal{L}}$ of $\mathcal{L}$-*multialgebras* with *multialgebra homomorphisms* between them. We note that this definition has not been universally used. Sometimes, instead of the commutativity of the diagram

$$\begin{tikzcd} F_\mathcal{L}(A) \arrow[r, "F_\mathcal{L}(h)"] \arrow[d, "\alpha"'] & F_\mathcal{L}(B) \arrow[d, "\beta"] \\ \mathcal{P}^*(A) \arrow[r, "\mathcal{P}^*(h)"'] & \mathcal{P}^*(B) \end{tikzcd}$$

only the relaxed condition that, for all $\lambda \in \Lambda, \vec{a} \in A^{r(\lambda)}$,

$$\mathcal{P}^*(h)(\alpha(\vec{a})) \subseteq \beta(F_\mathcal{L}(h)(\vec{a}))$$

is required from a multialgebra homomorphism [8]. But it has been used before in the literature, e.g., in [10], Section 6.

3 Dialgebra Homomorphisms and Bisimulations

From now on, whenever the functors F and G are fixed, the term **dialgebra** will be used in place of the more cumbersome $\langle F, G\rangle$-dialgebra.

Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be two dialgebras. A dialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is said to be an **isomorphism** if h has an inverse $h^{-1} : B \to A$, such that $h^{-1} : \mathbf{B} \to \mathbf{A}$ is also a dialgebra homomorphism. h is said to be a **monomorphism** if it is a mono in the category $\mathbf{Set}_G^F$, i.e., if, for every dialgebra $\mathbf{C} = \langle C, \gamma\rangle$ and all dialgebra homomorphisms $f, g : \mathbf{C} \to \mathbf{A}$,

$$h \circ f = h \circ g \quad \text{implies} \quad f = g.$$

In terms of commutative diagrams we have

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{A} \arrow[r, "h"] & \mathbf{B} \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} \mathbf{A} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{B} \end{tikzcd}$$

The dialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is called **injective** if $h : A \rightarrowtail B$ is injective in **Set**. If h is injective, then it is a monomorphism. On the other hand, $h : \mathbf{A} \to \mathbf{B}$ is said to be F-**injective,** G-**injective, biinjective,** if $F(h), G(h)$, both, respectively, are injective in **Set**.

A dialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is said to be an **epimorphism** if it is an epi in the category $\mathbf{Set}_G^F$ of $\langle F, G\rangle$-dialgebras, i.e., if, for every dialgebra $\mathbf{C} = \langle C, \gamma\rangle$ and for all dialgebra homomorphisms $f, g : \mathbf{B} \to \mathbf{C}$,

$$f \circ h = g \circ h \quad \text{implies} \quad f = g.$$

In diagrammatic form

$$\begin{tikzcd} \mathbf{A} \arrow[r, "h"] & \mathbf{B} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{C} \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} \mathbf{B} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{C} \end{tikzcd}$$

The homomorphism $h : \mathbf{A} \to \mathbf{B}$ is called **surjective** if $h : A \twoheadrightarrow B$ is surjective in **Set**. If h is surjective, then it is certainly an epimorphism. Furthermore, h is called F-**surjective**, G-**surjective**, **bisurjective,** if $F(h), G(h)$, both, respectively, are surjective.

Recall, from basic category theory, that a functor $F : \mathbf{Set} \to \mathbf{Set}$ is said to **preserve injectives**, respectively **surjectives**, if, whenever $h : A \to B$ is injective, respectively surjective, then $F(h) : F(A) \to F(B)$ is also injective, respectively surjective. In particular, if $h : A \rightarrowtail B$ is injective and $A \neq \emptyset$, then there exists $h' : B \to A$, such that $h'h = i_A$. But then $F(h')F(h) = i_{F(A)}$ and $F(h)$ is injective. Thus every endofunctor on **Set** preserves injectives with nonempty domains. Similarly, every endofunctor on **Set** preserves surjectives. Note that if h is injective, respectively surjective, and F (G, both) preserves injectives, respectively surjectives, then h is F- (G-, bi-)injective, respectively surjective.

1 Suppose $\mathcal{L}$ is an algebraic signature and $\mathcal{V}$ a variety of $\mathcal{L}$-algebras. Then, the functor $T_{\mathcal{V}} : \mathbf{Set} \to \mathbf{Set}$ preserves injectives and surjectives. Thus, every injective, respectively surjective, $\mathbf{T}_{\mathcal{V}}$-algebra homomorphism is biinjective, respectively bisurjective.

2 All most commonly used coalgebra functors, e.g., polynomial functors, the finite powerset functor and the powerset functor, preserve injectives and surjectives. Thus, every injective, respectively surjective, G-coalgebra homomorphism for one of these functors G is biinjective, respectively bisurjective.

A dialgebra $\mathbf{A} = \langle A, \alpha \rangle$ is a **subdialgebra** of a dialgebra $\mathbf{B} = \langle B, \beta \rangle$ if $A \subseteq B$ and the inclusion map $i : A \hookrightarrow B$ is a dialgebra homomorphism $i : \mathbf{A} \to \mathbf{B}$. A dialgebra $\mathbf{A}$ is a **homomorphic image** of a dialgebra $\mathbf{B}$ if there exists a surjective homomorphism $h : \mathbf{B} \twoheadrightarrow \mathbf{A}$.

1 $\langle F_{\mathcal{L}}, I_{\mathbf{Set}} \rangle$-subdialgebras in $\mathbf{Set}^{F_{\mathcal{L}}}$ are exactly the subalgebras in the universal algebraic sense.

2 $\langle I_{\mathbf{Set}}, G \rangle$-subdialgebras are exactly the G-subcoalgebras in the usual coalgebraic sense.

Analogous statements hold for homomorphic images of $\langle F_{\mathcal{L}}, I_{\mathbf{Set}} \rangle$-dialgebras and $\langle I_{\mathbf{Set}}, G \rangle$-dialgebras.

Proposition 1 *Every bijective dialgebra homomorphism is an isomorphism.*

Proof:

Suppose $h : \mathbf{A} \to \mathbf{B}$ is a dialgebra homomorphism, such that $h : A \to B$ has an inverse map $h^{-1} : B \to A$. It suffices to show that $h^{-1} : \mathbf{B} \to \mathbf{A}$ is a dialgebra homomorphism.

$$\begin{tikzcd}
F(B) \arrow[r, "F(h^{-1})"] \arrow[d, "\beta"'] & F(A) \arrow[r, "F(h)"] \arrow[d, "\alpha"] & F(B) \arrow[d, "\beta"] \\
G(B) \arrow[r, "G(h^{-1})"'] & G(A) \arrow[r, "G(h)"'] & G(B)
\end{tikzcd}$$

We have

$$\begin{aligned}
\alpha F(h^{-1}) &= G(h^{-1})G(h)\alpha F(h^{-1}) \\
&= G(h^{-1})\beta F(h)F(h^{-1}) \\
&= G(h^{-1})\beta.
\end{aligned}$$

■

As a corollary of Proposition 1 the corresponding universal algebraic result, Lemma 3.6 of [7], and the corresponding coalgebraic result, Proposition 2.3 of [30], may be obtained.

Proposition 2 *Let* $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ *and* $\mathbf{C} = \langle C, \gamma \rangle$ *be dialgebras and* $f : A \to B, g : A \to C$ *and* $h : C \to B$ *mappings such that* $f = h \circ g$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\
& C \arrow[ur, "h"'] &
\end{tikzcd}$$

1 If g is F-surjective and f, g are homomorphisms, then h is a homomorphism.

2 If h is G-injective and f, h are homomorphisms, then g is a homomorphism.

Proof:

$$\begin{tikzcd}
G(A) \arrow[rrrr, "G(f)"] \arrow[dddrr, "G(g)"'] & & & & G(B) \\
& F(A) \arrow[ul, "\alpha"'] \arrow[rr, "F(f)"] \arrow[dr, "F(g)"'] & & F(B) \arrow[ur, "\beta"] & \\
& & F(C) \arrow[ur, "F(h)"'] \arrow[d, "\gamma"'] & & \\
& & G(C) \arrow[uuurr, "G(h)"'] & &
\end{tikzcd}$$

1 Let $c \in F(C)$. Then, since g is F-surjective, there exists $a \in F(A)$, such that $c = F(g)(a)$. Thus

$$\begin{aligned}
G(h)\gamma(c) &= G(h)\gamma F(g)(a) \\
&= G(h)G(g)\alpha(a) \\
&= G(hg)\alpha(a) \\
&= G(f)\alpha(a) \\
&= \beta F(f)(a) \\
&= \beta F(h)F(g)(a) \\
&= \beta F(h)(c).
\end{aligned}$$

2
$$\begin{aligned} G(h)\gamma F(g) &= \beta F(h)F(g) \\ &= \beta F(hg) \\ &= \beta F(f) \\ &= G(f)\alpha \\ &= G(hg)\alpha \\ &= G(h)G(g)\alpha. \end{aligned}$$

Thus, since h is G-injective, i.e., $G(h)$ is injective and, therefore, mono, $\gamma F(g) = G(g)\alpha$.

■

Corresponding results from universal algebra and coalgebra may again be immediately obtained as corollaries. Next, the image and preimage constructions of [12], Section 3, and some related results are proven for dialgebras.

Lemma 3 *Let $\mathbf{A} = \langle A, \alpha\rangle$ be a dialgebra, S a set and $f : A \twoheadrightarrow S$ a surjective function. Then, there exists a dialgebra structure σ on S, such that, for every dialgebra $\mathbf{B} = \langle B, \beta\rangle$ and all $g : S \to B$, if $g \circ f$ is a homomorphism, so is g.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g \circ f"'] & & S \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

Proof:

Since $f : A \twoheadrightarrow S$ is surjective, there exists $f' : S \to A$, such that $f \circ f' = i_S$. Define $\sigma : F(S) \to G(S)$ by

$$F(S) \xrightarrow{F(f')} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{G(f)} G(S)$$
$$\sigma = G(f)\alpha F(f').$$

Now we have

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] & F(S) \arrow[r, "F(g)"] \arrow[d, "\sigma"'] & F(B) \arrow[d, "\beta"] \\ & G(S) \arrow[r, "G(g)"'] & G(B) \end{tikzcd}$$

$$\begin{aligned} G(g)\sigma F(f) &= G(g)G(f)\alpha F(f')F(f) \\ &= G(gf)\alpha F(f')F(f) \\ &= \beta F(gf)F(f')F(f) \\ &= \beta F(g)F(f)F(f')F(f) \\ &= \beta F(g)F(f) \end{aligned}$$

But f is surjective, whence $F(f)$ is surjective and, therefore, $G(g)\sigma = \beta F(g)$ and g is a homomorphism. ■

Similarly, we may show the following

Lemma 4 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be a dialgebra,* $S \neq \emptyset$ *a set and* $f : S \rightarrowtail A$ *an injective mapping. Then, there is a dialgebra structure* σ *on* S*, such that, for all dialgebras* $\mathbf{B} = \langle B, \beta \rangle$ *and all mappings* $g : B \to S$*, if* $f \circ g$ *is a homomorphism, the so is* g*.*

$$\begin{tikzcd} S \arrow[rr, "f"] & & A \\ & B \arrow[ul, "g"] \arrow[ur, "f \circ g"'] & \end{tikzcd}$$

If the functor F preserves the empty set, then the assumption that $S \neq \emptyset$ in Lemma 4 may be dropped.

Lemma 5 *Let* $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ *be dialgebras, with* $A \neq \emptyset$*, and* $h : \mathbf{A} \to \mathbf{B}$ *a homomorphism. Suppose that* $h = g \circ f$ *is an epi-mono factorization of* h *in* **Set**, *i.e.,* $f : A \twoheadrightarrow S$ *is an epimorphism,* $g : S \rightarrowtail B$ *is a monomorphism and the following commutes:*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \\ & S \arrow[ur, "g"'] & \end{tikzcd}$$

Then, there is a unique dialgebra structure σ *on* S*, such that* f *and* g *become homomorphisms.*

Proof:
Clearly, $S \neq \emptyset$. Thus, by Lemmas 3 and 4, there exist dialgebra structures $\sigma, \sigma' : F(S) \to G(S)$, such that the following rectangles commute

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(S) \arrow[d, "\sigma"] \\ G(A) \arrow[r, "G(f)"'] & G(S) \end{tikzcd} \qquad \begin{tikzcd} F(S) \arrow[r, "F(g)"] \arrow[d, "\sigma'"'] & F(B) \arrow[d, "\beta"] \\ G(S) \arrow[r, "G(g)"'] & G(B) \end{tikzcd}$$

Now it suffices to show that $\sigma = \sigma'$. We have

$$\begin{aligned} G(g)\sigma F(f) &= G(g)G(f)\alpha \\ &= G(gf)\alpha \\ &= \beta F(gf) \\ &= \beta F(g)F(f) \\ &= G(g)\sigma' F(f). \end{aligned}$$

But f is surjective, whence $F(f)$ is also surjective and g is injective, whence $G(g)$ is also injective and, therefore, $\sigma = \sigma'$. ■

Lemma 5 may be slightly modified to get rid of the assumption that $A \neq \emptyset$ at the expense of adding the hypothesis that G preserves monos. The following may then be obtained

Lemma 6 *Suppose that the functor $G : \mathbf{Set} \to \mathbf{Set}$ preserves monomorphisms and let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-dialgebras and $h : \mathbf{A} \to \mathbf{B}$ a homomorphism. Suppose, also, that $h = g \circ f$ is an epi-mono factorization of h in* **Set**, *i.e., $f : A \twoheadrightarrow S$ is an epimorphism, $g : S \rightarrowtail B$ is a monomorphism and the following commutes:*

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \\
& S \arrow[ur, "g"'] &
\end{tikzcd}$$

Then, there is a unique dialgebra structure σ on S, such that f and g become homomorphisms.

Proof:

By Lemma 3 there exists a dialgebra structure $\sigma = G(f)\alpha F(f') : F(S) \to G(S)$, such that the rectangle on the left below commutes. It is not hard to check that the rectangle on the right also commutes.

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(S) \arrow[r, "F(g)"] \arrow[d, "\sigma"] & F(B) \arrow[d, "\beta"] \\
G(A) \arrow[r, "G(f)"'] & G(S) \arrow[r, "G(g)"'] & G(B)
\end{tikzcd}$$

Uniqueness follows as in the proof of Lemma 5, since, in this case also, $F(f)$ is surjective and $G(g)$ is injective. ■

Lemma 7 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be dialgebras, with $A \neq \emptyset$, and $h : \mathbf{A} \to \mathbf{B}$ a homomorphism. Suppose that $h = g \circ f = g' \circ f'$ are epi-mono factorizations of h in* **Set**.

$$\begin{tikzcd}
& S \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "h"] \arrow[dr, "f'"'] & & B \\
& S' \arrow[ur, "g'"'] &
\end{tikzcd}$$

Then, the dialgebras $\mathbf{S} = \langle S, \sigma\rangle$ and $\mathbf{S'} = \langle S', \sigma'\rangle$ defined by Lemma 5 are isomorphic.

Proof:

It is easy to see that the kernel of f must be contained in the kernel of f'. But then, there exists in **Set** a unique (since f is surjective) mapping $p : S \to S'$, such that $pf = f'$. Now, by Proposition 2, we get that $p : \mathbf{S} \to \mathbf{S'}$ is a homomorphism, which is surjective since f' is surjective.

$$
\begin{tikzcd}
 & S \arrow[dd, "p"] \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & \\
 & S'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
S \arrow[dd, "q"'] \arrow[dr, "g"] & \\
 & B \\
S' \arrow[ur, "g'"'] &
\end{tikzcd}
$$

Similarly, it is easy to check that $g(S) \subseteq g'(S')$, whence there exists in **Set** a unique (since g' is injective) mapping $q : S \to S'$, such that $g'q = g$. Now, again by Proposition 2, we get that $q : \mathbf{S} \to \mathbf{S}'$ is a homomorphism and it is injective since g is. Now it suffices to show that $p = q$. We have

$$
\begin{aligned}
g'pf &= g'f' \\
&= gf \\
&= g'qf.
\end{aligned}
$$

But f is an epimorphism and g' a monomorphism whence $p = q$. ■

Lemmas 5 and 7 have the following easy corollary

Theorem 8 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-dialgebras, with $A \neq \emptyset$, and $h : \mathbf{A} \to \mathbf{B}$ a dialgebra homomorphism. h has a unique epi-mono factorization in $\mathbf{Set}^F_G$ as $\mathbf{A} \twoheadrightarrow h(\mathbf{A}) \hookrightarrow \mathbf{B}$.*

Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be two $\langle F, G \rangle$-dialgebras. A subset $R \subseteq A \times B$ is called an $\langle F, G \rangle$-**bisimulation** between $\mathbf{A}$ and $\mathbf{B}$ if there exists an $\langle F, G \rangle$-dialgebra structure $\rho : F(R) \to G(R)$ on R that makes the following diagram commute

$$
\begin{tikzcd}
F(A) \arrow[d, "\alpha"'] & F(R) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\rho"] & F(B) \arrow[d, "\beta"] \\
G(A) & G(R) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(B)
\end{tikzcd}
$$

where $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are the coordinate-wise projections. I.e., $R \subseteq A \times B$ is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ if there exists an $\langle F, G \rangle$-dialgebra structure ρ on R, such that the two projections become $\langle F, G \rangle$-dialgebra homomorphisms. Again, if $G = I_{\mathbf{Set}}$ and no confusion is possible, the term F-**bisimulation** is used in place of $\langle F, I_{\mathbf{Set}} \rangle$-bisimulation. Similarly, if $F = I_{\mathbf{Set}}$, the term G-**bisimulation** is used in place of $\langle I_{\mathbf{Set}}, G \rangle$-bisimulation. This terminology introduces ambiguity which will hopefully be dispersed by the context in which the terms will appear. An $\langle F, G \rangle$-**bisimulation on A** is an $\langle F, G \rangle$-bisimulation from $\mathbf{A}$ to itself. An $\langle F, G \rangle$-**bisimulation equivalence** on $\mathbf{A}$ is a bisimulation on $\mathbf{A}$ that is also an equivalence relation on A.

1 (See also [28], Section 2.) The example of varieties of universal algebras is revisited. A subset $R \subseteq A \times B$ of the cartesian product of the carriers of two $\mathcal{L}$-algebras $\mathbf{A}, \mathbf{B}$ is an $F_{\mathcal{L}}$-bisimulation in $\mathbf{Set}^{F_{\mathcal{L}}}$ if and only if R is a *substitutive* relation from $\mathbf{A}$ to $\mathbf{B}$, i.e., for all $\lambda \in \Lambda$ and $a_0, \ldots, a_{r(\lambda)-1} \in A, b_0, \ldots, b_{r(\lambda)-1} \in B$,

$$\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle \in R \text{ implies } \langle \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}), \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1})\rangle \in R.$$

To see this, suppose, first, that $R \subseteq A \times B$ is an $F_{\mathcal{L}}$-bisimulation in $\mathbf{Set}^{F_{\mathcal{L}}}$, i.e., there exists a $F_{\mathcal{L}}$-algebra structure $\rho : F_{\mathcal{L}}(R) \to R$, such that the following diagram commutes

$$\begin{tikzcd}
F_{\mathcal{L}}(A) \arrow[d, "\alpha"'] & F_{\mathcal{L}}(R) \arrow[l, "F_{\mathcal{L}}(\pi_1)"'] \arrow[r, "F_{\mathcal{L}}(\pi_2)"] \arrow[d, "\rho"] & F_{\mathcal{L}}(B) \arrow[d, "\beta"] \\
A & R \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

Now let $\lambda \in \Lambda, \langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle \in R$. Then by commutativity of the left rectangle above,

$$\pi_1(\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle))) = \alpha(F_{\mathcal{L}}(\pi_1)(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle)))$$

or, equivalently,

$$\pi_1(\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle))) = \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}).$$

Similarly, by commutativity of the second rectangle,

$$\pi_2(\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle))) = \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1}).$$

Thus,

$$\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle)) = \langle \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}), \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1})\rangle.$$

Therefore $\langle \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}), \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1})\rangle \in R$, as claimed. The converse is easier and is left to the reader.

2 If $F = I_{\mathbf{Set}}$, then a G-bisimulation between two $\langle I_{\mathbf{Set}}, G\rangle$-dialgebras is the usual notion of G-bisimulation of coalgebras (see, e.g., [30]). The same holds for G-bisimulation equivalence.

These notions restricted to transition systems give restrictions of the prototypical notions of bisimulation that first appeared in the context of concurrency theory [23, 24].

Theorem 9 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be two $\langle F, G\rangle$-dialgebras. A function $f : A \to B$ is a dialgebra homomorphism if and only if its graph $\Gamma(f)$ is an $\langle F, G\rangle$-bisimulation.*

Proof:

Suppose $f : A \to B$ is such that $\Gamma(f) \subseteq A \times B$ is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ with structure map $\gamma : F(\Gamma(f)) \to G(\Gamma(f))$, i.e., the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[d, "\alpha"'] & F(\Gamma(f)) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\gamma"] & F(B) \arrow[d, "\beta"] \\
G(A) & G(\Gamma(f)) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(B)
\end{tikzcd}$$

Since f is a function, $\pi_1 : \Gamma(f) \to A$ is bijective. Thus $\pi_1^{-1} : A \to \Gamma(f)$ exists and is a dialgebra homomorphism by Proposition 1. Hence $f = \pi_2 \circ \pi_1^{-1}$ is also a dialgebra homomorphism.

Conversely, suppose that $f : \mathbf{A} \to \mathbf{B}$ is a dialgebra homomorphism.

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(B) \arrow[d, "\beta"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

Endow $\Gamma(f) \subseteq A \times B$ with the dialgebra structure $\gamma = G(\pi_1^{-1})\alpha F(\pi_1) : F(\Gamma(f)) \to G(\Gamma(f))$.

$$F(\Gamma(f)) \xrightarrow{F(\pi_1)} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{G(\pi_1^{-1})} G(\Gamma(f))$$

Then

$$G(\pi_1)\gamma = G(\pi_1)G(\pi_1^{-1})\alpha F(\pi_1) = \alpha F(\pi_1)$$

and

$$\begin{aligned}
G(\pi_2)\gamma &= G(\pi_2)G(\pi_1^{-1})\alpha F(\pi_1) \\
&= G(\pi_2\pi_1^{-1})\alpha F(\pi_1) \\
&= G(f)\alpha F(\pi_1) \\
&= \beta F(f)F(\pi_1) \\
&= \beta F(f\pi_1) \\
&= \beta F(\pi_2).
\end{aligned}$$

■

Theorem 9 yields as corollaries Proposition 2.7 of [28] and Theorem 2.7 of [30].

4 Limits and Colimits of Dialgebras

For the categorical notions of *limit* and *colimit* of a given diagram $d : D \to \mathcal{C}$ with base graph $D = \langle V(D), E(D)\rangle$ in a category $\mathcal{C}$ the reader is referred to the general references [18, 3, 2, 5]. The notions of *product, equalizer, pullback* and the duals of *coproduct, coequalizer, pushout* will also be considered known and the same references may be consulted for their definitions and several illuminating examples. By the **type** of a limit or colimit we mean the isomorphism class, in the category **Grph** of graphs and graph homomorphisms, of the base graph of the limit or colimit, respectively. A functor $F : \mathbf{Set} \to \mathbf{Set}$ is said to **preserve** a type D of limit, respectively colimit, if, for every diagram $d : D \to \mathbf{Set}$ with limit, respectively colimit, $\langle L, \{f_d : d \in V(D)\}\rangle$, $\langle F(L), \{F(f_d) : d \in V(D)\}\rangle$ is the limit, respectively colimit, of the diagram $F \circ d : D \to \mathbf{Set}$. A **weak limit** (**weak colimit**) of a diagram in a category is defined similarly to a limit (colimit) except that uniqueness of the fill-in morphisms in the corresponding universal mapping properties is not required. A functor $F : \mathbf{Set} \to \mathbf{Set}$ is said to **preserve** a certain type of weak limit (colimit) if a weak limiting (colimiting) cone $\langle L, \{f_d : d \in V(D)\}\rangle$ of a diagram $d : D \to \mathbf{Set}$ is sent to a weak limiting (colimiting) cone $\langle F(L), \{F(f_d) : d \in V(D)\}\rangle$ of the diagram $F \circ d : D \to \mathbf{Set}$.

In this section a general study of limits and colimits in the category $\mathbf{Set}_G^F$ of $\langle F, G\rangle$-dialgebras is undertaken. The ultimate goal is to prove that $\mathbf{Set}_G^F$ has all limits that are preserved by G and all colimits that are preserved by F. Moreover, if G preserves a certain type of weak limit, then, for every diagram of that type in $\mathbf{Set}_G^F$, the corresponding diagram in $\mathbf{Set}$ has a limit that may be endowed with a dialgebra structure so that the resulting diagram consists of dialgebra homomorphisms that commute in $\mathbf{Set}_G^F$, as do in $\mathbf{Set}$ their underlying maps. Analogously, if F preserves a certain type of weak colimit, then, for every diagram of that type in $\mathbf{Set}_G^F$, the corresponding diagram in $\mathbf{Set}$ has a colimit that may be endowed with a dialgebra structure so that the resulting diagram consists of dialgebra homomorphisms that commute in $\mathbf{Set}_G^F$, as do in $\mathbf{Set}$ their underlying maps.

Let $U : \mathbf{Set}_G^F \to \mathbf{Set}$ be the underlying set forgetful functor from the category of $\langle F, G\rangle$-dialgebras to the category of sets. For every dialgebra $\mathbf{A} = \langle A, \alpha\rangle$, $U(\mathbf{A}) = A$ and, for every dialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$, $U(h) = h : A \to B$. The functor U is said to **create a type** D of **limit** (**colimit**) if, for every diagram $d : D \to \mathbf{Set}_G^F$, its limit (colimit) is constructed by first taking the limit (colimit) of $U \circ d : D \to \mathbf{Set}$ in $\mathbf{Set}$ and then supplying it in a unique way with an $\langle F, G\rangle$-dialgebra structure.

Theorem 10 *The forgetful functor $U : \mathbf{Set}_G^F \to \mathbf{Set}$ creates and preserves all types of limits that the functor $G : \mathbf{Set} \to \mathbf{Set}$ preserves.*

Proof:
Let $D = \langle V(D), E(D)\rangle$ be a graph and $d : D \to \mathbf{Set}_G^F$ a diagram in $\mathbf{Set}_G^F$. Suppose that G preserves limits of type D. Consider the diagram $U \circ d : D \to \mathbf{Set}$ in $\mathbf{Set}$. Since $\mathbf{Set}$ is complete, $U \circ d$ has a limit $\langle L, \{l_v : v \in V(D)\}\rangle$ in $\mathbf{Set}$, i.e., L is a set and, for all $v \in V(D)$, $l_v : L \to U(d(v))$ is a mapping, such that, for all $e : v_1 \to v_2$ in $E(D)$, the following diagram commutes

$$\begin{tikzcd} & L \arrow[dl, "l_{v_1}"'] \arrow[dr, "l_{v_2}"] & \\ U(d(v_1)) \arrow[rr, "U(d(e))"'] & & U(d(v_2)) \end{tikzcd}$$

and such that, for all other cones $\langle M, \{f_v : v \in V(D)\}\rangle, f_v : M \to U(d(v)), v \in V(D)$, such that, for all $e : v_1 \to v_2$ in $E(D)$, the following triangle commutes

$$\begin{tikzcd}
& M \arrow[dl, "f_{v_1}"'] \arrow[dr, "f_{v_2}"] & \\
U(d(v_1)) \arrow[rr, "U(d(e))"'] & & U(d(v_2))
\end{tikzcd}$$

there exists a unique mapping $f : M \to L$, such that the following triangle commutes, for all $v \in V(D)$,

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "f_v"'] & & L \arrow[dl, "l_v"] \\
& U(d(v)) &
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
& F(L) \arrow[dl, "F(l_{v_1})"'] \arrow[dr, "F(l_{v_2})"] & \\
F(U(d(v_1))) \arrow[rr, "F(U(d(e)))"'] \arrow[dd, "\delta_{d(v_1)}"'] & & F(U(d(v_2))) \arrow[dd, "\delta_{d(v_2)}"] \\
& G(L) \arrow[dl, "G(l_{v_1})"] \arrow[dr, "G(l_{v_2})"'] & \\
G(U(d(v_1))) \arrow[rr, "G(U(d(e)))"'] & & G(U(d(v_2)))
\end{tikzcd}$$

Since G preserves limits of type D, the cone $\langle G(L), \{G(l_v) : v \in V(D)\}\rangle$ is a limiting cone in **Set**. Therefore, since

$$\begin{array}{rcl}
G(U(d(e)))\delta_{d(v_1)}F(l_{v_1}) & = & \delta_{d(v_2)}F(U(d(e)))F(l_{v_1}) \\
& = & \delta_{d(v_2)}F(l_{v_2}),
\end{array}$$

there exists a unique map $\pi : F(L) \to G(L)$, such that the following diagram commutes in **Set** :

$$\begin{tikzcd}
& F(L) \arrow[dl, "F(l_{v_1})"'] \arrow[dr, "F(l_{v_2})"] \arrow[dd, "\pi"'] & \\
F(U(d(v_1))) \arrow[rr, "F(U(d(e)))"'] \arrow[dd, "\delta_{d(v_1)}"'] & & F(U(d(v_2))) \arrow[dd, "\delta_{d(v_2)}"] \\
& G(L) \arrow[dl, "G(l_{v_1})"'] \arrow[dr, "G(l_{v_2})"] & \\
G(U(d(v_1))) \arrow[rr, "G(U(d(e)))"'] & & G(U(d(v_2)))
\end{tikzcd}$$

Clearly, $\mathbf{L} = \langle L, \pi \rangle$ is a dialgebra and, for all $v \in V(d)$, $l_v : \mathbf{L} \to d(v)$ is a dialgebra homomorphism.

To show that $\langle \mathbf{L}, \{l_v : v \in V(D)\} \rangle$ is a limiting cone in $\mathbf{Set}_G^F$, consider any other cone $\langle \mathbf{M}, \{f_v : v \in V(D)\} \rangle$ in $\mathbf{Set}_G^F$:

$$\begin{tikzcd}
& U(M) \arrow[dl, "U(f_{v_1})"'] \arrow[dr, "U(f_{v_2})"] & \\
U(d(v_1)) \arrow[rr, "U(d(e))"'] & & U(d(v_2))
\end{tikzcd}$$

Since $\langle L, \{l_v : v \in V(D)\} \rangle$ is a limiting cone in **Set**, there exists a unique mapping $f : U(M) \to L$ in **Set**, such that, for all $v \in V(D)$, the following triangle commutes

$$\begin{tikzcd}
U(M) \arrow[rr, "f"] \arrow[dr, "U(f_v)"'] & & L \arrow[dl, "l_v"] \\
& U(d(v)) &
\end{tikzcd}$$

It suffices to show that $f : U(M) \to L$ is a dialgebra homomorphism $f : \mathbf{M} \to \mathbf{L}$. We have, for all $v \in V(D)$,

$$\begin{tikzcd}
F(U(M)) \arrow[r, "F(f)"] \arrow[d, "\mu"'] & F(L) \arrow[d, "\pi"] & \\
G(U(M)) \arrow[r, "G(f)"'] & G(L) \arrow[r, "G(l_v)"'] & G(U(d(v)))
\end{tikzcd}$$

$$\begin{aligned} G(l_v)\pi F(f) &= \delta_{d(v)}F(l_v)F(f) \\ &= \delta_{d(v)}F(l_v f) \\ &= \delta_{d(v)}F(U(f_v)) \\ &= G(U(f_v))\mu \\ &= G(l_v f)\mu \\ &= G(l_v)G(f)\mu, \end{aligned}$$

whence, by the universal mapping property of $\langle G(L), \{G(l_v) : v \in V(D)\}\rangle$, it now follows that $\pi F(f) = G(f)\mu$, as was to be shown. ■

This yields the following

Corollary 11 *The categories* $\mathbf{Set}^F$ *of* F*-algebras, for any functor* $F : \mathbf{Set} \to \mathbf{Set}$, $\mathbf{Set}^{F_{\mathcal{L}}}$ *of* $\mathcal{L}$*-algebras, and* $\vec{\mathcal{V}}$, *of any variety* $\mathcal{V}$, *are complete, i.e., have all small limits.*

Also, Theorem 4.6 of [30]

Corollary 12 *The category* $\mathbf{Set}_G$ *of* G*-coalgebras has all limits that* $G : \mathbf{Set} \to \mathbf{Set}$ *preserves. Moreover the forgetful functor* $U : \mathbf{Set}_G \to \mathbf{Set}$ *creates all these limits.*

Following along the lines of the proof of Theorem 10, the following may also be proved.

Theorem 13 *Let* D *be a graph,* $d : D \to \mathbf{Set}^F_G$ *be a diagram in* $\mathbf{Set}^F_G$ *and suppose that the functor* $G : \mathbf{Set} \to \mathbf{Set}$ *preserves weak limits of type* D*. Then the limit* $\langle L, \{l_v : v \in V(D)\}\rangle$ *of* $U \circ d : D \to \mathbf{Set}$ *in* $\mathbf{Set}$ *may be endowed with a dialgebra structure* $\mathbf{L} = \langle L, \pi\rangle$*, such that the following diagram commutes in* $\mathbf{Set}^F_G$*, for all* $e : v_1 \to v_2$ *in* $E(D)$*:*

$$\begin{tikzcd} & \mathbf{L} \arrow[dl, "l_{v_1}"'] \arrow[dr, "l_{v_2}"] & \\ d(v_1) \arrow[rr, "d(e)"'] & & d(v_2) \end{tikzcd}$$

The following results are the analogs of Theorems 10 and 13. Their proofs are also very similar to the proof of Theorem 10 and will, therefore, be omitted. First, the analog of Theorem 10.

Theorem 14 *The forgetful functor* $U : \mathbf{Set}^F_G \to \mathbf{Set}$ *creates and preserves all types of colimits that the functor* $F : \mathbf{Set} \to \mathbf{Set}$ *preserves.*

This yields Theorem 4.5 of [30]. (See also Theorem 4.2 of [12].)

Corollary 15 *The category* $\mathbf{Set}_G$ *of* G*-coalgebras, for any functor* $G : \mathbf{Set} \to \mathbf{Set}$, *is cocomplete, i.e., has all small colimits. Moreover, the forgetful functor* $U : \mathbf{Set}_G \to \mathbf{Set}$ *creates colimits.*

The following result concerning universal algebras may also be obtained.

Corollary 16 *The category* $\mathbf{Set}^{F_{\mathcal{L}}}$ *of* $F_{\mathcal{L}}$*-algebras has all colimits that* $F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ *preserves. Moreover, the forgetful functor* $U : \mathbf{Set}^{F_{\mathcal{L}}} \to \mathbf{Set}$ *creates all these colimits.*

Finally, the analog of Theorem 13 for colimits is given.

Theorem 17 *Let* D *be a graph,* $d : D \to \mathbf{Set}_G^F$ *be a diagram in* $\mathbf{Set}_G^F$ *and suppose that the functor* $F : \mathbf{Set} \to \mathbf{Set}$ *preserves weak colimits of type* D*. Then the colimit* $\langle C, \{c_v : v \in V(D)\}\rangle$ *of* $U \circ d : D \to \mathbf{Set}$ *in* $\mathbf{Set}$ *may be endowed with a dialgebra structure* $\mathbf{C} = \langle C, \kappa\rangle$*, such that the following diagram commutes in* $\mathbf{Set}_G^F$*, for all* $e : v_1 \to v_2$ *in* $E(D)$*:*

$$\begin{tikzcd} d(v_1) \arrow[rr, "d(e)"] \arrow[dr, "c_{v_1}"'] & & d(v_2) \arrow[dl, "c_{v_2}"] \\ & \mathbf{C} & \end{tikzcd}$$

The following Theorem, which is the analog of Theorem 4.3 of [30], exhibits some of the fruitful interaction of the notion of bisimulation with the weak preservation of limits by the functor G.

Theorem 18 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle, \mathbf{C} = \langle C, \gamma\rangle$ *be* $\langle F, G\rangle$*-dialgebras and* $f : \mathbf{A} \to \mathbf{C}, g : \mathbf{B} \to \mathbf{C}$ *be dialgebra homomorphisms. If* G *preserves weak pullbacks, then the pullback of* $f : A \to C, g : B \to C$ *in* $\mathbf{Set}$ *is a bisimulation from* $\mathbf{A}$ *to* $\mathbf{B}$.

Proof:
Let $f : \mathbf{A} \to \mathbf{C}$ and $g : \mathbf{B} \to \mathbf{C}$ be dialgebra homomorphisms.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(C) \arrow[d, "\gamma"] \\ G(A) \arrow[r, "G(f)"'] & G(C) \end{tikzcd} \qquad \begin{tikzcd} F(B) \arrow[r, "F(g)"] \arrow[d, "\beta"'] & F(C) \arrow[d, "\gamma"] \\ G(B) \arrow[r, "G(g)"'] & G(C) \end{tikzcd}$$

Consider the pullback diagram in $\mathbf{Set}$

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Then, since G preserves weak pullbacks, by Theorem 13, there exists a structure map $\pi : F(A \times_C B) \to G(A \times_C B)$, such that the following diagram commutes in $\mathbf{Set}_G^F$:

$$\begin{tikzcd} F(A) \arrow[d, "\alpha"'] & F(A \times_C B) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\pi"] & F(B) \arrow[d, "\beta"] \\ G(A) & G(A \times_C B) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(B) \end{tikzcd}$$

Hence $\mathbf{A} \times_C \mathbf{B} = \langle A \times_C B, \pi \rangle$ is a dialgebra and the projections π_1, π_2 become homomorphisms. ■

5 On Monos and Epis

Recall from category theory (see, e.g., [18] and [3]) that an arrow $a : A \to B$ in a category $\mathcal{C}$ is an epimorphism or epi if and only if the following is a pushout in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "a"'] & B \arrow[d, "i_B"] \\ B \arrow[r, "i_B"'] & B \end{tikzcd}$$

Dually, $a : A \to B$ in $\mathcal{C}$ is a monomorphism or mono if and only if the following diagram is a pullback in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "i_A"'] & A \arrow[d, "a"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

Since weak pullbacks of monos are regular pullbacks and weak pushouts of epis are regular pushouts, the following proposition follows:

Proposition 19 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be two $\langle F, G \rangle$-dialgebras and $f : \mathbf{A} \to \mathbf{B}$ an $\langle F, G \rangle$-dialgebra homomorphism.*

1 *If F preserves weak pushouts, then f is surjective if and only if it is epi.*

2 *If G preserves weak pullbacks, then f is injective if and only if it is mono.*

Proof:

1 By Theorem 14, $U : \mathbf{Set}_G^F \to \mathbf{Set}$ creates all types of colimits that F preserves. Thus, in particular, U creates pushouts of epis. Since, in addition, U preserves all colimits that it creates

$$\begin{tikzcd} \mathbf{A} \arrow[r, "f"] \arrow[d, "f"'] & \mathbf{B} \arrow[d, "i_B"] \\ \mathbf{B} \arrow[r, "i_B"'] & \mathbf{B} \end{tikzcd} \quad \text{is a pushout in } \mathbf{Set}_G^F \text{ iff} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "i_B"] \\ B \arrow[r, "i_B"'] & B \end{tikzcd} \quad \text{is a pushout in } \mathbf{Set}.$$

Thus, $f : \mathbf{A} \to \mathbf{B}$ is epi iff $f : A \to B$ is epi iff $f : A \to B$ is surjective iff $f : \mathbf{A} \to \mathbf{B}$ is surjective.

2 Note, similarly, that weak pullbacks of monos are ordinary pullbacks and that G preserves these. So U creates and preserves them as well. Now proceed as in 1. ■

As corollaries we obtain

Corollary 20 *In every variety of algebras injectives and monos coincide.*

and also Proposition 4.7 of [30]

Corollary 21 *For every functor $G : \mathbf{Set} \to \mathbf{Set}$, surjectives and epis coincide in $\mathbf{Set}_G$. Moreover, if G preserves weak pullbacks, then injectives and monos also coincide in $\mathbf{Set}_G$.*

6 More on Bisimulations

In this section, we elaborate more on dialgebra bisimulations. The goal is to show that several results concerning substitutive relations and congruences on the universal algebraic side and bisimulation relations and bisimulation equivalences, respectively, on the coalgebraic side may be jointly formulated in the framework of dialgebras.

Proposition 22 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be an $\langle F, G \rangle$-dialgebra. The diagonal Δ_A of A is a bisimulation equivalence on $\mathbf{A}$.*

Proof:
Note that $\Delta_A = \Gamma(i_A)$ and recall Theorem 9. ■

In the following theorem, it is shown that the converse relation of a bisimulation between two dialgebras is also a bisimulation.

Theorem 23 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-dialgebras and $R \subseteq A \times B$ a bisimulation between $\mathbf{A}$ and $\mathbf{B}$. The converse $R^{-1} \subseteq B \times A$ is a bisimulation between $\mathbf{B}$ and $\mathbf{A}$.*

Proof:
Suppose $R \subseteq A \times B$ is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ with dialgebra structure map $\rho : F(R) \to G(R)$, i.e., the following rectangles commutes:

$$\begin{tikzcd}
F(A) \arrow[d, "\alpha"'] & F(R) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\rho"] & F(B) \arrow[d, "\beta"] \\
G(A) & G(R) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(B)
\end{tikzcd}$$

Denote by $r : R \to R^{-1}$ the map sending $\langle a, b \rangle \in R$ to $\langle b, a \rangle \in R^{-1}$. Endow R^{-1} with the dialgebra structure

$$\begin{tikzcd}
F(R^{-1}) \arrow[r, "F(r^{-1})"] \arrow[d, "\rho'"'] & F(R) \arrow[d, "\rho"] \\
G(R^{-1}) & G(R) \arrow[l, "G(r)"]
\end{tikzcd}$$

$$\rho' = G(r)\rho F(r^{-1}) : F(R^{-1}) \to G(R^{-1}).$$

We then have

$$\begin{tikzcd}
F(B) \arrow[d, "\beta"'] & F(R^{-1}) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\rho'"] & F(A) \arrow[d, "\alpha"] \\
G(B) & G(R^{-1}) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(A)
\end{tikzcd}$$

$$\begin{aligned}
G(\pi_2)\rho' &= G(\pi_2)G(r)\rho F(r^{-1}) \\
&= G(\pi_2 r)\rho F(r^{-1}) \\
&= G(\pi_1)\rho F(r^{-1}) \\
&= \alpha F(\pi_1)F(r^{-1}) \\
&= \alpha F(\pi_1 r^{-1}) \\
&= \alpha F(\pi_2)
\end{aligned}$$

and, similarly, $G(\pi_1)\rho' = \beta F(\pi_1)$. ■

Next, given two dialgebra homomorphisms with the same domain, it is shown, roughly speaking, that the image of their product on the product of their codomains is a dialgebra bisimulation between the two codomains.

Lemma 24 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *and* $\mathbf{C} = \langle C, \gamma\rangle$ *be* $\langle F, G\rangle$*-dialgebras and* $f : \mathbf{A} \to \mathbf{B}, g : \mathbf{A} \to \mathbf{C}$ *dialgebra homomorphisms. Then the image* $\langle f, g\rangle(A) = \{\langle f(a), g(a)\rangle : a \in A\} \subseteq B \times C$ *is a bisimulation between* $\mathbf{B}$ *and* $\mathbf{C}$*.*

Proof:
Consider the diagram

$$\begin{tikzcd}
B & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, shift right, "\langle f,g\rangle"'] & C \\
 & \langle f,g\rangle(A) \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[u, shift right, "h"'] &
\end{tikzcd}$$

where $\langle f, g\rangle : A \to \langle f, g\rangle(A); a \mapsto \langle f(a), g(a)\rangle$ is surjective and, therefore, has a right inverse h. Thus $\langle f, g\rangle \circ h = i_A$. Define on $\langle f, g\rangle(A)$ the dialgebra structure

$$F(\langle f,g\rangle(A)) \xrightarrow{F(h)} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{G(\langle f,g\rangle)} G(\langle f,g\rangle(A))$$

$$\delta = G(\langle f, g\rangle)\alpha F(h) : F(\langle f, g\rangle(A)) \to G(\langle f, g\rangle(A)).$$

Then

$$\begin{tikzcd}
F(B) \arrow[d, "\beta"'] & F(\langle f,g\rangle(A)) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\delta"] & F(C) \arrow[d, "\gamma"] \\
G(B) & G(\langle f,g\rangle(A)) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(C)
\end{tikzcd}$$

$$\begin{aligned} G(\pi_2)\delta &= G(\pi_2)G(\langle f, g\rangle)\alpha F(h) \\ &= G(\pi_2\langle f, g\rangle)\alpha F(h) \\ &= G(g)\alpha F(h) \\ &= \gamma F(g)F(h) \\ &= \gamma F(gh) \\ &= \gamma F(\pi_2) \end{aligned}$$

and, similarly, $G(\pi_1)\delta = \beta F(\pi_1)$. ■

Under the hypothesis that G preserves weak pullbacks, it may be shown that the composition of two dialgebra bisimulations is also a dialgebra bisimulation between the appropriate dialgebras.

Theorem 25 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ and $\mathbf{C} = \langle C, \gamma\rangle$ be $\langle F, G\rangle$-dialgebras and $R \subseteq A \times B, Q \subseteq B \times C$ $\langle F, G\rangle$-bisimulations. If G preserves weak pullbacks, then $R \circ Q \subseteq A \times C$ is a bisimulation from $\mathbf{A}$ to $\mathbf{C}$.*

Proof:
The following pullback in **Set**

$$\begin{tikzcd} & & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & Q \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ A & & B & & C \end{tikzcd}$$

is such that $R \circ Q = \langle r_1\pi_1, q_2\pi_2\rangle(X)$ ([30], Section 18). Now, by Theorem 13, X may be endowed with a unique dialgebra structure such that the diagram above lifted to $\mathbf{Set}_G^F$ still commutes. Then, Lemma 24 shows that $R \circ Q$ is a bisimulation from $\mathbf{A}$ to $\mathbf{C}$. ■

Theorem 25 shows, in the algebraic side, that the composition of two substitutive relations is a substitutive relation and, in the coalgebraic side, yields as a corollary Theorem 5.4 or [30].

Regarding the assumptions in part *2* of the following theorem, note that if a functor $G : \mathbf{Set} \to \mathbf{Set}$ preserves generalized pullbacks, then it also preserves products. This is because preservation of the "empty" pullback yields preservation of the terminal object $1 = \{\emptyset\}$ and because the pullback of the diagram

$$\begin{tikzcd} & B \arrow[d, "t_B"] \\ A \arrow[r, "t_A"'] & 1 \end{tikzcd}$$

where $t_A : A \to 1$ and $t_B : B \to 1$ are the unique morphisms into the terminal object, gives the product of A and B.

Theorem 26 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-dialgebras and $R_i \subseteq A \times B, i \in I$, $\langle F, G\rangle$-bisimulations with dialgebra structures $\rho_i : F(R_i) \to G(R_i), i \in I$, respectively.*

1 If F preserves weak coproducts, then $\bigcup_{i\in I} R_i$ is a bisimulation.

2 If G preserves generalized pullbacks (or products and generalized weak pullbacks), then $\bigcap_{i\in I} R_i$ is a bisimulation.

Proof:

1 Consider the coproduct diagram in **Set**

$$\begin{tikzcd}
& R_i \arrow[dl, "\pi_1"'] \arrow[d, "k_i"] \arrow[dr, "\pi_2"] & \\
A & \sum_{i\in I} R_i \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

where $k_i : R_i \to \sum_{i\in I} R_i, i \in I$, are the injections and $\pi_1 : R_i \to A$, $\pi_2 : R_i \to B, p_1 : \sum_{i\in I} R_i \to A$ and $p_2 : \sum_{i\in I} R_i \to B$ are the coordinate-wise projections. Since F preserves weak coproducts, by Theorem 17, there is a dialgebra structure $\kappa : F(\sum_{i\in I} R_i) \to G(\sum_{i\in I} R_i)$, such that $\sum_{i\in I} \mathbf{R}_i = \langle \sum_{i\in I} R_i, \kappa\rangle$ is a dialgebra and $k_i : \mathbf{R}_i \to \sum_{i\in I} \mathbf{R}_i, i \in I$, are homomorphisms. It is not difficult to check by chasing the following diagram that $p_1 : \sum_{i\in I} \mathbf{R}_i \to \mathbf{A}$ and $p_2 : \sum_{i\in I} \mathbf{R}_i \to \mathbf{B}$ are also homomorphisms.

$$\begin{tikzcd}
F(R_i) \arrow[rr, "F(\pi_1)"] \arrow[d, "\rho_i"'] \arrow[ddr, "F(k_i)"] & & F(A) \arrow[d, "\alpha"] \\
G(R_i) \arrow[rr, "G(\pi_1)"] \arrow[ddr, "G(k_i)"'] & & G(A) \\
& F(\sum_{i\in I} R_i) \arrow[uur, "F(p_1)"] \arrow[d, "\kappa"'] & \\
& G(\sum_{i\in I} R_i) \arrow[uur, "G(p_1)"'] &
\end{tikzcd}$$

But, note that $\bigcup_{i\in I} R_i = \langle p_1, p_2\rangle(\sum_{i\in I} R_i)$. Thus, by Lemma 24, $\bigcup_{i\in I} R_i$ is also an $\langle F, G\rangle$-bisimulation.

2 Now consider the product diagram

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

Since G preserves pullbacks, it also preserves products (or products directly). Thus, by Theorem 10, there exists a dialgebra structure $\pi : F(A \times B) \to G(A \times B)$, such that

$\mathbf{A} \times \mathbf{B} = \langle A \times B, \pi \rangle$ is the product of $\mathbf{A}$ and $\mathbf{B}$ in $\mathbf{Set}_G^F$. Now, by the same theorem, and considering the diagram

$$\begin{tikzcd}
 & R_i \arrow[dl, "p_1"'] \arrow[d, "r_i"] \arrow[dr, "p_2"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

the inclusion $r_i : R_i \to A \times B$ is a dialgebra homomorphism from $\mathbf{R}_i$ to $\mathbf{A} \times \mathbf{B}, i \in I$. Finally, consider the generalized pullback diagram in $\mathbf{Set}$

$$\begin{tikzcd}
\prod_{i \in I_{A \times B}} R_i \arrow[r, "k_i"] \arrow[d, "k_j"'] & R_i \arrow[d, "r_i"] \\
R_j \arrow[r, "r_j"'] & A \times B
\end{tikzcd}$$

Note that $\prod_{i \in I_{A \times B}} R_i = \bigcap_{i \in I} R_i$. Since G preserves generalized weak pullbacks $\prod_{i \in I_{A \times B}} R_i$ may be endowed with a dialgebra structure, such that all projections $k_i, i \in I$, become homomorphisms. But then $\pi_1 r_i k_i : \bigcap_{i \in I} R_i \to A$ and $\pi_2 r_i k_i : \bigcap_{i \in I} R_i \to B$ become homomorphisms and $\bigcap_{i \in I} R_i$ is a bisimulation. ■

Given two $\langle F, G \rangle$-dialgebras $\mathbf{A}$ and $\mathbf{B}$, denote by $\mathrm{Bis}(\mathbf{A}, \mathbf{B})$ the collection of all $\langle F, G \rangle$-bisimulations between $\mathbf{A}$ and $\mathbf{B}$.

The following corollary has as special cases the statements that, in universal algebra, the collection of all substitutive relations between two algebras forms a complete lattice under inclusion with meet given by intersection and, in coalgebra, the collection of all bisimulations between two coalgebras also forms a complete lattice under inclusion with join given by union.

Corollary 27 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-dialgebras.*

1 If F preserves weak coproducts, then the collection of all bisimulations from $\mathbf{A}$ to $\mathbf{B}$ forms a complete lattice, such that, for all bisimulations $R_i, i \in I$,

$$\bigvee_{i \in I} R_i = \bigcup_{i \in I} R_i \quad \textit{and} \quad \bigwedge_{i \in I} R_i = \bigcup \{R \in \mathrm{Bis}(\mathbf{A}, \mathbf{B}) : R \subseteq \bigcap_{i \in I} R_i\}.$$

2 If G preserves generalized pullbacks (or products and generalized weak pullbacks), then the collection of all bisimulations from $\mathbf{A}$ to $\mathbf{B}$ forms a complete lattice, such that, for all bisimulations $R_i, i \in I$,

$$\bigwedge_{i \in I} R_i = \bigcap_{i \in I} R_i \quad \textit{and} \quad \bigvee_{i \in I} R_i = \bigcap \{R \in \mathrm{Bis}(\mathbf{A}, \mathbf{B}) : \bigcup_{i \in I} R_i \subseteq R\}.$$

As direct consequences of Corollary 27 the well-known characterizations of joins and meets in the lattice of universal algebraic congruences (see, e.g., [6], Section II.5, and [22], Sections 1.4 and 4.3) and universal coalgebraic bisimulations ([30], Corollary 5.6) may be obtained.

Given two $\langle F,G\rangle$-dialgebras $\mathbf{A}$ and $\mathbf{B}$, denote by $\mathbf{Bis}(\mathbf{A},\mathbf{B})$ the complete lattice of Corollary 27.1, in case F preserves weak coproducts, or Corollary 27.2, in case F preserves generalized pullbacks (or products and generalized weak pullbacks).

Let $\mathbf{A}=\langle A,\alpha\rangle,\mathbf{B}=\langle B,\beta\rangle$ be $\langle F,G\rangle$-dialgebras and $f:\mathbf{A}\to\mathbf{B}$ a homomorphism. Denote by $\mathrm{K}(f)\subseteq A^2$ the **kernel** of f, i.e.,

$$\mathrm{K}(f)=\{\langle a,a'\rangle\in A^2 : f(a)=f(a')\}.$$

Proposition 28 *Let $\mathbf{A}=\langle A,\alpha\rangle,\mathbf{B}=\langle B,\beta\rangle$ be $\langle F,G\rangle$-dialgebras and $f:\mathbf{A}\to\mathbf{B}$ a homomorphism. If G preserves weak pullbacks, then $\mathrm{K}(f)$ is a bisimulation equivalence on $\mathbf{A}$.*

Proof:
Clearly $\mathrm{K}(f)$ is an equivalence relation on A. Note that $\mathrm{K}(f)=\Gamma(f)\circ\Gamma(f)^{-1}$. Now, by Theorem 9, $\Gamma(f)$ is a bisimulation. Thus, by Theorem 23, $\Gamma(f)^{-1}$ is also a bisimulation. Finally, by Theorem 25, $\mathrm{K}(f)=\Gamma(f)\circ\Gamma(f)^{-1}$ is a bisimulation. ■

As a corollary we get that, in general, kernels of homomorphisms in universal algebra are congruences ([6], Theorem II.6.8, also [28], Proposition 2.6) and, in case G preserves weak pullbacks, kernels of G-coalgebra homomorphisms are bisimulation equivalences ([30], Proposition 5.7).

Proposition 29 *Let $\mathbf{A}=\langle A,\alpha\rangle$ be an $\langle F,G\rangle$-dialgebra and $R\subseteq A^2$ a bisimulation equivalence on $\mathbf{A}$. If F preserves weak coequalizers, then there exists a dialgebra structure $\alpha_R:F(A/R)\to G(A/R)$ on A/R, such that the quotient map $q_R:A\to A/R$ becomes a dialgebra homomorphism*

$$\begin{tikzcd}
F(A) \arrow[r, "F(q_R)"] \arrow[d, "\alpha"'] & F(A/R) \arrow[d, "\alpha_R"] \\
G(A) \arrow[r, "G(q_R)"'] & G(A/R)
\end{tikzcd}$$

and $\mathbf{A}/R=\langle A/R,\alpha_R\rangle$ becomes the coequalizer in $\mathbf{Set}_G^F$ of

$$\begin{tikzcd}
\mathbf{R} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbf{A}
\end{tikzcd}$$

Proof:
Note that

$$\begin{tikzcd}
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "q_R"] & A/R
\end{tikzcd}$$

is a coequalizer diagram in **Set** with both π_1 and π_2 dialgebra homomorphisms. Now q_R : $A\to A/R$ is surjective, whence $F(q_R):F(A)\to F(A/R)$ is also surjective. Therefore the weak coequalizer

$$\begin{tikzcd}
F(R) \arrow[r, shift left, "F(\pi_1)"] \arrow[r, shift right, "F(\pi_2)"'] & F(A) \arrow[r, "F(q_R)"] & F(A/R)
\end{tikzcd}$$

is actually a coequalizer. Thus F preserves this coequalizer. Hence, by Theorem 14, A/R may be endowed with a dialgebra structure $\alpha_R : F(A/R) \to G(A/R)$, such that $q_R : \mathbf{A} \to \mathbf{A}/R$ is the coequalizer in $\mathbf{Set}_G^F$ of

$$\begin{tikzcd} \mathbf{R} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbf{A} \end{tikzcd}$$

■

We may now obtain the analog of Proposition 5.9 of [30] for dialgebra bisimulations.

Proposition 30 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-dialgebras and $f : \mathbf{A} \to \mathbf{B}$ a dialgebra homomorphism. Suppose that G preserves weak pullbacks.*

1 If $R \subseteq A^2$ is a bisimulation on $\mathbf{A}$, then $f(R) = \{\langle f(a), f(a')\rangle : \langle a, a'\rangle \in R\}$ is a bisimulation on $\mathbf{B}$.

2 If $Q \subseteq B^2$ is a bisimulation on $\mathbf{B}$, then $f^{-1}(Q) = \{\langle a, a'\rangle \in A^2 : \langle f(a), f(a')\rangle \in Q\}$ is a bisimulation on $\mathbf{A}$.

Proof:
For *1*, note that $f(R) = \Gamma(f)^{-1} \circ R \circ \Gamma(f)$ and use Theorems 9, 23 and 25. For *2*, note that $f^{-1}(Q) = \Gamma(f) \circ Q \circ \Gamma(f)^{-1}$ and proceed as before. ■

7 Subdialgebras

Let $\mathbf{A} = \langle A, \alpha \rangle$ be an $\langle F, G \rangle$-dialgebra and $B \subseteq A$ with inclusion $i : B \hookrightarrow A$. If there exists a dialgebra structure $\beta : F(B) \to G(B)$ on B such that i is a homomorphism

$$\begin{tikzcd} F(B) \arrow[r, "F(i)"] \arrow[d, "\beta"'] & F(A) \arrow[d, "\alpha"] \\ G(B) \arrow[r, "G(i)"'] & G(A) \end{tikzcd}$$

then $\mathbf{B} = \langle B, \beta \rangle$ is said to be a **subdialgebra** of $\mathbf{A}$. This is denoted by $\mathbf{B} \leq \mathbf{A}$. If $B \neq \emptyset$ and such a dialgebra structure β on B exists, it is unique. To show this, recall that any endofunctor on **Set** preserves all monomorphisms whose domains are different from the empty set $\emptyset$ ([30], Proposition 18.1).

Proposition 31 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be a dialgebra and $\emptyset \neq B \subseteq A$ with inclusion $i : B \hookrightarrow A$. Let $\mathbf{B} = \langle B, \beta \rangle, \mathbf{B}' = \langle B, \beta' \rangle$ be two dialgebras such that $i : \mathbf{B} \to \mathbf{A}$ and $i : \mathbf{B}' \to \mathbf{A}$ are homomorphisms.*

$$\begin{tikzcd} F(B) \arrow[r, "F(i)"] \arrow[d, shift right, "\beta"'] \arrow[d, shift left, "\beta'"] & F(A) \arrow[d, "\alpha"] \\ G(B) \arrow[r, "G(i)"'] & G(A) \end{tikzcd}$$

Then $\beta = \beta'$, *i.e.,* $\mathbf{B} = \mathbf{B}'$.

Proof:
If $B \neq \emptyset$, then $G(i) : G(B) \to G(A)$ is mono. Thus, since $G(i)\beta = \alpha F(i) = G(i)\beta'$, we conclude $\beta = \beta'$. ■

Corollary 32 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be an* $\langle F, G \rangle$*-dialgebra. If* G *preserves monos, then, if* $B \subseteq A$ *admits a subdialgebra structure, it is unique. In particular, this holds if* G *preserves weak pullbacks.*

A dialgebra $\mathbf{A} = \langle A, \alpha \rangle$ is called **minimal** if it does not have a subdialgebra $\mathbf{B} = \langle B, \beta \rangle$, with $B \neq \emptyset, A$.

Proposition 33 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be a dialgebra and* $B \subseteq A$. B *is the universe of a subdialgebra of* $\mathbf{A}$ *iff* Δ_B *is a bisimulation on* $\mathbf{A}$.

Proof:
Suppose Δ_B is a bisimulation on $\mathbf{A}$ with structure map $\delta : F(\Delta_B) \to G(\Delta_B)$, i.e., such that the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[d, "\alpha"'] & F(\Delta_B) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\delta"] & F(A) \arrow[d, "\beta"] \\
G(A) & G(\Delta_B) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(A)
\end{tikzcd}$$

Define a dialgebra structure on B by

$$F(B) \xrightarrow{F(d^{-1})} F(\Delta_B) \xrightarrow{\delta} G(\Delta_B) \xrightarrow{G(d)} G(B)$$

$$\beta = G(d)\delta F(d^{-1}) : F(B) \to G(B),$$

where $d : \Delta_B \to B; \langle b, b \rangle \mapsto b$. Then

$$\begin{tikzcd}
F(B) \arrow[r, "F(i)"] \arrow[d, "\beta"'] & F(A) \arrow[d, "\alpha"] \\
G(B) \arrow[r, "G(i)"'] & G(A)
\end{tikzcd}$$

$$\begin{aligned}
G(i)\beta &= G(i)G(d)\delta F(d^{-1}) \\
&= G(id)\delta F(d^{-1}) \\
&= G(\pi_1)\delta F(d^{-1}) \\
&= \alpha F(\pi_1)F(d^{-1}) \\
&= \alpha F(\pi_1 d^{-1}) \\
&= \alpha F(i).
\end{aligned}$$

Conversely, if $\beta : F(B) \to G(B)$ is a dialgebra structure on B, such that $i : B \hookrightarrow A$ is a homomorphism, then, by Theorem 9, $\Delta_B = \Gamma(i) \subseteq B \times A$ is a bisimulation. Thus $\Delta_B \subseteq A^2$ is also a bisimulation. ■

Proposition 33 has as corollaries the well-known facts that a universal algebra $\mathbf{B}$ is a subalgebra of an algebra $\mathbf{A}$ if and only if $\Delta_B \subseteq A^2$ is a substitutive relation on $\mathbf{A}$ and Proposition 6.2 of [30].

Proposition 34 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-dialgebras, $R \subseteq A \times B$ a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ with structure map $\rho : F(R) \to G(R)$ and $\mathbf{Q} \leq \mathbf{R}$. Then $Q \subseteq A \times B$ is also a bisimulation between $\mathbf{A}$ and $\mathbf{B}$.*

Proof:
By assumption, the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[d, "\alpha"'] & F(R) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] \arrow[d, "\rho"] & F(B) \arrow[d, "\beta"] \\
G(A) & G(R) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] & G(B)
\end{tikzcd}$$

where $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are the two projections. Also, if $i : Q \hookrightarrow R$ is the inclusion, the following diagram commutes

$$\begin{tikzcd}
F(Q) \arrow[r, "F(i)"] \arrow[d, "\kappa"'] & F(R) \arrow[d, "\rho"] \\
G(Q) \arrow[r, "G(i)"'] & G(R)
\end{tikzcd}$$

where $\kappa : F(Q) \to G(Q)$ is the structure map of the subdialgebra $\mathbf{Q} \leq \mathbf{R}$. Then for $\pi'_1 : Q \to A, \pi'_2 : Q \to B$, defined by

$$\begin{tikzcd}
Q \arrow[r, "i"] \arrow[dr, "\pi'_1"'] & R \arrow[d, "\pi_1"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "i"] \arrow[dr, "\pi'_2"'] & R \arrow[d, "\pi_2"] \\
 & B
\end{tikzcd}$$

we have

$$\begin{tikzcd}
F(A) \arrow[d, "\alpha"'] & F(Q) \arrow[l, "F(\pi'_1)"'] \arrow[r, "F(\pi'_2)"] \arrow[d, "\kappa"] & F(B) \arrow[d, "\beta"] \\
G(A) & G(Q) \arrow[l, "G(\pi'_1)"] \arrow[r, "G(\pi'_2)"'] & G(B)
\end{tikzcd}$$

$$\begin{aligned} G(\pi_1')\kappa &= G(i\pi_1)\kappa \\ &= G(\pi_1)G(i)\kappa \\ &= G(\pi_1)\rho F(i) \\ &= \alpha F(\pi_1)F(i) \\ &= \alpha F(\pi_1') \end{aligned}$$

and, similarly, $G(\pi_2')\kappa = \beta F(\pi_2')$. ■

Theorem 35 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be dialgebras and $f : \mathbf{A} \to \mathbf{B}$ a dialgebra homomorphism. If G preserves weak pullbacks, then*

1. *If $C \subseteq A$ admits a subdialgebra structure, so does $f(C) \subseteq B$.*
2. *If $D \subseteq B$ admits a subdialgebra structure, so does $f^{-1}(D) \subseteq A$.*

Proof:

1. If $C \subseteq A$ admits a subdialgebra structure, then, by Proposition 33, Δ_C is a bisimulation on $\mathbf{A}$. Thus, by Proposition 30, $f(\Delta_C) = \Delta_{f(C)}$ is a bisimulation on $\mathbf{B}$. Thus, by Proposition 33 again, $f(C) \subseteq B$ admits a subdialgebra structure.
2. Similarly, if $D \subseteq B$ admits a subdialgebra structure, then, by Proposition 33, Δ_D is a bisimulation on $\mathbf{B}$. Thus, by Proposition 30, $f^{-1}(\Delta_D) = \Delta_{f^{-1}(D)}$ is a bisimulation on $\mathbf{A}$. Thus, by Proposition 33 again, $f^{-1}(D) \subseteq A$ admits a subdialgebra structure. ■

Theorem 35 immediately yields as corollaries Theorem II.6.3 of [6] and Theorem 6.3 of [30]. (See also Lemma 4.6 and Theorem 4.7 of [12] together with its corollaries.)

Theorem 36 *Let $\mathbf{A} = \langle A, \alpha\rangle$ be an $\langle F, G\rangle$-dialgebra.*

1. *If F preserves weak coproducts, then the collection of all subdialgebras of $\mathbf{A}$ forms a complete lattice such that, for all $\mathbf{B}_i \leq \mathbf{A}, i \in I$, the universe of $\bigvee_{i\in I} \mathbf{B}_i$ is $\bigcup_{i\in I} B_i$.*
2. *If G preserves generalized pullbacks (or products and generalized weak pullbacks), then the collection of all subdialgebras of $\mathbf{A}$ forms a complete lattice such that, for all $\mathbf{B}_i \leq \mathbf{A}, i \in I$, the universe of $\bigwedge_{i\in I} \mathbf{B}_i$ is $\bigcap_{i\in I} B_i$.*

Proof:

1. Let $\mathbf{B}_i = \langle B_i, \beta_i\rangle, i \in I$, be a collection of subdialgebras of $\mathbf{A}$. Then, by Proposition 33, $\Delta_{B_i}, i \in I$, are bisimulations on $\mathbf{A}$. By Theorem 26, $\Delta_{\cup_{i\in I} B_i} = \bigcup_{i\in I} \Delta_{B_i}$ is a bisimulation on $\mathbf{A}$. Thus, again by Proposition 33, $\bigcup_{i\in I} B_i$ admits a subdialgebra structure.
2. Similar to 1. ■

Part *2* of Theorem 36 has as corollary the fact that, for any universal algebra, the colection of all its subalgebras forms a complete lattice under inclusion with meet given by intersection. On the other hand parts *1* and *2* combined yield the statement that, given a functor G that preserves generalized pullbacks, the collection of all subcoalgebras of a given G-coalgebra forms a complete lattice with meet given by intersection and join given by union. In [30], this statement (Theorem 6.4) is proven with the relaxed assumption that G preserves generalized weak pullbacks. We were unable to relax the condition of preservetion of generalized ordinary pullbacks here.

Corollary 37 *1 The collection of subalgebras of an F-algebra forms a complete lattice with meet given by intersection.*

2 If G preserves generalized pullbacks (or products and generalized weak pullbacks), then the collection of subcoalgebras of a G-coalgebra forms a complete lattice with join given by union and meet given by intersection.

Corollary 37 justifies the following definitions. Let $\mathbf{A} = \langle A, \alpha \rangle$ be a dialgebra and $X \subseteq A$. In case G preserves generalized pullbacks (or products and generalized weak pullbacks), define the **subsystem $\langle \mathbf{X} \rangle = \langle \langle X \rangle, \hat{\xi} \rangle$ of $\mathbf{A}$ generated by** X by letting

$$\langle \mathbf{X} \rangle = \bigwedge \{ \mathbf{B} \leq \mathbf{A} : X \subseteq B \}.$$

If F preserves weak coproducts, define the **greatest subsystem $[\mathbf{X}] = \langle [X], \check{\xi} \rangle$ contained in** X by

$$[\mathbf{X}] = \bigvee \{ \mathbf{B} \leq \mathbf{A} : B \subseteq X \}.$$

We close this section with another result about intersections. Inspired by an analogous result for coalgebras ([13], Theorem 3.1), we show that, if the underlying sets of two subdialgebras of a given dialgebra have nonempty intersection, then this intersection admits a subdialgebra structure.

Theorem 38 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be an $\langle F, G \rangle$-dialgebra and $\mathbf{B} = \langle B, \beta \rangle, \mathbf{C} = \langle C, \gamma \rangle$ two subdialgebras of $\mathbf{A}$, such that $B \cap C \neq \emptyset$. Then the intersection $B \cap C$ admits a subdialgebra structure.*

Proof:

By assumption, if $i_B^A : B \hookrightarrow A$ and $i_C^A : C \hookrightarrow A$ are the inclusion maps, the following diagrams commute

$$\begin{tikzcd} F(B) \arrow[r, "F(i_B^A)"] \arrow[d, "\beta"'] & F(A) \arrow[d, "\alpha"] \\ G(B) \arrow[r, "G(i_B^A)"'] & G(A) \end{tikzcd} \qquad \begin{tikzcd} F(C) \arrow[r, "F(i_C^A)"] \arrow[d, "\gamma"'] & F(A) \arrow[d, "\alpha"] \\ G(C) \arrow[r, "G(i_C^A)"'] & G(A) \end{tikzcd}$$

Since $B \cap C \neq \emptyset$, assume $d \in B \cap C$ and define $p : B \to B \cap C$ and $q : A \to C$ by

$$p(b) = \begin{cases} b, & \text{if } b \in B \cap C \\ d, & \text{otherwise} \end{cases} \quad \text{and} \quad q(a) = \begin{cases} a, & \text{if } a \in C \\ d, & \text{otherwise} \end{cases}$$

for all $b \in B, a \in A$. Then

$$q \circ i_C^A = i_C \quad \text{and} \quad i_{B\cap C}^C \circ p = q \circ i_B^A.$$

Now endow $B \cap C$ with a structure map $\delta : F(B \cap C) \to G(B \cap C)$, with

$$F(B\cap C) \xrightarrow{F(i_{B\cap C}^B)} F(B) \xrightarrow{\beta} G(B) \xrightarrow{G(p)} G(B\cap C)$$

$$\delta = G(p) \circ \beta \circ F(i_{B\cap C}^B).$$

It suffices to show that the following rectangle commutes

$$\begin{tikzcd} F(B\cap C) \arrow[r, "F(i_{B\cap C}^A)"] \arrow[d, "\delta"'] & F(A) \arrow[d, "\alpha"] \\ G(B\cap C) \arrow[r, "G(i_{B\cap C}^A)"'] & G(A) \end{tikzcd}$$

We have

$$\begin{aligned}
G(i_{B\cap C}^A)\delta &= G(i_{B\cap C}^A)G(p)\beta F(i_{B\cap C}^B) \\
&= G(i_C^A i_{B\cap C}^C)G(p)\beta F(i_{B\cap C}^B) \\
&= G(i_C^A)G(i_{B\cap C}^C)G(p)\beta F(i_{B\cap C}^B) \\
&= G(i_C^A)G(q)G(i_B^A)\beta F(i_{B\cap C}^B) \\
&= G(i_C^A)G(q)\alpha F(i_B^A)F(i_{B\cap C}^B) \\
&= G(i_C^A)G(q)\alpha F(i_{B\cap C}^A) \\
&= G(i_C^A)G(q)\alpha F(i_C^A)F(i_{B\cap C}^C) \\
&= G(i_C^A)G(q)G(i_C^A)\gamma F(i_{B\cap C}^C) \\
&= G(i_C^A)G(i_C)\gamma F(i_{B\cap C}^C) \\
&= \alpha F(i_C^A)F(i_{B\cap C}^C) \\
&= \alpha F(i_{B\cap C}^A).
\end{aligned}$$

■

8 Isomorphism Theorems

Theorem 39 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-dialgebras and $f : \mathbf{A} \to \mathbf{B}$ a dialgebra homomorphism. If G preserves monomorphisms, then there exists the following commutative* *diagram of homomorphisms*

$$\begin{tikzcd} \mathbf{A} \arrow[rr, "f"] \arrow[dr, "f'"] \arrow[ddr, "q_{\mathrm{K}(f)}"'] & & \mathbf{B} \\ & f(\mathbf{A}) \arrow[ur, "i"'] & \\ & \mathbf{A}/\mathrm{K}(f) \arrow[u, "\cong"] \arrow[uur, "\mu"'] & \end{tikzcd}$$

where f', given by $a \mapsto f(a)$, is a surjection, $i : f(A) \hookrightarrow B$ is the inclusion, $q_{\mathrm{K}(f)}$ is the quotient map of the kernel $\mathrm{K}(f)$ *and μ is an injection.*

Proof:

Clearly, in **Set** f has the epi-mono factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f'"'] & & B \\ & f(A) \arrow[ur, "i"'] & \end{tikzcd}$$

where $f' : A \to f(A); a \mapsto f(a)$. By Lemma 6, there exists a unique dialgebra structure γ on $f(A)$, so that the upper triangle of the diagram in the statement of the Theorem commutes in $\mathbf{Set}^F_G$. Now, in **Set**, f' has the epi-mono factorization

$$\begin{tikzcd} A \arrow[rr, "f'"] \arrow[dr, "q_{K(f)}"'] & & f(A) \\ & A/K(f) \arrow[ur, "\cong"'] & \end{tikzcd}$$

Since G preserves monomorphisms, by Lemma 6, there exists a unique dialgebra structure δ on $A/K(f)$, so that the lower left triangle in the statement of the Theorem commutes in $\mathbf{Set}^F_G$. Therefore, if $\mu = i \circ \cong$, then μ is also an $\langle F, G\rangle$-homomorphism, it is an injection and the given diagram commutes. ∎

Theorem 39 is the analog of Theorem II.6.12 of [6] and Theorem 7.1 of [30].

Theorem 40 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-dialgebras, $f : \mathbf{A} \to \mathbf{B}$ a homomorphism and $R \subseteq A^2$ a bisimulation, such that $R \subseteq \mathrm{K}(f)$. If F preserves weak coequalizers, then there* *exists a unique homomorphism* $\bar{f} : \mathbf{A}/R \to \mathbf{B}$, *such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "q_R"] \arrow[dr, "f"'] & \mathbf{A}/R \arrow[d, "\bar{f}"] \\ & \mathbf{B} \end{tikzcd}$$

Proof:
Since $R \subseteq \mathrm{K}(f)$, there exists in **Set** $\bar{f} : A/R \to B$, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "q_R"] \arrow[dr, "f"'] & A/R \arrow[d, "\bar{f}"] \\ & B \end{tikzcd}$$

Since F preserves weak coequalizers, by Proposition 29, the following

$$\begin{tikzcd} \mathbf{R} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbf{A} \arrow[r, "q_R"] & \mathbf{A}/R \end{tikzcd}$$

is a coequalizer diagram in $\mathbf{Set}_G^F$. But, since $R \subseteq \mathrm{K}(f)$, $f \circ \pi_1 = f \circ \pi_2$. Thus, the existence of $\bar{f} : \mathbf{A}/R \to \mathbf{B}$ follows. ∎

Theorem 40 provides the dialgebraic analog of Lemma II.6.14 of [6] and Theorem 7.2 of [30]. Theorem 41 below is the analog of Theorem II.6.18 of [6] and (modulo weak versus regular pullbacks (see also the remarks after the proof of Theorem 36)) of Theorem 7.3 of [30].

Theorem 41 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be a dialgebra,* $\mathbf{B} = \langle B, \beta \rangle$ *a subdialgebra of* $\mathbf{A}$ *and* $R \subseteq A^2$ *a bisimulation equivalence on* $\mathbf{A}$. *Define* $B^R \subseteq A$ *by*

$$B^R = \{a \in A : \exists b \in B : \langle a, b \rangle \in R\}.$$

If F *preserves weak coequalizers and* G *preserves pullbacks, then*

1 $B^R \subseteq A$ *admits a subdialgebra structure.*

2 $Q = R \cap (B \times B)$ *is a bisimulation equivalence on* $\mathbf{B}$.

3 $\mathbf{B}/Q \cong \mathbf{B}^R/R$.

Proof:

1 Note that

$$\begin{tikzcd} \mathbf{A} & \mathbf{R} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbf{A} \end{tikzcd}$$

$B^R = \pi_1(\pi_2^{-1}(B))$. Thus, since G preserves weak pullbacks, by Theorem 35, $B^R \subseteq A$ admits a subdialgebra structure.

2 Since G preserves weak pullbacks, $\pi_1^{-1}(B) \subseteq R$ and $\pi_2^{-1}(B) \subseteq R$ admit both subdialgebra structures, by Theorem 35. Thus, by Theorem 36, since G preserves pullbacks, $\pi_1^{-1}(B) \cap \pi_2^{-1}(B) = R \cap (B \times B) \subseteq R$ also admits a subdialgebra structure. Now, by Proposition 34, $R \cap (B \times B)$ is a bisimulation on $\mathbf{B}$ and is clearly an equivalence since R is.

3 Since F preserves weak coequalizers, by Proposition 29,

$$\begin{tikzcd} \mathbf{R} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbf{A} \arrow[r, "q_R"] & \mathbf{A}/R \end{tikzcd}$$

is a coequalizer diagram in $\mathbf{Set}_G^F$. Consider the restriction $\sigma_R : B \to A/R$ of q_R to B in $\mathbf{Set}$.

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[dr, "\sigma_R"'] & A \arrow[d, "q_R"] \\ & A/R \end{tikzcd}$$

It is a homomorphism since $\sigma_R = q_R \circ i$, where $i : \mathbf{B} \to \mathbf{A}$ is the inclusion.

Now, since F preserves weak coequalizers, G preserves weak pullbacks, and since $\sigma_R(B) = q_R(i(B)) = q_R(B^R) = B^R/R$ and $\mathrm{K}(\sigma_R) = Q$, we obtain, by Theorem 39, $\mathbf{B}/Q \cong \mathbf{B}^R/R$.

■

The following is the dialgebraic analog of Theorem II.6.15 of [6] and of Theorem 7.4 of [30]. (See also Lemma 4.13 of [12].)

Theorem 42 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be a dialgebra and $R, Q \subseteq A^2$ be bisimulation equivalences on $\mathbf{A}$, such that $R \subseteq Q$. If F preserves weak coequalizers and G preserves weak pullbacks, then there is a unique homomorphism $\theta : \mathbf{A}/R \to \mathbf{A}/Q$, such that $\theta \circ q_R = q_Q$*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "q_R"] \arrow[dr, "q_Q"'] & \mathbf{A}/R \arrow[d, "\theta"] \\ & \mathbf{A}/Q \end{tikzcd}$$

Moreover $Q/R = \mathrm{K}(\theta)$ is a bisimulation equivalence on $\mathbf{A}/R$ and induces an isomorphism

$\theta' : (\mathbf{A}/R)/(Q/R) \to \mathbf{A}/Q$, *such that* $\theta = \theta' \circ q_{Q/R}$.

$$\begin{tikzcd}
\mathbf{A}/R \arrow[r, "q_{Q/R}"] \arrow[dr, "\theta"'] & (\mathbf{A}/R)/(Q/R) \arrow[d, "\theta'"] \\
& \mathbf{A}/Q
\end{tikzcd}$$

Proof:
Consider the diagram

$$\begin{tikzcd}
& \mathbf{A}/R \\
\mathbf{A} \arrow[ur, "q_R"] \arrow[dr, "q_Q"'] & \\
& \mathbf{A}/Q
\end{tikzcd}$$

which exists, by Proposition 29, since F preserves weak coequalizers. By Theorem 40, since F preserves weak coequalizers and $R \subseteq Q = \mathrm{K}(q_Q)$, there exists a unique $\theta = \bar{q}_Q : \mathbf{A}/R \to \mathbf{A}/Q$, such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "q_R"] \arrow[dr, "q_Q"'] & \mathbf{A}/R \arrow[d, "\theta"] \\
& \mathbf{A}/Q
\end{tikzcd}$$

Let $Q/R = \mathrm{K}(\theta)$. Since G preserves weak pullbacks, $\mathrm{K}(\theta)$ is a bisimulation equivalence on $\mathbf{A}/R$, by Theorem 28. Since F preserves weak coequalizers

$$\begin{tikzcd}
\mathrm{K}(\theta) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbf{A}/R \arrow[r, "q_{Q/R}"] & (\mathbf{A}/R)/(Q/R)
\end{tikzcd}$$

is a coequalizer diagram, by Proposition 29. Thus, by the universal mapping property of the coequalizer $(\mathbf{A}/R)/(Q/R)$, there exists unique $\theta' : (\mathbf{A}/R)/(Q/R) \to \mathbf{A}/Q$, such that

$$\begin{tikzcd}
\mathbf{A}/R \arrow[r, "q_{Q/R}"] \arrow[dr, "\theta"'] & (\mathbf{A}/R)/(Q/R) \arrow[d, "\theta'"] \\
& \mathbf{A}/Q
\end{tikzcd}$$

Now note that θ' is a bijection and use Proposition 1. ■

9 Simple, Initial and Final Dialgebras

In this section, the definitions of the analogs of simple algebras and coalgebras, initial or free algebras and final or cofree coalgebras are given for dialgebras. These play a major role in the development of universal algebra and the theory of coalgebras. Only some very straightforward facts are mentioned about those special dialgebras here and not much is said about the reasons why they enjoy such a central role in the corresponding theories. The reader is encouraged to consult, e.g., [6] or [22] for motivation and many more results on free algebras and [28, 16, 31, 32] for motivation and more results on cofree coalgebras. It is not clear yet if and how these results may be obtained under the umbrella of universal dialgebra.

An $\langle F, G\rangle$-dialgebra is called **simple** if it has no proper homomorphic images, i.e., if every surjection $f : \mathbf{A} \to \mathbf{B}$ is a dialgebra isomorphism.

Theorem 43 *Let $\mathbf{A} = \langle A, \alpha\rangle$ be an $\langle F, G\rangle$-dialgebra. If F preserves weak coequalizers and G preserves weak pullbacks, then $\mathbf{A}$ is simple iff Δ_A is the only bisimulation equivalence on $\mathbf{A}$.*

Proof:
Suppose $\mathbf{A}$ is simple and let $R \subseteq A \times A$ be a bisimulation equivalence on $\mathbf{A}$. Since F preserves weak coequalizers, by Proposition 29, there is a dialgebra structure on $\mathbf{A}/R$, such that

$$\begin{tikzcd} \mathbf{R} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbf{A} \arrow[r, "q_R"] & \mathbf{A}/R \end{tikzcd}$$

is a commuting diagram in $\mathbf{Set}_G^F$. Thus, q_R is an isomorphism and therefore $R = \Delta_A$.

Conversely, suppose that the only bisimulation equivalence on $\mathbf{A}$ is Δ_A and let $f : \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. Since G preserves weak pullbacks, by Proposition 28, $\mathrm{K}(f)$ is a bisimulation equivalence on $\mathbf{A}$. Thus, $\mathrm{K}(f) = \Delta_A$ and f is an isomorphism. ■

An $\langle F, G\rangle$-dialgebra $\mathbf{I} = \langle I, \iota\rangle$ is said to be **initial** if it is initial in the category $\mathbf{Set}_G^F$, i.e., if, for every dialgebra $\mathbf{A} = \langle A, \alpha\rangle$, there exists a unique homomorphism $f : \mathbf{I} \to \mathbf{A}$.

Dually, an $\langle F, G\rangle$-dialgebra $\mathbf{T} = \langle T, \tau\rangle$ is said to be **final** or **termimal** if it is final in the category $\mathbf{Set}_G^F$, i.e., if, for every dialgebra $\mathbf{A} = \langle A, \alpha\rangle$, there exists a unique homomorphism $g : \mathbf{A} \to \mathbf{T}$.

The following is easy to see.

Proposition 44 1 *If F preserves the initial object $\emptyset$ of $\mathbf{Set}$, then $\emptyset = \langle \emptyset, \emptyset\rangle$ is initial.*

2 *If G preserves the final object $\{\emptyset\}$ of $\mathbf{Set}$, then $\mathbf{1} = \langle \{\emptyset\}, 1_{F(\{\emptyset\})}\rangle$, where $1_{F(\{\emptyset\})} : F(\{\emptyset\}) \to \{\emptyset\}$ is the unique terminal map, is final.*

10 Comparing Categories of Dialgebras

Let $F, G, M, N : \mathbf{Set} \to \mathbf{Set}$ be endofunctors on $\mathbf{Set}$. Consider two natural transformations $\mu : M \to F$ and $\nu : G \to N$. These are collections of mappings $\mu_X : M(X) \to F(X)$ and $\nu_X : G(X) \to N(X)$, for all sets X, respectively, that make the following diagrams commute, for all mappings $f : X \to Y$ in $\mathbf{Set}$.

$$\begin{tikzcd} M(X) \arrow[r, "\mu_X"] \arrow[d, "M(f)"'] & F(X) \arrow[d, "F(f)"] \\ M(Y) \arrow[r, "\mu_Y"'] & F(Y) \end{tikzcd} \qquad \begin{tikzcd} G(X) \arrow[r, "\nu_X"] \arrow[d, "G(f)"'] & N(X) \arrow[d, "N(f)"] \\ G(Y) \arrow[r, "\nu_Y"'] & N(Y) \end{tikzcd}$$

Suppose $\mathbf{A} = \langle A, \alpha \rangle$ is an $\langle F, G \rangle$-dialgebra. Then

$$M(A) \xrightarrow{\mu_A} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{\nu_A} N(A)$$

$\mathbf{A}^\mu_\nu = \langle A, \nu_A \alpha \mu_A \rangle$ is an $\langle M, N \rangle$-dialgebra. Moreover, given two $\langle F, G \rangle$-dialgebras $\mathbf{A} = \langle A, \alpha \rangle$ and $\mathbf{B} = \langle B, \beta \rangle$ and a dialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$,

$$\begin{tikzcd} M(A) \arrow[r, "M(h)"] \arrow[d, "\mu_A"'] & M(B) \arrow[d, "\mu_B"] \\ F(A) \arrow[r, "F(h)"'] \arrow[d, "\alpha"'] & F(B) \arrow[d, "\beta"] \\ G(A) \arrow[r, "G(h)"'] \arrow[d, "\nu_A"'] & G(B) \arrow[d, "\nu_B"] \\ N(A) \arrow[r, "N(h)"'] & N(B) \end{tikzcd}$$

h is also an $\langle M, N \rangle$-dialgebra homomorphism $h : \mathbf{A}^\mu_\nu \to \mathbf{B}^\mu_\nu$.

If $R \subseteq A \times B$ is an $\langle F, G \rangle$-bisimulation with structure map $\rho : F(R) \to G(R)$, then $R \subseteq A \times B$ is also an $\langle M, N \rangle$-bisimulation with structure map $\nu_R \rho \mu_R : M(R) \to N(R)$:

$$\begin{tikzcd} M(A) \arrow[d, "\mu_A"'] & M(R) \arrow[l, "M(\pi_1)"'] \arrow[r, "M(\pi_2)"] \arrow[d, "\mu_R"] & M(B) \arrow[d, "\mu_B"] \\ F(A) \arrow[d, "\alpha"'] & F(R) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] \arrow[d, "\rho"] & F(B) \arrow[d, "\beta"] \\ G(A) \arrow[d, "\nu_A"'] & G(R) \arrow[l, "G(\pi_1)"] \arrow[r, "G(\pi_2)"'] \arrow[d, "\nu_R"] & F(B) \arrow[d, "\nu_B"] \\ N(A) & N(R) \arrow[l, "N(\pi_1)"] \arrow[r, "N(\pi_2)"'] & N(B) \end{tikzcd}$$

The following theorem has been established (See [30], Theorem 14.1.)

Theorem 45 *Let* $F, G, M, N : \mathbf{Set} \to \mathbf{Set}$ *be endomorphisms on* $\mathbf{Set}$ *and* $\mu : M \to F, \nu : G \to N$ *be natural transformations. Then* μ, ν *induce a functor* $\nu \circ (-) \circ \mu : \mathbf{Set}_G^F \to \mathbf{Set}_N^M$ *which maps an* $\langle F, G\rangle$*-dialgebra* $\mathbf{A} = \langle A, \alpha\rangle$ *to the* $\langle M, N\rangle$*-dialgebra* $\mathbf{A}_\nu^\mu = \langle A, \nu_A \alpha \mu_A\rangle$ *and an* $\langle F, G\rangle$*-dialgebra homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *to the* $\langle M, N\rangle$*-dialgebra homomorphism* $h : \mathbf{A}_\nu^\mu \to \mathbf{B}_\nu^\mu$*. Moreover,* $\nu \circ (-) \circ \mu$ *preserves bisimulations, i.e., if* $R \subseteq A \times B$ *is an* $\langle F, G\rangle$*-bisimulation between* $\mathbf{A}$ *and* $\mathbf{B}$*, then* $R \subseteq A \times B$ *is also an* $\langle M, N\rangle$*-bisimulation between* $\mathbf{A}_\nu^\mu$ *and* $\mathbf{B}_\nu^\mu$*.*

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