On Some Operations on Classes of Algebras and Coalgebras from a Bialgebraic Viewpoint

George Voutsadakis
Physical Science Laboratory
New Mexico State University

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Abstract

Let **Set** be the category of small sets and F, G two endofunctors on **Set**. An $\langle F, G\rangle$-bialgebra $\mathbf{A} = \langle A, \alpha\rangle$ consists of a set A and a mapping $\alpha : F(A) \to G(A)$. Given two operations $\mathbf{R}$ and $\mathbf{Q}$ on classes of $\langle F, G\rangle$-bialgebras, $\mathbf{QR}$ is their set-theoretic composition. Moreover $\mathbf{R} \leq \mathbf{Q}$ is defined to mean that, for every class K of $\langle F, G\rangle$-bialgebras, $\mathbf{R}(K) \subseteq \mathbf{Q}(K)$. $\leq$ is a partial ordering on the class of all operations on classes of bialgebras. Special cases of bialgebras include universal algebras and coalgebras. The result obtained by Pigozzi on the structure of the partially ordered monoid of operations on classes of algebras, generated by the operations of taking homomorphic images, subalgebras and direct products, is revisited from the point of view of operations on classes of bialgebras under the assumption that G preserves products and pullbacks. The result obtained by Mašulović and Tasić on the partially ordered monoid of operations on classes of coalgebras, generated by the operations of taking subcoalgebras, homomorphic images and sums, is also revisited from the point of view of the corresponding monoid of operations on classes of bialgebras under the assumption that F preserves coproducts and pushouts. The results here are not new, since the conditions imposed on G and F, respectively, reduce bialgebras to algebras and coalgebras, respectively, but the generalized framework allows the formulation of many more related problems, some of which are suggested in the closing section for further research, and the proofs in this new framework may help in suggesting solutions to these problems.

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1 Introduction

One of the best known results in universal algebra is Birkhoff's **HSP**-theorem, which states that a class of algebras is closed under the operations $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ of taking homomorphic images, subalgebras and products, respectively, if and only if it is equationally defined, i.e., if and only if there exits a set of equations over the type of the class that are satisfied by exactly the algebras of the class (see, e.g., [2] or [9]). These classes are called *varieties*. Set-theoretic composition on operators on classes of algebras is an operation on the class of all operators. Moreover, a natural ordering $\leq$ on operators on classes of algebras may be defined by stipulating that, given two operators $\mathbf{R}$ and $\mathbf{Q}$, $\mathbf{R} \leq \mathbf{Q}$ means that, for all classes K of algebras, $\mathbf{R}(K) \subseteq \mathbf{Q}(K)$. Starting from Birkhoff's result on varieties, Don Pigozzi [10] investigated the structure of the partially ordered monoid of the operations on classes of algebras generated by the operations $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$. He showed that there are 18 different operations in these monoid and that their partial ordering is the one given in Figure 1.

Recently, categorical structures dual to algebras, called *coalgebras*, have been defined and extensively used in theoretical computer science for the specification of data structures and programming languages and as a formalism in which models of computation may be developed (see, e.g., [13, 14, 15, 16]). In universal coalgebra, the role of products is assumed by sums. So closure of classes of algebras with respect to the operations of taking homomorphic images, subalgebras and direct products is translated in the domain of coalgebras as closure of classes of coalgebras with respect to the operations $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ of taking subcoalgebras, homomorphic images and sum coalgebras, respectively. Thus, keeping Pigozzi's result in mind, it is only natural to ask in the coalgebraic framework for the structure of the partially ordered monoid of operations on classes of coalgebras generated by $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$. The composition and ordering in this monoid are defined as for algebras. The structure of this monoid has been investigated by Mašulović and Tasić [8], who showed that the monoid contains 13 operations and its partial order is the one shown in Figure 2.

A common generalization of algebra and coalgebra, called *bialgebra*, has been introduced and studied in [20]. Bialgebraic constructs appeared for the first time in [19] under the name *generalized algebras* and, later, in [17, 18, 1], where the emphasis was on some properties of set functors and on the structure of categories of generalized algebras rather than on the generalized algebras themselves. Recently, bialgebras have been reintroduced from a more computational perspective under the name of *dialgebras* in [11]. Bialgebras include partial algebras and multi-algebras, that have been studied in universal algebra in parallel with ordinary algebras and have also been used in theoretical computer science as models of nondeterminism and parallelism (see, e.g., [3] and [7]). Bialgebras, once thoroughly developed, have the potential for an even wider applicability since they generalize algebras, multi-algebras and coalgebras that have already proven very useful in several applications.

Let **Set** be the category of small sets and $F, G : \mathbf{Set} \to \mathbf{Set}$ be two endofunctors on **Set**. An $\langle F, G\rangle$-*bialgebra* $\mathbf{A} = \langle A, \alpha\rangle$ consists of a set A and a mapping $\alpha : F(A) \to G(A)$. A *bialgebra homomorphism* $h : \mathbf{A} \to \mathbf{B}$ from a bialgebra $\mathbf{A}$ to a bialgebra $\mathbf{B}$ is a set mapping $h : A \to B$, such that the following rectangle commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{F(h)} & F(B) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ G(A) & \xrightarrow[G(h)]{} & G(B) \end{array}$$

The notions of a subbialgebra and of a homomorphic image make sense for bialgebras. The same holds for the notions of product and sum although these may not always exist as they do for universal algebras and coalgebras, respectively. However, appropriate conditions may be imposed on the functors F and G, so as to force the existence of products and/or sums. If these conditions are imposed, it is natural to ask about the structure of the partially ordered monoid of operations on classes of bialgebras that is generated by the operations of taking subbialgebras, homomorphic images, products and/or sums. In this paper, the aforementioned results of Pigozzi on algebras and Mašulović and Tasić on coalgebras are revisited from a bialgebraic viewpoint. Namely, it is shown, in the bialgebraic context, that, if G preserves products and pullbacks, then the partially ordered monoid of operations on classes of $\langle F, G\rangle$-bialgebras generated by the operations $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ is the partially ordered monoid of Pigozzi and, if F preserves coproducts and pushouts, then the partially ordered monoid of operations on classes of bialgebras generated by the operations $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ is the partially ordered monoid of Mašulović and Tasić. Although these results may sound new, they are restatements of the results of Pigozzi and Mašulović and Tasić, respectively. This is due to the fact that the conditions imposed on G and F, respectively, i.e., preservation of products and pullbacks and preservation of coproducts and pushouts, force $\langle F, G\rangle$-bialgebras to become $M \times F$-algebras and G^C-coalgebras for some sets M and C, respectively [17, 18][1]. However, viewing these two results in this generalized context naturally leads to several related open questions concerning genuine bialgebras and the proofs, in the present bialgebraic context, albeit similar to the proofs for algebras and coalgebras, may provide clues for solving those problems in the more general context. Some of these more general cases in which, either no conditions are imposed on the functors F or G, or all four operations

[1]The author acknowledges the help of an anonymous referee in bringing these facts to his attention.

are considered simultaneously, with or without conditions on F or G are left open for further research.

2 Basic Notions

Let **Set** be the category of all small sets and $F, G : \mathbf{Set} \to \mathbf{Set}$ two endofunctors on **Set**. An $\langle F, G\rangle$-**bialgebra A** is a pair $\mathbf{A} = \langle A, \alpha\rangle$, where A is a set and $\alpha : F(A) \to G(A)$ is a set function. An $\langle F, G\rangle$-**bialgebra homomorphism** $h : \mathbf{A} \to \mathbf{B}$ **from** a bialgebra $\mathbf{A} = \langle A, \alpha\rangle$ **to** a bialgebra $\mathbf{B} = \langle B, \beta\rangle$ is a mapping $h : A \to B$, such that the following diagram commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{F(h)} & F(B) \\ \alpha \downarrow & & \downarrow \beta \\ G(A) & \xrightarrow[G(h)]{} & G(B) \end{array}$$

Identity set morphisms are bialgebra homomorphisms and so is the composition of two homomorphisms. Thus, the collection of all $\langle F, G\rangle$-bialgebras with bialgebra homomorphisms between them forms a category, called the **category of $\langle F, G\rangle$-bialgebras** and denoted by $\mathbf{Set}_G^F$. Bialgebras were introduced in [20] as a construct unifying universal algebras and coalgebras and several of their elementary properties were investigated. Categories of bialgebras first appeared in [19] under the name of *generalized algebraic categories* as categories generalizing usual universal algebraic categories. Many of their properties were subsequently studied in [17, 18] and [1].

In what follows, use will be made of the underlying set forgetful functor $U : \mathbf{Set}_G^F \to \mathbf{Set}$. This is the functor that sends an $\langle F, G\rangle$-bialgebra $\mathbf{A} = \langle A, \alpha\rangle$ to its underlying set A and an $\langle F, G\rangle$-bialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ to the underlying set mapping $h : A \to B$.

Given a family $\mathbf{A}_i = \langle A_i, \alpha_i\rangle, i \in I$, of $\langle F, G\rangle$-bialgebras, their **product** $\prod_{i\in I} \mathbf{A}_i$ is defined to be their product in the category $\mathbf{Set}_G^F$, if this product exists. Similarly, their **sum** $\sum_{i\in I} \mathbf{A}_i$ is defined to be their coproduct or sum in $\mathbf{Set}_G^F$ if it exists. In [1] (see also [20]), the following sufficient conditions were given for the existence of limits and colimits in terms of properties of the functors F, G:

Theorem 1 *$U : \mathbf{Set}_G^F \to \mathbf{Set}$ creates and preserves all types of limits that $G : \mathbf{Set} \to \mathbf{Set}$ preserves.*

Theorem 2 *$U : \mathbf{Set}_G^F \to \mathbf{Set}$ creates and preserves all types of colimits that $F : \mathbf{Set} \to \mathbf{Set}$ preserves.*

In particular, if G preserves products, then all products in $\mathbf{Set}_G^F$ exist and are created and preserved by U, and, if F preserves sums, then all sums in $\mathbf{Set}_G^F$ exist and are created and preserved by U.

An $\langle F, G\rangle$-bialgebra $\mathbf{B} = \langle B, \beta\rangle$ is a **subbialgebra** of the bialgebra $\mathbf{A} = \langle A, \alpha\rangle$, denoted $\mathbf{B} \leq \mathbf{A}$, if $B \subseteq A$ and the inclusion map $i : B \hookrightarrow A$ is a homomorphism

$$\begin{array}{ccc} F(B) & \xrightarrow{F(i)} & F(A) \\ \beta \big\downarrow & & \big\downarrow \alpha \\ G(B) & \xrightarrow[G(i)]{} & G(A) \end{array}$$

$\mathbf{B} = \langle B, \beta\rangle$ is a **homomorphic image** of $\mathbf{A} = \langle A, \alpha\rangle$ if there exists a surjection $h : A \twoheadrightarrow B$, such that $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism.

Given a class K of $\langle F, G\rangle$-bialgebras, by $\mathbf{P}(K)$ will be denoted the class of all bialgebras isomorphic to a product of bialgebras in K, by $\mathbf{\Sigma}(K)$ the class of all bialgebras isomorphic to a sum of bialgebras in K, by $\mathbf{S}(K)$ the class of all bialgebras isomorphic to a subbialgebra of a bialgebra in K and, finally, by $\mathbf{H}(K)$ the class of all bialgebras that are homomorphic images of bialgebras in K. K will be said to be **closed** under $\mathbf{P}, \mathbf{\Sigma}, \mathbf{S}$ or $\mathbf{H}$ if $\mathbf{P}(K) \subseteq K, \mathbf{\Sigma}(K) \subseteq K, \mathbf{S}(K) \subseteq K$ or $\mathbf{H}(K) \subseteq K$, respectively.

The following lemma holds (see also [2], Section 9, [4], Lemma 7.2, and [5], Lemma 2.7)

Lemma 3 $\mathbf{P}, \mathbf{\Sigma}, \mathbf{S}$ *and* $\mathbf{H}$ *are closure operators, i.e., for all classes* K, K_1, K_2 *of* $\langle F, G\rangle$*-bialgebras and all* $\mathbf{O} \in \{\mathbf{P}, \mathbf{\Sigma}, \mathbf{S}, \mathbf{H}\}$,

1 $K \subseteq \mathbf{O}(K)$

2 $K_1 \subseteq K_2$ *implies* $\mathbf{O}(K_1) \subseteq \mathbf{O}(K_2)$

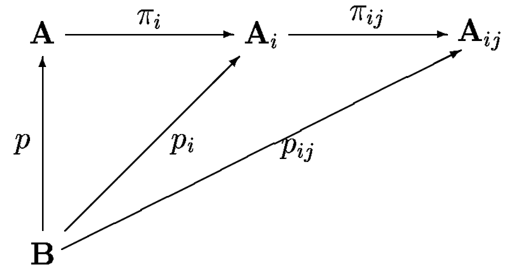
3 $\mathbf{O}(K) = \mathbf{O}(\mathbf{O}(K))$

Proof:

1 This is straightforward since, given a bialgebra $\mathbf{A} = \langle A, \alpha\rangle$, $\mathbf{A}$ is the product and coproduct of the set $\{\mathbf{A}\}$ and, also, it is a subbialgebra and a homomorphic image of itself via the identity map.

2 This follows directly from "set-theoretic" considerations.

3 For $\mathbf{S}$ and $\mathbf{H}$ the statement follows by the fact that the composition of two injections is an injection and the composition of two epimorphisms is an epimorphism. We prove the statement for $\mathbf{P}$. The proof for $\mathbf{\Sigma}$ is similar. Suppose that $\mathbf{A} = \langle A, \alpha \rangle \in \mathbf{P}(\mathbf{P}(K))$. Then, there exist $\mathbf{A}_i = \langle A_i, \alpha_i \rangle \in \mathbf{P}(K)$, such that $\mathbf{A} \cong \prod_{i \in I} \mathbf{A}_i$ with projections $\pi_i : \mathbf{A} \to \mathbf{A}_i, i \in I$, and, for each $i \in I$, there exist $\mathbf{A}_{ij}, j \in J_i$, such that $\mathbf{A}_i \cong \prod_{j \in J_i} \mathbf{A}_{ij}$ with projections $\pi_{ij} : \mathbf{A}_i \to \mathbf{A}_{ij}, i \in I, j \in J_i$. It is not difficult to check, then, that $\mathbf{A} \cong \prod_{i \in I, j \in J_i} \mathbf{A}_{ij}$ with projections $\pi'_{ij} = \pi_{ij} \circ \pi_i, i \in I, j \in J_i$.



For any bialgebra $\mathbf{B}$ and family of homomorphisms $p_{ij} : \mathbf{B} \to \mathbf{A}_{ij}, i \in I, j \in J_i$, the universal mapping property of $\mathbf{A}_i$ gives a collection of uniquely determined morphisms $p_i : \mathbf{B} \to \mathbf{A}_i, i \in I$, and the universal mapping property of $\mathbf{A}$ gives a unique morphism $p : \mathbf{B} \to \mathbf{A}$. ■

3 $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$

In this section, the structure of the partially ordered monoid generated by the operations $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ of taking homomorphic images, subbialgebras and product bialgebras on classes of $\langle F, G \rangle$-bialgebras is investigated. The multiplication operation of the monoid is the composition operation and the identity is the identity operator $\mathbf{I}$ on classes of bialgebras. The ordering is defined, for all operations $\mathbf{O}_1, \mathbf{O}_2$ on classes of bialgebras, by

$$\mathbf{O}_1 \leq \mathbf{O}_2 \text{ if and only if, for every class } K \text{ of } \langle F, G \rangle \text{ -bialgebras, } \mathbf{O}_1(K) \subseteq \mathbf{O}_2(K).$$

Since, in general, products may not exist in $\mathbf{Set}_G^F$ and in view of Theorem 1, we will restrict our attention to the special case in which the functor $G : \mathbf{Set} \to \mathbf{Set}$ preserves products and pullbacks. We will show that, in this case, the partially ordered monoid is the partially ordered monoid of operators on classes of universal algebras generated by the operators of taking homomorphic images, subalgebras and direct products, that was obtained by Don Pigozzi ([10], Theorem 2). Even though this result may seem to generalize the result of Pigozzi, since algebras are special cases of bialgebras in which the functor $F : \mathbf{Set} \to \mathbf{Set}$ is the functor Σ^*, studied in [12], Section 2.1, and $G : \mathbf{Set} \to \mathbf{Set}$ is the identity functor on **Set**, which preserves products and pullbacks, this is not the case. Since G preserves products and pullbacks, $G = (-)^M$ for a set M [17, 18], whence an $\langle F, G\rangle$-bialgebra is given by a map $\alpha : F(A) \to A^M$, and, therefore, reduces to an $M \times F$-algebra. Revisiting the proof in the bialgebraic context here, without using this fact, may help in discovering proofs in the genuine bialgebraic case, where these preservation properties will no more be in effect.

Lemma 4 *Let K be an arbitrary class of $\langle F, G\rangle$-bialgebras and suppose that G preserves products and pullbacks. Then*

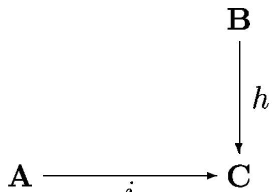
1 $\mathbf{SH} \leq \mathbf{HS}$

2 $\mathbf{PH} \leq \mathbf{HP}$

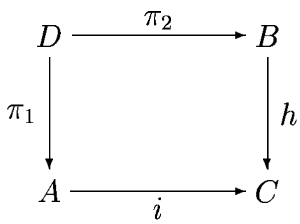
3 $\mathbf{PS} \leq \mathbf{SP}$

Proof:

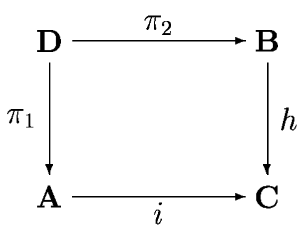
1 Suppose that $\mathbf{A} = \langle A, \alpha\rangle \in \mathbf{S}(\mathbf{H}(K))$. Then, there exists a bialgebra $\mathbf{B} \in K$, a surjective homomorphism $h : \mathbf{B} \twoheadrightarrow \mathbf{C}$ and an injection $i : \mathbf{A} \hookrightarrow \mathbf{C}$.



Since G preserves pullbacks, by Theorem 1, The pullback in **Set**



may be lifted to to a pullback in $\mathbf{Set}_G^F$



and $\pi_1 : \mathbf{D} \to \mathbf{A}$ is surjective and $\pi_2 : \mathbf{D} \to \mathbf{B}$ is injective. Thus, since $\mathbf{B} \in K$, $\mathbf{A} \in \mathbf{H}(\mathbf{S}(K))$ and $\mathbf{SH} \leq \mathbf{HS}$.

2 Suppose that $\mathbf{A} = \langle A, \alpha \rangle \in \mathbf{P}(\mathbf{H}(K))$. Then, there exists a collection $\mathbf{A}_i \in K$ and surjective homomorphisms $h_i : \mathbf{A}_i \twoheadrightarrow \mathbf{B}_i$, such that $\mathbf{A} = \prod_{i \in I} \mathbf{B}_i$, with projections $\pi_i : \mathbf{A} \to \mathbf{B}_i, i \in I$.

$$\begin{array}{ccc} & & \mathbf{A}_i \\ & & \downarrow{\scriptstyle h_i} \\ \prod_{i\in I} \mathbf{B}_i & \xrightarrow[\pi_i]{} & \mathbf{B}_i \end{array}$$

Since G preserves products, by Theorem 1, the product $\prod_{i \in I} \mathbf{A}_i$ exists in $\mathbf{Set}_G^F$. Denote by $\sigma_i : \prod_{i \in I} \mathbf{A}_i \to \mathbf{A}_i, i \in I$, its projections.

$$\begin{array}{ccc} \prod_{i\in I} \mathbf{A}_i & \xrightarrow{\sigma_i} & \mathbf{A}_i \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h_i} \\ \mathbf{A} = \prod_{i\in I} \mathbf{B}_i & \xrightarrow[\pi_i]{} & \mathbf{B}_i \end{array}$$

Then, by the universal mapping property of $\mathbf{A}$, there exists a unique homomorphism $f : \prod_{i \in I} \mathbf{A}_i \to \mathbf{A}$, such that $\pi_i f = h_i \sigma_i, i \in I$. f is surjective since it is surjective in $\mathbf{Set}$. Thus, since $\mathbf{A}_i \in K, i \in I$, $\mathbf{A} \in \mathbf{H}(\mathbf{P}(K))$ and $\mathbf{PH} \leq \mathbf{HP}$.

3 Suppose that $\mathbf{A} = \langle A, \alpha \rangle \in \mathbf{P}(\mathbf{S}(K))$. Then, there exists a collection $\mathbf{A}_i \in K, i \in I$, and injective homomorphisms $j_i : \mathbf{B}_i \rightarrowtail \mathbf{A}_i, i \in I$, such that $\mathbf{A} = \prod_{i \in I} \mathbf{B}_i$, with projections $\pi_i : \mathbf{A} \to \mathbf{B}_i, i \in I$.

$$\begin{array}{ccc} \prod_{i\in I} \mathbf{B}_i & \xrightarrow{\pi_i} & \mathbf{B}_i \\ & & \downarrow{\scriptstyle j_i} \\ & & \mathbf{A}_i \end{array}$$

Since G preserves products, by Theorem 1, the product $\prod_{i \in I} \mathbf{A}_i$ exists in $\mathbf{Set}_G^F$. Denote by $\sigma_i : \prod_{i \in I} \mathbf{A}_i \to \mathbf{A}_i, i \in I$, its projections.

$$\begin{array}{ccc} \mathbf{A} = \prod_{i\in I} \mathbf{B}_i & \xrightarrow{\pi_i} & \mathbf{B}_i \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j_i} \\ \prod_{i\in I} \mathbf{A}_i & \xrightarrow[\sigma_i]{} & \mathbf{A}_i \end{array}$$

Then, by the universal mapping property of $\prod_{i \in I} \mathbf{A}_i$, there exists a unique homomorphism $f : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$, such that $j_i \pi_i = \sigma_i f, i \in I$. f is injective since it is injective in **Set**. Thus, since $\mathbf{A}_i \in K, i \in I$, $\mathbf{A} \in \mathbf{S}(\mathbf{P}(K))$ and $\mathbf{PS} \leq \mathbf{SP}$. ■

Now Lemmas 3 and 4 impose the following nine relations on the generators of the partially ordered monoid $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$:

$$\begin{array}{ccc} \mathbf{I} \leq \mathbf{H} & \mathbf{I} \leq \mathbf{S} & \mathbf{I} \leq \mathbf{P} \\ \mathbf{H} = \mathbf{HH} & \mathbf{S} = \mathbf{SS} & \mathbf{P} = \mathbf{PP} \\ \mathbf{SH} \leq \mathbf{HS} & \mathbf{PH} \leq \mathbf{HP} & \mathbf{PS} \leq \mathbf{SP} \end{array}$$

But these are exactly the relations satisfied by the generators of the partially ordered monoid of operations on classes of algebras studied by Pigozzi in [10]. Thus Theorem 1 of [10] may be applied to obtain

Theorem 5 *Suppose G : **Set** $\to$ **Set** is a functor preserving products and pullbacks. Then every operation on classes of $\langle F, G\rangle$-bialgebras in the monoid generated by the operations $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ coincides with one of the 18 operations*

$$\begin{array}{ccccccccc} \mathbf{I} & \mathbf{H} & \mathbf{S} & \mathbf{P} & \mathbf{SH} & \mathbf{PH} & \mathbf{PS} & \mathbf{HS} & \mathbf{SP} \\ \mathbf{HP} & \mathbf{PSH} & \mathbf{PHS} & \mathbf{SPH} & \mathbf{HPS} & \mathbf{SHP} & \mathbf{SPHS} & \mathbf{SHPS} & \mathbf{HSP} \end{array}$$

Since, moreover, all universal algebras are $\langle F, G\rangle$-bialgebras, where G is the identity functor on **Set**, and, a fortiori, all commutative semigroups are bialgebras, Theorem 2 of [10] may also be applied to obtain

Theorem 6 *Suppose G : **Set** $\to$ **Set** is a functor preserving products and pullbacks. The structure of the partially ordered monoid of operations on classes of $\langle F, G\rangle$-bialgebras generated by $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ is given in Figure 1.*

4 $\mathbf{S}, \mathbf{H}$ and $\boldsymbol{\Sigma}$

In this section, the structure of the partially ordered monoid generated by the operations $\mathbf{S}, \mathbf{H}$ and $\boldsymbol{\Sigma}$ of taking subbialgebras, homomorphic images and sums on classes of $\langle F, G\rangle$-bialgebras is investigated. The multiplication operation of the monoid is again the composition operation and the identity is the identity operator $\mathbf{I}$ on classes of bialgebras. The ordering is defined as before. Since, in general, sums may not exist in $\mathbf{Set}_G^F$ and in view of Theorem 2, we will restrict our attention to the special case in which the functor F : **Set** $\to$ **Set** preserves coproducts and pushouts. We will show that, in this case, the partially ordered monoid is the partially ordered monoid of operators on classes of coalgebras generated by the operators of taking subcoalgebras,

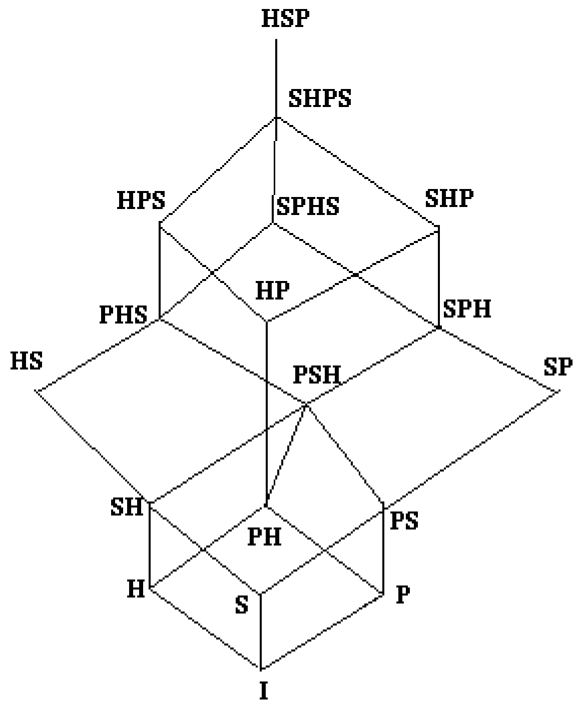


Figure 1: The partially ordered monoid of Theorem 6

homomorphic images and sum coalgebras, that was obtained by Dragan Mašulović and Boža Tasić ([8], Section 3). Similar to Section 3, despite its appearance, this result does not generalize the result of Mašulović and Tasić. This is due to the fact that, if F preserves coproducts and pushouts, then $F = C \times (-)$, for some set C [17, 18], whence an $\langle F, G\rangle$-bialgebra is given by a map $\alpha : C \times A \to G(A)$, and, therefore, reduces to a G^C-coalgebra. Once more, the reason for revisiting the proof in the bialgebraic context, without using this characterization of colimit-preserving functors, is the hope that it may provide clues as to how to proceed in the case of bialgebras, in which no preservation properties will be in effect.

Lemma 7 *Let K be an arbitrary class of $\langle F, G\rangle$-bialgebras and suppose that F preserves coproducts and pushouts. Then*

1 $\mathbf{HS} \leq \mathbf{SH}$

2 $\mathbf{\Sigma S} \leq \mathbf{S\Sigma}$

3 $\mathbf{\Sigma H} \leq \mathbf{H\Sigma}$

Parts *1,2* and *3* of Lemma 7 may be proven similarly to parts *1,2* and *3* of Lemma 4. So the proof will be omitted.

Now Lemmas 3 and 7 impose the following nine relations on the generators of the partially ordered monoid $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$:

$$\begin{array}{ccc} \mathbf{I} \leq \mathbf{H} & \mathbf{I} \leq \mathbf{S} & \mathbf{I} \leq \mathbf{P} \\ \mathbf{H} = \mathbf{HH} & \mathbf{S} = \mathbf{SS} & \mathbf{P} = \mathbf{PP} \\ \mathbf{HS} \leq \mathbf{SH} & \mathbf{\Sigma S} \leq \mathbf{S\Sigma} & \mathbf{\Sigma H} \leq \mathbf{H\Sigma} \end{array}$$

But, as is the case for coalgebras, the following two additional relations hold among the generators:

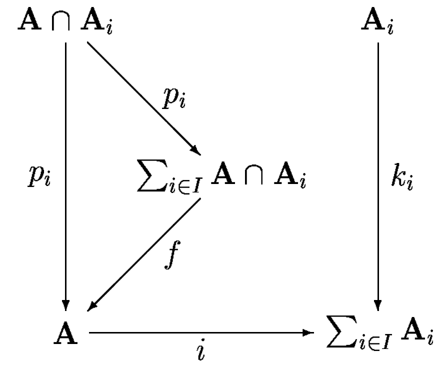
$$\mathbf{S\Sigma} \leq \mathbf{\Sigma S} \quad \text{and} \quad \mathbf{SH\Sigma} = \mathbf{HS\Sigma H}.$$

The proofs are given in Propositions 8 and 11, the first of which is essentially due to Gumm and Schröder (see [6], Lemma 3.2) and the second due to Mašulović and Tasić (see [8], Proposition 3.3), but are both adapted here for the case of bialgebras.

Proposition 8 *The operators* $\mathbf{S}$ *and* $\mathbf{\Sigma}$ *on classes of* $\langle F, G\rangle$*-bialgebras, with* F *coproduct preserving, commute.*

Proof:

By Lemma 7 we have $\mathbf{\Sigma S} \leq \mathbf{S\Sigma}$. So it suffices to show that $\mathbf{S\Sigma} \leq \mathbf{\Sigma S}$. Suppose that $\mathbf{A} = \langle A, \alpha\rangle \in \mathbf{S}(\mathbf{\Sigma}(K))$. If $A = \emptyset$, then $\mathbf{A} \in \mathbf{\Sigma}(\mathbf{S}(K))$ and we are done. So suppose that $A \neq \emptyset$. Then, there exist $\mathbf{A}_i = \langle A_i, \alpha_i\rangle \in K, i \in I$, such that $\mathbf{A} \leq \sum_{i\in I} \mathbf{A}_i$. Without loss of generality we may assume that $A \cap A_i \neq \emptyset$, for all $i \in I$. Denote by $\pi_i : \mathbf{A}_i \to \sum_{i\in I} \mathbf{A}_i, i \in I$, the injections and by $i : \mathbf{A} \to \sum_{i\in I} \mathbf{A}_i$ the inclusion morphism.



By Theorem 37 of [20], $A \cap A_i$ admits a subbialgebra structure $\rho_i : F(A \cap A_i) \to G(A \cap A_i)$, for all $i \in I$, i.e., the following rectangles commute

$$\begin{array}{ccc} F(A \cap A_i) & \xrightarrow{F(k_i)} & F(A) \\ \rho_i \downarrow & & \downarrow \alpha \\ G(A \cap A_i) & \xrightarrow[G(k_i)]{} & G(A) \end{array}$$

where $k_i : A \cap A_i \hookrightarrow A, i \in I$, are the inclusions. Since F preserves coproducts, the sum $\sum_{i \in I} A \cap A_i$ exists in $\mathbf{Set}_G^F$ by Theorem 2. Denote by $p_i : \mathbf{A} \cap \mathbf{A}_i \to \sum_{i \in I} \mathbf{A} \cap \mathbf{A}_i, i \in I$, the corresponding injections. By the universal mapping property of the coproduct $\sum_{i \in I} \mathbf{A} \cap \mathbf{A}_i$, there exists a unique homomorphism $f : \sum_{i \in I} \mathbf{A} \cap \mathbf{A}_i \to \mathbf{A}$, which coincides with the unique morphism from the coproduct $\sum_{i \in I} A \cap A_i$ to A in $\mathbf{Set}$, such that $fp_i = k_i, i \in I$. But $f : \sum_{i \in I} A \cap A_i \to A$ is a bijection, whence, by Proposition 1 of [20], it is a bialgebra isomorphism. Therefore $\mathbf{A} \cong \sum_{i \in I} \mathbf{A} \cap \mathbf{A}_i$ and, since $\mathbf{A}_i \in K$, for all $i \in I$, we have $\mathbf{A} \in \mathbf{\Sigma}(\mathbf{S}(K))$. Thus $\mathbf{S\Sigma} \leq \mathbf{\Sigma S}$. ■

In order to prove Proposition 11, the notion of a conjunct sum of $\langle F, G \rangle$-bialgebras will be used and two lemmas, correspoding to Lemmas 3.1 and 3.2 of [8], will be formulated. The proofs in [8] carry through here unchanged, but will be presented so as to give a full picture of the proof of Proposition 11 for bialgebras.

A bialgebra $\mathbf{A} = \langle A, \alpha \rangle$ is said to be the **conjunct sum** of bialgebras $\mathbf{A}_i = \langle A_i, \alpha_i \rangle, i \in I$, if there exist injective homomorphisms $e_i : \mathbf{A}_i \to \mathbf{A}, i \in I$, such that $A = \bigcup_{i \in I} e_i(A_i)$. In this case we write $\mathbf{A} = \sum_{i \in I}^c \mathbf{A}_i$. Given a class K of bialgebras, by $\mathbf{\Sigma}^c(K)$ is denoted the class of all bialgebras isomorphic to a conjunct sum of bialgebras in K. It is then easy to see that, if F preserves coproducts, then

$$\mathbf{\Sigma}^c \leq \mathbf{H\Sigma}. \tag{1}$$

The following two lemmas concern the relation of $\mathbf{\Sigma}^c$ with the operators $\mathbf{H}$ and $\mathbf{S}$ and will be used in the proof of Proposition 11.

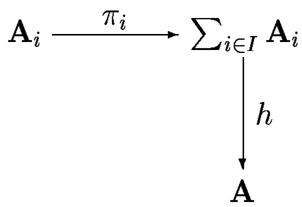
Lemma 9 *For any class K of $\langle F, G \rangle$-bialgebras, with F coproduct preserving,*

$$\mathbf{H}(\mathbf{\Sigma}(K)) = \mathbf{\Sigma}^c(\mathbf{H}(K)).$$

Proof:

First, $\mathbf{\Sigma}^c\mathbf{H} \overset{(1)}{\leq} \mathbf{H\Sigma H} \leq \mathbf{HH\Sigma} = \mathbf{H\Sigma}$.

Suppose, conversely, that $\mathbf{A} \in \mathbf{H}(\mathbf{\Sigma}(K))$. Then, there exist bialgebras $\mathbf{A}_i \in K, i \in I$, and a surjective homomorphism $h : \sum_{i \in I} \mathbf{A}_i \twoheadrightarrow \mathbf{A}$. Let $\pi_i : \mathbf{A}_i \to \sum_{i \in I} \mathbf{A}_i, i \in I$, denote the injections.

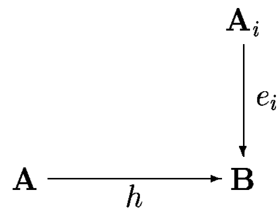


Now, by Theorem 7 of [20], the epi-mono factorization $\mathbf{A}_i \overset{h\pi_i}{\twoheadrightarrow} h(\pi_i(\mathbf{A}_i)) \hookrightarrow \mathbf{A}$ exists in $\mathbf{Set}_G^F$. Since h is a surjection, $A = \bigcup_{i\in I} h(\pi_i(A_i))$, whence $\mathbf{A} = \mathbf{\Sigma}^c_{i\in I} h(\pi_i(\mathbf{A}_i))$. Thus, $\mathbf{A} \in \mathbf{\Sigma}^c(\mathbf{H}(K))$.

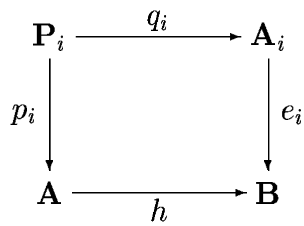
■

Lemma 10 *For any class K of $\langle F, G\rangle$-bialgebras,* $\mathbf{S}(\mathbf{\Sigma}^c(K)) \leq \mathbf{\Sigma}^c(\mathbf{S}(K))$.

Proof:
Suppose that $\mathbf{A} \in \mathbf{S}(\mathbf{\Sigma}^c(K))$. If $A = \emptyset$, then $\mathbf{A} \in \mathbf{\Sigma}^c(\mathbf{S}(K))$. So assume that $A \neq \emptyset$. Then, there exist $\mathbf{A}_i \in K, i \in I$, injections $e_i : \mathbf{A}_i \rightarrowtail \mathbf{B}, i \in I$, such that $B = \bigcup_{i\in I} e_i(A_i)$, and an injection $h : \mathbf{A} \rightarrowtail \mathbf{B}$.



Let $I_0 = \{i \in I : h(A) \cap e_i(A_i) \neq \emptyset\}$. Since $B = \bigcup_{i\in I} e_i(A_i)$, $I_0 \neq \emptyset$. Now, every endofunctor in **Set** preserves nonempty intersections, i.e., it preserves nonempty pullbacks of monos. Thus, by Theorem 1, The following pullback exists in $\mathbf{Set}_G^F$, for all $i \in I_0$,



and $p_i, q_i, i \in I_0$, are all injective. Thus $\mathbf{P}_i \in \mathbf{S}(K), i \in I_0$. Now, since all p_i's are injectives, to show that $\mathbf{A} \in \mathbf{\Sigma}^c(\mathbf{S}(K))$, it suffices to show that $A = \bigcup_{i\in I_0} p_i(P_i)$. But, if $a \in A$, then there exists $i \in I_0$ and $a_i \in A_i$, such that $e_i(a_i) = h(a)$. Therefore $\langle a, a_i\rangle \in P_i$ and $p_i(\langle a, a_i\rangle) = a$. ■

Proposition 11 *Suppose that F :* **Set** $\rightarrow$ **Set** *preserves coproducts and pushouts. Then* $\mathbf{SH\Sigma} = \mathbf{HS\Sigma H}$.

Proof:

$$\begin{array}{rcll} \mathbf{SH\Sigma} & = & \mathbf{S\Sigma^c H} & \text{(by Lemma 9)} \\ & \leq & \mathbf{\Sigma^c SH} & \text{(by Lemma 10)} \\ & \leq & \mathbf{H\Sigma SH} & \text{(by (1))} \\ & = & \mathbf{HS\Sigma H} & \text{(by Proposition 8)} \end{array}$$

On the other hand,

$$\begin{array}{rcll} \mathbf{HS\Sigma H} & \leq & \mathbf{HSH\Sigma} & \text{(by Lemma 7, Part } \mathit{3}) \\ & \leq & \mathbf{SHH\Sigma} & \text{(by Lemma 7, Part } \mathit{1}) \\ & = & \mathbf{SH\Sigma} & \text{(by Lemma 3, Part } \mathit{3}) \end{array}$$

■

It has now been shown that, for $\langle F, G\rangle$-bialgebras, with F coproduct and pushout preserving, the relations between the generators of the partially ordered monoid of operations generated by $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ are exactly the relations satisfied by the generators of the partially ordered monoid of operations on classes of coalgebras studied by Mašulović and Tasić in [8]. Thus their result may be applied to obtain

Theorem 12 *Suppose F : **Set** $\rightarrow$ **Set** is a functor preserving coproducts and pushouts. Then every operation on classes of $\langle F, G\rangle$-bialgebras in the monoid generated by the operations $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ coincides with one of the 13 operations*

$$\begin{array}{lllllll} \mathbf{I} & \mathbf{H} & \mathbf{S} & \mathbf{\Sigma} & \mathbf{HS} & \mathbf{S\Sigma} & \mathbf{\Sigma H} \\ \mathbf{SH} & \mathbf{H\Sigma} & \mathbf{\Sigma HS} & \mathbf{S\Sigma H} & \mathbf{HS\Sigma} & \mathbf{SH\Sigma} & \end{array}$$

Since, moreover, all coalgebras are $\langle F, G\rangle$-bialgebras, where F is the identity functor on **Set**, all examples given in [8] to demonstrate proper inclusions are valid in the case under consideration. Thus, we obtain

Theorem 13 *Suppose F : **Set** $\rightarrow$ **Set** is a functor preserving coproducts and pushouts. The structure of the partially ordered monoid of operations on classes of $\langle F, G\rangle$-bialgebras generated by $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ is given in Figure 2.*

5 Problems for Further Research

In this paper, the results of Pigozzi [10] on the structure of the partially ordered monoid of operators on classes of universal algebras generated by the operators $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ and of Mašulović and Tasić [8] on the structure of the partially ordered monoid of operators on classes of coalgebras generated by the operators $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ have been revisited from the point of view of operators on classes of $\langle F, G\rangle$-bialgebras

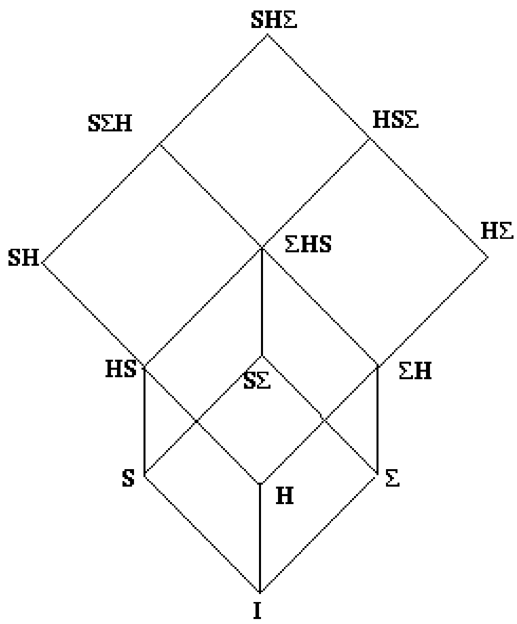


Figure 2: The partially ordered monoid of Theorem 13.

(a) generated by $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$, under the assumption that the functor G preserves products and pullbacks

(b) generated by $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$, under the assumption that F preserves coproducts and pushouts.

It is now natural to ask and try to give an answer to the following problems which are suggested for further research

Problem 1: Which is the structure of the partially ordered monoid of operations on classes of $\langle F, G\rangle$-bialgebras generated by $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ (where no restriction is imposed on G)?

Problem 2: Which is the structure of the partially ordered monoid of operations on classes of $\langle F, G\rangle$-bialgebras generated by $\mathbf{S}, \mathbf{H}$ and $\mathbf{\Sigma}$ (where no restriction is imposed on F)?

Problem 3: Which is the structure of the partially ordered monoid of operations on classes of bialgebras generated by $\mathbf{H}, \mathbf{S}, \mathbf{P}$ and $\mathbf{\Sigma}$ (with or without restrictions on the functors F and G)?

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