Categorical Abstract Algebraic Logic:
Compatibility Operators and Correspondence Theorems

George Voutsadakis[*]

October 11, 2015

To **Don Pigozzi** this work is dedicated
on the occasion of his 80th Birthday.

Abstract

Very recently Albuquerque, Font and Jansana, based on preceding work of Czelakowski on compatibility operators, introduced coherent compatibility operators and used Galois connections, formed by these operators, to provide a unified framework for the study of the Leibniz, the Suszko and the Tarski operators of abstract algebraic logic. Based on this work, we present a unified treatment of the operator approach to the categorical abstract algebraic logic hierarchy of π-institutions. This approach encompasses previous work by the author in this area, started under Don Pigozzi's guidance, and provides resources for new results on the semantic, i.e., operator-based, side of the hierarchy.

---

[*]School of Mathematics and Computer Science, Lake Superior State University, Sault Sainte Marie, MI 49783, USA, `gvoutsad@lssu.edu`

[0]*Keywords:* Leibniz operator, Tarski operator, Suszko operator, logical matrix, full model, reduced model, Leibniz filter, protoalgebraic logic, equivalential logic, algebraizable logic.

2010 AMS Subject Classification: 03G27

1 Introduction: The Three Operators of AAL

The operator approach in abstract algebraic logic (AAL) has born many fruits and forms the cornerstone of all three main directions in the field: The association of classes of algebras with logical systems, the correspondence between logical and algebraic properties and the study of specific classes of logical systems or specific classes of algebras per se in the context of algebraization and/or property correspondence. General surveys of the approach can be found in [17, 8, 20].

The operator approach has been extended by the author to a categorical framework starting with [29] and in a bulk of subsequent work [30, 31, 33, 34] and has provided equally intriguing results, while in the last few years, the levels of the AAL hierarchy of sentential logics have been extended, based on this approach, to the categorical abstract algebraic logic (CAAL) hierarchy of π-institutions [33, 32, 35, 37, 36].

Since the overviews cited above contain remarkably inclusive and relatively up-to-date presentations of the work in the field, we restrict the introduction to some of the essentials needed in placing the present work in context and in outlining some of the contents of the paper.

The operator approach in AAL was initiated by Blok and Pigozzi in their seminal "Memoirs" monograph [4], which was, to a certain extent, anticipated in the ground-breaking work of Czelakowski on equivalential logics [6, 7]. This was followed by vital and influential contributions by Hermann [22, 23, 24], Czelakowski and Pigozzi [11, 12, 13], Font and Jansana [17], Czelakowski and Jansana [10], Czelakowski [9] and Raftery [27].

Blok and Pigozzi introduced the Leibniz operator associating with a theory of a sentential logic the largest congruence on the formula algebra that is compatible with the theory. This may be generalized to an association with an arbitrary filter of the logic on any algebra of the same similarity type as the logic of the largest congruence on the algebra that is compatible with the filter. One of the main results of [4] was a characterization of algebraizability via a correspondence between theories and congruences. This was subsequently refined in many works, e.g., in [17], and, much more recently, in a unifying setting, encompassing many previously known results of this type, in [1].

The bulk of Blok and Pigozzi's and subsequent work focuses on establishing the main classes of the AAL Leibniz hierarchy, consisting of the protoalgebraic [3], equivalential [6, 7], truth-equational [27], weakly algebraizable [10] and algebraizable logics [4, 22, 23, 24], based on properties of the Leibniz operator, and on exploring various metalogical properties and related properties of their associated classes of algebras.

In the CAAL context, a categorical Leibniz operator, inspired by the work in AAL, was introduced in [33] with the goal of obtaining abstractions of the various results obtained in the AAL context using the Leibniz operator of Blok and Pigozzi; among them, perhaps most importantly, developing a CAAL hierarchy of π-institutions based on their algebraic character, indicating the strength of ties between the structure of the lattice of their theory families and that of the congruence systems on algebraic systems.

The second operator that was introduced historically was the Tarski operator [17] in seminal work carried out by Font and Jansana and ushering in a period of intense and fruitful investigations by the Barcelona School of AAL. The Tarski operator associates to a collection of filters on a specific algebra the largest congruence on the algebra that is compatible with all filters in the collection. The Tarski operator served the purpose of lifting the model theory of sentential logics from the level of logical matrices, which had been at the focus of the work of Czelakowski and Blok and Pigozzi, to the level of generalized matrices and of abstract logics (see, e.g., [17, 15]). A key aspect of this theory, playing a decisive role in the characterization of the classes in the Leibniz hierarchy, is the determination of full models of either single logical systems under consideration in a specific study or of classes of logical systems [16, 25, 2, 21]. These are the models that include all filters that are compatible with the Tarski congruence of the model.

Perhaps surprisingly, but understandably, if one takes into account the nature of closure systems defining π-institutions, the categorical Tarski operator [29] was introduced in CAAL before the corresponding abstraction of the Leibniz operator [33, 32]. It served the purpose of abstracting the theory of abstract logics of Font and Jansana to the level of models of π-institutions. Very important for our present work was the establishment of a General Correspondence Theorem [30, 31], which parallels a celebrated Correspondence Theorem in the context of sentential logics [17], in turn generalizing one of the original results of Blok and Pigozzi [4, 5].

In [9], Czelakowski introduced the last of the three major operators, the Suszko operator. Raftery [27] has employed the Suszko operator in the study of truth equational sentential logics. More recently, it has played a potent role in providing alternative characterizations and in studying properties of various classes of the Leibniz hierarchy [1], in addition to its critical role in the characterization of truth equationality. Following [1], we use in a similar way the corresponding CAAL operator, also termed the Suszko operator, which was introduced in [34], following [9].

The three operators are closely related, the Leibniz being in a sense the fundamental one. The Leibniz operator and the Suszko operator are applied to single filters of a logic and the Suszko congruence associated with a given filter is the intersection of all Leibniz congruences associated with filters of the logic that include the given filter. On the other hand the Tarski operator is applied to collections of filters of a logic and it gives the intersection of all Leibniz congruences of the filters in the collection. Thus, both the Suszko and the Tarski operators can be expressed in terms of the Leibniz operator in a straightforward way. One of the elegant contributions of [1] was the introduction of the unifying framework of compatibility operators in which all three operators can be treated uniformly to a far-reaching extent. We follow here [1] in treating the categorical operators in a similar way. We are able, as a result, on the one hand to both unify and simplify already known results from CAAL, and, on the other, to establish many hitherto unknown ones, that generalize to π-institutions corresponding known results from the AAL domain.

2 Abstract Compatibility Operators

In [1], Albuquerque, Font and Jansana developed the theory of $\mathcal{S}$-compatibility operators, encompassing and treating under a unified framework the three classical operators of AAL. We review briefly the basic components of the work in [1] since it forms the foundation for the work developed in the present paper.

We fix a sentential logic $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$. $\mathcal{S}$-compatibility operators are mappings $\nabla^{\boldsymbol{A}}$ from the set of all $\mathcal{S}$-filters $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ on an arbitrary algebra $\boldsymbol{A}$, of the similarity type $\mathcal{L}$ of $\mathcal{S}$, to the set of congruences $\mathrm{Con}(\boldsymbol{A})$ on the algebra. Such an operator $\nabla^{\boldsymbol{A}}$ maps an $\mathcal{S}$-filter F on $\boldsymbol{A}$ to a congruence $\nabla^{\boldsymbol{A}}(F)$ that is compatible with the filter. Since, by definition, the Leibniz congruence $\Omega^{\boldsymbol{A}}(F)$ is the largest congruence on $\boldsymbol{A}$ compatible with F [4], it follows that $\Omega^{\boldsymbol{A}}$ is the largest $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. Moreover, as shown by Czelakowski in [9], the Suszko operator $\widetilde{\Omega}^{\boldsymbol{A}}$ is the largest order-preserving $\mathcal{S}$-compatibility operator.

In the abstract theory, the Leibniz and Suszko operators form an example of another type of relationship. Namely, given an $\mathcal{S}$-compatibility operator $\nabla^{\boldsymbol{A}}$, two more "companion" operators are defined from it [1]:

- The *lifting* $\widetilde{\nabla}^{\boldsymbol{A}}$ is applied to arbitrary collections of $\mathcal{S}$-filters on $\boldsymbol{A}$; it associates with such a collection, the largest congruence on $\boldsymbol{A}$ that is compatible with all filters in the collection.

- The *relativization* $\widetilde{\nabla}_{\mathcal{S}}^{\boldsymbol{A}}$ is applied to an $\mathcal{S}$-filter and associates with it the largest congruence on $\boldsymbol{A}$ that is compatible with all $\mathcal{S}$-filters on $\boldsymbol{A}$ containing the given filter. Thus, its action is that of the lifting applied on the upset of the lattice of all $\mathcal{S}$-filters generated by the given filter.

Clearly, the Tarski operator is the lifting of the Leibniz operator and the Suszko operator is its relativization, and they constitute the prototypical examples of operators that motivate the general theory.

The springboard of the theory in [1] is the observation that $\widetilde{\nabla}^{\boldsymbol{A}}$ is part of a Galois connection between the powerset $\mathcal{P}(\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}))$ of the collection of $\mathcal{S}$-filters on $\boldsymbol{A}$ and the collection $\mathrm{Con}(\boldsymbol{A})$ of congruences on $\boldsymbol{A}$. The fixed points are the so-called $\nabla^{\boldsymbol{A}}$*-full sets of* $\mathcal{S}$*-filters* and the $\nabla^{\boldsymbol{A}}$*-full congruences*.

Another pair of important concepts consists of the $\nabla^{\boldsymbol{A}}$*-class* $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ of an $\mathcal{S}$-filter F, which is composed of all filters with which $\nabla^{\boldsymbol{A}}(F)$ is compatible, and the smallest element $F^{\nabla^{\boldsymbol{A}}}$ of this class. A filter F is termed a $\nabla^{\boldsymbol{A}}$*-filter* in [1] if $F = F^{\nabla^{\boldsymbol{A}}}$, i.e., if it is the smallest filter that is compatible with its $\nabla^{\boldsymbol{A}}$-associated congruence, again a concept that has been studied extensively in the traditional setting by Font and Jansana [18, 19] and Jansana [26].

If an $\mathcal{S}$-compatibility operator $\nabla^{\boldsymbol{A}}$ is defined for every algebra $\boldsymbol{A}$ of the same similarity type $\mathcal{L}$ as that of the sentential logic $\mathcal{S}$, then a family $\nabla = \{\nabla^{\boldsymbol{A}}\}_{\boldsymbol{A}\in\mathsf{Alg}(\mathcal{L})}$ is assembled. To relate the members of ∇ the increasing in strength notions of coherence, commutativity with inverse images of surjective homomorphisms and commutativity with inverse images of arbitrary homomorphisms are introduced. The first is novel in [1] whereas the latter two are well known in traditional AAL and play a critical role in the theory of protoalgebraic [3], equivalential [6, 7] and algebraizable [4] logics (see also [8, 20]).

Remarkably, taking advantage of coherence, a General Correspondence Theorem (Theorem 4.17 of [1]) is obtained to the effect that, for every surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ and every $\mathcal{S}$-filter F on $\boldsymbol{A}$, such that h is $\nabla^{\boldsymbol{A}}$-compatible with F, a technical condition, h induces an order isomorphism between the $\nabla^{\boldsymbol{A}}$-class of F and the $\nabla^{\boldsymbol{B}}$-class of $h(F)$, with inverse h^{-1}. Several well-known isomorphism theorems from the theory of protoalgebraic logics and beyond follow from this General Correspondence Theorem, including results of Blok and Pigozzi [4, 5], of Czelakowski [9] and of Font and Jansana [18].

Following the lead from the classical theory of AAL, based on $\nabla^{\boldsymbol{A}}, \widetilde{\nabla}^{\boldsymbol{A}}$

and $\widetilde{\nabla}_{\mathcal{S}}^{\boldsymbol{A}}$, classes of algebras are defined consisting of algebras that are reduced with respect to corresponding types of congruences. The abstract hypotheses of coherence and commutativity with inverse images of surjective homomorphisms imply various relationships between the classes analogous to those established in the traditional context in relation to the well-known classes $\mathsf{Alg}^*\mathcal{S}, \mathsf{Alg}^{\mathsf{Su}}\mathcal{S}$ and $\mathsf{Alg}\mathcal{S}$ (see Subsection 4.2 of [1]).

Using the concepts of full generalized matrix models, of the Leibniz operator, of the Suszko operator and of the aforementioned classes of algebras associated with $\mathcal{S}$, a wealth of characterizations of the classes in the AAL hierarchy is obtained in Section 6 of [1]. Some of these have been well-known in the AAL literature, some less well-known and some are new. What is remarkable, however, and motivated the present exposition, is the fact that they are all obtained as consequences of the treatment of abstract $\mathcal{S}$-compatibility operators and the basic Galois connection, as specialized in the context of the three main operators of AAL, essentially the Leibniz operator, since it is the fundamental among the three, and the Tarski and Suszko as the lifting and relativization of the Leibniz operator.

3 The Categorical Operators

Let **Sign** be a category, referred to as a **category of signatures**. Let, also, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor from the category of signatures, referred to as a **sentence functor**. A collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is called a **sentence family** of SEN.[1] A sentence family is a **sentence system** if it is invariant under **Sign**-morphisms, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(T_\Sigma) \subseteq T_{\Sigma'}$. An **equivalence family** $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN is a $|\mathbf{Sign}|$-indexed family of equivalence relations $\theta_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$. It is called an **equivalence system** if it is invariant under **Sign**-morphisms, i.e., if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)^2(\theta_\Sigma) \subseteq \theta_{\Sigma'}$. Signature-wise inclusion of both sentence families and equivalence families is denoted by $\leq$, i.e.,

$$T \leq T' \quad \text{iff} \quad T_\Sigma \subseteq T'_\Sigma, \text{ for all } \Sigma \in |\mathbf{Sign}|,$$

and

$$\theta \leq \theta' \quad \text{iff} \quad \theta_\Sigma \subseteq \theta'_\Sigma, \text{ for all } \Sigma \in |\mathbf{Sign}|.$$

Consider, in addition to a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a **category N of natural transformations** on SEN in the sense of, e.g., Section

[1] This was called an *axiom family* in CAAL before.

2 of [33]. The triple $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is called an **algebraic system**. An equivalence family θ on SEN is called a **congruence family** on $\boldsymbol{A}$ if it is invariant under N-morphisms, i.e., if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \psi_0, \dots, \varphi_{k-1}, \psi_{k-1} \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi_i, \psi_i\rangle \in \theta_\Sigma,\ i < k,\ \text{imply}\ \langle \sigma_\Sigma(\varphi_0, \dots, \varphi_{k-1}), \sigma_\Sigma(\psi_0, \dots, \psi_{k-1})\rangle \in \theta_\Sigma.$$

A **congruence system** is a congruence family that is an equivalence system, i.e., an equivalence family that is invariant under both **Sign**-morphisms and N-morphisms. The collection of all congruence systems on $\boldsymbol{A}$ is denoted by $\mathrm{ConSys}(\boldsymbol{A})$. Ordered by signature-wise inclusion $\le$, they form a complete lattice, denoted by $\mathbf{ConSys}(\boldsymbol{A}) = \langle \mathrm{ConSys}(\boldsymbol{A}), \le\rangle$.

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, termed the **base algebraic system**. An algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ is called an N-**algebraic system** if there exists a surjective functor $\ ' : N \to N'$ that preserves all projection natural transformations and, therefore, preserves also the arities of all natural transformations in N. We write σ' in N' to indicate the image in N' of a σ in N under the functor $\ '$. Given two N-algebraic systems $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ and $\boldsymbol{B} = \langle \mathbf{Sign}'', \mathrm{SEN}'', N''\rangle$, an N-(**algebraic system**) **morphism** $\langle H, \gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$ consists of

- a functor $H : \mathbf{Sign}' \to \mathbf{Sign}''$ and
- a natural transformation $\gamma : \mathrm{SEN}' \to \mathrm{SEN}'' \circ H$, such that, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}'|$ and all $\varphi_0, \dots, \varphi_{k-1} \in \mathrm{SEN}'(\Sigma)$,

$$\gamma_\Sigma(\sigma'_\Sigma(\varphi_0, \dots, \varphi_{k-1})) = \sigma''_{H(\Sigma)}(\gamma_\Sigma(\varphi_0), \dots, \gamma_\Sigma(\varphi_{k-1})).$$

Given an N-morphism $\langle H, \gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$, the **kernel** of $\langle H, \gamma\rangle$ is the congruence system $\mathrm{Ker}(\langle H, \gamma\rangle) = \{\mathrm{Ker}_\Sigma(\langle H, \gamma\rangle)\}_{\Sigma \in |\mathbf{Sign}'|}$, defined, for all $\Sigma \in |\mathbf{Sign}'|$, by

$$\mathrm{Ker}_\Sigma(\langle H, \gamma\rangle) = \{\langle \varphi, \psi\rangle \in \mathrm{SEN}'(\Sigma)^2 : \gamma_\Sigma(\varphi) = \gamma_\Sigma(\psi)\}.$$

Given an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and a congruence system θ on $\boldsymbol{A}$, one can define the **quotient algebraic system** $\boldsymbol{A}/\theta = \langle \mathbf{Sign}, \mathrm{SEN}^\theta, N^\theta\rangle$ of $\boldsymbol{A}$ by θ (see, e.g., [29]). In this case $\langle I_{\mathbf{Sign}}, \pi^\theta\rangle : \boldsymbol{A} \to \boldsymbol{A}/\theta$ denotes the projection morphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\theta$. Thus, given a class K of algebraic systems, it makes sense to consider the K-**relative congruence systems on** $\boldsymbol{A}$, i.e., those $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, such that $\boldsymbol{A}/\theta \in \mathsf{K}$. The class of all relative K-congruence systems on $\boldsymbol{A}$ is denoted by $\mathrm{ConSys}_{\mathsf{K}}(\boldsymbol{A})$.

Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a sentence family of SEN. A congruence system $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\boldsymbol{A}$ is **compatible with** T if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\varphi,\psi\rangle \in \theta_\Sigma \quad \text{and} \quad \varphi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

This condition is denoted $T \ \mathsf{comp}\ \theta$ and may be characterized in the following ways:

Lemma 1 *Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\theta \in \mathrm{ConSys}(\boldsymbol{A})$ and T a sentence family of* SEN. *The following conditions are equivalent:*

(i) *θ is compatible with T.*

(ii) *$\varphi \in T_\Sigma$ iff $\varphi/\theta_\Sigma \in T_\Sigma/\theta_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$.*

(iii) *$T = \pi^{\theta^{-1}}(\pi^\theta(T))$ $(\pi^{\theta^{-1}} := (\pi^\theta)^{-1})$.*

(iv) *$T_\Sigma = \bigcup_{\varphi\in T_\Sigma} \varphi/\theta_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, i.e., T_Σ is a union of θ_Σ-equivalence classes, for all $\Sigma \in |\mathbf{Sign}|$.*

As for the kernel of an N-morphism, we have:

Lemma 2 *Let $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ and $\boldsymbol{B} = \langle \mathbf{Sign}'', \mathrm{SEN}'', N''\rangle$ be N-algebraic systems and $\langle H,\gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$ an N-morphism.*

(1) *For all sentence families T of SEN', $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T iff $\gamma_\Sigma^{-1}(\gamma_\Sigma(T_\Sigma)) = T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$.*

(2) *For all $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, if $\mathrm{Ker}(\langle H,\gamma\rangle) \le \theta$, then $\gamma_\Sigma^{-1}(\gamma_\Sigma(\theta_\Sigma)) = \theta_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$.*

Consider again an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Given a sentence family T of SEN, there always exists a largest congruence system on $\boldsymbol{A}$ that is compatible with T (Proposition 2.2. of [33]). It is called the **Leibniz congruence system** of T on $\boldsymbol{A}$ and denoted $\Omega^{\boldsymbol{A}}(T) = \{\Omega_\Sigma^{\boldsymbol{A}}(T)\}_{\Sigma\in|\mathbf{Sign}|}$.

Given a collection $\mathcal{T}$ of sentence families of SEN, there always exists a largest congruence system on $\boldsymbol{A}$ that is compatible with every $T \in \mathcal{T}$. This is termed the **Tarski congruence system** of $\mathcal{T}$ on $\boldsymbol{A}$ and denoted by $\widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{T})$.

A π-**institution**[2] $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$ consists of

- an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and

[2] This is the same as a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, augmented with a category N of natural transformations on its sentence functor SEN, in traditional CAAL.

- a **closure system** C on SEN, i.e., a family of closure operators $C = \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ that satisfy, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,

$$\mathrm{SEN}(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)), \text{ for all } \Phi \subseteq \mathrm{SEN}(\Sigma),$$

  a condition known as **structurality**.

Given a π-institution $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$, a sentence family (system) $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of SEN is called a **theory family** (**system**) if each $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$ is a Σ-**theory**, i.e., a closed set under C: $C_\Sigma(T_\Sigma) = T_\Sigma$. The collection of all theory families of $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$. Ordered by signature wise inclusion $\leq$, the collection of all theory families forms a complete lattice that is denoted by $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq\rangle$.

Let $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$ be a π-institution. As a special case of the definition of the Tarski congruence system of a collection of sentence families, we obtain the **Tarski congruence system of** $\mathcal{I}$, i.e., the largest congruence system that is compatible with every theory family $T \in \mathrm{ThFam}(\mathcal{I})$. Ordinarily, instead of the notation $\widetilde{\Omega}^{\boldsymbol{A}}(\mathrm{ThFam}(\mathcal{I}))$, we use the notation $\widetilde{\Omega}^{\boldsymbol{A}}(C)$ or $\widetilde{\Omega}(\mathcal{I})$ for this congruence system.

Consider again a π-institution $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$ and a theory family $T \in \mathrm{ThFam}(\mathcal{I})$. The **Suszko congruence system of** T **in** $\mathcal{I}$, denoted $\widetilde{\Omega}^{\mathcal{I}}(T)$, is the largest congruence system that is compatible with all $T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Taking after similar notation in AAL, this set is usually denoted by

$$(\mathrm{ThFam}(\mathcal{I}))^T = \{T' \in \mathrm{ThFam}(\mathcal{I}) : T \leq T'\}.$$

Therefore, $\widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\Omega}^{\boldsymbol{A}}((\mathrm{ThFam}(\mathcal{I}))^T)$.

In summary, the three congruence systems $\Omega^{\boldsymbol{A}}(T)$, $\widetilde{\Omega}^{\mathcal{I}}(T)$ and $\widetilde{\Omega}^{\boldsymbol{A}}(C)$ are related by

$$\widetilde{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega^{\boldsymbol{A}}(T') : T' \in \mathrm{ThFam}(\mathcal{I}), T \leq T'\}$$

and

$$\widetilde{\Omega}(\mathcal{I}) = \bigcap\{\Omega^{\boldsymbol{A}}(T) : T \in \mathrm{ThFam}(\mathcal{I})\}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ an N-algebraic system. A pair $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ is an (**interpreted**) N-**algebraic system**[3] if $\boldsymbol{A}$ is an N-algebraic system and $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is an N-morphism.

[3]Hopefully, the overloading of terminology will not cause any confusion.

Let $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{B} = \langle \boldsymbol{B}, \langle G, \beta\rangle\rangle$ be two interpreted N-algebraic systems. An N-morphism $\langle H, \gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$ is called an **N-morphism from $\mathcal{A}$ to $\mathcal{B}$**, denoted $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$, if the following triangle commutes:

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle G,\beta\rangle"] & \\ \mathrm{SEN}' \arrow[rr, "\langle H,\gamma\rangle"'] & & \mathrm{SEN}'' \end{tikzcd}$$

Such an N-morphism is said to be **surjective** if both $H : \mathbf{Sign}' \to \mathbf{Sign}''$ and all $\gamma_{\Sigma'} : \mathrm{SEN}'(\Sigma') \to \mathrm{SEN}''(H(\Sigma'))$, $\Sigma' \in |\mathbf{Sign}'|$, are surjective.

An **N-matrix system** $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ is a pair consisting of an N-algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ and a sentence family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ of SEN'. An (**interpreted**) **N-matrix system**[3] $\mathfrak{A} = \langle \mathcal{A}, T'\rangle$ is a pair consisting of an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ and a sentence family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ of SEN'.

Fix a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, referred to as the **base π-institution**.[4] Then an interpreted N-matrix system $\mathfrak{A} = \langle \mathcal{A}, T'\rangle$ is called an **$\mathcal{I}$-matrix system** if T' is an **$\mathcal{I}$-filter family** of $\mathcal{A}$, i.e., for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$, and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}.$$

We denote by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ the collection of all $\mathcal{I}$-filter families of $\mathcal{A}$. Ordered by signature-wise inclusion $\leq$, this collection becomes a complete lattice, denoted by $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \leq\rangle$. Keeping in line with previously introduced notation, given $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we set

$$(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T'} = \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : T' \leq T''\}.$$

The following lemma provides some preservation properties of $\mathcal{I}$-filter families under the application of N-morphisms between the underlying N-algebraic systems.

Lemma 3 *Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution, $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle, \mathcal{B} = \langle \boldsymbol{B}, \langle G, \beta\rangle\rangle$ be N-algebraic systems, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ and $\boldsymbol{B} = \langle \mathbf{Sign}'', \mathrm{SEN}'', N''\rangle$, $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ an N-morphism and T'' a sentence family of $\mathcal{B}$.*

[4]The qualifying "base" is omitted whenever $\mathcal{I}$ is considered fixed in a specific context.

1. *If* $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, *then* $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.
2. *If* $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *then* $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.
3. *If* $\langle H, \gamma \rangle$ *is such that* H *is an isomorphism, and* $\mathrm{Ker}(\langle H, \gamma \rangle)$ *is compatible with* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *then* $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.

Proof:

1. Suppose $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$ and $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \gamma^{-1}_{F(\Sigma')}(T''_{H(F(\Sigma'))})$.

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dl, "{\langle F,\alpha\rangle}"'] \arrow[dr, "{\langle G,\beta\rangle}"] & \\ \mathrm{SEN}' \arrow[rr, "{\langle H,\gamma\rangle}"'] & & \mathrm{SEN}'' \end{tikzcd}$$

This holds iff

$$\begin{aligned} &\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq T''_{H(F(\Sigma'))} \\ &\qquad \text{iff} \quad \beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T''_{G(\Sigma')} \\ &\qquad \text{implies} \quad \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T''_{G(\Sigma')} \\ &\qquad \text{iff} \quad \gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))) \in T''_{H(F(\Sigma'))} \\ &\qquad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in \gamma^{-1}_{F(\Sigma')}(T''_{H(F(\Sigma'))}). \end{aligned}$$

2. Suppose $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$ and $\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T''_{G(\Sigma')}$. This holds iff

$$\begin{aligned} &\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq T''_{H(F(\Sigma'))} \\ &\qquad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \gamma^{-1}_{F(\Sigma')}(T''_{H(F(\Sigma'))}) \\ &\qquad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in \gamma^{-1}_{F(\Sigma')}(T''_{H(F(\Sigma'))}) \\ &\qquad \text{iff} \quad \gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))) \in T''_{H(F(\Sigma'))} \\ &\qquad \text{iff} \quad \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T''_{G(\Sigma')}. \end{aligned}$$

3. Note that compatibility of $\mathrm{Ker}(\langle H, \gamma \rangle)$ with $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ implies that, for all $\Sigma \in |\mathbf{Sign}'|$, $\gamma^{-1}_\Sigma(\gamma_\Sigma(T'_\Sigma)) = T'_\Sigma$, or, more compactly, $\gamma^{-1}(\gamma(T')) = T'$. Now assume $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such

that $\varphi \in C_\Sigma(\Phi)$ and $\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \gamma_{F(\Sigma')}(T'_{F(\Sigma')})$. This holds iff

$$\begin{array}{ll} \gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq \gamma_{F(\Sigma')}(T'_{F(\Sigma')}) & \\ \quad\text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \gamma^{-1}_{F(\Sigma')}(\gamma_{F(\Sigma')}(T'_{F(\Sigma')})) & \\ \quad\text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} & \\ \quad\text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T'_{F(\Sigma')} & \\ \quad\text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in \gamma^{-1}_{F(\Sigma')}(\gamma_{F(\Sigma')}(T'_{F(\Sigma')})) & \\ \quad\text{iff} \quad \gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))) \in \gamma_{F(\Sigma')}(T'_{F(\Sigma')}) & \\ \quad\text{iff} \quad \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in \gamma_{F(\Sigma')}(T'_{F(\Sigma')}). & \end{array}$$

■

Similar concepts and terminology may be applied to the so-called generalized matrix systems or gmatrix systems for short. An N-**gmatrix system** $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{T}' \rangle$ is a pair consisting of an N-algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ and a collection of sentence families $\mathcal{T}'$ of SEN'. An (**interpreted**) N-**gmatrix system**[3] $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}' \rangle$ is a pair consisting of an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ and a collection of sentence families $\mathcal{T}'$ of SEN'. An $\mathcal{I}$-**gmatrix system** $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}' \rangle$ is a tuple, such that every sentence family in $\mathcal{T}'$ is an $\mathcal{I}$-filter family of $\mathcal{A}$.

Note that, given an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$, the pair $\mathcal{I}' = \langle \boldsymbol{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is also a π-institution (in closure system form). In accordance, we define the **Suszko congruence of** $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, denoted $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T')$ by

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T') = \widetilde{\Omega}^{\mathcal{I}'}(T') = \bigcap \{ \Omega^{\boldsymbol{A}}(T'') : T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), T' \leq T'' \}.$$

We also extend the notation $\Omega^{\boldsymbol{A}}(T')$ and $\Omega^{\boldsymbol{A}}(\mathcal{T}')$ to interpreted N-algebraic systems, writing $\Omega^{\mathcal{A}}(T')$ and $\Omega^{\mathcal{A}}(\mathcal{T}')$, with the meaning that these are identical to those applied to the underlying N-algebraic system $\boldsymbol{A}$ of $\mathcal{A}$. The restriction of $\Omega^{\mathcal{A}}$ to $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is the **Leibniz operator on** $\mathcal{A}$. The restriction of $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ to $\mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$ is the **Suszko operator on** $\mathcal{A}$ and the restriction of $\widetilde{\Omega}^{\mathcal{A}}$ on $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is the **Tarski operator on** $\mathcal{A}$. The families

$$\begin{array}{rcl} \Omega & = & \{\Omega^{\mathcal{A}} : \mathcal{A} \text{ an } N\text{-algebraic system}\} \\ \widetilde{\Omega}^{\mathcal{I}} := \widetilde{\Omega}^{\bullet,\mathcal{I}} & = & \{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}} : \mathcal{A} \text{ an } N\text{-algebraic system}\} \\ \widetilde{\Omega} & = & \{\widetilde{\Omega}^{\mathcal{A}} : \mathcal{A} \text{ an } N\text{-algebraic system}\} \end{array}$$

are termed the **Leibniz**, the **Suszko** and the **Tarski operator**, respectively. Saying that one of those **has a property** P **globally** means that property P holds for every member of the family. E.g., the Leibniz operator is globally order preserving if $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\boldsymbol{A})$ is order preserving, for every N-algebraic system $\mathcal{A}$.

Concerning these operators, we have

Proposition 4 *Let $\mathcal{I}$ be a π-institution, $\mathcal{A},\mathcal{B}$ two N-algebraic systems and $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective N-morphism. For all $\mathcal{T}'' \cup \{T''\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,*

1. $\gamma^{-1}(\Omega^{\mathcal{B}}(T'')) = \Omega^{\mathcal{A}}(\gamma^{-1}(T''))$;
2. $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{B}}(\mathcal{T}'')) = \widetilde{\Omega}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}''))$.
3. $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}(T'')) = \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}((\gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})))^{\gamma^{-1}(T'')})$.

Proof: Property 1 is a well-known property of the categorical Leibniz operator (see, e.g., Lemma 5.4 of [33]). For Property 2,

$$\begin{array}{rcl}\gamma^{-1}(\widetilde{\Omega}^{\mathcal{B}}(\mathcal{T}'')) & = & \gamma^{-1}(\bigcap_{T''\in\mathcal{T}''}\Omega^{\mathcal{B}}(T'')) = \bigcap_{T''\in\mathcal{T}''}\gamma^{-1}(\Omega^{\mathcal{B}}(T'')) \\ & = & \bigcap_{T''\in\mathcal{T}''}\Omega^{\mathcal{A}}(\gamma^{-1}(T'')) = \widetilde{\Omega}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}'')).\end{array}$$

Finally, for Property 3, it suffices to notice that, because of surjectivity,

$$\gamma^{-1}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T''}) = (\gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})))^{\gamma^{-1}(T'')}$$

and, then, take advantage of Property 2. ■

4 Full Models, Algebras and the Hierarchy

The original definition of a full model in AAL was given by Font and Jansana in [17] and, it was, subsequently, adapted in CAAL in [30].

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$, with $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be a π-institution and $\mathcal{A} = \langle \boldsymbol{A}, \langle F,\alpha\rangle\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, an N-algebraic system. A collection $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is **full** if

$$\mathcal{T}' = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \leq \Omega^{\mathcal{A}}(T')\},$$

i.e., $\mathcal{T}'$ consists of all $\mathcal{I}$-filter families on $\mathcal{A}$ with which the Tarski congruence system $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ of $\mathcal{T}'$ is compatible.

If $\mathcal{T}'$ is full, then $\mathcal{T}'$ is a closure system on $\mathcal{A}$, whence the pair $\mathcal{I}' = \langle \boldsymbol{A}, \mathcal{T}'\rangle$ is a π-institution. We use the terminology **full $\mathcal{I}$-gmatrix system** for $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$ when $\mathcal{T}'$ is a full collection of $\mathcal{I}$-filter families.

Using the CAAL notion of a quotient algebraic system $\boldsymbol{A}/\theta = \boldsymbol{A}^{\theta} = \langle \mathbf{Sign}, \mathrm{SEN}^{\theta}, N^{\theta}\rangle$ of a given algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ modulo a congruence system θ on $\boldsymbol{A}$ [29], we may give several characterizations of full $\mathcal{I}$-gmatrix systems that parallel results from AAL (Proposition 2.7 of [1]).

Proposition 5 *Let* $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ *be an* N*-algebraic system, with* $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, let* $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\langle I_{\mathbf{Sign}'}, \pi\rangle : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ *be the canonical projection* N*-morphism. Then the following conditions are equivalent:*

(i) $\mathcal{T}'$ *is full.*

(ii) $\pi(\mathcal{T}') = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}'))$.

(iii) $\mathcal{T}' = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')))$.

(iv) $\mathcal{T}' = \gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$ *for some* N*-algebraic system* $\mathcal{B}$ *and some surjective* N*-morphism* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism.*

Proof:

(i)⇒(ii) Suppose that $\mathcal{T}'$ is full.

$$\begin{tikzcd} & \mathbf{F} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle F,\pi_F\circ\alpha\rangle"] & \\ \boldsymbol{A} \arrow[rr, "\langle I_{\mathbf{Sign}'},\pi\rangle"'] & & \boldsymbol{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \end{tikzcd}$$

If $T' \in \mathcal{T}'$, then we have $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \leq \Omega^{\mathcal{A}}(T')$. Thus, $\mathrm{Ker}(\langle I_{\mathbf{Sign}'}, \pi\rangle) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ is compatible with T', and, hence, by Lemma 3, $\pi(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}'))$. If, conversely, $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}'))$, consider $\pi^{-1}(T'')$. It is not difficult to see that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ is compatible with $\pi^{-1}(T'')$, whence, since $\mathcal{T}'$ is full, $\pi^{-1}(T'') \in \mathcal{T}'$. Moreover, $\pi(\pi^{-1}(T'')) = T''$ by surjectivity and, therefore, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')) \subseteq \pi(\mathcal{T}')$.

(ii)⇒(iii) Since every filter family $T' \in \mathcal{T}'$ is compatible with $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$, it follows that $\pi^{-1}(\pi(\mathcal{T}')) = \mathcal{T}'$, whence the hypothesis yields the conclusion.

(iii)⇒(iv) Obvious.

(iv)⇒(i) The inclusion $\mathcal{T}' \subseteq \{T' : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \leq \Omega^{\mathcal{A}}(T')\}$ is universally valid, since $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ is compatible with every $T' \in \mathcal{T}'$. For the converse, we note that the hypothesis that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ is compatible with every $T' \in \gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$ implies that there exists $\langle H, \widetilde{\gamma}\rangle : \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \to$

$\mathcal{B}/\widetilde{\Omega}^{\mathcal{B}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\langle H,\gamma\rangle"] \arrow[dd, "\langle I,\pi\rangle"'] & & \mathcal{B} \arrow[dd, "\langle I,\pi^{\mathcal{B}}\rangle"] \\
& \boldsymbol{F} \arrow[ul, "\langle F,\alpha\rangle"] \arrow[ur, "\langle HF,\gamma_F,\alpha\rangle"'] \arrow[dl, "\langle F,\pi_F,\alpha\rangle"] \arrow[dr, "\langle HF,\pi^{\mathcal{B}}_{HF}\gamma_F\alpha\rangle"'] & \\
\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \arrow[rr, "\langle H,\widetilde{\gamma}\rangle"'] & & \mathcal{B}/\widetilde{\Omega}^{\mathcal{B}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))
\end{tikzcd}$$

Now diagram chasing gives that, if $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ is compatible with T', then $T' \in \gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})) = \mathcal{T}'$ and, hence, $\mathcal{T}'$ is full. ■

Given two N-matrix systems $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ and $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$, an N-**matrix system morphism** $\langle H,\gamma\rangle : \mathfrak{A} \to \mathfrak{B}$ is a N-morphism $\langle H,\gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$, such that $\gamma^{-1}(T'') \leq T'$. It is called **strict** if $\gamma^{-1}(T'') = T'$. These definitions extend to interpreted systems with the proviso that N-morphisms must be replaced by morphisms between interpreted systems, i.e., algebraic morphisms commuting with the interpretations.

A N-matrix system $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ is said to be **Leibniz reduced** or simply **reduced** if $\Omega^{\boldsymbol{A}}(T') = \Delta^{\mathrm{SEN}'}$, where $\Delta^{\mathrm{SEN}'}$ is the identity congruence system of $\boldsymbol{A}$. This terminology applies also to interpreted N-matrix systems and to $\mathcal{I}$-matrix systems.

A gmatrix system $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{T}'\rangle$ is **Tarski reduced** or simply **reduced** if $\widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{T}') = \Delta^{\mathrm{SEN}'}$. This terminology also extends to interpreted N-gmatrix systems and to $\mathcal{I}$-gmatrix systems.

Finally, we call an $\mathcal{I}$-matrix system $\mathbb{A} = \langle \mathcal{A}, T'\rangle$ **Suszko reduced** if $\widetilde{\Omega}^{\boldsymbol{A},\mathcal{I}}(T') = \Delta^{\mathrm{SEN}'}$.

By analogy with the universal algebraic framework, reduced $\mathcal{I}$-matrix systems, Suszko reduced $\mathcal{I}$-matrix systems and Tarski reduced $\mathcal{I}$-gmatrix systems give rise to natural classes of N-algebraic systems that are associated to a given base π-institution $\mathcal{I}$.

$$\begin{array}{rcl}
\mathsf{AlgSys}^*(\mathcal{I}) & = & \{\mathcal{A} : (\exists T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T') = \Delta^{\mathrm{SEN}'})\} \\
\mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I}) & = & \{\mathcal{A} : (\exists T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T') = \Delta^{\mathrm{SEN}'})\} \\
\mathsf{AlgSys}(\mathcal{I}) & = & \{\mathcal{A} : (\exists \mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') = \Delta^{\mathrm{SEN}'})\} \\
& = & \{\mathcal{A} : \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}\}.
\end{array}$$

Analogously with the corresponding AAL classes and accompanying results, established in [4, 9, 17], we may obtain the following characterizations of these classes ($\mathbf{I}$ denotes the isomorphic copies operator for interpreted N-algebraic systems):

Lemma 6 *Let $\mathcal{I}$ be a π-institution.*

1. $\mathsf{AlgSys}^*(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\Omega^{\mathcal{A}}(T) : \mathcal{A}\ N\text{-}alg\ system, T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})$.
2. $\mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) : \mathcal{A}\ N\text{-}alg\ system, T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})$.
3. $\mathsf{AlgSys}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A}\ N\text{-}alg\ system, \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\ full\})$.
4. $\mathsf{AlgSys}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A}\ N\text{-}alg\ system, \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})$.
5. $\mathsf{AlgSys}(\mathcal{I}) = \mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I})$.

Adopting the operator approach in defining the main classes of a categorical abstract algebraic hierarchy of π-institutions, we have:

Definition 7 *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ a π-institution based on $\boldsymbol{F}$.*

- *$\mathcal{I}$ is* **protoalgebraic** ([3] in AAL and [33] in CAAL) *if Ω is globally order-preserving.*
- *$\mathcal{I}$ is* **equivalential** ([6, 7] in AAL and [35] in CAAL) *if Ω is globally order preserving and commutes with inverse images of N-morphisms.*
- *$\mathcal{I}$ is* **truth-equational** ([27] in AAL and [36] in CAAL) *if Ω is globally completely order reflecting.*
- *$\mathcal{I}$ is* **weakly algebraizable** ([10] in AAL and [37] in CAAL) *if it is protoalgebraic and truth-equational.*
- *$\mathcal{I}$ is* **algebraizable** ([4, 22] in AAL and [28] in CAAL) *if it is equivalential and truth-equational.*

These definitions preserved the structure of the AAL Leibniz hierarchy:

$$\begin{tikzcd}
 & \text{algebraizable} \arrow[dl] \arrow[dr] & & \\
\text{equivalential} \arrow[dr] & & \text{weakly algebraizable} \arrow[dl] \arrow[dr] & \\
 & \text{protoalgebraic} & & \text{truth-equational}
\end{tikzcd}$$

5 $\mathcal{I}$-Operators

Taking after [1], we define and study arbitrary $\mathcal{I}$-operators, which correspond in the CAAL framework to arbitrary $\mathcal{S}$-operators in AAL.

Definition 8 *Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a base π-institution and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an N-algebraic system.*

- *An $\mathcal{I}$-**operator** on $\mathcal{A}$ is a map $\nabla^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathbf{A})$.*

  *The $\mathcal{I}$-operator $\nabla^{\mathcal{A}}$ is called **order-preserving** if, for all $T', T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

  $$T' \leq T'' \quad \textit{implies} \quad \nabla^{\mathcal{A}}(T') \leq \nabla^{\mathcal{A}}(T'').$$

- *The **lifting** $\widetilde{\nabla}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{ConSys}(\mathbf{A})$ of $\nabla^{\mathcal{A}}$ is defined by*

  $$\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}') = \bigcap\{\nabla^{\mathcal{A}}(T') : T' \in \mathcal{T}'\}, \textit{ for all } \mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

- *The **relativization** $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathbf{A})$ of $\nabla^{\mathcal{A}}$ is defined, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by*

  $$\begin{array}{rcl}\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(T') & = & \bigcap\{\nabla^{\mathcal{A}}(T'') : T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), T' \leq T''\} \\ & = & \widetilde{\nabla}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T'}).\end{array}$$

- *The map $\nabla^{\mathcal{A}^{-1}} : \mathrm{ConSys}(\mathbf{A}) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is defined by*

  $$\nabla^{\mathcal{A}^{-1}}(\theta) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \leq \nabla^{\mathcal{A}}(T')\}, \textit{ for all } \theta \in \mathrm{ConSys}(\mathbf{A}).$$

Directly from these definitions we obtain

Lemma 9 *Let $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ be a base π-institution, $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ an N-algebraic system and $\nabla^{\mathcal{A}}$ an $\mathcal{I}$-operator on $\mathcal{A}$.*

1. $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}$ *is also an $\mathcal{I}$-operator.*
2. $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(T') \leq \nabla^{\mathcal{A}}(T')$*, for all* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.
3. $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}$ *is order-preserving.*
4. $\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}') \leq \nabla^{\mathcal{A}}(T')$*, for all* $T' \in \mathcal{T}'$.

By analogy with [1], the categorical Leibniz and Suszko operators are the prototypical examples of the general notion of $\mathcal{I}$-operator. The Suszko operator is the relativization of the Leibniz operator and is order-preserving. Finally, the Tarski operator is the lifting of the Leibniz operator.

In all subsequent results, when we say "let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-operator on $\mathcal{A}$" or quantify "for all $\mathcal{A}$", we implicitly make the assumption that $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is a base algebraic system, $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is a π-institution based on $\boldsymbol{F}$ and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ an N-algebraic system.

Proposition 10 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-operator on $\mathcal{A}$. The maps $\widetilde{\nabla}^{\mathcal{A}}$ and $\nabla^{\mathcal{A}^{-1}}$ establish a Galois connection between $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ and $\mathrm{ConSys}(\boldsymbol{A})$ with the first ordering being the subset relation $\subseteq$ and the second the signature-wise inclusion relation $\leq$.*

Proof: Suppose $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\theta \in \mathrm{ConSys}(\boldsymbol{A})$.

- Assume $\theta \leq \widetilde{\nabla}^{\mathcal{A}}(\mathcal{T})$. If $T \in \mathcal{T}$, then $\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}) \leq \nabla^{\mathcal{A}}(T)$. Thus, $\theta \leq \nabla^{\mathcal{A}}(T)$, whence $T \in \nabla^{\mathcal{A}^{-1}}(\theta)$. This proves that $\mathcal{T} \subseteq \nabla^{\mathcal{A}^{-1}}(\theta)$.
- If $\mathcal{T} \subseteq \nabla^{\mathcal{A}^{-1}}(\theta)$, then $\theta \leq \nabla^{\mathcal{A}}(T)$, for all $T \in \mathcal{T}$. Thus, $\theta \leq \widetilde{\nabla}^{\mathcal{A}}(\mathcal{T})$. ■

Applying now general results pertaining to Galois connections (see, e.g., p. 55 onwards of [14]), we may obtain the following statements as direct consequences of Proposition 10.

Corollary 11 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-operator on $\mathcal{A}$.*

1. *The maps* $\widetilde{\nabla}^{\mathcal{A}}$ *and* $\nabla^{\mathcal{A}^{-1}}$ *are order-reversing.*
2. *The map* $\nabla^{\mathcal{A}^{-1}} \circ \widetilde{\nabla}^{\mathcal{A}}$ *is a closure operator over* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

3. *The map* $\widetilde{\nabla}^{\mathcal{A}} \circ \nabla^{\mathcal{A}^{-1}}$ *is a closure operator on* $\mathrm{ConSys}(\boldsymbol{A})$.

4. *The set of fixed points of* $\nabla^{\mathcal{A}^{-1}} \circ \widetilde{\nabla}^{\mathcal{A}}$ *is* $\mathrm{Ran}(\nabla^{\mathcal{A}^{-1}})$.

5. *The set of fixed points of* $\widetilde{\nabla}^{\mathcal{A}} \circ \nabla^{\mathcal{A}^{-1}}$ *is* $\mathrm{Ran}(\widetilde{\nabla}^{\mathcal{A}})$.

6. *The maps* $\widetilde{\nabla}^{\mathcal{A}}$ *and* $\nabla^{\mathcal{A}^{-1}}$ *restrict to mutually inverse dual order isomorphisms between the set of fixed points of* $\nabla^{\mathcal{A}^{-1}} \circ \widetilde{\nabla}^{\mathcal{A}}$ *and the set of fixed points of* $\widetilde{\nabla}^{\mathcal{A}} \circ \nabla^{\mathcal{A}^{-1}}$.

We assign special names to the fixed points of the closure operators of Parts 2 and 3 of the preceding corollary. Both will be central to our subsequent analysis and to many of our results.

Definition 12 *Given an* $\mathcal{I}$*-operator* $\nabla^{\mathcal{A}}$ *on* $\mathcal{A}$,

- *a family* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is* $\nabla^{\mathcal{A}}$-**full** *if* $\mathcal{T} = \nabla^{\mathcal{A}^{-1}}(\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}))$;
- *a congruence system* $\theta \in \mathrm{ConSys}(\boldsymbol{A})$ *is* $\nabla^{\mathcal{A}}$-**full** *if* $\theta = \widetilde{\nabla}^{\mathcal{A}}(\nabla^{\mathcal{A}^{-1}}(\theta))$.

Then, Part 6 of the corollary asserts that $\widetilde{\nabla}^{\mathcal{A}}$ and $\nabla^{\mathcal{A}^{-1}}$ restrict to mutually inverse dual order isomorphisms between the sets of $\nabla^{\mathcal{A}}$-full $\mathcal{I}$-gmatrices on $\mathcal{A}$ and $\nabla^{\mathcal{A}}$-full congruence systems on $\boldsymbol{A}$.

Another consequence of the previously described Galois connection is the following

Proposition 13 *Let* $\nabla^{\mathcal{A}}$ *be an* $\mathcal{I}$*-operator on* $\mathcal{A}$.

1. $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is* $\nabla^{\mathcal{A}}$*-full iff it is the largest* $\mathcal{U} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *such that* $\widetilde{\nabla}^{\mathcal{A}}(\mathcal{U}) = \widetilde{\nabla}^{\mathcal{A}}(\mathcal{T})$.

2. $\theta \in \mathrm{ConSys}(\boldsymbol{A})$ *is* $\nabla^{\mathcal{A}}$*-full iff it is the largest* $\eta \in \mathrm{ConSys}(\boldsymbol{A})$, *such that* $\nabla^{\mathcal{A}^{-1}}(\eta) = \nabla^{\mathcal{A}^{-1}}(\theta)$.

Focusing, next, on the Leibniz operator $\Omega^{\mathcal{A}}$, whose lifting is the Tarski operator $\widetilde{\Omega}^{\mathcal{A}}$, we note, first, that, if $\theta \in \mathrm{ConSys}(\boldsymbol{A})$,

$$\begin{array}{rcl}\Omega^{\mathcal{A}^{-1}}(\theta) & = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \leq \Omega^{\mathcal{A}}(T)\} \\ & = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : T \ \mathsf{comp}\ \theta\} \\ & \subseteq & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).\end{array}$$

We obtain

Proposition 14 *Let* $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ *be an* N*-algebraic system, with* $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$, $\theta \in \mathrm{ConSys}(\boldsymbol{A})$ *and* $\langle I_{\mathbf{Sign}}, \pi \rangle \coloneqq \langle I_{\mathbf{Sign}}, \pi^\theta \rangle : \mathrm{SEN} \to \mathrm{SEN}^\theta$ *the corresponding projection* N*-morphism.*

1. $\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$ *and* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\Omega^{\mathcal{A}^{-1}}(\theta))$.

2. *The natural transformations* $\pi : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}^\theta$ *and* $\pi^{-1} : \mathcal{P}\mathrm{SEN}^\theta \to \mathcal{P}\mathrm{SEN}$ *restrict to order-isomorphisms between the sets* $\Omega^{\mathcal{A}^{-1}}(\theta)$ *and* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$.

Proof:

1. Suppose that $T \in \Omega^{\mathcal{A}^{-1}}(\theta)$. Then θ is compatible with T. By Lemma 3, $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$. Since, by compatibility, $T = \pi^{-1}(\pi(T))$, we obtain $T \in \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$.

   If, conversely, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, we get $\pi^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and, by the surjectivity of π, $T' = \pi(\pi^{-1}(T'))$. This implies $\pi^{-1}(T') = \pi^{-1}(\pi(\pi^{-1}(T')))$, showing that θ is compatible with $\pi^{-1}(T')$, or, equivalently, $\pi^{-1}(T') \in \Omega^{\mathcal{A}^{-1}}(\theta)$.

   Taking into account the surjectivity of π, we get the second equality.

2. By Part 1, both π and π^{-1} are onto their respective codomains. Note, in addition, that

   - by the surjectivity of π, $\pi\pi^{-1} = \mathrm{I}_{\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)}$ and
   - by the definition of $\Omega^{\mathcal{A}^{-1}}(\theta)$, $\pi^{-1}\pi = \mathrm{I}_{\Omega^{\mathcal{A}^{-1}}(\theta)}$.

   These show that π and π^{-1} are mutually inverse bijections and, therefore, being order preserving, must be order isomorphisms.

■

Taking into account that isomorphisms preserve least elements, we get

Corollary 15 *Let* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, *such that* θ *is compatible with* T. *Then* T *is the least element of* $\Omega^{\mathcal{A}^{-1}}(\theta)$ *iff* T/θ *is the least element of* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$.

Using the characterization of full $\mathcal{I}$-gmatrix systems of Proposition 5, we get

Corollary 16 *For all* $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, *the set* $\Omega^{\mathcal{A}^{-1}}(\theta)$ *is full and, hence, a closure system.*

Specifically for the $\Omega^{\mathcal{A}}$-full sets of $\mathcal{I}$-filter families and the $\Omega^{\mathcal{A}}$-full congruence systems on $\boldsymbol{A}$, we have the following characterizations, which form an abstraction to the categorical level of Propositions 3.10 and 3.11 of [1]:

Proposition 17 *Let* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\theta \in \mathrm{ConSys}(\boldsymbol{A})$.

- $\mathcal{T}$ *is* $\Omega^{\mathcal{A}}$*-full iff it is full.*
- θ *is* $\Omega^{\mathcal{A}}$*-full iff* $\theta \in \mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\boldsymbol{A})$.

Proof:

- In general, $\Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(\mathrm{T})\}$. On the other hand, $\mathcal{T}$ is $\Omega^{\mathcal{A}}$-full iff $\mathcal{T} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))$ and it is full iff $\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(\mathrm{T})\}$. Thus, the two notions coincide.

- Assume θ is $\Omega^{\mathcal{A}}$-full. Then, $\mathcal{A}/\theta = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\Omega^{\mathcal{A}^{-1}}(\theta))$. By Corollary 16, $\Omega^{\mathcal{A}^{-1}}(\theta)$ is full. By Part 3 of Lemma 6, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\Omega^{\mathcal{A}^{-1}}(\theta)) \in \mathsf{AlgSys}(\mathcal{I})$. Therefore, we conclude that $\theta \in \mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{A})}(\mathcal{A})$.

  If, conversely, $\mathcal{A}/\theta \in \mathsf{AlgSys}(\mathcal{I})$, then $\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \Delta^{\mathrm{SEN}'^{\theta}}$, whence

$$\begin{array}{rcl} \theta & = & \mathrm{Ker}(\langle I_{\mathbf{Sign}'}, \pi^{\theta}\rangle) = \pi^{\theta^{-1}}(\Delta^{\mathrm{SEN}'^{\theta}}) \\ & = & \pi^{\theta^{-1}}(\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) \\ & \stackrel{\text{Prop. 4}}{=} & \widetilde{\Omega}^{\mathcal{A}}(\pi^{\theta^{-1}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) \stackrel{\text{Prop. 14}}{=} \widetilde{\Omega}(\Omega^{\mathcal{A}^{-1}}(\theta)). \end{array}$$

  Hence θ is $\Omega^{\mathcal{A}}$-full.

■

Using Proposition 17 we obtain the following statement on the Galois connection established in Proposition 10 and Corollary 11 as pertaining to the special case of the Tarksi operator, viewed as the lifting of the Leibniz operator:

Corollary 18 *The maps* $\widetilde{\Omega}^{\mathcal{A}}$ *and* $\Omega^{\mathcal{A}^{-1}}$ *establish a Galois connection between* $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ *and* $\mathrm{ConSys}(\boldsymbol{A})$, *and they restrict to mutually inverse dual order isomorphisms between the poset of all full* $\mathcal{I}$*-gmatrix systems on* $\mathcal{A}$ *and the poset* $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\boldsymbol{A})$.

The isomorphism of Corollary 18 is actually the one established a decade ago as Theorem 13 of [31], taking after the Isomorphism Theorem 2.30 of

[17]. The form in Corollary 18, expressed in terms of Galois connections in the context of $\mathcal{I}$-operators, is Corollary 3.12 of [1].

Finally, putting together the equivalence between fullness of $\mathcal{I}$-gmatrix systems and $\nabla^{\mathcal{A}}$-fullness given in Proposition 17 and the general characterization of $\nabla^{\mathcal{A}}$-fullness given in Proposition 13, we obtain

Proposition 19 *A subset* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is full iff* $\mathcal{T}$ *is the largest* $\mathcal{U} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{U})$*.*

Leaving, once more, aside the special case of the Leibniz operator and returning to arbitrary $\mathcal{I}$-operators, and still following the ideas in [1], we introduce the concept of a $\nabla^{\mathcal{A}}$-class of a theory family T and, based on it, that of a $\nabla^{\mathcal{A}}$-filter family (see Subsection 3.3 of [1]).

Definition 20 *Let* $\nabla^{\mathcal{A}}$ *be an* $\mathcal{I}$*-operator on* $\mathcal{A}$ *and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. The* $\nabla^{\mathcal{A}}$**-class of** T *is the set*

$$[\![T]\!]^{\nabla^{\mathcal{A}}} = \Omega^{\mathcal{A}^{-1}}(\nabla^{\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

In other words, the $\nabla^{\mathcal{A}}$-class of a filter family T of $\mathcal{A}$ consists of all those filter families of $\mathcal{A}$ with which the $\nabla^{\mathcal{A}}$-congruence system of T is compatible.

Exploiting Corollary 16, with $\theta = \nabla^{\mathcal{A}}(T)$, we get

Proposition 21 *Let* $\nabla^{\mathcal{A}}$ *be an* $\mathcal{I}$*-operator on* $\mathcal{A}$ *and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. The* $\nabla^{\mathcal{A}}$*-class* $[\![T]\!]^{\nabla^{\mathcal{A}}}$ *of* T *is full. Thus, it is a closure system and* $[\![T]\!]^{\nabla^{\mathcal{A}}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\nabla^{\mathcal{A}}}))$*.*

As a consequence it makes sense to consider the $\leq$-smallest $\mathcal{I}$-filter family in the $\nabla^{\mathcal{A}}$-class of T:

Definition 22 *Given an* $\mathcal{I}$*-operator* $\nabla^{\mathcal{A}}$ *on* $\mathcal{A}$ *and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, the smallest element of the* $\nabla^{\mathcal{A}}$*-class* $[\![T]\!]^{\nabla^{\mathcal{A}}}$ *is denoted by* $T^{\nabla^{\mathcal{A}}} = \bigcap [\![T]\!]^{\nabla^{\mathcal{A}}}$*. We call* T *a* $\nabla^{\mathcal{A}}$**-filter family** *if* $T = T^{\nabla^{\mathcal{A}}}$ *and we denote the set of all* $\nabla^{\mathcal{A}}$*-filter families of* $\mathcal{A}$ *by* $\mathrm{FiFam}^{\nabla^{\mathcal{A}}}(\mathcal{A})$*.*

The first result asserts the injectivity of the $\mathcal{I}$-operator $\nabla^{\mathcal{A}}$ on the collection of $\nabla^{\mathcal{A}}$-filter families:

Proposition 23 *Every* $\mathcal{I}$*-operator* $\nabla^{\mathcal{A}}$ *on* $\mathcal{A}$ *is order-reflecting and, thus, injective, on* $\mathrm{FiFam}^{\nabla^{\mathcal{A}}}(\mathcal{A})$*.*

Proof: Suppose $T', T'' \in \mathrm{FiFam}^{\nabla^{\mathcal{A}}}(\mathcal{A})$, with $\nabla^{\mathcal{A}}(T') \leq \nabla^{\mathcal{A}}(T'')$. Then, clearly, $[\![T'']\!]^{\nabla^{\mathcal{A}}} \subseteq [\![T']\!]^{\nabla^{\mathcal{A}}}$, whence $T' = \bigcap [\![T']\!]^{\nabla^{\mathcal{A}}} \leq \bigcap [\![T'']\!]^{\nabla^{\mathcal{A}}} = T''$. ∎

One can now state the following properties relating to $\nabla^{\mathcal{A}}$-filter families:

Lemma 24 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-operator on $\mathcal{A}$. For all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

1. $[\![T]\!]^{\nabla^{\mathcal{A}}} \subseteq (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T^{\nabla^{\mathcal{A}}}}$*;*
2. *If $[\![T]\!]^{\nabla^{\mathcal{A}}} \subseteq [\![T']\!]^{\nabla^{\mathcal{A}}}$, then $T'^{\nabla^{\mathcal{A}}} \leq T^{\nabla^{\mathcal{A}}}$.*

If, moreover, $\nabla^{\mathcal{A}}$ is order-preserving, then:

3. *If $T \leq T'$, then $[\![T']\!]^{\nabla^{\mathcal{A}}} \subseteq [\![T]\!]^{\nabla^{\mathcal{A}}}$ and $T^{\nabla^{\mathcal{A}}} \leq T'^{\nabla^{\mathcal{A}}}$.*
4. $(\mathrm{FiFam}^{\nabla^{\mathcal{A}}}(\mathcal{A}))^{T} \subseteq [\![T]\!]^{\nabla^{\mathcal{A}}}$*;*

Proof: Part 1 follows from $T^{\nabla^{\mathcal{A}}} = \bigcap [\![T]\!]^{\nabla^{\mathcal{A}}}$. For Part 2, we have $T'^{\nabla^{\mathcal{A}}} = \bigcap [\![T']\!]^{\nabla^{\mathcal{A}}} \leq \bigcap [\![T]\!]^{\nabla^{\mathcal{A}}} = T^{\nabla^{\mathcal{A}}}$. For Part 3, taking into account order preservation, if $T \leq T'$, then $\nabla^{\mathcal{A}}(T) \leq \nabla^{\mathcal{A}}(T')$, whence $[\![T']\!]^{\nabla^{\mathcal{A}}} \subseteq [\![T]\!]^{\nabla^{\mathcal{A}}}$ and $T^{\nabla^{\mathcal{A}}} \leq T'^{\nabla^{\mathcal{A}}}$. For Part 4, suppose $T \leq T' = \bigcap [\![T']\!]^{\nabla^{\mathcal{A}}}$. By order preservation and by Proposition 21, $\nabla^{\mathcal{A}}(T) \leq \nabla^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T')$, whence $T' \in [\![T]\!]^{\nabla^{\mathcal{A}}}$. ∎

In concluding this section on $\mathcal{I}$-operators and their properties, we prove a lemma, relating the property of a filter family being a $\nabla^{\mathcal{A}}$-filter family with the form of its $\nabla^{\mathcal{A}}$-class, for an order preserving $\mathcal{I}$-operator $\nabla^{\mathcal{A}}$ that is dominated by the Leibniz operator on $\mathcal{A}$. Lemma 25 also helps usher in the material of Section 6.

Lemma 25 *Let $\nabla^{\mathcal{A}}$ be an order-preserving $\mathcal{I}$-operator, such that $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then $[\![T]\!]^{\nabla^{\mathcal{A}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T}$ iff $T = T^{\nabla^{\mathcal{A}}}$, i.e., iff T is a $\nabla^{\mathcal{A}}$-filter family.*

Proof: Suppose, first, that $[\![T]\!]^{\nabla^{\mathcal{A}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T}$. Then, we have $T^{\nabla^{\mathcal{A}}} = \bigcap [\![T]\!]^{\nabla^{\mathcal{A}}} = \bigcap (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T} = T$.

Conversely, if $T = T^{\nabla^{\mathcal{A}}}$, then by Part 1 of Lemma 24, we have $[\![T]\!]^{\nabla^{\mathcal{A}}} \subseteq (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T}$. On the other hand, if $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, then, by the hypotheses, $\nabla^{\mathcal{A}}(T) \leq \nabla^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T')$, whence $T' \in [\![T]\!]^{\nabla^{\mathcal{A}}}$. ∎

6 $\mathcal{I}$-Compatibility Operators and Coherence

We focus next on $\mathcal{I}$-operators that associate to a given filter family on an algebraic system $\mathcal{A}$ a congruence system that is compatible with the filter family. In the AAL context of [1], such operators are termed *$\mathcal{S}$-compatibility operators*.

Definition 26 *An $\mathcal{I}$-**compatibility operator** on $\mathcal{A}$ is an $\mathcal{I}$-operator $\nabla^{\mathcal{A}}$ on $\mathcal{A}$, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the congruence system $\nabla^{\mathcal{A}}(T)$ is compatible with T, i.e., $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$.*

This is equivalent to saying that an $\mathcal{I}$-operator is an $\mathcal{I}$-compatibility operator iff $T \in [\![T]\!]^{\nabla^{\mathcal{A}}}$, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By definition, the largest $\mathcal{I}$-compatibility operator is $\Omega^{\mathcal{A}}$ and the smallest one is the one sending every $\mathcal{I}$-filter family to the identity congruence system $\Delta^{\mathrm{SEN}'}$ on SEN'. As has been shown in Theorem 4 of [33] (see, also, [34] and Theorem 1.6 of [9] for the progenitor in AAL), the Suszko operator $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ has the distinction of being the largest order-preserving $\mathcal{I}$-compatibility operator on $\mathcal{A}$.

Some easy properties of $\mathcal{I}$-compatibility operators, refining those properties of $\mathcal{I}$-operators enumerated in Lemma 24, follow. Note, also, that Lemma 25 dealt with an $\mathcal{I}$-compatibility operator.

Lemma 27 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-compatibility operator on $\mathcal{A}$. For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

1. $T \in [\![T]\!]^{\nabla^{\mathcal{A}}}$;
2. $T^{\nabla^{\mathcal{A}}} \leq T$.

If $\nabla^{\mathcal{A}}$ is order-preserving, then:

3. $[\![T]\!]^{\nabla^{\mathcal{A}}} \subseteq [\![T^{\nabla^{\mathcal{A}}}]\!]^{\nabla^{\mathcal{A}}}$;
4. *Every $\nabla^{\mathcal{A}}$-full class of $\mathcal{I}$-filter families is an upset of* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: Part 1 follows by the remark following Definition 26. Part 2 follows by Part 1 and the definition of $T^{\nabla^{\mathcal{A}}}$. Part 3 follows by Part 2 and Part 3 of Lemma 24. Finally, Part 4 follows by the definition $\nabla^{\mathcal{A}}$-fullness and the order preservation of $\nabla^{\mathcal{A}}$. ■

Compatibility of the $\mathcal{I}$-operators allows the following rewriting of Corollary 15 characterizing $\nabla^{\mathcal{A}}$-filter families:

Corollary 28 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-compatibility operator on $\mathcal{A}$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, T is a $\nabla^{\mathcal{A}}$-filter family of $\mathcal{A}$ iff $T/\nabla^{\mathcal{A}}(T)$ is the least $\mathcal{I}$-filter family of $\mathcal{A}/\nabla^{\mathcal{A}}(T)$.*

Proof: Set $\theta = \nabla^{\mathcal{A}}(T)$ in Corollary 15. ∎

The following corollary characterizes the property of an N-algebraic system having all filter families being $\nabla^{\mathcal{A}}$-filter families:

Corollary 29 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-compatibility operator on $\mathcal{A}$. The the following are equivalent:*

(i) *Every $\mathcal{I}$-filter family of $\mathcal{A}$ is a $\nabla^{\mathcal{A}}$-filter family.*

(ii) *For all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$.*

Proof:

(i)⇒(ii) Suppose $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then $T' \in [\![T]\!]^{\nabla^{\mathcal{A}}}$. But, by hypothesis, T is the smallest theory family in $[\![T]\!]^{\nabla^{\mathcal{A}}}$, whence $T \leq T'$.

(ii)⇒(i) By Lemma 27, Part 2, we have $T^{\nabla^{\mathcal{A}}} \leq T$. On the other hand, by the definition of $T^{\nabla^{\mathcal{A}}}$, we get that $\nabla^{\mathcal{A}}(T)$ is compatible with $T^{\nabla^{\mathcal{A}}}$, whence $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\nabla^{\mathcal{A}}})$. But, then, by hypothesis, $T \leq T^{\nabla^{\mathcal{A}}}$. So $T = T^{\nabla^{\mathcal{A}}}$. ∎

A **family of $\mathcal{I}$-compatibility operators** (see Subsection 4.1 of [1])

$$\nabla := \{\nabla^{\mathcal{A}} : \mathcal{A} \text{ an } N\text{-algebraic system}\}$$

is a collection, where $\nabla^{\mathcal{A}}$ is an $\mathcal{I}$-compatibility operator on $\mathcal{A}$, for every N-algebraic system $\mathcal{A}$.

Definition 30 *Let $\nabla^{\mathcal{A}}$ and $\nabla^{\mathcal{B}}$ be $\mathcal{I}$-compatibility operators on $\mathcal{A}$ and $\mathcal{B}$. The pair $\langle \nabla^{\mathcal{A}}, \nabla^{\mathcal{B}} \rangle$* **commutes with inverse (surjective) N-morphisms** *if, for all (surjective) $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ and all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,*

$$\nabla^{\mathcal{A}}(\gamma^{-1}(T'')) = \gamma^{-1}(\nabla^{\mathcal{B}}(T'')).$$

A family ∇ of $\mathcal{I}$-compatibility operators **commutes with inverse (surjective) N-morphisms** *if, for all N-algebraic systems $\mathcal{A}$ and $\mathcal{B}$, the pair $\langle \nabla^{\mathcal{A}}, \nabla^{\mathcal{B}} \rangle$ commutes with inverse (surjective) N-morphisms.*

The following notions of compatibility of morphisms with filter families and with collections of filter families will prove helpful. It abstracts to the categorical context Definition 4.7 of [1].

Definition 31 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-compatibility operator on $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$, and assume T is a $\mathcal{I}$-filter family on* SEN' *and $\mathcal{T}$ is a collection of $\mathcal{I}$-filter families on* SEN'.

- *An N-morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is* $\nabla^{\mathcal{A}}$-**compatible with** *T if*
$$\mathrm{Ker}(\langle H, \gamma \rangle) \le \nabla^{\mathcal{A}}(T).$$
- *An N-morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is* $\nabla^{\mathcal{A}}$-**compatible with** *$\mathcal{T}$ if it is $\nabla^{\mathcal{A}}$-compatible with every $T \in \mathcal{T}$.*

Note that $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is $\Omega^{\mathcal{A}}$-compatible with a filter family T on SEN' iff the congruence $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T and, in case H is an isomorphism, this happens if and only if the matrix system morphism $\langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, \gamma(T) \rangle$ is strict. Also note that, in case H is an isomorphism, $\langle \mathcal{A}, T \rangle \overset{\langle H,\gamma \rangle}{\to} \langle \mathcal{B}, \gamma(T) \rangle$ is a deductive matrix system morphism if and only if $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T. Czelakowski used the corresponding sentential concept in his study of the Suszko operator in [9] to obtain a general Correspondence Theorem that was generalized in Theorem 4.17 of [1]. Using the abstract version encapsulated in Definition 31, we will obtain a similar general correspondence result in Theorem 40 as an analog of Theorem 4.17 of [1].

For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the projection N-morphism $\langle I_{\mathbf{Sign}'}, \pi \rangle : \mathcal{A} \to \mathcal{A}/\nabla^{\mathcal{A}}(T)$ is always $\nabla^{\mathcal{A}}$-compatible with T. In addition, since $\nabla^{\mathcal{A}}$ is an $\mathcal{I}$-compatibility operator, if $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is $\nabla^{\mathcal{A}}$-compatible with T, then it is also $\Omega^{\mathcal{A}}$-compatible with T, i.e., $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T. In case H is an isomorphism, this implies that $T = \gamma^{-1}(\gamma(T))$ and $\nabla^{\mathcal{A}}(T) = \gamma^{-1}(\gamma(\nabla^{\mathcal{A}}(T)))$.

Definition 32 *A family ∇ of $\mathcal{I}$-compatibility operators is called* (**weakly**) **coherent** *if, for all surjective N-morphisms $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ (with H an isomorphism) and all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,*
$$\langle H, \gamma \rangle\ \nabla^{\mathcal{A}}\text{-compatible with } \gamma^{-1}(T'') \quad \text{implies} \quad \nabla^{\mathcal{A}}(\gamma^{-1}(T'')) = \gamma^{-1}(\nabla^{\mathcal{B}}(T'')).$$

Since the reverse implication of the defining condition is universally valid, a family ∇ of $\mathcal{I}$-compatibility operators is (weakly) coherent iff, for every surjective N-morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ (with H an isomorphism) and every $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, $\langle H, \gamma \rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\gamma^{-1}(T'')$ *if and only if* $\nabla^{\mathcal{A}}(\gamma^{-1}(T'')) = \gamma^{-1}(\nabla^{\mathcal{B}}(T''))$.

Lemma 33 *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators. Then, for every surjective N-morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and every $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H,\gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with T', then*

$$\gamma(\nabla^{\mathcal{A}}(T')) = \nabla^{\mathcal{B}}(\gamma(T')).$$

Proof: Suppose ∇ is weakly coherent, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ surjective, with H an isomorphism, and compatible with T'. By compatibility of ∇ and Lemma 3, we get $T' = \gamma^{-1}(\gamma(T'))$ and $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Thus, by weak coherence $\nabla^{\mathcal{A}}(T') = \nabla^{\mathcal{A}}(\gamma^{-1}(\gamma(T'))) = \gamma^{-1}(\nabla^{\mathcal{B}}(\gamma(T')))$. Finally, by surjectivity, $\gamma(\nabla^{\mathcal{A}}(T')) = \nabla^{\mathcal{B}}(\gamma(T'))$. ■

Corollary 34 *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators and $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ an N-isomorphism. Then, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,*

$$\gamma(\nabla^{\mathcal{A}}(T')) = \nabla^{\mathcal{B}}(\gamma(T')) \quad \textit{and} \quad \nabla^{\mathcal{A}}(\gamma^{-1}(T'')) = \gamma^{-1}(\nabla^{\mathcal{B}}(T'')).$$

Proof: The first property follows immediately by Lemma 33. For the second property, note that the kernel of an isomorphism is the identity congruence system, whence every isomorphism is $\nabla^{\mathcal{A}}$-compatible with all sentence families of $\mathcal{A}$, for any $\mathcal{I}$-operator $\nabla^{\mathcal{A}}$. Therefore, the property holds by weak coherence. ■

Putting together Definitions 30 and 32, we obtain

Proposition 35 *If ∇ is a family of $\mathcal{I}$-compatibility operators that commutes with inverse surjective N-morphisms, then it is coherent.*

Since Ω satisfies this property (see [33]), we obtain that the Leibniz operator (viewed as a family of operators) is indeed a coherent family of $\mathcal{I}$-compatibility operators.

For weakly coherent families ∇ of $\mathcal{I}$-compatibility operators, the $\nabla^{\mathcal{A}}$-full congruence systems on $\mathcal{A}$ and the $\nabla^{\mathcal{A}}$-full collections of filter families, introduced in Definition 12 as the fixed-points of $\widetilde{\nabla}^{\mathcal{A}} \circ \nabla^{\mathcal{A}^{-1}}$ and $\nabla^{\mathcal{A}^{-1}} \circ \widetilde{\nabla}^{\mathcal{A}}$, respectively, can be characterized more elegantly (see Corollary 4.14 and Proposition 4.16 of [1] for the original characterizations in the AAL context).

Proposition 36 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, for all $\theta \in \mathrm{ConSys}(\boldsymbol{A})$ (denoting $\langle I_{\mathbf{Sign}'},\pi\rangle := \langle I_{\mathbf{Sign}'},\pi^{\theta}\rangle : \mathcal{A} \to \mathcal{A}/\theta$),*

$$\nabla^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(\nabla^{\mathcal{A}/\theta}(T')) = \nabla^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

Proof: If $T \in \nabla^{\mathcal{A}^{-1}}(\theta)$, then, by definition, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\theta \leq \nabla^{\mathcal{A}}(T)$. Thus, $T = \pi^{-1}(\pi(T))$, whence, by Lemma 33, $\pi(\nabla^{\mathcal{A}}(T)) = \nabla^{\mathcal{A}/\theta}(\pi(T))$. Set $T' = \pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$. Then $T = \pi^{-1}(T')$ and $\nabla^{\mathcal{A}}(\pi^{-1}(T')) = \nabla^{\mathcal{A}}(T) = \pi^{-1}(\pi(\nabla^{\mathcal{A}}(T))) = \pi^{-1}(\nabla^{\mathcal{A}/\theta}(\pi(T))) = \pi^{-1}(\nabla^{\mathcal{A}/\theta}(T'))$.

Conversely, suppose that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that $\pi^{-1}(\nabla^{\mathcal{A}/\theta}(T')) = \nabla^{\mathcal{A}}(\pi^{-1}(T'))$. Then $\langle I_{\mathbf{Sign}'}, \pi\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\pi^{-1}(T')$, or, equivalently, $\pi^{-1}(T') \in \nabla^{\mathcal{A}^{-1}}(\theta)$. ∎

Now, Definition 12 immediately yields

Corollary 37 *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then $\mathcal{T}$ is $\nabla^{\mathcal{A}}$-full iff*

$$\mathcal{T} = \pi^{\theta^{-1}}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{\theta^{-1}}(\nabla^{\mathcal{A}/\theta}(T')) = \nabla^{\mathcal{A}}(\pi^{\theta^{-1}}(T'))\}),$$

for some $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, which can be taken equal to $\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T})$.

Recall that, by Proposition 14, $\Omega^{\mathcal{A}}$-full filter families are of the form $\pi^{\theta^{-1}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$, for some $\theta \in \mathrm{ConSys}(\boldsymbol{A})$. But since, given an $\mathcal{I}$-compatibility operator $\nabla^{\mathcal{A}}$, for every filter family T, $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$, we get $\nabla^{\mathcal{A}^{-1}}(\theta) \subseteq \Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{\theta^{-1}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$. Thus, $\nabla^{\mathcal{A}^{-1}}(\theta)$ must be of the form $\pi^{\theta^{-1}}(\mathcal{T}')$, for some $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$.

Lemma 38 *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators and $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ be a surjective N-morphism, with H an isomorphism.*

1. *For all $\mathcal{T}'' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, if $\langle H, \gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\gamma^{-1}(\mathcal{T}'')$, then $\widetilde{\nabla}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}'')) = \gamma^{-1}(\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T}''))$.*
2. *For all $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H, \gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\mathcal{T}'$, then $\gamma(\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}')) = \widetilde{\nabla}^{\mathcal{B}}(\gamma(\mathcal{T}'))$.*

Proof:

1. If $\langle H, \gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\gamma^{-1}(\mathcal{T}'')$, then, by definition, it is $\nabla^{\mathcal{A}}$-compatible with every $\gamma^{-1}(T'')$, for $T'' \in \mathcal{T}''$. Using weak coherence, we get $\widetilde{\nabla}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}'')) = \bigcap_{T'' \in \mathcal{T}''} \nabla^{\mathcal{A}}(\gamma^{-1}(T'')) = \bigcap_{T'' \in \mathcal{T}''} \gamma^{-1}(\nabla^{\mathcal{B}}(T'')) = \gamma^{-1}(\bigcap_{T'' \in \mathcal{T}''} \nabla^{\mathcal{B}}(T'')) = \gamma^{-1}(\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T}''))$.
2. If $\langle H, \gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\mathcal{T}'$, then $\langle H, \gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with all $T' \in \mathcal{T}'$, by definition of $\nabla^{\mathcal{A}}$-compatibility, whence $\gamma^{-1}(\gamma(T')) = T'$,

for all $T' \in \mathcal{T}'$, i.e., $\gamma^{-1}(\gamma(\mathcal{T}')) = \mathcal{T}'$. This implies that $\langle H, \gamma \rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\gamma^{-1}(\gamma(\mathcal{T}'))$. Now using Part 1, we get

$$\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}') = \widetilde{\nabla}^{\mathcal{A}}(\gamma^{-1}(\gamma(\mathcal{T}'))) = \gamma^{-1}(\widetilde{\nabla}^{\mathcal{B}}(\gamma(\mathcal{T}')))$$

and, finally, using surjectivity, $\gamma(\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}')) = \widetilde{\nabla}^{\mathcal{B}}(\gamma(\mathcal{T}'))$. ■

A characterization of $\nabla^{\mathcal{A}}$-full congruence systems analogous to that of a $\nabla^{\mathcal{A}}$-full collection of filter families, given in Corollary 37, is as follows:

Proposition 39 *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators and $\theta \in \mathrm{ConSys}(\boldsymbol{A})$. Then θ is $\nabla^{\mathcal{A}}$-full iff (denoting $\langle I_{\mathbf{Sign}'}, \pi \rangle := \langle I_{\mathbf{Sign}'}, \pi^{\theta} \rangle : \mathcal{A} \to \mathcal{A}/\theta$)*

$$\widetilde{\nabla}^{\mathcal{A}/\theta}(\{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(\nabla^{\mathcal{A}/\theta}(T'')) = \nabla^{\mathcal{A}}(\pi^{-1}(T''))\}) = \Delta^{\mathrm{SEN}'/\theta}.$$

Proof: Let $\mathcal{T}'' = \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(\nabla^{\mathcal{A}/\theta}(T'')) = \nabla^{\mathcal{A}}(\pi^{-1}(T''))\}$. Then $\langle I_{\mathbf{Sign}'}, \pi \rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\pi^{-1}(\mathcal{T}'')$, whence, by Proposition 36 and Lemma 38, we get θ $\nabla^{\mathcal{A}}$-full iff

$$\theta = \widetilde{\nabla}^{\mathcal{A}}(\nabla^{\mathcal{A}^{-1}}(\theta)) = \widetilde{\nabla}^{\mathcal{A}}(\pi^{-1}(\mathcal{T}'')) = \pi^{-1}(\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}'')). \tag{1}$$

If θ $\nabla^{\mathcal{A}}$-full, then, by Condition (1) and the surjectivity of π, $\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}'') = \pi(\pi^{-1}(\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}''))) = \pi(\theta) = \Delta^{\mathrm{SEN}'/\theta}$.

If, on the other hand, $\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}'') = \Delta^{\mathrm{SEN}'/\theta}$, then $\theta = \pi^{-1}(\Delta^{\mathrm{SEN}'/\theta}) = \pi^{-1}(\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}''))$ and, therefore, θ is $\nabla^{\mathcal{A}}$-full, by Condition (1). ■

Since the Leibniz operator commutes with all surjective N-morphisms, when $\nabla \equiv \Omega$ in Proposition 39, the family $\mathcal{T}'' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, whence $\Omega^{\mathcal{A}^{-1}}(\theta) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, as was shown in Proposition 14.

Since $\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathcal{T}'') = \Delta^{\mathrm{SEN}'/\theta}$ is equivalent to $\theta \in \mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$, we also obtain the result proven in Proposition 17.

We are now ready to lift the General Correspondence Theorem 4.17 of [1] to CAAL.

Theorem 40 (General Correspondence Theorem) *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators. For every surjective N-morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H, \gamma \rangle$ is $\nabla^{\mathcal{A}}$-compatible with T, then $\langle H, \gamma \rangle$ induces an order isomorphism between $[\![T]\!]^{\nabla^{\mathcal{A}}}$ and $[\![\gamma(T)]\!]^{\nabla^{\mathcal{B}}}$, whose inverse is given by γ^{-1}.*

Proof: Since $\langle H,\gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with T, by Lemma 1, $\gamma^{-1}(\gamma(T)) = T$ and, in addition, by Lemma 3, taking into account the fact that H is postulated to be an isomorphism, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.

Let, first, $U \in [\![T]\!]^{\nabla^{\mathcal{A}}}$. Then $\mathrm{Ker}(\langle H,\gamma\rangle) \leq \nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(U)$. Thus, by Lemma 3, $\gamma^{-1}(\gamma(U)) = U$ and $\gamma(U) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Since $\langle H,\gamma\rangle$ is $\Omega^{\mathcal{A}}$-compatible with U and $\nabla^{\mathcal{A}}$-compatible with T, and both operators are weakly coherent, Lemma 33 yields $\nabla^{\mathcal{B}}(\gamma(T)) = \gamma(\nabla^{\mathcal{A}}(T)) \leq \gamma(\Omega^{\mathcal{A}}(U)) = \Omega^{\mathcal{B}}(\gamma(U))$. Therefore, $\gamma(U) \in [\![\gamma(T)]\!]^{\nabla^{\mathcal{B}}}$.

Next, assume $U' \in [\![\gamma(T)]\!]^{\nabla^{\mathcal{B}}}$. Thus, $\nabla^{\mathcal{B}}(\gamma(T)) \leq \Omega^{\mathcal{B}}(U')$. It is the case, by Lemma 3, that $\gamma^{-1}(U') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by surjectivity, that $\gamma(\gamma^{-1}(U')) = U'$. Moreover, $\langle H,\gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $T = \gamma^{-1}(\gamma(T))$. Therefore, by weak coherence,

$$\nabla^{\mathcal{A}}(T) = \nabla^{\mathcal{A}}(\gamma^{-1}(\gamma(T))) = \gamma^{-1}(\nabla^{\mathcal{B}}(\gamma(T))) \leq \gamma^{-1}(\Omega^{\mathcal{B}}(U')) = \Omega^{\mathcal{A}}(\gamma^{-1}(U')),$$

proving that $\gamma^{-1}(U') \in [\![T]\!]^{\nabla^{\mathcal{A}}}$.

Thus, $\langle H,\gamma\rangle$ induces a bijection between $[\![T]\!]^{\nabla^{\mathcal{A}}}$ and $[\![\gamma(T)]\!]^{\nabla^{\mathcal{B}}}$ with inverse γ^{-1}. Both mappings are order-preserving, whence we obtain the asserted order isomorphism. ∎

This isomorphism also shows that the least elements of the corresponding isomorphic complete lattices correspond.

Corollary 41 *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators. For every surjective N-morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H,\gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with T, then*

$$T \in \mathrm{FiFam}^{\nabla^{\mathcal{A}}}(\mathcal{A}) \quad \textit{iff} \quad \gamma(T) \in \mathrm{FiFam}^{\nabla^{\mathcal{B}}}(\mathcal{B}).$$

To obtain an analogous correspondence theorem for the relativized operators $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}$ (see Theorem 4.20 of [1]), we first show that relativization preserves weak coherence:

Proposition 42 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then the family*

$$\widetilde{\nabla}^{\bullet,\mathcal{I}} = \{\widetilde{\nabla}^{\mathcal{A},\mathcal{I}} : \mathcal{A} \textit{ an } N\textit{-algebraic system}\}$$

is also a weakly coherent family of $\mathcal{I}$-compatibility operators.

Proof: By definition of $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}$, if ∇ is a family of $\mathcal{I}$-compatibility operators, then $\nabla^{\bullet,\mathcal{I}}$ is one also. To show that it is also weakly coherent, let $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ surjective, with H an isomorphism, $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}$-compatible with $\gamma^{-1}(T'')$. Then $\mathrm{Ker}(\langle H,\gamma\rangle) \leq \widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(\gamma^{-1}(T''))$. Let $T' \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{\gamma^{-1}(T'')}$, i.e., $\gamma^{-1}(T'') \leq T'$. Then $\mathrm{Ker}(\langle H,\gamma\rangle) \leq \widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(\gamma^{-1}(T'')) \leq \widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(T') \leq \nabla^{\mathcal{A}}(T')$. Therefore, $\langle H,\gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with T' and, hence, $T' = \gamma^{-1}(\gamma(T'))$ and $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Now we get

$$\nabla^{\mathcal{A}}(T') = \nabla^{\mathcal{A}}(\gamma^{-1}(\gamma(T'))) = \gamma^{-1}(\nabla^{\mathcal{B}}(\gamma(T'))). \tag{2}$$

Claim: $\gamma((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{\gamma^{-1}(T'')}) = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T''}$.
If $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $\gamma^{-1}(T'') \leq T'$, then we have already shown that $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $T'' = \gamma(\gamma^{-1}(T'')) \leq \gamma(T')$. Conversely, suppose that $U'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, with $T'' \leq U''$. Then $U'' = \gamma(\gamma^{-1}(U''))$, $\gamma^{-1}(U'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\gamma^{-1}(T'') \leq \gamma^{-1}(U'')$. This finishes the proof of the claim. ▲

Using Equation (2), we now get

$$\begin{array}{rcl}
\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(\gamma^{-1}(T'')) & = & \bigcap\{\nabla^{\mathcal{A}}(T') : T' \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{\gamma^{-1}(T'')}\} \\
& \overset{\text{Eq. (2)}}{=} & \bigcap\{\gamma^{-1}(\nabla^{\mathcal{B}}(\gamma(T'))) : T' \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{\gamma^{-1}(T'')}\} \\
& = & \gamma^{-1}(\bigcap\{\nabla^{\mathcal{B}}(\gamma(T')) : T' \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{\gamma^{-1}(T'')}\}) \\
& \overset{\text{Claim}}{=} & \gamma^{-1}(\bigcap\{\nabla^{\mathcal{B}}(U'') : U'' \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T''}\}) \\
& = & \gamma^{-1}(\widetilde{\nabla}^{\mathcal{B},\mathcal{I}}(T'')).
\end{array}$$

Therefore, the family $\widetilde{\nabla}^{\bullet,\mathcal{I}}$ is weakly coherent, as claimed. ■

Recall that to simplify notation, we sometimes use $\widetilde{\nabla}^{\mathcal{I}} := \widetilde{\nabla}^{\bullet,\mathcal{I}}$ for the family of the relativized operators corresponding to the $\mathcal{I}$-operator ∇.

Theorem 43 (Relativized Correspondence) *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators. For every surjective N-morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H,\gamma\rangle$ is $\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}$-compatible with T, then $\langle H,\gamma\rangle$ induces an order isomorphism between $[\![T]\!]^{\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}}$ and $[\![\gamma(T)]\!]^{\widetilde{\nabla}^{\mathcal{B},\mathcal{I}}}$, whose inverse is given by γ^{-1}.*

Proof: Immediately follows by Theorem 40, taking into account the weak coherence property of $\widetilde{\nabla}^{\bullet,\mathcal{I}}$, established in Proposition 42. ■

The ordinary reduction processes of AAL, using the Leibniz and Suszko operators, were abstracted to the case of an arbitrary family of $\mathcal{S}$-operators in Definition 4.21 of [1]. In a parallel treatment, the reductions with respect to the categorical Leibniz and Suszko operators, which give rise to the CAAL algebraic system classes, can be lifted to arbitrary $\mathcal{I}$-operators.

Definition 44 *Let ∇ be a family of $\mathcal{I}$-operators. Define*

$$\begin{aligned}
\mathsf{AlgSys}^{\nabla}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A}/\nabla^{\mathcal{A}}(T) : \mathcal{A} \textit{ an } N\textit{-algebraic system}, \ T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})\\
\mathsf{AlgSys}_{\nabla}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A} : \textit{exists } T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \textit{ such that } \nabla^{\mathcal{A}}(T) = \Delta^{\mathrm{SEN}'}\})\\
\mathsf{AlgSys}^{\widetilde{\nabla}^{\mathcal{I}}}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A}/\widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(T) : \mathcal{A} \textit{ an } N\textit{-algebraic system}, \ T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})\\
\mathsf{AlgSys}_{\widetilde{\nabla}^{\mathcal{I}}}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A} : \textit{exists } T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \textit{ such that } \widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(T) = \Delta^{\mathrm{SEN}'}\}).
\end{aligned}$$

We undertake, first, the task of showing that each pair of identically sup- and sub-scripted classes of algebraic systems, i.e., classes referring to the same weakly coherent family of $\mathcal{I}$-compatibility operators, consists of identical classes of N-algebraic systems. The key observation is the well-known (in both AAL and CAAL) fact that the congruence system corresponding to a reduced matrix system is the identity congruence system, i.e., "reduction always produces a reduced system".

Lemma 45 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, if $\theta \leq \nabla^{\mathcal{A}}(T)$, then $\nabla^{\mathcal{A}/\theta}(T/\theta) = \nabla^{\mathcal{A}}(T)/\theta$. In particular, $\nabla^{\mathcal{A}/\nabla^{\mathcal{A}}(T)}(T/\nabla^{\mathcal{A}}(T)) = \Delta^{\mathrm{SEN}'/\nabla^{\mathcal{A}}(T)}$.*

Proof: For the first equality, noting that $\langle I_{\mathbf{Sign}'}, \pi^{\theta}\rangle$ is $\nabla^{\mathcal{A}}$-compatible with T, by the hypothesis, and using weak coherence and Lemma 33, we get that

$$\nabla^{\mathcal{A}/\theta}(T/\theta) = \nabla^{\mathcal{A}/\theta}(\pi^{\theta}(T)) = \pi^{\theta}(\nabla^{\mathcal{A}}(T)) = \nabla^{\mathcal{A}}(T)/\theta.$$

For $\theta = \nabla^{\mathcal{A}}(T)$, then, we obtain $\nabla^{\mathcal{A}/\nabla^{\mathcal{A}}(T)}(T/\nabla^{\mathcal{A}}(T)) = \nabla^{\mathcal{A}}(T)/\nabla^{\mathcal{A}}(T) = \Delta^{\mathrm{SEN}'/\nabla^{\mathcal{A}}(T)}$. ∎

Proposition 46 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then $\mathsf{AlgSys}^{\nabla}(\mathcal{I}) = \mathsf{AlgSys}_{\nabla}(\mathcal{I})$. Moreover, the class*

$$\{\mathcal{A} : \textit{exists } T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \textit{ such that } \nabla^{\mathcal{A}}(T) = \Delta^{\mathrm{SEN}'}\}$$

is closed under isomorphic copies.

Proof: If $\mathcal{A} \in \mathsf{AlgSys}_{\nabla}(\mathcal{I})$, then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\nabla^{\mathcal{A}}(T) = \Delta^{\mathrm{SEN}'}$. Thus, $\mathcal{A}/\nabla^{\mathcal{A}}(T) \cong \mathcal{A}$ and $\mathcal{A} \in \mathsf{AlgSys}^{\nabla}(\mathcal{I})$.

If, conversely, $\mathcal{A} \in \mathsf{AlgSys}^{\nabla}(\mathcal{I})$, then, there exists $\mathcal{B}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathcal{A} \cong \mathcal{B}/\nabla^{\mathcal{B}}(T)$. But then, by Lemma 45, $\nabla^{\mathcal{B}/\nabla^{\mathcal{B}}(T)}(T/\nabla^{\mathcal{B}}(T)) = \Delta^{\mathrm{SEN}''/\nabla^{\mathcal{B}}(T)}$, whence, $\mathcal{B}/\nabla^{\mathcal{B}}(T) \in \mathsf{AlgSys}_{\nabla}(\mathcal{I})$. Therefore, $\mathcal{A} \in \mathsf{AlgSys}_{\nabla}(\mathcal{I})$. ∎

Corollary 47 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then* $\mathsf{AlgSys}^{\widetilde{\nabla}^{\mathcal{I}}}(\mathcal{I}) = \mathsf{AlgSys}_{\widetilde{\nabla}^{\mathcal{I}}}(\mathcal{I})$*. Moreover, the class*

$$\{\mathcal{A} : \textit{exists } T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \textit{ such that } \widetilde{\nabla}^{\mathcal{A},\mathcal{I}}(T) = \Delta^{\mathrm{SEN}'}\}$$

is closed under isomorphic copies.

Proof: By putting together Proposition 42, asserting that $\widetilde{\nabla}^{\bullet,\mathcal{I}}$ is also a weakly coherent family of compatibility operators, and Proposition 46. ■

As special cases of Proposition 46 and Corollary 47, we get $\mathsf{AlgSys}^{\Omega}(\mathcal{I}) = \mathsf{AlgSys}_{\Omega}(\mathcal{I}) = \mathsf{AlgSys}^{*}(\mathcal{I})$ and $\mathsf{AlgSys}^{\widetilde{\Omega}^{\bullet,\mathcal{I}}}(\mathcal{I}) = \mathsf{AlgSys}_{\widetilde{\Omega}^{\bullet,\mathcal{I}}}(\mathcal{I}) = \mathsf{AlgSys}^{\mathrm{Su}}(\mathcal{I})$, equalities that were asserted in Lemma 6.

Relating to the lifting of an $\mathcal{I}$-operator ∇, we consider the following corresponding classes of N-algebraic systems.

Definition 48 *For a π-institution $\mathcal{I}$ and family ∇ of $\mathcal{I}$-operators, define*

$$\begin{aligned}\mathsf{AlgSys}^{\widetilde{\widetilde{\nabla}}}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A}/\widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \textit{ an } N\textit{-algebraic system}, \ \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})\\ \mathsf{AlgSys}_{\widetilde{\widetilde{\nabla}}}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A} : \textit{exists } \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \textit{ such that } \widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathrm{SEN}'}\}).\end{aligned}$$

Like before, each of these two classes may be obtained by considering exclusively the ∇-full $\mathcal{I}$-gmatrix systems. Moreover, in the case of $\mathsf{AlgSys}_{\widetilde{\widetilde{\nabla}}}(\mathcal{I})$, we may consider the largest $\mathcal{I}$-gmatrix system, which is always ∇-full.

Lemma 49 *Let ∇ be a family of $\mathcal{I}$-operators. The following hold:*

1. $\mathsf{AlgSys}^{\widetilde{\widetilde{\nabla}}}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \textit{ an } N\textit{-algebraic system}, \ \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \ \nabla\textit{-full}\})$

2. $$\begin{aligned}\mathsf{AlgSys}_{\widetilde{\widetilde{\nabla}}}(\mathcal{I}) &= \mathbf{I}(\{\mathcal{A} : \textit{exists } \nabla\textit{-full } \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \\ &\qquad \textit{such that } \widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathrm{SEN}'}\})\\ &= \mathbf{I}(\{\mathcal{A} : \widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}\}).\end{aligned}$$

Proof:

1. The right-to-left inclusion is obvious. Suppose that $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Corollary 11 and Definition 12, the congruence system $\widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T})$ is a ∇-full congruence system. Thus, there exists a ∇-full $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}') = \widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T})$. Therefore, $\mathcal{A}/\widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T}) = \mathcal{A}/\widetilde{\widetilde{\nabla}}^{\mathcal{A}}(\mathcal{T}') \in \mathsf{AlgSys}^{\widetilde{\widetilde{\nabla}}}(\mathcal{I})$.

2. The first equality repeats the argument in Part 1. For the second, the right-to-left inclusion is obvious and, for the reverse, if $\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathrm{SEN}'}$, for some $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, then $\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathrm{SEN}'}$, which yields the conclusion. ■

We establish some connections between the algebraic system classes associated with the lifting and those associated with the relativization of a family of $\mathcal{I}$-operators.

Proposition 50 *Let ∇ be a family of $\mathcal{I}$-operators. Then*

$$\mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I}) = \mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}^{\mathcal{I}}}(\mathcal{I}) \quad \textit{and} \quad \mathsf{AlgSys}^{\widetilde{\overleftarrow{\nabla}}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathsf{AlgSys}^{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I}).$$

Proof: Since, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\overleftarrow{\nabla}}^{\mathcal{A},\mathcal{I}}(T) = \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T)$, we obtain that $\mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I})$ and, also, that $\mathsf{AlgSys}^{\widetilde{\overleftarrow{\nabla}}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathsf{AlgSys}^{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I})$. To show equality in the first case, suppose that $\mathcal{A} \in \mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I})$. By Lemma 49, we get $\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}$. Setting $T' = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get

$$\widetilde{\overleftarrow{\nabla}}^{\mathcal{A},\mathcal{I}}(T') = \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T'}) = \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}.$$

Therefore, $\mathcal{A} \in \mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}^{\mathcal{I}}}(\mathcal{I})$. ■

Lemma 51 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}/\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})) = \Delta^{\mathrm{SEN}'/\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})}.$$

Proof: Let $\theta = \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})$. Note that $\langle I_{\mathbf{Sign}'}, \pi^\theta \rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\mathcal{T}$, by the hypothesis. Thus, using weak coherence and Lemma 38, we get

$$\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}/\theta}(\mathcal{T}/\theta) = \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}/\theta}(\pi^\theta(\mathcal{T})) = \pi^\theta(\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})) = \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})/\theta = \Delta^{\mathrm{SEN}'/\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathcal{T})}.$$

■

Proposition 52 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then $\mathsf{AlgSys}^{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I}) = \mathsf{AlgSys}_{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I})$.*

Moreover, the class $\{\mathcal{A} : \widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}\}$ is closed under isomorphic images and

$$\mathsf{AlgSys}^{\widetilde{\overleftarrow{\nabla}}}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\overleftarrow{\nabla}}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) : \mathcal{A} \textit{ an } N\textit{-algebraic system}\}).$$

Proof: If $\mathcal{A} \in \mathsf{AlgSys}_{\widetilde{\nabla}}(\mathcal{I})$, then, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathrm{SEN}'}$. Thus, $\mathcal{A}/\widetilde{\nabla}^{\mathcal{A}}(\mathcal{T}) \cong \mathcal{A}$ and $\mathcal{A} \in \mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})$. If, conversely, $\mathcal{A} \in \mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})$, then, $\mathcal{A} \cong \mathcal{B}/\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T})$, for some $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. But then, by Lemma 51, $\widetilde{\nabla}^{\mathcal{B}/\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T})}(\mathcal{T}/\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T})) = \Delta^{\mathrm{SEN}''/\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T})}$, whence, $\mathcal{B}/\widetilde{\nabla}^{\mathcal{B}}(\mathcal{T}) \in \mathsf{AlgSys}_{\widetilde{\nabla}}(\mathcal{I})$. Therefore, $\mathcal{A} \in \mathsf{AlgSys}_{\widetilde{\nabla}}(\mathcal{I})$.

The displayed equality now follows by Lemma 49. ■

Taking into account Corollary 47 and Proposition 50, Proposition 52 yields that, under weak coherence, four of our six classes of N-algebraic systems actually coincide.

Corollary 53 *If ∇ is a weakly coherent family of $\mathcal{I}$-compatibility operators, then*

$$\mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I}) = \mathsf{AlgSys}_{\widetilde{\nabla}}(\mathcal{I}) = \mathsf{AlgSys}^{\widetilde{\nabla}^{\mathcal{I}}}(\mathcal{I}) = \mathsf{AlgSys}_{\widetilde{\nabla}^{\mathcal{I}}}(\mathcal{I}).$$

Proposition 54 *Let ∇ be a family of $\mathcal{I}$-compatibility operators that commutes with inverse surjective N-morphisms. For every N-algebraic system $\mathcal{A}$ and $\theta \in \mathrm{ConSys}(\mathbf{A})$,*

$$\theta \text{ is } \nabla^{\mathcal{A}}\text{-full} \quad \text{iff} \quad \theta \in \mathrm{ConSys}_{\mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})}(\mathcal{A}).$$

Proof: Suppose $\theta \in \mathrm{ConSys}(\boldsymbol{A})$ is $\nabla^{\mathcal{A}}$-full. By Corollary 11, $\theta = \widetilde{\nabla}^{\mathcal{A}}(\mathcal{T})$, for some $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, $\mathcal{A}/\theta \in \mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})$.

Conversely, if $\theta \in \mathrm{ConSys}_{\mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})}(\mathcal{A})$, then $\mathcal{A}/\theta \in \mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})$. By Proposition 52, there exists $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that $\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}') = \Delta^{\mathrm{SEN}'/\theta}$. Let $\langle I_{\mathbf{Sign}'}, \pi\rangle := \langle I_{\mathbf{Sign}'}, \pi^{\theta}\rangle : \mathcal{A} \to \mathcal{A}/\theta$. Then $\pi^{-1}(\mathcal{T}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by commutativity,

$$\theta = \mathrm{Ker}(\langle I_{\mathbf{Sign}'}, \pi\rangle) = \pi^{-1}(\Delta^{\mathrm{SEN}'/\theta}) = \pi^{-1}(\widetilde{\nabla}^{\mathcal{A}/\theta}(\mathcal{T}')) = \widetilde{\nabla}^{\mathcal{A}}(\pi^{-1}(\mathcal{T}')).$$

Thus, by Corollary 11, θ is $\nabla^{\mathcal{A}}$-full. ■

Proposition 54, taking into account the isomorphism in Corollary 11, gives a natural generalization of the isomorphism of Corollary 18.

Corollary 55 *Let ∇ be a family of $\mathcal{I}$-compatibility operators that commutes with inverse surjective N-morphisms. For every $\mathcal{A}$, the maps $\widetilde{\nabla}^{\mathcal{A}}$ and $\nabla^{\mathcal{A}^{-1}}$ are mutually inverse dual order isomorphisms betwen the lattice of $\nabla^{\mathcal{A}}$-full $\mathcal{I}$-gmatrix systems and the lattice* $\mathrm{ConSys}_{\mathsf{AlgSys}^{\widetilde{\nabla}}(\mathcal{I})}(\mathcal{A})$.

We will continue our developments along the line of the general theory presented here, establishing more analogs of results obtained in the AAL framework in [1], pertaining to characterizations of classes in the Leibniz hierarchy, in a forthcoming companion to the present work.

Acknowledgements

The author is heavily indebted to many people for their pioneering work in this field, which has inspired and guided his own work, as well as for having offered support and encouragement for the best part of almost two decades of work in the field. **Don Pigozzi** has made the beginning, the end and everything in between possible, since, without him, I would not have entered the field in the first place and, without his ongoing support, I would not have had the opportunities that I was afforded to have kept working in it, both physically and mentally. *Janusz Czelakowski* has performed pioneering work and, both during his visits in Ames, Iowa, and in subsequent years through his work has been a source of major inspiration. The Barcelona group, under the tireless guidance of *Josep Maria Font* and *Ramon Jansana*, based on the work of *Wim Blok*, Don and Janusz have contributed some of the most fascinating and beautiful recent developments in this area, including work on which the developments in the present paper are based, and have also been a source of inspiration and support for my own work. Last, but not less importantly, I would like to mention as other sources of motivation and stimulus the work of *James Raftery* and of *Manuel António Martins*.

Finally, outside the field, *Charles Wells* has provided critical support and encouragement and *Giora Slutzki* has been with me from the very beginning and, recently, became an "official advisor" of mine as well.

References

[1] Albuquerque. H., Font, J.M., and Jansana, R., *Compatibility Operators in Abstract Algebraic Logic*, to appear in The Journal of Symbolic Logic

[2] Babyonyshev, S.V., *Fully Fregean Logics*, Reports on Mathematical Logic, Vol. 37 (2003), pp. 59-77

[3] Blok, W.J., and Pigozzi, D., *Protoalgebraic Logics*, Studia Logica, Vol. 45 (1986), pp. 337-369

[4] Blok, W.J., and Pigozzi, D., *Algebraizable Logics*, Memoirs of the American Mathematical Society, Vol. 77, No. 396 (1989)

[5] Blok, W.J., and Pigozzi, D., *Algebraic Semantics for Universal Horn Logic Without Equality*, in Universal Algebra and Quasigroup Theory, A. Romanowska and J.D.H. Smith, Eds., Heldermann Verlag, Berlin 1992

[6] Czelakowski, J., *Equivalential Logics I*, Studia Logica, Vol. 40 (1981), pp. 227-236

[7] Czelakowski, J., *Equivalential Logics II*, Studia Logica, Vol. 40 (1981), pp. 355-372

[8] Czelakowski, J., *Protoalgebraic Logics*, Trends in Logic-Studia Logica Library 10, Kluwer, Dordrecht, 2001

[9] Czelakowski, J., *The Suszko Operator Part I*, Studia Logica, Vol. 74 (2003), pp. 181-231

[10] Czelakowski, J., and Jansana, R., *Weakly Algebraizable Logics*, Journal of Symbolic Logic, Vol. 64 (2000), pp. 641-668

[11] Czelakowski, J. and Pigozzi, D., *Amalgamation and Interpolation in Abstract Algebraic Logic*, Models, Algebras and Proofs, in Lecture Notes in Pure and Applied Mathematics, Vol. 203, Dekker, New York, 1999, pp. 187-265

[12] Czelakowski, J., and Pigozzi, D., *Fregean Logics*, Annals of Pure and Applied Logic, Vol. 127 (2004), pp. 17-76

[13] Czelakowski, J., and Pigozzi, D., *Fregean Logics with the Multiterm Deduction Theorem and Their Algebraization*, Studia Logica, Vol. 78 (2004), pp. 171-212

[14] Davey, B.A., and Priestley, H.A., *Introduction to Lattices and Order, Second Edition*, Cambridge University Press, Cambridge 2002

[15] Font, J.M., *Generalized Matrices in Abstract Algebraic Logic*, in V. F. Hendriks, and J. Malinowski, (eds.), Trends in Logic. 50 years of Studia Logica, Trends in Logic - Studia Logica Library, Vol. 21, Kluwer, Dordrecht, 2003, pp. 57-86

[16] Font, J.M., and Jansana, R., *Full Models for Sentential Logics*, Bulletin of the Section of Logic, Vol. 24, No. 3 (1995), pp. 123-131

[17] Font, J.M., and Jansana, R., *A General Algebraic Semantics for Sentential Logics*, Lecture Notes in Logic, Vol. 332, No. 7 (1996), Springer-Verlag, Berlin Heidelberg, 1996

[18] Font, J.M., and Jansana, R., *Leibniz Filters and the Strong Version of a Protoalgebraic Logic*, Archive for Mathematical Logic, Vol. 40 (2001), pp. 437-465

[19] Font, J.M., and Jansana, R., *Leibniz-linked Pairs of Deductive Systems*, Studia Logica, Vol. 99, No. 1-3 (2011), pp. 171-20

[20] Font, J.M., Jansana, R., and Pigozzi, D., *A Survey of Abstract Algebraic Logic*, Studia Logica, Vol. 74, No. 1/2 (2003), pp. 13-97

[21] Font, J.M., Jansana, R., and Pigozzi, D., *On the Closure Properties of the Class of Full G-models of a Deductive System*, Studia Logica, Vol. 83 (2006), pp. 215-278

[22] Herrmann, B., *Equivalential Logics and Definability of Truth,* Ph.D. Dissertation, Freie Universität Berlin, Berlin 1993

[23] Herrmann, B., *Equivalential and Algebraizable Logics*, Studia Logica, Vol. 57 (1996), pp. 419-436

[24] Herrmann, B., *Characterizing Equivalential and Algebraizable Logics by the Leibniz Operator*, Studia Logica, Vol. 58 (1997), pp. 305-323

[25] Jansana, R., *Full Models for Positive Modal Logic*, Mathematical Logic Quarterly, Vol. 48, No. 3 (2002), pp. 427-445

[26] Jansana, R., *Leibniz Filters Revisited*, Studia Logica, Vol. 75 (2003), pp. 305-317

[27] Raftery, J.G., *The Equational Definability of Truth Predicates*, Reports on Mathematical Logic, Vol. 41 (2006), pp. 95-149

[28] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Algebraizable Institutions*, Applied Categorical Structures, Vol. 10, No. 6 (2002), pp. 531-568

[29] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Tarski Congruence Systems, Logical Morphisms and Logical Quotients*, Journal of Pure and Applied Mathematics: Advances and Applications, Vol. 13, No. 1 (2015), pp. 27-73

[30] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Models of π-institutions*, Notre Dame Journal of Formal Logic, Vol. 46, No. 4 (2005), pp. 439-460

[31] Voutsadakis, G., *Categorical Abstract Algebraic Logic: $(\mathcal{I}, N)$-Algebraic Systems*, Applied Categorical Structures, Vol. 13, No. 3 (2005), pp. 265-280

[32] Voutsadakis, G., *Categorical Abstract Algebraic Logic: More on Protoalgebraicity*, Notre Dame Journal of Formal Logic, Vol. 47, No. 4 (2006), pp. 487-514

[33] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Prealgebraicity and Protoalgebraicity*, Studia Logica, Vol. 85, No. 2 (2007), pp. 217-251

[34] Voutsadakis, G., *Categorical Abstract Algebraic Logic: The Categorical Suszko Operator*, Mathematical Logic Quarterly, Vol. 53, No. 6 (2007), pp. 616-635

[35] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Equivalential π-Institutions*, Australasian Journal of Logic, Vol. 6 (2008), 24pp

[36] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Truth Equational π-Institutions*, Notre Dame Journal of Formal Logic, Vol. 56, No. 2 (2015), pp 351-378

[37] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Weakly Algebraizable π-Institutions*, Preprint available at `http://www.voutsadakis.com/RESEARCH/papers.html`