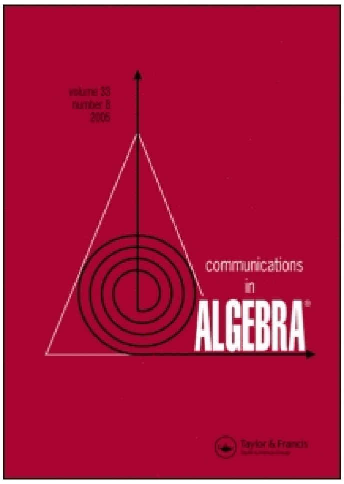

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Categorical Abstract Algebraic Logic: Local Characterization Theorems for Classes of Systems

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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: LOCAL CHARACTERIZATION THEOREMS FOR CLASSES OF SYSTEMS

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Let $\mathcal{L} = \langle \mathbf{F}, R, \rho \rangle$ be a system language. Given a class of $\mathcal{L}$-systems $\mathtt{K}$ and an $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathbf{SEN}, \langle \mathbf{N}, F \rangle \rangle$, i.e., a functor $\mathbf{SEN}: \mathbf{Sign} \to \mathbf{Set}$, with $\mathbf{N}$ a category of natural transformations on $\mathbf{SEN}$, and $\mathbf{F} : \mathbf{F} \to \mathbf{N}$ a surjective functor preserving all projections, define the collection $\mathtt{K}_{\mathbf{A}}$ of $\mathbf{A}$-systems in $\mathtt{K}$ as the collection of all members of $\mathtt{K}$ of the form $\mathfrak{A} = \langle \mathbf{SEN}, \langle \mathbf{N}, F \rangle, R^{\mathfrak{A}} \rangle$, for some set of relation systems $R^{\mathfrak{A}}$ on $\mathbf{SEN}$. Taking after work of Czelakowski and Elgueta in the context of the model theory of equality-free first-order logic, several relationships between closure properties of the class $\mathtt{K}$, on the one hand, and local properties of $\mathtt{K}_{\mathbf{A}}$ and global properties connecting $\mathtt{K}_{\mathbf{A}}$ and $\mathtt{K}_{\mathbf{A}'}$, whenever there exists an $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$, on the other, are investigated. In the main result of the article, it is shown, roughly speaking, that $\mathtt{K}_{\mathbf{A}}$ is an algebraic closure system, for every $\mathcal{L}$-algebraic system $\mathbf{A}$, provided that $\mathtt{K}$ is closed under subsystems and reduced products.

Key Words: κ-Filtered direct product; κ-Filtered intersection; Lyndon classes; Quasivarieties; Subdirect product; Upward κ-directed poset; Union of $\langle \kappa, \sqsubseteq \rangle$-directed system.

2000 Mathematics Subject Classification: Primary 03G99, 18C15; Secondary 68N30.

1. INTRODUCTION

Logical matrices serve as models of sentential logics. As a consequence, they play a key role in the theory of abstract algebraic logic, one of whose main goals is the discovery of conditions under which a sentential logic has a distinctive algebraic character. Of particular interest in the theory of logical matrices from the point of view of abstract algebraic logic have been certain operations on classes of logical matrices, like closure under subdirect products, submatrices, κ-reduced products, etc., that help characterize different classes of logics that form steps in the abstract algebraic hierarchy of sentential logics (see, e.g., Blok and Pigozzi, 1986, 1992; Czelakowski, 2001).

Starting with the work of Bloom (1975), it was made clear that the theory of logical matrices can be perceived as part of the model theory of equality-free first-order logic; more specifically, of universal Horn logic without equality and with a single unary predicate standing for the truth predicate for sentential formulas.

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Subsequent work of Blok and Pigozzi (1992), in the context of their theory of protoalgebraic (Blok and Pigozzi, 1986) and algebraizable logics (Blok and Pigozzi, 1989), showed that, in fact, in many respects, much of the theory of logical matrices, including the study of some of the most useful closure properties on classes of logical matrices, may be carried out, in a general way, inside the framework of universal Horn logic without equality. Moreover, in many instances, this point of view has many advantages in terms of clarity and simplicity over the traditional, more restrictive, treatment.

Elgueta (Czelakowski and Elgueta, 1999; Elgueta, 1997, 1998, 1999a,b; Elgueta and Jansana, 1999c) and Dellunde (Casanovas et al., 1996; Dellunde, 1999, 2000; Dellunde and Jansana, 1996), together with their collaborators, explored this idea further and developed the model theory of equality-free first-order logic as an independent branch of the model theory of first-order logic. The motivation for the development of their theory came from the theory of sentential logics and logical matrices, but they were able to obtain much more general and abstract results. Their point of view, however, remained close to that of abstract algebraic logic and this enabled them to demonstrate very successfully many of the deep interconnections between these two branches of logic.

It was not, however, until very recently that some of the fundamental notions of the theory of algebraizability of sentential logics, as developed by Czelakowski (1981), Blok and Pigozzi (1986, 1989), and Font and Jansana (1996), among others, have been successfully adapted to cover the case of logical systems formalized as π-institutions (see Fiadeiro and Sernadas, 1988; Goguen and Burstall, 1984, 1992). This class of systems includes logical systems, such as multisignature equational logic and first-order logic, which cannot be treated in a very elegant way in the older framework, but also some other systems, whose syntax is not string-based, that could not be treated directly at all with previously existing techniques. Major steps in this development include the introduction of a categorical Tarski operator (Voutsadakis, Preprint), of a model theory of π-institutions along the lines of the model theory of sentential logics (Voutsadakis, 2005a), of algebraic counterparts for π-institutions (Voutsadakis, 2005b) and, finally, of a categorical Leibniz operator (Voutsadakis, 2007a), that led to the exploration of classes of π-institutions analogous to the main classes of the abstract algebraic hierarchy of sentential logics. The development of this theory leads naturally to the idea of adapting the model theory of equality-free first-order logic to cover the case of models, whose algebraic components, rather than being universal algebras, parallel more closely the algebraic counterparts of π-institutional logics. These structures, called *structure systems*, were introduced in Voutsadakis (2007b) and their theory further developed along the lines of the theory of Elgueta in the series of articles Voutsadakis (2006a,b, 2007c).

The focus of the present work is the work of Czelakowski and Elgueta (1999), in which the authors study conditions under which a class of first-order structures is closed under some common algebraic operations, like, for instance, taking substructures and direct products. More specifically, given a class $\mathcal{K}$ of first-order structures and an algebra $\mathbf{A}$, if $\mathcal{K}_{\mathbf{A}}$ denotes the set of members of $\mathcal{K}$ whose underlying algebra is $\mathbf{A}$, Czelakowski and Elgueta reveal local properties of $\mathcal{K}_{\mathbf{A}}$ and global properties relating $\mathcal{K}_{\mathbf{A}}$ with $\mathcal{K}_{\mathbf{B}}$ whenever there exists an algebra homomorphism from $\mathbf{A}$ to $\mathbf{B}$, that ensure the closure of the class $\mathcal{K}$ under a variety of algebraic operators. In the present work, some of the main results of Czelakowski and Elgueta (1999) are adapted and shown to still hold in the more general framework of structure systems.

2. PRELIMINARIES

In this section, we review briefly some of the notions and notational conventions that were adopted in reference to structure systems in previous work (see Voutsadakis, 2006a,b, 2007b,c).

The triple $\mathscr{L} = \langle \mathbf{F}, R, \rho \rangle$ stands for a system language (equality-free with a nonempty set R of relation symbols). That is $\mathscr{L} = \langle \mathbf{F}, R, \rho \rangle$, consists of a clone category $\mathbf{F}$, a nonempty set of relation symbols R and an arity function $\rho : R \to \omega$. Recall from Voutsadakis (2007b) that, in this context, a clone category is a category $\mathbf{F}$ with objects all finite natural numbers that is isomorphic to the category of natural transformations $\mathbf{N}$ on a given functor SEN: $\mathbf{Sign} \to \mathbf{Set}$ via an isomorphism that preserves projections.

Given a system language $\mathscr{L}$, $\mathscr{L}$-(structure) systems were defined in Section 2 of Voutsadakis (2007b). They are triples $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, consisting of:

(i) A functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Set}$;
(ii) A category of natural transformations $\mathbf{N}^{\mathfrak{A}}$ on $\mathrm{SEN}^{\mathfrak{A}}$, such that $F^{\mathfrak{A}} : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}}$ is a surjective functor that preserves all projections $p^{kl} : k \to 1, k \in \omega, l < k$; and
(iii) $R^{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$ a family of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ indexed by R, such that $r^{\mathfrak{A}}$ is n-ary if $\rho(r) = n$.

Given an $\mathscr{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, the pair $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle \rangle$ is said to be the *underlying $\mathscr{L}$-algebraic system of* $\mathfrak{A}$. Thus, *$\mathscr{L}$-algebraic systems* are exactly the underlying $\mathscr{L}$-algebraic systems of $\mathscr{L}$-systems. Intrinsically, an $\mathscr{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle \rangle$ may be defined as a functor SEN, with a category $\mathbf{N}$ of natural transformations on SEN, together with a surjective functor $F : \mathbf{F} \to \mathbf{N}$, that preserves all projections. When the term *class of $\mathscr{L}$-systems* is used, it will always mean a nonempty class.

$\mathscr{L}$-terms and $\mathscr{L}$-formulas were also defined in Section 2 of Voutsadakis (2007b). We are going to follow, however, the modified version of the syntax for $\mathscr{L}$-systems that deals also with individual variables from a denumerable set V, as introduced in Section 3 of Voutsadakis (2006a) (see also Section 2 of Voutsadakis, 2007c). Because $\mathbf{F}$ is assumed to be a category of natural transformations, one may switch in this context between these two forms of syntax without affecting the expressibility of the terms. The variable form is more convenient because it provides greater flexibility in reusing subterms to form more complex terms.

Given an $\mathscr{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ and an $\mathscr{L}$-formula $\alpha(\vec{x})$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$ (n the length of $\vec{x}$), we write $\mathfrak{A} \models_\Sigma \alpha(\vec{x})[\vec{\phi}]$ to indicate that $\langle \Sigma, \vec{\phi} \rangle$ satisfies $\alpha(\vec{x})$ in $\mathfrak{A}$ in the sense of Section 2 of Voutsadakis (2007b) (with the necessary minor modifications when variables are included in the language). We sometimes say also that $\vec{\phi}$ Σ-satisfies $\alpha(\vec{x})$ in $\mathfrak{A}$ in this case. $\mathfrak{A} \models \alpha(\vec{x})$ means that, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, $\mathfrak{A} \models_\Sigma \alpha(\vec{x})[\vec{\phi}]$. If $\mathtt{K}$ is a class of $\mathscr{L}$-systems, $\mathtt{K} \models \alpha(\vec{x})$ means that $\mathfrak{A} \models \alpha(\vec{x})$, for all $\mathfrak{A} \in \mathtt{K}$. Finally, given a set Γ of $\mathscr{L}$-sentences, $\mathrm{Mod}(\Gamma)$ denotes the class of all $\mathscr{L}$-systems $\mathfrak{A}$, such that $\mathfrak{A} \models \gamma$, for all $\gamma \in \Gamma$.

The notions of an $\mathscr{L}$-subsystem and of a filter extension of a given $\mathscr{L}$-system are defined in Section 3.1 of Voutsadakis (2007b).

Now suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ are two $\mathscr{L}$-systems. An $(\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{B}})$-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN}^{\mathfrak{A}} \to^{se} \mathrm{SEN}^{\mathfrak{B}}$ is said to be an *$\mathscr{L}$-morphism* from $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle \rangle$ to $\mathbf{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle \rangle$, written $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{B}$, if the following triangle commutes

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "F^{\mathfrak{A}}"'] \arrow[dr, "F^{\mathfrak{B}}"] & \\
\mathbf{N}^{\mathfrak{A}} \arrow[rr, dashed, no head] & & \mathbf{N}^{\mathfrak{B}}
\end{tikzcd}$$

where the dashed line represents the two-way correspondence established by the $(\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{B}})$-epimorphic property. It is said to be an *$\mathscr{L}$-morphism* from $\mathfrak{A}$ to $\mathfrak{B}$, written $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$, if it is an *$\mathscr{L}$-morphism* from $\mathbf{A}$ to $\mathbf{B}$ and, in addition, for every n-ary relation symbol $r \in R$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r_{\Sigma}^{\mathfrak{A}} \quad \text{implies } \alpha_{\Sigma}(\vec{\phi}) \in r_{F(\Sigma)}^{\mathfrak{B}}.$$

This definition, broken up in two stages, first for $\mathscr{L}$-algebraic systems and then for $\mathscr{L}$-structure systems, here, coincides with the corresponding definition of Section 3.2 of Voutsadakis (2007b).

An $\mathscr{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$, as above, is *injective* if both the functor $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ is injective (both on objects and on morphisms) and, for every $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, the mapping $\alpha_{\Sigma} : \mathrm{SEN}^{\mathfrak{A}}(\Sigma) \to \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma))$ is injective. Similarly, $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$ is *surjective* if both the functor $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ is surjective and, for every $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, the mapping $\alpha_{\Sigma} : \mathrm{SEN}^{\mathfrak{A}}(\Sigma) \to \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma))$ is surjective. We use the usual conventional notations $\langle F, \alpha \rangle : \mathfrak{A} \rightarrowtail \mathfrak{B}$ and $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ to signify that $\langle F, \alpha \rangle$ is injective and surjective, respectively.

An $\mathscr{L}$ morphism $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$ is called *strict* or *strong*, written $\langle F, \alpha \rangle : \mathfrak{A} \to_s \mathfrak{B}$, if, for all $r \in R$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r_{\Sigma}^{\mathfrak{A}} \quad \text{iff } \alpha_{\Sigma}(\vec{\phi}) \in r_{F(\Sigma)}^{\mathfrak{B}}.$$

$\mathfrak{B}$ is said to be a *contraction* of $\mathfrak{A}$ and $\mathfrak{A}$ an *expansion* of $\mathfrak{B}$ if there exists a strict surjective $\mathscr{L}$-morphism, called a *reductive $\mathscr{L}$-morphism*, $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ from $\mathfrak{A}$ onto $\mathfrak{B}$. A class $\mathtt{K}$ of $\mathscr{L}$-systems is an *abstract class* if it is closed under expansions and contractions and contains an $\mathscr{L}$-system with at least one nonempty relation system.

If $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ are $\mathscr{L}$-systems and $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{B}$ is an $\mathscr{L}$-(algebra) morphism, then $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathscr{L}$-morphism if and only if, for all atomic $\mathscr{L}$-formulas $\gamma(\vec{x})$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, where n is the length of $\vec{x}$,

$$\mathfrak{A} \models_{\Sigma} \gamma(\vec{x})[\vec{\phi}] \quad \text{implies } \mathfrak{B} \models_{F(\Sigma)} \gamma(\vec{x})[\alpha_{\Sigma}(\vec{\phi})].$$

Similarly, $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ is a strict $\mathscr{L}$-morphism if and only if, for all atomic $\mathscr{L}$-formulas $\gamma(\vec{x})$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, where n is the length of $\vec{x}$,

$$\mathfrak{A} \models_\Sigma \gamma(\vec{x})[\vec{\phi}] \quad \text{iff } \mathfrak{B} \models_{F(\Sigma)} \gamma(\vec{x})[\alpha_\Sigma(\vec{\phi})].$$

These properties entail that, if $\mathfrak{A}, \mathfrak{B}$ are $\mathscr{L}$-systems and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ a reductive $\mathscr{L}$-morphism, then, for all equality-free $\mathscr{L}$-formulas $\gamma(\vec{x})$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\mathfrak{A} \models_\Sigma \gamma(\vec{x})[\vec{\phi}] \quad \text{iff } \mathfrak{B} \models_{F(\Sigma)} \gamma(\vec{x})[\alpha_\Sigma(\vec{\phi})].$$

This was the content of Proposition 7 of Voutsadakis (2007b). An interesting consequence of this result is that all classes of $\mathscr{L}$-systems axiomatized by equality-free sentences must be abstract classes.

Similarly with the case of first-order structures, given two $\mathscr{L}$-systems $\mathfrak{A}$ and $\mathfrak{B}$ and an $\mathscr{L}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$, one may define an $\mathscr{L}$-system $\alpha^{-1}(\mathfrak{B})$. This construction was carried out in detail in Section 3.2 of Voutsadakis (2007b). More specifically, it was shown in Lemma 5 of Voutsadakis (2007b) that the restriction $\langle F, \alpha\rangle \restriction_{\alpha^{-1}(\mathfrak{B})}: \alpha^{-1}(\mathfrak{B}) \to_s \mathfrak{B}$ is a strong $\mathscr{L}$-morphism and that, if, in addition, $\langle F, \alpha\rangle$ is surjective, then $\langle F, \alpha\rangle \restriction_{\alpha^{-1}(\mathfrak{B})}: \alpha^{-1}(\mathfrak{B}) \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathscr{L}$-morphism.

Moreover, if $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathscr{L}$-morphism, $\mathfrak{C}$ is an $\mathscr{L}$-subsystem of $\mathfrak{A}$ and $\mathfrak{D}$ is an $\mathscr{L}$-subsystem of $\mathfrak{B}$, then we may define $\alpha^{-1}(\mathfrak{D})$ and, under some restrictions on F, we may also define $\alpha(\mathfrak{C})$, as in Section 3.2 of Voutsadakis (2007b), which turn out to be $\mathscr{L}$-subsystems of $\mathfrak{A}$ and of $\mathfrak{B}$, respectively. Namely, we have the following lemma.

Lemma 1 (Lemma 6 of Voutsadakis, 2007b). *Let* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *be* $\mathscr{L}$*-systems and* $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ *a strong* $\mathscr{L}$*-morphism.*

1. *If* $\mathfrak{D} \subseteq \mathfrak{B}$, *then* $\alpha^{-1}(\mathfrak{D}) \subseteq \mathfrak{A}$.
2. *If* $\mathfrak{C} \subseteq \mathfrak{A}$ *and* $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ *is injective, then* $\alpha(\mathfrak{C}) \subseteq \mathfrak{B}$.

Recall from Section 2 of Voutsadakis (2006a) that, given an $\mathscr{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, a binary relation system $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ on $\mathrm{SEN}^{\mathfrak{A}}$ is a congruence system of $\mathfrak{A}$ if it is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$ and, for every $r \in R$, with $\rho(r) = n$, and all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r_\Sigma^{\mathfrak{A}} \quad \text{and} \quad \vec{\phi}\theta_\Sigma^n\vec{\psi} \quad \text{imply } \vec{\psi} \in r_\Sigma^{\mathfrak{A}}.$$

If $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathscr{L}$-morphism, its kernel $\theta^{\langle F,\alpha\rangle} = \mathrm{Ker}(\langle F, \alpha\rangle) = \{\theta_\Sigma^{\langle F,\alpha\rangle}\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ is given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, by

$$\theta_\Sigma^{\langle F,\alpha\rangle} = \{\langle \phi, \psi\rangle \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^2 : \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)\}.$$

If $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ is a strict $\mathscr{L}$-morphism, then $\theta^{\langle F,\alpha\rangle}$ is a congruence system of $\mathfrak{A}$ (see Lemma 2 of Voutsadakis, 2006a). On the other hand, given a congruence system

θ of $\mathfrak{A}$, the quotient of $\mathfrak{A}$ by θ, denoted $\mathfrak{A}/\theta$ or $\mathfrak{A}^\theta$, is the $\mathscr{L}$-system whose underlying $\mathscr{L}$-algebraic system is $\mathbf{A}/\theta = \langle \mathrm{SEN}^{\mathfrak{A}^\theta}, \langle \mathbf{N}^{\mathfrak{A}^\theta}, F^{\mathfrak{A}^\theta}\rangle\rangle$ and whose relation systems $r^{\mathfrak{A}/\theta}$, for $r \in R$, with $\rho(r) = n$, are given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, by

$$\vec{\phi}/\theta_\Sigma \in r_\Sigma^{\mathfrak{A}/\theta} \quad \text{iff } \vec{\phi} \in r_\Sigma^{\mathfrak{A}}.$$

Quotient systems were defined in detail in Section 4 of Voutsadakis (2006a). It is easy to see that $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\theta\rangle : \mathfrak{A} \to \mathfrak{A}/\theta$, given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, by $\pi^\theta_\Sigma(\phi) = \phi/\theta_\Sigma$, is a reductive $\mathscr{L}$-morphism with $\mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\theta\rangle) = \theta$. Hence, congruence systems of $\mathscr{L}$-systems amount exactly to kernels of strict $\mathscr{L}$-morphisms. In Voutsadakis (2006a), it was also shown that analogs of the well-known Homomorphism Theorems of Universal Algebra hold for $\mathscr{L}$-systems. Specifically, the Homomorphism Theorem states the following.

Theorem 2 (Homomorphism Theorem 10 of Voutsadakis, 2006a). *Let* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *be two* $\mathscr{L}$*-systems and* $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *a reductive system morphism. Then, there exists a reductive system morphism* $\langle F, \beta\rangle : \mathfrak{A}^{\mathrm{Ker}(\langle F,\alpha\rangle)} \twoheadrightarrow_s \mathfrak{B}$, *such that the following triangle commutes*:

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "{\langle F, \alpha\rangle}"] \arrow[dr, "{\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\langle F,\alpha\rangle}\rangle}"'] & & \mathfrak{B} \\
& \mathfrak{A}/\mathrm{Ker}(\langle F, \alpha\rangle) \arrow[ur, "{\langle F, \beta\rangle}"'] &
\end{tikzcd}$$

3. OPERATIONS ON STRUCTURES

Let $\kappa \geq \omega$ be a cardinal, I a set and $\mathscr{F}$ a κ-complete filter or ultrafilter on I. If $\mathfrak{A}_i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i\rangle, R^i\rangle$, $i \in I$, is a collection of $\mathscr{L}$-systems, define the direct product $\prod_{i\in I} \mathfrak{A}_i$ and the κ-reduced product or ultraproduct $\prod_{i\in I} \mathfrak{A}_i/\mathscr{F}$ as in Section 3.3 of Voutsadakis (2007b). If $I = \emptyset$, then $\prod_{i\in I} \mathfrak{A}_i$ is the $\mathscr{L}$-system with the trivial sentence functor (its signature category is the trivial one-object category, and the functor maps the single object to a one-element set) all of whose relation systems have nonempty components. By $\langle P^j, \pi^j\rangle : \prod_{i\in I} \mathfrak{A}_i \to \mathfrak{A}_j$, $j \in I$, is denoted the projection $\mathscr{L}$-morphism and by $\langle \mathrm{I}, \pi^{\mathscr{F}}\rangle : \prod_{i\in I} \mathfrak{A}_i \to \prod_{i\in I} \mathfrak{A}_i/\mathscr{F}$ the quotient $\mathscr{L}$-morphism onto the κ-filtered product or ultraproduct. The pre-image $\pi^{\mathscr{F}^{-1}}(\prod_{i\in I} \mathfrak{A}_i/\mathscr{F})$ is denoted by $\prod_{i\in I}^{\mathscr{F}} \mathfrak{A}_i$ and is called the *κ-filtered direct product of* $\mathfrak{A}_i$, $i \in I$, by $\mathscr{F}$. It clearly satisfies, for all $r \in R$, with $\rho(r) = n$, all $\Sigma_i \in |\mathbf{Sign}^i|$ and all $\vec{\phi}_i \in \mathrm{SEN}^i(\Sigma_i)^n$, $i \in I$,

$$\vec{\phi} \in r_{\prod_{i\in I}\Sigma_i}^{\prod_{i\in I}^{\mathscr{F}}\mathfrak{A}_i} \quad \text{iff } \{i \in I : \vec{\phi}_i \in r^i_{\Sigma_i}\} \in \mathscr{F}.$$

We will write $\vec{\phi}/\mathcal{F}$ instead of $\vec{\phi}/\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}$, for all $\Sigma_i \in |\mathbf{Sign}^i|$, $\phi_i \in \mathrm{SEN}^i(\Sigma_i)$, $i \in I$, to denote the equivalence class of $\vec{\phi}$ modulo the congruence system

$$\equiv^{\mathcal{F}} = \left\{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}\right\}_{\prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|}.$$

If $\mathfrak{A}$ is an $\mathcal{L}$-system, $\mathfrak{A}$ is a *subdirect product* of the $\mathcal{L}$-systems $\mathfrak{A}_i, i\in I$, written $\mathfrak{A} \subseteq_{\mathrm{sd}} \prod_{i\in I}\mathfrak{A}_i$, if $\mathfrak{A}$ is an $\mathcal{L}$-subsystem of the direct product $\prod_{i\in I}\mathfrak{A}_i$ and, for every $i\in I$, the projection $\langle P^i, \pi^i\rangle : \prod_{i\in I}\mathfrak{A}_i \to \mathfrak{A}_i$, restricted to $\mathfrak{A}$ is a surjective $\mathcal{L}$-morphism. An injective $\mathcal{L}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail \prod_{i\in I}\mathfrak{A}_i$ is a *subdirect embedding*, in symbols $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I}\mathfrak{A}_i$, if $\alpha(\mathfrak{A}) \subseteq_{\mathrm{sd}} \prod_{i\in I}\mathfrak{A}_i$, where implicit in the notation is that $\alpha(\mathfrak{A})$ is well defined. Obviously, subdirect embeddings are strict $\mathcal{L}$-morphisms.

Suppose that $\mathfrak{A}_i = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^i\rangle, i \in I$, is a collection of $\mathcal{L}$-systems over the same underlying $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$. The *$\kappa$-filtered intersection of* $\mathfrak{A}_i$, $i \in I$, by the filter $\mathcal{F}$ on I, in symbols $\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i$, is defined to be the $\mathcal{L}$-system over $\mathbf{A}$, such that, for every $r \in R$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$,

$$\vec{\phi} \in r_{\Sigma}^{\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i} \qquad \text{iff } \left\{i \in I : \vec{\phi} \in r^i_{\Sigma}\right\} \in \mathcal{F}.$$

Proposition 3. *Given $\mathcal{L}$-systems $\mathfrak{A}_i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i\rangle, R^i\rangle, i \in I$, and a filter $\mathcal{F}$ on I, $\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i$ is also an $\mathcal{L}$-system.*

Proof. We only have to show that, for all $r \in R$, with $\rho(r) = n$, and all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\mathrm{SEN}(f)\left(r_{\Sigma_1}^{\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i}\right) \subseteq r_{\Sigma_2}^{\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i}.$$

Suppose, to this end that $\vec{\phi} \in \mathrm{SEN}(\Sigma_1)^n$, such that $\vec{\phi} \in r_{\Sigma_1}^{\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i}$. Then, by definition, $\{i \in I : \vec{\phi} \in r^i_{\Sigma_1}\} \in \mathcal{F}$. But, since $\mathfrak{A}_i$ is an $\mathcal{L}$-system, for all $i \in I$, we also have that $\{i \in I : \vec{\phi} \in r^i_{\Sigma_1}\} \subseteq \{i \in I : \mathrm{SEN}(f)(\vec{\phi}) \in r^i_{\Sigma_2}\}$, whence, since $\mathcal{F}$ is a filter, we obtain that $\{i \in I : \mathrm{SEN}(f)(\vec{\phi}) \in r^i_{\Sigma_2}\} \in \mathcal{F}$, and, therefore, $\mathrm{SEN}(f)(\vec{\phi}) \in r_{\Sigma_2}^{\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i}$ and $\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i$ is, in fact, an $\mathcal{L}$-system. □

If $\mathcal{F} = \{I\}$, then $\bigcap^{\mathcal{F}}_{i\in I}\mathfrak{A}_i$ is the usual intersection of the $\mathfrak{A}_i$, $i \in I$.

From the definitions of κ-filtered direct products and of κ-filtered intersections, it follows that

$$\prod^{\mathcal{F}}_{i\in I}\mathfrak{A}_i = \bigcap^{\mathcal{F}}_{i\in I}\pi^{i^{-1}}(\mathfrak{A}_i). \tag{1}$$

We have indeed, for all $r \in R$, with $\rho(r) = n$, and all $\Sigma_i \in |\mathbf{Sign}^i|$, $\phi_i \in \mathrm{SEN}^i(\Sigma_i)$, $i \in I$,

$$\begin{aligned}\vec{\phi} \in r^{\prod^{\mathcal{F}}_{i\in I}\mathfrak{A}_i}_{\prod_{i\in I}\Sigma_i} \quad &\text{iff } \left\{i \in I : \phi_i \in r^i_{\Sigma_i}\right\} \in \mathcal{F} \\ &\text{iff } \left\{i \in I : \pi^i_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) \in r^i_{\Sigma_i}\right\} \in \mathcal{F}\end{aligned}$$

$$\text{iff } \left\{i \in I : \vec{\phi} \in \left(\pi^i_{\prod_{i\in I}\Sigma_i}\right)^{-1}(r^i_{\Sigma_i})\right\} \in \mathscr{F}$$

$$\text{iff } \vec{\phi} \in \bigcap_{i\in I}^{\mathscr{F}} \left(\pi^i_{\prod_{i\in I}\Sigma_i}\right)^{-1}(r^i_{\Sigma_i})$$

$$\text{iff } \vec{\phi} \in r^{\bigcap^{\mathscr{F}}_{i\in I}\pi^{i^{-1}}(\mathfrak{A}_i)}_{\prod_{i\in I}\Sigma_i}.$$

Therefore, κ-filtered direct products may be expressed as κ-filtered intersections. And, hence, since κ-reduced products are contractions of κ-filtered direct products, we also get that κ-reduced products may be expressed as contractions of κ-filtered intersections.

An *upward κ-directed poset* is a poset $\langle P, \leq\rangle$ with the property that, if $X \subseteq P$ is such that $|X| < \kappa$, then, there exists an $r \in P$, such that $p \leq r$, for all $p \in X$. A *$\langle\kappa, \sqsubseteq\rangle$-directed system in $\mathscr{L}$* consists of an upward κ-directed poset $\langle P, \leq\rangle$ and an $\mathscr{L}$-system $\mathfrak{A}_p = \langle \text{SEN}, \langle \mathbf{N}, F\rangle, R^p\rangle$, for all $p \in P$, such that, for all $p, q \in P$,

$$\text{if } p \leq q, \qquad \text{then } \mathfrak{A}_p \sqsubseteq \mathfrak{A}_q,$$

where $\sqsubseteq$ denotes the filter-extension relation, as defined in Section 3 of Voutsadakis (2007b). Since all $\mathscr{L}$-systems in a $\langle\kappa, \sqsubseteq\rangle$-directed system have the same underlying $\mathscr{L}$-algebraic system $\mathbf{A} = \langle \text{SEN}, \langle \mathbf{N}, F\rangle\rangle$, the *union of the system* $\{\mathfrak{A}_p : p \in P\}$, in symbols $\bigcup_{p\in P}\mathfrak{A}_p$, may be defined as the system on $\mathbf{A}$, given, for all $r \in R$, with $\rho(r) = n$, and all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \text{SEN}(\Sigma)^n$,

$$\vec{\phi} \in r_\Sigma^{\bigcup_{p\in P}\mathfrak{A}_p} \qquad \text{iff } \vec{\phi} \in \bigcup_{p\in P} r^p_\Sigma.$$

If $\kappa = \omega$, $\{\mathfrak{A}_p : p \in P\}$ is called a *$\sqsubseteq$-directed system* in $\mathscr{L}$.

The following lemma shows that closure of a class $\mathtt{K}$ of $\mathscr{L}$-systems under κ-filtered intersections is equivalent to closure under intersections and unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems. Lemma 4 is an analog of Lemma 5 of Czelakowski and Elgueta (1999) and it is established by using the same arguments. The proof is included here for the sake of completeness.

Lemma 4. *Let $\kappa \geq \omega$ be a cardinal and $\mathtt{K}$ a class of $\mathscr{L}$-systems, all of which have the same underlying $\mathscr{L}$-algebraic system. Then $\mathtt{K}$ is closed under κ-filtered intersections if and only if it is closed under intersections and unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems.*

Proof. Suppose, first, that $\mathtt{K}$ is closed under intersections and unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems. Let $\mathfrak{A}_i = \langle \text{SEN}, \langle \mathbf{N}, F\rangle, R^i\rangle$, $i \in I$, be a collection of $\mathscr{L}$-systems in $\mathtt{K}$ and $\mathscr{F}$ a κ-complete filter on I. Note that, if $\mathscr{F}$ is ordered by reverse inclusion, then the system $\{\bigcap_{i\in X}\mathfrak{A}_i : X \in \mathscr{F}\}$ is a $\langle\kappa, \sqsubseteq\rangle$-directed system of $\mathscr{L}$-systems in $\mathtt{K}$, since $\mathtt{K}$ is closed under intersections and $\mathscr{F}$ is κ-complete. Therefore, since $\mathtt{K}$ is closed under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, we get that $\bigcup_{X\in\mathscr{F}}\bigcap_{i\in X}\mathfrak{A}_i \in \mathtt{K}$. Now it suffices to show that

$$\bigcap_{i\in I}^{\mathscr{F}}\mathfrak{A}_i = \bigcup_{X\in\mathscr{F}}\bigcap_{i\in X}\mathfrak{A}_i. \tag{2}$$

To this end, suppose that $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$. Then

$$\begin{aligned}
\vec{\phi} \in r_\Sigma^{\bigcap_{i\in I}^{\mathcal{F}} \mathfrak{A}_i} \quad & \text{iff } \{i \in I : \vec{\phi} \in r_\Sigma^i\} \in \mathcal{F} \\
& \text{iff } (\exists X \in \mathcal{F})(\forall i \in X)(\vec{\phi} \in r_\Sigma^i) \\
& \text{iff } (\exists X \in \mathcal{F})\Big(\vec{\phi} \in \bigcap_{i\in X} r_\Sigma^i\Big) \\
& \text{iff } \vec{\phi} \in \bigcup_{X\in\mathcal{F}} \bigcap_{i\in X} r_\Sigma^i \\
& \text{iff } \vec{\phi} \in \bigcup_{X\in\mathcal{F}} r_\Sigma^{\bigcap_{i\in X} \mathfrak{A}_i} \\
& \text{iff } \vec{\phi} \in r_\Sigma^{\bigcup_{X\in\mathcal{F}} \bigcap_{i\in X} \mathfrak{A}_i}.
\end{aligned}$$

Suppose, conversely, that K is closed under κ-filtered intersections. Then it is closed under intersections because of the remark following the definition of κ-filtered intersections. So it suffices to show that K is also closed under unions of $\langle \kappa, \sqsubseteq \rangle$-directed systems. To see this, let $\langle P, \leq \rangle$ be an upward κ-directed poset and $\mathfrak{A}_p = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle, R^p \rangle$, $p \in P$, a collection of $\mathcal{L}$-systems in K, such that $\mathfrak{A}_p \sqsubseteq \mathfrak{A}_q$, if $p \leq q$.

Let $\mathcal{F}_P$ be the filter over P generated by the sets $[p) = \{q \in P : p \leq q\}$. First, note that $\mathcal{F}_P$ is κ-complete. Indeed, if $D \subseteq \mathcal{F}_P$, with $|D| < \kappa$, then, for all $X \in D$, there exist finitely many $p_0^X, \ldots, p_{m_X-1}^X \in P$, such that $[p_0^X) \cap \cdots \cap [p_{m_X-1}^X) \subseteq X$. Since $\kappa \geq \omega$, we have that $|\{p_j^X : j < m_X, X \in D\}| < \kappa$. Thus, since P is upward κ-directed, there exists $q \in P$, such that $p_j^X < q$, for all $j < m_X, X \in D$. Hence $[q) \subseteq X$, for all $X \in D$ and, therefore, $[q) \subseteq \bigcap D$. Thus $\bigcap D \in \mathcal{F}_P$.

Finally, it is shown that $\bigcup_{p\in P} \mathfrak{A}_p = \bigcap_{p\in P}^{\mathcal{F}_P} \mathfrak{A}_p$. This will show that K is closed under unions of $\langle \kappa, \sqsubseteq \rangle$-directed systems, since, by closedness under κ-filtered intersections, $\bigcap_{p\in P}^{\mathcal{F}_P} \mathfrak{A}_p \in$ K. We have that, if $X \in \mathcal{F}_P$, then, there exist $p_0, \ldots, p_{m-1} \in P$, such that $[p_0) \cap \cdots \cap [p_{m-1}) \subseteq X$. Thus, since P is upward κ-directed, there exists a $p_X \in P$, such that $[p_X) \subseteq X$. Hence, $\bigcap_{p\in X} \mathfrak{A}_p \sqsubseteq \mathfrak{A}_{p_X}$. Thus, we have

$$\begin{aligned}
\bigcup_{p\in P} \mathfrak{A}_p &= \bigcup_{p\in P} \bigcap_{q\in[p)} \mathfrak{A}_q \qquad (\mathfrak{A}_p, p \in P, \text{ is } \langle \kappa, \sqsubseteq \rangle\text{-directed}) \\
&\sqsubseteq \bigcup_{X\in\mathcal{F}_P} \bigcap_{p\in X} \mathfrak{A}_p \qquad (\text{since } [p) \in \mathcal{F}_P, \text{ for all } p \in P) \\
&\sqsubseteq \bigcup_{X\in\mathcal{F}_P} \mathfrak{A}_{p_X} \qquad (\text{by the choice of } p_X) \\
&\sqsubseteq \bigcup_{p\in P} \mathfrak{A}_p.
\end{aligned}$$

Therefore, finally, $\bigcap_{p\in P}^{\mathcal{F}_P} \mathfrak{A}_p \overset{(2)}{=} \bigcup_{X\in\mathcal{F}_P} \bigcap_{p\in X} \mathfrak{A}_p = \bigcup_{p\in P} \mathfrak{A}_p$. □

A *κ-directed diagram in $\mathcal{L}$*, or simply a *directed diagram in $\mathcal{L}$*, if $\kappa = \omega$, is an upward κ-directed poset $\langle P, \leq \rangle$ together with:

(i) An $\mathcal{L}$-system $\mathfrak{A}_p = \langle \mathrm{SEN}^p, \langle \mathbf{N}^p, F^p \rangle, R^p \rangle$, with $\mathrm{SEN}^p : \mathbf{Sign} \to \mathbf{Set}$, for all $p \in P$; and

(ii) An $\mathscr{L}$-morphism $\langle I_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$, for all $p, q \in P$, such that the following conditions hold:

1. $\langle I_{\mathbf{Sign}}, \alpha^{pp}\rangle = \langle I_{\mathbf{Sign}}, \iota^p\rangle : \mathfrak{A}_p \to \mathfrak{A}_p$ is the identity $\mathscr{L}$-morphism; and
2. If $p \le q \le r$, then $\langle I_{\mathbf{Sign}}, \alpha^{pr}\rangle = \langle I_{\mathbf{Sign}}, \alpha^{qr}\rangle \circ \langle I_{\mathbf{Sign}}, \alpha^{pq}\rangle$.

$$\begin{tikzcd} & \mathfrak{A}_p \arrow[dl, "\langle I_{\mathbf{Sign}}, \alpha^{pq}\rangle"'] \arrow[dr, "\langle I_{\mathbf{Sign}}, \alpha^{pr}\rangle"] & \\ \mathfrak{A}_q \arrow[rr, "\langle I_{\mathbf{Sign}}, \alpha^{qr}\rangle"'] & & \mathfrak{A}_r \end{tikzcd}$$

If, for all $p, q \in P$, $\langle I_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$ is surjective, then the directed diagram is also called *surjective*. The (*surjective*) *κ-direct limit* of the (surjective) κ-directed diagram, given above, denoted $\lim_{p\in P} \mathfrak{A}_p = \langle \mathrm{SEN}^l, \langle \mathbf{N}^l, F^l\rangle, R^l\rangle$, is defined as follows.

For all $\Sigma \in |\mathbf{Sign}|$, let $\bigcup_{p\in P}\{p\} \times \mathrm{SEN}^p(\Sigma)$ denote the disjoint union of $\mathrm{SEN}^p(\Sigma)$, for all $p \in P$. Define a binary relation $\sim_\Sigma$ on $\bigcup_{p\in P}\{p\} \times \mathrm{SEN}^p(\Sigma)$ by setting, for all $p, q \in P$ and all $\phi \in \mathrm{SEN}^p(\Sigma)$, $\psi \in \mathrm{SEN}^q(\Sigma)$,

$$\langle p, \phi\rangle \sim_\Sigma \langle q, \psi\rangle \quad \text{if and only if there exists } r \in P, \text{ such that}$$
$$p \le r, q \le r \quad \text{and} \quad \alpha^{pr}_\Sigma(\phi) = \alpha^{qr}_\Sigma(\psi).$$

It is not difficult to check that, for all $\Sigma \in |\mathbf{Sign}|$, $\sim_\Sigma$ is an equivalence relation on $\bigcup_{p\in P}\{p\} \times \mathrm{SEN}^p(\Sigma)$. Define now

$$\mathrm{SEN}^l(\Sigma) = \left(\bigcup_{p\in P}\{p\} \times \mathrm{SEN}^p(\Sigma)\right)\Big/{\sim_\Sigma}, \qquad \text{for all } \Sigma \in |\mathbf{Sign}|.$$

Given $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, let $\mathrm{SEN}^l(f) : \mathrm{SEN}^l(\Sigma_1) \to \mathrm{SEN}^l(\Sigma_2)$ be defined, for all $p \in P$ and all $\phi \in \mathrm{SEN}^p(\Sigma_1)$, by

$$\mathrm{SEN}^l(f)(\langle p, \phi\rangle/{\sim_{\Sigma_1}}) = \langle p, \mathrm{SEN}^p(f)(\phi)\rangle/{\sim_{\Sigma_2}}.$$

It is not difficult to see that $\mathrm{SEN}^l(f)$ is a well-defined mapping and that SEN^l, thus defined on objects and morphisms of **Sign**, is a functor $\mathrm{SEN}^l : \mathbf{Sign} \to \mathbf{Set}$.

Next, for every $p \in P$, let $\langle I_{\mathbf{Sign}}, \alpha^p\rangle : \mathrm{SEN}^p \to \mathrm{SEN}^l$ be given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}^p(\Sigma)$, by

$$\alpha^p_\Sigma(\phi) = \langle p, \phi\rangle/{\sim_\Sigma}.$$

Then $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle : \mathrm{SEN}^p \to \mathrm{SEN}^l$ is a translation and we have,

$$\begin{tikzcd}
\mathrm{SEN}^p \arrow[rr, "\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle"] \arrow[dr, "\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{p}\rangle"'] & & \mathrm{SEN}^q \arrow[dl, "\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{q}\rangle"] \\
& \mathrm{SEN}^l &
\end{tikzcd}$$

$$\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^q\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle = \langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle, \qquad \text{for all } p, q \in P, \quad \text{such that } p \le q.$$

Indeed, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}^p(\Sigma)$,

$$\begin{aligned} \alpha^q_\Sigma(\alpha^{pq}_\Sigma(\phi)) &= \langle q, \alpha^{pq}_\Sigma(\phi)\rangle/{\sim_\Sigma} \\ &= \langle p, \phi\rangle/{\sim_\Sigma} \\ &= \alpha^p_\Sigma(\phi). \end{aligned}$$

Now, let $F^l : \mathbf{F} \to \mathbf{N}^l$ be defined by setting, for all n-ary σ in $\mathbf{F}$ and all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi}/{\sim_\Sigma} \in \mathrm{SEN}^l(\Sigma)^n$,

$$F^l(\sigma)_\Sigma\big(\vec{\phi}/{\sim_\Sigma}\big) = \alpha^p_\Sigma\big(\sigma^p_\Sigma(\vec{\phi})\big),$$

where $p \in P$ is such that $\vec{\phi} \in \mathrm{SEN}^p(\Sigma)^n$, which exists by the κ-directedness of $\langle P, \le\rangle$. Similarly, if $r \in R$, with $\rho(r) = n$, then, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi}/{\sim_\Sigma} \in \mathrm{SEN}^l(\Sigma)^n$,

$$\vec{\phi}/{\sim_\Sigma} \in r^l_\Sigma \qquad \text{iff } (\exists p \in P)(\exists \vec{\psi} \in \mathrm{SEN}^p(\Sigma)^n)\big(\vec{\psi}/{\sim_\Sigma} = \vec{\phi}/{\sim_\Sigma} \text{ and } \vec{\psi} \in r^p_\Sigma\big).$$

The following proposition may now be proven.

Proposition 5. *Suppose that $\langle P, \le\rangle$ is an upward κ-directed poset and $\mathfrak{A}_p = \langle \mathrm{SEN}^p, \langle \mathbf{N}^p, F^p\rangle, R^p\rangle, p \in P$, with $\mathrm{SEN}^p : \mathbf{Sign} \to \mathbf{Set}$, for all $p \in P$, and $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$, $p, q \in P$, a κ-directed diagram in $\mathscr{L}$. Then:*

1. $\lim_{p\in P} \mathfrak{A}_p = \langle \mathrm{SEN}^l, \langle \mathbf{N}^l, F^l\rangle, R^l\rangle$ *is an $\mathscr{L}$-system*;
2. $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle : \mathfrak{A}_p \to \lim_{p\in P} \mathfrak{A}_p$ *is an $\mathscr{L}$-morphism, for all $p \in P$*;
3. *If $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$ is strict or injective or surjective, for all $p, q \in P$, then $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle : \mathfrak{A}_p \to \lim_{p\in P} \mathfrak{A}_p$ is strict or injective or surjective, respectively, for all $p \in P$.*

Proof. We only provide a sketch of the proof, focusing on the most important points.

1. Since it is not difficult to see that $\mathrm{SEN}^l : \mathbf{Sign} \to \mathbf{Set}$ is indeed a functor, we only show that, for every $r \in R$, with $\rho(r) = n$, $r^l = \{r^l_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a relation system on SEN^l. Suppose, to this end, that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$. Let

also $\phi_i \in \mathrm{SEN}^{p_i}(\Sigma_1)$, for all $i < n$, such that $\langle\langle p_0, \phi_0\rangle, \ldots, \langle p_{n-1}, \phi_{n-1}\rangle\rangle/{\sim_{\Sigma_1}} \in r^l_{\Sigma_1}$. Thus, there exist $p \in P$ and $\vec{\psi} \in \mathrm{SEN}^p(\Sigma_1)^n$, such that $\vec{\psi} \in r^p_{\Sigma_1}$ and $\langle p_i, \phi_i\rangle/{\sim_{\Sigma_1}} = \langle p, \psi_i\rangle/{\sim_{\Sigma_1}}$, for all $i < n$. Since r^p is a relation system on SEN^p, the first condition implies that $\mathrm{SEN}^p(f)(\vec{\psi}) \in \mathrm{SEN}^p(\Sigma_2)^n$ is such that $\mathrm{SEN}^p(f)(\vec{\psi}) \in r^p_{\Sigma_2}$. The second condition implies that, for all $i < n$, there exists an $r_i \in P$, with $p_i \leq r_i$ and $p \leq r_i$, such that $\alpha^{p_ir_i}_{\Sigma_1}(\phi_i) = \alpha^{pr_i}_{\Sigma_1}(\psi_i)$. This yields that $\mathrm{SEN}^{r_i}(f)(\alpha^{p_ir_i}_{\Sigma_1}(\phi_i)) = \mathrm{SEN}^{r_i}(f)(\alpha^{pr_i}_{\Sigma_1}(\psi_i))$. Thus, we obtain that $\alpha^{p_ir_i}_{\Sigma_2}(\mathrm{SEN}^{p_i}(f)(\phi_i)) = \alpha^{pr_i}_{\Sigma_2}(\mathrm{SEN}^p(f)(\psi_i))$, which gives $\langle p_i, \mathrm{SEN}^{p_i}(f)(\phi_i)\rangle/{\sim_{\Sigma_2}} = \langle p, \mathrm{SEN}^p(f)(\psi_i)\rangle/{\sim_{\Sigma_2}}$. This proves that

$$\langle\langle p_0, \mathrm{SEN}^{p_0}(f)(\phi_0)\rangle/{\sim_{\Sigma_2}}, \ldots, \langle p_{n-1}, \mathrm{SEN}^{p_{n-1}}(f)(\phi_{n-1})\rangle/{\sim_{\Sigma_2}}\rangle \in r^l_{\Sigma_2},$$

i.e., that $\mathrm{SEN}^l(\langle\langle p_0, \phi_0\rangle/{\sim_{\Sigma_1}}, \ldots, \langle p_{n-1}, \phi_{n-1}\rangle/{\sim_{\Sigma_1}}\rangle) \in r^l_{\Sigma_2}$, showing that r^l is indeed a relation system on SEN^l.

2. We start this part by showing that $\alpha^p : \mathrm{SEN}^p \to \mathrm{SEN}^l$ is a natural transformation, for every $p \in P$.

$$\begin{tikzcd}
\mathrm{SEN}^p(\Sigma_1) \arrow[r, "\alpha^p_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}^p(f)"'] & \mathrm{SEN}^l(\Sigma_1) \arrow[d, "\mathrm{SEN}^l(f)"] \\
\mathrm{SEN}^p(\Sigma_2) \arrow[r, "\alpha^p_{\Sigma_2}"'] & \mathrm{SEN}^l(\Sigma_2)
\end{tikzcd}$$

Indeed, we have, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $\phi \in \mathrm{SEN}^p(\Sigma_1)$,

$$\begin{aligned}
\mathrm{SEN}^l(f)(\alpha^p_{\Sigma_1}(\phi)) &= \mathrm{SEN}^l(f)(\langle p, \phi\rangle/{\sim_{\Sigma_1}}) \\
&= \langle p, \mathrm{SEN}^p(f)(\phi)\rangle/{\sim_{\Sigma_2}} \\
&= \alpha^p_{\Sigma_2}(\mathrm{SEN}^p(f)(\phi)).
\end{aligned}$$

Next, we need to show that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle$ preserves all natural transformations in $\mathbf{F}$. But this is straightforward, since by the definition of $\mathbf{N}^l$, for every n-ary σ in $\mathbf{F}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}^p(\Sigma)^n$, we have that

$$\sigma^l_\Sigma\big(\alpha^p_\Sigma(\phi_0), \ldots, \alpha^p_\Sigma(\phi_{n-1})\big) = \sigma^l_\Sigma(\langle p, \phi_0\rangle/{\sim_\Sigma}, \ldots, \langle p, \phi_{n-1}\rangle/{\sim_\Sigma}) = \alpha^p_\Sigma\big(\sigma^p_\Sigma(\vec{\phi})\big).$$

Finally, it remains to show that all relations in R are preserved by $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle$. Let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}^p(\Sigma)^n$. If $\vec{\phi} \in r^p_\Sigma$, then it follows trivially by the definition of r^l_Σ, that $\alpha^p_\Sigma(\vec{\phi}) \in r^l_\Sigma$. Thus, $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle : \mathfrak{A}_p \to \lim_{p\in P} \mathfrak{A}_p$ is in fact an $\mathscr{L}$-morphism, for all $p \in P$.

3. We start by showing that, if $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$ is strict, for all $p, q \in P$, then $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle : \mathfrak{A}_p \to \lim_{p\in P} \mathfrak{A}_p$ is also strict, for all $p \in P$. Let, to this end, $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}^p(\Sigma)^n$, such that $\vec{\phi}/{\sim_\Sigma} = \alpha^p_\Sigma(\vec{\phi}) \in r^l_\Sigma$. Thus, there exist $q \in P$ and $\vec{\psi} \in \mathrm{SEN}^q(\Sigma)^n$, such that $\vec{\psi} \in r^q_\Sigma$ and $\vec{\phi}/{\sim_\Sigma} = \vec{\psi}/{\sim_\Sigma}$. Hence,

by the definition of $\sim_\Sigma$ and by the directedness of P, we get that there exists $s \in P$, such that $\alpha^{ps}_\Sigma(\phi_i) = \alpha^{qs}_\Sigma(\psi_i)$, for all $i < n$. This shows that $\alpha^{ps}_\Sigma(\vec{\phi}) = \alpha^{qs}_\Sigma(\vec{\psi}) \in r^s_\Sigma$, since $\vec{\psi} \in r^q_\Sigma$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{qs}\rangle$ is an $\mathscr{L}$-morphism, by Part 1. But, by the hypothesis, $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{ps}\rangle$ is strict, whence we get that $\vec{\phi} \in r^p_\Sigma$, showing that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{p}\rangle$ is also strict.

Next, it is shown that if $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$ is injective, for all $p, q \in P$, then $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{p}\rangle : \mathfrak{A}_p \to \lim_{p\in P} \mathfrak{A}_p$ is injective, for all $p \in P$. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}^p(\Sigma)$, such that $\alpha^p_\Sigma(\phi) = \alpha^p_\Sigma(\psi)$. Then $\langle p, \phi\rangle/{\sim_\Sigma} = \langle p, \psi\rangle/{\sim_\Sigma}$. Therefore, there exists $q \in P$, with $p \leq q$, such that $\alpha^{pq}_\Sigma(\phi) = \alpha^{pq}_\Sigma(\psi)$. But $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle$ is injective, by the hypothesis, whence we get that $\phi = \psi$ and, therefore, $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{p}\rangle$ is also injective.

Finally, it is shown that, if $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$ is surjective, for all $p, q \in P$, then $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{p}\rangle : \mathfrak{A}_p \to \lim_{p\in P} \mathfrak{A}_p$ is also surjective, for all $p \in P$. Suppose that $q \in P$, $\Sigma \in |\mathbf{Sign}|$ and $\psi \in \mathrm{SEN}^q(\Sigma)$. We must show that there exists $\phi \in \mathrm{SEN}^p(\Sigma)$, such that $\alpha^p_\Sigma(\phi) = \langle q, \psi\rangle/{\sim_\Sigma}$. Since P is directed, there exists $s \in P$, such that $q \leq s$ and $p \leq s$. Then, by the definition of $\sim_\Sigma$, $\langle q, \psi\rangle/{\sim_\Sigma} = \langle s, \alpha^{qs}_\Sigma(\psi)\rangle/{\sim_\Sigma}$. But, by the hypothesis, $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{ps}\rangle$ is surjective, whence, there exists $\phi \in \mathrm{SEN}^p(\Sigma)$, such that $\alpha^{ps}_\Sigma(\phi) = \alpha^{qs}_\Sigma(\psi)$. Using once more the definition of $\sim_\Sigma$, we now get that $\langle q, \psi\rangle/{\sim_\Sigma} = \langle p, \phi\rangle/{\sim_\Sigma}$. Thus, we finally obtain $\langle q, \psi\rangle/{\sim_\Sigma} = \langle p, \phi\rangle/{\sim_\Sigma} = \alpha^p_\Sigma(\phi)$, showing that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{p}\rangle$ is, in fact, surjective. □

4. MAIN THEOREMS

Let $\mathtt{K}$ be a class of $\mathscr{L}$-systems and $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ an $\mathscr{L}$-algebraic system. The collection $\mathtt{K}_\mathbf{A}$ of $\mathbf{A}$-systems in $\mathtt{K}$ is the collection of members of $\mathtt{K}$ with underlying $\mathscr{L}$-algebraic system $\mathbf{A}$. $\mathtt{K}_\mathbf{A}$ is partially ordered by the filter extension relation $\sqsubseteq$, for every $\mathscr{L}$-algebraic system $\mathbf{A}$. The purpose of this section is to characterize the closure of abstract classes of $\mathscr{L}$-systems under certain algebraic operations in terms of combinatorial properties concerning the collections $\mathtt{K}_\mathbf{A}$, for $\mathbf{A}$ ranging over all $\mathscr{L}$-algebraic systems, in the spirit of Czelakowski and Elgueta (1999).

The first two theorems form an analog of Theorem 1 of Czelakowski and Elgueta (1999) and deal with intersections and with unions of $\langle \kappa, \sqsubseteq\rangle$-directed systems in $\mathtt{K}_\mathbf{A}$, respectively. As Czelakowski and Elgueta point out, the first theorem may be traced back at least to the work of Mal'cev (1971).

Theorem 6. *Let $\mathtt{K}$ be an abstract class of $\mathscr{L}$-systems. $\mathtt{K}_\mathbf{A}$ is closed under arbitrary intersections, for every $\mathscr{L}$-algebraic system $\mathbf{A}$, if and only if $\mathtt{K}$ is closed under subdirect products.*

Proof. Assume, first, that $\mathtt{K}_\mathbf{A}$ is closed under intersections, for every $\mathbf{A}$. Suppose that $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{A}_i$, with $\mathfrak{A}_i \in \mathtt{K}$, for all $i \in I$. We must show that $\mathfrak{A} \in \mathtt{K}$. Since $\langle P^i, \pi^i\rangle \circ \langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{A}_i$ is surjective, for all $i \in I$, we have, by Lemma 5 of Voutsadakis (2007b), that $\langle P^i, \pi^i\rangle \circ \langle F, \alpha\rangle : (\pi^i \circ \alpha)^{-1}(\mathfrak{A}_i) \twoheadrightarrow_s \mathfrak{A}_i$ is a reductive $\mathscr{L}$-morphism, for all $i \in I$. Thus, since $\mathfrak{A}_i \in \mathtt{K}$ and $\mathtt{K}$ is abstract, we get that $(\pi^i \circ \alpha)^{-1}(\mathfrak{A}_i) \in \mathtt{K}$, for all $i \in I$. Now it suffices to show that $\mathfrak{A} = \bigcap_{i\in I}(\pi^i \circ \alpha)^{-1}(\mathfrak{A}_i)$. To see that this holds, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then, we have

$$\begin{aligned}
\vec{\phi} \in r_{\Sigma}^{\bigcap_{i\in I}(\pi^i\circ\alpha)^{-1}(\mathfrak{A}_i)} \quad & \text{iff } \vec{\phi} \in \bigcap_{i\in I} r_{\Sigma}^{(\pi^i\circ\alpha)^{-1}(\mathfrak{A}_i)} \\
& \text{iff } \pi^i_{F(\Sigma)}(\alpha_{\Sigma}(\vec{\phi})) \in r^{\mathfrak{A}_i}_{P^i(F(\Sigma))}, \quad \text{for all } i \in I, \\
& \text{iff } \alpha_{\Sigma}(\vec{\phi}) \in r^{\prod_{i\in I}\mathfrak{A}_i}_{F(\Sigma)} \\
& \text{iff } \vec{\phi} \in r^{\mathfrak{A}}_{\Sigma} \quad (\text{since } \langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \textstyle\prod_{i\in I}\mathfrak{A}_i).
\end{aligned}$$

Suppose, conversely, that $\mathtt{K}$ is closed under subdirect products. If $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ is an $\mathscr{L}$-algebraic system and $\mathfrak{A}_i = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^i\rangle$, $i \in I$, is a collection of $\mathscr{L}$-systems in $\mathtt{K}_{\mathbf{A}}$, then the intersection $\bigcap_{i\in I}\mathfrak{A}_i$ is an $\mathscr{L}$-system with underlying $\mathscr{L}$-algebraic system $\mathbf{A}$ and $\langle F, \alpha\rangle : \bigcap_{i\in I}\mathfrak{A}_i \to \prod_{i\in I}\mathfrak{A}_i$, defined by $F(\Sigma) = \prod_{i\in I}\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and, similarly, for morphisms, and

$$\alpha_{\Sigma}(\phi) = \prod_{i\in I}\phi, \qquad \text{for all } \Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma),$$

is a subdirect embedding of $\bigcap_{i\in I}\mathfrak{A}_i$ into $\prod_{i\in I}\mathfrak{A}_i$. In fact, for all $r \in R$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$,

$$\begin{aligned}
\vec{\phi} \in r_{\Sigma}^{\bigcap_{i\in I}\mathfrak{A}_i} \quad & \text{iff } \vec{\phi} \in \bigcap_{i\in I} r^i_{\Sigma} \\
& \text{iff } \vec{\phi} \in r^i_{\Sigma}, \text{ for all } i \in I, \\
& \text{iff } \langle \vec{\phi}_0, \ldots, \vec{\phi}_{n-1}\rangle \in r^{\prod_{i\in I}\mathfrak{A}_i}_{\prod_{i\in I}\Sigma},
\end{aligned}$$

where, of course, $\vec{\phi}_k = \prod_{i\in I}\phi_k$, for all $k < n$. □

To show that, for an abstract class $\mathtt{K}$ of $\mathscr{L}$-systems, closure under surjective κ-direct limits amounts to closure of $\mathtt{K}_{\mathbf{A}}$ under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, for every $\mathscr{L}$-algebraic system $\mathbf{A}$, a lemma is needed, showing that unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems of $\mathscr{L}$-systems constitute special cases of surjective κ-direct limits of $\mathscr{L}$-systems.

Lemma 7. *Let $\langle P, \leq\rangle$ be a κ-directed poset and $\mathfrak{A}_p = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^p\rangle$, $p \in P$, a collection of $\mathscr{L}$-systems over the same underlying $\mathscr{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$, such that $\mathfrak{A}_p \sqsubseteq \mathfrak{A}_q$, if $p \leq q$. Consider also the κ-directed diagram in $\mathscr{L}$ formed by the $\mathscr{L}$-morphisms $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathfrak{A}_p \to \mathfrak{A}_q$, $p, q \in P$, with $p \leq q$. Then $\bigcup_{p\in P}\mathfrak{A}_p \cong \lim_{p\in P}\mathfrak{A}_p$.*

Proof. It is easy to see that the pair $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \bigcup_{p\in P}\mathfrak{A}_p \to \lim_{p\in P}\mathfrak{A}_p$, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\alpha_{\Sigma}(\phi) = \langle p, \phi\rangle/\!\sim_{\Sigma} \qquad (p \text{ fixed but arbitrary})$$

is a well-defined $\mathscr{L}$-morphism from $\bigcup_{p\in P}\mathfrak{A}_p$ to $\lim_{p\in P}\mathfrak{A}_p$, which is also an isomorphism. □

Theorem 8. *Let* $\mathtt{K}$ *be an abstract class of* $\mathscr{L}$*-systems. If* $\kappa \geq \omega$ *is a cardinal, then* $\mathtt{K}_{\mathbf{A}}$ *is closed under unions of* $\langle\kappa, \sqsubseteq\rangle$*-directed systems, for every* $\mathscr{L}$*-algebraic system* $\mathbf{A}$*, if and only if* $\mathtt{K}$ *is closed under surjective* κ*-directed limits.*

Proof. Suppose, first, that $\mathtt{K}$ is closed under surjective κ-directed limits. Then, $\mathtt{K}_{\mathbf{A}}$ is closed under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, for every $\mathscr{L}$-algebraic system $\mathbf{A}$, by Lemma 7.

Suppose, conversely, that $\mathtt{K}_{\mathbf{A}}$ is closed under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, for every $\mathscr{L}$-algebraic system $\mathbf{A}$. Let $\langle P, \leq\rangle$ be an upward κ-directed poset, $\mathfrak{A}_p = \langle \mathrm{SEN}^p, \langle \mathbf{N}^p, F^p\rangle, R^p\rangle \in \mathtt{K}, p \in P$, with $\mathrm{SEN}^p : \mathbf{Sign} \to \mathbf{Set}$, for all $p\in P$, and $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle : \mathfrak{A}_p \twoheadrightarrow \mathfrak{A}_q, p, q \in P$, surjective $\mathscr{L}$-morphisms, that satisfy the two conditions of the definition of a surjective κ-directed diagram. We must show that $\mathfrak{A}^l = \langle \mathrm{SEN}^l, \langle \mathbf{N}^l, F^l\rangle, R^l\rangle := \lim_{p\in P}\mathfrak{A}_p \in \mathtt{K}$.

Set $\mathfrak{B}_p = \bigcup_{s\geq p}\alpha^{ps^{-1}}(\mathfrak{A}_s)$. It is shown now that $\mathfrak{B}_p = \alpha^{p^{-1}}(\mathfrak{A}^l)$. Both $\mathscr{L}$-systems $\mathfrak{B}_p$ and $\alpha^{p^{-1}}(\mathfrak{A}^l)$ have the same underlying $\mathscr{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}^p, \langle \mathbf{N}^p, F^p\rangle\rangle$. Suppose that $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}^p(\Sigma)^n$. If $\vec{\phi} \in r_\Sigma^{\alpha^{p^{-1}}(\mathfrak{A}^l)}$, then, there exists $q \in P$ and $\vec{\psi} \in \mathrm{SEN}^q(\Sigma)^n$, such that $\vec{\phi}/{\sim_\Sigma} = \vec{\psi}/{\sim_\Sigma}$ and $\vec{\psi} \in r_\Sigma^q$. Hence, there exists $s \in P$, such that $p \leq s$, $q \leq s$ and $\alpha_\Sigma^{ps}(\vec{\phi}) = \alpha_\Sigma^{qs}(\vec{\psi})$. Thus, we get that $\alpha_\Sigma^{ps}(\vec{\phi}) \in r_\Sigma^s$, for some $s \geq p$. This proves that $\vec{\phi} \in r_\Sigma^{\mathfrak{B}_p}$ and, therefore, $r^{\alpha^{p^{-1}}(\mathfrak{A}^l)} \leq r^{\mathfrak{B}_p}$. Suppose, conversely, that $\vec{\phi} \in r_\Sigma^{\mathfrak{B}_p}$. Then, there exists $s \geq p$, such that $\alpha_\Sigma^{ps}(\vec{\phi}) \in r_\Sigma^s$. Hence, since $(\{p\} \times \vec{\phi})/{\sim_\Sigma} = (\{s\} \times \alpha_\Sigma^{ps}(\vec{\phi}))/{\sim_\Sigma}$, we have that $(\{p\} \times \vec{\phi})/{\sim_\Sigma} \in r_\Sigma^l$, whence $\vec{\phi} \in r_\Sigma^{\alpha^{p^{-1}}(\mathfrak{A}^l)}$ and this proves that $r^{\mathfrak{B}_p} \leq r^{\alpha^{p^{-1}}(\mathfrak{A}^l)}$.

Since $\mathfrak{B}_p = \alpha^{p^{-1}}(\mathfrak{A}^l)$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^p\rangle$ is a surjective $\mathscr{L}$-morphism, we have that $\alpha^{p^{-1}}(\mathfrak{A}^l)$ is an expansion of $\mathfrak{A}^l$. Thus, if $\mathfrak{B}_p$ is shown to be in $\mathtt{K}$, we will also have that $\mathfrak{A}^l \in \mathtt{K}$, since $\mathtt{K}$ is an abstract class.

We focus now in showing that $\mathfrak{B}_p \in \mathtt{K}$. Since $\mathtt{K}$ is assumed closed under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, $\mathfrak{A}_p \in \mathtt{K}$, for all $p \in P$, and $\mathtt{K}$ is abstract, to do this, it suffices to show that $\alpha^{ps^{-1}}(\mathfrak{A}_s)$, $s \geq p$, is a $\langle\kappa, \sqsubseteq\rangle$-directed system. To this end, let $X \subseteq P$ be such that $|X| < \kappa$ and assume $p \leq s$, for all $s \in X$. Since P is upward κ-directed, there exists $q \in P$, such that $s \leq q$, for all $s \in X$. But, for all $s \in X$, we have that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{pq}\rangle = \langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{ps}\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}}, \alpha^{sq}\rangle$, whence, for all $s \in X$, $\alpha^{ps^{-1}}(\mathfrak{A}_s) \sqsubseteq \alpha^{pq^{-1}}(\mathfrak{A}_q)$. Thus $\alpha^{pq^{-1}}(\mathfrak{A}_q)$ is a filter extension of $\alpha^{ps^{-1}}(\mathfrak{A}_s)$, for all $s \in X$. Therefore, $\alpha^{ps^{-1}}(\mathfrak{A}_s)$, $s \geq p$, is indeed a $\langle\kappa, \sqsubseteq\rangle$-directed system in $\mathtt{K}$. □

The next theorem is an analog of Theorem 2 of Czelakowski and Elgueta (1999) for $\mathscr{L}$-systems. It provides a characterization of closedness under subsystems of an abstract class $\mathtt{K}$ of $\mathscr{L}$-systems in terms of global properties relating $\mathtt{K}_{\mathbf{A}}$ and $\mathtt{K}_{\mathbf{B}}$ for two $\mathscr{L}$-algebraic systems $\mathbf{A}$ and $\mathbf{B}$ whenever there exists an $\mathscr{L}$-morphism $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$. Its corollary, Corollary 10, that follows, provides a similar characterization for the property of being closed under subsystems and direct products.

Theorem 9. *Let* $\mathtt{K}$ *be an abstract class of* $\mathcal{L}$*-systems.* $\mathtt{K}$ *is closed under subsystems if and only if, for every* $\mathcal{L}$*-morphism* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ *from an* $\mathcal{L}$*-algebraic system* $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle$ *to an* $\mathcal{L}$*-algebraic system* $\mathbf{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle$*, with* $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ *injective, if* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}_{\mathbf{B}}$*, then* $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}_{\mathbf{A}}$.

Proof. Suppose, first, that $\mathtt{K}$ is closed under subsystems. Let $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ be an $\mathcal{L}$-morphism from an $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle$ to an $\mathcal{L}$-algebraic system $\mathbf{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle$, with $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ injective, and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}_{\mathbf{B}}$. Using both parts of Lemma 6 of Voutsadakis (2007b), we obtain that $\langle F, \alpha\rangle : \alpha^{-1}(\mathfrak{B}) \twoheadrightarrow_s \alpha(\alpha^{-1}(\mathfrak{B}))$ is a reductive $\mathcal{L}$-morphism. Since $\alpha(\alpha^{-1}(\mathfrak{B})) \subseteq \mathfrak{B} \in \mathtt{K}$, we get that $\alpha(\alpha^{-1}(\mathfrak{B})) \in \mathtt{K}$, since $\mathtt{K}$ is closed under subsystems, and, hence, $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}$, since $\mathtt{K}$ is abstract. Therefore, $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}_{\mathbf{A}}$.

Suppose, conversely, that for every $\mathcal{L}$-morphism $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ from an $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle$ to an $\mathcal{L}$-algebraic system $\mathbf{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle$, with $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ injective, if $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}_{\mathbf{B}}$, then $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}_{\mathbf{A}}$. Let $\mathfrak{A} = \langle \mathbf{Sign}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle \subseteq \mathfrak{B} = \langle \mathbf{Sign}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$, with $\langle J, j\rangle : \mathfrak{A} \to_s \mathfrak{B}$ the injection $\mathcal{L}$-morphism (which is strict by the definition of a subsystem). Since $\mathfrak{B} \in \mathtt{K}_{\mathbf{B}}$, we get, by the hypothesis, that $\mathfrak{A} = j^{-1}(\mathfrak{B}) \in \mathtt{K}_{\mathbf{A}}$ and this yields that $\mathfrak{A} \in \mathtt{K}$. So $\mathtt{K}$ is in fact closed under subsystems. □

Corollary 10. *Let* $\mathtt{K}$ *be an abstract class of* $\mathcal{L}$*-systems.* $\mathtt{K}$ *is closed under subsystems and direct products if and only if it satisfies the following conditions*:

1. *For every* $\mathcal{L}$*-morphism* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ *from an* $\mathcal{L}$*-algebraic system* $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle$ *to an* $\mathcal{L}$*-algebraic system* $\mathbf{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle$*, with* $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ *injective, if* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}_{\mathbf{B}}$*, then* $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}_{\mathbf{A}}$;
2. $\mathtt{K}_{\mathbf{A}}$ *is closed under arbitrary intersections, for every* $\mathcal{L}$*-algebraic system* $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle$.

Proof. The statement follows directly by Theorems 6 and 9. □

Theorem 3 of Czelakowski and Elgueta (1999) adapts the analog of Corollary 10 to fit the hypothesis of closure under κ-filtered products instead of that of closure under direct products. Unfortunately, in the present context, we were only able to obtain one of the two directions of the analog of Theorem 3 of Czelakowski and Elgueta (1999). So only a partial analog is presented and the reverse implication is left as an open problem. More precisely, if in Corollary 10, closure under direct products is replaced by closure under κ-filtered products, for some cardinal $\kappa \geq \omega$, we obtain the following result, which forms only a partial analog of Theorem 3 of Czelakowski and Elgueta (1999).

Theorem 11. *Let* $\mathtt{K}$ *be an abstract class of* $\mathcal{L}$*-systems and* $\kappa \geq \omega$ *a regular cardinal. Then the following statements are related by* $1 \to (2 \leftrightarrow 3)$:

1. $\mathtt{K}$ *is closed under subsystems and* κ*-reduced products*;
2. $\mathtt{K}_{\mathbf{A}}$ *is closed under intersections and unions of* $\langle \kappa, \sqsubseteq\rangle$*-directed systems, for every* $\mathcal{L}$*-algebraic system* $\mathbf{A}$;
3. $\mathtt{K}_{\mathbf{A}}$ *is closed under* κ*-filtered intersections, for every* $\mathcal{L}$*-algebraic system* $\mathbf{A}$.

Proof. $1 \to 2$ Suppose that $\mathtt{K}$ is closed under subsystems and κ-reduced products. Then $\mathtt{K}$ is also closed under subdirect products and, hence, by Theorem 6, $\mathtt{K}_{\mathbf{A}}$ is closed under arbitrary intersections, for every $\mathscr{L}$-algebraic system $\mathbf{A}$. To show that $\mathtt{K}_{\mathbf{A}}$ is closed under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, let $\langle P, \leq\rangle$ be an upward κ-directed poset and $\mathfrak{A}_p = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^p\rangle$ over $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ an $\mathscr{L}$-system in $\mathtt{K}$, for every $p \in P$, such that $\mathfrak{A}_p \sqsubseteq \mathfrak{A}_q$, if $p \leq q$.

Let $\mathscr{F}_P$ be the filter over P generated by the sets $[p) = \{q \in P : p \leq q\}$, $p \in P$. Consider the direct product $\prod_{p\in P} \mathfrak{A}_p$ and the reduced product $\prod_{p\in P} \mathfrak{A}_p/\mathscr{F}_P$. Let $X = \{X_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ be the axiom system on $\prod_{p\in P} \mathrm{SEN}$ defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$X_\Sigma = \left\{\vec{\phi} \in \prod_{p\in P} \mathrm{SEN}(\Sigma) : (\exists s \in P)(\forall p, q \geq s)(\phi_p = \phi_q)\right\}.$$

Further, let $\mathfrak{B} = [X]$ be the subsystem of $\prod_{p\in P} \mathfrak{A}_p$ generated by the axiom system X as in Section 3.1 of Voutsadakis (2007b). Note that X is closed under all natural transformations in $\prod_{p\in P} \mathbf{N}$, whence we have that $[X] = X$.

Finally, consider the subsystem $[X/\mathscr{F}_P]$ of $\mathfrak{B}/\mathscr{F}_P$ generated by the axiom system $X/\mathscr{F}_P = \{X_\Sigma/\mathscr{F}_P\}_{\Sigma\in|\mathbf{Sign}|}$ as in Section 3.1 of Voutsadakis (2007b), which is also a subsystem of $\prod_{p\in P} \mathfrak{A}_p/\mathscr{F}_P$, since $\mathfrak{B}$ is a subsystem of $\prod_{p\in P} \mathfrak{A}_p$.

Since $\mathtt{K}$ is closed under subsystems and under κ-reduced products and since, as was shown in the Proof of Lemma 4, $\mathscr{F}_P$ is a κ-complete filter, we have that $[X/\mathscr{F}_P] \in \mathtt{K}$, whence, to prove that $\mathtt{K}$ is closed under unions of $\langle\kappa, \sqsubseteq\rangle$-directed systems, it suffices to show that $[X/\mathscr{F}_P] \cong \bigcup_{p\in P} \mathfrak{A}_p$.

If $\vec{\phi} \in X_\Sigma$, then, there exists a unique $\phi \in \mathrm{SEN}(\Sigma)$, such that, for some $s \in P$, $\phi_p = \phi$, for all $p \geq s$. Since $\langle P, \leq\rangle$ is upward directed, we also have that, for all $n \in \omega$ and all $\vec{\phi}^0, \ldots, \vec{\phi}^{n-1} \in X_\Sigma$, there exists $s \in P$, such that $\phi^0_p = \phi^0, \ldots, \phi^{n-1}_p = \phi^{n-1}$, for all $p \geq s$. Thus, we may define $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : X/\mathscr{F}_P \to \mathrm{SEN}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma(\vec{\phi}/\mathscr{F}_P) = \phi, \qquad \text{for all } \vec{\phi} \in X_\Sigma.$$

If $\vec{\phi}/\mathscr{F}_P = \vec{\psi}/\mathscr{F}_P$, then $\{p \in P : \phi_p = \psi_p\} \in \mathscr{F}_P$, whence, there exist $p_0, \ldots, p_{n-1} \in P$, such that $[p_0) \cap \cdots \cap [p_{n-1}) \subseteq \{p \in P : \phi_p = \psi_p\}$. Thus, since $\langle P, \leq\rangle$ is upward directed, there exists $s \in P$, such that $\phi_p = \psi_p$, whenever $p \geq s$. So $\phi = \phi_p = \psi_p = \psi$, for all $p \geq s$ and, therefore, α_Σ is well-defined, for all $\Sigma \in |\mathbf{Sign}|$.

Suppose next that $\alpha_\Sigma(\vec{\phi}/\mathscr{F}_P) = \alpha_\Sigma(\vec{\psi}/\mathscr{F}_P)$. Then $\phi = \psi$, whence, there exists $s \in P$, such that $\phi_p = \phi = \psi = \psi_p$, for all $p \geq s$. Thus $\vec{\phi}/\mathscr{F}_P = \vec{\psi}/\mathscr{F}_P$ and α_Σ is an injection, and, since it is obviously surjective, also a bijection for all $\Sigma \in |\mathbf{Sign}|$.

We proceed, finally, to show that it is an $\mathscr{L}$-algebraic system morphism and also an $\mathscr{L}$-structure system morphism. Let σ in $\mathbf{F}$ be n-ary and $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi}^0, \ldots, \vec{\phi}^{n-1} \in X_\Sigma$. Then we have

$$\begin{aligned}
\alpha_\Sigma\big(\sigma^{[X/\mathscr{F}_P]}_\Sigma(\vec{\phi}^0/\mathscr{F}_P, \ldots, \vec{\phi}^{n-1}/\mathscr{F}_P)\big) &= \alpha_\Sigma\big(\sigma^{\mathfrak{B}}_\Sigma(\vec{\phi}^0, \ldots, \vec{\phi}^{n-1})/\mathscr{F}_P\big)\\
&= \sigma^{\mathbf{A}}_\Sigma(\phi^0, \ldots, \phi^{n-1})\\
&= \sigma^{\mathbf{A}}_\Sigma\big(\alpha_\Sigma(\vec{\phi}^0/\mathscr{F}_P), \ldots, \alpha_\Sigma(\vec{\phi}^{n-1}/\mathscr{F}_P)\big).
\end{aligned}$$

Now, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi}^0, \ldots, \vec{\phi}^{n-1} \in X_\Sigma$. Then, we have

$$\begin{aligned}
&\langle \vec{\phi}^0/\mathscr{F}_P, \ldots, \vec{\phi}^{n-1}/\mathscr{F}_P\rangle \in r_\Sigma^{[X/\mathscr{F}_P]} \\
&\qquad \text{iff } \langle \vec{\phi}^0/\mathscr{F}_P, \ldots, \vec{\phi}^{n-1}/\mathscr{F}_P\rangle \in r_{\prod_{i\in I}\Sigma}^{\prod_{p\in P}\mathfrak{A}_p/\mathscr{F}_P} \\
&\qquad \text{iff } \{p \in P : \langle \phi_p^0, \ldots, \phi_p^{n-1}\rangle \in r_\Sigma^p\} \in \mathscr{F}_P \\
&\qquad \text{iff } (\exists p_0, \ldots, p_{m-1} \in P)\left(\bigcap_{i=0}^{m-1}[p_i) \subseteq \{p \in P : \langle \phi_p^0, \ldots, \phi_p^{n-1}\rangle \in r_\Sigma^p\}\right) \\
&\qquad \text{iff } (\exists s \in P)(\forall p \geq s)(\langle \phi_p^0, \ldots, \phi_p^{n-1}\rangle \in r_\Sigma^p) \\
&\qquad \text{iff } \langle \alpha_\Sigma(\vec{\phi}^0/\mathscr{F}_P), \ldots, \alpha_\Sigma(\vec{\phi}^{n-1}/\mathscr{F}_P)\rangle \in r_\Sigma^{\bigcup_{p\in P}\mathfrak{A}_p}.
\end{aligned}$$

$2 \leftrightarrow 3$ This is the content of Lemma 4. □

For $\kappa = \omega$ we obtain the following corollary of Theorem 11, which forms a partial analog of the Corollary of Theorem 11 of Czelakowski and Elgueta (1999).

Corollary 12. *Let* $\mathtt{K}$ *be an abstract class of* $\mathscr{L}$*-systems. The poset* $\langle \mathtt{K}_\mathbf{A}, \sqsubseteq\rangle$ *is an algebraic closure system on the poset of all* $\mathscr{L}$*-systems on* $\mathbf{A}$, *for every* $\mathscr{L}$*-algebraic system* $\mathbf{A}$, *if* $\mathtt{K}$ *is closed under subsystems and reduced products.*

Finally, a partial analog of Theorem 4 of Czelakowski and Elgueta (1999) is also provided. It states, roughly speaking that closure of an abstract class $\mathtt{K}$ of $\mathscr{L}$-systems under subsystems, direct products, and homomorphic images implies that $\mathtt{K}_\mathbf{A}$ is a principal filter of the poset of all $\mathscr{L}$-systems on $\mathbf{A}$, for every $\mathscr{L}$-algebraic system $\mathbf{A}$. The proof of the converse of this statement would follow from the $2 \to 1$ direction of Theorem 11. Therefore the equivalence in Theorem 13 is also left as an open problem.

Theorem 13. *If an abstract class* $\mathtt{K}$ *of* $\mathscr{L}$*-systems is closed under subsystems, direct products and* $\mathscr{L}$*-morphic images, then* $\mathtt{K}_\mathbf{A}$ *is a principal filter of the poset of all* $\mathscr{L}$*-systems on* $\mathbf{A}$, *for every* $\mathscr{L}$*-algebraic system* $\mathbf{A}$.

Proof. Suppose that $\mathtt{K}$ is closed under subsystems, direct products and $\mathscr{L}$-morphic images. Let $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ be an $\mathscr{L}$-algebraic system. By Theorem 6, $\mathtt{K}_\mathbf{A}$ is closed under arbitrary intersections. Thus, $\mathtt{K}_\mathbf{A}$ has a least element $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^\mathfrak{A}\rangle$. Therefore, every filter extension $\mathfrak{B}$ of $\mathfrak{A}$ is an $\mathscr{L}$-morphic image of $\mathfrak{A}$ and, hence, $\mathfrak{B} \in \mathtt{K}$, whence $\mathfrak{B} \in \mathtt{K}_\mathbf{A}$. Thus $\mathtt{K}_\mathbf{A}$ is the principal filter of the poset of all $\mathscr{L}$-systems on $\mathbf{A}$ generated by $\mathfrak{A}$. □

We refer the reader to Czelakowski and Elgueta (1999) for some comparisons between their original results, that provided the inspiration for the results developed here, and some results that had been obtained before in a similar context by Gorbunov and Tumanov (1982) (see also Gorbunov, 1994).

5. OPEN PROBLEM

Is it true in Theorem 11 that all three statements are equivalent? If this can be answered to the affirmative, then both equivalences in Corollary 12 and Theorem 13 would also be established.

The equivalence in Corollary 12 is obvious given the direction $2 \to 1$ of Theorem 11.

For the equivalence in Theorem 13, we could work as follows: Suppose that $\mathtt{K}_{\mathbf{A}}$ is a principal filter on the poset of all $\mathscr{L}$-systems on $\mathbf{A}$, for every $\mathscr{L}$-algebraic system $\mathbf{A}$. Then $\mathtt{K}$ is closed under subsystems and κ-reduced products, under the hypothesis of the validity of $2 \to 1$ of Theorem 11, whence it is closed under direct products as well. To see that it is also closed under $\mathscr{L}$-morphic images, suppose that $\mathfrak{A} \in \mathtt{K}$ and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is a surjective $\mathscr{L}$-morphism. Then $\mathfrak{A} \sqsubseteq \alpha^{-1}(\mathfrak{B})$, whence $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}_{\mathbf{A}}$, since $\mathtt{K}_{\mathbf{A}}$ is a filter. Hence $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}$. Moreover, $\langle F, \alpha\rangle : \alpha^{-1}(\mathfrak{B}) \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathscr{L}$-morphism, whence, since $\alpha^{-1}(\mathfrak{B}) \in \mathtt{K}$ and $\mathtt{K}$ is abstract, $\mathfrak{B} \in \mathtt{K}$. This shows that $\mathtt{K}$ would be closed under subsystems, direct products and $\mathscr{L}$-morphic images, in case $2 \to 1$ of Theorem 11 was shown to hold.

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