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Categorical abstract algebraic logic: Gentzen π-institutions and the deduction-detachment property

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Given a π-institution $\mathcal{I}$, a hierarchy of π-institutions $\mathcal{I}^{(n)}$ is constructed, for $n \geq 1$. We call $\mathcal{I}^{(n)}$ the *n-th order counterpart of* $\mathcal{I}$. The second-order counterpart of a deductive π-institution is a Gentzen π-institution, i. e. a π-institution associated with a structural Gentzen system in a canonical way. So, by analogy, the second order counterpart $\mathcal{I}^{(2)}$ of $\mathcal{I}$ is also called the "Gentzenization" of $\mathcal{I}$. In the main result of the paper, it is shown that $\mathcal{I}$ is *strongly Gentzen*, i. e. it is deductively equivalent to its Gentzenization via a special deductive equivalence, if and only if it has the *deduction-detachment property*.

1 Introduction

Categorical abstract algebraic logic [28] is the area of abstract algebraic logic (see [13] for an excellent overview) that abstracts ideas and results developed in the universal algebraic framework of deductive systems and sentential logics [12, 11] to the categorical framework of institutions and π-institutions [15, 16, 10]. This abstraction includes incorporating the substitution operations in the language in the form of morphisms, rather than treating them in the metalanguage, as is done in the classical universal algebraic framework. Inclusion of morphisms allows treating logics over multiple signatures and logics with quantifiers in a very systematic, rather than ad-hoc, way (see [29, 30] for a detailed explanation of the advantages of this approach). Equational and first-order logic, the main two paradigms in this framework, have been algebraized using the categorical method in [33] and [35], respectively, and an investigation into the nature of their categorical algebraic counterparts has been carried out in [32, 34]. The process of algebraization of deductive systems of Blok and Pigozzi [3], that was later adapted to cover various other logics (for instance [19, 20, 21, 12, 23, 1]), has been generalized in [28] (see also [29, 30]) to the categorical process of algebraization of π-institutions. At both levels one of the key notions is the notion of equivalence. Equivalence of k-deductive systems was introduced in [4] and was the inspirational force behind the development of the notion of deductive equivalence of π-institutions [28, 29]. This notion is also based on ideas developed in the institution domain, especially different forms of institution morphisms (see [15, 16, 18] for an overview).

The development of these algebraization frameworks gave a solid basis on which to study the correspondence between metalogical properties and algebraic properties. In the universal algebraic side, this direction of abstract algebraic logic is widespread and may be found in the study of various properties, for instance, in [5, 9, 12], a detailed study of the *deduction-detachment property* for deductive systems is undertaken. Some other examples

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include [7] that studies the amalgamation property and [8] on the Maehara interpolation property. On the categorical algebraic side, some properties of institutions and π-institutions have been studied in the literature before the development of the categorical algebraization process [27]. However, a systematic study inside this framework was initiated in [31]. Several metalogical properties were introduced in that paper and they were all shown to be stable under deductive equivalence. One of these properties was the deduction-detachment property, which had been investigated extensively before in the deductive system framework in [5, 9, 12] and will also be the focus in this paper.

On a different direction, investigations on extending the Blok-Pigozzi framework to the algebraization of Gentzen systems were initiated in [24, 25] and a systematic and complete account, exploiting the Tarski congruence of an abstract logic, has been presented in [12]. It turns out that this investigation of second-order deduction, as the deductive apparatus of Gentzen systems is sometimes thought of intuitively, provides new insights and helps in understanding several properties of deductive systems. Most notably, it has been of great importance in understanding some of the properties of non protoalgebraic deductive systems (those in the lowest end of the algebraic hierarchy but still amenable to algebraic logic techniques [13]), which are not so easily understood in the first-order deductive system framework.

Motivated by the considerations presented above, given an institution $\mathcal{I}$, its *n-th order counterpart* $\mathcal{I}^{(n)}$, $n \geq 2$, is defined and the relation of its deductive power with that of $\mathcal{I}$ in the special case when $\mathcal{I}$ has the deduction-detachment property is investigated. In the main result of the paper it is shown that having the deduction-detachment property is equivalent to $\mathcal{I}^{(n)}$ being deductively equivalent to $\mathcal{I}$ via a special deductive equivalence, for all $n \geq 2$.

2 Higher order counterparts

For all categorical concepts and unexplained categorical notation the reader is referred to any of [2], [6] or [22].

A *π-institution* (see [10]) $\mathcal{I} = \langle \mathrm{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathrm{Sign}|}\rangle$ is a triple consisting of

(i) a category Sign, whose objects are called *signatures* and whose morphisms are called *assignments*;

(ii) a functor SEN : $\mathrm{Sign} \longrightarrow \mathrm{Set}$ from the category of signatures to the category of small sets, giving, for each $\Sigma \in |\mathrm{Sign}|$, the set of Σ-*sentences* $\mathrm{SEN}(\Sigma)$ and mapping an assignment $f : \Sigma_1 \longrightarrow \Sigma_2$ to a *substitution* $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma_1) \longrightarrow \mathrm{SEN}(\Sigma_2)$;

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \longrightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathrm{Sign}|$, called Σ-*closure,* such that

(a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathrm{Sign}|$, $A \subseteq \mathrm{SEN}(\Sigma)$,

(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathrm{Sign}|$, $A \subseteq \mathrm{SEN}(\Sigma)$,

(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathrm{Sign}|$, $A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

A family

$$\{C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \longrightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))\}_{\Sigma\in|\mathrm{Sign}|}$$

will be referred to as a *closure system on* SEN if it satisfies (iii)(a) – (d) above.

It is well-known that given an institution $\mathcal{I}$, a π-institution $\pi(\mathcal{I})$ results by taking the semantic closure relations of $\mathcal{I}$ as the closure relations of $\pi(\mathcal{I})$ (see [10]). Therefore the abundance of examples of institutions in the literature (see, for instance, [15, 16, 17, 26, 29, 30, 31]) immediately yields, via this construction, many important examples of logics formulated as π-institutions. We will not present any more examples here.

In what follows, let $\mathcal{P}$ denote the power set functor and $\cdot^2$ denote the Cartesian square functor. Also by capital Greek letters, like Γ, Δ, Φ, will be denoted ordered pairs of subsets of sentences and by the same letter with the subscripts 1 and 2 will be denoted the first and second subset in the pair, respectively. In other words, we will always have $\Gamma = \langle \Gamma_1, \Gamma_2\rangle$, etc. By capital boldfaced Greek letters, like $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Phi}$, will be denoted collections of pairs of subsets. This follows a similar convention adopted for sequents and sets of sequents of Gentzen systems in [14].

Given a π-institution $\mathcal{I} = \langle \mathrm{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathrm{Sign}|}\rangle$, define the π-institution

$$\mathcal{I}^{(2)} = \langle \mathrm{Sign}, \mathrm{SEN}^{(2)}, \{C^{(2)}_\Sigma\}_{\Sigma\in|\mathrm{Sign}|}\rangle$$

as follows:

1. $\mathrm{SEN}^{(2)} = (\mathcal{P} \circ \mathrm{SEN})^2$,

2. $C^{(2)}_\Sigma : \mathcal{P}(\mathcal{P}(\mathrm{SEN}(\Sigma))^2) \longrightarrow \mathcal{P}(\mathcal{P}(\mathrm{SEN}(\Sigma))^2)$ is defined, for all $\Sigma \in |\mathrm{Sign}|$, by $\Gamma \in C^{(2)}_\Sigma(\mathbf{\Phi})$ if and only if, for all $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$,

$$\begin{array}{l}\mathrm{SEN}(f)(\Phi_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi_1)) \text{ for all } \Phi = \langle \Phi_1, \Phi_2\rangle \in \mathbf{\Phi} \\ \text{implies } \mathrm{SEN}(f)(\Gamma_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma_1)).\end{array}$$

Sometimes, the last condition of 2. will be abbreviated to

$$\mathrm{SEN}(f)(\mathbf{\Phi}) \subseteq C_{\Sigma'} \text{ implies } \mathrm{SEN}(f)(\Gamma) \in C_{\Sigma'}.$$

Proposition 2.1 *Let $\mathcal{I}$ be a π-institution. Then $\mathcal{I}^{(2)}$ is also a π-institution.*

Proof. It is obvious that $\mathrm{SEN}^{(2)} = \mathcal{P}\mathrm{SEN}^2 : \mathrm{Sign} \longrightarrow \mathrm{Set}$ is a functor. Also, for

$$C^{(2)} : \mathcal{P}(\mathcal{P}\mathrm{SEN}^2) \longrightarrow \mathcal{P}(\mathcal{P}\mathrm{SEN}^2),$$

we may prove conditions (iii)(a) – (d) of the definition of a π-institution.

(iii)(a) Suppose that $\Phi = \langle \Phi_1, \Phi_2\rangle \in \mathbf{\Phi} \subseteq \mathcal{P}\mathrm{SEN}(\Sigma)^2$. Then, if $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$ are such that $\mathrm{SEN}(f)(\mathbf{\Phi}) \subseteq C_{\Sigma'}$, then it is obvious that $\mathrm{SEN}(f)(\Phi) \in C_{\Sigma'}$.

(iii)(b) Now suppose that $\Gamma \in C^{(2)}_\Sigma(C^{(2)}_\Sigma(\mathbf{\Phi}))$. Thus, for every $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$,

$$\text{if } \mathrm{SEN}^{(2)}(f)(C^{(2)}_\Sigma(\mathbf{\Phi})) \subseteq C_{\Sigma'}, \text{ then } \mathrm{SEN}^{(2)}(f)(\Gamma) \in C_{\Sigma'}.$$

Let $\Sigma'' \in |\mathrm{Sign}|$, $g \in \mathrm{Sign}(\Sigma, \Sigma'')$ be such that $\mathrm{SEN}^{(2)}(g)(\mathbf{\Phi}) \subseteq C_{\Sigma''}$. Then $\mathrm{SEN}^{(2)}(g)(C^{(2)}_\Sigma(\mathbf{\Phi})) \subseteq C_{\Sigma''}$, whence $\mathrm{SEN}^{(2)}(g)(\Gamma) \in C_{\Sigma''}$.

(iii)(c) Let $\mathbf{\Phi} \subseteq \mathbf{\Psi}$ and suppose $\Gamma \in C^{(2)}_\Sigma(\mathbf{\Phi})$. Then, if $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$ are such that

$$\mathrm{SEN}^{(2)}(f)(\mathbf{\Psi}) \subseteq C_{\Sigma'},$$

then $\mathrm{SEN}^{(2)}(f)(\mathbf{\Phi}) \subseteq C_{\Sigma'}$, whence $\mathrm{SEN}^{(2)}(f)(\Gamma) \in C_{\Sigma'}$, and, therefore, $C^{(2)}_\Sigma(\mathbf{\Phi}) \subseteq C^{(2)}_\Sigma(\mathbf{\Psi})$.

(iii)(d) Suppose that $\Sigma_1, \Sigma_2 \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma_1, \Sigma_2)$, $\mathbf{\Phi} \subseteq \mathrm{SEN}^{(2)}(\Sigma_1)$. Let $\Gamma \in C^{(2)}_{\Sigma_1}(\mathbf{\Phi})$. Now suppose that $\Sigma' \in |\mathrm{Sign}|$, $g \in \mathrm{Sign}(\Sigma_2, \Sigma')$,

$$\begin{tikzcd} \Sigma_1 \arrow[r, "f"] \arrow[dr, "gf"'] & \Sigma_2 \arrow[d, "g"] \\ & \Sigma' \end{tikzcd}$$

such that $\mathrm{SEN}^{(2)}(g)(\mathrm{SEN}^{(2)}(f)(\mathbf{\Phi})) \subseteq C_{\Sigma'}$. Then we have $\mathrm{SEN}^{(2)}(gf)(\mathbf{\Phi}) \subseteq C_{\Sigma'}$, whence we obtain

$$\mathrm{SEN}^{(2)}(gf)(\Gamma) \in C_{\Sigma'},$$

i. e. $\mathrm{SEN}^{(2)}(g)(\mathrm{SEN}^{(2)}(f)(\Gamma)) \in C_{\Sigma'}$, and, therefore, $\mathrm{SEN}^{(2)}(f)(\Gamma) \in C^{(2)}_{\Sigma_2}(\mathrm{SEN}^{(2)}(f)(\mathbf{\Phi}))$. □

Observe that, for all $\Sigma \in |\mathrm{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \text{ implies } \langle \Phi, \{\varphi\}\rangle \in C^{(2)}_\Sigma(\emptyset).$$

Lemma 2.2 *Let* $\mathcal{I} = \langle \mathrm{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathrm{Sign}|}\rangle$ *be a* π*-institution. The closure system of* $\mathcal{I}^{(2)}$ *satisfies the following properties:*

1. $\langle \Phi, \Phi\rangle \in C^{(2)}_\Sigma(\emptyset)$, *for all* $\Phi \subseteq \mathrm{SEN}(\Sigma)$.
2. $\langle \Gamma \cup \Delta, \Phi\rangle \in C^{(2)}_\Sigma(\{\langle \Gamma, \Phi\rangle\})$, *for all* $\Gamma, \Delta, \Phi \subseteq \mathrm{SEN}(\Sigma)$.
3. $\langle \Gamma, \Delta\rangle \in C^{(2)}_\Sigma(\{\langle \Gamma, \Phi\rangle, \langle \Gamma \cup \Phi, \Delta\rangle\})$, *for all* $\Gamma, \Delta, \Phi \subseteq \mathrm{SEN}(\Sigma)$.
4. $\langle \Gamma, \{\delta\}\rangle \in C^{(2)}_\Sigma(\mathbf{\Phi})$, *for all* $\delta \in \Delta$, *implies* $\langle \Gamma, \Delta\rangle \in C^{(2)}_\Sigma(\mathbf{\Phi})$, *for all* $\Sigma \in |\mathrm{Sign}|$, $\Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$ *and* $\mathbf{\Phi} \subseteq \mathcal{P}\mathrm{SEN}(\Sigma)^2$.

P r o o f. All four properties follow directly from the definition of the closure system $C^{(2)}$ and the properties of the closure system C. □

In this context, we will usually denote a pair $\Phi = \langle \Phi_1, \Phi_2\rangle \in \mathrm{SEN}^{(2)}(\Sigma)$ by $\Phi_1 \vdash_\Sigma \Phi_2$. Furthermore, if $\Phi_1 = \emptyset$, we will write $\vdash_\Sigma \Phi_2$. If it so happens that Φ_1 or Φ_2 is a single sentence, we will follow common practice to omit parentheses, thus identifying the singleton set with the single element that it contains. Moreover, instead of $\Phi \in C^{(2)}_\Sigma(\mathbf{\Phi})$ we will sometimes use the notation $\mathbf{\Phi} \mid\!\sim_\Sigma \Phi$. With this notation the first three properties in Lemma 2.2 take the following forms, recognizable as generalized forms of the structural rules of Axiom, Weakening and Cut, respectively, of Gentzen calculi:

Axiom	$\mid\!\sim_\Sigma \Phi \vdash_\Sigma \Phi$,
Weakening	$\Gamma \vdash_\Sigma \Phi \mid\!\sim_\Sigma \Gamma, \quad \Delta \vdash_\Sigma \Phi$,
Cut	$\Gamma \vdash_\Sigma \Phi, \quad \Gamma, \Phi \vdash_\Sigma \Delta \mid\!\sim_\Sigma \Gamma \vdash_\Sigma \Delta$.

Using the construction of $\mathcal{I}^{(2)}$ out of $\mathcal{I}$ we may further define inductively

$$\mathcal{I}^{(1)} = \mathcal{I}, \quad \mathcal{I}^{(n+1)} = (\mathcal{I}^{(n)})^{(2)} \quad \text{for all } n \geq 1.$$

$\mathcal{I}^{(n)}$ is said to be the *n-th order counterpart of $\mathcal{I}$*.

3 Deductive and Gentzen π-institutions

Recall from [29, 30] the definition of the deductive institution associated with the 1-deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S}\rangle$.

Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S}\rangle$ a deductive system over $\mathcal{L}$, i. e. $\vdash_\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}_\mathcal{L}(V)) \times \mathrm{Fm}_\mathcal{L}(V)$ is a structural consequence operator on the set $\mathrm{Fm}_\mathcal{L}(V)$ of $\mathcal{L}$-formulas with variables in V. We construct the π-institution $\mathcal{I}_\mathcal{S} = \langle \mathrm{Sign}_\mathcal{S}, \mathrm{SEN}_\mathcal{S}, \{C_\Sigma\}_{\Sigma \in |\mathrm{Sign}_\mathcal{S}|}\rangle$ as follows:

(i) $\mathrm{Sign}_\mathcal{S}$ is the one-object category with object V and morphisms all assignments $f : V \longrightarrow V$, i. e. set maps $f : V \longrightarrow \mathrm{Fm}_\mathcal{L}(V)$. The identity morphism is the inclusion $i_V : V \longrightarrow \mathrm{Fm}_\mathcal{L}(V)$. Composition $g \circ f$ of two assignments f and g is defined by $g \circ f = g^* f$, where $g^* : \mathrm{Fm}_\mathcal{L}(V) \longrightarrow \mathrm{Fm}_\mathcal{L}(V)$ denotes the substitution extending the assignment g.

(ii) $\mathrm{SEN}_\mathcal{S} : \mathrm{Sign}_\mathcal{S} \longrightarrow \mathrm{Set}$ maps V to $\mathrm{Fm}_\mathcal{L}(V)$ and $f : V \longrightarrow V$ to $f^* : \mathrm{Fm}_\mathcal{L}(V) \longrightarrow \mathrm{Fm}_\mathcal{L}(V)$. It is easy to see that $\mathrm{SEN}_\mathcal{S}$ is a functor.

(iii) $C_V : \mathcal{P}(\mathrm{Fm}_\mathcal{L}(V)) \longrightarrow \mathcal{P}(\mathrm{Fm}_\mathcal{L}(V))$ is the standard closure operator $C_\mathcal{S} : \mathcal{P}(\mathrm{Fm}_\mathcal{L}(V)) \longrightarrow \mathcal{P}(\mathrm{Fm}_\mathcal{L}(V))$ associated with the deductive system $\mathcal{S}$, i. e.

$$C_V(\Phi) = \{\varphi \in \mathrm{Fm}_\mathcal{L}(V) : \Phi \vdash_\mathcal{S} \varphi\} \quad \text{for all } \Phi \subseteq \mathrm{Fm}_\mathcal{L}(V).$$

C_V, defined in this way, satisfies the conditions imposed in the definition of a π-institution. Thus, $\mathcal{I}_\mathcal{S}$ is a π-institution. It will be called the *deductive π-institution associated with the deductive system $\mathcal{S}$*. See [30] for more details. Note that the same exact construction works for general k-deductive systems in the sense of Blok and Pigozzi [4].

Again, let $\mathcal{L}$ be a language type and $\mathcal{G}$ be a structural Gentzen system, i. e. a pair $\mathcal{G} = \langle \mathcal{L}, \mid\!\sim_{\mathcal{G}} \rangle$, where $\mid\!\sim_{\mathcal{G}}$ is a structural consequence relation on the collection of sequents of formulas $\mathrm{Seq}(\mathrm{Fm}_{\mathcal{L}}(V))$, that, in addition, satisfies the structural rules

1. $\emptyset \mid\!\sim_{\mathcal{G}} \varphi \vdash \varphi$, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,
2. $\Gamma \vdash \varphi \mid\!\sim_{\mathcal{G}} \Gamma, \psi \vdash \varphi$, for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, and
3. $\{\Gamma \vdash \varphi, \Gamma, \varphi \vdash \psi\} \mid\!\sim_{\mathcal{G}} \Gamma \vdash \psi$, for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$.

See e. g. [12] for more details on Gentzen systems and some of their important applications in abstract algebraic logic. The *Gentzen π-institution $\mathcal{I}_{\mathcal{G}} = \langle \mathrm{Sign}_{\mathcal{G}}, \mathrm{SEN}_{\mathcal{G}}, \{C_\Sigma\}_{\Sigma \in |\mathrm{Sign}_{\mathcal{G}}|} \rangle$ associated with the Gentzen system $\mathcal{G}$* is defined as follows:

(i) $\mathrm{Sign}_{\mathcal{G}}$ is identical with $\mathrm{Sign}_{\mathcal{S}}$, defined above.

(ii) $\mathrm{SEN}_{\mathcal{G}} : \mathrm{Sign}_{\mathcal{G}} \longrightarrow \mathbf{Set}$ maps V to $\mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2$, the collection of all pairs of sets of $\mathcal{L}$-formulas with variables in V, and $f : V \longrightarrow V$ to $\mathcal{P}(f^*)^2 : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2 \longrightarrow \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2$, i. e. the application of the mapping f^* to a pair of sets of $\mathcal{L}$-formulas is done "element-wise". It is easy to see that $\mathrm{SEN}_{\mathcal{G}}$ is a functor.

(iii) Finally, $C_V : \mathcal{P}(\mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2) \longrightarrow \mathcal{P}(\mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2)$ is the closure operator associated with the Gentzen system $\mathcal{G}$ in the following way: for all $\mathbf{\Phi} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2$,

$$\begin{aligned} C_V(\mathbf{\Phi}) = \{\Psi = \langle \Psi_1, \Psi_2 \rangle \in \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))^2 :& \\ \textstyle\bigcup_{\Phi \in \mathbf{\Phi}} \{\Phi_1 \vdash \varphi : \varphi \in \Phi_2\} \mid\!\sim_{\mathcal{G}} \Psi_1 \vdash \psi,& \text{ for all } \psi \in \Psi_2\}. \end{aligned}$$

C_V, defined in this way, satisfies the conditions imposed in the definition of a π-institution. Thus, $\mathcal{I}_{\mathcal{G}}$ is a π-institution.

If we term a π-institution a Gentzen π-institution if it is of the form $\mathcal{I}_{\mathcal{G}}$, for some Gentzen system $\mathcal{G}$, then it is not very difficult to see that

Proposition 3.1 *Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, the second-order counterpart $\mathcal{I}_{\mathcal{S}}^{(2)}$ of the π-institution $\mathcal{I}_{\mathcal{S}}$ associated with $\mathcal{S}$ is a Gentzen π-institution.*

Because of Proposition 3.1, the second order counterpart $\mathcal{I}^{(2)}$ of an arbitrary given institution $\mathcal{I}$ may also be called, by analogy with deductive π-institutions, the *Gentzenization of $\mathcal{I}$*.

4 Gentzenization and the deduction-detachment property

In this section we prove the main theorem of the paper which may be viewed either as a characterization of those π-institutions that are equivalent to their Gentzenizations in terms of the deduction-detachment property or, equivalently, as a characterization of those π-institutions having the deduction-detachment property in terms of being deductively equivalent to their Gentzenizations.

A π-institution $\mathcal{I}$ is said to be *essentially Gentzen* or *essentially second order* if it is deductively equivalent (see [29]) to its Gentzenization, written $\mathcal{I} \dashv\vdash \mathcal{I}^{(2)}$. If, in addition, the interpretation $\langle F, \alpha \rangle : \mathcal{I} \longrightarrow \mathcal{I}^{(2)}$ witnessing the deductive equivalence has $F = \mathrm{I}_{\mathbf{Sign}}$, the identity signature functor, and $\alpha_\Sigma(\varphi) = \{\vdash_\Sigma \varphi\}$, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, and the adjoint equivalence $\langle F, G, \eta, \varepsilon \rangle : \mathbf{Sign} \longrightarrow \mathbf{Sign}$ is the identity, then $\mathcal{I}$ is said to be *strongly Gentzen*.

Recall from [31] that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ is said to have the *deduction-detachment property* if there exists a natural transformation $E : \mathcal{P}\mathrm{SEN}^2 \longrightarrow \mathcal{P}\mathrm{SEN}$, called a *deduction-detachment* or *implication system*, such that, for all $\Sigma \in |\mathbf{Sign}|$, and all $\Phi \cup \Gamma \cup \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$\Phi \subseteq C_\Sigma(\Gamma \cup \Delta) \quad \text{iff} \quad E_\Sigma(\Delta, \Phi) \subseteq C_\Sigma(\Gamma).$$

It is shown next that, for any π-institution, having the deduction-detachment property is equivalent to being strongly Gentzen.

Theorem 4.1 *A π-institution $\mathcal{I}$ is strongly Gentzen if and only if it has the deduction-detachment property.*

P r o o f. Suppose, first, that $\mathcal{I}$ has the deduction-detachment property with the deduction-detachment system $E : \mathcal{P}\mathrm{SEN}^2 \longrightarrow \mathcal{P}\mathrm{SEN}$. Let $F, G : \mathrm{Sign} \longrightarrow \mathrm{Sign}$ be the identity functors $F = G = \mathrm{I}_{\mathrm{Sign}}$. Recall that $\mathrm{SEN}^{(2)} = \mathcal{P}\mathrm{SEN}^2$ and define the natural transformations $\alpha : \mathrm{SEN} \longrightarrow \mathcal{P}(\mathcal{P}\mathrm{SEN}^2)$ by

$$\alpha_\Sigma(\varphi) = \{\vdash_\Sigma \varphi\} \quad \text{for all } \Sigma \in |\mathrm{Sign}|,\ \varphi \in \mathrm{SEN}(\Sigma),$$

and $\beta : \mathcal{P}\mathrm{SEN}^2 \longrightarrow \mathcal{P}\mathrm{SEN}$ by

$$\beta_\Sigma(\Phi \vdash_\Sigma \Psi) = E_\Sigma(\Phi, \Psi) \quad \text{for all } \Sigma \in |\mathrm{Sign}|,\ \Phi, \Psi \subseteq \mathrm{SEN}(\Sigma).$$

It is not difficult to check that α is a natural transformation: Indeed for $f \in \mathrm{Sign}(\Sigma_1, \Sigma_2)$, $\varphi \in \mathrm{SEN}(\Sigma_1)$, we have

$$\begin{tikzcd} \mathrm{SEN}(\Sigma_1) \arrow[r, "\alpha_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathcal{P}(\mathrm{SEN}(\Sigma_1))^2 \arrow[d, "\mathcal{P}(\mathrm{SEN}(f))^2"] \\ \mathrm{SEN}(\Sigma_2) \arrow[r, "\alpha_{\Sigma_2}"'] & \mathcal{P}(\mathrm{SEN}(\Sigma_2))^2 \end{tikzcd}$$

and

$$\alpha_{\Sigma_2}(\mathrm{SEN}(f)(\varphi)) = \{\vdash_{\Sigma_2} \mathrm{SEN}(f)(\varphi)\}\mathcal{P}(\mathrm{SEN}(f))^2(\vdash_{\Sigma_1} \varphi)\mathcal{P}(\mathrm{SEN}(f))^2(\alpha_{\Sigma_1}(\varphi)).$$

β is a natural transformation, since E is, by hypothesis. Obviously, F and G, being identities, are part of a natural equivalence with identity unit and counit and, therefore, to prove that $\mathcal{I}$ and $\mathcal{I}^{(2)}$ are deductively equivalent, it suffices by a result of [29] to show the following relations, for all $\Sigma \in |\mathrm{Sign}|$:

(1) $\varphi \in C_\Sigma(\Phi)$ implies $\alpha_\Sigma(\varphi) \subseteq C_\Sigma^{(2)}(\alpha_\Sigma(\Phi))$ for all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$.

(2) $C_\Sigma^{(2)}(\Phi) = C_\Sigma^{(2)}(\alpha_\Sigma(\beta_\Sigma(\Phi)))$ for all $\Phi = \langle \Phi_1, \Phi_2 \rangle \in \mathcal{P}\mathrm{SEN}(\Sigma)^2$.

For (1), we need to show that if $\varphi \in C_\Sigma(\Phi)$, then $\vdash_\Sigma \varphi \in C_\Sigma^{(2)}(\vdash_\Sigma \Phi)$. Suppose $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$ are such that $\mathrm{SEN}(f)(\Phi) \subseteq C_{\Sigma'}(\emptyset)$. Then, since $\varphi \in C_\Sigma(\Phi)$, we get $\mathrm{SEN}(f)(\varphi) \in C_{\Sigma'}(\mathrm{SEN}(f)(\Phi))$, whence $\mathrm{SEN}(f)(\varphi) \in C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq C_{\Sigma'}(\emptyset)$, as required.

For (2) we need to show that, for all $\Phi = \langle \Phi_1, \Phi_2 \rangle \in \mathcal{P}\mathrm{SEN}(\Sigma)^2$, $C_\Sigma^{(2)}(\Phi) = C_\Sigma^{(2)}(\vdash_\Sigma E_\Sigma(\Phi_1, \Phi_2))$. We have, for all $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle \in \mathcal{P}\mathrm{SEN}(\Sigma)^2$,

$$\begin{aligned} \Gamma \in C_\Sigma^{(2)}(\Phi) \quad \text{iff} \quad & \mathrm{SEN}(f)(\Phi_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi_1)) \\ & \text{implies } \mathrm{SEN}(f)(\Gamma_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma_1)), \text{ for all } \Sigma' \in |\mathrm{Sign}|,\ f \in \mathrm{Sign}(\Sigma, \Sigma') \\ \text{iff} \quad & E_{\Sigma'}(\mathrm{SEN}(f)(\Phi_1), \mathrm{SEN}(f)(\Phi_2)) \subseteq C_{\Sigma'}(\emptyset) \\ & \text{implies } \mathrm{SEN}(f)(\Gamma_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma_1)), \text{ for all } \Sigma' \in |\mathrm{Sign}|,\ f \in \mathrm{Sign}(\Sigma, \Sigma') \end{aligned}$$

iff

$$\begin{tikzcd} \mathcal{P}\mathrm{SEN}(\Sigma)^2 \arrow[r, "E_\Sigma"] \arrow[d, "\mathcal{P}\mathrm{SEN}(f)^2"'] & \mathcal{P}\mathrm{SEN}(\Sigma) \arrow[d, "\mathcal{P}\mathrm{SEN}(f)"] \\ \mathcal{P}\mathrm{SEN}(\Sigma')^2 \arrow[r, "E_{\Sigma'}"'] & \mathcal{P}\mathrm{SEN}(\Sigma'). \end{tikzcd}$$

$\mathrm{SEN}(f)(E_\Sigma(\Phi_1, \Phi_2)) \subseteq C_{\Sigma'}(\emptyset)$ implies $\mathrm{SEN}(f)(\Gamma_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma_1))$, for all $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$, if and only if $\Gamma_1 \vdash_\Sigma \Gamma_2 \in C_\Sigma^{(2)}(\vdash_\Sigma E_\Sigma(\Phi_1, \Phi_2))$. Therefore $C_\Sigma^{(2)}(\Phi) = C_\Sigma^{(2)}(\vdash_\Sigma E_\Sigma(\Phi_1, \Phi_2))$.

Suppose, conversely, that $\mathcal{I}$ and $\mathcal{I}^{(2)}$ are deductively equivalent π-institutions via the interpretations

$$\langle \mathrm{I}_{\mathrm{Sign}}, \alpha \rangle : \mathcal{I} \longrightarrow \mathcal{I}^{(2)},$$

where $\alpha_\Sigma(\varphi) = \{\vdash_\Sigma \varphi\}$, for all $\Sigma \in |\mathrm{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, $\langle \mathrm{I}_{\mathrm{Sign}}, \beta \rangle : \mathcal{I}^{(2)} \longrightarrow \mathcal{I}$ and the identity natural equivalence. Then define the natural transformation $E : \mathcal{P}\mathrm{SEN}^2 \longrightarrow \mathcal{P}\mathrm{SEN}$ by

$$E_\Sigma(\Delta, \Phi) = \beta_\Sigma(\Delta \vdash_\Sigma \Phi) \quad \text{for all } \Sigma \in |\mathrm{Sign}|,\ \Delta, \Phi \subseteq \mathrm{SEN}(\Sigma).$$

It suffices now to show that $E : \mathcal{P}\mathrm{SEN}^2 \longrightarrow \mathcal{P}\mathrm{SEN}$ is a natural transformation and that, for all $\Sigma \in |\mathrm{Sign}|$, and all $\Phi \cup \Gamma \cup \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$\Phi \subseteq C_\Sigma(\Gamma \cup \Delta) \quad \text{iff} \quad E_\Sigma(\Delta, \Phi) \subseteq C_\Sigma(\Gamma).$$

The naturality of E, i. e. the commutativity of the diagram

$$\begin{tikzcd} \mathcal{P}\mathrm{SEN}(\Sigma_1)^2 \arrow[r, "E_\Sigma"] \arrow[d, "\mathcal{P}\mathrm{SEN}(f)^2"'] & \mathcal{P}\mathrm{SEN}(\Sigma_1) \arrow[d, "\mathcal{P}\mathrm{SEN}(f)"] \\ \mathcal{P}\mathrm{SEN}(\Sigma_2)^2 \arrow[r, "E_{\Sigma_2}"'] & \mathcal{P}\mathrm{SEN}(\Sigma_2) \end{tikzcd}$$

for all $\Sigma_1, \Sigma_2 \in |\mathrm{Sign}|$ and all $f \in \mathrm{Sign}(\Sigma_1, \Sigma_2)$, follows from the naturality of β. Finally, for the deduction-detachment property, it is first shown that

(3) $$\Phi \subseteq C_\Sigma(\Gamma \cup \Delta) \quad \text{iff} \quad \Delta \vdash_\Sigma \Phi \in C^{(2)}_\Sigma(\vdash_\Sigma \Gamma).$$

Suppose that $\Phi \subseteq C_\Sigma(\Gamma \cup \Delta)$ and that, for all $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$, we have $\mathrm{SEN}(f)(\Gamma) \subseteq C_{\Sigma'}(\emptyset)$. Then, we have, for all $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$, that $\mathrm{SEN}(f)(\Gamma \cup \Delta) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Delta))$, whence we have

$$\mathrm{SEN}(f)(\Phi) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma \cup \Delta)) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Delta)),$$

whence $\Delta \vdash_\Sigma \Phi \in C^{(2)}_\Sigma(\vdash_\Sigma \Gamma)$.

Suppose that $\Delta \vdash_\Sigma \Phi \in C^{(2)}_\Sigma(\vdash_\Sigma \Gamma)$. Thus, for all $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$,

(4) if $\mathrm{SEN}(f)(\Gamma) \subseteq C_{\Sigma'}(\emptyset)$, then $\mathrm{SEN}(f)(\Phi) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Delta))$.

Now suppose that, for all $\Sigma' \in |\mathrm{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$, we have $\mathrm{SEN}(f)(\Gamma \cup \Delta) \subseteq C_{\Sigma'}(\emptyset)$. Therefore, $\mathrm{SEN}(f)(\Gamma) \subseteq C_{\Sigma'}(\emptyset)$, whence, by (4), $\mathrm{SEN}(f)(\Phi) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Delta))$, and, hence,

$$\mathrm{SEN}(f)(\Phi) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Delta)) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma \cup \Delta)) \subseteq C_{\Sigma'}(\emptyset).$$

Thus $\vdash_\Sigma \Phi \in C^{(2)}_\Sigma(\vdash_\Sigma \Gamma \cup \Delta)$, i. e. $\alpha_\Sigma(\Phi) \subseteq C^{(2)}_\Sigma(\alpha_\Sigma(\Gamma \cup \Delta))$ and, therefore, $\Phi \subseteq C_\Sigma(\Gamma \cup \Delta)$.

Now, we have, for all $\Sigma \in |\mathrm{Sign}|$, and all $\Phi \cup \Gamma \cup \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \Phi \subseteq C_\Sigma(\Gamma \cup \Delta) & \text{iff} & \Delta \vdash_\Sigma \Phi \in C^{(2)}_\Sigma(\vdash_\Sigma \Gamma) \qquad \text{(by (3))} \\ & \text{iff} & \Delta \vdash_\Sigma \Phi \in C^{(2)}_\Sigma(\alpha_\Sigma(\Gamma)) \\ & \text{iff} & \beta_\Sigma(\Delta \vdash_\Sigma \Phi) \subseteq C_\Sigma(\beta_\Sigma(\alpha_\Sigma(\Gamma))) \\ & \text{iff} & E_\Sigma(\Delta, \Phi) \subseteq C_\Sigma(\Gamma). \end{array}$$ □

Theorem 4.2 *If a π-institution $\mathcal{I}$ has the deduction-detachment property, then the Gentzenization of $\mathcal{I}$ also has the deduction-detachment property.*

P r o o f. If $\mathcal{I}$ has the deduction-detachment property, then, by Theorem 4.1, $\mathcal{I}$ is strongly Gentzen, whence it is deductively equivalent to its Gentzenization $\mathcal{I}^{(2)}$. Therefore, by [31, Theorem 2.17], we get that $\mathcal{I}^{(2)}$ also has the deduction-detachment property. □

By induction on n, the following may be obtained from Theorem 4.2.

Corollary 4.3 *If a π-institution $\mathcal{I}$ has the deduction-detachment property, then the n-th order counterpart $\mathcal{I}^{(n)}$ of $\mathcal{I}$ also has the deduction-detachment property, for all $n \geq 2$.*

Now the following theorem is a consequence of Theorem 4.1 and it provides a criterion for collectively testing all n-th order counterparts with respect to the deduction-detachment property.

Theorem 4.4 *A π-institution $\mathcal{I}$ is strongly Gentzen if and only if $\mathcal{I}^{(n)}$ has the deduction-detachment property, for every $n \geq 1$.*

P r o o f. This follows directly by combining Theorem 4.1 with Corollary 4.3. □

5 Open problem

It would be very interesting to investigate if there exists a metalogical property of a π-institution, similar to having the deduction-detachment property, that is equivalent to it being essentially, rather than strongly, Gentzen. We conjecture that an even more relaxed form of the deduction-detachment property than ours may be what is needed in that case.

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