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Dedekind–MacNeille Completion of *n*-ordered Sets

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Abstract A completion of an n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ is defined, by analogy with the case of posets (2-ordered sets), as a pair $\langle e, \mathbf{Q} \rangle$, where $\mathbf{Q}$ is a complete n-lattice and $e : \mathbf{P} \to \mathbf{Q}$ is an n-order embedding. The Basic Theorem of Polyadic Concept Analysis is exploited to construct a completion of an arbitrary n-ordered set. The completion reduces to the Dedekind–MacNeille completion in the dyadic case, the case of posets. A characterization theorem is provided, analogous to the well-known dyadic one, for the case of joined n-ordered sets. The condition of joinedness is trivial in the dyadic case and, therefore, this characterization theorem generalizes the uniqueness theorem for the Dedekind–MacNeille completion of an arbitrary poset.

Keywords Formal concept analysis · Formal contexts · Triadic concept analysis · Triordered sets · Complete trilattices · Trilattices · Polyadic concept analysis · n-ordered sets · n-lattices · Completions

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1 Introduction

In [9, 10] Wille introduced Formal Concept Analysis in order to exploit the powerful machinery of Lattice Theory in a wide variety of applications. The book [4] gives an overview of Formal Concept Analysis. The basic notion is that of a binary relation

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between a set of *objects* and a set of *attributes*, called a *formal context*, which gives rise to a Galois connection whose collection of closed sets forms a complete lattice. This lattice is termed the *concept lattice* of the formal context. Every complete lattice arises in this way as the concept lattice of a formal context. In [11], Wille generalized this framework to a ternary relation between objects, attributes and *situations*. It is appropriately termed a *triadic context*. In a fashion similar to the dyadic case, a triadic context gives rise to *triadic concepts*. Endowed with three quasi-orderings, modeling the inclusions in each of the three components, triadic concepts form a 3-quasi-ordered structure satisfying the conditions of *uniqueness* and *antiordinality*. It is termed a *complete trilattice* by Wille. In [11] the *Basic Theorem of Triadic Concept Analysis* is proved, which states, roughly speaking, that every complete trilattice is the trilattice of triadic concepts of some formal triadic context. As is the case with lattices, besides the formulation of the concept of a complete trilattice in an order-theoretic way, there exists a formulation in terms of six infinitary operations, called *joins*.

Biedermann [1] considered algebraic structures with six operations similar in nature to the operations of complete trilattices but of finite ranks. Thus arose, in analogy with the case of lattices and complete lattices, the structure of a *trilattice*. In [1] an equational basis for the theory of trilattices was provided making trilattices an intriguing part of the theory of universal algebraic varieties.

The author, inspired by Wille, generalized the Triadic Concept Analysis to n dimensions for arbitrary n, giving rise to *Polyadic Concept Analysis* [7]. An n-ary relation replaces the ternary relation of triadic contexts. The *n-adic contexts* give rise, in a way analogous to the triadic case, to *n-adic formal concepts*. They form an n-quasi-ordered structure, also satisfying uniqueness and antiordinality (an n-dimensional version), called a *complete n-lattice*. Complete n-lattices have, in analogy to the triadic case, formulations in terms of $n!$ algebraic operations, generalizing the 6 joins of complete trilattices, also called joins. In [7], an analog of the Basic Theorem of Wille for the n-adic case, the *Basic Theorem of Polyadic Concept Analysis*, was proved. It states, roughly speaking, that every complete n-lattice is the n-lattice of n-adic concepts of some formal n-adic context. Inspired by Biedermann, the author studied the structures arising from complete n-lattices by restricting these joins to finite ranks. The ensuing structures are called *n-lattices* and in [8] an equational basis for n-lattices, generalizing the basis of Biedermann for trilattices, was given.

Dyadic formal contexts can be used to provide a Dedekind–MacNeille completion [6] (see also [3], Definition 2.31, [2], Section V.9) of an arbitrary partially ordered set (see Theorem 4 of [4]). In the present paper an analog of this construction for the polyadic case is studied. More specifically, it is shown that, in a way analogous to the dyadic case, the Basic Theorem of Polyadic Concept Analysis may be used to provide a Dedekind–MacNeille completion of n-ordered sets. This is a complete n-lattice in which the original n-ordered set may be embedded via an n-order embedding. This construction covers the case of posets, since 2-ordered sets reduce to posets. An analog of the characterization theorem of the Dedekind–MacNeille completion of a poset is also provided in the n dimensions for the special class of joined n-ordered sets. The condition of joinedness is trivially satisfied in two dimensions, whence, the n-dimensional characterization theorem provided here properly generalizes the corresponding theorem on posets.

2 *n*-ordered Sets and *n*-lattices

For more details on the material provided in this section the reader is referred to [11] and [1] for the triadic case and to [7, 8] for the general n-adic case.

An *ordinal structure* $\mathbf{P} = \langle P, \lesssim_1, \lesssim_2, \ldots, \lesssim_n \rangle$ is a relational structure whose n relations are quasiorders. Let $\sim_i = \lesssim_i \cap \gtrsim_i$, for $i = 1, 2, \ldots, n$. An *n-ordered set* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ is an ordinal structure, such that, for all $x, y \in P$ and all $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$,

1. $x \sim_{i_1} y, \ldots, x \sim_{i_n} y$ imply $x = y$ (*Uniqueness Condition*)
2. $x \lesssim_{i_1} y, \ldots, x \lesssim_{i_{n-1}} y$ imply $x \gtrsim_{i_n} y$ (*Antiordinal Dependency*)

Each quasiorder $\lesssim_i$ induces in the standard way an order $\leq_i$ on the set of equivalence classes $P/{\sim_i} = \{[x]_i : x \in P\}$, $i = 1, 2, \ldots, n$, where $[x]_i = \{y \in P : x \sim_i y\}$.

Let $\mathbf{P} = \langle P, \lesssim_1, \lesssim_2, \ldots, \lesssim_n \rangle$ be an n-ordered set, $j_1, j_2, \ldots, j_{n-1} \in \{1, 2, \ldots, n\}$ be distinct and $X_1, X_2, \ldots, X_{n-1} \subseteq P$.

- An element $b \in P$ is called a $(j_{n-1}, \ldots, j_1)$-*bound* of $(X_{n-1}, X_{n-2}, \ldots, X_1)$ if $x_i \lesssim_{j_i} b$, for all $x_i \in X_i$ and all $i = 1, \ldots, n-1$. The set of all $(j_{n-1}, \ldots, j_1)$-bounds of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$.
- A $(j_{n-1}, \ldots, j_1)$-bound $l \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$ of $(X_{n-1}, \ldots, X_1)$ is called a $(j_{n-1}, \ldots, j_1)$-*limit* of $(X_{n-1}, \ldots, X_1)$ if $l \gtrsim_{j_n} b$, for all $(j_{n-1}, \ldots, j_1)$-bounds $b \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$. The set of all $(j_{n-1}, \ldots, j_1)$-limits of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}}$.

The following proposition is Proposition 4 of [7]. When applied to an element x of a quasi-ordered set, the term “largest” in the quasi-ordering $\lesssim$ means greater than or equal to all other elements with respect to $\lesssim$ without, of course, necessarily implying uniqueness. A similar comment holds for the term “smallest.”

Proposition 1 *Let $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ be an n-ordered set, $X_1, \ldots, X_{n-1} \subseteq P$ and $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. Then, there exists at most one $(j_{n-1}, \ldots, j_1)$-limit $\bar{l}$ of $(X_{n-1}, \ldots, X_1)$ satisfying*

(C) *$\bar{l}$ is the largest in $\lesssim_{j_2}$ among the largest limits in $\lesssim_{j_3}$ among … among the largest limits in $\lesssim_{j_{n-1}}$ among the largest limits in $\lesssim_{j_n}$ or, equivalently,*

(C’) *$\bar{l}$ is the smallest in $\lesssim_{j_1}$ among the largest limits in $\lesssim_{j_3}$ among … among the largest limits in $\lesssim_{j_{n-1}}$ among the largest limits in $\lesssim_{j_n}$.*

If it exists, a $(j_{n-1}, \ldots, j_1)$-limit satisfying the statement in Proposition 1 is called the $(j_{n-1}, \ldots, j_1)$-*join* of $(X_{n-1}, \ldots, X_1)$ and denoted by $\nabla_{j_{n-1}, \ldots, j_1}(X_{n-1}, \ldots, X_1)$.

A *complete n-lattice* $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ is an n-ordered set in which all $(j_{n-1}, \ldots, j_1)$-joins $\nabla_{j_{n-1}, \ldots, j_1}(X_{n-1}, \ldots, X_1)$ exist, for all $X_1, \ldots, X_{n-1} \subseteq L$ and all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. A complete n-lattice is *bounded* by $0_{j_n} := \nabla_{j_{n-1}, \ldots, j_1}(L, L, \ldots, L)$, where $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. This element does not depend on the order of the j_i’s. Indeed, if $0'_{j_n} = \nabla_{k_{n-1}, \ldots, k_1}(L, \ldots, L)$, with $\{k_1, \ldots, k_{n-1}\} = \{j_1, \ldots, j_{n-1}\}$, then, by the bound property of the joins, we get that $0_{j_n} \lesssim_{j_i} 0'_{j_n}$ and, also $0'_{j_n} \lesssim_{j_i} 0_{j_n}$, for all $i < n$. Thus, $0_{j_n} \sim_{j_i} 0'_{j_n}$, for all $i < n$, which, combined with antiordinality, yields that $0_{j_n} \sim_{j_i} 0'_{j_n}$, for all $i = 1, \ldots, n$. Therefore, by uniqueness, we obtain that $0_{j_n} = 0'_{j_n}$.

One may restrict attention to the case where $(j_{n-1}, \ldots, j_1)$-joins exist only for finite subsets of an n-ordered set. If, for all $k = 1, \ldots, n-1$, i_k is a positive integer and $\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,i_k})$, then the $(i_{n-1}, \ldots, i_1)$*-ary* $(j_{n-1}, \ldots, j_1)$*-join* of $(\vec{x}_{n-1}, \ldots, \vec{x}_1)$ is defined to be

$$\nabla^{i_{n-1},\ldots,i_1}_{j_{n-1},\ldots,j_1}(\vec{x}_{n-1}, \ldots, \vec{x}_1) := \nabla_{j_{n-1},\ldots,j_1}(\{x_{n-1,1}, \ldots, x_{n-1,i_{n-1}}\}, \ldots, \{x_{1,1}, \ldots, x_{1,i_1}\}).$$

The tuple $(i_{n-1}, \ldots, i_1)$ is then called the *arity* of $\nabla^{i_{n-1},\ldots,i_1}_{j_{n-1},\ldots,j_1}$. The following theorem is one of the key results of [8].

Theorem 2 (Reduction of Arity Theorem) *If in an n-ordered set* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$ *all* $(j_{n-1}, \ldots, j_1)$*-joins of arity* $(2, 2, \ldots, 2)$ *exist, then all finitary* $(j_{n-1}, \ldots, j_1)$*-joins* $\nabla^{i_{n-1}\ldots i_1}_{j_{n-1},\ldots,j_1}$ *exist, where* $\nabla^{i_{n-1}\ldots i_1}_{j_{n-1},\ldots,j_1}$ *is* $(i_{n-1}, \ldots, i_1)$*-ary.*

An *n-lattice* $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n\rangle$ is an n-ordered set in which all the $(2, \ldots, 2)$-ary $(j_{n-1}, \ldots, j_1)$-joins exist. The following notation will be used for these joins:

$$\begin{aligned}\nabla_{j_{n-1},\ldots,j_1}(\vec{x}_{n-1}, \ldots, \vec{x}_1) &:= \nabla_{j_{n-1},\ldots,j_1}(x_{n-1,1}, x_{n-1,2}, \ldots, x_{1,1}, x_{1,2})\\ &:= \nabla_{j_{n-1},\ldots,j_1}((x_{n-1,1}, x_{n-1,2}), \ldots, (x_{1,1}, x_{1,2})),\end{aligned}$$

where $\vec{x}_i = (x_{i,1}, x_{i,2})$, $x_{i,1}, x_{i,2} \in L$, $i = 1, \ldots, n-1$. The *derived operations* of arities $(\epsilon_{n-1}, \ldots, \epsilon_1)$, where $\epsilon_i \in \{1, 2\}$, $i = 1, \ldots, n-1$, are all the operations defined by the joins above by taking the argument corresponding to $\vec{x}_i$ to be $\vec{x}_i = (x_{i,1}, x_{i,2})$, if $\epsilon_i = 2$, and $\vec{x}_i = (x_i, x_i)$, if $\epsilon_i = 1$, $i = 1, \ldots, n-1$.

The main theorem of [8] provides an equational basis for n-lattices. This basis was first worked out and presented for trilattices by Biedermann [1].

3 The Dedekind–MacNeille Completion

A *completion* of an n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$ is a pair $\langle e, \mathbf{Q}\rangle$, where $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n\rangle$ is a complete n-lattice and $e : \mathbf{P} \to \mathbf{Q}$ is an *n-order embedding*, i.e., a mapping $e : P \to Q$, such that for all $i = 1, \ldots, n$ and all $x, y \in P$,

$$x \lesssim_i y \quad \text{if and only if} \quad e(x) \lesssim_i e(y).$$

An *i-order filter* of an n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$ is a subset $F_i \subseteq P$, such that, for all $x, y \in P$, $x \lesssim_i y$ and $x \in F_i$ imply $y \in F_i$. Dually, an *i-order ideal* of $\mathbf{P}$ is a subset $I_i \subseteq P$, such that, for all $x, y \in P$, $x \lesssim_i y$ and $y \in I_i$ imply $x \in I_i$. F_i is a *principal i-order filter* if

$$F_i = \{x \in P : p \lesssim_i x\},$$

for some $p \in P$, and I_i is a *principal i-order ideal* if

$$I_i = \{x \in P : x \lesssim_i p\},$$

for some $p \in P$.

The $\lesssim_i$*-order filter generated* by $p_1, \dots, p_m \in P$ will be denoted by

$$[p_1, \dots, p_m)_i := \{x \in P : p_j \lesssim_i x, \text{ for some } 1 \leq j \leq m\}$$

and the $\lesssim_i$*-order ideal generated* by $p_1, \dots, p_m \in P$ will be denoted by

$$(p_1, \dots, p_m]_i := \{x \in P : x \lesssim_i p_j, \text{ for some } 1 \leq j \leq m\}.$$

In particular, the principal i-order filter generated by $p \in P$ will be denoted by $[p)_i$ and the principal i-order ideal generated by $p \in P$ will be denoted by $(p]_i$. If $\mathbf{Q}$ is an n-ordered set, $P \subseteq Q$ and $q \in Q$, the notation

$$(q]_i^P = \{p \in P : p \lesssim_i q\}$$

will be used for the $\lesssim_i$-order ideal in P "generated" by $q \in Q$.

Definition 3 Given an n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \dots, \lesssim_n \rangle$, define the n-ary *completion relation* $R(\mathbf{P}) \subseteq P^n$ of $\mathbf{P}$ to be the set of all ordered n-tuples $\langle p_1, \dots, p_n \rangle \in P^n$, for which there exists $p \in P$, such that $p_i \lesssim_i p$, for all $i = 1, \dots, n$.

If $\langle p_1, \dots, p_n \rangle \in R(\mathbf{P})$, it will be said that $\langle p_1, \dots, p_n \rangle$ is a *bounded* n-tuple in $\mathbf{P}$.

The completion relation of $\mathbf{P}$ may also be characterized in terms of the principal filters of $\mathbf{P}$ as follows:

Proposition 4 *Let* $\mathbf{P} = \langle P, \lesssim_1, \dots, \lesssim_n \rangle$ *be an n-ordered set. Then, for all elements* $p_1, \dots, p_n \in P$,

$$\langle p_1, \dots, p_n \rangle \in R(\mathbf{P}) \quad \textit{iff} \quad \bigcap_{i=1}^{n} [p_i)_i \neq \emptyset.$$

Proof Obvious from the definition of $R(\mathbf{P})$. $\square$

By the Basic Theorem of Polyadic Concept Analysis (Theorem 6 of [7]) the concept n-lattice $\mathcal{C}(\mathbb{P}) = \langle \mathcal{C}(\mathbb{P}), \subseteq_1, \dots, \subseteq_n \rangle$ of the n-adic context $\mathbb{P} = \langle P, \dots, P, R(\mathbf{P}) \rangle$ is a complete n-lattice, whose $(j_{n-1}, \dots, j_1)$-joins are described, for all $\mathcal{A}_i \subseteq \mathcal{C}(\mathbb{P})$, $i = 1, \dots, n-1$, by

$$\begin{aligned} &\nabla_{j_{n-1}, \dots, j_1} (\mathcal{A}_{n-1}, \dots, \mathcal{A}_1) \\ &\quad = \mathfrak{b}_{j_{n-1}, \dots, j_1} \left(\left\{ \bigcup \left\{ A_{j_i} : (A_1, \dots, A_n) \in \mathcal{A}_i \right\} : i = n-1, \dots, 1 \right\} \right). \end{aligned}$$

The operation $\mathfrak{b}_{j_{n-1}, \dots, j_1}$ is described in detail in Proposition 3 of [7]. It corresponds, roughly speaking, to first "closing" with respect to the j_n-th component, followed by "closing" with respect to the j_{n-1}-st component and so on down to the j_1-st component.

Given $A_j \subseteq P$, $j \neq i$, the notation

$$\begin{aligned} &R(\mathbf{P})_i(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n) \\ &\quad = \{p \in P : (p_1, \dots, p_{i-1}, p, p_{i+1}, \dots, p_n) \in R(\mathbf{P}), \text{ for all } p_j \in A_j, j \neq i\} \end{aligned}$$

will be used for the closure with respect to the i-th component, i.e., for the image of $(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ under $R(\mathbf{P})$.

It is shown next that, given an n-ordered set $\mathbf{P}$ and an element $p \in P$, the n-tuple $\langle (p]_1, \ldots, (p]_n \rangle \in \mathcal{P}(P)^n$ is in $\mathcal{C}(\mathbb{P})$.

Lemma 5 *Suppose that* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *is an n-ordered set and* $p \in P$. *Then* $\langle (p]_1, \ldots, (p]_n \rangle \in \mathcal{C}(\mathbb{P})$.

Proof It must be shown that, for all $i = 1, \ldots, n$,

$$(p]_i = R(\mathbf{P})_i((p]_1, \ldots, (p]_{i-1}, (p]_{i+1}, \ldots, (p]_n).$$

We do this for $i = n$; the remaining equalities will then follow by symmetry.

First, if $x_n \lesssim_n p$, then, for all $x_1, \ldots, x_{n-1} \in P$, such that $x_i \lesssim_i p$, we have $\langle x_1, \ldots, x_n \rangle \in R(\mathbf{P})$, since $p \in P$ is such that $x_i \lesssim_i p$, for all $i = 1, \ldots, n$.

Suppose conversely, that $x_n \notin (p]_n$, i.e., that $x_n \not\lesssim_n p$. Suppose, for the sake of obtaining a contradiction, that there exists $z \in P$, such that $p \lesssim_i z$, for all $i = 1, \ldots, n-1$, and $x_n \lesssim_n z$. Then, by antiordinality, $x_n \lesssim_n z \lesssim_n p$, which contradicts our hypothesis.

Thus $\langle (p]_1, \ldots, (p]_n \rangle \in \mathcal{C}(\mathbb{P})$. □

In the next lemma, it is shown that every n-ordered set $\mathbf{P}$ can be embedded into the complete n-lattice $\mathcal{C}(\mathbb{P})$ of the n-adic concepts of the n-adic context $\mathbb{P}$.

Lemma 6 *Let* $= \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *be an n-ordered set. The mapping* $e : \mathbf{P} \to \mathcal{C}(\mathbb{P})$, *defined by*

$$e(p) = \langle (p]_1, \ldots, (p]_n \rangle, \quad \textit{for all } p \in P,$$

is an n-order embedding of $\mathbf{P}$ *into* $\mathcal{C}(\mathbb{P})$.

Proof It was shown in Lemma 5 that $e(p) \in \mathcal{C}(\mathbb{P})$, for all $p \in P$.

It remains, therefore, to show that $e : \mathbf{P} \to \mathcal{C}(\mathbb{P})$ is an n-order embedding.

Assume, first, that $p, q \in P$, such that $p \lesssim_i q$. Then $(p]_i \subseteq (q]_i$, whence $e(p) \subseteq_i e(q)$. Suppose, conversely, that $p, q \in P$, such that $e(p) \subseteq_i e(q)$. Therefore $(p]_i \subseteq (q]_i$. Thus $p \in (q]_i$, i.e., $p \lesssim_i q$. □

One obtains from Lemma 6 the following

Theorem 7 *Every n-ordered set* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ *has a completion* $\langle e, \mathcal{C}(\mathbb{P}) \rangle$, *where* $\mathcal{C}(\mathbb{P}) = \langle \mathcal{C}(\mathbb{P}), \subseteq_1, \ldots, \subseteq_n \rangle$ *is the complete n-lattice of n-adic concepts of the n-adic context* $\mathbb{P} = \langle P, \ldots, P, R(\mathbf{P}) \rangle$, *where* $R(\mathbf{P})$ *is the completion relation of* $\mathbf{P}$.

Proof Lemma 6 shows that the mapping $e : \mathbf{P} \to \mathcal{C}(\mathbb{P})$, defined by

$$e(p) = \langle (p]_1, \ldots, (p]_n \rangle, \quad \text{for all } p \in P,$$

is an n-order embedding of $\mathbf{P}$ into $\mathcal{C}(\mathbb{P})$. □

The completion of Theorem 7 is termed the *Dedekind–MacNeille Completion* of the n-ordered set $\mathbf{P}$ since, in the dyadic case, it reduces to the well-known Dedekind–MacNeille completion of a poset (see [4] for more details on the dyadic case).

The notation $\mathbf{DM}(\mathbf{P})$, introduced in [3], is used to denote the Dedekind–MacNeille completion of $\mathbf{P}$. So $\mathrm{DM}(\mathbf{P}) := \mathcal{C}(\mathbb{P})$ and $\mathbf{DM}(\mathbf{P}) := \langle \mathcal{C}(\mathbb{P}), \subseteq_1, \ldots, \subseteq_n \rangle$.

In the sequel the n-order embedding $e : \mathbf{P} \to \mathcal{C}(\mathbb{P})$, defined by

$$e(p) = \langle (p]_1, \ldots, (p]_n \rangle, \quad \text{for all } p \in P,$$

will be referred to as the *standard embedding* of $\mathbf{P}$ into $\mathbf{DM}(\mathbf{P})$ and the letter e will be reserved to denote this standard embedding.

Important Remark We note that, unfortunately, it has not been possible to prove that finitary n-adic joins that exist in $\mathbf{P}$ are preserved by the standard embedding for $n \geq 3$. It is, moreover, conjectured that joins are not preserved in general for $n \geq 3$.

In the next section, three key properties of the Dedekind–MacNeille completion of n-ordered sets will be studied. The first, called *density*, generalizes a well-known 2-dimensional analog to n dimensions. Density is the key property used in the characterization of the Dedekind–MacNeille completion of a poset. It will also be key in the characterization of the Dedekind–MacNeille completion of a joined n-ordered set, that will be provided in the last section of the paper. The second condition, called *closure*, is a consequence of density in two dimensions, but this does not seem to be the case in arbitrary dimensions. Roughly speaking, a subset P of an n-ordered set $\mathbf{Q}$ is closed in $\mathbf{Q}$, if, for all $1 \leq i \leq n$, every element $p \in P$ that forms as the ith coordinate with jth coordinates all elements in P $\lesssim_j$-below a fixed element $q \in Q$, $j \neq i$, a joined n-tuple, i.e., such that

$$(p_1, \ldots, p_{i-1}, p, p_{i+1}, \ldots, p_n) \text{ is joined for all } p_j \in P, \text{ such that } p_j \lesssim_j q, \; j \neq i,$$

has to be $\lesssim_i$-below q. The third condition to be studied in the following section is *separation*. Separation is also a consequence of density in 2 dimensions, but this does not seem to be the case in arbitrary dimensions either. Therefore, as is the case with closure, this condition does not appear explicitly in the 2-dimensional characterization theorem, but it is also a key condition in our n-dimensional analog.

4 Properties of the Completion: Density, Closure, Separation and Joinedness

Let $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n \rangle$ be an n-ordered set, $P \subseteq Q$ and $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. Then P is said to be $(j_{n-1}, \ldots, j_1)$-*join dense* in $\mathbf{Q}$ if, for every element $q \in Q$, there are subsets $A_1, \ldots, A_{n-1} \subseteq P$, such that $q = \nabla_{j_{n-1},\ldots,j_1}(A_{n-1}, \ldots, A_1)$. The subset P is said to be *dense* in $\mathbf{Q}$ if it is $(j_{n-1}, \ldots, j_1)$-dense, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$.

This condition is an adaptation to n dimensions of the corresponding condition for posets, given, e.g., in Definition 2.34 of [3]. Lemma 8 is the analog in n dimensions of the 2-dimensional Lemma 2.35 of [3].

Lemma 8 *Let $\mathbf{Q}$ be an n-ordered set and $P \subseteq Q$. The following are related by* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *in general and are equivalent if $\mathbf{Q}$ is a complete n-lattice:*

1. *P is $(j_{n-1}, \ldots, j_1)$-join-dense in $\mathbf{Q}$;*
2. *$q = \nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}}, \ldots, (q]^P_{j_1})$, for every $q \in Q$;*
3. *for all $q, q' \in Q$, with $q <_{j_n} q'$, there exists $i \neq n$ and $p \in P$, such that $p \lesssim_{j_i} q$ and $p \not\lesssim_{j_i} q'$.*

Proof

(1) $\Rightarrow$ (2) Assume that P is $(j_{n-1},\ldots,j_1)$-join-dense in $\mathbf{Q}$ and let $q\in Q$. Then, there exist $A_{n-1},\ldots,A_1\subseteq P$, such that $q=\nabla_{j_{n-1},\ldots,j_1}(A_{n-1},\ldots,A_1)$. Clearly, q is a $(j_{n-1},\ldots,j_1)$-bound of $((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$. If b is any $(j_{n-1},\ldots,j_1)$-bound of $((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, then, for all $p_{j_i}\in(q]^P_{j_i}$, $p_{j_i}\lesssim_{j_i} b$, $i=1,\ldots,n-1$, whence, for all $a_i\in A_i$, $a_i\lesssim_{j_i} b$ and, therefore, by the limit property of joins, $b\lesssim_{j_n}\nabla_{j_{n-1},\ldots,j_1}(A_{n-1},\ldots,A_1)=q$. Thus q is a $(j_{n-1},\ldots,j_1)$-limit of $((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$. Now suppose that l is a $(j_{n-1},\ldots,j_1)$-limit of $((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$. Then, since q is also a $(j_{n-1},\ldots,j_1)$-limit of $((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, we get that $l\sim_{j_n} q=\nabla_{j_{n-1},\ldots,j_1}(A_{n-1},\ldots,A_1)$. Thus l is a $(j_{n-1},\ldots,j_1)$-bound of $(A_{n-1},\ldots,A_1)$ and $\sim_{j_n}$-equivalent to $q=\nabla_{j_{n-1},\ldots,j_1}(A_{n-1},\ldots,A_1)$, which implies, by the join property, that $l\lesssim_{j_{n-1}}\nabla_{j_{n-1},\ldots,j_1}(A_{n-1},\ldots,A_1)=q$. Similarly, it may be shown that, for all $k=1,\ldots,n-1$, if both l and q are $(j_{n-1},\ldots,j_1)$-bounds of $((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, that are largest in $\lesssim_{j_{k+1}}$ among the largest in $\lesssim_{j_{k+2}}$, among ... among the largest in $\lesssim_{j_n}$, then $l\lesssim_{j_k} q$. This proves that

$$q=\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1}).$$

(2) $\Rightarrow$ (1) This part is obvious by the definition of $(j_{n-1},\ldots,j_1)$-density.

(2) $\Rightarrow$ (3) Suppose that, for all $q\in Q$, $q=\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$ and let $q,q'\in Q$, with $q<_{j_n} q'$. If, for all $i\neq n$ and all $p\in P$, $p\lesssim_{j_i} q$ implies $p\lesssim_{j_i} q'$, we have that $(q]^P_{j_i}\subseteq(q']^P_{j_i}$, $i=1,\ldots,n-1$, whence, by the limit property of joins, $\nabla_{j_{n-1},\ldots,j_1}((q']^P_{j_{n-1}},\ldots,(q']^P_{j_1})\lesssim_{j_n}\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, i.e., that $q'\lesssim_{j_n} q$, which contradicts the hypothesis.

Finally, assume that $\mathbf{Q}$ is a complete n-lattice and suppose that for all $q,q'\in Q$, with $q<_{j_n} q'$, there exists $i\neq n$ and $p\in P$, such that $p\lesssim_{j_i} q$ and $p\not\lesssim_{j_i} q'$. Then, suppose, for the sake of obtaining a contradiction, that $q\in Q$, is such that $q\neq\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$. Then, there exists k, $1\le k\le n$, such that $q\sim_{j_l}\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, for all $l>k$, and $q<_{j_k}\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$. Thus, by the hypothesis, there exists $i\neq k$ and $p\in P$, such that $p\lesssim_{j_i} q$ but $p\not\lesssim_{j_i}\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$. Since $q\sim_{j_l}\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, for all $l>k$, this cannot happen for $i>k$, whence $i<k$. But $p\lesssim_{j_i} q$ and $p\not\lesssim_{j_i}\nabla_{j_{n-1},\ldots,j_1}((q]^P_{j_{n-1}},\ldots,(q]^P_{j_1})$, for some $i<k$, contradicts the bound property of a join! □

We prove now a lemma characterizing the $(j_{n-1},\ldots,j_1)$-joins in the Dedekind–MacNeille completion of an n-ordered set $\mathbf{P}$ of the form $\nabla_{j_{n-1},\ldots,j_1}(e(A_{n-1}),\ldots,e(A_1))$ for some $A_1,\ldots,A_{n-1}\subseteq P$.

Lemma 9 *Let* $\mathbf{P}$ *be an n-ordered set and let* $e:\mathbf{P}\to\mathbf{DM}(\mathbf{P})$ *be the standard n-order embedding of* $\mathbf{P}$ *into its Dedekind–MacNeille completion. Then, for all* $A_1,\ldots,A_{n-1}\subseteq P$,

$$\nabla_{j_{n-1},\ldots,j_1}(e(A_{n-1}),\ldots,e(A_1))=\mathfrak{b}_{j_{n-1},\ldots,j_1}((A_{n-1}]_{j_{n-1}},\ldots,(A_1]_{j_1}).$$

Proof The equation holds since, for all $a_i \in A_i$, $i = 1, \ldots, n-1$, $e(a_i) = \langle (a_i]_1, \ldots, (a_i]_n \rangle$, whence

$$\nabla_{j_{n-1},\ldots,j_1}(e(A_{n-1}), \ldots, e(A_1)) = \mathfrak{b}_{j_{n-1},\ldots,j_1}\left(\bigcup_{a_{n-1}\in A_{n-1}} (a_{n-1}]_{j_{n-1}}, \ldots, \bigcup_{a_1 \in A_1} (a_1]_{j_1} \right),$$

by the definition of $\nabla_{j_{n-1},\ldots,j_1}$ in the Dedekind–MacNeille completion of $\mathbf{P}$. □

Next, it is shown that the image of an n-ordered set into its Dedekind–MacNeille completion via the standard embedding is a dense subset of the completion. Theorem 8 generalizes to n dimensions the corresponding result for posets (see, e.g., Theorem 2.36 (i) of [3]).

Theorem 10 (Density) *Let* $\mathbf{P}$ *be an* n*-ordered set and let* $e : \mathbf{P} \to \mathbf{DM}(\mathbf{P})$ *be the standard* n*-order embedding of* $\mathbf{P}$ *into its Dedekind–MacNeille completion. The image* $e(P)$ *is dense in* $\mathbf{DM}(\mathbf{P})$.

Proof Since $\mathbf{DM}(\mathbf{P})$ is a complete n-lattice, Lemma 8 (3) may be used to show that $e(P)$ is dense in $\mathbf{DM}(\mathbf{P})$.

We must show that $e(P)$ is $(j_{n-1}, \ldots, j_1)$-join dense in $\mathbf{DM}(\mathbf{P})$, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. We do this only for $(n-1, \ldots, 1)$-join density since all other cases follow by symmetry.

To this end, suppose that $\langle A_1, \ldots, A_n\rangle, \langle B_1, \ldots, B_n\rangle \in \mathrm{DM}(\mathbf{P})$, are such that $\langle A_1, \ldots, A_n\rangle <_n \langle B_1, \ldots, B_n\rangle$. Then $A_n \subset B_n$. Therefore, by antiordinality in $\mathbf{DM}(\mathbf{P})$, there exists $i \neq n$, such that $A_i \not\subseteq B_i$. Hence, there exists $p \in P$, such that $p \in A_i$ and $p \notin B_i$. But this shows that, for this p, $e(p) \subseteq_i \langle A_1, \ldots, A_n\rangle$ and $e(p) \not\subseteq_i \langle B_1, \ldots, B_n\rangle$. Now Lemma 8 may be invoked to conclude that $e(P)$ is $(n-1, \ldots, 1)$-join dense in $\mathbf{DM}(\mathbf{P})$. □

Suppose that $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n\rangle$ is an n-ordered set, $P \subseteq Q$ and $1 \leq i \leq n$. P is said to be *i-closed in* $\mathbf{Q}$ if, for all $q \in Q$ and $x \in P$, $\langle p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_n\rangle \in R(\mathbf{P})$, for all $p_j \in P$, with $p_j \lesssim_j q$, $j \neq i$, implies $x \lesssim_i q$. It is said to be *closed in* $\mathbf{Q}$ if it is i-closed in $\mathbf{Q}$, for all $i = 1, \ldots, n$.

Note that, in the dyadic case, every dense subset of a poset is also a closed subset.

It is now shown that, for every n-ordered set $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$, $e(P)$ is closed in the Dedekind–MacNeille completion $\mathbf{DM}(\mathbf{P})$.

Theorem 11 (Closure) *Let* $\mathbf{P}$ *be an* n*-ordered set and let* $e : \mathbf{P} \to \mathbf{DM}(\mathbf{P})$ *be the standard* n*-order embedding of* $\mathbf{P}$ *into its Dedekind–MacNeille completion. The image* $e(P)$ *is closed in* $\mathbf{DM}(\mathbf{P})$.

Proof We show n-closure; the remaining closures follow by symmetry. Suppose that $x \in P$, and that, for all $p_1, \ldots, p_{n-1} \in P$, such that

$$e(p_1) \subseteq_1 \langle A_1, \ldots, A_n\rangle, \ldots, e(p_{n-1}) \subseteq_{n-1} \langle A_1, \ldots, A_n\rangle,$$

$\langle p_1, \ldots, p_{n-1}, x\rangle \in R(\mathbf{P})$. This yields that, for all $p_1, \ldots, p_{n-1} \in P$, such that $p_1 \in A_1, \ldots, p_{n-1} \in A_{n-1}$, we have that $\langle p_1, \ldots, p_{n-1}, x\rangle \in R(\mathbf{P})$. Thus, by the meaning of $\langle A_1, \ldots, A_n\rangle \in \mathrm{DM}(\mathbf{P}) := \mathcal{C}(\mathbb{P})$, we must have $x \in A_n$ and, therefore, $e(x) \subseteq_n \langle A_1, \ldots, A_n\rangle$, i.e., $e(P)$ is n-closed in $\mathbf{DM}(\mathbf{P})$. □

Let $\mathbf{Q}$ be an n-ordered set with $P \subseteq Q$, as before. For $i = 1, \ldots, n$, P is said to be *i-separating* in $\mathbf{Q}$ if, for all $x, y \in Q$, such that $x \not\lesssim_i y$, there exists $p \in P$, such that $p \lesssim_i x$ and $p \not\lesssim_i y$. It is said to be *separating in* $\mathbf{Q}$ if it is i-separating, for all $i = 1, \ldots, n$.

We note that, in the dyadic case (posets) separability follows from density, since the condition defining separability becomes equivalent to Condition 3 of Lemma 8.

In the next result, it is shown that $e(P)$ is separating in $\mathbf{DM}(\mathbf{P})$, for every n-ordered set $\mathbf{P}$.

Theorem 12 (Separation) *Let* $\mathbf{P}$ *be an n-ordered set and let* $e : \mathbf{P} \to \mathbf{DM}(\mathbf{P})$ *be the standard n-order embedding of* $\mathbf{P}$ *into its Dedekind–MacNeille completion. The image* $e(P)$ *is separating in* $\mathbf{DM}(\mathbf{P})$.

Proof Suppose that $\langle A_1, \ldots, A_n\rangle, \langle B_1, \ldots, B_n\rangle \in \mathrm{DM}(\mathbf{P})$ are such that $\langle A_1, \ldots, A_n\rangle \not\subseteq_i \langle B_1, \ldots, B_n\rangle$. Then $A_i \not\subseteq B_i$, whence, there exists $p \in P$, such that $p \in A_i$ and $p \notin B_i$. Hence $(p]_i \subseteq A_i$ and $(p]_i \not\subseteq B_i$, which shows that $e(p) \subseteq_i \langle A_1, \ldots, A_n\rangle$ and $e(p) \not\subseteq_i \langle B_1, \ldots, B_n\rangle$. Therefore $e(P)$ is i-separating in $\mathbf{DM}(\mathbf{P})$, for all $i = 1, \ldots, n$. □

Finally, we introduce the notion of joinedness. The absolute version refers to an arbitrary n-ordered set standing on its own. The relative version refers to a subset of a given n-ordered set $\mathbf{Q}$ being joined in $\mathbf{Q}$. We urge the reader to notice that in the dyadic case absolute joinedness is a trivial condition, i.e., all posets are joined 2-ordered sets.

Let $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$ be an n-ordered set and $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. $\mathbf{P}$ is said to be $(j_{n-1}, \ldots, j_1)$-*joined* if, for all $p_{n-1}, \ldots, p_1 \in P$, $(p_{n-1}, \ldots, p_1)$ has a $(j_{n-1}, \ldots, j_1)$-limit in $\mathbf{P}$. $\mathbf{P}$ is said to be *joined* if it is $(j_{n-1}, \ldots, j_1)$-joined, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$.

Next, let $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n\rangle$ be an n-ordered set and $P \subseteq Q$. For fixed $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$, P is said to be $(j_{n-1}, \ldots, j_1)$-*joined in* $\mathbf{Q}$ if, for all $p_1, \ldots, p_{n-1} \in P$, there exists $p \in P$, such that p is a $(j_{n-1}, \ldots, j_1)$-limit of $(\{p_{n-1}\}, \ldots, \{p_1\})$ in $\mathbf{Q}$. P is said to be *joined in* $\mathbf{Q}$ if it is $(j_{n-1}, \ldots, j_1)$-joined in $\mathbf{Q}$, for all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$.

It is now shown that if $\mathbf{P}$ is a joined n-ordered set, then it is joined in its Dedekind–MacNeille completion $\mathbf{DM}(\mathbf{P})$.

Proposition 13 *If* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$ *is a joined n-ordered set, then* $e(P)$ *is joined in* $\mathbf{DM}(\mathbf{P})$.

Proof Suppose that $\mathbf{P}$ is joined. To show that in that case $e(P)$ is joined in $\mathbf{DM}(\mathbf{P})$, it suffices to show, by symmetry, that, for all $p_1, \ldots, p_{n-1} \in P$, if l is an $(n-1, \ldots, 1)$-limit of $(p_{n-1}, \ldots, p_1)$ in $\mathbf{P}$, then $e(l)$ is an $(n-1, \ldots, 1)$-limit of $(e(p_{n-1}), \ldots, e(p_1))$ in $\mathbf{DM}(\mathbf{P})$. Obviously, since $p_i \lesssim_i l, i = 1, \ldots, n-1$, we get that $e(p_i) \subseteq_i e(l), i = 1, \ldots, n-1$. Therefore, by the limit property of joins,

$$e(l) \subseteq_n \nabla_{n-1,\ldots,1}(e(p_{n-1}), \ldots, e(p_1)).$$

To show that $\nabla_{n-1,\dots,1}(e(p_{n-1}),\dots,e(p_1)) \subseteq_n e(l)$, it suffices to show, by Lemma 9 and the definition of $\mathfrak{b}_{n-1,\dots,1}$, that $R(\mathbf{P})_n((p_1]_1,\dots,(p_{n-1}]_{n-1}) \subseteq (l]_n$. To this end, suppose that $x_n \in R(\mathbf{P})_n((p_1]_1,\dots,(p_{n-1}]_{n-1})$. Then there exists $p \in P$, such that $p_i \lesssim_i p, i = 1,\dots,n-1$, and $x_n \lesssim_n p$. Thus, since l is an $(n-1,\dots,1)$-limit of $(p_{n-1},\dots,p_1)$ in $\mathbf{P}$, we get that $p \lesssim_n l$. Hence $x_n \lesssim_n l$ and $x_n \in (l]_n$. □

In the next section a characterization of the Dedekind–MacNeille completion of an n-ordered set, similar to the one existing for the dyadic case (i.e., the one for posets) will be provided. Details for the dyadic case may be found, for instance, in Theorem 2.36 of [3].

5 A Uniqueness Theorem

In this section, a uniqueness theorem for the Dedekind–MacNeille completion of a joined n-ordered set is proved. This theorem (Theorem 15) generalizes the 2-dimensional Theorem 2.36 of [3] to n dimensions. Before Theorem 15, a few intermediate results will be presented. They all constitute steps in the proof of the main theorem and treat more general cases and, as a consequence, have weaker conclusions.

First, it is shown that if P is a subset of an n-ordered set $\mathbf{Q}$ that is joined, dense and closed in $\mathbf{Q}$, then, there exists an n-order homomorphism $h : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$, i.e., a mapping $h : Q \to \mathrm{DM}(\mathbf{P})$, such that, for all $x, y \in Q$ and all $i = 1,\dots,n$,

$$x \lesssim_i y \quad \text{implies} \quad h(x) \subseteq_i h(y),$$

that agrees with the standard embedding $e : P \to \mathrm{DM}(\mathbf{P})$ on P.

Theorem 14 (Joined, Dense and Closed Subsets) *Let $\mathbf{Q} = \langle Q, \lesssim_1,\dots,\lesssim_n\rangle$ be an n-ordered set, $\mathbf{P} = \langle P, \lesssim_1,\dots,\lesssim_n\rangle$ an induced sub-n-ordered set of $\mathbf{Q}$, such that P is joined, dense and closed in $\mathbf{Q}$, and $e : \mathbf{P} \to \mathbf{DM}(\mathbf{P})$ the standard n-order embedding of $\mathbf{P}$ into its Dedekind–MacNeille completion. Then, there exists an n-order homomorphism $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$. Moreover ϕ agrees with e on P, that is $\phi(p) = e(p)$, for all $p \in P$.*

Proof Suppose, first, that $\mathbf{Q} = \langle Q, \lesssim_1,\dots,\lesssim_n\rangle$ is an n-ordered set and $P \subseteq Q$ is joined, dense and closed in $\mathbf{Q}$. Define the mapping $\phi : Q \to \mathrm{DM}(\mathbf{P})$ by setting, for all $q \in Q$,

$$\phi(q) = \langle (q]_1^P,\dots,(q]_n^P\rangle.$$

It will be shown that $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$ is an n-order homomorphism and that it agrees with e on P.

First, to see that ϕ is well-defined, it must be shown that $\phi(q) \in \mathrm{DM}(\mathbf{P})$, for all $q \in Q$. To this end, it must be shown that, for all $i = 1,\dots,n$,

$$(q]_i^P = R(\mathbf{P})_i\left((q]_1^P,\dots,(q]_{i-1}^P,(q]_{i+1}^P,\dots,(q]_n^P\right).$$

It is just shown that $(q]_n^P = R(\mathbf{P})_n\left((q]_1^P,\dots,(q]_{n-1}^P\right)$, since the remaining relations, then, follow by symmetry.

For the left-to-right inclusion, suppose that $x_n \in (q]^P_n$ and that $p_i \in (q]^P_i$, for all $i = 1, \ldots, n-1$. By $(n-1, \ldots, 1)$-joinedness, there exists $p \in P$, such that p is an $(n-1, \ldots, 1)$-limit of $(p_{n-1}, \ldots, p_1)$ in Q. Thus $p_i \lesssim_i p, i = 1, \ldots, n-1$, and, by $(j_{n-1}, \ldots, j_1)$-join density, $q \lesssim_n p$. Therefore $p_i \lesssim_i p$, for all $i = 1, \ldots, n-1$, and $x_n \lesssim_n q \lesssim_n p$. Hence $\langle p_1, \ldots, p_{n-1}, x_n \rangle \in R(\mathbf{P})$, i.e., $x_n \in R(\mathbf{P})_n\left((q]^P_1, \ldots, (q]^P_{n-1}\right)$.

For the right-to-left inclusion, suppose that $x_n \in P$, such that, for all $p_i \in (q]^P_i, i = 1, \ldots, n-1$, $\langle p_1, \ldots, p_{n-1}, x_n \rangle \in R(\mathbf{P})$. Hence, for all $p_i \in P$, such that $p_i \lesssim_i q, i = 1, \ldots, n-1$, $\langle p_1, \ldots, p_{n-1}, x_n \rangle \in R(\mathbf{P})$, whence, by the closure property, we get that $x_n \lesssim_n q$ and $x_n \in (q]^P_n$.

Next, we show that $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$ is an n-order homomorphism.

If $x, y \in Q$, such that $x \lesssim_i y$, and $p \in (x]^P_i$, then $p \lesssim_i x$, whence $p \lesssim_i y$ and $p \in (y]^P_i$, yielding that $(x]^P_i \subseteq (y]^P_i$ and, therefore $\phi(x) \subseteq_i \phi(y)$.

The last statement is obvious, since, for all $p \in P$, we have

$$\phi(p) = \langle (p]^P_1, \ldots, (p]^P_n \rangle = e(p).$$

□

Since n-lattices are joined n-ordered sets, if in Theorem 14 the n-ordered set $\mathbf{P}$ happens to be an n-lattice, then the hypothesis of joinedness may be dropped.

Corollary 15 (Dense and Closed *n*-lattices) *Let $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n \rangle$ be an n-ordered set, $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ an induced sub-n-ordered set of $\mathbf{Q}$, that is an n-lattice and is dense and closed in $\mathbf{Q}$, and $e : \mathbf{L} \to \mathbf{DM}(\mathbf{L})$ the standard n-order embedding of $\mathbf{L}$ into its Dedekind–MacNeille completion. Then, there exists an n-order homomorphism $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{L})$. Moreover ϕ agrees with e on L, that is $\phi(x) = e(x)$, for all $x \in L$.*

Proof First, notice that, since $\mathbf{L}$ is an n-lattice, it is a joined n-ordered set. Therefore, by Proposition 13, L is also joined in $\mathbf{Q}$. Thus, all hypotheses of Theorem 14 are satisfied and, therefore, there is an n-order homomorphism ϕ of $\mathbf{Q}$ into $\mathbf{DM}(\mathbf{L})$ that agrees with e on L. □

Next, it is shown that, if the hypothesis of separation is added in Theorem 14, then the n-order homomorphism guaranteed by the conclusion of the theorem is actually an n-order embedding $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$. Separation is the property that forces injectivity of ϕ. This condition is similar to the one used for Boolean algebras in the dyadic case (see [5], Definition 4.15).

Theorem 16 (Joined, Dense, Closed and Separating Subsets) *Let $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n \rangle$ be an n-ordered set, $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ an induced sub-n-ordered set of $\mathbf{Q}$, such that P is joined, dense, closed and separating in $\mathbf{Q}$, and $e : \mathbf{P} \to \mathbf{DM}(\mathbf{P})$ the standard n-order embedding of $\mathbf{P}$ into its Dedekind–MacNeille completion. Then, there exists an n-order embedding $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$ that agrees with e on P.*

Proof Since P is joined, dense and closed in $\mathbf{Q}$, there exists, by Theorem 14 an n-order homomorphism $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$ that agrees with e on P. Moreover ϕ is given by

$$\phi(q) = \langle (q]^P_1, \ldots, (q]^P_n \rangle, \quad \text{for all} \quad q \in Q.$$

To show that ϕ is an n-order embedding, it suffices to show that, for all $x, y \in Q$, and all $i = 1, \ldots, n$, if $\phi(x) \subseteq_i \phi(y)$, then $x \lesssim_i y$.

To this end, let $x, y \in Q$, such that $\phi(x) \subseteq_i \phi(y)$. Then, by the definition of ϕ, $(x]_i^P \subseteq (y]_i^P$. Thus

$$p \lesssim_i x, \quad \text{implies} \quad p \lesssim_i y, \quad \text{for all } p \in P. \tag{1}$$

Assume, for the sake of obtaining a contradiction, that $x \not\lesssim_i y$. Then, by i-separation, there exists $p \in P$, such that $p \lesssim_i x$ and $p \not\lesssim_i y$. This contradicts Implication (1) and concludes the proof that ϕ is also an n-order embedding. □

Analogously with the case of a joined, dense and closed subsets of an n-ordered set, if the n-ordered set $\mathbf{P}$ in the hypothesis of Theorem 16 happens to be an n-lattice, then it is automatically joined and, therefore, Theorem 16 assumes the form of

Corollary 17 (Dense, Closed and Separating *n*-lattices) *Let $\mathbf{Q} = \langle Q, \lesssim_1, \ldots, \lesssim_n\rangle$ be an n-ordered set, $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n\rangle$ an induced sub-n-ordered set of $\mathbf{Q}$, that is an n-lattice and such that L is dense, closed and separating in $\mathbf{Q}$, and $e : \mathbf{L} \to \mathbf{DM}(\mathbf{L})$ the standard n-order embedding of $\mathbf{L}$ into its Dedekind–MacNeille completion. Then, there exists an n-order embedding $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{L})$ that agrees with e on L.*

Proof From Theorem 16 and Proposition 13, given that an n-lattice is a joined n-ordered set. □

Finally, it is shown that, if the n-ordered set $\mathbf{Q}$ in the hypothesis of Theorem 16 happens to be a complete n-lattice, then the n-order embedding $\phi : \mathbf{Q} \to \mathbf{DM}(\mathbf{P})$ guaranteed by the conclusion of the theorem becomes an n-order isomorphism.

Theorem 18 (Joined, Dense, Closed and Separating Subsets in Complete *n*-lattices) *Let $\mathbf{C} = \langle C, \lesssim_1, \ldots, \lesssim_n\rangle$ be a complete n-lattice, $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n\rangle$ an induced sub-n-ordered set of $\mathbf{C}$, such that P is joined, dense, closed and separating in $\mathbf{C}$, and $e : \mathbf{P} \to \mathbf{DM}(\mathbf{P})$ the standard n-order embedding of $\mathbf{P}$ into its Dedekind–MacNeille completion. Then $\mathbf{C} \cong \mathbf{DM}(\mathbf{P})$ via an n-order isomorphism which agrees with e on P.*

Proof Suppose that $\mathbf{C}$ is a complete n-lattice and let P be a subset of C which is joined, dense, closed and separating in $\mathbf{C}$. Then $\phi : \mathbf{C} \to \mathbf{DM}(\mathbf{P})$, defined by

$$\phi(q) = \langle (q]_1^P, \ldots, (q]_n^P\rangle, \quad \text{for all } q \in C,$$

is, by Theorem 16, an n-order embedding which agrees with e on P. Thus, it suffices to show that $\phi : \mathbf{C} \to \mathbf{DM}(\mathbf{P})$ is onto. To see this, let $\langle A_1, \ldots, A_n\rangle \in \mathrm{DM}(\mathbf{P})$. Consider the element $q \in C$, defined by

$$q = \nabla_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1).$$

It will be shown that $\phi(q) := \langle (q]_1^P, \ldots, (q]_n^P\rangle = \langle A_1, \ldots, A_n\rangle$.

Clearly, for $i = 1, \ldots, n-1$, if $p_i \in A_i$, then, by the bound property of joins, $p_i \lesssim_i \nabla_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1) = q$. Thus $\langle A_1, \ldots, A_n\rangle \subseteq_i \phi(q)$, for all $i = 1, \ldots, n-1$.

To conclude the proof, it suffices, by the uniqueness condition, to show that $\langle A_1, \ldots, A_n \rangle \subseteq_n \phi(q)$, i.e., that

$$A_n \subseteq (\nabla_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)]_n^P.$$

If $x_n \in A_n$, then, by the concept closure of $\langle A_1, \ldots, A_n \rangle$ with respect to $R(\mathbf{P})$, for all $p_i \in P$, such that $p_i \lesssim_i \nabla_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)$, $i = 1, \ldots, n-1$,, we have that, $\langle p_1, \ldots, p_{n-1}, x_n \rangle \in R(\mathbf{P})$. Hence, by the closure of P in $\mathbf{C}$, we get that $x_n \lesssim_n \nabla_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)$, i.e., that $A_n \subseteq (\nabla_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)]_n^P$. □

Corollary 19 (Dense, Closed and Separating sub-lattices of Complete *n*-lattices) *Let* $\mathbf{C} = \langle C, \lesssim_1, \ldots, \lesssim_n \rangle$ *be a complete n-lattice,* $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ *a sub-n-lattice of* $\mathbf{C}$*, such that* L *is dense, closed and separating in* $\mathbf{C}$*, and* $e : \mathbf{L} \to \mathbf{DM}(\mathbf{L})$ *the standard n-order embedding of* $\mathbf{L}$ *into its Dedekind–MacNeille completion. Then* $\mathbf{C} \cong \mathbf{DM}(\mathbf{L})$ *via an n-order isomorphism which agrees with e on L.*

Proof It suffices to notice that, in Theorem 15, if the n-ordered set $\mathbf{P}$ happens to be an n-lattice, then P is automatically joined and, therefore, by Proposition 13, it is also joined in $\mathbf{C}$. □

An important question that remains open is whether the Dedekind–MacNeille completion is the smallest completion of an n-ordered set. We close by formally posing it as an open problem.

Open Problem Does every n-order embedding $h : \mathbf{P} \to \mathbf{C}$ of an n-ordered set $\mathbf{P}$ into a complete n-lattice $\mathbf{C}$ extend to an embedding $\phi : \mathbf{DM}(\mathbf{P}) \to \mathbf{C}$ of its Dedekind–MacNeille completion $\mathbf{DM}(\mathbf{P})$ into $\mathbf{C}$?

$$\begin{tikzcd} \mathbf{P} \arrow[rr, "e"] \arrow[dr, "h"'] & & \mathbf{DM}(\mathbf{P}) \arrow[dl, "\phi"] \\ & \mathbf{C} & \end{tikzcd}$$

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