Categorical Abstract Algebraic Logic:
Generalized Tarski Congruence Systems

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Abstract

Blok and Pigozzi introduced the Leibniz operator, mapping filters of an algebra to congruences of the algebra, in order to provide a characterization of algebraizable deductive systems, i.e., those deductive systems whose entailment relation is very closely connected to the algebraic consequence operation associated with a quasi-variety of universal algebras. Font and Jansana generalized the Leibniz operator to the Tarski operator, that maps closure systems on a given algebra to congruences of the algebra. In previous work by the author, Tarski congruence systems were introduced as a generalization of the Tarski operator in order to cover logical systems formalized as π-institutions. Tarski congruence systems consist, roughly speaking, of a system of equivalence relations, one for each set of sentences, of the given π-institution, that are preserved both by the signature morphisms and by given collections of natural transformations from tuples of sentences to sentences. In this paper, Tarski congruence systems are generalized so as to accommodate also equivalences at the signature level. In the main result of the paper a characterization of these generalized Tarski congruence systems is obtained, which abstracts the known characterization of Tarski congruence systems.

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[0] *Keywords:* abstract algebraic logic, deductive systems, institutions, equivalent deductive systems, algebraizable deductive systems, adjunctions, equivalent institutions, algebraizable institutions, Leibniz congruence, Tarski congruence, algebraizable sentential logics.

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1 Introduction

In [3], Blok and Pigozzi introduced the concept of the Leibniz congruence associated with the theories of a deductive system. Leibniz congruences are, more generally, associated with filters of logical matrices; the case of theories, i.e., filters on formula algebras, being a special case. More specifically, given a logical matrix $\mathcal{A} = \langle \mathbf{A}, F \rangle$, the Leibniz congruence associated with $\mathcal{A}$ is the largest congruence on the algebra $\mathbf{A}$ that is compatible with F, in the sense that F is the union of equivalence classes of the congruence. Properties of the Leibniz congruence give rise to the abstract algebraic hierarchy of logics, consisting of the major classes of protoalgebraic, equivalential and algebraizable logics [6], [2], [3] (see also [10] for an overview). In subsequent work, Font and Jansana [9] generalized the work of Blok and Pigozzi by considering the notion of a Tarski congruence of an abstract logic. An abstract logic, or generalized matrix, $\mathbb{L} = \langle \mathbf{A}, C \rangle$ consists of an algebra $\mathbf{A}$ together with a closure operator on A, the universe of $\mathbf{A}$. The Tarski congruence associated with the abstract logic $\mathbb{L}$ is the largest congruence that is compatible with all closed sets of the closure C. Both the Leibniz and the Tarksi congruence of a logic provide significant tools for the investigation of the algebraizability of a logic and for the study of the connections between metalogical properties of logics and corresponding algebraic properties. Except for [9] and [10], [7] is another exposition of the rôle that congruences with compatibility properties play in studying the interaction between logical and algebraic properties.

In the dissertation [16] and accompanying subsequent work [17, 18] the author generalized the theory of algebraizability to a categorical, more abstract, level, able to cover the case of institutional logics. The class of logics formalized as π-institutions includes, besides all sentential logics, logics with multiple signatures and quantifiers as well as some logics whose syntax is not string-based like, for instance, the graph-based equational logic presented in [18]. In recent work [19], the notion of a congruence system, as pertaining to π-institutions, which allows carrying some of the universal algebraic results to the level of π-institutions was introduced. Congruence systems are special kinds of systems of equivalence relations on the sentences of a π-institution. Roughly speaking, they have the additional feature that they are preserved by sentence morphisms and they are also preserved by selected collections of finitary natural transformations from tuples of sentences to sentences. Based on this notion of a congruence system, the notion of a Tarksi congruence system associated with a given π-institution was defined. Again roughly speaking, congruence systems are ordered and, thus, endowed with the structure of a complete lattice. The Tarski congruence system is the largest congruence system in this ordering that is compatible with all theories over all signatures, in a way analogous to the Tarski congruences of Font and Jansana.

In the present paper the notion of the Tarski congruence system, introduced in [19], is generalized to encompass congruences on the category of signatures which were absent in [19]. The development here follows along similar lines as that of [19]. First, the notion of a generalized congruence system is introduced. These generalized congruence systems are ordered and the ordering proves to be a complete lattice ordering. The logical congruence systems, i.e., those that are compatible with all theories of the π-institution $\mathcal{I}$, form a complete lattice under this ordering. The largest one is singled out and called the (generalized) Tarski congruence system of $\mathcal{I}$. A characterization of the Tarksi congruence systems that generalizes the one given for the special Tarksi congruence systems of [19] is provided.

The reader is referred to either of [1], [4] or [15] for all unexplained categorical notation, to [11], [12] for the introduction and the basic concepts pertaining to institutions and to [8] for those on π-institutions, and, finally, to [17] for the introduction of translations and interpretations between π-institutions. In [14], a comparison is given of many of the different notions of morphisms that have been introduced in the theory of institutions, some of which are related to the ones used here.

2 Sentential Logics and π-Institutions

In this section, bits of the theory of the Tarski operator of sentential logics, as introduced by Font and Jansana in [9], that serves as the paradigm for the categorical theory and may be viewed as the primary motivation for its development, are presented. Discussing some relevant aspects of the theory of sentential logics and introducing their analogs in the theory of π-institutions will facilitate the understanding of the theory developed in later sections, where references to and comparisons between these two theories will be frequently made. The primary reference source for the material on sentential logics is [9], but the reader is also referred to [3] and [7].

Recall that a *logical matrix* $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is a pair consisting of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and a subset $F \subseteq A$, called the *filter* of $\mathcal{A}$. A congruence θ of $\mathbf{A}$ is said to be *compatible* with F if F is a union of θ-equivalence classes, i.e., if, for all $a, b \in A$, $\langle a, b \rangle \in \theta$ and $a \in F$ imply $b \in F$. In this case θ is called a *matrix congruence* of $\mathcal{A}$. The collection of all congruences of $\mathbf{A}$

forms a lattice under inclusion. The collection of all matrix congruences of $\mathcal{A}$ forms a principal ideal of this lattice and its maximum element is called the *Leibniz congruence* of $\mathcal{A}$ and denoted by $\Omega(\mathcal{A})$ or $\Omega_{\mathbf{A}}(F)$. Blok and Pigozzi [3] introduced this congruence and they proved that, for all $a, b \in A$,

$$\langle a,b\rangle \in \Omega_{\mathbf{A}}(F) \qquad \text{iff} \qquad \begin{array}{l} \forall \phi(p,\vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \forall \vec{c} \in A^k, \\ \phi^{\mathbf{A}}(a,\vec{c}) \in F \Longleftrightarrow \phi^{\mathbf{A}}(b,\vec{c}) \in F, \end{array} \tag{1}$$

where, by $\mathrm{Fm}_{\mathcal{L}}(V)$ is denoted the set of $\mathcal{L}$-formulas in a fixed denumerable set of variables V and k is the length of $\vec{q}$. Extensive study of the properties of $\Omega_{\mathbf{A}}$ viewed as an operator from the lattice of filters on $\mathbf{A}$ to the lattice of congruences of $\mathbf{A}$ has given rise to an algebraic hierarchy of logics, which constitutes the backbone of the area of abstract algebraic logic. (A very good reference is [7], where the interested reader may find, apart from a description of the most important classes of this hierarchy, many more references to original works.)

Recall from [9] that an *abstract logic* $\mathbb{L} = \langle \mathbf{A}, C\rangle$ consists of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ together with a closure operator C on A. In [22], an abstract logic was called a *generalized matrix*. A congruence θ of $\mathbf{A}$ is said to be a *logical congruence* of $\mathbb{L}$, if, for all $a, b \in A$,

$$\langle a,b\rangle \in \theta \quad \text{implies} \quad C(a) = C(b).$$

This is equivalent to θ being compatible with all C-closed sets of A. As in the case of a logical matrix, it is also the case here that the lattice of all logical congruences of $\mathbb{L}$ is a principal ideal of the lattice of all congruences of $\mathbf{A}$ and its largest element is called the *Tarski congruence* of $\mathbb{L}$ and denoted by $\widetilde{\Omega}(\mathbb{L})$ or $\widetilde{\Omega}_{\mathbf{A}}(C)$. The Tarski congruence of an abstract logic is the main tool of the theory developed in [9], where it is noted that the characterization of the Leibniz congruence (1) immediately yields the following characterization of the Tarski congruence.

$$\begin{array}{l} \langle a,b\rangle \in \widetilde{\Omega}_{\mathbf{A}}(C) \quad \text{iff} \\ \qquad\qquad \forall \phi(p,\vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \forall \vec{c} \in A^k, C(\phi^{\mathbf{A}}(a,\vec{c})) = C(\phi^{\mathbf{A}}(b,\vec{c})). \end{array}$$

Finally, before introducing the basic analogs of these notions for logics formalized as π-institutions, the definition of a π-institution is provided, which will be the central object of our investigations. For many more details on institutions the reader is referred to the original sources [11] and [12], where many examples may also be found. For π-institutions, the work of Fiadeiro and Sernadas [8] is the original reference. For other examples of a logical nature the reader may consult [18] and [17]. A lot of examples pertaining to theoretical computer science may be found in the literature, e.g., in [13] and [14].

A **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ consists of

(i) a category **Sign** whose objects are called **signatures**,

(ii) a functor SEN : **Sign** $\rightarrow$ **Set**, from the category **Sign** of signatures into the category **Set** of sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or Σ-**sentences** and

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called Σ-**closure,** such that

  (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

  (b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

  (c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

  (d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

Sometimes the focus will be on just the signature category and the sentence functor. In that case, we will suppress **Sign** and only speak of SEN : **Sign** $\rightarrow$ **Set** with the signature category being understood from context.

The **clone of all natural transformations on** SEN is the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \rightarrow \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau : \mathrm{SEN}^\alpha \rightarrow \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category with objects all objects of the form SEN^k for $k < \omega$, that contains all projection morphisms $p^{k,i} : \mathrm{SEN}^k \rightarrow \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \rightarrow \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and is such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN (see, also, [19, 20, 21]).

In [19] parts of the theory of the Tarski congruences for abstract logics were generalized to cover the case of π-institutions. We describe briefly this theory since it forms the basis for the developments in the present paper and its basic concepts are special cases of the concepts that will be developed here. Given a category **Sign** and a functor SEN : **Sign** $\to$ **Set**, an **equivalence system** θ on SEN is a collection $\theta = \{\theta_\Sigma : \Sigma \in |\mathbf{Sign}|\}$, where θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and every $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2}$. Given a category N of natural transformations on SEN, an equivalence system θ is an N-**congruence system** if, in addition, it satisfies, for all $\Sigma \in |\mathbf{Sign}|, \sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$\vec{\phi}\ \theta_\Sigma^n\ \vec{\psi} \quad \text{implies} \quad \sigma_\Sigma(\vec{\phi})\ \theta_\Sigma\ \sigma_\Sigma(\vec{\psi}).$$

Congruence systems on SEN may be ordered by signature-wise inclusion. It is shown in [19] that the collection of all N-congruence systems on SEN, ordered in this way, forms a complete lattice.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$, an N-congruence system on SEN is called a **logical** N**-congruence system**, if, for all $\Sigma \in |\mathbf{Sign}|$,

$$\langle \phi, \psi\rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

Logical N-congruence systems also form a complete lattice under the previously described ordering. The largest logical N-congruence system of $\mathcal{I}$ is the one termed the **Tarski** N**-congruence system** of the π-institution $\mathcal{I}$. It is denoted by $\widetilde{\Omega}^N(\mathcal{I})$. In [19] the following characterization of the Tarski N-congruence system, inspired by the one given by Font and Jansana for the Tarski congruence of an abstract logic, is proved: $\langle \phi, \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, all natural transformations $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$ and all $i = 0, \ldots, k-1$,

$$\begin{array}{l} C_{\Sigma'}(\tau_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\phi), \chi_i, \ldots, \chi_{k-2})) \quad = \\ \qquad\qquad C_{\Sigma'}(\tau_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\psi), \chi_i, \ldots, \chi_{k-2})). \end{array} \tag{2}$$

This condition will sometimes be abbreviated by

$$C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})),$$

with the understanding that $\mathrm{SEN}(f)(\phi)$ or $\mathrm{SEN}(f)(\psi)$ may occur in any of the k positions of $\tau_{\Sigma'}$, the only restriction being that they occur in the same position on both sides of Equation (2). Hopefully, this notational convention, that takes after a similar convention adopted in the theory of deductive systems, will not cause any confusion. Special care will be taken if, in some particular context, extra transparency is required on this point.

The reader is invited to notice the absence of a category component from the equivalence systems and the N-congruence systems described above. Only identification of sentences over the same signature is allowed. It is not possible to identify signatures and then to identify sentences over two different, but identified, signatures. The present paper will be an attempt to bend this rigidity. A more general notion of a congruence system will be introduced, where identification of signatures will be allowed. Then a characterization of the resulting generalized Tarski congruence system will be given along the lines of the characterization (2) for the special case reviewed above.

3 Category Congruences

Let **Sign** be a category. A (**category**) **congruence** $S = \langle S_1, S_2 \rangle$ on **Sign** consists of an equivalence relation $S_1 \subseteq |\mathbf{Sign}|^2$ together with an equivalence relation $S_2 \subseteq \mathrm{Mor}(\mathbf{Sign})^2$, such that the following conditions are satisfied:

- If $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$ are such that $f \; S_2 \; g$, then necessarily $\Sigma_1 \; S_1 \; \Sigma'_1$ and $\Sigma_2 \; S_1 \; \Sigma'_2$,
- If $\Sigma \; S_1 \; \Sigma'$, then $i_\Sigma \; S_2 \; i_{\Sigma'}$,
- If $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma_2, \Sigma_3)$, $f' \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$, and $g' \in \mathbf{Sign}(\Sigma'_2, \Sigma'_3)$ and $f \; S_2 \; f', g \; S_2 \; g'$, then $g \circ f \; S_2 \; g' \circ f'$.

The notation $\overline{\Sigma}$ and $\overline{f}$ will be used to denote the equivalence classes of a $\Sigma \in |\mathbf{Sign}|$ and $f \in \mathrm{Mor}(\mathbf{Sign})$ when the congruence S is clear from context. Otherwise, to make S explicit, we write $\overline{\Sigma}_S$ and $\overline{f}_S$, respectively.

Congruences are exactly kernels of functors. This is the content of the following proposition.

Proposition 1 *Given a category* **Sign**, $S = \langle S_1, S_2 \rangle$ *is a congruence on* **Sign** *if and only if there exists a category* **C** *and a functor* $F : \mathbf{Sign} \to \mathbf{C}$, *such that*

- $\Sigma \; S_1 \; \Sigma'$ *if and only if* $F(\Sigma) = F(\Sigma')$, *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$,

- *$f\ S_2\ g$ if and only if $F(f) = F(g)$, for all $f, g \in \mathrm{Mor}(\mathbf{Sign})$.*

Proof: (Sketch) First, suppose that $F : \mathbf{Sign} \to \mathbf{C}$ is a functor and that $S = \langle S_1, S_2\rangle$ is defined by the two given conditions. Then it is clear that the three conditions defining a congruence are satisfied by S.

Conversely, if $S = \langle S_1, S_2\rangle$ is a congruence on **Sign**, let C be the digraph defined by taking as its vertices the collection of S_1-equivalence classes of objects in $|\mathbf{Sign}|$ and as its directed arcs from $\overline{\Sigma_1}$ to $\overline{\Sigma_2}$ the collection of all equivalence classes of morphisms $f \in \mathbf{Sign}(\Sigma_1', \Sigma_2')$, with $\Sigma_1' \in \overline{\Sigma_1}$ and $\Sigma_2' \in \overline{\Sigma_2}$. This definition makes sense because of the first condition of the definition of the congruence S. Let D be the collection of all diagrams of the form

$$\begin{tikzcd} & \overline{\Sigma_2} \arrow[dr, "\overline{g}"] & \\ \overline{\Sigma_1} \arrow[ur, "\overline{f}"] \arrow[rr, "\overline{g \circ f}"'] & & \overline{\Sigma_3} \end{tikzcd}$$

for all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma_2, \Sigma_3)$. The third condition on S implies that D is well-defined. Create the free category $\mathbf{C}_D$ on C that respects the commutativity of all diagrams in D. For the reader that is familiar with sketch theory, this is tantamount to creating the theory of the linear sketch $\langle C, D\rangle$ with graph C and collection of diagrams D (see, e.g., [1]). Now define $F : \mathbf{Sign} \to \mathbf{C}_D$, by $F(\Sigma) = \overline{\Sigma}$, for all $\Sigma \in |\mathbf{Sign}|$, and $F(f) = \overline{f}$, for all $f \in \mathrm{Mor}(\mathbf{Sign})$. It is easy to check, using the second property of a congruence and the property of the identities in **Sign**, on the one hand, and the commutativity of the diagrams in D, on the other, that F is a functor. That $\Sigma\ S_1\ \Sigma'$ if and only if $F(\Sigma) = F(\Sigma')$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, and $f\ S_2\ g$ if and only if $F(f) = F(g)$, for all $f, g \in \mathrm{Mor}(\mathbf{Sign})$ are now straightforward consequences of the definition of F. ■

The category $\mathbf{C}_D$ will be denoted by $\mathbf{Sign}/S$ and referred to as the **quotient category** of **Sign** by the congruence S. Similarly, the functor $F : \mathbf{Sign} \to \mathbf{Sign}/S$ will be denoted by $\Pi^S : \mathbf{Sign} \to \mathbf{Sign}/S$ and referred to as the **canonical projection functor** associated with the congruence S.

The reader should be cautioned that, despite its name, this functor is *not* necessarily surjective. A simple, but illustrative, example is provided. Consider the category **Sign**, all of whose objects and morphisms, except for the identity morphisms, are pictured below:

$$\begin{tikzcd} \Sigma_1 \arrow[r, "f"] & \Sigma_2 & \\ & \Sigma_2' \arrow[r, "g"'] & \Sigma_3 \end{tikzcd}$$

Suppose that $S = \langle S_1, S_2\rangle$ is the congruence on **Sign**, that identifies Σ_2 and Σ_2' and, also, i_{Σ_2} and $i_{\Sigma_2'}$. Then **Sign**$/S$ is the category shown below (except for identities):

$$\begin{tikzcd} \overline{\Sigma_1} \arrow[rr, "\overline{g} \circ \overline{f}"] \arrow[dr, "\overline{f}"'] & & \overline{\Sigma_3} \\ & \overline{\Sigma_2} \arrow[ur, "\overline{g}"'] & \end{tikzcd}$$

Clearly, the canonical projection functor Π^S : **Sign** $\rightarrow$ **Sign**$/S$ is not surjective on morphisms, since $\overline{g} \circ \overline{f}$ does not have a preimage in **Sign**.

Next, a natural partial ordering on the category congruences of a given category is introduced. Then, it is shown that category congruences form a complete lattice under this ordering.

Let **Sign** be a category. For all congruences $S = \langle S_1, S_2\rangle, R = \langle R_1, R_2\rangle$ on **Sign**, define

$$S \leq R \quad \text{iff} \quad S_1 \subseteq R_1 \quad \text{and} \quad S_2 \subseteq R_2.$$

Theorem 2 *The collection of all category congruences* Con(**Sign**) *on the category* **Sign** *forms a complete lattice under the ordering* $\leq$.

Proof: Since $\leq$ is obviously a partial ordering, it suffices to show that Con(**Sign**) has a largest element and is closed under component-wise intersections. It is easy to check either directly, based on the definition of a congruence, or indirectly, based on Proposition 1, that $\nabla = \langle \nabla_1, \nabla_2\rangle$, where $\nabla_1 = |\mathbf{Sign}| \times |\mathbf{Sign}|$ and $\nabla_2 = \mathrm{Mor}(\mathbf{Sign}) \times \mathrm{Mor}(\mathbf{Sign})$, is a category congruence on **Sign** and it is obviously the largest such. On the other hand, suppose that $\{S^i = \langle S_1^i, S_2^i\rangle : i \in I\}$ is a collection of category congruences on **Sign**. Clearly, both $\bigcap_{i\in I} S_1^i$ and $\bigcap_{i\in I} S_2^i$ are equivalence relations on $|\mathbf{Sign}|$ and Mor(**Sign**), respectively. Now, it is not very difficult to see that $\bigcap_{i\in I} S^i = \langle \bigcap_{i\in I} S_1^i, \bigcap_{i\in I} S_2^i\rangle$ is a category congruence on **Sign**. In fact, the three conditions defining a congruence may be verified as follows:

- If $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$ are such that $f \bigcap_{i \in I} S^i_2 \; g$, then $f \; S^i_2 \; g$, for all $i \in I$, whence, necessarily $\Sigma_1 \; S^i_1 \; \Sigma'_1$, and $\Sigma_2 \; S^i_1 \; \Sigma'_2$, for all $i \in I$, and, therefore, $\Sigma_1 \bigcap_{i \in I} S^i_1 \; \Sigma'_1$ and $\Sigma_2 \bigcap_{i \in I} S^i_1 \; \Sigma'_2$.
- If $\Sigma \bigcap_{i \in I} S^i_1 \; \Sigma'$, then $\Sigma \; S^i_1 \; \Sigma'$, for all $i \in I$, whence $i_\Sigma \; S^i_2 \; i_{\Sigma'}$, for all $i \in I$, and, therefore, $i_\Sigma \bigcap_{i \in I} S^i_2 \; i_{\Sigma'}$.
- If $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma_2, \Sigma_3)$, $f' \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$, and $g' \in \mathbf{Sign}(\Sigma'_2, \Sigma'_3)$ and $f \bigcap_{i \in I} S^i_2 \; f', g \bigcap_{i \in I} S^i_2 \; g'$, then $f \; S^i_2 \; f', g \; S^i_2 \; g'$, for all $i \in I$, whence $g \circ f \; S^i_2 \; g' \circ f'$, for all $i \in I$, and, therefore, $g \circ f \bigcap_{i \in I} S^i_2 \; g' \circ f'$.

■

4 Equivalence Systems

Let **Sign** be a category and SEN : **Sign** $\rightarrow$ **Set** be a functor. A pair $\langle S, \theta \rangle$ is said to be an **equivalence system** on SEN if S is a congruence on **Sign** and $\theta = \{\langle \overline{\Sigma}, \theta_{\overline{\Sigma}} \rangle : \Sigma \in |\mathbf{Sign}|\}$ is a collection of binary relations $\theta_{\overline{\Sigma}} \subseteq (\bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma'))^2$, such that the following conditions are satisfied:

- $\theta_{\overline{\Sigma}}$ is an equivalence relation on $\bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma')$, for all $\Sigma \in |\mathbf{Sign}|$,
- if $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$ with $f \; S_2 \; g$, and $\phi \in \mathrm{SEN}(\Sigma_1)$, $\psi \in \mathrm{SEN}(\Sigma'_1)$, with $\langle \phi, \psi \rangle \in \theta_{\overline{\Sigma_1}}$, then $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(g)(\psi) \rangle \in \theta_{\overline{\Sigma_2}}$.

Sometimes, when the equivalence class $\overline{\Sigma}$ is clear from context, the $\overline{\Sigma}$-equivalence $\langle \overline{\Sigma}, \theta_{\overline{\Sigma}} \rangle$ will be denoted simply by $\theta_{\overline{\Sigma}}$.

Let **Sign**, **Sign**$'$ be two categories and SEN : **Sign** $\rightarrow$ **Set**, SEN$'$: **Sign**$'$ $\rightarrow$ **Set** two functors. A **translation** $\langle F, \alpha \rangle$: SEN $\rightarrow$ SEN$'$ from SEN to SEN$'$ [17] is a pair consisting of a functor F : **Sign** $\rightarrow$ **Sign**$'$ and a natural transformation $\alpha : \mathrm{SEN} \rightarrow \mathcal{P}\mathrm{SEN}'F$. A translation is said to be a **singleton translation**, denoted $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^s \mathrm{SEN}'$, if, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma), |\alpha_\Sigma(\phi)| = 1$. In that case, the set $\alpha_\Sigma(\phi)$ will be identified with the element it contains and α will be treated as a natural transformation $\alpha : \mathrm{SEN} \rightarrow \mathrm{SEN}'F$.

It is shown next that, given a category **Sign** and a functor SEN : **Sign** $\rightarrow$ **Set**, kernels of singleton translations $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^s \mathrm{SEN}'$, for some category **Sign**$'$ and some functor SEN$'$: **Sign**$'$ $\rightarrow$ **Set** are equivalence systems on SEN.

Proposition 3 *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *be a functor.* $\langle S, \theta\rangle$ *is an equivalence system on* SEN *if there exists a category* $\mathbf{Sign}'$, *a functor* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *and a singleton translation* $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$, *such that*

- $\Sigma \; S_1 \; \Sigma'$ *if and only if* $F(\Sigma) = F(\Sigma')$, *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$,
- $f \; S_2 \; g$ *if and only if* $F(f) = F(g)$, *for all* $f, g \in \mathrm{Mor}(\mathbf{Sign})$,
- *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, *with* $\Sigma \; S_1 \; \Sigma'$ *and all* $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$, $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma}}$ *if and only if* $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$.

Proof: Suppose that $\mathbf{Sign}'$ is a category, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a functor and $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ a singleton translation. Define $S = \langle S_1, S_2\rangle$ by setting

- $\Sigma \; S_1 \; \Sigma'$ iff $F(\Sigma) = F(\Sigma')$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, and
- $f \; S_2 \; g$ iff $F(f) = F(g)$, for all $f, g \in \mathrm{Mor}(\mathbf{Sign})$.

By Proposition 1, S is a well-defined congruence on the category **Sign**. Next, define $\theta = \{\langle \overline{\Sigma}, \theta_{\overline{\Sigma}}\rangle : \Sigma \in |\mathbf{Sign}|\}$ by letting, for all $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$, with $\Sigma \; S_1 \; \Sigma'$,

$$\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma}} \quad \text{iff} \quad \alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi).$$

Clearly, for all $\Sigma \in |\mathbf{Sign}|, \theta_{\overline{\Sigma}}$ is an equivalence relation on $\bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma')$. Thus, it suffices to show that $\langle S, \theta\rangle$ is an equivalence system on SEN, i.e., that, if $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$ with $f \; S_2 \; g$, and $\phi \in \mathrm{SEN}(\Sigma_1)$, $\psi \in \mathrm{SEN}(\Sigma'_1)$, with $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma_1}}$, then $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(g)(\psi)\rangle \in \theta_{\overline{\Sigma_2}}$. So suppose that $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma_1}}$. Then $\alpha_{\Sigma_1}(\phi) = \alpha_{\Sigma'_1}(\psi)$. Thus, since $f \; S_2 \; g$, we get that $\mathrm{SEN}'(F(f))(\alpha_{\Sigma_1}(\phi)) = \mathrm{SEN}'(F(g))(\alpha_{\Sigma'_1}(\psi))$. Therefore

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma_1) & \xrightarrow{\alpha_{\Sigma_1}} & \mathrm{SEN}'(F(\Sigma_1)) \\
{\scriptstyle \mathrm{SEN}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}(\Sigma_2) & \xrightarrow[\alpha_{\Sigma_2}]{} & \mathrm{SEN}'(F(\Sigma_2))
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{SEN}(\Sigma'_1) & \xrightarrow{\alpha_{\Sigma'_1}} & \mathrm{SEN}'(F(\Sigma'_1)) \\
{\scriptstyle \mathrm{SEN}(g)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(g))} \\
\mathrm{SEN}(\Sigma'_2) & \xrightarrow[\alpha_{\Sigma'_2}]{} & \mathrm{SEN}'(F(\Sigma'_2))
\end{array}$$

$\alpha_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) = \alpha_{\Sigma'_2}(\mathrm{SEN}(g)(\psi))$. But this, by the definition of θ, gives $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(g)(\psi)\rangle \in \theta_{\overline{\Sigma_2}}$. ■

The next result is a partial converse of Proposition 3. It yields the construction of a singleton translation $\langle F, \alpha\rangle$ from a given equivalence system under the hypothesis that the equivalence system is *confluent*. Confluence is a property that will be introduced formally below. It says, roughly speaking, that all equivalence classes of sentences over an equivalence class of signatures have at least one representative over each of the constituent signatures in the class. Confluence of equivalence relations results also in a corresponding property of the natural transformation α of the constructed translation.

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a functor. An equivalence system $\langle S, \theta\rangle$ on SEN is said to be **confluent** if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, with $\Sigma\ S_1\ \Sigma'$, and $\phi \in \mathrm{SEN}(\Sigma)$, there exists $\psi \in \mathrm{SEN}(\Sigma')$, such that $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma}}$.

Let $\mathbf{Sign}, \mathbf{Sign}'$ be two categories and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ two functors. A singleton translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ from SEN to SEN' is said to be **confluent**, if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, such that $F(\Sigma) = F(\Sigma')$, and all $\phi \in \mathrm{SEN}(\Sigma)$, there exists $\psi \in \mathrm{SEN}(\Sigma')$, such that $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$.

An easy observation is contained in the following lemma.

Lemma 4 *Let $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ be a singleton translation. If α_Σ is surjective, for all $\Sigma \in |\mathbf{Sign}|$, then $\langle F, \alpha\rangle$ is confluent.*

Proof: Suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, such that $F(\Sigma) = F(\Sigma')$, and let $\phi \in \mathrm{SEN}(\Sigma)$. Then $\alpha_\Sigma(\phi) \in \mathrm{SEN}'(F(\Sigma)) = \mathrm{SEN}'(F(\Sigma'))$. Hence, since $\alpha_{\Sigma'} : \mathrm{SEN}(\Sigma') \to \mathrm{SEN}'(F(\Sigma'))$ is surjective, there exists $\psi \in \mathrm{SEN}(\Sigma')$, such that $\alpha_{\Sigma'}(\psi) = \alpha_\Sigma(\phi)$ and $\langle F, \alpha\rangle$ is confluent. ■

Next, it is shown that confluent equivalence systems correspond exactly to confluent translations.

Proposition 5 *Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor. $\langle S, \theta\rangle$ is a confluent equivalence system on SEN if and only if there exists a category $\mathbf{Sign}'$, a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and a confluent translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$, such that*

- *$\Sigma\ S_1\ \Sigma'$ if and only if $F(\Sigma) = F(\Sigma')$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$,*
- *$f\ S_2\ g$ if and only if $F(f) = F(g)$, for all $f, g \in \mathrm{Mor}(\mathbf{Sign})$,*
- *for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, with $\Sigma\ S_1\ \Sigma'$ and all $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$, $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma}}$ if and only if $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$.*

Proof: Suppose that $\langle F,\alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ is a confluent translation. Define $\langle S,\theta\rangle$ as in the statement of the proposition. By Proposition 3, $\langle S,\theta\rangle$ is an equivalence system. It therefore suffices to show that it is confluent. To this end, let $\Sigma,\Sigma' \in |\mathbf{Sign}|$, such that $\Sigma\ S_1\Sigma'$ and $\phi \in \mathrm{SEN}(\Sigma)$. By the definition of S_1, we have $F(\Sigma) = F(\Sigma')$, whence, since $\langle F,\alpha\rangle$ is confluent, there exists $\psi \in \mathrm{SEN}(\Sigma')$, such that $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$. But, by the definition of θ, this means that $\langle\phi,\psi\rangle \in \theta_{\overline{\Sigma}}$ and $\langle S,\theta\rangle$ is confluent.

Suppose, conversely, that $\langle S,\theta\rangle$ is a confluent equivalence system on SEN. Let $\mathbf{Sign}' = \mathbf{Sign}/S$ and $F = \Pi^S : \mathbf{Sign} \to \mathbf{Sign}'$. By Proposition 1, $\mathbf{Sign}'$ is a well-defined category and $F : \mathbf{Sign} \to \mathbf{Sign}'$ is a well-defined functor. Next, define $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ at the object level by letting

$$\mathrm{SEN}'(\overline{\Sigma}) = (\bigcup_{\Sigma'\in\overline{\Sigma}} \mathrm{SEN}(\Sigma'))/\theta_{\overline{\Sigma}}, \quad \text{for all} \quad \Sigma \in |\mathbf{Sign}|.$$

At the morphism level, given $\overline{\Sigma_1},\overline{\Sigma_2} \in |\mathbf{Sign}'|$, $\overline{f} \in \mathbf{Sign}'(\overline{\Sigma_1},\overline{\Sigma_2})$, with $f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$, and $\phi/\theta_{\overline{\Sigma_1}} \in \overline{\Sigma_1}$, with $\phi \in \mathrm{SEN}(\Sigma_1')$, such that $\Sigma_1'\ S_1\ \Sigma_1$, let

$$\mathrm{SEN}'(\overline{f})(\phi/\theta_{\overline{\Sigma_1}}) = \mathrm{SEN}(f)(\psi)/\theta_{\overline{\Sigma_2}},$$

where $\psi \in \mathrm{SEN}(\Sigma_1)$, with $\langle\psi,\phi\rangle \in \theta_{\overline{\Sigma_1}}$. Such a $\psi \in \mathrm{SEN}(\Sigma_1)$ exists by the confluence of $\langle S,\theta\rangle$. Note that, since $\langle S,\theta\rangle$ is an equivalence system on SEN, the definition of $\mathrm{SEN}'(\overline{f})$ is independent of the choice of ψ. So SEN' is well-defined on morphisms that are in the range of F. The definition of SEN' at the morphism level is extended to all of the morphisms in $\mathbf{Sign}'$ by first extending it to all compositions of morphisms of the form $\overline{f}$ for f in $\mathrm{Mor}(\mathbf{Sign})$ and then dividing out by the commutativity relations "inherited" by $\mathbf{Sign}$. Since $\overline{gf} = \overline{g}\overline{f}$, when g,f are composing, the extension is well-defined. To see that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a functor, note that, for all $\Sigma \in |\mathbf{Sign}|$, all $\Sigma' \in |\mathbf{Sign}|$, with $\Sigma\ S_1\ \Sigma'$, and all $\phi \in \mathrm{SEN}(\Sigma')$, if $\psi \in \mathrm{SEN}(\Sigma)$, with $\langle\phi,\psi\rangle \in \theta_{\overline{\Sigma}}$,

$$\begin{aligned}\mathrm{SEN}'(\overline{i_\Sigma})(\phi/\theta_{\overline{\Sigma}}) &= \mathrm{SEN}(i_\Sigma)(\psi)/\theta_{\overline{\Sigma}}\\ &= \psi/\theta_{\overline{\Sigma}}\\ &= \phi/\theta_{\overline{\Sigma}},\end{aligned}$$

and if $f \in \mathbf{Sign}'(\overline{\Sigma_1},\overline{\Sigma_2}), g \in \mathbf{Sign}'(\overline{\Sigma_2},\overline{\Sigma_3})$, such that $f = \overline{f_k f_{k-1}} \ldots \overline{f_1}$, $g = \overline{g_l g_{l-1}} \ldots \overline{g_1}$, then

$$\begin{aligned}\mathrm{SEN}'(g \circ f) &= \mathrm{SEN}'(\overline{g_l g_{l-1}} \ldots \overline{g_1}\overline{f_k f_{k-1}} \ldots \overline{f_1})\\ &= \mathrm{SEN}'(\overline{g_l g_{l-1}} \ldots \overline{g_1})\mathrm{SEN}'(\overline{f_k f_{k-1}} \ldots \overline{f_1})\\ &= \mathrm{SEN}'(g)\mathrm{SEN}'(f).\end{aligned}$$

Last, define $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$, by letting $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ be given by

$$\alpha_\Sigma(\phi) = \phi/\theta_{\overline{\Sigma}}, \quad \text{for all} \quad \phi \in \mathrm{SEN}(\Sigma).$$

$\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$ is a natural transformation, since, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\phi \in \mathrm{SEN}(\Sigma_1)$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma_1) \arrow[r, "\alpha_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'(F(\Sigma_1)) \arrow[d, "\mathrm{SEN}'(F(f))"] \\
\mathrm{SEN}(\Sigma_2) \arrow[r, "\alpha_{\Sigma_2}"'] & \mathrm{SEN}'(F(\Sigma_2))
\end{tikzcd}$$

$$\begin{array}{rcl}
\mathrm{SEN}'(F(f))(\alpha_{\Sigma_1}(\phi)) & = & \mathrm{SEN}'(\overline{f})(\phi/\theta_{\overline{\Sigma_1}}) \\
& = & \mathrm{SEN}(f)(\phi)/\theta_{\overline{\Sigma_2}} \\
& = & \alpha_{\Sigma_2}(\mathrm{SEN}(f)(\phi)).
\end{array}$$

$\langle F, \alpha \rangle$ is confluent, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, such that $F(\Sigma) = F(\Sigma')$, and $\phi \in \mathrm{SEN}(\Sigma)$, we get that $\Sigma \ S_1 \ \Sigma'$, whence, since $\langle S, \theta \rangle$ is confluent, we get a $\psi \in \mathrm{SEN}(\Sigma')$, such that $\langle \phi, \psi \rangle \in \theta_{\overline{\Sigma}}$, whence, by definition of α, $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$, i.e., $\langle F, \alpha \rangle$ is confluent.

Clearly, $\langle F, \alpha \rangle$ and $\langle S, \theta \rangle$ are related as postulated in the itemized clauses of the statement. ■

It was not possible to obtain a full converse of Proposition 3 without the property of confluence for the equivalence systems and the corresponding singleton translations. The difficulty lies in the fact that SEN' may not be definable at the morphism level if $\langle S, \theta \rangle$ is not confluent. We call $\langle S, \theta \rangle$ *extendable* if defining SEN' is possible.

More formally, given a category **Sign** and a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, an equivalence system $\langle S, \theta \rangle$ on SEN is said to be **extendable** if, given the object mapping $\mathrm{SEN}' : |\mathbf{Sign}/S| \to |\mathbf{Set}|$, with

$$\mathrm{SEN}'(\overline{\Sigma}) = \bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma')/\theta_{\overline{\Sigma}},$$

the mapping $\mathrm{SEN}'(\overline{f})$ for $\overline{f} \in \mathrm{Mor}(\mathbf{Sign}/S)$, which is a partial set function, with $\mathrm{SEN}'(\overline{f}) : \bigcup_{\Sigma' \in \overline{\Sigma_1}} \mathrm{SEN}(\Sigma')/\theta_{\overline{\Sigma_1}} \to \bigcup_{\Sigma' \in \overline{\Sigma_2}} \mathrm{SEN}(\Sigma')/\theta_{\overline{\Sigma_2}}$, given, for all $\phi \in \Sigma_1'$, such that $\Sigma_1' \ S_1 \ \Sigma_1$, by

$$\mathrm{SEN}(\overline{f})(\phi/\theta_{\overline{\Sigma_1}}) = \mathrm{SEN}(f)(\psi)/\theta_{\overline{\Sigma_2}},$$

if there exists $\psi \in \mathrm{SEN}(\Sigma_1)$, such that $\langle\phi,\psi\rangle \in \theta_{\overline{\Sigma_1}}$, may be extended to a total set function so that $\mathrm{SEN}' : \mathbf{Sign}/S \to \mathbf{Set}$ becomes a functor.

It is now clear from the proof of Proposition 5 that extendable equivalence systems correspond exactly to kernels of singleton translations.

Proposition 6 *Let* $\mathbf{Sign}$ *be a category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor.* $\langle S,\theta\rangle$ *is an extendable equivalence system on* SEN *if and only if there exists a category* $\mathbf{Sign}'$, *a functor* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *and a singleton translation* $\langle F,\alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$, *such that*

- $\Sigma\ S_1\ \Sigma'$ *if and only if* $F(\Sigma) = F(\Sigma')$, *for all* $\Sigma,\Sigma' \in |\mathbf{Sign}|$,
- $f\ S_2\ g$ *if and only if* $F(f) = F(g)$, *for all* $f,g \in \mathrm{Mor}(\mathbf{Sign})$,
- *for all* $\Sigma,\Sigma' \in |\mathbf{Sign}|$, *with* $\Sigma\ S_1\ \Sigma'$ *and all* $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$, $\langle\phi,\psi\rangle \in \theta_{\overline{\Sigma}}$ *if and only if* $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$.

Next, given a category **Sign** and a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a partial ordering is introduced on the collection Eqv(SEN) of all equivalence systems on SEN. It is then shown that Eqv(SEN) forms a complete lattice under this ordering. This ordering, restricted to the class $\mathrm{Eqv}_{\mathrm{ID}}(\mathrm{SEN})$ of all equivalence systems of the form $\langle\Delta_{\mathbf{Sign}},\theta\rangle$, where $\Delta_{\mathbf{Sign}}$ is the identity congruence on **Sign**, also yields a complete lattice structure. Thus, the following increasing hierarchy of complete lattice structures is formed

$$\begin{array}{c}\mathrm{Eqv}(\mathrm{SEN})\\ \uparrow\\ \mathrm{Eqv}_{\mathrm{ID}}(\mathrm{SEN})\end{array}$$

The collection $\mathrm{Eqv}_{\mathrm{ID}}(\mathrm{SEN})$ is singled out since it is the collection used in [19] to define the Tarski congruence system of a given π-institution. Since in that case only the identity category congruence is allowed as the first component of the equivalence systems, the equivalence systems considered in [19] are special cases of those developed here. This fact accounts for the "generalized" in the "generalized Tarski congruence systems" of the title.

More formally, given a category **Sign** and a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a partial ordering $\leq$ is defined on the collection of all equivalence systems on SEN by setting

$$\langle S,\theta\rangle \leq \langle R,\eta\rangle \quad \text{iff} \quad S \leq R \;\; \text{and} \;\; \theta_{\overline{\Sigma}_S} \subseteq \eta_{\overline{\Sigma}_R}, \quad \text{for all} \;\; \Sigma \in |\mathbf{Sign}|.$$

Theorem 7 *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *be a functor. The collection* Eqv(SEN) *of all equivalence systems on* SEN *forms a complete lattice under the partial ordering* $\leq$.

Proof: It is clear that $\nabla = \langle \nabla_{\mathbf{Sign}}, \nabla_{\mathrm{SEN}} \rangle$, where

$$\nabla_{\mathbf{Sign}} = \langle \nabla_{|\mathbf{Sign}|}, \nabla_{\mathrm{Mor}(\mathbf{Sign})} \rangle$$

is the congruence on **Sign** defined by $\nabla_{|\mathbf{Sign}|} = |\mathbf{Sign}| \times |\mathbf{Sign}|$ and by $\nabla_{\mathrm{Mor}(\mathbf{Sign})} = \mathrm{Mor}(\mathbf{Sign}) \times \mathrm{Mor}(\mathbf{Sign})$, and

$$\nabla_{\mathrm{SEN}} = \{\langle |\mathbf{Sign}|, (\bigcup_{\Sigma \in |\mathbf{Sign}|} \mathrm{SEN}(\Sigma))^2 \rangle\}$$

is an equivalence system of SEN. So it is the maximum element of Eqv(SEN) under the inclusion $\leq$.

To prove the statement, it suffices, thus, to show that the collection Eqv(SEN) is closed under greatest lower bounds. These are given by signature congruence intersections in the signature component and by domain-restricted signature respecting intersections in the second component. Suppose to this end that $\langle S^i, \theta^i \rangle, i \in I$, is a collection of equivalence systems of SEN. By Theorem 2, $\bigcap_{i \in I} S^i = \langle \bigcap_{i \in I} S^i_1, \bigcap_{i \in I} S^i_2 \rangle$ is a category congruence on **Sign**. Suppose, now, that $\Sigma \in |\mathbf{Sign}|$. Let $\overline{\Sigma}$ be the equivalence class of Σ with respect to $\bigcap_{i \in I} S^i_1$ and $\overline{\Sigma}_{S^i}$ its equivalence class with respect to the equivalence S^i_1, for all $i \in I$. Consider the system

$$\langle \bigcap_{i \in I} S^i, \{\langle \overline{\Sigma}, \bigcap_{i \in I} \theta^i_{\overline{\Sigma}_{S^i}} \restriction_{\bigcup_{\Sigma' \in \bigcap_{i \in I} \overline{\Sigma}_{S^i}} \mathrm{SEN}(\Sigma')} \rangle : \Sigma \in |\mathbf{Sign}|\} \rangle.$$

By $\langle \overline{\Sigma}, \theta_{\overline{\Sigma}} \rangle$ is denoted the pair $\langle \overline{\Sigma}, \bigcap_{i \in I} \theta^i_{\overline{\Sigma}_{S^i}} \restriction_{\bigcup_{\Sigma' \in \bigcap_{i \in I} \overline{\Sigma}_{S^i}} \mathrm{SEN}(\Sigma')} \rangle$. It is shown that this is a valid equivalence system on SEN, whence Eqv(SEN) is closed under infima, which will prove that Eqv(SEN) is endowed with the structure of a complete lattice.

First, it must be shown that $\bigcap_{i \in I} \theta^i_{\overline{\Sigma}_{S^i}} \restriction_{\bigcup_{\Sigma' \in \bigcap_{i \in I} \overline{\Sigma}_{S^i}} \mathrm{SEN}(\Sigma')}$ is an equivalence relation on $\bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma')$. By its definition, it is in fact a relation over the set of sentences $\bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma')$. Furthermore, it is an equivalence relation, since it is the intersection of the restriction of the equivalences $\theta^i_{\overline{\Sigma}_{S^i}}$ to the set of sentences $\bigcup_{\Sigma' \in \overline{\Sigma}} \mathrm{SEN}(\Sigma')$.

Finally, it suffices to show that, if $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$, such that $f \bigcap_{i \in I} S^i_2\ g$, and $\langle \phi, \psi \rangle \in \theta_{\overline{\Sigma_1}}$, we have $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(g)(\psi) \rangle \in$

$\theta_{\overline{\Sigma_2}}$. Indeed, if $\langle\phi,\psi\rangle \in \theta_{\overline{\Sigma_1}}$, then we get $\langle\phi,\psi\rangle \in \bigcap_{i\in I}\theta^i_{\overline{\Sigma_1}_{S^i}}$ and $\phi,\psi \in \bigcup_{\Sigma'\in\bigcap_{i\in I}\overline{\Sigma_1}_{S^i}}\mathrm{SEN}(\Sigma')$, whence $\langle\phi,\psi\rangle \in \theta^i_{\overline{\Sigma_1}_{S^i}}$, for all $i\in I$, and $\phi,\psi \in \bigcup_{\Sigma'\in\bigcap_{i\in I}\overline{\Sigma_1}_{S^i}}\mathrm{SEN}(\Sigma')$. Hence, since, for all $i\in I$, $\langle S^i,\theta^i\rangle$ is an equivalence system on SEN, $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(g)(\psi)\rangle \in \theta^i_{\overline{\Sigma_2}_{S^i}}$, for all $i\in I$, and $\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi) \in \bigcup_{\Sigma'\in\bigcap_{i\in I}\overline{\Sigma_2}_{S^i}}\mathrm{SEN}(\Sigma')$. Therefore, we obtain $\langle\mathrm{SEN}(f)(\phi),\ \mathrm{SEN}(g)(\psi)\rangle \in \bigcap_{i\in I}\theta^i_{\overline{\Sigma_2}_{S^i}}$, together with

$$\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi) \in \bigcup_{\Sigma'\in\bigcap_{i\in I}\overline{\Sigma_2}_{S^i}}\mathrm{SEN}(\Sigma'),$$

which give that $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(g)(\psi)\rangle \in \theta_{\overline{\Sigma_2}}$. Thus

$$\langle\bigcap_{i\in I}S^i,\{\langle\overline{\Sigma},\bigcap_{i\in I}\theta^i_{\overline{\Sigma}_{S^i}}\restriction_{\bigcup_{\Sigma'\in\bigcap_{i\in I}\overline{\Sigma}_{S^i}}\mathrm{SEN}(\Sigma')}\rangle : \Sigma\in|\mathbf{Sign}|\}\rangle$$

is an equivalence system on SEN. ■

5 Congruence Systems

Let again **Sign** be a category and SEN : **Sign** $\to$ **Set** be a functor. In addition, let N be a category of natural transformations on SEN. An equivalence system $\langle S,\theta\rangle$ on SEN is said to be an N-**congruence system** on SEN if, for all $\sigma:\mathrm{SEN}^k\to\mathrm{SEN}$ in N, it satisfies, for all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, with $\Sigma\ S_1\ \Sigma'$, and all $\vec{\phi}\in\mathrm{SEN}(\Sigma)^k,\vec{\psi}\in\mathrm{SEN}(\Sigma')^k$,

$$\vec{\phi}\ \theta^k_{\overline{\Sigma}}\ \vec{\psi}\quad\text{imply}\quad\sigma_\Sigma(\vec{\phi})\ \theta_{\overline{\Sigma}}\ \sigma_{\Sigma'}(\vec{\psi}).$$

Given two functors SEN : **Sign** $\to$ **Set**, SEN$'$: **Sign$'$** $\to$ **Set** and categories of natural transformations N,N', respectively, on SEN, SEN$'$, a singleton translation $\langle F,\alpha\rangle$ from SEN to SEN$'$ is said to be (N,N')-**homomorphic** if, for every natural transformation $\sigma:\mathrm{SEN}^k\to\mathrm{SEN}$ in N, there exists a natural transformation $\tau:\mathrm{SEN}'^k\to\mathrm{SEN}'$ in N', such that, for every $\Sigma\in|\mathbf{Sign}|$ and every $\vec{\phi}\in\mathrm{SEN}(\Sigma)^k$,

$$\begin{array}{ccc}\mathrm{SEN}(\Sigma)^k & \xrightarrow{\alpha^k_\Sigma} & \mathrm{SEN}'(F(\Sigma))^k\\ {\scriptstyle\sigma_\Sigma}\downarrow & & \downarrow{\scriptstyle\tau_{F(\Sigma)}}\\ \mathrm{SEN}(\Sigma) & \xrightarrow[\alpha_\Sigma]{} & \mathrm{SEN}'(F(\Sigma))\end{array}$$

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \tau_{F(\Sigma)}(\alpha^k_\Sigma(\vec{\phi})). \tag{3}$$

It is said to be (N, N')-**epimorphic** if it is (N, N')-homomorphic and, in addition, for every $\tau : \mathrm{SEN}'^k \to \mathrm{SEN}'$ in N', there exists $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, such that Equation (3) holds, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^k$. We denote homomorphic by the superscript ^{h} and epimorphic by the superscript ^{e}, respectively, assuming that the categories N and N' are clear from context.

The following results are analogs of Propositions 3 and 5, lifting results pertaining to equivalence systems and translations to corresponding results pertaining to N-congruence systems and (N, N')-epimorphic translations, respectively.

Proposition 8 *Let* $\mathbf{Sign}$ *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor and* N *a category of natural transformations on* SEN. $\langle S, \theta\rangle$ *is an* N*-congruence system on* SEN *if there exists a category* $\mathbf{Sign}'$, *a functor* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, *a category* N' *of natural transformations on* SEN' *and an* (N, N')*-homomorphic translation* $\langle F, \alpha\rangle : \mathrm{SEN} \to^h \mathrm{SEN}'$, *such that*

- $\Sigma\ S_1\ \Sigma'$ *if and only if* $F(\Sigma) = F(\Sigma')$, *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$,
- $f\ S_2\ g$ *if and only if* $F(f) = F(g)$, *for all* $f, g \in \mathrm{Mor}(\mathbf{Sign})$,
- *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, *with* $\Sigma\ S_1\ \Sigma'$ *and all* $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$, $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma}}$ *if and only if* $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$.

Proof: Suppose there exists a category $\mathbf{Sign}'$, a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, a category N' of natural transformations on SEN' and an (N, N')-homomorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^h \mathrm{SEN}'$, such that the three conditions of the statement are satisfied. Then, by Proposition 3, $\langle S, \theta\rangle$ is an equivalence system on SEN. To show that it is an N-congruence system, let $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ be a natural transformation in N and let $\tau : \mathrm{SEN}'^k \to \mathrm{SEN}'$ be the corresponding transformation in N', given by the (N, N')-homomorphic property, such that, for all $\Sigma \in |\mathbf{Sign}|$, the rectangle

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\alpha^k_\Sigma} & \mathrm{SEN}'(F(\Sigma))^k \\
{\scriptstyle \sigma_\Sigma}\downarrow & & \downarrow{\scriptstyle \tau_{F(\Sigma)}} \\
\mathrm{SEN}(\Sigma) & \xrightarrow[\alpha_\Sigma]{} & \mathrm{SEN}'(F(\Sigma))
\end{array}$$

commutes. Then, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, with $F(\Sigma) = F(\Sigma')$, and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k, \vec{\psi} \in \mathrm{SEN}(\Sigma')^k$, such that $\vec{\phi}\ \theta^k_{\overline{\Sigma}}\ \vec{\psi}$, we have $\alpha^k_\Sigma(\vec{\phi}) = \alpha^k_{\Sigma'}(\vec{\psi})$, whence $\tau_{F(\Sigma)}(\alpha^k_\Sigma(\vec{\phi})) = \tau_{F(\Sigma')}(\alpha^k_{\Sigma'}(\vec{\psi}))$ and, therefore, by the commutativity of the diagram, $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \alpha_{\Sigma'}(\sigma_{\Sigma'}(\vec{\psi}))$. But this yields $\sigma_\Sigma(\vec{\phi})\ \theta_{\overline{\Sigma}}\ \sigma_{\Sigma'}(\vec{\psi})$, whence $\langle S, \theta\rangle$ is indeed an N-congruence system on SEN. ■

Proposition 9 *Let* **Sign** *be a category,* SEN : **Sign** $\rightarrow$ **Set** *be a functor and N a category of natural transformations on* SEN. *$\langle S, \theta\rangle$ is a confluent N-congruence system on* SEN *if and only if there exists a category* **Sign′**, *a functor* SEN′ : **Sign′** $\rightarrow$ **Set**, *a category N' of natural transformations on* SEN′ *and a confluent (N, N')-epimorphic translation $\langle F, \alpha\rangle$:* SEN $\rightarrow^e$ SEN′, *such that*

- $\Sigma\ S_1\ \Sigma'$ *if and only if* $F(\Sigma) = F(\Sigma')$, *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$,
- $f\ S_2\ g$ *if and only if* $F(f) = F(g)$, *for all* $f, g \in \mathrm{Mor}(\mathbf{Sign})$,
- *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, *with* $\Sigma\ S_1\ \Sigma'$ *and all* $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$, $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma}}$ *if and only if* $\alpha_\Sigma(\phi) = \alpha_{\Sigma'}(\psi)$.

Proof: Sufficiency is provided by Proposition 8 together with the observation that, since $\langle F, \alpha\rangle$ is a confluent (N, N')-epimorphic translation, $\langle S, \theta\rangle$, as constructed in the proof of Proposition 8, is a confluent equivalence system and, as a result, also a confluent N-congruence system.

For necessity, if $\langle S, \theta\rangle$ is a confluent N-congruence system on SEN, let $\mathbf{Sign}', \mathrm{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$ and $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow \mathrm{SEN}'$ be defined as in the proof of Proposition 5. Then, by Proposition 5, it suffices to show the existence of a category of natural transformations N' on SEN′ such that $\langle F, \alpha\rangle$ is (N, N')-epimorphic.

To this end, given $\sigma : \mathrm{SEN}^k \rightarrow \mathrm{SEN}$ in N, let $\overline{\sigma} : \mathrm{SEN}'^k \rightarrow \mathrm{SEN}'$ be defined, for all $\overline{\Sigma} \in |\mathbf{Sign}'|, \phi_i \in \mathrm{SEN}(\Sigma_i), i = 0, \ldots, k-1$, with $\Sigma_i\ S_1\ \Sigma$, for all $i = 0, \ldots, k-1$, by

$$\overline{\sigma}_{\overline{\Sigma}}(\vec{\phi}/\theta_{\overline{\Sigma}}) = \sigma_\Sigma(\phi'_0, \ldots, \phi'_{k-1})/\theta_{\overline{\Sigma}},$$

where $\phi'_i \in \mathrm{SEN}(\Sigma)$ with $\phi'_i\ \theta_{\overline{\Sigma}}\ \phi_i, i = 0, \ldots, k-1$, are sentences whose existence is guaranteed by confluence. This definition is independent of the representatives $\phi'_i, i = 0, \ldots, k-1$, since, if $\Sigma' \in |\mathbf{Sign}|$, with $\Sigma\ S_1\ \Sigma'$, and $\vec{\psi} \in \mathrm{SEN}(\Sigma')^k$, such that $\psi_i\ \theta_{\overline{\Sigma}}\ \phi'_i$, $i = 0, \ldots, k-1$, then, since $\langle S, \theta\rangle$ is an N-congruence system, $\sigma_\Sigma(\vec{\phi}')\ \theta_{\overline{\Sigma}}\ \sigma_{\Sigma'}(\vec{\psi})$. $\overline{\sigma} : \mathrm{SEN}'^k \rightarrow \mathrm{SEN}'$ is a natural transformation, since, for all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, we have

$$\begin{tikzcd}
\mathrm{SEN}'(\overline{\Sigma}_1)^k \arrow[r, "\overline{\sigma}_{\overline{\Sigma}_1}"] \arrow[d, "\mathrm{SEN}'(\overline{f})^k"'] & \mathrm{SEN}'(\overline{\Sigma}_1) \arrow[d, "\mathrm{SEN}'(\overline{f})"] \\
\mathrm{SEN}'(\overline{\Sigma}_2)^k \arrow[r, "\overline{\sigma}_{\overline{\Sigma}_2}"'] & \mathrm{SEN}'(\overline{\Sigma}_2)
\end{tikzcd}$$

$$\begin{array}{rcl}
\overline{\sigma}_{\overline{\Sigma}_2}(\mathrm{SEN}'(\overline{f})^k(\vec{\phi}/\theta_{\overline{\Sigma}_1})) & = & \overline{\sigma}_{\overline{\Sigma}_2}(\mathrm{SEN}(f)^k(\vec{\phi}')/\theta_{\overline{\Sigma}_2}) \\
& = & \sigma_{\Sigma_2}(\mathrm{SEN}(f)^k(\vec{\phi}'))/\theta_{\overline{\Sigma}_2} \\
& = & \mathrm{SEN}(f)(\sigma_{\Sigma_1}(\vec{\phi}'))/\theta_{\overline{\Sigma}_2} \\
& = & \mathrm{SEN}'(\overline{f})(\sigma_{\Sigma_1}(\vec{\phi}')/\theta_{\overline{\Sigma}_2}) \\
& = & \mathrm{SEN}'(\overline{f})(\overline{\sigma}_{\overline{\Sigma}_1}(\vec{\phi}'/\theta_{\overline{\Sigma}_1})) \\
& = & \mathrm{SEN}'(\overline{f})(\overline{\sigma}_{\overline{\Sigma}_1}(\vec{\phi}/\theta_{\overline{\Sigma}_1})).
\end{array}$$

Now, define $N' = \{\overline{\sigma} : \sigma \text{ in } N\}$. Then we have

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^k \arrow[r, "\alpha_\Sigma^k"] \arrow[d, "\sigma_\Sigma"'] & \mathrm{SEN}'(F(\Sigma))^k \arrow[d, "\overline{\sigma}_{\overline{\Sigma}}"] \\
\mathrm{SEN}(\Sigma) \arrow[r, "\alpha_\Sigma"'] & \mathrm{SEN}'(F(\Sigma))
\end{tikzcd}$$

$$\begin{array}{rcl}
\overline{\sigma}_{\overline{\Sigma}}(\alpha_\Sigma^k(\vec{\phi})) & = & \overline{\sigma}_{\overline{\Sigma}}(\vec{\phi}/\theta_{\overline{\Sigma}}) \\
& = & \sigma_\Sigma(\vec{\phi})/\theta_{\overline{\Sigma}} \\
& = & \alpha_\Sigma(\sigma_\Sigma(\vec{\phi})),
\end{array}$$

whence, $\langle F, \alpha\rangle$ is in fact an (N, N')-epimorphic translation. ■

Next, given a category **Sign**, a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a category N of natural transformations on SEN, the partial ordering $\leq$, introduced previously on the collection of all equivalence systems, is restricted to the collection $\mathrm{Con}^N(\mathrm{SEN})$ of all N-congruence systems on SEN. It is then shown that $\mathrm{Con}^N(\mathrm{SEN})$ forms a complete lattice under this ordering. The same ordering, restricted to the class $\mathrm{Con}^N_{\mathrm{ID}}(\mathrm{SEN})$ of all congruence systems of the form $\langle \Delta_{\mathbf{Sign}}, \theta\rangle$, where $\Delta_{\mathbf{Sign}}$ is the identity congruence on **Sign**, also yields a complete lattice structure. Thus, the following increasing hierarchy of complete lattice structures is formed

$$\begin{tikzcd}
\mathrm{Con}^N(\mathrm{SEN}) \\
\mathrm{Con}^N_{\mathrm{ID}}(\mathrm{SEN}) \arrow[u]
\end{tikzcd}$$

The collection $\mathrm{Con}^N_{\mathrm{ID}}(\mathrm{SEN})$ is the one used in [19] to define the Tarski congruence system of a given π-institution. Thus, with respect to N-congruence systems as well, the concepts introduced in [19] constitute special cases of the concepts presented in the present work.

More formally, given a category **Sign**, a functor SEN : **Sign** $\to$ **Set** and a category of natural transformations N on SEN, consider the partial ordering $\leq$, that was defined on the collection of all equivalence systems on SEN by

$$\langle S,\theta\rangle \leq \langle R,\eta\rangle \quad \text{iff} \quad S \leq R \quad \text{and} \quad \theta_{\overline{\Sigma}_S} \subseteq \eta_{\overline{\Sigma}_R}, \quad \text{for all} \quad \Sigma \in |\mathbf{Sign}|,$$

restricted to the collection $\mathrm{Con}^N(\mathrm{SEN})$ of all N-congruence systems on SEN.

Theorem 10 *Let* **Sign** *be a category,* SEN : **Sign** $\to$ **Set** *a functor and N a category of natural transformations on* SEN*. The collection* $\mathrm{Con}^N(\mathrm{SEN})$ *of all N-congruence systems on* SEN *forms a complete lattice under the partial ordering $\leq$.*

Proof: $\langle \mathrm{Con}^N(\mathrm{SEN}), \leq\rangle$ may easily be seen to be a complete sublattice of the complete lattice $\langle \mathrm{Eqv}(\mathrm{SEN}), \leq\rangle$ of all equivalence systems on SEN. ∎

6 Logical and Tarski Congruence Systems

Let now $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. An equivalence system $\langle S,\theta\rangle$ on SEN is called a **logical equivalence system** of $\mathcal{I}$ if, for all $\Sigma_1, \Sigma'_1 \in |\mathbf{Sign}|$, with $\Sigma_1\ S_1\ \Sigma'_1$, all $\phi \in \mathrm{SEN}(\Sigma_1), \psi \in \mathrm{SEN}(\Sigma'_1)$, with $\langle \phi,\psi\rangle \in \theta_{\overline{\Sigma_1}}$, we have

$$\begin{tikzcd} \Sigma_1 \arrow[dr, "f"'] & & \Sigma'_1 \arrow[dl, "g"] \\ & \Sigma_2 & \end{tikzcd}$$

$$C_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) = C_{\Sigma_2}(\mathrm{SEN}(g)(\psi)),$$

for all $\Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1,\Sigma_2), g \in \mathbf{Sign}(\Sigma'_1,\Sigma_2)$, with $f\ S_2\ g$.

An N-congruence system of SEN is a **logical N-congruence system** of $\mathcal{I}$ if it is logical as an equivalence system of $\mathcal{I}$.

Theorem 11 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution and N a category of natural transformations on* SEN*. The collection* $\mathrm{LCon}^N(\mathcal{I})$*, sometimes also denoted by* $\mathrm{LCon}^N(C)$*, of all logical N-congruence systems of $\mathcal{I}$ forms a complete lattice under component-wise inclusion.*

Proof: If $\langle S^i, \theta^i\rangle, i \in I$, is a collection of logical N-congruence systems on SEN, then, using the conventions adopted in the proof of Proposition 7, we obtain that

$$\langle \bigcap_{i\in I} S^i, \{\langle \overline{\Sigma}, \bigcap_{i\in I} \theta^i_{\overline{\Sigma}_{S^i}} \restriction_{\bigcup_{\Sigma' \in \bigcap_{i\in I} \overline{\Sigma}_{S^i}} \mathrm{SEN}(\Sigma')}\rangle : \Sigma \in |\mathbf{Sign}|\}\rangle \in \mathrm{LCon}^N(\mathcal{I}),$$

since, by Theorem 10, the displayed pair is an N-congruence system on SEN, and it is a logical N-congruence system. To see this, suppose that $\Sigma_1, \Sigma'_1 \in |\mathbf{Sign}|$, with $\Sigma_1 \ S_1 \ \Sigma'_1$ and $\phi \in \mathrm{SEN}(\Sigma_1), \psi \in \mathrm{SEN}(\Sigma'_1)$, with $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma_1}}$. Then $\langle \phi, \psi\rangle \in \bigcap_{i\in I} \theta_{\overline{\Sigma_1}_{S^i}}$ and $\phi, \psi \in \bigcup_{\Sigma' \in \overline{\Sigma_1}} \mathrm{SEN}(\Sigma')$. Thus, $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma_1}_{S^i}}$, for all $i \in I$, and $\phi, \psi \in \bigcup_{\Sigma' \in \overline{\Sigma_1}} \mathrm{SEN}(\Sigma')$. Now, suppose that $\Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma_2)$, such that $f \ S_2 \ g$.

$$\begin{tikzcd} \Sigma_1 \arrow[dr, "f"'] & & \Sigma'_1 \arrow[dl, "g"] \\ & \Sigma_2 & \end{tikzcd}$$

Thus, since $S_2 = \bigcap_{i\in I} S^i_2$, $\langle \phi, \psi\rangle \in \theta_{\overline{\Sigma_1}_{S^i}}$ and $\langle S^i, \theta^i\rangle$ is logical, we have $C_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) = C_{\Sigma_2}(\mathrm{SEN}(g)(\psi))$. Therefore

$$\langle \bigcap_{i\in I} S^i, \{\langle \overline{\Sigma}, \bigcap_{i\in I} \theta^i_{\overline{\Sigma}_{S^i}} \restriction_{\bigcup_{\Sigma' \in \bigcap_{i\in I} \overline{\Sigma}_{S^i}} \mathrm{SEN}(\Sigma')}\rangle : \Sigma \in |\mathbf{Sign}|\}\rangle$$

is also logical.

It remains now to show that $\mathrm{LCon}^N(\mathcal{I})$ has a greatest element. To this end, consider a directed subset $\{\langle S^i, \theta^i\rangle : i \in I\}$ of $\mathrm{LCon}^N(\mathcal{I})$. It is not difficult to check that $\langle \bigcup_{i\in I} S^i, \bigcup_{i\in I} \theta^i\rangle$, where $\bigcup_{i\in I} S^i = \langle \bigcup_{i\in I} S^i_1, \bigcup_{i\in I} S^i_2\rangle$, is a logical N-congruence system on SEN, whence it is an upper bound for $\{\langle S^i, \theta^i\rangle : i \in I\}$ in $\mathrm{LCon}^N(\mathcal{I})$. So, by Zorn's Lemma, $\mathrm{LCon}^N(\mathcal{I})$ has a maximal element. If $\langle S, \theta\rangle \neq \langle R, \eta\rangle$ are two such maximal elements, then, it is not difficult to verify that their join $\langle V, \zeta\rangle$ as N-congruence systems of SEN is a logical N-congruence system of $\mathcal{I}$. This, however, contradicts their maximality, since, clearly, $\langle S, \theta\rangle < \langle V, \zeta\rangle$ and $\langle R, \eta\rangle < \langle V, \zeta\rangle$. Therefore, the maximal element of $\mathrm{LCon}^N(\mathcal{I})$, provided by Zorn's Lemma, is in fact a largest element. ■

The largest logical N-congruence system is called the **Tarski N-congruence system** of $\mathcal{I}$ and is denoted by $\langle \Omega^N(\mathcal{I}), \omega^N(\mathcal{I})\rangle$ or $\langle \Omega^N(C), \omega^N(C)\rangle$. In the context where the π-institution $\mathcal{I}$ and the category N are fixed, we may simply write $\langle \Omega, \omega\rangle$ if confusion is not likely. The reader is cautioned about this choice of notation. The capital omega is referring to the category congruence and the small omega to the sentence component. Ordinarily, in the special case considered in [19], since the category congruence is always the identity congruence, the capital omega is used for the sentence component, following the long standing notational tradition in abstract algebraic logic, as originated by Blok and Pigozzi [3].

The following theorem is an adaptation of a characterization result of Font and Jansana ([9], Proposition 1.2) of the Tarski congruence of an abstract logic, which is, in turn, a generalization of a characterization result of Blok and Pigozzi [3] of the Leibniz congruence of a logical matrix. It is also a generalization of a result in [19] concerning the largest logical N-congruence system of a π-institution with an identity category congruence, which will be denoted here by $\langle \Delta_{\mathbf{Sign}}, \omega^N_{\mathrm{ID}}(\mathcal{I})\rangle$. Once more, it is noted that, in [19], the notation $\widetilde{\Omega}^N(\mathcal{I}) := \omega^N_{\mathrm{ID}}(\mathcal{I})$ was used. Some lemmas will be presented first that will be subsequently used to prove the main theorem. The proof is broken into several steps and its format goes roughly as follows:

1. For a fixed category congruence S, a characterization is given for the largest logical N-congruence system $\langle S, \omega^N_S(\mathcal{I})\rangle$ with signature component S.

2. It is shown, based on the result of the first step, that, for every category congruence S of **Sign**,
$$\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} \subseteq \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma.$$

3. This implies that the category congruence $\Omega^N(\mathcal{I})$ is the largest category congruence S, with $\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$.

4. By step 3, it follows that the category congruence $\Omega^N(\mathcal{I})$ is the join of the collection of all category congruences S, such that, for all $\Sigma \in |\mathbf{Sign}|$,
$$\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma.$$

5. Finally, $\omega^N(\mathcal{I}) = \omega^N_{\Omega^N(\mathcal{I})}(\mathcal{I})$.

The first lemma below, Lemma 12, is a technical lemma used to prove the first step in the sequence outlined above. The first step itself is presented in Lemma 13, that immediately follows Lemma 12.

Lemma 12 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ *be a* π*-institution,* N *a category of natural transformations on* SEN *and* $\langle S, \theta\rangle$ *a logical* N*-congruence system of* $\mathcal{I}$*. Let* $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma_1)$ *and* $f, g \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$*, such that* $f\ S_2\ g$*. Then, for all natural transformations* $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ *in* N *and all* $\vec{\chi} \in \mathrm{SEN}(\Sigma_2)^{k-1}$,

$$C_{\Sigma_2}(\tau_{\Sigma_2}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma_2}(\tau_{\Sigma_2}(\mathrm{SEN}(g)(\phi), \vec{\chi})). \tag{4}$$

Proof: We have $\langle \phi, \phi\rangle \in \theta_{\overline{\Sigma_1}}$, since $\theta_{\overline{\Sigma_1}}$ is an equivalence relation. Thus, since $f\ S_2\ g$ and $\langle S, \theta\rangle$ is an equivalence system, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(g)(\phi)\rangle \in \theta_{\overline{\Sigma_2}}$. Hence, since $\vec{\chi}\ \theta^{k-1}_{\overline{\Sigma_2}}\ \vec{\chi}$ and $\langle S, \theta\rangle$ is an N-congruence system on SEN, we get

$$\langle \tau_{\Sigma_2}(\mathrm{SEN}(f)(\phi), \vec{\chi}), \tau_{\Sigma_2}(\mathrm{SEN}(g)(\phi), \vec{\chi})\rangle \in \theta_{\overline{\Sigma_2}}.$$

Finally, since $\langle S, \theta\rangle$ is a logical congruence system, we get Equation (4). ■

Lemma 13 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ *be a* π*-institution,* N *a category of natural transformations on* SEN *and* S *a category congruence on* **Sign***. Let* $\Sigma_1, \Sigma'_1 \in |\mathbf{Sign}|$*, such that* $\Sigma_1\ S_1\ \Sigma'_1$*, and* $\phi \in \mathrm{SEN}(\Sigma_1), \psi \in \mathrm{SEN}(\Sigma'_1)$*. Then* $\langle \phi, \psi\rangle \in \omega^N_S(\mathcal{I})_{\overline{\Sigma_1}}$ *if and only if, for all* $\Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma'_1, \Sigma_2)$*, such that* $f\ S_2\ g$*, all natural transformations* $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ *in* N *and all* $\vec{\chi} \in \mathrm{SEN}(\Sigma_2)^{k-1}$,

$$C_{\Sigma_2}(\tau_{\Sigma_2}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma_2}(\tau_{\Sigma_2}(\mathrm{SEN}(g)(\psi), \vec{\chi})). \tag{5}$$

Proof: Consider the π-institution $\mathcal{I}$, the category N of natural transformations on SEN and suppose that $S = \langle S_1, S_2\rangle$ is a category congruence on **Sign**. Define, for all $\Sigma \in |\mathbf{Sign}|$, the equivalence relation $R_{\overline{\Sigma}}$ on the collection $\bigcup_{\Sigma'\in\overline{\Sigma}} \mathrm{SEN}(\Sigma')$ as the collection of all pairs $\langle \phi, \psi\rangle$ that satisfy Equation (5), where $\overline{\Sigma}$ denotes the S_1-class of Σ. $R_{\overline{\Sigma}}$ is an equivalence relation on $\bigcup_{\Sigma'\in\overline{\Sigma}} \mathrm{SEN}(\Sigma')$ since it is reflexive by Lemma 12 and it is obviously symmetric and transitive.

R is also an equivalence system of SEN, since, for all $\Sigma_1, \Sigma_2, \Sigma'_1, \Sigma'_2 \in |\mathbf{Sign}|$, with $\Sigma_1\ S_1\ \Sigma'_1, \Sigma_2\ S_1\ \Sigma'_2$, $h \in \mathbf{Sign}(\Sigma_1, \Sigma_2), k \in \mathbf{Sign}(\Sigma'_1, \Sigma'_2)$, with $h\ S_2\ k$, and all $\phi \in \mathrm{SEN}(\Sigma_1), \psi \in \mathrm{SEN}(\Sigma'_1)$, with $\langle \phi, \psi\rangle \in R_{\overline{\Sigma_1}}$, we get, for all $\Sigma_3 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_2, \Sigma_3),\ g \in \mathbf{Sign}(\Sigma_2', \Sigma_3)$, with $f\ S_2\ g$,

$$\begin{tikzcd}
\Sigma_1 \arrow[d, "h"'] & & \Sigma_1' \arrow[d, "k"] \\
\Sigma_2 \arrow[dr, "f"'] & & \Sigma_2' \arrow[dl, "g"] \\
& \Sigma_3 &
\end{tikzcd}$$

$\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma_3)^{k-1}$,

$$\begin{array}{l} C_{\Sigma_3}(\tau_{\Sigma_3}(\mathrm{SEN}(f)(\mathrm{SEN}(h)(\phi)), \vec{\chi})) = \\ \qquad\qquad\qquad \begin{array}{rl} = & C_{\Sigma_3}(\tau_{\Sigma_3}(\mathrm{SEN}(fh)(\phi), \vec{\chi})) \\ = & C_{\Sigma_3}(\tau_{\Sigma_3}(\mathrm{SEN}(gk)(\psi), \vec{\chi})) \\ = & C_{\Sigma_3}(\tau_{\Sigma_3}(\mathrm{SEN}(g)(\mathrm{SEN}(k)(\psi)), \vec{\chi})), \end{array} \end{array}$$

where, passing from the first to the second lines above, we have used the fact that $\langle \phi, \psi \rangle \in R_{\overline{\Sigma_1}}$ and $fh\ S_2\ gk$. Therefore, we get that $\langle \mathrm{SEN}(h)(\phi), \mathrm{SEN}(k)(\psi) \rangle \rangle \in R_{\overline{\Sigma_2}}$.

Furthermore, R is an N-congruence system on SEN. To see this, let $\Sigma_1, \Sigma_1' \in |\mathbf{Sign}|$, with $\Sigma_1\ S_1\ \Sigma_1'$, $\vec{\phi} \in \mathrm{SEN}(\Sigma_1)^n, \vec{\psi} \in \mathrm{SEN}(\Sigma_1')^n$, with $\vec{\phi}\ R^n_{\overline{\Sigma_1}}\ \vec{\psi}$ and $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ be in N. Then, for all $\Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), g \in \mathbf{Sign}(\Sigma_1', \Sigma_2)$, with $f\ S_2\ g$, $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma_2)^{k-1}$,

$$\begin{tikzcd}
\Sigma_1 \arrow[dr, "f"'] & & \Sigma_1' \arrow[dl, "g"] \\
& \Sigma_2 &
\end{tikzcd}$$

$$\begin{array}{rcl} C_{\Sigma_2}(\tau_{\Sigma_2}(\mathrm{SEN}(f)(\sigma_{\Sigma_1}(\vec{\phi})), \vec{\chi})) & = & C_{\Sigma_2}(\tau_{\Sigma_2}(\sigma_{\Sigma_2}(\mathrm{SEN}(f)^n(\vec{\phi})), \vec{\chi})) \\ & = & C_{\Sigma_2}(\tau_{\Sigma_2}(\sigma_{\Sigma_2}(\mathrm{SEN}(g)^n(\vec{\psi})), \vec{\chi})) \\ & = & C_{\Sigma_2}(\tau_{\Sigma_2}(\mathrm{SEN}(g)(\sigma_{\Sigma_1'}(\vec{\psi})), \vec{\chi})), \end{array}$$

where, passing from the first to the second row, we have used the fact that $\tau(\sigma, \ldots) : \mathrm{SEN}^{n+k-1} \to \mathrm{SEN}$ is in N and $\vec{\phi}\ R^n_{\overline{\Sigma_1}}\ \vec{\psi}$. Therefore, we obtain $\langle \sigma_{\Sigma_1}(\vec{\phi}), \sigma_{\Sigma_1'}(\vec{\psi}) \rangle \in R_{\overline{\Sigma_1}}$ and R is an N-congruence system of SEN. Finally, it is straightforward, taking the identity natural transformation ι :

SEN $\to$ SEN (which is in N) for τ in Equation (5), that R is a logical N-congruence system of $\mathcal{I}$. Therefore, by the definition of $\omega_S^N(\mathcal{I})_{\overline{\Sigma}}$, we get that $R_{\overline{\Sigma}} \subseteq \omega_S^N(\mathcal{I})_{\overline{\Sigma}}$, for all $\Sigma \in |\mathbf{Sign}|$.

Conversely, if $\Sigma_1\ S_1\ \Sigma_1'$ and $\phi \in \mathrm{SEN}(\Sigma_1), \psi \in \mathrm{SEN}(\Sigma_1')$, with $\langle \phi, \psi\rangle \in \omega_S^N(\mathcal{I})_{\overline{\Sigma_1}}$, then, since $\langle S, \omega_S^N(\mathcal{I})\rangle$ is an equivalence system of SEN, we get, for every $\Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$, and $g \in \mathbf{Sign}(\Sigma_1',\Sigma_2)$, such that $f\ S_2\ g$,

$$\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(g)(\psi)\rangle \in \omega_S^N(\mathcal{I})_{\overline{\Sigma_2}}.$$

Now, since $\omega_S^N(\mathcal{I})_{\overline{\Sigma_2}}$ is an equivalence relation on $\bigcup_{\Sigma \in \overline{\Sigma_2}} \mathrm{SEN}(\Sigma)$, we get, for every $\vec{\chi} \in \mathrm{SEN}(\Sigma_2)^{k-1}$, $\vec{\chi}\ (\omega_S^N(\mathcal{I})_{\overline{\Sigma_2}})^{k-1}\ \vec{\chi}$, whence, since $\langle S, \omega_S^N(\mathcal{I})\rangle$ is an N-congruence system, we get, for every $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N,

$$\langle \sigma_{\Sigma_2}(\mathrm{SEN}(f)(\phi), \vec{\chi}), \sigma_{\Sigma_2}(\mathrm{SEN}(g)(\psi), \vec{\chi})\rangle \in \omega_S^N(\mathcal{I})_{\overline{\Sigma_2}}.$$

Therefore, since $\langle S, \omega_S^N(\mathcal{I})\rangle$ is a logical N-congruence system of $\mathcal{I}$, we have

$$C_{\Sigma_2}(\sigma_{\Sigma_2}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma_2}(\sigma_{\Sigma_2}(\mathrm{SEN}(g)(\psi), \vec{\chi})).$$

Hence $\langle \phi, \psi\rangle \in R_{\overline{\Sigma_1}}$ and $\omega_S^N(\mathcal{I})_{\overline{\Sigma}} \subseteq R_{\overline{\Sigma}}$, for every $\Sigma \in |\mathbf{Sign}|$. ■

Lemma 13 yields immediately the following corollary characterizing the restriction $\omega_S^N(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)}$ of the N-congruence $\omega_S^N(\mathcal{I})_{\overline{\Sigma}}$ on the sentences of one member of its signature class.

Corollary 14 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution,* N *a category of natural transformations on* SEN *and* S *a category congruence on* **Sign**. *Let* $\Sigma \in |\mathbf{Sign}|$ *and* $\phi, \psi \in \mathrm{SEN}(\Sigma)$. *Then* $\langle \phi, \psi\rangle \in \omega_S^N(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)}$ *if and only if, for all* $\Sigma' \in |\mathbf{Sign}|$, $f, g \in \mathbf{Sign}(\Sigma, \Sigma')$, *such that* $f\ S_2\ g$, *all natural transformations* $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ *in* N *and all* $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(g)(\psi), \vec{\chi})). \tag{6}$$

Based on this corollary, the second step of the outlined proof may now be accomplished. Namely, it is shown that, for every category congruence S of **Sign**, the largest N-congruence system $\langle \Delta_{\mathbf{Sign}}, \omega_{\mathrm{ID}}^N(\mathcal{I})\rangle$ corresponding to the identity category congruence on **Sign** is at least as large on the sentences of each individual signature of $\mathcal{I}$ as the restriction of the largest logical N-congruence system $\langle S, \omega_S^N(\mathcal{I})\rangle$ corresponding to the category congruence S on **Sign**.

Lemma 15 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution,* N *a category of natural transformations on* SEN *and* S *a category congruence on* **Sign**. *For every* $\Sigma \in |\mathbf{Sign}|$,

$$\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} \subseteq \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma.$$

Proof: Suppose that $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)}$. Then, by Corollary 14, for all $\Sigma' \in |\mathbf{Sign}|$, $f, g \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $f \; S_2 \; g$, all natural transformations $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, $C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(g)(\psi), \vec{\chi}))$. Thus, a fortiori, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, all natural transformations $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, $C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}))$. But this shows that $\langle \phi, \psi\rangle \in \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$. ∎

Lemma 15, together with the existence result of the Tarski congruence system of a π-institution, immediately implies

Corollary 16 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *The category congruence* $\Omega^N(\mathcal{I})$ *is the largest category congruence* S*, with* $\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$*, for all* $\Sigma \in |\mathbf{Sign}|$.

Corollary 16 brings us one step closer to our goal, since it may now be shown that the category congruence $\Omega^N(\mathcal{I})$ is the join in $\langle \mathrm{Con}(\mathbf{Sign}), \leq\rangle$ of the collection of all category congruences S, such that $\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$.

Lemma 17 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *The category congruence* $\Omega^N(\mathcal{I})$ *is the join in the complete lattice* $\mathrm{Con}(\mathbf{Sign})$ *of the collection of all category congruences* S*, with* $\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$*, for all* $\Sigma \in |\mathbf{Sign}|$.

Proof: The Tarski N-congruence system is the join in the lattice of all N-congruence systems $\mathrm{Con}^N(\mathrm{SEN})$ of all logical N-congruence systems. Since, by Corollary 16, we have that $\omega^N_{\Omega^N(\mathcal{I})}(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$, it follows that the Tarski N-congruence system is also the join in the lattice of all N-congruence systems $\mathrm{Con}^N(\mathrm{SEN})$ of all logical N-congruence systems $\langle S, \theta\rangle$, such that $\omega^N_S(\mathcal{I})_{\overline{\Sigma}} \restriction_{\mathrm{SEN}(\Sigma)} = \omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$. Taking now the signature part of this join yields the statement. ∎

Finally, combining Lemma 13 with Lemma 17, we obtain a full characterization of the Tarski N-congruence system of a given π-institution. Note that this characterization includes as a special case the characterization of the special Tarski N-congruence system $\langle \Omega^N_{\mathrm{ID}}(\mathcal{I}), \omega^N_{\mathrm{ID}}(\mathcal{I})\rangle$ that had already been obtained in [19].

Theorem 18 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution and N a category of natural transformations on* SEN. *The Tarski N-congruence system $\langle \Omega^N(\mathcal{I}), \omega^N(\mathcal{I})\rangle$ of $\mathcal{I}$ is the N-congruence system constructed in three steps as follows:*

1. *$\omega^N_{\mathrm{ID}}(\mathcal{I})_\Sigma$ is constructed using Lemma 13.*
2. *$\Omega^N(\mathcal{I})$ is constructed using Part 1 and Lemma 17.*
3. *$\omega^N(\mathcal{I})$ is finally constructed using Part 2 and Lemma 13.*

7 Discussion and Open Problems

In this paper, we were able to generalize the notion of a congruence of an algebra to that of a congruence system of a π-institution. Congruence systems were endowed with a complete lattice ordering, which gave rise to a generalized Tarski congruence system for π-institutions. This is the largest logical congruence system of the π-institution in a way analogous to the Tarski congruence of an abstract logic, as introduced by Josep Maria Font and Ramon Jansana in [9]. The notion presented here generalizes also the notion introduced in [19]. The present, more general, notion is not as satisfactory as the special case, introduced in [19]. First, it has not been possible to prove Proposition 6 for arbitrary equivalence systems. Furthermore, it has not been possible to discover a simple property of an equivalence system equivalent to extendability. Therefore, we do not know whether the correspondence between confluent N-congruence systems on SEN and confluent (N, N')-epimorphic translations of Proposition 5 extends to all N-congruence systems and all (N, N')-epimorphic translations. The most important issue that makes this a serious deficiency is that, despite the fact that the notion of confluence solves the problem of defining the quotient functor $\mathrm{SEN}' : \mathbf{Sign}/S \to \mathbf{Set}$ and the category N' on SEN', confluent congruence systems are not closed under intersections and therefore do not form a lattice. Therefore, there is no hope of defining a confluent Tarski N-congruence system or to guarantee that the Tarski N-congruence system of a π-institution will be a confluent congruence system. It seems, at present, that the solution to this problem may likely come from strengthening confluence to obtain a new property of N-congruence systems that would allow us, at the same time, to define a category of natural transformations N' on the quotient and to obtain a complete lattice structure on those N-congruences that satisfy this property.

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