# Categorical Abstract Algebraic Logic: Categorical Algebraization of Equational Logic

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To the memory of Wim Blok.

## Abstract

This paper deals with the algebraization of multi-signature equational logic in the context of the modern theory of categorical abstract algebraic logic. Two are the novelties compared to traditional treatments: First, interpretations between different algebraic types are handled in the object language rather than the metalanguage. Second, rather than constructing the type of the algebraizing class of algebras explicitly in an ad-hoc universal algebraic way, the whole clone is naturally constructed using categorical algebraic techniques.

*Keywords*: algebraic logic, equivalent deductive systems, algebraizable logics, institutions, equivalent institutions, algebraizable institutions, algebraic theories, monads, triples, adjunctions, equational logic, clone algebras, substitution algebras

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## 1 Introduction

Equational logic has been one of the most popular and best studied logics in both mathematics and computer science. [26] and [23] provide an overview of equational logic and of results pertaining to both some of its logical and some of its computational aspects. In [1] classical results from the model theory of algebras and some results from the theory of partial algebras and relational structures were generalized to categories satisfying a variety of conditions simulating conditions holding in model categories. These results were further unified to one general axiomatizability result in [24].

Recent work in computer science, especially the areas of logic-based specifications and logic programming, has focused on different versions of equational logic and other equational based logics. In [6, 7], for instance, weak inclusion systems were introduced as a way of lifting the notion of inclusion from the category of sets to arbitrary categories. Inclusion systems are similar to the classical factorization systems. However, when inclusion systems are used, the factorizations obtained are literally unique and not unique only up to isomorphisms. Roşu, inspired by [1] and [24], used inclusion systems, a strengthening of weak inclusion systems, in [25] to provide a categorical framework for equational logic together with versions of Birkhoff-style axiomatizabil-

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ity results in the spirit of Andréka, Németi and Sain. A different abstraction of equational logic, category-based equational logic, was given in [13] and [18] with a view towards applications to the theory and the methods of logic programming. Both [25] and [18] include formalizations of the equational systems studied as institutions.

Besides formalizing a logical system as an institution, some work has been focused on connecting different institutions together via institution morphisms. Institution morphisms are involved in a substantial way in relating institutions with their algebraic counterparts in the context of categorical abstract algebraic logic. A survey of the most popular kinds of institution morphisms that have been introduced in the institution literature is given in [19]. [8] presents a nice account of how components of one logical system, formalized as an institution-like structure, may be borrowed by another system when a morphism that connects parts of the two systems exists.

As is common with many other popular logics, equational logic has also been extensively discussed and studied in the context of algebraic logic. [14] presents an algebraization of equational logic, which is further pursued and refined in the context of first-order logic in [9]. In the main framework of [14], variables play the role of term operations and the actual operations are the coordinate-wise projections and the substitutions of one operation for a variable in another. As a consequence, it is assumed that the algebraic type is countably infinite, fixed but arbitrary, and all operations are infinitary. The substitution operators are chosen in an ad-hoc way and satisfy several axioms that reflect the properties of the metamathematical substitution operators of algebra.

In [4], Blok and Pigozzi, following work of Czelakowski [10] and their own previous work [3], made for the first time precise the notion of an algebraizable logic. Their effort spearheaded the development of a bulk of work that came to be known under the name of abstract algebraic logic. Instead of considering the algebraization of individual specific logics one at a time in an ad-hoc fashion, classes of sentential logics or deductive systems are handled collectively in a systematic way and their properties both with respect to algebraizability and as related to the properties of their algebraizing classes of algebras are studied. A comprehensive overview of the main results of this field is presented in [11]. In this new context, the effort to devise a more satisfactory algebraization of equational logic has been renewed. In [12] the work of [14] is exploited to develop a system of equational logic that is amenable to the general algebraization techniques of [4]. This system is called hyperequational logic. The fact that the algebraization of equational logic of [14] is used to form a system of equational logic that is amenable to the algebraization techniques of [4] is an indication of the inadequacy of the theory of [4] to deal with multi-signature logics in a satisfactory way. Instead of dealing directly with a "natural" deductive system representing equational logic, a modification of the system, based on the knowledge of a previous algebraization, has to take place. This problem also appears when one deals with first-order logic, as is shown by considering the system in Appendix C of [4] in comparison with the algebraization of first-order logic using cylindric algebras [20]. Pigozzi had been aware of this problem very soon after the publication of [4] and this led to his directing the author's doctoral dissertation [27] that set out to address this and other similar problems and restrictions of the algebraization framework of [4].

The new categorical algebraization framework, developed in [27], has been presented in [28, 29]. A strengthening of one of the main results of [28] has been recently given in [30]. Among the main examples of this modern categorical treatment of algebraizability, presented in [29], was the algebraization of multi-signature equational logic. The main two distinctive features of this algebraization, as compared to the ones presented in [14] and [12], are, first, the treatment of substitution operations in the object language rather than the metalanguage and, second, the use of categorical algebraic techniques rather than universal algebraic methods to construct, in a natural way, the whole clone of operations of the algebraizing class of algebras rather than choosing basic operations and axioms in an ad-hoc way, as is done in the traditional treatment. In [29], it was promised that the details of the constructions and the proofs on this novel algebraization of equational logic were relegated to a forthcoming paper. This paper fulfills that promise with a slight twist. As the basic category in which equational logic and its signatures are developed is taken here the category of chain sets rather than the category of $\omega$-sets as was done in [29]. The development and the details otherwise are completely parallel. In [31], a follow-up to the present work, the connections of the algebraic theory that is used in the categorical algebraization of equational logic in the present paper with varieties of algebras that have been used in the traditional algebraizations of equational logic is explored in more detail. In a similar vein, in [32] the categorical algebraization process is applied to a system of first-order logic without terms and an algebraic theory is obtained. That theory is then used to algebraize the system. Connections of the theory with the variety of cylindric algebras, which have been used to algebraize equational logic without terms in the traditional way have been explored in some detail in [33].

Next, a summary is given of the background needed to understand the modern algebraization process. For categorical prerequisites and notation the reader is referred to any of [2, 5, 21]. More specifically, for background information on the theory of algebraic theories (or monads or triples) and how it relates to universal algebra, [22] and Volume 2 of [5] are excellent references.

Replacing deductive systems in the categorical framework are the notions of an institution, introduced by Goguen and Burstall [16, 17], and of a $\pi$-institution, a modification of institution, introduced by Fiadeiro and Sernadas [15].

An **institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ is a quadruple consisting of

(i) a category **Sign** whose objects are called **signatures** and whose morphisms are called **assignments**,

(ii) a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ from the category of signatures to the category of small sets, giving, for each $\Sigma \in |\mathbf{Sign}|$, the set of **$\Sigma$-sentences** $\mathrm{SEN}(\Sigma)$ and mapping an assignment $f : \Sigma_1 \to \Sigma_2$ to a **substitution** $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma_1) \to \mathrm{SEN}(\Sigma_2)$,

(iii) a functor $\mathrm{MOD} : \mathbf{Sign} \to \mathbf{CAT}^{\mathrm{op}}$ from the category of signatures to the opposite of the category of categories giving, for each signature $\Sigma$, the category of **$\Sigma$-models** $\mathrm{MOD}(\Sigma)$,

(iv) for each signature $\Sigma$, a **satisfaction relation** $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times \mathrm{SEN}(\Sigma)$, such that, for all $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign}), \phi \in \mathrm{SEN}(\Sigma_1)$ and $m \in |\mathrm{MOD}(\Sigma_2)|$, the following **satisfaction condition** holds

$$\mathrm{MOD}(f)(m) \models_{\Sigma_1} \phi \quad \text{iff} \quad m \models_{\Sigma_2} \mathrm{SEN}(f)(\phi).$$

Pictorially, this condition may be illustrated as follows:

$$\begin{tikzcd} \mathrm{MOD}(f)(m) \arrow[r, phantom, "\models_{\Sigma_1}"] & \phi \arrow[d, "\mathrm{SEN}(f)"] \\ m \arrow[u, "\mathrm{MOD}(f)"] \arrow[r, phantom, "\models_{\Sigma_2}"] & \mathrm{SEN}(f)(\phi) \end{tikzcd}$$

A $\pi$-**institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$, on the other hand, is a triple with its first two components exactly the same as the first two components of an institution and, for every $\Sigma \in |\mathbf{Sign}|$, a closure operator $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, such that, for every $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign})$,

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\Gamma)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\Gamma)), \quad \text{for all } \Gamma \subseteq \mathrm{SEN}(\Sigma_1).$$

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, define, for all $\Sigma \in |\mathbf{Sign}|$, $\Gamma \subseteq \mathrm{SEN}(\Sigma)$, $M \subseteq |\mathrm{MOD}(\Sigma)|$,

$$\Gamma^* = \{m \in |\mathrm{MOD}(\Sigma)| : m \models_\Sigma \Gamma\} \quad \text{and} \quad M^* = \{\phi \in \mathrm{SEN}(\Sigma) : M \models_\Sigma \phi\}$$

and set $C_\Sigma(\Gamma) = \Gamma^{**}$, for all $\Sigma \in |\mathbf{Sign}|, \Gamma \subseteq \mathrm{SEN}(\Sigma)$. Then $\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ is a $\pi$-institution, called the $\pi$-**institution associated with** the institution $\mathcal{I}$ and denoted by $\pi(\mathcal{I})$, or, sometimes, also by $\mathcal{I}$, for simplicity. In the sequel, instead of $C_\Sigma(\Gamma)$ to denote the closure of a set $\Gamma$ of $\Sigma$-sentences of an institution or of a $\pi$-institution, the simplifying notation $\Gamma^c$ will be used. Since the signature $\Sigma$ is usually clear from context, this notation will not cause any confusion.

Let $\mathbf{C}$ be a category, $\mathbf{T} = \langle T, \eta, \mu\rangle$ an algebraic theory in monoid form in $\mathbf{C}$, $\mathbf{L}$ a full subcategory of $\mathbf{C_T}$, $\Xi : \mathbf{C} \to \mathbf{Set}$ a functor and $\mathbf{Q}$ a subcategory of $\mathbf{C^T}$. Define the $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-**algebraic institution** $\mathcal{I}_{\langle \mathbf{L},\Xi,\mathbf{Q}\rangle} = \langle \mathbf{L}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ as follows

(i) $\mathrm{EQ} : \mathbf{L} \to \mathbf{Set}$ is given by $\mathrm{EQ} = ((\Xi \circ U_\mathbf{T}) \restriction_\mathbf{L})^2$, i.e.,

$$\mathrm{EQ}(L) = \Xi(T(L))^2, \quad \text{for every } L \in |\mathbf{L}|,$$

and, given $f : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$,

$$\mathrm{EQ}(f)(\langle s,t\rangle) = (\Xi(\mu_K T(f))(s), \Xi(\mu_K T(f))(t)), \quad \text{for all } \langle s,t\rangle \in \Xi(T(L))^2.$$

$$\Xi(T(L)) \xrightarrow[\Xi(T(f))]{} \Xi(T(T(K))) \xrightarrow[\Xi(\mu_K)]{} \Xi(T(K))$$

(ii) $\mathrm{ALG} : \mathbf{L} \to \mathbf{CAT}^{\mathrm{op}}$ is the functor that sends an object $L \in |\mathbf{L}|$ to the category $\mathrm{ALG}(L)$ with objects triples of the form $\langle\langle X,\xi\rangle, f\rangle, \langle X,\xi\rangle \in |\mathbf{Q}|, f : L \rightharpoonup X \in \mathrm{Mor}(\mathbf{C_T})$, and morphisms $h : \langle\langle X,\xi\rangle, f\rangle \to \langle\langle Y,\zeta\rangle, g\rangle$ $\mathbf{Q}$-morphisms $h : \langle X,\xi\rangle \to \langle Y,\zeta\rangle$, such that $g = T(h)f$.

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ T(X) \arrow[rr, "T(h)"'] & & T(Y) \end{tikzcd}$$

Moreover, given $k : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$, $\mathrm{ALG}(k) : \mathrm{ALG}(K) \to \mathrm{ALG}(L)$ is the functor that sends $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(K)|$ to $\langle\langle X, \xi\rangle, f \circ k\rangle \in |\mathrm{ALG}(L)|$ and $h : \langle\langle X, \xi\rangle, f\rangle \to \langle\langle Y, \zeta\rangle, g\rangle \in \mathrm{Mor}(\mathrm{ALG}(K))$ to

$$\mathrm{ALG}(k)(h) = h : \langle\langle X, \xi\rangle, f \circ k\rangle \to \langle\langle Y, \zeta\rangle, g \circ k\rangle \in \mathrm{Mor}(\mathrm{ALG}(L)).$$

(iii) $\models_L \subseteq |\mathrm{ALG}(L)| \times \mathrm{EQ}(L)$ is defined by

$$\langle\langle X, \xi\rangle, f\rangle \models_L \langle s, t\rangle \text{ iff } \Xi(\xi\mu_X T(f))(s) = \Xi(\xi\mu_X T(f))(t),$$

$$T(L) \xrightarrow[T(f)]{} T(T(X)) \xrightarrow[\mu_X]{} T(X) \xrightarrow[\xi]{} X$$

for all $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(L)|, \langle s, t\rangle \in \mathrm{EQ}(L)$.

By the **$\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic $\pi$-institution**, we will understand the $\pi$-institution (also denoted by $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle}$) associated with the institution $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle}$

Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ be two $\pi$-institutions. A translation of $\mathcal{I}_1$ in $\mathcal{I}_2$ is a pair $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ consisting of a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ and a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F$.

A translation is called an interpretation if, in addition, for all $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\phi \in C_{\Sigma_1}(\Phi) \quad \text{if and only if} \quad \alpha_{\Sigma_1}(\phi) \subseteq C_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi)).$$

$\mathcal{I}_1$ and $\mathcal{I}_2$ are called deductively equivalent if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that

1. $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is an adjoint equivalence
2. for all $\Sigma_1 \in |\mathbf{Sign}_1|, \Sigma_2 \in |\mathbf{Sign}_2|, \phi \in \mathrm{SEN}_1(\Sigma_1), \psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$C_{G(F(\Sigma_1))}(\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)) = C_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi)))$$
$$C_{\Sigma_2}(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))) = C_{\Sigma_2}(\psi).$$

A $\pi$-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ is **algebraizable** if it is deductively equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic $\pi$-institution. Similarly, an institution $\mathcal{I}$ is **algebraizable** if it is deductively equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic institution.

In Section 2, the institution of equational logic is defined in detail. This differs from the institution of Goguen and Burstall in several ways. First, it is restricted to single sorted algebras rather than handling the general case of multiple sorts. On the other hand, the institution of Section 2 allows for the substitution of term operations of one signature for basic operations of another whereas the one presented in [16] is restricted to substitutions of basic operations for basic operations. The added generality, in this respect, is crucial for our algebraization framework. The presentation is split to syntax, semantics and the interaction between them via the Tarski-style satisfaction relations. In Section 3, the algebraic institution of equational algebras, corresponding to the substitution algebras of Feldman, is constructed. The adjunction that gives rise to the algebraic theory is developed first. The theory is then described as it is naturally extracted from the adjunction in the usual way. They both form the basis of the algebraic institution of equational algebra. Finally, in Section 4, the actual algebraization process is given. The functors and the natural transformations are first constructed and, then, the conditions that show that the ensuing translations are inverse interpretations are proven in detail, concluding the presentation.

## 2 Equational Logic

### *The Underlying Category*

Definition 2.1
By an **ascending chain of sets** or, simply, a **chain set** A, we mean a family of sets $A = \{A_k : k \in \omega\}$, such that $A_k \subseteq A_{k+1}$, for every $k \in \omega$. By a **chain set morphism** $f : A \to B$, we mean a family of set maps $f = \{f_k : A_k \to B_k : k \in \omega\}$, such that the following diagram commutes, for every $k \in \omega$,

$$\begin{tikzcd} A_{k+1} \arrow[r, "f_{k+1}"] & B_{k+1} \\ A_k \arrow[u, "i"] \arrow[r, "f_k"'] & B_k \arrow[u, "i"'] \end{tikzcd}$$

where by $i : A_k \to A_{k+1}$ and $i : B_k \to B_{k+1}$ we denote the inclusion maps.

Given two chain set morphisms $f : A \to B$ and $g : B \to C$ we define their **composite** $gf : A \to C$ to be the collection of maps $gf = \{g_k f_k : A_k \to C_k : k \in \omega\}$. With this composition the collection of chain sets with chain set morphisms between them forms a category. It is called the **category of chain sets** and denoted by **CSet**.

### *The Signatures*

To faithfully represent algebraic systems, any formalization of equational logic must handle in a satisfactory way types of algebraic structures and the possible interpretations of one type in another. Use of the institution structure as the underlying formalism encourages viewing algebraic signatures as objects in a category and the interpretations between them as morphisms in this category. This category, called **Sign**, will now be defined.

A countable set $V = \{v_0, v_1, \ldots\}$, called **set of variables,** is fixed in advance and well-ordered and by **Set** is denoted the category of small sets.

Definition 2.2
Let $X \in |\mathbf{CSet}|$. The **chain set of $X$-terms**

$$\mathrm{Tm}_X(V) = \{\mathrm{Tm}_X(V)_k : k \in \omega\} \in |\mathbf{CSet}|$$

is defined by letting $\mathrm{Tm}_X(V)_k$ be the smallest set with

- $v_i \in \mathrm{Tm}_X(V)_k, i < k$,
- $x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V)_k$, for all $n \in \omega, x \in X_n - X_{n-1}, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$.

Definition 2.3
Let $X \in |\mathbf{CSet}|$. Define a doubly indexed collection of functions

$$R_{X_{k,l}} : \mathrm{Tm}_X(V)_k \times \mathrm{Tm}_X(V)_l^k \to \mathrm{Tm}_X(V)_l$$

by recursion on the structure of $X$-terms as follows:

- $R_{X_{k,l}}(v_i, \vec{s}) = s_i$, for all $i < k, \vec{s} \in \mathrm{Tm}_X(V)_l^k$,
- $R_{X_{k,l}}(x(t_0, \ldots, t_{n-1}), \vec{s}) = x(R_{X_{k,l}}(t_0, \vec{s}), \ldots, R_{X_{k,l}}(t_{n-1}, \vec{s}))$, for all $n \in \omega, x \in X_n - X_{n-1}, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k, \vec{s} \in \mathrm{Tm}_X(V)_l^k$.

It can be shown by an easy induction on the structure of $X$-terms that, for all $k, l \in \omega, t \in \mathrm{Tm}_X(V)_k, \vec{s} \in \mathrm{Tm}_X(V)_l^k$,

$$R_{X_{k,l}}(t, \vec{s}) = R_{X_{k,l+1}}(t, \vec{s}) \quad \text{and} \quad R_{X_{k,l}}(t, \vec{s}) = R_{X_{k+1,l}}(t, \vec{s}). \tag{2.1}$$

This fact will be used repeatedly in the sequel without being explicitly mentioned.

It is also not very difficult to see, by induction on the structure of $X$-terms, that, for all $k \in \omega, t \in \mathrm{Tm}_X(V)_k$,

$$R_{X_{k,k}}(t, \langle v_0, v_1, \ldots, v_{k-1} \rangle) = t. \tag{2.2}$$

Given two chain sets $X$ and $Y$, any chain set morphism $f$ from $X$ into the chain set $\mathrm{Tm}_Y(V)$ may be extended to a chain set morphism $f^*$ from $\mathrm{Tm}_X(V)$ into $\mathrm{Tm}_Y(V)$. The definition of this extension is given next.

Definition 2.4
Let $X, Y \in |\mathbf{CSet}|, f : X \to \mathrm{Tm}_Y(V) \in \mathrm{Mor}(\mathbf{CSet})$. Define $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$, with $f_k^* : \mathrm{Tm}_X(V)_k \to \mathrm{Tm}_Y(V)_k$, for every $k \in \omega$, by recursion on the structure of $X$-terms as follows:

- $f_k^*(v_i) = v_i, i < k$,
- $f_k^*(x(t_0, \ldots, t_{n-1})) = R_{Y_{n,k}}(f_n(x), \langle f_k^*(t_0), \ldots, f_k^*(t_{n-1}) \rangle)$, for all $n \in \omega, x \in X_n - X_{n-1}, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$.

It is not hard to check that $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$, as defined in 2.4, is a chain set morphism. In the sequel, we write $f : X \rightharpoondown Y$ to denote a **CSet**-map $f : X \to \mathrm{Tm}_Y(V)$. Given two such maps $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$, their **composition** $g \circ f : X \rightharpoondown Z$ is defined to be the **CSet**-map

$$g \circ f = g^* f.$$

We now proceed to show that the composition $\circ$ is associative, i.e., that given three morphisms $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ and $h : Z \rightharpoondown W$ we have $(h \circ g) \circ f = h \circ (g \circ f)$. Some technical lemmas are needed first that will also be of use later. The proofs can be carried out by a routine induction on the structure of $X$-terms and are therefore omitted.

Lemma 2.5
Let $k, l, m \in \omega, t \in \mathrm{Tm}_X(V)_k, \vec{u} \in \mathrm{Tm}_X(V)_l^k$ and $\vec{s} \in \mathrm{Tm}_X(V)_m^l$. Then

$$R_{X_{l,m}}(R_{X_{k,l}}(t, \vec{u}), \vec{s}) = R_{X_{k,m}}(t, \langle R_{X_{l,m}}(u_0, \vec{s}), \ldots, R_{X_{l,m}}(u_{k-1}, \vec{s}) \rangle).$$

Using Lemma 2.5 the following may be shown:

Lemma 2.6
Let $f : X \rightharpoondown Y, k, l \in \omega, t \in \mathrm{Tm}_X(V)_k, \vec{s} \in \mathrm{Tm}_X(V)_l^k$. Then

$$f_l^*(R_{X_{k,l}}(t, \vec{s})) = R_{Y_{k,l}}(f_k^*(t), f_l^*(\vec{s})).$$

With the help of Lemma 2.6 the following may now be easily proved:

LEMMA 2.7
Let $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$. Then $(g \circ f)^* = g^* f^*$.

**Proof:**
It suffices to show that, for every $k \in \omega, t \in \mathrm{Tm}_X(V)_k$, $(g \circ f)^*_k(t) = g^*_k(f^*_k(t))$. We use induction on the structure of the $X$-term $t$.

If $t = v_j, j < k$, $(g \circ f)^*_k(v_j) = v_j = g^*_k(f^*_k(v_j))$.

Next, if $n \in \omega, x \in X_n - X_{n-1}, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$,

$$(g \circ f)^*_k(x(t_0, \ldots, t_{n-1})) =$$

$$\begin{array}{ll} = & R_{Z_{n,k}}(g^*_n(f_n(x)), \langle (g \circ f)^*_k(t_0), \ldots, (g \circ f)^*_k(t_{n-1})\rangle) \\ = & R_{Z_{n,k}}(g^*_n(f_n(x)), \langle g^*_k(f^*_k(t_0)), \ldots, g^*_k(f^*_k(t_{n-1}))\rangle) \\ = & g^*_k(R_{Y_{n,k}}(f_n(x), \langle f^*_k(t_0), \ldots, f^*_k(t_{n-1})\rangle)) \\ = & g^*_k(f^*_k(x(t_0, \ldots, t_{n-1}))). \end{array}$$

■

If $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ and $h : Z \rightharpoondown W$ we have

$$\begin{array}{rcll} h \circ (g \circ f) & = & h^*(g \circ f) & \\ & = & h^*(g^* f) & \\ & = & (h^* g^*) f & \\ & = & (h \circ g)^* f & \text{(by Lemma 2.7)} \\ & = & (h \circ g) \circ f, & \end{array}$$

whence $\circ$ is associative as claimed.

Now define $j_X : X \rightharpoondown X$, given by $j_{X_k} : X_k \to \mathrm{Tm}_X(V)_k$, with

$$j_{X_k}(x) = x(v_0, \ldots, v_{k-1}), \text{ for all } x \in X_k - X_{k-1}.$$

Note, that, for all $k \in \omega, t \in \mathrm{Tm}_X(V)_k$,

$$j^*_{X_k}(t) = t. \tag{2.3}$$

It is not hard to prove, using (2.2) and (2.3), that, given $f : X \rightharpoondown Y$ and $g : Z \rightharpoondown X$ we have $f \circ j_X = f$ and $j_X \circ g = g$.

The discussion above shows that **Sign**, having collection of objects $|\mathbf{CSet}|$ and collections of morphisms

$$\mathbf{Sign}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{CSet}(X, \mathrm{Tm}_Y(V)\},$$

for all $X, Y \in |\mathbf{CSet}|$, with composition $\circ$ and $X$-identity $j_X$, is a category.

## *The Syntax*

In formalizing a logical system one has first to define its syntactic component. In classical deductive systems this consists of defining the well-formed formulas and the substitutions of formulas for individual variables. In a multi-signature system, like equational logic, a set of well-formed formulas for each type has to be defined and the effect of the different possible interpretations of one type into another on formulas specified. The use of the institution structure as the underlying formalism in this context makes it possible to unify the setting by considering a functor SEN : **Sign** $\rightarrow$ **Set**, whose object part gives the $X$-formulas, for each chosen type $X$, and whose morphism part specifies the effect of type interpretations on formulas.

At the object level, for every $X \in |\mathbf{Sign}|$, we define

$$\mathrm{SEN}(X) = (\bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)_k)^2.$$

We call an $s \approx t \in \mathrm{SEN}(X)$ an $X$-**equation.** At the morphism level, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$, we define $\mathrm{SEN}(f) : \mathrm{SEN}(X) \rightarrow \mathrm{SEN}(Y)$ by letting, for all $s, t \in \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)_k$,

$$\mathrm{SEN}(f)(s \approx t) = f_k^*(s) \approx f_k^*(t), \quad \text{if} \quad s, t \in \mathrm{Tm}_X(V)_k.$$

$\mathrm{SEN}(f)$ is well-defined, because, if $s \in \mathrm{Tm}_X(V)_k \cap \mathrm{Tm}_X(V)_l$, then $f_k^*(s) = f_l^*(s)$ and the same holds for $t$, by the definition of a **CSet**-morphism. It therefore remains to show that SEN is indeed a functor. If $f : X \rightharpoondown Y, g : Y \rightharpoondown Z \in \mathrm{Mor}(\mathbf{Sign})$, we have

$$\begin{array}{rcl} \mathrm{SEN}(g \circ f) & = & [(g \circ f)^*]^2 \\ & = & [g^* f^*]^2 \quad \text{(by Lemma 2.7)} \\ & = & \mathrm{SEN}(g)\mathrm{SEN}(f). \end{array}$$

## *The Semantics*

The second component that has to be specified in the description of a logic is its semantics. For the case of a multi-signature logical system a collection of models has to be specified for each of the different types. Moreover, the effect of interpreting one type into another on models has to be described. The use of the institution formalism imposes a certain uniformity on the collections of models. A functor MOD : **Sign** $\rightarrow$ $\mathbf{CAT}^{\mathrm{op}}$ has to be defined. Its object part determines, for each given type $X$, a category $\mathrm{MOD}(X)$ whose objects are the $X$-models and whose morphisms represent the possible transformations of one model into another that "preserve the structure" of the models that is of interest. Its morphism part is the one that specifies how the admissible interpretations of types, i.e., morphisms in **Sign**, affect the models.

We start by describing first the functor MOD : **Sign** $\rightarrow$ $\mathbf{CAT}^{\mathrm{op}}$ at the object level. Let $A$ be a set. By $\mathrm{Cl}(A) \in |\mathbf{CSet}|$ we denote the chain set whose $k$-th level $\mathrm{Cl}_k(A)$ consists of all functions $f : A^\omega \rightarrow A$ that depend only on the first $k$ variables. Let $X \in |\mathbf{CSet}|$. By an $X$-**algebra** $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$ we mean a pair consisting of a set $A$ and a **CSet**-morphism $X^{\mathbf{A}} : X \rightarrow \mathrm{Cl}(A)$. Given two $X$-algebras $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle, \mathbf{B} = \langle B, X^{\mathbf{B}} \rangle$, by an $X$-**algebra homomorphism** $h : \mathbf{A} \rightarrow \mathbf{B}$ we mean a **Set**-map $h : A \rightarrow B$, such that, for all $n \in \omega, x \in X_n, \vec{a} \in A^\omega$,

$$h(x^{\mathbf{A}}(\vec{a})) = x^{\mathbf{B}}(h(\vec{a})).$$

$X$-algebras with $X$-algebra homomorphisms between them form a category, denoted by $\mathrm{MOD}(X)$.

Given an $X$-algebra $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle$, one defines a **CSet**-map ${}^{\mathbf{A}} : \mathrm{Tm}_X(V) \to \mathrm{Cl}(A)$ by induction on the structure of $X$-terms as follows:

- $v_i^{\mathbf{A}} = p_i$, with $p_i(\vec{a}) = a_i$, for all $\vec{a} \in A^\omega, i < k$.
- $[x(t_0, \ldots, t_{n-1})]^{\mathbf{A}}(\vec{a}) = x^{\mathbf{A}}(t_0^{\mathbf{A}}(\vec{a}), \ldots, t_{n-1}^{\mathbf{A}}(\vec{a}))$, for all $\vec{a} \in A^\omega, n \in \omega, x \in X_n - X_{n-1}, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$.

The following lemma may now be proved by an easy induction on the structure of $X$-terms.

Lemma 2.8
Let $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle, \mathbf{B} = \langle B, X^{\mathbf{B}}\rangle \in |\mathrm{MOD}(X)|$, and $h : \mathbf{A} \to \mathbf{B} \in \mathrm{Mor}(\mathrm{MOD}(X))$ and $t \in \mathrm{Tm}_X(V)_k$. Then

$$h(t^{\mathbf{A}}(\vec{a})) = t^{\mathbf{B}}(h(\vec{a})).$$

Next, we define MOD at the morphism level. Let $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$. $\mathrm{MOD}(f) : \mathrm{MOD}(Y) \to \mathrm{MOD}(X)$ is the functor defined as follows: Given $\mathbf{A} = \langle A, Y^{\mathbf{A}}\rangle \in |\mathrm{MOD}(Y)|$,

$$\mathrm{MOD}(f)(\mathbf{A}) = \langle A, X^{\mathrm{MOD}(f)(\mathbf{A})}\rangle, \quad \text{with}$$

$$x^{\mathrm{MOD}(f)(\mathbf{A})}(\vec{a}) = f(x)^{\mathbf{A}}(\vec{a}), \quad \text{for all} \quad x \in X, \vec{a} \in A^\omega.$$

Moreover, given a morphism $h : \langle A, Y^{\mathbf{A}}\rangle \to \langle B, Y^{\mathbf{B}}\rangle \in \mathrm{Mor}(\mathrm{MOD}(Y))$, $\mathrm{MOD}(f)(h) : \langle A, X^{\mathrm{MOD}(f)(\mathbf{A})}\rangle \to \langle B, X^{\mathrm{MOD}(f)(\mathbf{B})}\rangle$ is given by

$$\mathrm{MOD}(f)(h) = h.$$

Lemma 2.8 may be used to show that $\mathrm{MOD}(f)$ is well defined at the morphism level. It is then immediate that $\mathrm{MOD} : \mathbf{Sign} \to \mathbf{CAT}^{\mathrm{op}}$, as defined above, is a functor.

## *Syntax, Semantics and Satisfaction*

The syntax and the semantics components of equational logic having been defined , it remains to see how these two interact. This is the most important feature of the logic, since it allows the specification of a deductive apparatus in the case of a semantically defined logic. This interaction takes the form of a satisfaction relation between models and sentences. Following Tarski, one has to specify when a sentence of the logic is satisfied by a given model. Since a multi-signature system is under consideration, a collection of such satisfaction relations has to be defined. More precisely, for each type $X$, one has to define what it means for an $X$-algebra $\mathbf{A}$ to satisfy an $X$-equation $s \approx t$. Using the institution framework we proceed by completing the definition of the appropriate institution.

Define $\mathcal{EQ} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ by letting **Sign** be the category defined in "The Signatures" subsection, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be the functor defined in "The Syntax" subsection, MOD be the functor defined in "The Semantics" subsection and, for every $X \in |\mathbf{Sign}|$, $\models_X \subseteq |\mathrm{MOD}(X)| \times \mathrm{SEN}(X)$ be defined by

$$\langle A, X^{\mathbf{A}}\rangle \models_X s \approx t \quad \text{iff} \quad s^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(\vec{a}) \text{ for every } \vec{a} \in A^\omega,$$

for all $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle \in |\mathrm{MOD}(X)|, s \approx t \in \mathrm{SEN}(X)$.

The institution formalism requires showing that "truth is invariant under change of notation". This means that, if one type gets interpreted into another, the induced transformations on the sentences and the models will be such that satisfaction will not be affected. Formally, if $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign}), s \approx t \in \mathrm{SEN}(X)$ and $\mathbf{A} = \langle A, Y^{\mathbf{A}} \rangle \in |\mathrm{MOD}(Y)|$, we must have

$$\langle A, X^{\mathrm{MOD}(f)(\mathbf{A})} \rangle \models_X s \approx t \quad \text{iff} \quad \langle A, Y^{\mathbf{A}} \rangle \models_Y \mathrm{SEN}(f)(s \approx t). \tag{2.4}$$

To show that this equivalence holds in the present context, two technical lemmas are needed first. They may again be proved by simple induction on the structure of $X$-terms, so their proofs will be omitted.

Lemma 2.9
Let $X \in |\mathbf{Sign}|, k, l \in \omega, t \in \mathrm{Tm}_X(V)_k, \vec{s} \in \mathrm{Tm}_X(V)_l^k, \mathbf{A} = \langle A, X^{\mathbf{A}} \rangle \in |\mathrm{MOD}(X)|$, $\vec{a} \in A^\omega$. Then

$$R_{X_{k,l}}(t, \vec{s})^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(s_0^{\mathbf{A}}(\vec{a}), \ldots, s_{k-1}^{\mathbf{A}}(\vec{a})).$$

With the help of Lemma 2.9, the following may now be proved.

Lemma 2.10
Let $X, Y \in |\mathbf{Sign}|, f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign}), k \in \omega, t \in \mathrm{Tm}_X(V)_k, \mathbf{A} = \langle A, Y^{\mathbf{A}} \rangle \in |\mathrm{MOD}(Y)|$. Then

$$t^{\mathrm{MOD}(f)(\mathbf{A})}(\vec{a}) = f_k^*(t)^{\mathbf{A}}(\vec{a}), \quad \text{for every } \vec{a} \in A^\omega.$$

Lemma 2.10 may now be used to prove that the satisfaction relation (2.4) holds. We have

$$\begin{array}{lll}
\langle A, X^{\mathrm{MOD}(f)(\mathbf{A})} \rangle \models_X s \approx t & \text{iff} & s^{\mathrm{MOD}(f)(\mathbf{A})}(\vec{a}) = t^{\mathrm{MOD}(f)(\mathbf{A})}(\vec{a}),\ \vec{a} \in A^\omega, \\
& \text{iff} & f_k^*(s)^{\mathbf{A}}(\vec{a}) = f_k^*(t)^{\mathbf{A}}(\vec{a}),\ \text{for all } \vec{a} \in A^\omega, \\
& \text{iff} & \langle A, Y^{\mathbf{A}} \rangle \models_Y f^*(s) \approx f^*(t) \\
& \text{iff} & \langle A, Y^{\mathbf{A}} \rangle \models_Y \mathrm{SEN}(f)(s \approx t),
\end{array}$$

as claimed. $\mathcal{EQ}$ is called the **equational institution.**

## 3 Equational Algebra

Roughly speaking, the aim of this paper is to show how one can construct in a very natural way an algebraic theory whose algebras may be used to "simulate" algebras of arbitrary types. In the equational institution, to each chosen type $X$, there are associated $X$-equations and $X$-algebras that are related to each other via the $X$-satisfaction relation. Each type is related to the remaining types via the signature interpretations, i.e., mappings in **Sign**, which also affect the equations and the algebras accordingly. This relations between the types impose a certain uniformity. This makes it possible to "unite" the different algebra types in one, by exploiting the common features in the construction. Surveying the types individually, one may notice that all the equations, regardless of type, use common variables in $V$ and substitution operations of terms of one type for basic operations of another are performed uniformly. The most important difference between types is the number and arity of operation symbols used to construct the terms of each type. Grouping the common features together, an algebraic theory in **Sign**, representing "algebras of a single type" will be obtained, such that, the sets of variables, used to construct this single type terms, will correspond to algebraic types of the equational institution. Moreover, substitution of a term for a variable in this single type context will correspond to interpreting a type into another type in the equational institution context.

### *The Adjunction*

It is well-known from categorical algebra that to each algebraic theory $\mathbf{T} = \langle T, \eta, \mu \rangle$ in monoid form in a category $\mathbf{C}$, there correspond two very important adjunctions. One is the Kleisli adjunction between $\mathbf{C}$ and the Kleisli category $\mathbf{C}_{\mathbf{T}}$, with objects the free algebras of the theory $\mathbf{T}$, and the other is the Eilenberg-Moore adjunction between $\mathbf{C}$ and the Eilenberg-Moore category $\mathbf{C}^{\mathbf{T}}$, with objects all $\mathbf{T}$-algebras. Conversely, one of the most natural ways in which an algebraic theory may arise, is through the construction of an adjunction $\langle F, G, \eta, \epsilon \rangle : \mathbf{C} \to \mathbf{D}$, between a category $\mathbf{C}$ and a category $\mathbf{D}$. Such an adjunction gives rise to an algebraic theory $\mathbf{T} = \langle T, \eta, \mu \rangle$ in $\mathbf{C}$, by setting $T = UF$ and $\mu = U\epsilon_F$. The fact that an algebraic theory is obtained in such a natural way via an adjunction is, besides generality, the main feature that makes the modern, categorical treatment of algebraic logic so much more successful in dealing with multi-signature logical systems, like equational logic, than the old, traditional one. In this context, algebraic counterparts are constructed naturally without artificial, ad-hoc manipulation of the logical system.

In this subsection, we show how to naturally extract an adjunction out of the construction of the signature category **Sign** of the equational institution.

First, define a functor $F : \mathbf{CSet} \to \mathbf{Sign}$ by

$$F(X) = X, \quad \text{for every} \quad X \in |\mathbf{CSet}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{CSet})$,

$$F(f) = j_Y f : X \rightharpoondown Y.$$

If $f : X \to Y, g : Y \to Z \in \mathrm{Mor}(\mathbf{CSet})$, then, for all $k \in \omega, x \in X_k$, with $f_k(x) \in Y_l - Y_{l-1}$

$$\begin{array}{rcl}
F(gf)_k(x) & = & j_{Z_k}((gf)_k(x)) \\
& = & j_{Z_k}((g_k f_k)(x)) \\
& = & (j_{Z_k} g_k)(f_k(x)) \\
& = & (j_Z g)_l(f_k(x)) \\
& = & R_{Z_{l,k}}((j_Z g)_l(f_k(x)), \langle (j_Z g)^*_k(v_0), \ldots, (j_Z g)^*_k(v_{l-1}) \rangle) \\
& = & (j_Z g)^*_k(f_k(x)(v_0, \ldots, v_{l-1})) \\
& = & (j_Z g)^*_k((j_Y f)_k(x)) \\
& = & F(g)^*_k(F(f)^*_k(x)) \\
& = & (F(g)_k \circ F(f)_k)(x),
\end{array}$$

i.e., $F$ is indeed a functor.

Next define a functor $U : \mathbf{Sign} \to \mathbf{CSet}$ by

$$U(X) = \mathrm{Tm}_X(V), \quad \text{for every} \quad X \in |\mathbf{Sign}|,$$

and, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$,

$$U(f) = f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V).$$

Then, if $f : X \rightharpoondown Y, g : Y \rightharpoondown Z \in \mathrm{Mor}(\mathbf{Sign})$, we have

$$\begin{array}{rcl} U(g \circ f) & = & (g \circ f)^* \\ & = & g^* f^* \quad \text{(by Lemma 2.7)} \\ & = & U(g)U(f), \end{array}$$

i.e., $U$ is indeed a functor, as claimed.

Finally, define natural transformations $\eta : I_{\mathbf{CSet}} \to UF$ by $\eta_X : X \to \mathrm{Tm}_X(V)$, with

$$\eta_X = j_X, \quad \text{for every} \quad X \in |\mathbf{CSet}|,$$

and $\epsilon : FU \to I_{\mathbf{Sign}}$ by $\epsilon_X : \mathrm{Tm}_X(V) \rightharpoondown X$, with

$$\epsilon_X = i_{\mathrm{Tm}_X(V)}, \quad \text{for every} \quad X \in |\mathbf{Sign}|.$$

We show that $\eta$ and $\epsilon$ are in fact natural transformations. To this end, let $f : X \to Y \in \mathrm{Mor}(\mathbf{CSet})$. We need to show that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathrm{Tm}_X(V) \arrow[d, "(j_Y f)^*"] \\ Y \arrow[r, "\eta_Y"'] & \mathrm{Tm}_Y(V) \end{tikzcd}$$

We have, for all $x \in X_k - X_{k-1}$,

$$\begin{array}{rcl} (j_Y f)^*_k(\eta_{X_k}(x)) & = & (j_Y f)^*_k(x(v_0, \ldots, v_{k-1})) \\ & = & R_{Y_{k,k}}((j_Y f)_k(x), \langle (j_Y f)^*_k(v_0), \ldots, (j_Y f)^*_k(v_{k-1})\rangle) \\ & = & R_{Y_{k,k}}((j_Y f)_k(x), \langle v_0, \ldots, v_{k-1}\rangle) \\ & = & (j_Y f)_k(x) \quad \text{by (2.2).} \end{array}$$

For $\epsilon$, let $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$. We have to show that the following diagram commutes

$$\begin{tikzcd} \mathrm{Tm}_X(V) \arrow[r, "\epsilon_X"] \arrow[d, "j_{\mathrm{Tm}_Y(V)} f^*"'] & X \arrow[d, "f"] \\ \mathrm{Tm}_Y(V) \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

We have

$$\begin{array}{rcl} f \circ \epsilon_X & = & f^* \epsilon_X \\ & = & f^* \\ & = & \epsilon_Y f^* \\ & = & \epsilon^*_Y j_{\mathrm{Tm}_Y(V)} f^* \\ & = & \epsilon_Y \circ F(U(f)). \end{array}$$

The next theorem may now be proved, that shows that the functors $F : \mathbf{CSet} \to \mathbf{Sign}$ and $U : \mathbf{Sign} \to \mathbf{CSet}$ are adjoints with unit $\eta$ and counit $\epsilon$. This will conclude the first stage of the algebraization, i.e., the construction of the adjunction that will help create the algebraic theory used to algebraize equational logic.

THEOREM 3.1
$\langle F, U, \eta, \epsilon\rangle : \mathbf{CSet} \to \mathbf{Sign}$ is an adjunction.

**Proof:**
The proof boils down in showing that the following diagrams commute, for all $X \in |\mathbf{CSet}|, Y \in |\mathbf{Sign}|$,

$$\begin{tikzcd}
\mathrm{Tm}_X(V) \arrow[r, "\eta_{\mathrm{Tm}_X(V)}"] \arrow[dr, "i_{\mathrm{Tm}_X(V)}"'] & \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[d, "i^*_{\mathrm{Tm}_X(V)}"] \\
& \mathrm{Tm}_X(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\eta_{\mathrm{Tm}_Y(V)}\eta_Y"] \arrow[dr, "\eta_Y"'] & \mathrm{Tm}_Y(V) \arrow[d, "i_{\mathrm{Tm}_Y(V)}"] \\
& Y
\end{tikzcd}$$

Commutativity of the first triangle follows directly from the fact that the morphism $j_{\mathrm{Tm}_X(V)}$ is the identity morphism of $\circ$. For the second diagram, we have

$$\begin{aligned}
i_{\mathrm{Tm}_Y(V)} \circ (j_{\mathrm{Tm}_Y(V)} j_Y) &= i^*_{\mathrm{Tm}_Y(V)}(j_{\mathrm{Tm}_Y(V)} j_Y) \\
&= (i^*_{\mathrm{Tm}_Y(V)} j_{\mathrm{Tm}_Y(V)}) j_Y \\
&= i_{\mathrm{Tm}_Y(V)} j_Y \\
&= j_Y,
\end{aligned}$$

as required. ■

## *The Algebraic Theory*

In this subsection, we review in some detail how the adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{CSet} \to \mathbf{Sign}$ gives rise to an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in monoid form in $\mathbf{CSet}$. As already mentioned, this process of extracting the algebraic theory $\mathbf{T}$ out of the adjunction relating the signature category of the multi-signature logic with its "underlying category" is the gist of the modern, categorical theory of algebraizability. It is the process of natural, automatic abstraction of the common features present in the different syntax components of the logical system that are, however, interrelated via the **Sign**-morphisms, which may be viewed as uniformly applicable substitution operations. Once this is done, to complete the algebraization, it only remains to investigate whether the semantical deduction of the logical system can be simulated via the semantical deduction induced by some class of $\mathbf{T}$-algebras.

To create the algebraic theory, we set $T = UF$ and $\mu = U\epsilon_F$. $T : \mathbf{CSet} \to \mathbf{CSet}$ is a functor, since it is the composite of two functors, and $\mu : TT \to T$ is a natural transformation, since $\epsilon$ is a natural transformation. Furthermore, the triangular identities of the adjunction induce the commutativity of the following diagrams, that are the prerequisites for $\mathbf{T} = \langle T, \eta, \mu \rangle$ to be an algebraic theory in **CSet**.

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "i_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T(\eta)"'] \arrow[dl, "i_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "T(\mu)"] \arrow[d, "\mu_T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd}$$

Moreover there exists a unique functor $K : \mathbf{CSet_T} \to \mathbf{Sign}$ from the Kleisli category of the theory $\mathbf{T}$ to **Sign**, called the Kleisli comparison functor of the adjunction, that makes the $F$- and $U$-paths of the following diagrams commute.

$$\begin{tikzcd} \mathbf{CSet_T} \arrow[rr, "K"] & & \mathbf{Sign} \\ & \mathbf{CSet} \arrow[ul, "F_{\mathbf{T}}"] \arrow[ur, "F"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{CSet_T} \arrow[rr, "K"] \arrow[dr, "U_{\mathbf{T}}"'] & & \mathbf{Sign} \arrow[dl, "U"] \\ & \mathbf{CSet} & \end{tikzcd}$$

Since the Kleisli category of an algebraic theory has, by definition, as objects the same objects with the underlying category of the theory and as morphisms from an object $X$ to an object $Y$ all the morphisms in the underlying category from $X$ to $T(Y)$, with composition $\circ_K$ given by $g \circ_K f = \mu_Y T(g) f$, for all $f : X \to T(Y), g : Y \to T(Z) \in \mathrm{Mor}(\mathbf{CSet})$, it is easy to see that in this case $\mathbf{CSet_T} = \mathbf{Sign}$ and $K = I_{\mathbf{Sign}}$. In fact

$$\begin{aligned} g \circ_K f &= \mu_Z T(g) f \\ &= U(\epsilon_{F(Z)}) U(F(g)) f \\ &= U(\epsilon_{F(Z)} \circ F(g)) f \\ &= U(i^*_{\mathrm{Tm}_{F(Z)}(V)} j_Z g) f \\ &= U(g) f \\ &= g^* f \\ &= g \circ f. \end{aligned}$$

Therefore **Sign** is the category of all free algebras of the algebraic theory $\mathbf{T}$ over **CSet**.

A $\mathbf{T}$-algebra $\mathbf{X} = \langle X, \xi \rangle$ in **CSet** is now a pair consisting of a chain set $X$ together with a **CSet**-morphism $\xi : \mathrm{Tm}_X(V) \to X$, such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[dr, "i_X"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[r, "(j_X \xi)^*"] \arrow[d, "i^*_{\mathrm{Tm}_X(V)}"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ \mathrm{Tm}_X(V) \arrow[r, "\xi"'] & X \end{tikzcd}$$

The chain set $X$ is called the carrier of $\mathbf{X}$ and the morphism $\xi$ is called the structure map of $\mathbf{X}$. Moreover, given two $\mathbf{T}$-algebras $\mathbf{X} = \langle X, \xi \rangle$ and $\mathbf{Y} = \langle Y, \zeta \rangle$, a $\mathbf{T}$-algebra homomorphism $h : \mathbf{X} \to \mathbf{Y}$ is a **CSet**-morphism $h : X \to Y$, such that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Tm}_X(V) \arrow[r, "(j_Y h)^*"] \arrow[d, "\xi"'] & \mathrm{Tm}_Y(V) \arrow[d, "\zeta"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

The category with collection of objects the collection of all **T**-algebras and with morphisms all **T**-algebra homomorphisms between them is known as the Eilenberg-Moore category of **T**-algebras in **CSet** and is denoted by $\mathbf{CSet}^{\mathbf{T}}$.

## *The Clone Algebras*

In this subsection, it is shown how, given a set $A$, a **T**-algebra $\mathbf{A}^*$ may be associated with it. This association is very important for several reasons. First, it gives a concrete example of what a **T**-algebra looks like. Intuitively speaking, $\mathbf{A}^*$ will have as universe the clone of all finitary functions on $A$ and its structure map will show how these operations behave under composition of functions. Second, it will be shown that, roughly speaking, any $X$-algebra $\mathbf{A}$, with universe $A$, satisfies the same $X$-equations in the equational institution as $\mathbf{A}^*$ will satisfy in an algebraic institution based on the algebraic theory **T**, once we "make fundamental operations in $\mathbf{A}^*$ agree with those of $\mathbf{A}$". We call $\mathbf{A}^*$ the **clone algebra of** $A$. This process will allow the construction of a class of clone algebras, whose semantical equational entailment will then be used to algebraize equational logic.

Recall that, given a set $A$, $\mathrm{Cl}(A)$ denotes the chain set whose $k$-th level $\mathrm{Cl}_k(A)$ consists of all functions $f : A^\omega \to A$ that depend only on the first $k$ variables. Given such a set $A$, we define $\mathbf{A}^* = \langle \mathrm{Cl}(A), \xi_A \rangle$, where $\xi_A : \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \to \mathrm{Cl}(A) \in \mathrm{Mor}(\mathbf{CSet})$ is determined by $\xi_{A_k} : \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k \to \mathrm{Cl}_k(A)$, defined by recursion on the structure of $\mathrm{Cl}(A)$-terms over $V$ as follows.

- $\xi_{A_k}(v_i) = p_i$, for every $i < k$, where $p_i : A^\omega \to A$ is the $i$-th projection map.
- $\xi_{A_k}(f(t_0, \ldots, t_{n-1})) = f(\xi_{A_k}(t_0), \ldots, \xi_{A_k}(t_{n-1}))$, for all $n \in \omega, f \in \mathrm{Cl}_n(A) - \mathrm{Cl}_{n-1}(A), t_0, \ldots, t_{n-1} \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k$.

It will be shown next that $\mathbf{A}^*$ is a **T**-algebra. A technical lemma is needed first, whose proof is by an easy induction on the structure of $\mathrm{Cl}(A)$-terms and will therefore be omitted.

Lemma 3.2
Let $A$ be a set, $k, l \in \omega, t \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k, \vec{s} \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_l^k$. Then

$$\xi_{A_l}(R_{\mathrm{Cl}(A)_{k,l}}(t, \vec{s})) = \xi_{A_k}(t)(\xi_{A_l}(s_0), \ldots, \xi_{A_l}(s_{k-1})).$$

With the help of Lemma 3.2 it is now easy to show

Theorem 3.3
$\mathbf{A}^* = \langle \mathrm{Cl}(A), \xi_A \rangle$ is a **T**-algebra.

**Proof:**

By the definition of a **T**-algebra, we need to check the commutativity of the following diagrams

$$\begin{tikzcd} \mathrm{Cl}(A) \arrow[r, "\eta_{\mathrm{Cl}(A)}"] \arrow[dr, "i_{\mathrm{Cl}(A)}"'] & \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \arrow[d, "\xi_A"] \\ & \mathrm{Cl}(A) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V) \arrow[r, "(\eta_{\mathrm{Cl}(A)}\xi_A)^*"] \arrow[d, "i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}"'] & \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \arrow[d, "\xi_A"] \\ \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \arrow[r, "\xi_A"'] & \mathrm{Cl}(A) \end{tikzcd}$$

For the triangle, let $k \in \omega, f \in \mathrm{Cl}_k(A) - \mathrm{Cl}_{k-1}(A)$. Then

$$\begin{array}{rcl} \xi_{A_k}(\eta_{\mathrm{Cl}(A)_k}(f)) & = & \xi_{A_k}(f(v_0, \ldots, v_{k-1})) \\ & = & f(\xi_{A_k}(v_0), \ldots, \xi_{A_k}(v_{k-1})) \\ & = & f(p_0, \ldots, p_{k-1}) \\ & = & f. \end{array}$$

For the rectangle, we work by induction on the structure of a $\mathrm{Tm}_{\mathrm{Cl}(A)}(V)$-term $t$.

For $t = v_i, i < k$, $\xi_{A_k}((\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(v_i)) = \xi_{A_k}(v_i) = \xi_{A_k}(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(v_i))$.

Next, if $n \in \omega, t \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_n - \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_{n-1}, \vec{s} \in \mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)^n_k$, such that, for all $i < n$, $\xi_{A_k}((\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(s_i)) = \xi_{A_k}(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(s_i))$, then

$$\xi_{A_k}((\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(t(s_0, \ldots, s_{n-1}))) =$$

$$\begin{array}{rl} = & \xi_{A_k}(R_{\mathrm{Cl}(A)_{n,k}}(\eta_{\mathrm{Cl}(A)_n}(\xi_{A_n}(t)), \langle(\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(s_0), \ldots, (\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(s_{n-1})\rangle)) \\ = & \xi_{A_n}(\eta_{\mathrm{Cl}(A)_n}(\xi_{A_n}(t)))(\xi_{A_k}((\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(s_0)), \ldots, \xi_{A_k}((\eta_{\mathrm{Cl}(A)}\xi_A)^*_k(s_{n-1}))) \\ = & \xi_{A_n}(t)(\xi_{A_k}(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(s_0)), \ldots, \xi_{A_k}(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(s_{n-1}))) \\ = & \xi_{A_k}(R_{\mathrm{Cl}(A)_{n,k}}(t, \langle i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(s_0), \ldots, i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(s_{n-1})\rangle)) \\ = & \xi_{A_k}(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k}(t(s_0, \ldots, s_{n-1}))). \end{array}$$

■

## 4 Algebraization of Equational Logic

Roughly speaking, algebraizing a logical system means associating with it an algebraic system in such a way that, first, each system may be syntactically interpreted in the other and, second, the entailment of each system may be simulated by the entailment of the other under the chosen syntactical interpretation. More specifically, the algebraization process of a multi-signature logic consists of two main components. A type of algebras has to be chosen that abstracts the syntactical features of the logic common to all its signature components. This choice makes possible the syntactical interpretation of the logic into the algebraic system and vice-versa. Once the type has been chosen, a class of algebras of that type has to be selected in such a way that the semantical consequence relation induced by it may simulate and be simulated by the consequence relation of the logical system under the previously chosen syntactical interpretations.

In the institution context, given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, that represents the multi-signature logical system to be algebraized, the following steps have to be carried out. An algebraic theory $\mathbf{T}$ in a category $\mathbf{C}$ has to be chosen, that corresponds to the choice of a single type of algebras. A full subcategory $\mathbf{L}$ of the Kleisli category $\mathbf{C_T}$ and a subcategory $\mathbf{Q}$ of the Eilenberg-Moore category $\mathbf{C^T}$ of $\mathbf{T}$-algebras have to be selected in such a way that an institution $\mathcal{I}^{\mathbf{L}}_{\mathbf{Q}} = \langle \mathbf{L}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ may be constructed, that is deductively equivalent to $\mathcal{I}$. This means that there exist functors $F : \mathbf{Sign} \to \mathbf{L}, G : \mathbf{L} \to \mathbf{Sign}$, that are components of a natural equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign} \to \mathbf{L}$ modeling the syntactic interpretations, and natural transformations $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}F, \beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}G$, such that, for every choice of $\Sigma \in |\mathbf{Sign}|, L \in |\mathbf{L}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma), \Psi \cup \{\psi\} \subseteq \mathrm{EQ}(L)$, the following relations hold

$$\phi \in \Phi^c \quad \text{iff} \quad \alpha_\Sigma(\phi) \subseteq \alpha_\Sigma(\Phi)^c, \tag{4.1}$$

$$\psi \in \Psi^c \quad \text{iff} \quad \beta_L(\psi) \subseteq \beta_L(\Psi)^c \tag{4.2}$$

$$\mathrm{SEN}(\eta_\Sigma)(\phi)^c = \beta_{F(\Sigma)}(\alpha_\Sigma(\phi))^c \quad \text{and} \quad \mathrm{EQ}(\epsilon_L)(\alpha_{G(L)}(\beta_L(\psi)))^c = \{\psi\}^c \tag{4.3}$$

Roughly speaking, $\alpha$ and $\beta$ simulate the deduction mechanism of $\mathcal{I}$ into that of $\mathcal{I}^{\mathbf{L}}_{\mathbf{Q}}$ and vice-versa and are inverses of each other.

For the special case of the equational institution $\mathcal{EQ} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ we choose the algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in $\mathbf{CSet}$, that was constructed in the previous section. We set $\mathbf{L} = \mathbf{Sign} = \mathbf{CSet_T}$ and let $\mathbf{Q}$ be the full subcategory of $\mathbf{CSet^T}$ with collection of objects

$$\{\mathbf{A}^* = \langle \mathrm{Cl}(A), \xi_A\rangle : A \in |\mathbf{Set}|\}.$$

Construct the institution $\mathcal{I}_{\mathbf{Q}} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ as follows:

(i) $\mathrm{EQ} = \mathrm{SEN}$.

(ii) For every $X \in |\mathbf{Sign}|$, $\mathrm{ALG}(X)$ is the category with objects pairs $\langle \mathbf{A}^*, f\rangle, \mathbf{A}^* \in |\mathbf{Q}|, f : X \rightharpoondown \mathrm{Cl}(A) \in \mathrm{Mor}(\mathbf{Sign})$, and morphisms $h : \langle \mathbf{A}^*, f\rangle \to \langle \mathbf{B}^*, g\rangle$, $\mathbf{T}$-algebra homomorphisms $h : \mathbf{A}^* \to \mathbf{B}^*$, such that $g = h \circ f$. Moreover, given $k : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign}), \mathrm{ALG}(k) : \mathrm{ALG}(Y) \to \mathrm{ALG}(X)$ is the functor that maps an object $\langle \mathbf{A}^*, f\rangle \in |\mathrm{ALG}(Y)|$ to $\langle \mathbf{A}^*, f \circ k\rangle \in |\mathrm{ALG}(X)|$ and a morphism $h : \langle \mathbf{A}^*, f\rangle \to \langle \mathbf{B}^*, g\rangle$ to the morphism $\mathrm{ALG}(k)(h) : \langle \mathbf{A}^*, f \circ k\rangle \to \langle \mathbf{B}^*, g \circ k\rangle$, with $\mathrm{ALG}(k)(h) = h$.

(iii) Finally, satisfaction in $\mathcal{I}_{\mathbf{Q}}$ is defined, for every $X \in |\mathbf{Sign}|$, by

$$\langle \mathbf{A}^*, f\rangle \models_X s \approx t \quad \text{iff} \quad \xi_A(f^*(s)) = \xi_A(f^*(t)),$$

for all $\langle \mathbf{A}^*, f\rangle \in |\mathrm{ALG}(X)|, s \approx t \in \mathrm{EQ}(X)$.

Before stating and proving the main result of the paper on the deductive equivalence of $\mathcal{EQ}$ and $\mathcal{I}_{\mathbf{Q}}$, the satisfaction of an equation by an $X$-algebra in the equational institution $\mathcal{EQ}$ and the satisfaction of the same equation by a clone algebra in the algebraic institution $\mathcal{I}_{\mathbf{Q}}$ have to be related. Some preliminary work is done in the following two lemmas. The proofs are, once more, by routine induction on the structure of terms and therefore omitted.

Lemma 4.1
Let $X \in |\mathbf{Sign}|, k \in \omega, t \in \mathrm{Tm}_X(V)_k$ and $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle \in |\mathrm{MOD}(X)|$. Then

$$t^{\mathbf{A}} = \xi_{A_k}((\eta_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*_k(t)).$$

Lemma 4.2
Let $X \in |\mathbf{Sign}|, k \in \omega, t \in \mathrm{Tm}_X(V)_k$ and $\langle \langle \mathrm{Cl}(A), \xi_A \rangle, f \rangle \in |\mathrm{ALG}(X)|$. Then, if $\mathbf{A} = \langle A, \xi_A f \rangle$,

$$t^{\mathbf{A}} = \xi_{A_k}(f^*_k(t)).$$

With the help of Lemmas 4.1 and 4.2, the following theorem may now be proved

Theorem 4.3
$\mathcal{EQ} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ and $\mathcal{I}_{\mathbf{Q}} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ are deductively equivalent institutions.

**Proof:**
We take $F = G = I_{\mathbf{Sign}}$ as the signature functors and define the natural transformations $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$ and $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$, by $\alpha_X : \mathrm{SEN}(X) \to \mathcal{P}(\mathrm{EQ}(X))$, with

$$\alpha_X(s \approx t) = \{s \approx t\}, \quad \text{for every} \quad s \approx t \in \mathrm{SEN}(X),$$

and $\beta_X : \mathrm{EQ}(X) \to \mathcal{P}(\mathrm{SEN}(X))$, with

$$\beta_X(s \approx t) = \{s \approx t\}, \quad \text{for every} \quad s \approx t \in \mathrm{EQ}(X).$$

It is straightforward to check that $\alpha$ and $\beta$ are indeed natural transformations. We now need to show that (4.1),(4.2) and (4.3) hold.

For (4.1) and (4.2), let $X \in |\mathbf{Sign}|, E \cup \{s \approx t\} \subseteq \mathrm{SEN}(X)$. We need to show that

$$s \approx t \in E^{c_{\mathcal{EQ}}} \quad \text{iff} \quad s \approx t \in E^{c_{\mathcal{I}_{\mathbf{Q}}}}.$$

We first show that, if $s \approx t \in E^{c_{\mathcal{EQ}}}$, then $s \approx t \in E^{c_{\mathcal{I}_{\mathbf{Q}}}}$. Suppose that $s \approx t \in E^{c_{\mathcal{EQ}}}$. Then, for every $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle \in |\mathrm{MOD}(X)|$,

$$e_0^{\mathbf{A}}(\vec{a}) = e_1^{\mathbf{A}}(\vec{a}), \quad \text{for every } e_0 \approx e_1 \in E, \vec{a} \in A^\omega, \quad \text{implies} \quad s^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(\vec{a}). \tag{4.4}$$

Now assume that $\langle \mathbf{A}^*, f \rangle \in |\mathrm{ALG}(X)|$, such that $\langle \mathbf{A}^*, f \rangle \models_X e_0 \approx e_1$, for all $e_0 \approx e_1 \in E$. Then $\xi_A(f^*(e_0)) = \xi_A(f^*(e_1))$, for all $e_0 \approx e_1 \in E$. Thus, by Lemma 4.2, $e_0^{\mathbf{A}} = e_1^{\mathbf{A}}$, for all $e_0 \approx e_1 \in E$, whence, by (4.4), $s^{\mathbf{A}} = t^{\mathbf{A}}$ and, by reversing the steps in the deduction above, $\langle \mathbf{A}^*, f \rangle \models_X s \approx t$. Hence $s \approx t \in E^{c_{\mathcal{I}_{\mathbf{Q}}}}$, as was to be shown.

Suppose, conversely, that $s \approx t \in E^{c_{\mathcal{I}_{\mathbf{Q}}}}$. Then, for every $\langle \mathbf{A}^*, f \rangle \in |\mathrm{ALG}(X)|$,

$$\langle \mathbf{A}^*, f \rangle \models_X e_0 \approx e_1, \quad \text{for all } e_0 \approx e_1 \in E, \quad \text{implies} \quad \langle \mathbf{A}^*, f \rangle \models_X s \approx t. \tag{4.5}$$

Now assume that $\langle A, X^{\mathbf{A}} \rangle \in |\mathrm{MOD}(X)|$, such that $e_0^{\mathbf{A}}(\vec{a}) = e_1^{\mathbf{A}}(\vec{a})$, for all $e_0 \approx e_1 \in E, \vec{a} \in A^\omega$. Then, by Lemma 4.1, $\xi_A((\eta_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*(e_0)) = \xi_A((\eta_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*(e_1))$, for all $e_0 \approx e_1 \in E$, i.e., $\langle \mathbf{A}^*, \eta_{\mathrm{Cl}(A)} X^{\mathbf{A}} \rangle \models_X e_0 \approx e_1$, for every $e_0 \approx e_1 \in E$. Thus, by (4.5), $\langle \mathbf{A}^*, \eta_{\mathrm{Cl}(A)} X^{\mathbf{A}} \rangle \models_X s \approx t$ and reversing the steps in the deduction above $s^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(\vec{a})$, for all $\vec{a} \in A^\omega$. Therefore $s \approx t \in E^{c_{\mathcal{EQ}}}$, as required.

Since $\beta_X(\alpha_X(s \approx t)) = \{s \approx t\}$ and $\alpha_X(\beta_X(s \approx t)) = \{s \approx t\}$, (4.3) obviously holds. ■

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