Math. Log. Quart. **52**, No. 3, 288 – 294 (2006) / **DOI** 10.1002/malq.200510034

Categorical abstract algebraic logic: The largest theory system included in a theory family

George Voutsadakis*

School of Mathematics and Computer Science, Lake Superior State University,
Sault Sainte Marie, MI 49783, USA

Received 8 August 2005, revised 16 March 2006, accepted 20 March 2006
Published online 1 May 2006

Key words Algebraic logic, equivalent deductive systems, equivalent institutions, algebraizable logics, algebraizable institutions, lattice of theories, Leibniz operator, Leibniz theory systems, protoalgebraic logics, protoalgebraicity, prealgebraicity.

MSC (2000) 03G99, 18C15, 08C05, 08B05, 68N30

In this note, it is shown that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, every theory family T of $\mathcal{I}$ includes a unique largest theory system $\overleftarrow{T}$ of $\mathcal{I}$. $\overleftarrow{T}$ satisfies the important property that its N-Leibniz congruence system always includes that of T. As a consequence, it is shown, on the one hand, that the relation $\Omega^N(\overleftarrow{T}) = \Omega^N(T)$ characterizes N-protoalgebraicity inside the class of N-prealgebraic π-institutions and, on the other, that all N-Leibniz theory families associated with theory families of a protoalgebraic π-institution $\mathcal{I}$ are in fact N-Leibniz theory systems.

1 Introduction

The present note continues recent investigations in the categorical abstract algebraic logic theory of prealgebraicity and protoalgebraicity of π-institutions. Unlike other recent articles by the author that were based directly on attempts to abstract or generalize parts of the theory of algebraizability of deductive systems [2, 3] and of sentential logics [6], the present work has a unique flavor stemming from a peculiarity special in the categorical context.

In [18], an attempt was made to lift the notion of a protoalgebraic logic of Blok and Pigozzi [2] to the level of π-institutions. This attempt was primarily based on a novel notion of a Leibniz operator for π-institutions, the N-Leibniz operator, first introduced in [12] and further elaborated on in [13, 14, 15, 16, 17, 18, 19, 20]. Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, a *theory family of* $\mathcal{I}$ is a collection $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$ is a Σ-theory of $\mathcal{I}$, for all $\Sigma \in |\mathbf{Sign}|$. On the other hand, a *theory system* T *of* $\mathcal{I}$ is a theory family of $\mathcal{I}$ that is preserved by all signature morphisms, i. e., such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}$. As it became clear in the development of [18], in place of the monotonicity property of Blok and Pigozzi's Leibniz operator [3] on the theories of a deductive system or of a sentential logic, two alternatives may be used in the π-institution framework: either monotonicity of the N-Leibniz operator on theory families or monotonicity of the N-Leibniz operator on theory systems. Since all theory systems are theory families, monotonicity on theory systems seems to define a class of π-institutions, called the N-prealgebraic π-institutions in [18], that is wider than the one obtained by monotonicity of the N-Leibniz operator on theory families, called the class of N-protoalgebraic π-institutions in [18]. In fact it was shown via an example in [18] that the inclusion of N-protoalgebraic π-institutions inside N-prealgebraic π-institutions is proper in general. More specifically, if $\mathcal{I}_\mathcal{S}$ is the canonical π-institution associated with a sentential logic $\mathcal{S}$ in the sense of [10, Section 3] or [11, Section 2.1], then the notions of N-prealgebraicity and N-protoalgebraicity for π-institutions of this form do not coincide, since theory families of $\mathcal{I}_\mathcal{S}$ are just theories of the sentential logic $\mathcal{S}$

* e-mail: gvoutsad@lssu.edu

whereas theory systems of $\mathcal{I}_{\mathcal{S}}$ are theories of $\mathcal{S}$ that are closed under substitutions. For instance, it is known that the $\{\wedge, \vee\}$-fragment of classical propositional logic $\mathrm{CPC}_{\wedge\vee}$ is not protoalgebraic and hence is not N-protoalgebraic. But it is N-prealgebraic, since there are only two theory systems, $\{\emptyset\}$ and $\{\mathrm{Fm}_{\mathcal{L}}(V)\}$, which renders Ω^N monotonic on theory systems. The present work further clarifies the reasons why these two classes are different.

Besides this desire to further clarify the prealgebraicity-protoalgebraicity question, one more difference between the sentential and the categorical framework motivated the current work. In [20], the notion of a Leibniz theory of a sentential logic [7, 8] was adapted in order to obtain the corresponding notion of an N-Leibniz theory system of a π-institution. Since in π-institutions the dichotomy between theory families and theory systems is ever present, it may be possible, to each given theory family T in an N-protoalgebraic π-institution, to associate either its N-Leibniz theory family or its N-Leibniz theory system by taking intersections of theory families or theory systems, respectively, having the same Leibniz N-congruence system. Which of these two should then be chosen to play the role that Leibniz theories play in the sentential framework? This question is settled in this note by showing that, if the π-institution is N-protoalgebraic, then all N-Leibniz theory families are actually N-Leibniz theory systems so that no choice is necessary in this case.

Both the prealgebraicity-protoalgebraicity issue and the Leibniz theory family-theory system question are addressed via a common construction which seems to have some importance of its own. More precisely, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a theory family T of $\mathcal{I}$, T is shown to contain a largest theory system $\overleftarrow{T}$. If, in addition N is a category of natural transformations on SEN, $\overleftarrow{T}$ is shown to have the remarkable property that $\Omega^N(T) \leq \Omega^N(\overleftarrow{T})$. Note that this goes in the opposite direction of $\overleftarrow{T} \leq T$, which holds by the definition of $\overleftarrow{T}$. Once this relationship between the two Leibniz N-congruence systems is established, it is not very difficult to obtain the two results dealing with N-protoalgebraicity and with N-Leibniz theory families. Theorem 4.1 gives a characterization of the class of N-protoalgebraic π-institutions inside the class of N-prealgebraic π-institutions. Namely, it is shown that an N-prealgebraic π-institution $\mathcal{I}$ is N-protoalgebraic if and only if the Leibniz N-congruence system corresponding to $\overleftarrow{T}$ is identical with the Leibniz N-congruence system corresponding to T, for every theory family T of $\mathcal{I}$. This latter condition is not universally true for all π-institutions, all categories of natural transformations N on SEN and all theory families T, whence Theorem 4.1 provides a reason why N-prealgebraicity is a concept different, in general, than N-protoalgebraicity. Theorem 4.3, on the other hand, shows that given a theory family T of an N-protoalgebraic π-institution $\mathcal{I}$, the unique N-Leibniz theory family $T^{(N)}$ included in T coincides with the unique N-Leibniz theory system $\overleftarrow{T}^N$ included in $\overleftarrow{T}$, i. e., the unique N-Leibniz theory family included in any theory family of $\mathcal{I}$ (including all theory systems) is itself a theory system. Thus, the extra parenthesis $^{(N)}$ in the superscript for N-Leibniz theory families is not needed.

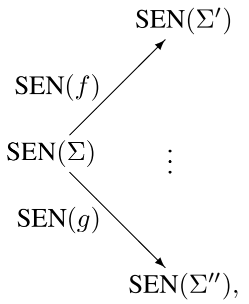
For general background on abstract algebraic logic, the reader is referred to the book [5] and the monograph [6]. For all unexplained categorical notation any of [1], [4] or [9] may be consulted.

2 Largest theory system in a theory family

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on SEN. We show in this section that, for every theory family T of $\mathcal{I}$, there exists a unique $\leq$-largest theory system $\overleftarrow{T}$ of $\mathcal{I}$ $\leq$-included in T. This result has two very important consequences for the theory of categorical abstract algebraic logic. On the one hand, it provides a characterization of the class of N-protoalgebraic π-institutions inside the class of N-prealgebraic π-institutions, as introduced in [18], and, on the other hand, it yields the interesting property that the N-Leibniz theory family $T^{(N)}$ corresponding to a given theory family T of an N-protoalgebraic π-institution $\mathcal{I}$ is an N-Leibniz theory system, as introduced in [20].

Let T be a theory family of the π-institution $\mathcal{I}$. Define the collection $\overleftarrow{T} = \{\overleftarrow{T}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overleftarrow{T}_\Sigma = \bigcap\{\mathrm{SEN}(f)^{-1}(T_{\Sigma'}) \,:\, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')\},$$



that is $\overleftarrow{T}_\Sigma$ is defined by pulling back along all signature morphisms with domain Σ. This description accounts for the choice of $\overleftarrow{}$ in our notation.

Proposition 2.1 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* T *a theory family of* $\mathcal{I}$*. Then* $\overleftarrow{T}$ *is a theory system of* $\mathcal{I}$*, such that* $\overleftarrow{T} \leq T$*.*

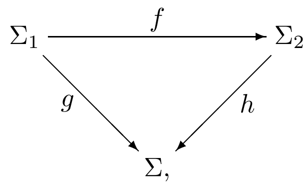
Proof. First, it is shown that $\overleftarrow{T}_\Sigma$ is a Σ-theory of $\mathcal{I}$, for all $\Sigma \in |\mathbf{Sign}|$. We do have, by definition,

$$\overleftarrow{T}_\Sigma = \bigcap\{\mathrm{SEN}(f)^{-1}(T_{\Sigma'}) \,:\, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')\},$$

whence $\overleftarrow{T}_\Sigma$ is a theory because it is the intersection of inverse images of theories, which are themselves theories.

That $\overleftarrow{T} \leq T$ is easily seen, since one of the theories in the intersection in the definition of $\overleftarrow{T}_\Sigma$ is the theory $\mathrm{SEN}(i_\Sigma)^{-1}(T_\Sigma) = T_\Sigma$.

So, it only remains to show that $\overleftarrow{T}$ is a theory system, rather than just a theory family. Suppose, to this end, that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$. Then, we have



$$\begin{aligned}
\overleftarrow{T}_{\Sigma_1} &= \bigcap\{\mathrm{SEN}(g)^{-1}(T_\Sigma) \,:\, \Sigma \in |\mathbf{Sign}|, g \in \mathbf{Sign}(\Sigma_1, \Sigma)\}\\
&\subseteq \bigcap\{\mathrm{SEN}(hf)^{-1}(T_\Sigma) \,:\, \Sigma \in |\mathbf{Sign}|, h \in \mathbf{Sign}(\Sigma_2, \Sigma)\}\\
&= \bigcap\{\mathrm{SEN}(f)^{-1}(\mathrm{SEN}(h)^{-1}(T_\Sigma)) \,:\, \Sigma \in |\mathbf{Sign}|, h \in \mathbf{Sign}(\Sigma_2, \Sigma)\}\\
&= \mathrm{SEN}(f)^{-1}(\bigcap\{\mathrm{SEN}(h)^{-1}(T_\Sigma) \,:\, \Sigma \in |\mathbf{Sign}|, h \in \mathbf{Sign}(\Sigma_2, \Sigma)\})\\
&= \mathrm{SEN}(f)^{-1}(\overleftarrow{T}_{\Sigma_2}),
\end{aligned}$$

and, therefore, $\mathrm{SEN}(f)(\overleftarrow{T}_{\Sigma_1}) \subseteq \overleftarrow{T}_{\Sigma_2}$, and $\overleftarrow{T}$ is a theory system of $\mathcal{I}$, as was to be shown. □

Abstractly, $\overleftarrow{T}$ may be characterized as the largest theory system of $\mathcal{I}$ $\leq$-included in the theory family T.

Proposition 2.2 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* T *a theory family of* $\mathcal{I}$*. Then* $\overleftarrow{T}$ *is the largest theory system of* $\mathcal{I}$ *that is* $\leq$*-included in* T*.*

Proof. Suppose that T' is a theory system of $\mathcal{I}$, such that $T' \leq T$. Let $\Sigma \in |\mathbf{Sign}|$, and $\varphi \in \mathrm{SEN}(\Sigma)$, such that $\varphi \in T'_\Sigma$. Therefore, since T' is, by hypothesis, a theory system of $\mathcal{I}$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\varphi) \in T'_{\Sigma'}$. But, also by hypothesis, $T' \leq T$, whence we obtain $\mathrm{SEN}(f)(\varphi) \in T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. But this is equivalent to $\varphi \in \mathrm{SEN}(f)^{-1}(T_{\Sigma'})$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, i. e., that

$$\varphi \in \bigcap\{\mathrm{SEN}(f)^{-1}(T_{\Sigma'}) : \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')\} = \overleftarrow{T}_\Sigma.$$

Therefore $T'_\Sigma \subseteq \overleftarrow{T}_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and, hence $T' \leq \overleftarrow{T}$. □

It is now obvious that the following holds:

Corollary 2.3 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* T *a theory family of* $\mathcal{I}$. T *is a theory system of* $\mathcal{I}$ *if and only if* $\overleftarrow{T} = T$.

P r o o f. The largest theory system that is included in a given theory family is the theory family itself if and only if the theory family is a theory system. Now use Proposition 2.2. □

Corollary 2.4 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* T, T' *theory families of* $\mathcal{I}$. *If* $T \leq T'$, *then* $\overleftarrow{T} \leq \overleftarrow{T'}$.

P r o o f. It suffices to notice that, if $T \leq T'$, then $\overleftarrow{T}$ is a theory system that is included in T', whence by Proposition 2.2, $\overleftarrow{T} \leq \overleftarrow{T'}$. □

3 Leibniz congruence systems

It will now be shown, in what is the main result of this note, that the Leibniz N-congruence system corresponding to the theory system $\overleftarrow{T}$ $\leq$-includes the Leibniz N-congruence system corresponding to the theory family T. But, first, an auxiliary technical lemma is needed to facilitate the main proof.

Lemma 3.1 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution,* N *a category of natural transformations on* SEN *and* T *a theory family of* $\mathcal{I}$. *Let* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. *Then, for all* $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ *in* N, *all* $\varphi \in \mathrm{SEN}(\Sigma)$ *and all* $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$$

if and only if, for all $\Sigma'' \in |\mathbf{Sign}|$ *and all* $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, $\sigma_{\Sigma''}(\mathrm{SEN}(gf)(\varphi), \mathrm{SEN}(g)^{n-1}(\vec{\chi})) \in T_{\Sigma''}$.

P r o o f. Suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. If $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ is a natural transformation in N, $\varphi \in \mathrm{SEN}(\Sigma)$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$, we have

$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$

iff $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}) \in \mathrm{SEN}(g)^{-1}(T_{\Sigma''})$, for all $\Sigma'' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$

iff $\mathrm{SEN}(g)(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})) \in T_{\Sigma''}$, for all $\Sigma'' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$

iff

$$\begin{array}{ccc} \mathrm{SEN}(\Sigma')^n & \xrightarrow{\sigma_{\Sigma'}} & \mathrm{SEN}(\Sigma') \\ {\scriptstyle \mathrm{SEN}(g)^n}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}(g)} \\ \mathrm{SEN}(\Sigma'')^n & \xrightarrow[\sigma_{\Sigma''}]{} & \mathrm{SEN}(\Sigma'') \end{array}$$

and $\sigma_{\Sigma''}(\mathrm{SEN}(gf)(\varphi), \mathrm{SEN}(g)^{n-1}(\vec{\chi})) \in T_{\Sigma''}$, for all $\Sigma'' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$. □

Lemma 3.1 is now used to prove the central theorem of the present work which states that, for all theory families T of a π-institution $\mathcal{I}$, the Leibniz N-congruence system associated with T is included in the Leibniz N-theory system associated with $\overleftarrow{T}$. In other words, we have $\Omega^N(T) \leq \Omega^N(\overleftarrow{T})$, despite the fact that $\overleftarrow{T} \leq T$!

Theorem 3.2 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. *If* T *is a theory family of* $\mathcal{I}$, *then* $\Omega^N(T) \leq \Omega^N(\overleftarrow{T})$.

Proof. To show that $\Omega^N(T) \leq \Omega^N(\overleftarrow{T})$, suppose that $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \text{SEN}(\Sigma)$, and $\langle \varphi, \psi \rangle \in \Omega^N_\Sigma(T)$. Recall from the characterization of Ω^N, given in [18, Proposition 2.4], that, for all $\sigma : \text{SEN}^n \longrightarrow \text{SEN}$ in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \text{SEN}(\Sigma')^{n-1}$,

$$(1) \qquad \sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\text{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Now let $\sigma : \text{SEN}^n \longrightarrow \text{SEN}$ be in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \text{SEN}(\Sigma')^{n-1}$. We have

$$\sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$$

if and only if, by Lemma 3.1, $\sigma_{\Sigma''}(\text{SEN}(gf)(\varphi), \text{SEN}(g)^{n-1}(\vec{\chi})) \in T_{\Sigma''}$, for all $\Sigma'' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, if and only if, by equivalence (1),

$$\sigma_{\Sigma''}(\text{SEN}(gf)(\psi), \text{SEN}(g)^{n-1}(\vec{\chi})) \in T_{\Sigma''}, \quad \text{for all } \Sigma'' \in |\mathbf{Sign}|, g \in \mathbf{Sign}(\Sigma', \Sigma''),$$

if and only if, again by Lemma 3.1, $\sigma_{\Sigma'}(\text{SEN}(f)(\psi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$. Thus, using [18, Proposition 2.4] once more, we get that $\langle \varphi, \psi \rangle \in \Omega^N_\Sigma(\overleftarrow{T})$. Therefore $\Omega^N(T) \leq \Omega^N(\overleftarrow{T})$. □

4 Important consequences

Two important consequences of Theorem 3.2 are obtained in this section. First, a characterization theorem for the class of N-protoalgebraic π-institutions, as it sits inside the class of N-prealgebraic π-institutions, is given in terms of the relation between $\Omega^N(T)$ and $\Omega^N(\overleftarrow{T})$. This characterization fills in a gap that was left open in the development of the theory of prealgebraicity and protoalgebraicity of [18]. As another important consequence of Theorem 3.2, it is shown that, in the context of N-protoalgebraic π-institutions, the N-Leibniz theory family $T^{(N)}$ corresponding to a given theory family T, defined in a way analogous to the N-Leibniz theory system T^N corresponding to a given theory system T in [20], is an N-Leibniz theory system. Therefore all N-Leibniz theory families are N-Leibniz theory systems and the theory of [20] covers these families in their full generality.

Theorem 4.1 (Characterization of protoalgebraicity) *Let $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, C \rangle$, with N a category of natural transformations on* SEN*, be an N-prealgebraic π-institution. Then $\mathcal{I}$ is N-protoalgebraic if and only if, for all theory families $T \in \text{ThFam}(\mathcal{I})$, $\Omega^N(T) = \Omega^N(\overleftarrow{T})$.*

Proof. If $\mathcal{I}$ is N-protoalgebraic, then, since $\overleftarrow{T} \leq T$, we obtain that $\Omega^N(\overleftarrow{T}) \leq \Omega^N(T)$. The reverse inclusion holds by Theorem 3.2.

Suppose, conversely, that, for all theory families $T \in \text{ThFam}(\mathcal{I})$, $\Omega^N(T) = \Omega^N(\overleftarrow{T})$. Then, if

$$T, T' \in \text{ThFam}(\mathcal{I})$$

are such that $T \leq T'$, we have

$$\begin{aligned} \Omega^N(T) &= \Omega^N(\overleftarrow{T}) && \text{(by hypothesis)} \\ &\leq \Omega^N(\overleftarrow{T'}) && \text{(by } N\text{-prealgebraicity and Corollary 2.4)} \\ &= \Omega^N(T') && \text{(by hypothesis).} \end{aligned}$$

Therefore $\mathcal{I}$ is N-protoalgebraic. □

In the remainder of this section $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, C \rangle$, with N a category of natural transformations on SEN, will be an N-protoalgebraic π-institution. Recall from [20] the concept of the N-Leibniz theory system T^N associated with a given theory system T of $\mathcal{I}$. In a way analogous to the way T^N was defined for T a theory system, $T^{(N)}$ may be defined for an arbitrary theory family T. Namely, $T^{(N)}$ is the least theory family of $\mathcal{I}$ that has the same Leibniz N-congruence system as the theory family T. Such a theory family always exists. Existence of $T^{(N)}$ may be shown in a way very similar to the way existence of T^N was shown for a theory system T in [20, Proposition 1].

Proposition 4.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be an N-protoalgebraic π-institution. For every theory family T of $\mathcal{I}$, there exists a unique N-Leibniz theory family $T^{(N)}$ of $\mathcal{I}$, such that $\Omega^N(T^{(N)}) = \Omega^N(T)$. It is given by*

$$T^{(N)} = \bigcap\{T' \in \mathrm{ThFam}(\mathcal{I}) \,:\, \Omega^N(T') = \Omega^N(T)\}.$$

In the second important consequence of Theorem 3.2, it is now shown that, given a theory family T of an N-protoalgebraic π-institution $\mathcal{I}$, the N-Leibniz theory family $T^{(N)}$ associated with T coincides with the N-Leibniz theory system $\overleftarrow{T}^N$ associated with $\overleftarrow{T}$. Hence studying N-Leibniz theory systems in this context exhausts the study of all N-Leibniz theory families.

Theorem 4.3 (Leibniz theory systems) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-protoalgebraic π-institution. For every theory family T of $\mathcal{I}$, $T^{(N)} = \overleftarrow{T}^N$.*

Proof. We first show that $T^{(N)} \le \overleftarrow{T}^N$. In fact we have

$$\begin{aligned} T^{(N)} &= \bigcap\{T' \in \mathrm{ThFam}(\mathcal{I}) \,:\, \Omega^N(T') = \Omega^N(T)\} && \text{(by the definition of } T^{(N)}) \\ &\le \bigcap\{T' \in \mathrm{ThSys}(\mathcal{I}) \,:\, \Omega^N(T') = \Omega^N(T)\} && \text{(since } \mathrm{ThSys}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})) \\ &= \bigcap\{T' \in \mathrm{ThSys}(\mathcal{I}) \,:\, \Omega^N(T') = \Omega^N(\overleftarrow{T})\} && \text{(by Theorem 4.1)} \\ &= \overleftarrow{T}^N && \text{(by the definition of } \overleftarrow{T}^N). \end{aligned}$$

For the opposite inclusion we have

$$\begin{aligned} \overleftarrow{T}^N &= \bigcap\{T' \in \mathrm{ThSys}(\mathcal{I}) \,:\, \Omega^N(T') = \Omega^N(\overleftarrow{T})\} && \text{(by the definition of } \overleftarrow{T}^N) \\ &= \bigcap\{\overleftarrow{T'} \,:\, T' \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \Omega^N(T') = \Omega^N(T)\} && \text{(equal sets by Theorem 4.1)} \\ &\le \bigcap\{T' \,:\, T' \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \Omega^N(T') = \Omega^N(T)\} && \text{(by Proposition 2.2)} \\ &= T^{(N)} && \text{(by the definition of } T^{(N)}). \end{aligned}$$ □

Similarly to the way T^N was defined for a theory system T and $T^{(N)}$ was defined for a theory family T, one may define the notion of the N-Leibniz theory system T^N associated with an arbitrary theory family T of an N-protoalgebraic π-institution $\mathcal{I}$ as the least theory system of $\mathcal{I}$ that has the same Leibniz N-congruence system as T. With this definition, then, Theorem 4.3 yields immediately:

Corollary 4.4 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-protoalgebraic π-institution. For every theory family T of $\mathcal{I}$, $T^{(N)} = T^N$, i. e., the N-Leibniz theory family associated with T equals the N-Leibniz theory system associated with T.*

References

[1] M. Barr and C. Wells, Category Theory for Computing Science, Third Edition (Les Publications CRM, 1999).
[2] W. J. Blok and D. Pigozzi, Protoalgebraic logics. Studia Logica **45**, 337 – 369 (1986).
[3] W. J. Blok and D. Pigozzi, Algebraizable logics. Mem. Amer. Math. Soc. **77**, No. 396 (1989).
[4] F. Borceux, Handbook of Categorical Algebra. Encyclopedia of Mathematics and its Applications 50 (Cambridge University Press, 1994).
[5] J. Czelakowski, Protoalgebraic Logics. Studia Logica Library 10 (Kluwer, 2001).
[6] J. M. Font and R. Jansana, A General Algebraic Semantics for Sentential Logics. Lecture Notes in Logic 7 (Springer-Verlag, 1996).
[7] J. M. Font and R. Jansana, Leibniz filters and the strong version of a protoalgebraic logic. Arch. Math. Logic **40**, 437 – 465 (2001).
[8] R. Jansana, Leibniz filters revisited. Studia Logica **75**, 305 – 317 (2003).
[9] S. Mac Lane, Categories for the Working Mathematician (Springer-Verlag, 1971).
[10] G. Voutsadakis, Categorical abstract algebraic logic: Equivalent institutions. Studia Logica **74**, 275 – 311 (2003).
[11] G. Voutsadakis, Categorical abstract algebraic logic: Algebraizable institutions. Applied Categorical Structures **10**, 531 – 568 (2002).

[12] G. Voutsadakis, Categorical abstract algebraic logic: Tarski congruence systems, logical morphisms and logical quotients. Submitted to Annals of Pure and Applied Logic.
Preprint available at `http://pigozzi.lssu.edu/WWW/research/papers.html`.

[13] G. Voutsadakis, Categorical abstract algebraic logic: Models of π-institutions. Notre Dame J. Formal Logic **46**, 439 – 460 (2005).

[14] G. Voutsadakis, Categorical abstract algebraic logic: Generalized Tarski congruence systems. Submitted to Theory and Applications of Categories.
Preprint available at `http://pigozzi.lssu.edu/WWW/research/papers.html`.

[15] G. Voutsadakis, Categorical abstract algebraic logic: $(\mathcal{I}, N)$-algebraic systems. Applied Categorical Structures **13**, 265 – 280 (2005).

[16] G. Voutsadakis, Categorical abstract algebraic logic: Gentzen π-institutions. Submitted to Mathematica Scandinavica.
Preprint available at `http://pigozzi.lssu.edu/WWW/research/papers.html`.

[17] G. Voutsadakis, Categorical abstract algebraic logic: Full models, Frege systems and metalogical properties. To appear in Reports on Mathematical Logic.
Preprint available at `http://pigozzi.lssu.edu/WWW/research/papers.html`.

[18] G. Voutsadakis, Categorical abstract algebraic logic: Prealgebraicity and protoalgebraicity. To appear in Studia Logica.
Preprint available at `http://pigozzi.lssu.edu/WWW/research/papers.html`.

[19] G. Voutsadakis, Categorical abstract algebraic logic: More on protoalgebraicity. To appear in the Notre Dame Journal of Formal Logic. Preprint available at `http://pigozzi.lssu.edu/WWW/research/papers.html`.

[20] G. Voutsadakis, Categorical abstract algebraic logic: Protoalgebraicity and Leibniz theory systems. Scientiae Mathematicae Japonicae **18**, 109 – 117 (2005).