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Categorical abstract algebraic logic

by

George Voutsadakis

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Mathematics

Major Professor: Don L. Pigozzi

Iowa State University

Ames, Iowa

1998

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To Don Pigozzi.

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1 INTRODUCTION

In 1989 Blok and Pigozzi [6], following in the footsteps of Czelakowski [13] and their own previous work [5], made precise, for the first time, the notion of *algebraizable logic*. A bulk of work has been published since, influenced by this "Memoirs monograph", and a new area in algebraic logic has emerged that has come to be known as *abstract algebraic logic*.

To Blok and Pigozzi, it became clear later [8] that their definition of algebraizability fell into a more general framework. Thus, they defined the notion of *equivalence* for two deductive systems and showed how algebraizability can be perceived to be an instance of this equivalence. In their work they dealt with sentential logics over a fixed signature. Thus, *universal algebra* was the natural, necessary and indispensable algebraic tool for handling algebraizability and promoting this marriage of logic and algebra.

In an appendix of the "Memoirs monograph", Blok and Pigozzi showed how one can use the traditional "cylindrification" process of [28] to transform first-order logic to a sentential structural system and, thus, make it amenable to their algebraization techniques. To Pigozzi this two-step algebraization of first-order logic did not seem very satisfactory. The "cylindrification" step, although necessary if the currently available tools were to be effective at all, was clumsy and unnatural and suggested that a more direct handling of varying signatures required a more advanced and sophisticated framework.

In the meantime, in 1984, Goguen and Burstall [26], in a completely different context, following work of Barwise on abstract model theory [1], had introduced the model-theoretic structure of *institution*. This structure had proved very useful in handling logics with varying signatures, like equational and first-order logic. In 1988, Fiadeiro and Sernadas [21] used Goguen and Burstall's idea to obtain an institution-like syntax-oriented structure which they called *π-institution.*

In 1994, Diskin [18], having seen the work of Blok and Pigozzi and realizing its limitations in dealing with logics with varying signatures, suggested the use of institutions and the algebraization of institutions as the appropriate framework for generalizing Blok and Pigozzi's work. The institution formalism uses concepts and tools borrowed from category theory. Thus, it was only natural that the marriage of logic and algebra at this higher level would require the tools and techniques of *categorical algebra*, namely the powerful machinery of the theory of *algebraic theories.*

This is the point where the present work comes into play. It is our belief that a new "section" of abstract algebraic logic is emerging, its main, distinctive feature being the replacement of universal by categorical algebraic techniques. Our main goal in this thesis is to make a contribution in this newly emerging area by advancing this line of research and to present a, hopefully, convincing argument for the need to pursue further research in order to deepen our understanding of what *categorical abstract algebraic logic* has to offer and what are its limitations.

In the remainder of this chapter we will give a brief outline of the contents of the thesis and a short review of past developments in this area that are our starting points and have influenced our work significantly.

Thesis Outline

In the remainder of this chapter, some previous work on abstract algebraic logic and categorical algebra is reviewed. On the abstract algebraic logic side, the definition of a *deductive system* [51] and its generalization to that of a *k-deductive system* [7] are recalled. The notion of *equivalence* for k-deductive systems [8] is then introduced and the theory leading to the well known characterization of equivalence (Theorem 1.1) overviewed. The *algebraization* of deductive systems [6, 7] comes next and the *intrinsic characterization* of algebraizability (Theorem 1.3), which has been the starting point for abstract algebraic logic, is briefly discussed. On the categorical algebra side, the notion of an *algebraic theory* [38, 24], given in both its clone and its monoid form, starts the development. The description of the *Kleisli category* [32] of the free algebras of a theory follows. The notions of a **T**-*algebra* and **T**-*homomorphism* and the *Eilenberg-Moore category* [19] of a theory come next. The central notion of an *adjunction* [31] is then introduced and the well known *Freyd adjoint functor theorem* (Theorem 1.8) [25] and the *special adjoint functor theorem* (Theorem 1.9) are briefly discussed. The fundamental relationship between algebraic theories and adjunctions [32, 19], which is crucial throughout the thesis, comes afterwards. In this context, the *comparison functors* play a key role (Theorems 1.11 and 1.12) and lead naturally to *Beck's* famous *characterization of algebraic functors* (Theorem 1.13), concluding the first chapter of the thesis.

In the second chapter, the main development of categorical abstract algebraic logic begins. First, the notions of an *institution* (Definition 2.1) [26, 27] and of a π-*institution* (Definition 2.3) [21], which form the basis of our formalism throughout the thesis, are introduced. The central notion of a *term* π-*institution* (Definition 2.7) is given and some examples provided. The *category of theories* of an institution is then described. The structure of this category plays as an important a role in categorical abstract algebraic logic as the lattice of theories in classical abstract algebraic logic. Categories of theories of two π-institutions can be related via *functors* from one into the other. The strength of the ties with which such a functor binds these two categories can be estimated by looking at some of its *abstract properties*. Those are presented next (Definition 2.19). The notions of a *translation* and that of an *interpretation* between two institutions are then introduced (Definition 2.21). The description of *interpretability, quasi-equivalence,* *strong quasi-equivalence* and *deductive equivalence*, the central notions of this chapter, follows (Definition 2.22). They are used to compare both the syntax and the deductive apparatuses of two π-institutions. Each one is stronger than the preceding one in the list. Because of the generality of the notion of institution, it is very difficult to prove useful results for the most general case. One has to restrict to special classes of institutions with features that fit particular applications. Thus, focusing on term π-institutions, *necessary and sufficient conditions* are given for each of the three relations of equivalence to hold between two term π-institutions in terms of their categories of theories (Theorems 2.29, 2.31 and 2.36), which readily extend to institutions. Turning to the special case of deductive equivalence, the *logical interdependence* of the characterizing conditions is also investigated. It turns out that a very concise and elegant *characterization of deductive equivalence*, that parallels the characterization of equivalence of deductive systems [7], can be obtained for term institutions (Theorem 2.41). The special case of *deductive autoequivalence* is then introduced, which naturally relates this result to the corresponding one for deductive systems. This relation is described in detail in the last paragraph of the chapter (Theorem 2.48). The requirement of equivalence for the signature categories of two deductively equivalent π-institutions appears to be too strict. This is the main motive for investigating the weaker notions of quasi-equivalence and strong quasi-equivalence, which relate the signature categories of the two institutions more loosely.

The necessary framework having been laid in Chapter 2, Chapter 3 deals directly with the *algebraization issue*. Based on the notion of an algebraic theory, which generalizes that of a variety, the concept of an *algebraic institution* is described (Definition 3.1), which corresponds to the notion of a 2-deductive system based on the equational consequence relation of some class of algebras. Corresponding to the notions of interpretability, quasi-equivalence, strong quasi-equivalence and deductive equivalence are the notions of *pre-algebraizability, quasi-algebraizability, strong quasi-algebraizability* and *algebraiz-*

ability (Definition 3.4), which seem to form an algebraic hierarchy of π-institutions, based on both their syntax and their deductive power. As immediate consequences of the characterization theorems of Chapter 2, one obtains, here, *characterizations for the different levels of algebraizability* in terms of the categories of theories of the π-institutions involved (Corollary 3.5). These parallel the characterization of the algebraizability of deductive systems of [6]. Three very interesting and important applications are discussed in length closing Chapter 3. First, two institutions based on an algebraic theory in a category $\mathcal{K}$, one with richer syntax structure than the other, but very similar in deductive power, are defined and it is shown that they are quasi-equivalent (Theorem 3.8). Second, inspired by the theory of deductive systems, a "universal algebraic" π-institution and its corresponding categorical counterpart are shown to be deductively equivalent (Theorem 3.11). Finally, the algebraization of the, so called, *equational institution*, an institution that naturally represents a somewhat nonstandard version of equational logic, is described (Theorem 3.27). The detailed development of the algebraic theory in **SET**, on which the algebraic institution used for this algebraization is built, is delegated to Chapter 6, although some of its essential features are described here, as the need arises.

The characterization of algebraizability of a π-institution, provided in Chapter 3, is not intrinsic in the sense that it requires a priori knowledge of the algebraic institution that will be used as the algebraic counterpart in the algebraization. Following [6], one hopes to discover a set of intrinsic necessary and sufficient conditions as concise and elegant as possible. This task is undertaken in Chapter 4. Again the most general case, and, in fact, even the term case, seem to be very difficult to handle. Investigation is restricted, thus, further, to a subclass of term π-institutions, the, so called, *theory institutions* (Definition 4.1). The syntax of these institutions is very nice, already algebraic in nature. The focus now is on the deductive apparatus. In this context, a *generalized Leibniz operator* from theories to generalized equational theories can be defined (Definition 4.6). Further restricting both the class of institutions to the, so called, *Blok-Pigozzi* *institutions* (Definition 4.18) and the type of algebraizability to the, so called, *auto-algebraizability* (Definition 4.3), requiring the syntax component to remain invariant, makes it possible to give *intrinsic necessary and sufficient conditions* (Theorem 4.19) similar to the ones presented in the main theorem of [6]. The chapter concludes with a detailed description of the connection between Blok-Pigozzi theory institutions and deductive systems in the sense of [6] (Theorem 4.23), which, in addition, justifies the name chosen for those institutions. The end of Chapter 4 signals the end of the first main section of the thesis dealing with the study of the algebraization process of institutions itself.

As is the case with classical abstract algebraic logic, two other directions are of equal interest. One is the study of metalogical properties and how these properties are related to corresponding algebraic properties of the algebraizing counterparts and the other is the study of the classes of algebras that are used as algebraic counterparts of logical systems. These are the two directions that are pursued in the remainder of the thesis.

In Chapter 5, several *metalogical properties* are introduced and it is shown that they are preserved under deductive equivalence. More precisely, if two institutions are deductively equivalent, then one has the property if and only if the other does. The properties studied are the deduction-detachment property (Definition 5.1), the conjunction and the disjunction properties (Definitions 5.6 and 5.3, respectively), negation (Definition 5.9), the Craig interpolation property (Definition 5.12), the Robinson consistency property (Definition 5.14) and the Lindenbaum property (Definition 5.17). Those properties have been introduced long ago for deductive systems (see [23]) and adapted later for institutions [50]. They are here formulated in a somewhat non-standard form, appropriate for our purposes, but their essential features are, hopefully, preserved.

In Chapter 6, a detailed study of the *algebraic theory of abstract clone algebras*, used in Chapter 3 to algebraize the equational institution, is undertaken. The main theorem (Theorem 6.14) gives a *universal algebraic characterization* of the Eilenberg-

Moore category of the algebras of this theory. Some connections with the variety of representable substitution algebras of [20] are also investigated (Theorem 6.19).

On the Abstract Algebraic Logic Side

In this section, some of the most important notions and results that have appeared in the literature of abstract algebraic logic and that have significantly influenced our development of categorical abstract algebraic logic, presented in the main body of the thesis, are briefly reviewed. In the first subsection, the main definition of a k-deductive system is given. In the second, the notion of equivalence for deductive systems is described. Finally, in the third subsection, the work of Blok and Pigozzi on the algebraization of k-deductive systems is summarized and a generalization of the main result due to Herrmann is stated.

Deductive Systems

For more details on the material that is reviewed in this section the reader is referred to [6, 7]. Given a set X, we denote by $\overline{X}$ a disjoint copy of X constructed in some canonical way, e.g., $\overline{X} = X \times \{\emptyset\}$, and by $\overline{x}$ the copy of $x \in X$ in $\overline{X}$.

Let $\Lambda = \{\lambda_i : i \in I\}$ be a set of symbols and $\rho : \Lambda \to \omega$ a rank function. $\mathcal{L} = \langle \Lambda, \rho \rangle$ is called a **language type.** If $\lambda \in \Lambda$, we call λ a **connective of rank** $\rho(\lambda)$ or a $\rho(\lambda)$**-ary function symbol.** If $\rho(\lambda) = 0$, then λ is called a **propositional constant** or, simply, a **constant.** Let V be an arbitrary countably infinite set, called the **set of variables.** The set of **formulas of type $\mathcal{L}$ over** V, sometimes called $\mathcal{L}$**-terms over** V, is denoted by $\mathrm{Tm}_{\mathcal{L}}(V)$ and is the smallest set satisfying

- $\overline{v} \in \mathrm{Tm}_{\mathcal{L}}(V)$, for every $v \in V$, and
- $\lambda(t_0, \ldots, t_{\rho(\lambda)-1}) \in \mathrm{Tm}_{\mathcal{L}}(V)$, for all $\lambda \in \Lambda, t_0, \ldots, t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(V)$.

From the logic point of view, one thinks of the elements of Λ as connectives and the elements of $\mathrm{Tm}_{\mathcal{L}}(V)$ as formulas built using these connectives, whereas from the algebra point of view, the elements of Λ are thought of as operation symbols and the elements of $\mathrm{Tm}_{\mathcal{L}}(V)$ as terms built using these operation symbols. This dual interpretation explains the use of $\mathrm{Tm}_{\mathcal{L}}(V)$ to denote the set of formulas. As usual an $\mathcal{L}$-algebra structure can be associated with $\mathrm{Tm}_{\mathcal{L}}(V)$ by setting

$$\lambda^{\mathbf{Tm}_{\mathcal{L}}(V)}(t_0,\ldots,t_{\rho(\lambda)-1}) = \lambda(t_0,\ldots,t_{\rho(\lambda)-1}),$$

for every $\lambda \in \Lambda, t_0,\ldots,t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(V)$. We denote the resulting $\mathcal{L}$-algebra by $\mathbf{Tm}_{\mathcal{L}}(V)$ and call it the **algebra of $\mathcal{L}$-formulas** or the **algebra of $\mathcal{L}$-terms over V.** An **assignment** is a mapping $\sigma : V \to \mathrm{Tm}_{\mathcal{L}}(V)$. Every assignment extends uniquely to a homomorphism $\sigma^* : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{Tm}_{\mathcal{L}}(V)$, which is called a **substitution.** Conversely, every such homomorphism gives rise to an assignment by restricting to variables and the two mappings $\sigma \mapsto \sigma^*$ and $h \mapsto h|_{\overline{V}} \circ \overline{}$ are inverses of each other, where $\overline{} : V \to \overline{V}$ maps $v \in V$ to its copy $\overline{v}$. Thus, without any possibility of confusion we use σ to denote both the assignment and the corresponding substitution.

Given $k \in \omega$, a **k-deductive system $\mathcal{S}$** over $\mathcal{L}$ [7] is a pair $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}}\rangle$, where $\mathbf{Tm}_{\mathcal{L}}(V)$ is the $\mathcal{L}$-formula algebra and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Tm}_{\mathcal{L}}(V)^k) \times \mathrm{Tm}_{\mathcal{L}}(V)^k$ is a relation satisfying the following conditions, for all $\Gamma, \Delta \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k$ and $\phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k$,

(i) $\phi \in \Gamma$ implies $\Gamma \vdash_{\mathcal{S}} \phi$

(ii) $\Gamma \vdash_{\mathcal{S}} \phi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash_{\mathcal{S}} \phi$

(iii) $\Gamma \vdash_{\mathcal{S}} \phi$ and $\Delta \vdash_{\mathcal{S}} \gamma$, for every $\gamma \in \Gamma$, imply $\Delta \vdash_{\mathcal{S}} \phi$

(iv) $\Gamma \vdash_{\mathcal{S}} \phi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma(\phi)$, for every substitution σ,

where, obviously, if $\phi = \langle \phi_0,\ldots,\phi_{k-1}\rangle$, $\sigma(\phi) = \langle \sigma(\phi_0),\ldots,\sigma(\phi_{k-1})\rangle$, and $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$.

Let κ be an infinite cardinal. A k-deductive system $\mathcal{S}$ is said to be **κ-ary** if, for every $\Gamma \cup \{\phi\} \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k$,

(v) $\Gamma \vdash_{\mathcal{S}} \phi$ implies $\Gamma_0 \vdash_{\mathcal{S}} \phi$, for some $\Gamma_0 \subseteq \Gamma$, with $|\Gamma_0| < \kappa$.

In particular, if $\kappa = \omega$, then $\mathcal{S}$ is called **finitary.**

If we define $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Tm}_{\mathcal{L}}(V)^k) \to \mathcal{P}(\mathrm{Tm}_{\mathcal{L}}(V)^k)$ by

$$C_{\mathcal{S}}(\Gamma) = \{\phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k : \Gamma \vdash_{\mathcal{S}} \phi\}, \text{ for every } \Gamma \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k,$$

then (i)-(v) above take the following form

(i') $\Gamma \subseteq C_{\mathcal{S}}(\Gamma)$

(ii') $\Gamma \subseteq \Delta$ implies $C_{\mathcal{S}}(\Gamma) \subseteq C_{\mathcal{S}}(\Delta)$

(iii') $C_{\mathcal{S}}(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(\Gamma)$

(iv') $\sigma(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(\sigma(\Gamma))$

(v') $C_{\mathcal{S}}(\Gamma) = \bigcup\{C_{\mathcal{S}}(\Gamma_0) : \Gamma_0 \subseteq \Gamma, |\Gamma_0| < \kappa\}$,

i.e., they become the well-known **Tarski closure axioms** [51].

A subset $T \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k$ will be called an **$\mathcal{S}$-theory** if, for every $\phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k$, $T \vdash_{\mathcal{S}} \phi$ implies $\phi \in T$, or, equivalently, if $C_{\mathcal{S}}(T) = T$. The collection of all $\mathcal{S}$-theories is denoted by $\mathrm{Th}_{\mathcal{S}}$. Ordered by inclusion, they form a complete lattice which will be denoted by $\mathbf{Th}_{\mathcal{S}}$.

Equivalent Deductive Systems

In [8], Blok and Pigozzi define the notion of equivalence for k-deductive systems. Their basic definitions and results are the paradigms for the deductive equivalence of institutions that we will introduce in this thesis.

Let $\mathcal{S}_1 = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}_1} \rangle$ be a finitary k-deductive system and $\mathcal{S}_2 = \langle \mathbf{Tm}_{\mathcal{L}}(V)^l, \vdash_{\mathcal{S}_2} \rangle$ a finitary l-deductive system, both over the same language type $\mathcal{L}$. By a (k,l)-**translation** we mean a finite set τ of l-formulas in a single k-variable v, i.e., $v \in V^k$ with $v_i \neq v_j, i < j < k$, and $\tau \subseteq \mathrm{Tm}_{\mathcal{L}}(\{v_0, \dots, v_{k-1}\})^l$. Thus, for some $n \in \omega$,

$$\tau(v) = \{\tau_i(v) : i < n\}.$$

For $\Gamma \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k$, let $\tau(\Gamma) = \bigcup_{\phi \in \Gamma} \tau(\phi)$.

A (k,l)-translation τ is a (k,l)-**interpretation** of $\mathcal{S}_1$ in $\mathcal{S}_2$, written $\tau : \mathcal{S}_1 \to \mathcal{S}_2$, if, for all $\Gamma \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k, \phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k$,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \quad \text{iff} \quad \tau(\Gamma) \vdash_{\mathcal{S}_2} \tau(\phi), \tag{1.1}$$

or, equivalently, using the closure operator notation,

$$\phi \in C_{\mathcal{S}_1}(\Gamma) \quad \text{iff} \quad \tau(\phi) \subseteq C_{\mathcal{S}_2}(\tau(\Gamma)).$$

$\mathcal{S}_1$ and $\mathcal{S}_2$ are **equivalent** if there exists a (k,l)-interpretation $\tau : \mathcal{S}_1 \to \mathcal{S}_2$ and an (l,k)-interpretation $\rho : \mathcal{S}_2 \to \mathcal{S}_1$ that are **inverses** of each other in the sense that, for every $\phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k$,

$$\phi \dashv\vdash_{\mathcal{S}_1} \rho(\tau(\phi)) \tag{1.2}$$

and, for every $\psi \in \mathrm{Tm}_{\mathcal{L}}(V)^l$,

$$\psi \dashv\vdash_{\mathcal{S}_2} \tau(\rho(\psi)), \tag{1.3}$$

where $\Gamma \dashv\vdash_{\mathcal{S}_1} \Delta$ means $\Gamma \vdash_{\mathcal{S}_1} \delta$, for every $\delta \in \Delta$, and $\Delta \vdash_{\mathcal{S}_1} \gamma$, for every $\gamma \in \Gamma$, and similarly for $\dashv\vdash_{\mathcal{S}_2}$.

It is proved in [8] that the existence of τ and ρ together with conditions (1.1) and (1.3) are sufficient for the equivalence of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Recall that, given a k-deductive system $\mathcal{S}$ over $\mathcal{L}$ and an $\mathcal{L}$-algebra $\mathbf{A}$, an **$\mathcal{S}$-filter** F on $\mathbf{A}$ is a subset of A^k, such that, for all $\Gamma \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k, \phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k$,

$$\Gamma \vdash_{\mathcal{S}} \phi \quad \text{implies} \quad \phi^{\mathbf{A}}(\vec{a}) \in F \quad \text{whenever} \quad \Gamma^{\mathbf{A}}(\vec{a}) \subseteq F, \quad \text{for all} \quad \vec{a} : V \to A,$$

where $\Gamma^{\mathbf{A}}(\vec{a}) = \{\gamma^{\mathbf{A}}(\vec{a}) : \gamma \in \Gamma\}$. The collection of all $\mathcal{S}$-filters on $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Under set inclusion, they form a complete lattice, which will be denoted by $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$.

Given an interpretation $\tau : \mathcal{S}_1 \to \mathcal{S}_2$, an $\mathcal{L}$-algebra $\mathbf{A}$ and an $\mathcal{S}_1$-filter F on $\mathbf{A}$, define

$$\tau_{\mathcal{S}_2}(F) = \mathrm{Fg}^{\mathbf{A}}_{\mathcal{S}_2}(\tau^{\mathbf{A}}(F)), \tag{1.4}$$

where $\mathrm{Fg}^{\mathbf{A}}_{\mathcal{S}_2}(G)$ denotes the $\mathcal{S}_2$-filter on $\mathbf{A}$ generated by G and $\tau^{\mathbf{A}}(G) = \bigcup\{\tau^{\mathbf{A}}(\vec{a}) : \vec{a} \in G\}$, for every $G \subseteq A^k$. In the particular case where $\mathbf{A} = \mathbf{Tm}_{\mathcal{L}}(V)$ and $T \in \mathrm{Th}_{\mathcal{S}_1}$, (1.4) assumes the form

$$\tau_{\mathcal{S}_2}(T) = C_{\mathcal{S}_2}(\tau(T)). \tag{1.5}$$

If $\mathcal{S}_1, \mathcal{S}_2$ are equivalent via interpretations $\tau : \mathcal{S}_1 \to \mathcal{S}_2$ and $\rho : \mathcal{S}_2 \to \mathcal{S}_1$, then, for any $\mathcal{L}$-algebra $\mathbf{A}$,

$$\tau_{\mathcal{S}_2} : \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A}) \to \mathrm{Fi}_{\mathcal{S}_2}(\mathbf{A}) \text{ and } \rho_{\mathcal{S}_1} : \mathrm{Fi}_{\mathcal{S}_2}(\mathbf{A}) \to \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A})$$

are lattice isomorphisms and inverses of each other. Moreover, for all endomorphisms $h : \mathbf{A} \to \mathbf{A}$, the following diagrams commute:

$$\begin{tikzcd}
\mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A}) \arrow[r, "\tau_{\mathcal{S}_2}"] \arrow[d, "h_{\mathcal{S}_1}"'] & \mathrm{Fi}_{\mathcal{S}_2}(\mathbf{A}) \arrow[d, "h_{\mathcal{S}_2}"] \\
\mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A}) \arrow[r, "\tau_{\mathcal{S}_2}"'] & \mathrm{Fi}_{\mathcal{S}_2}(\mathbf{A})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fi}_{\mathcal{S}_2}(\mathbf{A}) \arrow[r, "\rho_{\mathcal{S}_1}"] \arrow[d, "h_{\mathcal{S}_2}"'] & \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A}) \arrow[d, "h_{\mathcal{S}_1}"] \\
\mathrm{Fi}_{\mathcal{S}_2}(\mathbf{A}) \arrow[r, "\rho_{\mathcal{S}_1}"'] & \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A})
\end{tikzcd}$$

where $h_{\mathcal{S}_1}(F) = \mathrm{Fg}^{\mathbf{A}}_{\mathcal{S}_1}(h(F))$, for every filter $F \in \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A})$, and similarly for $h_{\mathcal{S}_2}$. This property has come to be known as **commutativity with endomorphisms** and, specialized to formula algebras, as **commutativity with substitutions.**

The main result of [8] is the following:

THEOREM 1.1 *Assume $\mathcal{S}_1$ is a finitary k-deductive system and $\mathcal{S}_2$ is a finitary l-deductive system over the same language type $\mathcal{L}$. The following are equivalent*

(i) $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent.

(ii) There exists an isomorphism from $\mathbf{Th}_{\mathcal{S}_1}$ to $\mathbf{Th}_{\mathcal{S}_2}$ that commutes with substitutions.

(iii) For each $\mathcal{L}$-algebra $\mathbf{A}$, there exists an isomorphism from $\mathbf{Fi}_{\mathcal{S}_1}(\mathbf{A})$ to $\mathbf{Fi}_{\mathcal{S}_2}(\mathbf{A})$ that commutes with endomorphisms.

Algebraizability of Deductive Systems

Finally, we review some notions and results on the algebraizability of k-deductive systems, our main sources being [6, 7, 8] and [29].

Let K be a class of $\mathcal{L}$-algebras. Define the 2-deductive system $\mathcal{S}_K = \langle \mathbf{Tm}_{\mathcal{L}}(V)^2, \models_K \rangle$ as follows:

For every $E \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^2$ and $\langle t_0, t_1 \rangle \in \mathrm{Tm}_{\mathcal{L}}(V)^2$,

$$E \models_K \langle t_0, t_1 \rangle \quad \text{iff, for all} \quad \mathbf{A} \in K, \quad h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A},$$
$$h^{\mathbf{A}}(e_0) = h^{\mathbf{A}}(e_1), \text{ for every } \langle e_0, e_1 \rangle \in E, \text{ implies } h^{\mathbf{A}}(t_0) = h^{\mathbf{A}}(t_1).$$

$\mathcal{S}_K$ is the **2-deductive system associated with** K. In case K is a quasivariety axiomatized by a known set of quasi-identities, one can express $\models_K$ in terms of axioms and rules of inference, see, e.g., [8].

A finitary k-deductive system $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}} \rangle$ will be said to be **algebraizable** with **equivalent algebraic semantics** K if $\mathcal{S}_K$ is finitary and $\mathcal{S}$ is equivalent to $\mathcal{S}_K$ in the sense of the previous subsection, i.e., if there exists a $(k, 2)$-interpretation $\tau : \mathcal{S} \to \mathcal{S}_K$ and a $(2, k)$-interpretation $\rho : \mathcal{S}_K \to \mathcal{S}$ that are inverses of each other. The τ will be called **defining equations** and the ρ **equivalence formulas** for $\mathcal{S}$ and K. Blok and Pigozzi prove

THEOREM 1.2 *Let $\mathcal{S}$ be a finitary k-deductive system over $\mathcal{L}$ and K a class of $\mathcal{L}$-algebras such that $\models_K$ is finitary. $\mathcal{S}_K$ is equivalent to $\mathcal{S}$ iff there exists an isomorphism from $\mathbf{Th}_{\mathcal{S}}$ onto $\mathbf{Th}_{\mathcal{S}_K}$ that commutes with substitutions.*

Given a k-deductive system $\mathcal{S}$ over $\mathcal{L}$ and an $\mathcal{S}$-theory $T \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k$, the **Leibniz congruence** $\Omega(T)$ associated with T is defined to be the largest congruence Θ on $\mathbf{Tm}_{\mathcal{L}}(V)$ that is compatible with T, i.e., such that, for every $\phi, \psi \in \mathrm{Tm}_{\mathcal{L}}(V)^k$, $\phi_i \,\Theta\, \psi_i, i < k$, and $\phi \in T$ imply $\psi \in T$. The main result of [6], generalized to k-deductive systems [8], is the following

THEOREM 1.3 *Let $\mathcal{S}$ be a finitary k-deductive system over $\mathcal{L}$. $\mathcal{S}$ is algebraizable iff*

(i) Ω is injective on $\mathbf{Th}_{\mathcal{S}}$ and

(ii) Ω preserves unions of directed subsets of $\mathbf{Th}_{\mathcal{S}}$.

In [29] Hermann generalized slightly the results of Blok and Pigozzi by considering infinitary deductive systems and allowing the sets of defining equations and equivalence formulas in an algebraic equivalence to be infinite. The following is his main result, given here for k-deductive systems

THEOREM 1.4 *Let $\mathcal{S}$ be a k-deductive system over $\mathcal{L}$. $\mathcal{S}$ is algebraizable in the sense of Herrmann iff*

(i) Ω is injective on $\mathbf{Th}_{\mathcal{S}}$

(ii) Ω is meet-continuous on $\mathbf{Th}_{\mathcal{S}}$ and

(iii) Ω commutes with inverse substitutions.

On the Categorical Algebra Side

In this section we review some basic definitions and results of the theory of algebraic theories. The interested reader may consult [39, 43, 2] or [9] for a more detailed account. First, some basic notational conventions used throughout the thesis are presented.

Given a category $\mathcal{K}$, by $|\mathcal{K}|$ is denoted the collection of objects of $\mathcal{K}$ and by $\mathrm{Mor}(\mathcal{K})$ the collection of morphisms of $\mathcal{K}$. Given $A, B \in |\mathcal{K}|$, by $\mathcal{K}(A, B)$ is denoted the set of all morphisms in $\mathcal{K}$ with domain A and codomain B. By $\mathcal{P} : \mathbf{SET} \to \mathbf{SET}$ is denoted the powerset functor. Following a usual convention, given a morphism $f : A \to B$ in **SET** and a subset $X \subseteq A$, we write $f(X)$ instead of $\mathcal{P}(f)(X)$. Finally, given a category $\mathcal{K}$ and $A \in |\mathcal{K}|$, by $(A \mid \mathcal{K})$ is denoted the category of *objects under* A, i.e., the category with objects all pairs $\langle f, K\rangle, K \in |\mathcal{K}|, f \in \mathcal{K}(A, K)$, and arrows $h : \langle f, K\rangle \to \langle g, L\rangle$ those arrows $h \in \mathcal{K}(K, L)$, such that $g = hf$, i.e., such that the following diagram commutes

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ K \arrow[rr, "h"'] & & L \end{tikzcd}$$

Composition in $(A \mid \mathcal{K})$ is inherited from the composition in $\mathcal{K}$.

Algebraic Theories

In the sequel $\mathcal{K}$ will be a fixed category, called the **base category.**

An **algebraic theory in clone form** in (or over) $\mathcal{K}$ is a triple $\mathbf{T} = \langle T, \eta, \circ\rangle$, where

(1) $T : |\mathcal{K}| \to |\mathcal{K}|$ is an object function

(2) $\eta : |\mathcal{K}| \to \mathrm{Mor}(\mathcal{K})$ is a mapping such that $\eta_A : A \to T(A)$

(3) $\circ$ is a mapping assigning to every $(A, B, C) \in |\mathcal{K}|^3$ a function $\circ : \mathcal{K}(B, T(C)) \times \mathcal{K}(A, T(B)) \to \mathcal{K}(A, T(C))$ such that

  (i) $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$, for all $\alpha : A \to T(B), \beta : B \to T(C), \gamma : C \to T(D)$,

  (ii) $\eta_B \circ \alpha = \alpha$, for every $\alpha : A \to T(B)$,

  (iii) $\alpha \circ (\eta_B f) = \alpha f$, for all $f : A \to B$ and $\alpha : B \to T(C)$.

The category having as collection of objects $|\mathcal{K}|$ and, for all $A, B \in |\mathcal{K}|$, as collection of morphisms from A to B all $\mathcal{K}$-maps $f : A \to T(B)$, with composition $\circ$ and identities $\eta_A, A \in |\mathcal{K}|$, is called the **Kleisli category of T** and denoted by $\mathcal{K}_{\mathbf{T}}$. The notation $f : A \rightharpoonup B$ is used to denote an arrow $f : A \to T(B)$ in $\mathcal{K}_{\mathbf{T}}$.

An **algebraic theory in monoid form** in (or over) $\mathcal{K}$ is a triple $\mathbf{T} = \langle T, \eta, \mu\rangle$, where

(1) $T : \mathcal{K} \to \mathcal{K}$ is a functor

(2) $\eta : I_{\mathcal{K}} \to T$ is a natural transformation

(3) $\mu : TT \to T$ is a natural transformation such that, for every $A \in |\mathcal{K}|$, the following diagrams commute

$$\begin{tikzcd} T(A) \arrow[r, "\eta_{T(A)}"] \arrow[dr, "i_{T(A)}"'] & T(T(A)) \arrow[d, "\mu_A"] & T(A) \arrow[l, "T(\eta_A)"'] \arrow[dl, "i_{T(A)}"] \\ & T(A) & \end{tikzcd}$$

$$\begin{tikzcd} T(T(T(A))) \arrow[r, "T(\mu_A)"] \arrow[d, "\mu_{T(A)}"'] & T(T(A)) \arrow[d, "\mu_A"] \\ T(T(A)) \arrow[r, "\mu_A"'] & T(A) \end{tikzcd}$$

If, given a category $\mathcal{K}$ and an algebraic theory $\mathbf{T} = \langle T, \eta, \circ \rangle$ in clone form over $\mathcal{K}$, one defines $T : \mathrm{Mor}(\mathcal{K}) \to \mathrm{Mor}(\mathcal{K})$, for every $f : A \to B \in \mathrm{Mor}(\mathcal{K})$, by $T(f) : T(A) \to T(B)$ with $T(f) = \eta_B f \circ i_{T(A)}$, and, for every $A \in |\mathcal{K}|, \mu_A = i_{T(A)} \circ i_{T(T(A))}$, then $\mathbf{T} = \langle T, \eta, \mu \rangle$ is an algebraic theory in monoid form.

Conversely, if, given $\mathcal{K}$ and an algebraic theory $\mathbf{T} = \langle T, \eta, \mu \rangle$ in monoid form over $\mathcal{K}$, one defines, for every $(A, B, C) \in |\mathcal{K}|^3, \circ : \mathcal{K}(B, T(C)) \times \mathcal{K}(A, T(B)) \to \mathcal{K}(A, T(C))$, by $\alpha \circ \beta = \mu_C T(\beta)\alpha$,

$$A \xrightarrow{\alpha} T(B) \xrightarrow{T(\beta)} T(T(C)) \xrightarrow{\mu_C} T(C)$$

for every $\alpha : A \to T(B)$ and $\beta : B \to T(C)$, then $\mathbf{T} = \langle T, \eta, \circ \rangle$ is an algebraic theory in clone form and the two translations, just described, from algebraic theories in one form to algebraic theories in the other are inverses of one another.

Let $\mathcal{K}$ be a category and $\mathbf{T} = \langle T, \eta, \circ, \mu \rangle$ an algebraic theory over $\mathcal{K}$. A **T-algebra** is a pair $\langle A, \alpha \rangle$, where $A \in |\mathcal{K}|$ and $\alpha : T(A) \to A \in \mathrm{Mor}(\mathcal{K})$ satisfying commutativity of the following diagrams

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "i_A"'] & T(A) \arrow[d, "\alpha"] \\
 & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T(T(A)) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\
T(A) \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

If $\langle A, \alpha \rangle, \langle B, \beta \rangle$ are $\mathbf{T}$-algebras, a **T-homomorphism** from $\langle A, \alpha \rangle$ to $\langle B, \beta \rangle$ is a map $f : A \to B \in \mathrm{Mor}(\mathcal{K})$, such that the following diagram commutes

$$
\begin{tikzcd}
T(A) \arrow[r, "T(f)"] \arrow[d, "\alpha"'] & T(B) \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

It is not difficult to see that $i_A : \langle A, \alpha \rangle \to \langle A, \alpha \rangle$ is a $\mathbf{T}$-homomorphism and, if $f : \langle A, \alpha \rangle \to \langle B, \beta \rangle$ and $g : \langle B, \beta \rangle \to \langle C, \gamma \rangle$ are $\mathbf{T}$-homomorphisms, then so is $gf : \langle A, \alpha \rangle \to \langle C, \gamma \rangle$. Thus, $\mathbf{T}$-algebras together with $\mathbf{T}$-homomorphisms form a category $\mathcal{K}^{\mathbf{T}}$, called the **Eilenberg-Moore category** of the algebraic theory $\mathbf{T}$.

Adjoints

Let $\mathcal{A}, \mathcal{K}$ be categories, $U : \mathcal{A} \to \mathcal{K}$ be a functor and $K \in |\mathcal{K}|$. A **free $\mathcal{A}$-object over K with respect to U** is a pair $\langle F, \eta \rangle$, where $F \in |\mathcal{A}|$ and $\eta : K \to U(F) \in \mathrm{Mor}(\mathcal{K})$ such that if $\langle A, f \rangle$ is another pair with $A \in |\mathcal{A}|, f : K \to U(A) \in \mathrm{Mor}(\mathcal{K})$, then there exists unique $f^{\#} : F \to A \in \mathrm{Mor}(\mathcal{A})$ such that $U(f^{\#})\eta = f$. In this case, η is said to be **universal** to U from K.

Pictorially, we have

$$
\begin{tikzcd}
F \\
A \arrow[u, leftarrow, "f^{\#}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K \arrow[r, "\eta"] \arrow[dr, "f"'] & U(F) \arrow[d, "U(f^{\#})"] \\
& U(A)
\end{tikzcd}
$$

Dual to the notion of a universal arrow from an object to a functor is the notion of a universal arrow from a functor to an object. Let $\mathcal{A}, \mathcal{K}$ be two categories, $F : \mathcal{K} \to \mathcal{A}$ a functor and $A \in |\mathcal{A}|$. Given an object $U \in |\mathcal{K}|$, a mapping $\epsilon : F(U) \to A \in \mathrm{Mor}(\mathcal{A})$ is called **universal** to A from F if, for all $K \in |\mathcal{K}|, f : F(K) \to A$, there exists unique $f^{\#} : K \to U \in \mathrm{Mor}(\mathcal{K})$ such that $\epsilon F(f^{\#}) = f$.

Pictorially, we have

$$
\begin{tikzcd}
U \\
K \arrow[u, "f^{\#}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(U) \arrow[r, "\epsilon"] & A \\
F(K) \arrow[u, "F(f^{\#})"] \arrow[ur, "f"']
\end{tikzcd}
$$

Let $\mathcal{A}, \mathcal{K}$ be categories. An **adjunction from $\mathcal{K}$ to $\mathcal{A}$** is a triple $\langle F, U, \phi \rangle : \mathcal{K} \to \mathcal{A}$, where $F : \mathcal{K} \to \mathcal{A}, U : \mathcal{A} \to \mathcal{K}$ are functors and ϕ is a function assigning to each pair of objects $K \in |\mathcal{K}|, A \in |\mathcal{A}|$ a bijection

$$\phi = \phi_{K,A} : \mathcal{A}(F(K), A) \cong \mathcal{K}(K, U(A))$$

which is natural in K and A.

Given such an adjunction, F is said to be a **left adjoint** for U while U is called a **right adjoint** for F. Moreover, the image of an arrow $f : F(K) \to A$ under $\phi_{K,A}$ is its **right adjunct** and the image of an arrow $g : K \to U(A)$ under $\phi^{-1}_{K,A}$ is its **left adjunct.**

In [39] the following theorems are proved:

THEOREM 1.5 *An adjunction $\langle F, U, \phi \rangle : \mathcal{K} \to \mathcal{A}$ determines* *(i)* *a natural transformation* $\eta : I_{\mathcal{K}} \to UF$ *such that, for every object* $K \in |\mathcal{K}|$, *the arrow* η_K *is universal to* U *from* K, *while the right adjunct of each* $f : F(K) \to A$ *is* $\phi(f) = U(f)\eta_K : K \to U(A)$

(ii) *a natural transformation* $\epsilon : FU \to I_{\mathcal{A}}$ *such that each arrow* ϵ_A *is universal to* A *from* F, *while each* $g : K \to U(A)$ *has left adjunct* $\phi^{-1}(g) = \epsilon_A F(g) : F(K) \to A$.

Moreover the following triangles commute

$$\begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, "i_U"'] & UFU \arrow[d, "U\epsilon"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "i_F"'] & FUF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd}$$

η is called the **unit** and ϵ the **counit** of the given adjunction $\langle F, U, \phi\rangle : \mathcal{K} \to \mathcal{A}$.

THEOREM 1.6 *Each adjunction* $\langle F, U, \phi\rangle : \mathcal{K} \to \mathcal{A}$ *is completely determined by the functors* F, U *and the natural transformations* $\eta : I_{\mathcal{K}} \to UF$ *and* $\epsilon : FU \to I_{\mathcal{A}}$ *satisfying the commutativity of the above triangles.*

Given theorems 1.5 and 1.6 we feel free to switch between the notations $\langle F, U, \phi\rangle : \mathcal{K} \to \mathcal{A}$ and $\langle F, U, \eta, \epsilon\rangle : \mathcal{K} \to \mathcal{A}$ for the given adjunction.

The following theorem relating free objects and adjoints is proved in [43].

THEOREM 1.7 *Let* $\mathcal{K}, \mathcal{A}$ *be categories and* $U : \mathcal{A} \to \mathcal{K}$ *be a functor. Then there exists an adjunction of the form* $\langle F, U, \eta, \epsilon\rangle : \mathcal{K} \to \mathcal{A}$ *if and only if, for every* $K \in |\mathcal{K}|$, *there exists a free* $\mathcal{A}$*-object over* K *with respect to* U.

In both [39] and [43] one can find the following theorems

THEOREM 1.8 (THE FREYD ADJOINT FUNCTOR THEOREM) *Given a small-complete category* $\mathcal{A}$ *with small hom-sets, a functor* $U : \mathcal{A} \to \mathcal{K}$ *has a left adjoint if and only if it preserves all small limits and satisfies the*

Solution set condition: *For each* $K \in |\mathcal{K}|$, *there exists a small set* I, *an* I*-indexed family of objects* $A_i \in |\mathcal{A}|$ *and an* I*-indexed family of morphisms* $f_i : K \to U(A_i) \in$ Mor($\mathcal{K}$) *such that every* $h : K \to U(A)$ *can be factored as* $h = U(t)f_i$, *for some* $i \in I$ *and* $t : A_i \to A$.

$$\begin{tikzcd} K \arrow[rr, "h"] \arrow[dr, "f_i"'] & & U(A) \\ & U(A_i) \arrow[ur, "U(t)"'] & \end{tikzcd}$$

The following notions of a subobject and a cogenerating set are necessary for the formulation of the special adjoint functor theorem and its corollary.

Given a category $\mathcal{A}$ and two monics $f : B \to A$, $g : C \to A \in \mathrm{Mor}(\mathcal{A})$ with common codomain A we write $f \leq g$ when f factors through g, i.e., when $f = gf'$, for some $f' : B \to C$.

$$\begin{tikzcd} B \arrow[dr, "f"] \arrow[dd, dashed, "f'"'] & \\ & A \\ C \arrow[ur, "g"'] & \end{tikzcd}$$

When $f \leq g$ and $g \leq f$, we write $f \equiv g$. The relation $\equiv$ is an equivalence relation on the monics with codomain A and its equivalence classes are called the **subobjects** of A. Following common practice we sometimes identify a representative $f : B \to A$ with the subobject represented by f. $\mathcal{A}$ will be called **well-powered** when the subobjects of each $A \in |\mathcal{A}|$ can be indexed by a small set.

Given a category $\mathcal{A}$, a set $\mathcal{C} \subseteq |\mathcal{A}|$ will be called a **cogenerating set** for $\mathcal{A}$ if to every parallel pair $g \neq g' : A \to B \in \mathrm{Mor}(\mathcal{A})$ there exists $C \in \mathcal{C}$ and $f : B \to C$ with $fg \neq fg'$.

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & B \arrow[r, dashed, "f"] & C \end{tikzcd}$$

THEOREM 1.9 (THE SPECIAL ADJOINT FUNCTOR THEOREM) *Let $\mathcal{A}$ be a small-complete category, with small hom-sets and a small cogenerating set $\mathcal{C}$, such that every set of subobjects of an object $A \in |\mathcal{A}|$ has a pullback, and let $\mathcal{K}$ be a category with small hom-sets. Then a functor $U : \mathcal{A} \to \mathcal{K}$ has a left adjoint if and only if it preserves all small limits and all pullbacks of families of monics.*

Finally, if $\mathcal{A}$ happens to be well-powered the Special Adjoint Functor Theorem assumes the following form

COROLLARY 1.10 *If $\mathcal{A}$ is small-complete, well-powered, with small hom-sets and a small cogenerating set while $\mathcal{K}$ has small hom-sets then a functor $U : \mathcal{A} \to \mathcal{K}$ has a left adjoint if and only if it preserves all small limits.*

Theories and Adjoints

Let $\mathcal{A}, \mathcal{K}$ be categories and $\langle F, U, \eta, \epsilon \rangle : \mathcal{K} \to \mathcal{A}$ an adjunction. Define $T : \mathcal{K} \to \mathcal{K}$ by $T = UF$ and $\mu : UFUF \to UF$ by $\mu = U\epsilon_F$, where $\epsilon : FU \to I_{\mathcal{A}}$ is the counit of the given adjunction. The definition of composition of natural transformations and that of natural transformations and functors give $\epsilon \circ \epsilon = \epsilon(FU\epsilon) = \epsilon(\epsilon_{FU})$, i.e., commutativity of

$$
\begin{tikzcd}
FUFU \arrow[r, "FU\epsilon"] \arrow[d, "\epsilon_{FU}"'] & FU \arrow[d, "\epsilon"] \\
FU \arrow[r, "\epsilon"'] & I_{\mathcal{A}}
\end{tikzcd}
$$

and the triangular identities of the adjunction give

$$
\begin{tikzcd}
U \arrow[r, "\eta_U"] \arrow[dr, "i_U"'] & UFU \arrow[d, "U\epsilon"] \\
& U
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, "i_F"'] & FUF \arrow[d, "\epsilon_F"] \\
& F
\end{tikzcd}
$$

These three diagrams show that $\langle T, \eta, \mu \rangle$ is an algebraic theory in monoid form in $\mathcal{K}$, since they immediately yield commutativity of the following three diagrams

$$
\begin{tikzcd}
UFUFUF \arrow[r, "UFU\epsilon_F"] \arrow[d, "U\epsilon_{FUF}"'] & UFUF \arrow[d, "U\epsilon_F"] \\
UFUF \arrow[r, "U\epsilon_F"'] & UF
\end{tikzcd}
$$

$$\begin{tikzcd} UF \arrow[r, "\eta_{UF}"] \arrow[dr, "i_{UF}"'] & UFUF \arrow[d, "U\epsilon_F"] & UF \arrow[l, "UF\eta"'] \arrow[dl, "i_{UF}"] \\ & UF & \end{tikzcd}$$

$\langle T, \eta, \mu\rangle = \langle UF, \eta, U\epsilon F\rangle$ will be called the **theory of the adjunction** $\langle F, U, \eta, \epsilon\rangle : \mathcal{K} \to \mathcal{A}$.

Suppose next that $\langle T, \eta, \mu\rangle$ is an algebraic theory in monoid form over $\mathcal{K}$. Consider the Eilenberg-Moore category $\mathcal{K}^{\mathbf{T}}$ of $\mathbf{T}$-algebras and define functors $U^{\mathbf{T}} : \mathcal{K}^{\mathbf{T}} \to \mathcal{K}$ and $F^{\mathbf{T}} : \mathcal{K} \to \mathcal{K}^{\mathbf{T}}$ as follows:

$U^{\mathbf{T}}(\langle X, \xi\rangle) = X$ and, if $h : \langle X, \xi\rangle \to \langle Y, \zeta\rangle \in \mathrm{Mor}(\mathcal{K}^{\mathbf{T}})$, then $U^{\mathbf{T}}(h) = h$. Further, $F^{\mathbf{T}}(X) = \langle T(X), \mu_X\rangle$ and, if $f : X \to Y \in \mathrm{Mor}(\mathcal{K})$, then $F^{\mathbf{T}}(f) = T(f)$. Finally, define natural transformations $\eta^{\mathbf{T}} : I_{\mathcal{K}} \to U^{\mathbf{T}}F^{\mathbf{T}}$ and $\epsilon^{\mathbf{T}} : F^{\mathbf{T}}U^{\mathbf{T}} \to I_{\mathcal{K}^{\mathbf{T}}}$ by $\eta^{\mathbf{T}} = \eta$ and $\epsilon^{\mathbf{T}}_{\langle X, \xi\rangle} = \xi$. It then turns out that $\langle F^{\mathbf{T}}, U^{\mathbf{T}}, \eta^{\mathbf{T}}, \epsilon^{\mathbf{T}}\rangle : \mathcal{K} \to \mathcal{K}^{\mathbf{T}}$ is an adjunction and, moreover that the theory of this adjunction is the original theory $\langle T, \eta, \mu\rangle$.

Let us now return for a while to the Kleisli category $\mathcal{K}_{\mathbf{T}}$ of a given theory $\langle T, \eta, \mu\rangle$ over a category $\mathcal{K}$. Recall that $\mathcal{K}_{\mathbf{T}}$ has as objects the objects of $\mathcal{K}$ and as morphisms $f : X \rightharpoonup Y$, $\mathcal{K}$-morphisms $f : X \to T(Y)$. Moreover composition is defined by

$$g \circ f = \mu_Z T(g) f, \text{ for all } f : X \rightharpoonup Y, g : Y \rightharpoonup Z \in \mathrm{Mor}(\mathcal{K}_{\mathbf{T}}).$$

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T(T(Z)) \xrightarrow{\mu_Z} T(Z)$$

Define functors $U_{\mathbf{T}} : \mathcal{K}_{\mathbf{T}} \to \mathcal{K}$ and $F_{\mathbf{T}} : \mathcal{K} \to \mathcal{K}_{\mathbf{T}}$ as follows:

$U_{\mathbf{T}}(X) = T(X)$, for every $X \in |\mathcal{K}_{\mathbf{T}}|$, and, if $f : X \rightharpoonup Y \in \mathrm{Mor}(\mathcal{K}_{\mathbf{T}})$, then $U_{\mathbf{T}}(f) = \mu_Y T(f) : T(X) \to T(Y)$. Further $F_{\mathbf{T}}(X) = X$, for every $X \in |\mathcal{K}|$, and, if $f : X \to Y \in \mathrm{Mor}(\mathcal{K})$, then $F_{\mathbf{T}}(f) = \eta_Y f : X \rightharpoonup Y \in \mathrm{Mor}(\mathcal{K}_{\mathbf{T}})$. Finally, define $\eta_{\mathbf{T}} : I_{\mathcal{K}} \to U_{\mathbf{T}}F_{\mathbf{T}}$ by $\eta_{\mathbf{T}} = \eta$ and $\epsilon_{\mathbf{T}} : F_{\mathbf{T}}U_{\mathbf{T}} \rightharpoonup I_{\mathcal{K}_{\mathbf{T}}}$ by $\epsilon_{\mathbf{T}\,X} = i_{T(X)}$. It then turns out that $\langle F_{\mathbf{T}}, U_{\mathbf{T}}, \eta_{\mathbf{T}}, \epsilon_{\mathbf{T}}\rangle :$

$\mathcal{K} \to \mathcal{K}_{\mathbf{T}}$ is an adjunction and that the theory of this adjunction is the given theory $\langle T, \eta, \mu \rangle$ as well.

The Comparison Functors

In this paragraph we consider the reverse problem. Instead of starting with a given algebraic theory and comparing it with the theory of the adjunction constructed from the original theory, we start with an adjunction $\langle F, U, \eta, \epsilon \rangle : \mathcal{K} \to \mathcal{A}$ and investigate its relationship with the adjunctions constructed as before by the theory of the given adjunction.

In this direction, the following theorem is proved in [39].

THEOREM 1.11 (COMPARING ADJUNCTIONS WITH ALGEBRAS) *Let $\langle F, U, \eta, \epsilon \rangle : \mathcal{K} \to \mathcal{A}$ be an adjunction, $\mathbf{T} = \langle UF, \eta, U\epsilon F \rangle$ the theory it defines in $\mathcal{K}$. Then there exists a unique functor $K' : \mathcal{A} \to \mathcal{K}^{\mathbf{T}}$ with $U^{\mathbf{T}} K' = U$ and $K'F = F^{\mathbf{T}}$.*

The functor K', whose existence and uniqueness is asserted in the theorem and which makes the F and U paths of the following diagrams commute

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "K'"] & & \mathcal{K}^{\mathbf{T}} \\
& \mathcal{K} \arrow[ul, "F"] \arrow[ur, "F^{\mathbf{T}}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A} \arrow[rr, "K'"] \arrow[dr, "U"'] & & \mathcal{K}^{\mathbf{T}} \arrow[dl, "U^{\mathbf{T}}"] \\
& \mathcal{K} &
\end{tikzcd}
$$

is defined as follows:

$K'(A) = \langle U(A), U\epsilon_A \rangle$, for every $A \in |\mathcal{A}|$, and, if $f : A \to B \in \mathrm{Mor}(\mathcal{A})$, then $K'(f) = U(f) : \langle U(A), U\epsilon_A \rangle \to \langle U(B), U\epsilon_B \rangle$.

A similar result holds with the Kleisli category $\mathcal{K}_{\mathbf{T}}$ of $\mathbf{T}$ in place of the category $\mathcal{K}^{\mathbf{T}}$ of the $\mathbf{T}$–algebras.

THEOREM 1.12 (COMPARING ADJUNCTIONS WITH FREE ALGEBRAS) *Let $\langle F, U, \eta, \epsilon \rangle : \mathcal{K} \to \mathcal{A}$ be an adjunction and $\mathbf{T} = \langle UF, \eta, U\epsilon F \rangle$ the theory it defines in $\mathcal{K}$. Then there exists a unique functor $L : \mathcal{K}_{\mathbf{T}} \to \mathcal{A}$, with $UL = U_{\mathbf{T}}$ and $LF_{\mathbf{T}} = F$, whose restriction gives an equivalence of categories $\mathcal{K}_{\mathbf{T}} \to F(\mathcal{K})$.*

$$\begin{tikzcd} \mathcal{K}_{\mathbf{T}} \arrow[rr, "L"] & & \mathcal{A} \\ & \mathcal{K} \arrow[ul, "F_{\mathbf{T}}"] \arrow[ur, "F"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}_{\mathbf{T}} \arrow[rr, "L"] \arrow[dr, "U_{\mathbf{T}}"'] & & \mathcal{A} \arrow[dl, "U"] \\ & \mathcal{K} & \end{tikzcd}$$

We now set out to present Beck's Theorem which provides a characterization of algebras, i.e., gives necessary and sufficient conditions for the comparison functor K : $\mathcal{A} \to \mathcal{K}^{\mathbf{T}}$ to be an isomorphism of categories.

First we need some new concepts.

Let $\mathcal{C}$ be a category. A **fork** in $\mathcal{C}$ is a diagram

$$A \overset{f_1}{\underset{f_2}{\longrightarrow}} B \xrightarrow{g} C$$

with $gf_1 = gf_2$. A **coequalizer** g of the parallel pair f_1, f_2 is then a fork as above such that, for any $g' : B \to D$, with $g'f_1 = g'f_2$, there exists unique $h : C \to D$ with $g' = hg$.

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & B \arrow[r, "g"] \arrow[dr, "g'"'] & C \arrow[d, dashed, "h"] \\ & & D \end{tikzcd}$$

An arrow g is called an **absolute coequalizer** of f_1, f_2 if, for every category $\mathcal{K}$ and for every functor $T : \mathcal{C} \to \mathcal{K}$ the fork

$$T(A) \overset{T(f_1)}{\underset{T(f_2)}{\longrightarrow}} T(B) \xrightarrow{T(g)} T(C)$$

has still $T(g)$ as a coequalizer.

A **split fork** in $\mathcal{C}$ is a fork with two additional arrows $h_1 : C \to B, h_2 : B \to A$ as follows

$$\begin{tikzcd} A \arrow[r, shift left=2, "f_1"] \arrow[r, "f_2"] & B \arrow[r, shift left, "g"] \arrow[l, bend left, "h_2"] & C \arrow[l, shift left, "h_1"] \end{tikzcd}$$

that satisfy $gf_1 = gf_2, gh_1 = i_C, f_1h_2 = i_B, f_2h_2 = h_1g$. h_1, h_2 are said to **split** the fork.

The above conditions imply that

- g is a split epi with right inverse h_1
- g is the coequalizer of f_1, f_2
- g is an absolute coequalizer of f_1, f_2.

Given $f_1, f_2 : A \to B$ in $\mathcal{C}$, $g : B \to C$ will be called a **split coequalizer** of f_1, f_2, if there exists a split fork with f_1, f_2, g as above.

Finally, a functor $U : \mathcal{A} \to \mathcal{K}$ is said to **create coequalizers** for a parallel pair $f_1, f_2 : A \to B$ in $\mathcal{A}$ when, to each coequalizer $u : U(B) \to X$ of $U(f_1), U(f_2)$ in $\mathcal{K}$ there is unique C and $h : B \to C$ with $U(C) = X$ and $U(h) = u$, and, moreover, h is the coequalizer of f_1, f_2.

THEOREM 1.13 (BECK'S CHARACTERIZATION OF ALGEBRAIC FUNCTORS) *Let* $\langle F, U, \eta, \epsilon\rangle : \mathcal{K} \to \mathcal{A}$ *be an adjunction,* $\langle T, \eta, \mu\rangle$ *the theory of this adjunction,* $\mathcal{K}^{\mathbf{T}}$ *the category of* $\mathbf{T}$*-algebras and* $\langle F^{\mathbf{T}}, U^{\mathbf{T}}, \eta^{\mathbf{T}}, \epsilon^{\mathbf{T}}\rangle : \mathcal{K} \to \mathcal{K}^{\mathbf{T}}$ *the adjunction of* $\mathbf{T}$. *Then the following are equivalent*

(i) The comparison functor $K : \mathcal{A} \to \mathcal{K}^{\mathbf{T}}$ *is an isomorphism*

(ii) $U : \mathcal{A} \to \mathcal{K}$ *creates coequalizers for those parallel pairs* f_1, f_2 *in* $\mathcal{A}$ *for which* $U(f_1), U(f_2)$ *has an absolute coequalizer in* $\mathcal{K}$

(iii) $U : \mathcal{A} \to \mathcal{K}$ *creates coequalizers for those parallel pairs* f_1, f_2 *in* $\mathcal{A}$ *for which* $U(f_1), U(f_2)$ *has a split coequalizer in* $\mathcal{K}$.

For a proof of this theorem the reader is also referred to [39] and [43].

2 EQUIVALENT INSTITUTIONS

The notion of a term π-institution is introduced. Then the notions of quasi-equivalence, strong quasi-equivalence and deductive equivalence are defined for π-institutions. Necessary and sufficient conditions are given for the quasi-equivalence and the deductive equivalence of two term π-institutions, based on the relationship between their categories of theories. The results carry over without any complications to institutions, via their associated π-institutions. An application is also given.

Introduction

In [6], Blok and Pigozzi presented the theory of algebraizable deductive systems. They called a deductive system *algebraizable* if there exists a quasivariety, over the same signature, and translations from the sentences of the system into equations and vice-versa that, roughly speaking, simulate the deduction over the system in the equational deduction over the quasivariety and vice-versa and are inverses of each other. In [8] they realized that this notion of algebraizability presents a specific example of the notion of *equivalence* of deductive systems. It is simply the equivalence of a deductive system with another very special system, namely the 2-deductive system that is associated with the chosen quasivariety.

In this chapter, inspired by the work of Blok and Pigozzi, and in an attempt to set a framework for the algebraization of institutions, the notions of quasi-equivalence and deductive equivalence for π-institutions are introduced.

Roughly speaking, a π-*institution* consists of an arbitrary category of *signatures* together with a functor SEN that gives, for each signature object Σ, a set of Σ-*sentences*. For each Σ, a mapping C_Σ, mapping sets of Σ-sentences to sets of Σ-sentences, called the Σ-*closure*, is defined, satisfying the usual Tarski closure axioms.

An *institution*, on the other hand, consists of an arbitrary category of *signatures* together with two functors SEN and MOD that give, respectively, for each signature object Σ, a set of Σ-*sentences* and a category of Σ-*models*. For each signature object Σ, sentences and models are related via a Σ-*satisfaction relation*. The main axiom formalizes the slogan that "truth is invariant under change of notation", see [26]. The Σ-satisfaction relation induces in the standard way a Σ-consequence relation on the set of Σ-sentences. The axiom above, then, may be interpreted as giving a structurality condition for these induced consequence relations. Thus, every institution gives rise in a natural way to a π-institution.

Following [21] and [26, 27], the *category of theories* of a π-institution and that of an institution are considered, i.e., the category with objects theories (closed sets of sentences) with respect to either the sentence closures, in the π-institution framework, or the induced consequence relations, in the institution framework. This category plays the role of the theory lattice of a deductive system in this broader context.

Inspired by [6, 8, 18], the notions of *quasi-equivalence* and *deductive equivalence* for two π-institutions are then defined. Generally speaking, two π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ are *quasi-equivalent* if the sentence closures of the first can be interpreted in the corresponding closures of the second and vice versa. This notion of quasi-equivalence generalizes the notion of equivalence for deductive systems introduced in [8]. Attention is subsequently restricted to a special, but yet wide, class of π-institutions, the, so-called, *term π-institutions*. Some examples of term π-institutions are provided. Using the theory categories of π-institutions, necessary and sufficient conditions for the quasi-equivalence and the deductive equivalence of two term π-institutions are given. Namely, it is proved that two term π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ are quasi-equivalent if and only if their categories of theories are adjoint categories via an adjunction satisfying some additional, relatively simple and quite natural, conditions. A similar characterization for deductive equivalence is also provided. More precisely, it is shown that two term π-institutions are deductively equivalent if and only if their categories of theories are naturally equivalent (in the usual category theoretical sense) via an equivalence satisfying some of the same conditions. These results carry over without any complications to the institution framework.

Finally, as an application of the theory, the special case of deductive institutions, that naturally correspond to deductive systems, is explored in some detail. As another application, two institutions based on an algebraic theory $\mathbf{T}$ in a category $\mathcal{K}$, that have very similar deductive apparatuses, will be constructed in the next chapter and it will be shown that they are quasi-equivalent but, in general, not deductively equivalent, institutions.

Institutions and π-Institutions

DEFINITION 2.1 (GOGUEN AND BURSTALL) *An* **institution**

$$\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$$

consists of

(i) A category **SIGN** *whose objects are called* **signatures**

(ii) A functor SEN : **SIGN** $\rightarrow$ **SET**, *from the category* **SIGN** *of signatures into the category* **SET** *of sets, called the* **sentence functor** *and giving, for each signature* Σ, *a set whose elements are called* **sentences over** *that signature* Σ *or* **Σ-sentences.**

(iii) A functor MOD : **SIGN** $\rightarrow$ $\mathbf{CAT}^{op}$ *from the category of signatures into the opposite of the category of categories, called the* **model functor** *and giving, for each signature* Σ, *a category whose objects are called* **Σ-models** *and whose morphisms are called* **Σ-morphisms.**

(iv) A relation $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times \mathrm{SEN}(\Sigma)$, *for each* $\Sigma \in |\mathbf{SIGN}|$, *called* Σ-**satisfaction,** *such that for every morphism* $f : \Sigma_1 \to \Sigma_2$ *in* **SIGN** *the* **satisfaction condition**

$$m_2 \models_{\Sigma_2} \mathrm{SEN}(f)(\phi_1) \quad \textit{if and only if} \quad \mathrm{MOD}(f)(m_2) \models_{\Sigma_1} \phi_1$$

holds, for every $m_2 \in |\mathrm{MOD}(\Sigma_2)|$ *and every* $\phi_1 \in \mathrm{SEN}(\Sigma_1)$.

The defining categories and functors of an institution together with their interconnections are illustrated by the following diagram:

$$\begin{tikzcd} & \mathbf{SET} \\ \mathbf{SIGN} \arrow[ur, "\mathrm{SEN}"] \arrow[dr, "\mathrm{MOD}"'] & \models \\ & \mathbf{CAT}^{op} \end{tikzcd}$$

Furthermore, the satisfaction condition can be given pictorially as follows:

If $f : \Sigma_1 \to \Sigma_2$ is a morphism in **SIGN**, then,

$$\begin{tikzcd} \mathrm{MOD}(\Sigma_1) & \models_{\Sigma_1} & \mathrm{SEN}(\Sigma_1) \arrow[d, "\mathrm{SEN}(f)"] \\ \mathrm{MOD}(\Sigma_2) \arrow[u, "\mathrm{MOD}(f)"] & \models_{\Sigma_2} & \mathrm{SEN}(\Sigma_2) \end{tikzcd}$$

Given an institution $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle, \Sigma \in |\mathbf{SIGN}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$ and $M \subseteq |\mathrm{MOD}(\Sigma)|$, we define

$$\Phi^* = \{m \in |\mathrm{MOD}(\Sigma)| : m \models_\Sigma \phi \quad \text{for every} \quad \phi \in \Phi\}$$

and

$$M^* = \{\phi \in \mathrm{SEN}(\Sigma) : m \models_\Sigma \phi \quad \text{for every} \quad m \in M\}.$$

Moreover we set $\Phi^c = \Phi^{**}$ and $M^c = M^{**}$.

From now on when the "c" symbol is used, its scope will be the largest possible well-formed expression to its left. For instance, in $\mathrm{SEN}(f)(\Phi)^c$ the scope of "c" is

$\mathrm{SEN}(f)(\Phi)$ and not just (Φ), and in $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))^c$ the scope of the second "c" is $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))$ and not just $\mathrm{SEN}(f)^{-1}(\Phi^c)$.

Goguen and Burstall [27], prove the following very useful lemma that is used below to obtain the π-institution associated with a given institution $\mathcal{I}$.

LEMMA 2.2 (CLOSURE LEMMA) *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ *be an institution,* $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{SIGN})$ *and* $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$. *Then*

$$\mathrm{SEN}(f)(\Phi^c) \subseteq \mathrm{SEN}(f)(\Phi)^c.$$

DEFINITION 2.3 (FIADEIRO AND SERNADAS) *A* π-**institution**

$$\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$$

consists of the following ((i) and (ii) are the same as those for institution)

(i) A category **SIGN** *whose objects are called* **signatures**

(ii) A functor SEN : **SIGN** $\to$ **SET**, *from the category* **SIGN** *of signatures into the category* **SET** *of sets, called the* **sentence functor** *and giving, for each signature* Σ, *a set whose elements are called* **sentences over** *that signature* Σ *or* Σ-**sentences.**

(iii) A mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, *for each* $\Sigma \in |\mathbf{SIGN}|$, *called* Σ-**closure,** *such that*

(a) $A \subseteq C_\Sigma(A)$, *for all* $\Sigma \in |\mathbf{SIGN}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, *for all* $\Sigma \in |\mathbf{SIGN}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, *for all* $\Sigma \in |\mathbf{SIGN}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, *for all* $\Sigma_1, \Sigma_2 \in |\mathbf{SIGN}|, f \in \mathbf{SIGN}(\Sigma_1, \Sigma_2), A \subseteq \mathrm{SEN}(\Sigma_1)$.

Given an institution $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$, define

$$\pi(\mathcal{I}) = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle,$$

by setting

$$C_\Sigma(\Phi) = \Phi^c, \text{ for all } \Sigma \in |\mathbf{SIGN}|, \Phi \subseteq \mathrm{SEN}(\Sigma).$$

It is easy to verify, using Lemma 2.2, that $\pi(\mathcal{I})$ is a π-institution. We will refer to $\pi(\mathcal{I})$ as to the **π-institution associated with** the institution $\mathcal{I}$.

From now on, given a π-institution $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$, a signature Σ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we will use the simplified notation Φ^c to denote $C_\Sigma(\Phi)$. Usually the signature Σ is clear from context and therefore this simplified notation does not cause any confusion.

COROLLARY 2.4 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a π-institution. Then*

$$\mathrm{SEN}(f)(\Phi^c)^c = \mathrm{SEN}(f)(\Phi)^c \textit{ for all } f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{SIGN}), \Phi \subseteq \mathrm{SEN}(\Sigma_1).$$

Proof:

Clearly $\mathrm{SEN}(f)(\Phi)^c \subseteq \mathrm{SEN}(f)(\Phi^c)^c$. For the reverse inclusion

$$\mathrm{SEN}(f)(\Phi^c)^c \subseteq (\mathrm{SEN}(f)(\Phi)^c)^c = \mathrm{SEN}(f)(\Phi)^c,$$

the inclusion being valid by (iii)(d) of Definition 2.3, as required. ■

Another lemma will also be of utmost importance for our subsequent considerations.

LEMMA 2.5 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a π-institution, $f : \Sigma_1 \to \Sigma_2$ a morphism in **SIGN** and $\Phi \subseteq \mathrm{SEN}(\Sigma_2)$. Then*

$$\mathrm{SEN}(f)^{-1}(\Phi^c)^c = \mathrm{SEN}(f)^{-1}(\Phi^c).$$

Proof:

Clearly, $\mathrm{SEN}(f)^{-1}(\Phi^c) \subseteq \mathrm{SEN}(f)^{-1}(\Phi^c)^c$. For the reverse inclusion, let

$$\phi \in \mathrm{SEN}(f)^{-1}(\Phi^c)^c.$$

Then $\mathrm{SEN}(f)(\phi) \in \mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c)^c)$, whence, by (iii)(d) of Definition 2.3,

$$\mathrm{SEN}(f)(\phi) \in \mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))^c,$$

and therefore $\mathrm{SEN}(f)(\phi) \in (\Phi^c)^c$, i.e., $\mathrm{SEN}(f)(\phi) \in \Phi^c$. Hence $\phi \in \mathrm{SEN}(f)^{-1}(\Phi^c)$, as required. ∎

COROLLARY 2.6 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a π-institution. $f : \Sigma_1 \to \Sigma_2$ an isomorphism in* **SIGN** *and $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$. Then*

$$\mathrm{SEN}(f)(\Phi^c)^c = \mathrm{SEN}(f)(\Phi^c).$$

The definition of a term π-institution is now given. Some examples follow in the next section.

DEFINITION 2.7 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a π-institution, $A \in |\mathbf{SIGN}|$ and $p \in \mathrm{SEN}(A)$. $\langle A, p\rangle$ is called a* **source signature-variable pair** *if there exists a function $f : \{\langle \Sigma, \phi\rangle : \Sigma \in |\mathbf{SIGN}|, \phi \in \mathrm{SEN}(\Sigma)\} \to |(A \,|\, \mathbf{SIGN})|$, such that, for all $\Sigma \in |\mathbf{SIGN}|$ and for all $\phi \in \mathrm{SEN}(\Sigma)$, $f_{\langle\Sigma,\phi\rangle} : A \to \Sigma$ and $\mathrm{SEN}(f_{\langle\Sigma,\phi\rangle})(p) = \phi$ and*

$$\forall \Sigma' \in |\mathbf{SIGN}|\ \forall g : \Sigma \to \Sigma' \quad (g f_{\langle\Sigma,\phi\rangle} = f_{\langle\Sigma',\mathrm{SEN}(g)(\phi)\rangle}).$$

A π-institution is called **term** *if it has a source signature-variable pair. An institution $\mathcal{I}$ is called* **term** *if its associated π-institution $\pi(\mathcal{I})$ is a term π-institution.*

A **SIGN**-*object such as A will be called a* **source signature** *and a sentence such as p will be called a* **source variable** *or, simply, a* **variable.**

The following diagrams illustrate the definition:

$$\begin{tikzcd} A \arrow[rr, "f_{\langle\Sigma,\phi\rangle}"] \arrow[dr, "f_{\langle\Sigma',\mathrm{SEN}(g)(\phi)\rangle}"'] & & \Sigma \arrow[dl, "g"] \\ & \Sigma' & \end{tikzcd} \qquad \begin{tikzcd} p \arrow[rr, "\mathrm{SEN}(f_{\langle\Sigma,\phi\rangle})"] \arrow[dr, "\mathrm{SEN}(f_{\langle\Sigma',\mathrm{SEN}(g)(\phi)\rangle})"'] & & \phi \arrow[dl, "\mathrm{SEN}(g)"] \\ & SEN(g)(\phi) & \end{tikzcd}$$

Examples

Two examples of term institutions are provided. The first is borrowed by the quantifier-free first-order theory of n-ary relations and the second by the theory of finite state automata.

n-ary Relations

The reader is referred to [27] for a more general construction of a multi-sorted institution for first-order logic. Let **SET** denote as usual the category of all small sets. Given $X \in |\mathbf{SET}|$, let $\overline{X}$ denote a disjoint copy of X constructed in some canonical way. $\overline{X}$ could be, e.g., the set $X \times \{\emptyset\}$.

Given $X \in |\mathbf{SET}|$, define $R(X)$, the **propositional language of n-ary relations in** X, or, more simply, X-**relational formulas,** to be the smallest set, such that

- $\overline{X} \subseteq R(X)$
- $\neg r \in R(X)$, for every $r \in R(X)$, and
- $r_0 \wedge r_1 \in R(X)$, for all $r_0, r_1 \in R(X)$.

Given $f : X \rightarrow R(Y)$, define $f^{\bullet} : R(X) \rightarrow R(Y)$ by recursion on the structure of X-relational formulas as follows:

- $f^{\bullet}(\overline{x}) = f(x)$, for every $x \in X$,
- $f^{\bullet}(\neg r) = \neg f^{\bullet}(r)$, for every $r \in R(X)$, and
- $f^{\bullet}(r_0 \wedge r_1) = f^{\bullet}(r_0) \wedge f^{\bullet}(r_1)$, for all $r_0, r_1 \in R(X)$.

A map $f : X \rightarrow R(Y)$ will be denoted by $f : X \rightharpoondown Y$. Define **SIGN** to be the category with collection of objects $|\mathbf{SET}|$ and morphisms f from X to Y all set maps $f : X \rightharpoondown Y$, i.e.,

$$\mathbf{SIGN}(X, Y) = \{f : X \rightarrow R(Y) \in \mathrm{Mor}(\mathbf{SET})\}, \text{ for all } X, Y \in |\mathbf{SET}|.$$

Composition $\circ$ of $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ in **SIGN** is defined by

$$g \circ f = g^{\bullet} f.$$

Given $X \in |\mathbf{SIGN}|$, let $\mathbf{RM}_X$ be the category whose objects are all first-order relational structures $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$, where each symbol $x \in X$ is interpreted as an n-ary relation symbol $x^{\mathbf{A}}$, and whose morphisms are all first-order structure homomorphisms $h : \mathbf{A} \to \mathbf{B} \in \mathrm{Mor}(\mathbf{RM}_X)$ such that

$$\langle a_0, \ldots, a_{n-1} \rangle \in x^{\mathbf{A}} \quad \text{if and only if} \quad \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in x^{\mathbf{B}}, \tag{2.1}$$

for all $a_0, \ldots, a_{n-1} \in A$.

The following lemma holds

LEMMA 2.8 *Let* $X \in |\mathbf{SET}|, \langle A, X^{\mathbf{A}} \rangle, \langle B, X^{\mathbf{B}} \rangle \in |\mathbf{RM}_X|$ *and* $h : \langle A, X^{\mathbf{A}} \rangle \to \langle B, X^{\mathbf{B}} \rangle \in \mathrm{Mor}(\mathbf{RM}_X)$. *Then, for all* $r \in R(X), a_0, \ldots, a_{n-1} \in A$,

$$\langle a_0, \ldots, a_{n-1} \rangle \in r^{\mathbf{A}} \quad \textit{iff} \quad \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in r^{\mathbf{B}}.$$

Proof:

By induction on the structure of $r \in R(X)$.

If $r = \overline{x}$, for some $x \in X$, then

$$\begin{aligned}
\langle a_0, \ldots, a_{n-1} \rangle \in \overline{x}^{\mathbf{A}} \quad &\text{iff} \quad \langle a_0, \ldots, a_{n-1} \rangle \in x^{\mathbf{A}} \quad (\text{by defin. of } \overline{x}^{\mathbf{A}}) \\
&\text{iff} \quad \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in x^{\mathbf{B}} \quad (\text{since } h \in \mathrm{Mor}(\mathbf{RM}_X)) \\
&\text{iff} \quad \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in \overline{x}^{\mathbf{B}} \quad (\text{by defin. of } \overline{x}^{\mathbf{B}})
\end{aligned}$$

If $r \in R(X)$, such that $\langle a_0, \ldots, a_{n-1} \rangle \in r^{\mathbf{A}}$ iff $\langle h(a_0), \ldots, h(a_{n-1}) \rangle \in r^{\mathbf{B}}$, then

$$\begin{aligned}
\langle a_0, \ldots, a_{n-1} \rangle \in (\neg r)^{\mathbf{A}} \quad &\text{iff} \quad \langle a_0, \ldots, a_{n-1} \rangle \notin r^{\mathbf{A}} \quad (\text{by defin. of } (\neg r)^{\mathbf{A}}) \\
&\text{iff} \quad \langle h(a_0), \ldots, h(a_{n-1}) \rangle \notin r^{\mathbf{B}} \quad (\text{by induction hypothesis}) \\
&\text{iff} \quad \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in (\neg r)^{\mathbf{B}} \quad (\text{by defin. of } (\neg r)^{\mathbf{B}}).
\end{aligned}$$

The remaining case can be handled similarly. ■

Now, let $X \in |\mathbf{SIGN}|, r \in R(X)$ and $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle \in |\mathbf{RM}_X|$. Define, as usual the **interpretation** $r^{\mathbf{A}}$ of r in $\mathbf{A}$ by recursion on the structure of the X-relational formula r as follows

- $\overline{x}^{\mathbf{A}}$ is the interpretation of x in $\mathbf{A}$, for every $x \in X$,
- $(\neg r)^{\mathbf{A}} = A^n - r^{\mathbf{A}}$, for every $r \in R(X)$, and
- $(r_0 \wedge r_1)^{\mathbf{A}} = r_0^{\mathbf{A}} \cap r_1^{\mathbf{A}}$, for all $r_0, r_1 \in R(X)$.

Finally, given $f : X \rightarrow Y \in \mathrm{Mor}(\mathbf{SIGN})$ and $\mathbf{A} = \langle A, Y^{\mathbf{A}} \rangle \in |\mathbf{RM}_Y|$, define $f^{\#}(\mathbf{A}) = \langle A, X^{f^{\#}(\mathbf{A})} \rangle \in |\mathbf{RM}_X|$, by setting

$$\langle a_0, \ldots, a_{n-1} \rangle \in x^{f^{\#}(\mathbf{A})} \quad \text{iff} \quad \langle a_0, \ldots, a_{n-1} \rangle \in f(x)^{\mathbf{A}}, \tag{2.2}$$

for all $x \in X, a_0, \ldots, a_{n-1} \in A$.

DEFINITION 2.9 *Define $\mathcal{REL} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$, as follows:*

(i) SEN : **SIGN** $\rightarrow$ **SET** *is defined by*

$$\mathrm{SEN}(X) = R(X), \quad \textit{for every} \quad X \in |\mathbf{SIGN}|,$$

and, given $f : X \rightarrow Y \in \mathrm{Mor}(\mathbf{SIGN}), \mathrm{SEN}(f) : R(X) \rightarrow R(Y)$ *is given by* $\mathrm{SEN}(f) = f^{*}$.

(ii) MOD : **SIGN** $\rightarrow$ $\mathbf{CAT}^{op}$ *is defined as follows: For every* $X \in |\mathbf{SIGN}|, \mathrm{MOD}(X)$ *is the category with objects all pairs of the form* $\langle \mathbf{A}, \vec{a} \rangle$, *where* $\mathbf{A} \in |\mathbf{RM}_X|$ *and* $\vec{a} \in A^{\omega}$ *and morphisms* $h : \langle \mathbf{A}, \vec{a} \rangle \rightarrow \langle \mathbf{B}, \vec{b} \rangle, \mathbf{RM}_X$*-morphisms* $h : \mathbf{A} \rightarrow \mathbf{B}$, *such that* $\vec{b} = h(\vec{a})$, *i.e.,* $b_i = h(a_i)$, *for every* $i \in \omega$.

Given $k : X \rightarrow Y \in \mathrm{Mor}(\mathbf{SIGN})$ *the functor* $\mathrm{MOD}(k) : \mathrm{MOD}(Y) \rightarrow \mathrm{MOD}(X)$ *sends* $\langle \mathbf{A}, \vec{a} \rangle$ *to* $\langle f^{\#}(\mathbf{A}), \vec{a} \rangle$ *and a morphism* $h : \langle \mathbf{A}, \vec{a} \rangle \rightarrow \langle \mathbf{B}, \vec{b} \rangle$ *to the morphism* $\mathrm{MOD}(k)(h) : \langle f^{\#}(\mathbf{A}), \vec{a} \rangle \rightarrow \langle f^{\#}(\mathbf{B}), \vec{b} \rangle$ *with* $\mathrm{MOD}(k)(h) = h$.

(iii) For all $X \in |\mathbf{SIGN}|, r \in R(X)$ *and* $\langle \mathbf{A}, \vec{a} \rangle \in |\mathrm{MOD}(X)|$,

$$\langle \mathbf{A}, \vec{a} \rangle \models_X r \quad \textit{iff} \quad \langle a_0, \ldots, a_{n-1} \rangle \in r^{\mathbf{A}}.$$

Next, it is shown that the previous construction gives an institution. A lemma is needed first.

LEMMA 2.10 *Let* $X, Y \in |\mathbf{SET}|, f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{SIGN})$ *and* $\mathbf{A} = \langle A, Y^{\mathbf{A}} \rangle \in |\mathbf{RM}_Y|$. *Then, for all* $r \in R(X), a_0, \ldots, a_{n-1} \in A$,

$$\langle a_0, \ldots, a_{n-1} \rangle \in r^{f^{\#}(\mathbf{A})} \quad \textit{iff} \quad \langle a_0, \ldots, a_{n-1} \rangle \in f^{\bullet}(r)^{\mathbf{A}}.$$

Proof:

By induction on the structure of $r \in R(X)$.

If $r = \overline{x}$, for some $x \in X$, then

$$\begin{aligned} \langle a_0, \ldots, a_{n-1} \rangle \in \overline{x}^{f^{\#}(\mathbf{A})} & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \in x^{f^{\#}(\mathbf{A})} \quad (\text{by defin. of } \overline{x}^{f^{\#}(\mathbf{A})}) \\ & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \in f(x)^{\mathbf{A}} \quad (\text{by } (2.2)) \\ & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \in f^{\bullet}(\overline{x})^{\mathbf{A}} \quad (\text{by defin. of } f^{\bullet}) \end{aligned}$$

If $r \in R(X)$, such that $\langle a_0, \ldots, a_{n-1} \rangle \in r^{f^{\#}(\mathbf{A})}$ iff $\langle a_0, \ldots, a_{n-1} \rangle \in f^{\bullet}(r)^{\mathbf{A}}$, then

$$\begin{aligned} \langle a_0, \ldots, a_{n-1} \rangle \in (\neg r)^{f^{\#}(\mathbf{A})} & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \notin r^{f^{\#}(\mathbf{A})} \quad (\text{by defin. of } (\neg r)^{f^{\#}(\mathbf{A})}) \\ & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \notin f^{\bullet}(r)^{\mathbf{A}} \quad (\text{by the ind. hypothesis}) \\ & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \in (\neg f^{\bullet}(r))^{\mathbf{A}} \quad (\text{by defin. of } (\neg f^{\bullet}(r))^{\mathbf{A}}) \\ & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \in f^{\bullet}(\neg r)^{\mathbf{A}} \quad (\text{by defin. of } f^{\bullet}). \end{aligned}$$

The remaining case can be treated similarly. ■

THEOREM 2.11 $\mathcal{REL} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ *is an institution.*

Proof:

We only show that MOD is well-defined on morphisms and then verify that the satisfaction condition holds.

First, let $k : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{SIGN})$ and $h : \langle \mathbf{A}, \vec{a} \rangle \to \langle \mathbf{B}, \vec{b} \rangle \in \mathrm{Mor}(\mathrm{MOD}(Y))$. Then $h : \langle k^{\#}(\mathbf{A}), \vec{a} \rangle \to \langle k^{\#}(\mathbf{B}), \vec{b} \rangle \in \mathrm{Mor}(\mathrm{MOD}(X))$, since, for all $a_0, \ldots, a_{n-1} \in A$,

$$\begin{aligned} \langle a_0, \ldots, a_{n-1} \rangle \in x^{k^{\#}(\mathbf{A})} & \text{ iff } \langle a_0, \ldots, a_{n-1} \rangle \in k(x)^{\mathbf{A}} \quad (\text{by } (2.2)) \\ & \text{ iff } \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in k(x)^{\mathbf{B}} \quad (\text{by Lemma 2.8}) \\ & \text{ iff } \langle h(a_0), \ldots, h(a_{n-1}) \rangle \in x^{k^{\#}(\mathbf{B})}. \quad (\text{by } (2.2)) \end{aligned}$$

Finally, let $k : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{SIGN}), r \in R(X)$ and $\langle\langle A, Y^{\mathbf{A}}\rangle, \vec{a}\rangle \in |\mathrm{MOD}(Y)|$. Then

$$\begin{aligned}
\mathrm{MOD}(k)(\langle\langle A, Y^{\mathbf{A}}\rangle, \vec{a}\rangle) \models_X r \quad & \text{iff} \quad \langle\langle A, k^{\#}(Y^{\mathbf{A}})\rangle, \vec{a}\rangle \models_X r \\
& \text{iff} \quad \langle a_0, \dots, a_{n-1}\rangle \in r^{k^{\#}(\mathbf{A})} \\
& \text{iff} \quad \langle a_0, \dots, a_{n-1}\rangle \in k^{\text{"}}(r)^{\mathbf{A}} \quad \text{(by Lemma 2.10)} \\
& \text{iff} \quad \langle\langle A, Y^{\mathbf{A}}\rangle, \vec{a}\rangle \models_Y k^{\text{"}}(r),
\end{aligned}$$

as required. ■

Finally, it is shown that $\mathcal{REL} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ is a term institution.

THEOREM 2.12 *$\mathcal{REL} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ is a term institution.*

Proof:

Let $A = \{a\} \in |\mathbf{SIGN}|$ be a one-element set and $p = \bar{a} \in \mathrm{SEN}(A) = R(A)$. Define $f : \{\langle X, r\rangle : X \in |\mathbf{SIGN}|, r \in R(X)\} \to |(A \downarrow \mathbf{SIGN})|$ by

$$f_{\langle X,r\rangle} : A \rightharpoondown X, \quad \text{with} \quad f_{\langle X,r\rangle}(a) = r.$$

A straightforward computation verifies that, for every $g : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{SIGN})$, $g \circ f_{\langle X,r\rangle} = f_{\langle Y,\mathrm{SEN}(g)(r)\rangle}$, as required. ■

In [27], Goguen and Burstall construct an institution for first-order logic with terms. Although $\mathcal{REL}$ represents the quantifier-free fragment of first-order logic without terms having only relational symbols of a single arity, it is not a special case of the construction in [27]. The main reason is that in the present development relational symbols of one signature may be mapped to complex relational formulas of another signature whereas in [27] the morphisms in the signature category map relational symbols of one signature only to relational symbols of another signature rather than to more complex formulas. The present treatment, although more general in this respect, has the drawback that it can only handle first-order structure homomorphisms satisfying (2.1) and not all first-order structure homomorphisms. This is because Lemma 2.8 fails for an arbitrary first-order structure homomorphism $h : \mathbf{A} \to \mathbf{B}$.

Automata

Given $X \in |\mathbf{SET}|$, define $W(X)$, the set of all **words in** X, or, more simply, X-**words,** to be the smallest set, such that

- $\overline{X} \cup \{\lambda\} \subseteq W(X)$ and
- $w_1 w_2 \in W(X)$, for all $w_1, w_2 \in W(X)$.

Given $f : X \to W(Y)$, define $f^{\bullet} : W(X) \to W(Y)$ by recursion on the structure of X-words as follows:

- $f^{\bullet}(\lambda) = \lambda$,
- $f^{\bullet}(\overline{x}) = f(x)$, for every $x \in X$, and
- $f^{\bullet}(w_1 w_2) = f^{\bullet}(w_1) f^{\bullet}(w_2)$, for all $w_1, w_2 \in W(X)$.

A map $f : X \to W(Y)$ will be denoted by $f : X \rightharpoondown Y$. Define **SIGN** to be the category with collection of objects $|\mathbf{SET}|$ and morphisms f from X to Y all set maps $f : X \rightharpoondown Y$, i.e.,

$$\mathbf{SIGN}(X, Y) = \{f : X \to W(Y) \in \mathrm{Mor}(\mathbf{SET})\}, \text{ for all } X, Y \in |\mathbf{SET}|.$$

Composition $\circ$ of $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ in **SIGN** is defined by

$$g \circ f = g^{\bullet} f.$$

Given $X \in |\mathbf{SIGN}|$, let **AUT** be the category with objects all finite state automata (see [35, 44]) $M = \langle Q, \Sigma, q_0, \delta, A\rangle$, and morphisms $h : \langle Q, \Sigma, q_0, \delta, A\rangle \to \langle P, T, p_0, \epsilon, B\rangle$ pairs $h = \langle h_S, h_I\rangle$ of **SET**-functions $h_S : Q \to P$ and $h_I : \Sigma \to T$, such that

- $h_S(q_0) = p_0$
- $h_S(\delta(q, \sigma)) = \epsilon(h_S(q), h_I(\sigma))$, for all $q \in Q, \sigma \in \Sigma$, and

- $h_S(A) = B$.

Definition 2.13 *Define* $\mathcal{AUT} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$, *as follows:*

(i) $\mathrm{SEN} : \mathbf{SIGN} \to \mathbf{SET}$ *is defined by*

$$\mathrm{SEN}(X) = W(X), \quad \textit{for every} \quad X \in |\mathbf{SIGN}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{SIGN}), \mathrm{SEN}(f) : W(X) \to W(Y)$ *is given by* $\mathrm{SEN}(f) = f^*$.

(ii) $\mathrm{MOD} : \mathbf{SIGN} \to \mathbf{CAT}^{op}$ *is defined as follows: For every* $X \in |\mathbf{SIGN}|$, $\mathrm{MOD}(X)$ *is the category with objects all pairs of the form* $\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle$, *where* $\langle Q, \Sigma, q_0, \delta, A\rangle \in |\mathbf{AUT}|$ *and* $f : X \to \Sigma \in \mathrm{Mor}(\mathbf{SIGN})$ *and morphisms* $h : \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \to \langle\langle P, T, p_0, \epsilon, B\rangle, g\rangle$ **AUT**-*morphisms* $h : \langle Q, \Sigma, q_0, \delta, A\rangle \to \langle P, T, p_0, \epsilon, B\rangle$, *such that* $g = h_I^* f$.

Given $k : X \to Y \in \mathrm{Mor}(\mathbf{SIGN})$ *the functor* $\mathrm{MOD}(k) : \mathrm{MOD}(Y) \to \mathrm{MOD}(X)$ *sends* $\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle$ *to* $\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f \circ k\rangle$ *and a morphism* $h : \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \to \langle\langle P, T, p_0, \epsilon, B\rangle, g\rangle$ *to the morphism* $\mathrm{MOD}(k)(h) : \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f \circ k\rangle \to \langle\langle P, T, p_0, \epsilon, B\rangle, g \circ k\rangle$ *with* $\mathrm{MOD}(k)(h) = h$.

(iii) For all $X \in |\mathbf{SIGN}|, w \in W(X)$ *and* $\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \in |\mathrm{MOD}(X)|$,

$$\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \models_X w \quad \textit{iff} \quad \delta^*(q_0, f^*(w)) \in A.$$

Next, it is shown that the previous construction gives an institution.

Theorem 2.14 $\mathcal{AUT} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ *is an institution.*

Proof:

We only show that MOD is well-defined on morphisms and then verify that the satisfaction condition holds.

First, let $k : X \to Y \in \mathrm{Mor}(\mathbf{SIGN})$ and $h : \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \to \langle\langle P, T, p_0, \epsilon, B\rangle, g\rangle \in \mathrm{Mor}(\mathrm{MOD}(Y))$. Then

$$h : \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f \circ k\rangle \to \langle\langle P, T, p_0, \epsilon, B\rangle, g \circ k\rangle \in \mathrm{Mor}(\mathrm{MOD}(X)),$$

since

$$\begin{aligned} g^{*}k &= (h^{*}_{S}f)^{*}k &= (h^{*}_{S}f^{*})k \\ &= h^{*}_{S}(f^{*}k) &= h^{*}_{S}(f \circ k), \end{aligned}$$

as required.

Finally, let $k : X \rightharpoonup Y \in \text{Mor}(\mathbf{SIGN}), w \in W(X)$ and $\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \in |\text{MOD}(Y)|$. Then

$$\begin{aligned} \text{MOD}(k)(\langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle) \models_X w \quad &\text{iff} \quad \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f \circ k\rangle \models_X w \\ &\text{iff} \quad \delta^{*}(q_0, (f \circ k)^{*}(w)) \in A \\ &\text{iff} \quad \delta^{*}(q_0, f^{*}(k^{*}(w))) \in A \\ &\text{iff} \quad \langle\langle Q, \Sigma, q_0, \delta, A\rangle, f\rangle \models_Y k^{*}(w), \end{aligned}$$

as required. ■

Finally, it is shown that $\mathcal{AUT} = \langle \mathbf{SIGN}, \text{SEN}, \text{MOD}, \models\rangle$ is a term institution.

THEOREM 2.15 $\mathcal{AUT} = \langle \mathbf{SIGN}, \text{SEN}, \text{MOD}, \models\rangle$ *is a term institution.*

Proof:

Let $A = \{a\} \in |\mathbf{SIGN}|$ be a one-element set and $p = \overline{a} \in \text{SEN}(A) = W(A)$. Define $f : \{\langle X, w\rangle : X \in |\mathbf{SIGN}|, w \in W(X)\} \to |(A \,|\, \mathbf{SIGN})|$ by

$$f_{\langle X,w\rangle} : A \rightharpoonup X, \quad \text{with} \quad f_{\langle X,w\rangle}(a) = w.$$

A straightforward computation verifies that, for every $g : X \rightharpoonup Y \in \text{Mor}(\mathbf{SIGN})$, $g \circ f_{\langle X,w\rangle} = f_{\langle Y,\text{SEN}(g)(w)\rangle}$, as required. ■

The Category of Theories

Let $\mathcal{I} = \langle \mathbf{SIGN}, \text{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|}\rangle$ be a π-institution. Following [21] we define its **category of theories** $\mathbf{TH}(\mathcal{I})$, as follows:

The objects of $\mathbf{TH}(\mathcal{I})$ are pairs $\langle \Sigma, T\rangle$, where $\Sigma \in |\mathbf{SIGN}|$ and $T \subseteq \text{SEN}(\Sigma)$ with $T^c = T$. The morphisms $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle$ are $\mathbf{SIGN}$−morphisms $f : \Sigma_1 \to \Sigma_2$, such that $\text{SEN}(f)(T_1) \subseteq T_2$.

Given an institution $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ define $\mathbf{TH}(\mathcal{I}) = \mathbf{TH}(\pi(\mathcal{I}))$, i.e., its **category of theories** is the category of theories of its associated π-institution. It is straightforward to verify that this notion coincides with the notion defined directly in [27].

Now, coming back to the π-institution framework, define a functor $\mathrm{SIG} : \mathbf{TH}(\mathcal{I}) \to \mathbf{SIGN}$ by

$$\mathrm{SIG}(\langle \Sigma, T\rangle) = \Sigma, \quad \text{for every} \quad \langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|,$$

and

$$\mathrm{SIG}(f) = f, \quad \text{for every} \quad f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I})).$$

Then the following holds.

LEMMA 2.16 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ *be a* π*-institution and* $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$ *an isomorphism. Then* $\mathrm{SEN}(\mathrm{SIG}(f))(T_1) = T_2$.

Proof:

Since $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, $\mathrm{SEN}(\mathrm{SIG}(f))(T_1) \subseteq T_2$. Since $f^{-1} : \langle \Sigma_2, T_2\rangle \to \langle \Sigma_1, T_1\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, we also have

$$\mathrm{SEN}(\mathrm{SIG}(f))^{-1}(T_2) = \mathrm{SEN}(\mathrm{SIG}(f^{-1}))(T_2) \subseteq T_1.$$

Thus, $T_2 \subseteq \mathrm{SEN}(\mathrm{SIG}(f))(T_1)$, whence $\mathrm{SEN}(\mathrm{SIG}(f))(T_1) = T_2$, as was to be shown. ■

Next, define a functor $\mathrm{THY} : \mathbf{SIGN} \to \mathbf{TH}(\mathcal{I})$ by

$$\mathrm{THY}(\Sigma) = \langle \Sigma, \emptyset^c\rangle, \quad \text{for every} \quad \Sigma \in |\mathbf{SIGN}|,$$

and $\mathrm{THY}(f) : \langle \Sigma_1, \emptyset^c\rangle \to \langle \Sigma_2, \emptyset^c\rangle$, with

$$\mathrm{SIG}(\mathrm{THY}(f)) = f, \quad \text{for every} \quad f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{SIGN}),$$

which is well-defined, since, by (iii)(d) of Definition 2.3, $\mathrm{SEN}(f)(\emptyset^c)^c \subseteq \mathrm{SEN}(f)(\emptyset)^c = \emptyset^c$.

Finally, define natural transformations $\eta : I_{\mathbf{SIGN}} \to \mathrm{SIG} \circ \mathrm{THY}$ by

$$\eta_\Sigma : \Sigma \to \mathrm{SIG}(\mathrm{THY}(\Sigma)) \in \mathrm{Mor}(\mathbf{SIGN}),$$

with

$$\eta_\Sigma = i_\Sigma, \quad \text{for every} \quad \Sigma \in |\mathbf{SIGN}|,$$

and $\epsilon : \mathrm{THY} \circ \mathrm{SIG} \to I_{\mathbf{TH}(\mathcal{I})}$ by $\epsilon_{\langle \Sigma, T\rangle} : \langle \Sigma, \emptyset^c\rangle \to \langle \Sigma, T\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I})))$, with

$$\mathrm{SIG}(\epsilon_{\langle \Sigma, T\rangle}) = i_\Sigma, \quad \text{for every} \quad \langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|.$$

Then, the following theorem ([21], Proposition 3.32) holds.

THEOREM 2.17 $\langle \mathrm{THY}, \mathrm{SIG}, \eta, \epsilon\rangle : \mathbf{SIGN} \to \mathbf{TH}(\mathcal{I})$ *is an adjunction.*

Proof:

By the preceding discussion η and ϵ are natural transformations. Thus, it suffices to show that the following triangles commute:

$$\begin{tikzcd}
\mathrm{SIG}(\langle \Sigma, T\rangle) \arrow[r, "\eta_{\mathrm{SIG}(\langle \Sigma, T\rangle)}"] \arrow[dr, "i_{\mathrm{SIG}(\langle \Sigma, T\rangle)}"'] & \mathrm{SIG}(\mathrm{THY}(\mathrm{SIG}(\langle \Sigma, T\rangle))) \arrow[d, "\mathrm{SIG}(\epsilon_{\langle \Sigma, T\rangle})"] \\
& \mathrm{SIG}(\langle \Sigma, T\rangle)
\end{tikzcd}$$

$$\mathrm{SIG}(\epsilon_{\langle \Sigma, T\rangle}) \circ \eta_{\mathrm{SIG}(\langle \Sigma, T\rangle)} = i_\Sigma i_\Sigma = i_\Sigma = i_{\mathrm{SIG}(\langle \Sigma, T\rangle)},$$

as required, and

$$\begin{tikzcd}
\mathrm{THY}(\Sigma) \arrow[r, "\mathrm{THY}(\eta_\Sigma)"] \arrow[dr, "i_{\mathrm{THY}(\Sigma)}"'] & \mathrm{THY}(\mathrm{SIG}(\mathrm{THY}(\Sigma))) \arrow[d, "\epsilon_{\mathrm{THY}(\Sigma)}"] \\
& \mathrm{THY}(\Sigma)
\end{tikzcd}$$

$$\epsilon_{\mathrm{THY}(\Sigma)} \circ \mathrm{THY}(\eta_\Sigma) = \epsilon_{\mathrm{THY}(\Sigma)} \circ \mathrm{THY}(i_\Sigma) = \epsilon_{\mathrm{THY}(\Sigma)} \circ i_{\mathrm{THY}(\Sigma)} =$$

$$= \epsilon_{\mathrm{THY}(\Sigma)} = i_{\langle\Sigma,\emptyset^c\rangle} = i_{\mathrm{THY}(\Sigma)},$$

as required. ∎

In the sequel we will denote by $\mathbf{TH}_\emptyset(\mathcal{I})$ the full subcategory of $\mathbf{TH}(\mathcal{I})$ with objects all theories of the form $\langle\Sigma, \emptyset^c\rangle, \Sigma \in |\mathbf{SIGN}|$.

Then the proof of Theorem 2.17 gives

THEOREM 2.18 $\mathrm{THY}_\emptyset : \mathbf{SIGN} \to \mathbf{TH}_\emptyset(\mathcal{I})$ *acting as* $\mathrm{THY} : \mathbf{SIGN} \to \mathbf{TH}(\mathcal{I})$ *is an isomorphism of categories with inverse* $\mathrm{SIG}_\emptyset : \mathbf{TH}_\emptyset(\mathcal{I}) \to \mathbf{SIGN}$ *given by* $\mathrm{SIG}_\emptyset = \mathrm{SIG}|_{\mathbf{TH}_\emptyset(\mathcal{I})}$.

Relating Categories of Theories

Let $\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$ be two π-institutions. Properties of functors relating the categories of theories $\mathbf{TH}(\mathcal{I}_1)$ and $\mathbf{TH}(\mathcal{I}_2)$ will now be introduced, that will be used in the sequel to give the main characterization theorems of the relations of quasi-equivalence and deductive equivalence between the π-institutions themselves.

Denote by $\pi_2 : |\mathbf{TH}(\mathcal{I}_1)| \to |\mathbf{SET}|$ the second projection, defined by $\pi_2(\langle\Sigma_1, T_1\rangle) = T_1$, for every $\langle\Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, and, similarly, $\pi_2 : |\mathbf{TH}(\mathcal{I}_2)| \to |\mathbf{SET}|$, given by $\pi_2(\langle\Sigma_2, T_2\rangle) = T_2$, for every $\langle\Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$.

DEFINITION 2.19 *A functor* $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *will be called*

(i) **signature-respecting** *if there exists a functor* $F' : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, *such that the following rectangle commutes*

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{SIGN}_1 \arrow[r, "F'"'] & \mathbf{SIGN}_2 \end{tikzcd}$$

If this is the case, it is easy to verify that F' *is necessarily unique.*

(ii) **(strongly) monotonic** *if, for all* $\langle\Sigma_1, T_1\rangle, \langle\Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$T_1 \subseteq T_1' \quad \textit{(if and) only if} \quad \pi_2(F(\langle\Sigma_1, T_1\rangle)) \subseteq \pi_2(F(\langle\Sigma_1, T_1'\rangle)),$$

(iii) **join-continuous** *if, for all* $\Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$(\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c = \pi_2(F(\langle\Sigma_1, \Phi^c\rangle)).$$

Finally, a signature-respecting functor $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *will be said to* **commute with substitutions** *if, for every* $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{SIGN}_1)$,

$$\mathrm{SEN}_2(F'(f))(\pi_2(F(\langle\Sigma_1, T_1\rangle)))^c = \pi_2(F(\langle\Sigma_1', \mathrm{SEN}_1(f)(T_1)^c\rangle)),$$

for every $\langle\Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, *where* $F' : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ *is the (necessarily unique) functor of (i).*

The properties above may be extended to the case where the two categories of theories $\mathbf{TH}(\mathcal{I}_1)$ and $\mathbf{TH}(\mathcal{I}_2)$ are related via an adjunction. The following definition then applies

DEFINITION 2.20 *An adjunction* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *will be called*

(i) **signature-respecting** *if both* F *and* G *are signature-respecting,*

(ii) **(strongly) monotonic** *if both* F *and* G *are (strongly) monotonic,*

(iii) **join-continuous** *if both* F *and* G *are join-continuous.*

Finally, a signature-respecting adjunction will be said to **commute with substitutions** *if both* F *and* G *commute with substitutions.*

Relating Institutions

In this section the notion of a translation and that of an interpretation between two π-institutions are introduced. Based on these notions, the relations of quasi-equivalence, strong quasi-equivalence and deductive equivalence, increasing in strength, can be defined between two π-institutions. These relations provide the necessary means for comparing their deductive apparatuses. The weakest notion is introduced first and the rest are then developed in increasing order of strength. Characterizations of these relations will be provided in the following sections of this chapter, in terms of the strength of the ties that they impose between the categories of theories of the two π-institutions they relate.

DEFINITION 2.21 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions.

- *A* **translation** *of $\mathcal{I}_1$ in $\mathcal{I}_2$ is a pair $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ consisting of*
  - *(i) a functor $F : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ and*
  - *(ii) a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2\, F$.*
- *A translation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an* **interpretation of $\mathcal{I}_1$ in $\mathcal{I}_2$** *if, for all $\Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,*

$$\phi \in \Phi^c \quad \textit{if and only if} \quad \alpha_{\Sigma_1}(\phi) \subseteq \alpha_{\Sigma_1}(\Phi)^c. \tag{2.3}$$

Using these notions the following relations on π-institutions can be defined.

DEFINITION 2.22 *Let $\mathcal{I}_1, \mathcal{I}_2$ be two π-institutions, as above.*

- *$\mathcal{I}_1$ will be said to be* **interpretable in** *$\mathcal{I}_2$ if there exists an interpretation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$.*
- *$\mathcal{I}_1$ will be said to be* **left quasi-equivalent to** *$\mathcal{I}_2$ and $\mathcal{I}_2$ is* **right quasi-equivalent to** *$\mathcal{I}_1$ if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that*
  1. *$\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ is an adjunction*
  2. *for all $\Sigma_1 \in |\mathbf{SIGN}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1)$,*

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)^c \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c \tag{2.4}$$

and, for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))^c \subseteq \{\psi\}^c. \tag{2.5}$$

In this case $\langle F,\alpha\rangle$ *is a* **left quasi-inverse** *of* $\langle G,\beta\rangle$ *and* $\langle G,\beta\rangle$ *a* **right quasi-inverse** *of* $\langle F,\alpha\rangle$.

- $\mathcal{I}_1$ *will be said to be* **strongly left quasi-equivalent to** $\mathcal{I}_2$ *and* $\mathcal{I}_2$ **strongly right quasi-equivalent to** $\mathcal{I}_1$ *if there exist interpretations* $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, $\langle G,\beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, *such that 1 and 2 above hold, but in 2 the inclusions are replaced by equalities.*

  In this case $\langle F,\alpha\rangle$ *is a* **strong left quasi-inverse** *of* $\langle G,\beta\rangle$ *and* $\langle G,\beta\rangle$ *a* **strong right quasi-inverse** *of* $\langle F,\alpha\rangle$.

- $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are* **deductively equivalent** *if there exist an interpretation* $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *and an interpretation* $\langle G,\beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, *such that* $\langle F,\alpha\rangle$ *and* $\langle G,\beta\rangle$ *are* **inverses** *of one another meaning that* $\langle F,\alpha\rangle$ *is a strong left quasi-inverse of* $\langle G,\beta\rangle$ *and in 1 above the adjunction is replaced by an adjoint equivalence.*

Note that, if $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent via the interpretations $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G,\beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F,G,\eta,\epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, then, for all $\Sigma_2 \in |\mathbf{SIGN}_2|$ and $\psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$\{\psi\}^c = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi))^c), \tag{2.6}$$

and, for all $\Sigma_1 \in |\mathbf{SIGN}_1|$ and $\phi \in \mathrm{SEN}_1(\Sigma_1)$,

$$\{\phi\}^c = \mathrm{SEN}_1(\eta_{\Sigma_1})^{-1}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c). \tag{2.7}$$

In this case (2.6) and (2.7) are equivalent to (2.5) and (2.4), respectively, in view of Corollaries 2.4 and 2.6 and the fact that η_{Σ_1} and ϵ_{Σ_2} are isomorphisms.

We define the corresponding notions for institutions using their associated π-institutions.

DEFINITION 2.23 *Let* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *be two institutions.*

- $\mathcal{I}_1$ *is* **interpretable in** $\mathcal{I}_2$ *if* $\pi(\mathcal{I}_1)$ *is interpretable in* $\pi(\mathcal{I}_2)$.

- $\mathcal{I}_1$ *is* **(strong) left quasi-equivalent to** $\mathcal{I}_2$ *if* $\pi(\mathcal{I}_1)$ *is (strong) left quasi-equivalent to* $\pi(\mathcal{I}_2)$ *and, similarly, for* **(strong) right quasi-equivalence.**

- $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are* **deductively equivalent** *if* $\pi(\mathcal{I}_1)$ *and* $\pi(\mathcal{I}_2)$ *are deductively equivalent.*

Note that if $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent and $\langle F,\alpha\rangle$, $\langle G,\beta\rangle$ inverses of each other, then each is both left and right strong quasi-equivalent to the other and the unit and counit of the quasi-invertibility relations are natural isomorphisms.

A technical lemma that will be used very often in what follows is given first.

LEMMA 2.24 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π*-institutions and* $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *an interpretation. Then*

$$\alpha_{\Sigma_1}(\Phi^c)^c = \alpha_{\Sigma_1}(\Phi)^c, \quad \textit{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1). \tag{2.8}$$

Proof:

Clearly, $\alpha_{\Sigma_1}(\Phi)^c \subseteq \alpha_{\Sigma_1}(\Phi^c)^c$. Since α is an interpretation, $\alpha_{\Sigma_1}(\Phi^c) \subseteq \alpha_{\Sigma_1}(\Phi)^c$, whence $\alpha_{\Sigma_1}(\Phi^c)^c \subseteq (\alpha_{\Sigma_1}(\Phi)^c)^c$, i.e., $\alpha_{\Sigma_1}(\Phi^c)^c \subseteq \alpha_{\Sigma_1}(\Phi)^c$, as required. ■

A lemma giving a property of the quasi-invertibility relations follows.

LEMMA 2.25 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π*-institutions such that there exist translations* $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G,\beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and an adjunction* $\langle F,G,\eta,\epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, *such that, for all* $\Sigma_1 \in |\mathbf{SIGN}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1)$, *condition (2.4) holds. Then,*

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(\Phi)^c \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi))^c \quad \textit{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1).$$

Similarly, if, for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \psi \in \mathrm{SEN}_2(\Sigma_2)$, *condition (2.5) holds, then*

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Psi)))^c \subseteq \Psi^c \quad \textit{for all} \quad \Sigma_2 \in |\mathbf{SIGN}_2|, \Psi \subseteq \mathrm{SEN}_2(\Sigma_2).$$

Proof:

$$\begin{aligned}
\mathrm{SEN}_1(\eta_{\Sigma_1})(\Phi)^c &= (\textstyle\bigcup_{\phi\in\Phi}\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi))^c\\
&= (\textstyle\bigcup_{\phi\in\Phi}\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)^c)^c\\
&\subseteq (\textstyle\bigcup_{\phi\in\Phi}\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c)^c \text{ (by hypothesis)}\\
&= (\textstyle\bigcup_{\phi\in\Phi}\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi)))^c\\
&= \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi))^c,
\end{aligned}$$

as required. The second assertion can be proved similarly. ∎

COROLLARY 2.26 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions such that there exist translations $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G,\beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and an adjunction $\langle F,G,\eta,\epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, such that, for all $\Sigma_1 \in |\mathbf{SIGN}_1|$ and all $\phi \in \mathrm{SEN}_1(\Sigma_1)$, $\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)^c = \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c$, i.e., (2.4) holds with equality in place of the inclusion. Then

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(\Phi)^c = \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi))^c \quad \textit{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1).$$

Similarly, if, for all $\Sigma_2 \in |\mathbf{SIGN}_2|$ and all $\psi \in \mathrm{SEN}_2(\Sigma_2)$, $\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))^c = \{\psi\}^c$, then

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Psi)))^c = \Psi^c \quad \textit{for all} \quad \Sigma_2 \in |\mathbf{SIGN}_2|, \Psi \subseteq \mathrm{SEN}_2(\Sigma_2).$$

Proof:

In the proof of Lemma 2.25 replace inclusions by equalities. ∎

We next prove a theorem showing that the existence of an adjoint equivalence together with conditions (2.3) and (2.6) are sufficient for deductive equivalence.

THEOREM 2.27 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be π-institutions. $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent if there exist translations $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G,\beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that $\langle F,G,\eta,\epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ is an adjoint equivalence, $\langle F,\alpha\rangle$ is an interpretation and, for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$\{\psi\}^c = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi))^c).$$

Proof:

We first need to verify that $\langle G, \beta\rangle$ is also an interpretation. To this end, let $\Sigma_2 \in |\mathbf{SIGN}_2|, \Psi \cup \{\psi\} \subseteq \mathrm{SEN}_2(\Sigma_2)$. We have

$$\psi \in \Psi^c \text{ iff}$$
$$\{\psi\}^c \subseteq \Psi^c \text{ iff, by Corollary 2.26,}$$
$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi))^c) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Psi))^c) \text{ iff, since } \epsilon_{\Sigma_2} \text{ is iso,}$$
$$\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)) \subseteq \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Psi))^c \text{ iff, since } \alpha \text{ is an interpretation,}$$
$$\beta_{\Sigma_2}(\psi) \subseteq \beta_{\Sigma_2}(\Psi)^c,$$

as required. Thus, $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ is also an interpretation.

Next, let $\Sigma_1 \in |\mathbf{SIGN}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1)$. We need to show that condition (2.7) holds. We have

$$\{\phi\}^c = \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c) \text{ iff, since } \alpha \text{ is an interpretation,}$$

$$\alpha_{\Sigma_1}(\phi)^c = \alpha_{\Sigma_1}(\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c))^c$$

iff, since α is a natural transformation (see diagram below),

$$\begin{tikzcd}
\mathrm{SEN}_1(G(F(\Sigma_1))) \arrow[r, "\alpha_{G(F(\Sigma_1))}"] \arrow[d, "\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})"'] & \mathrm{SEN}_2(F(G(F(\Sigma_1)))) \arrow[d, "\mathrm{SEN}_2(F(\eta_{\Sigma_1}^{-1}))"] \\
\mathrm{SEN}_1(\Sigma_1) \arrow[r, "\alpha_{\Sigma_1}"'] & \mathrm{SEN}_2(F(\Sigma_1))
\end{tikzcd}$$

$$\alpha_{\Sigma_1}(\phi)^c = \mathrm{SEN}_2(F(\eta_{\Sigma_1}^{-1}))(\alpha_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c))^c$$

iff, since $\langle F, G, \eta, \epsilon\rangle$ is an equivalence (see diagram below),

$$\begin{tikzcd}
F(\Sigma_1) \arrow[r, "F(\eta_{\Sigma_1})"] \arrow[dr, "i_{F(\Sigma_1)}"'] & F(G(F(\Sigma_1))) \arrow[d, "\epsilon_{F(\Sigma_1)}"] \\
& F(\Sigma_1)
\end{tikzcd}$$

$$\begin{aligned}
\alpha_{\Sigma_1}(\phi)^c &= \mathrm{SEN}_2(\epsilon_{F(\Sigma_1)})(\alpha_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c))^c \\
&= \mathrm{SEN}_2(\epsilon_{F(\Sigma_1)})(\alpha_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c)^c)^c \text{ (by Corollary 2.4)} \\
&= \mathrm{SEN}_2(\epsilon_{F(\Sigma_1)})(\alpha_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c)^c) \text{ (by Corollary 2.6)} \\
&= \mathrm{SEN}_2(\epsilon_{F(\Sigma_1)})(\alpha_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi)))^c) \text{ (by Lemma 2.24)}
\end{aligned}$$

which holds, by assumption and Corollary 2.26. ■

Interpretability

We start by giving a characterization of the existence of a translation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, from a term π-institution $\mathcal{I}_1$ to a π-institution $\mathcal{I}_2$.

LEMMA 2.28 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions. If there exists a translation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, then there exists a signature-respecting functor $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$.

Moreover, if $\mathbf{SIGN}_1 = \mathbf{SIGN}_2 = \mathbf{SIGN}$ and $F = I_{\mathbf{SIGN}}$, then F' makes the following diagram commute

$$\begin{tikzcd}
\mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F'"] \arrow[dr, "\mathsf{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathsf{SIG}"] \\
& \mathbf{SIGN} &
\end{tikzcd}$$

Proof:

Suppose that $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is a translation. Define $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ as follows.

$$F'(\langle\Sigma_1, T_1\rangle) = \langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle, \quad \text{for every} \quad \langle\Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|,$$

and, given $f : \langle\Sigma_1, T_1\rangle \to \langle\Sigma_1', T_1'\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_1))$, $F'(f) : \langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle \to \langle F(\Sigma_1'), \alpha_{\Sigma_1'}(T_1')^c\rangle$ is determined by

$$\mathrm{SIG}(F'(f)) = F(\mathrm{SIG}(f)).$$

We have

$$\begin{aligned}\mathrm{SEN}_2(F(f))(\alpha_{\Sigma_1}(T_1)^c)^c &= \mathrm{SEN}_2(F(f))(\alpha_{\Sigma_1}(T_1))^c \text{ (by Corollary 2.4)}\\ &= \alpha_{\Sigma_1'}(\mathrm{SEN}_1(f)(T_1))^c \text{ (since } \alpha \text{ is a natural transf.)}\\ &\subseteq \alpha_{\Sigma_1'}(T_1')^c \text{ (since } f:\langle\Sigma_1,T_1\rangle\to\langle\Sigma_1',T_1'\rangle\in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_1))),\end{aligned}$$

whence $F'(f)$ is a well-defined theory morphism. Since F is a functor, F', which agrees with F on morphisms, is also a functor. For signature-respectability, we must show that the following diagram commutes:

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F'"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{SIGN}_1 \arrow[r, "F"'] & \mathbf{SIGN}_2 \end{tikzcd}$$

For every $\langle\Sigma_1,T_1\rangle\in|\mathbf{TH}(\mathcal{I}_1)|$,

$$\begin{aligned}\mathrm{SIG}(F'(\langle\Sigma_1,T_1\rangle)) &= \mathrm{SIG}(\langle F(\Sigma_1),\alpha_{\Sigma_1}(T_1)^c\rangle)\\ &= F(\Sigma_1)\\ &= F(\mathrm{SIG}(\langle\Sigma_1,T_1\rangle)),\end{aligned}$$

as required, and, for every $f:\langle\Sigma_1,T_1\rangle\to\langle\Sigma_1',T_1'\rangle\in\mathrm{Mor}(\mathbf{TH}(\mathcal{I}_1))$, we have, by definition of F',

$$\mathrm{SIG}(F'(f)) = F(\mathrm{SIG}(f)),$$

as required. The final assertion of the lemma is straightforward. ∎

THEOREM 2.29 *Let* $\mathcal{I}_1 = \langle\mathbf{SIGN}_1,\mathrm{SEN}_1,\{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle$ *be a term* π*-institution and* $\mathcal{I}_2 = \langle\mathbf{SIGN}_2,\mathrm{SEN}_2,\{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$ *be a* π*-institution.*

(i) There exists a translation $\langle F',\alpha\rangle:\mathcal{I}_1\to\mathcal{I}_2$ *if and only if there exists a signature-respecting functor* $F:\mathbf{TH}(\mathcal{I}_1)\to\mathbf{TH}(\mathcal{I}_2)$.

(ii) Moreover, in case $\mathbf{SIGN}_1=\mathbf{SIGN}_2=\mathbf{SIGN}$, *there exists a translation* $\langle I_{\mathbf{SIGN}},\alpha\rangle$ $:\mathcal{I}_1\to\mathcal{I}_2$ *if and only if there exists a functor* $F:\mathbf{TH}(\mathcal{I}_1)\to\mathbf{TH}(\mathcal{I}_2)$ *that makes*

the following diagram commute

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

Proof:

A stronger "only if", without the requirement that $\mathcal{I}_1$ be term, was proved in Lemma 2.28. For the "if" direction, suppose that $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is a signature-respecting functor. Then, there exists a unique functor $F' : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, such that the following rectangle commutes

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{SIGN}_1 \arrow[r, "F'"'] & \mathbf{SIGN}_2 \end{tikzcd}$$

Moreover, if the given triangle commutes, then $F' = I_{\mathbf{SIGN}}$. Since $\mathcal{I}_1$ is term, there exists a source signature $A \in |\mathbf{SIGN}_1|$ and a variable $p \in \mathrm{SEN}_1(A)$. Set

$$\Theta = \pi_2(F(\langle A, \{p\}^c\rangle)).$$

Define $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2F'$ by $\alpha_{\Sigma_1} : \mathrm{SEN}_1(\Sigma_1) \to \mathcal{P}(\mathrm{SEN}_2(F'(\Sigma_1)))$, with

$$\alpha_{\Sigma_1}(\phi) = \mathrm{SEN}_2(F'(f_{\langle\Sigma_1,\phi\rangle}))(\Theta), \quad \text{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1).$$

It suffices to show that $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2F'$ is a natural transformation, i.e., that the following diagram commutes, for every $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{SIGN}_1)$.

$$\begin{tikzcd} \mathrm{SEN}_1(\Sigma_1) \arrow[r, "\alpha_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}_1(f)"'] & \mathcal{P}\mathrm{SEN}_2(F'(\Sigma_1)) \arrow[d, "\mathcal{P}\mathrm{SEN}_2(F'(f))"] \\ \mathrm{SEN}_1(\Sigma_1') \arrow[r, "\alpha_{\Sigma_1'}"'] & \mathcal{P}\mathrm{SEN}_2(F'(\Sigma_1')) \end{tikzcd}$$

For every $\phi \in \mathrm{SEN}_1(\Sigma_1)$, we have

$$\begin{aligned}
\mathcal{P}\mathrm{SEN}_2(F'(f))(\alpha_{\Sigma_1}(\phi)) &= \mathrm{SEN}_2(F'(f))(\mathrm{SEN}_2(F'(f_{\langle\Sigma_1,\phi\rangle}))(\Theta)) \text{ (by defin. of } \alpha_{\Sigma_1}) \\
&= \mathrm{SEN}_2(F'(ff_{\langle\Sigma_1,\phi\rangle}))(\Theta) \text{ (since } \mathrm{SEN}_2F' \text{ is a functor)} \\
&= \mathrm{SEN}_2(F'(f_{\langle\Sigma_1',\mathrm{SEN}_1(f)(\phi)\rangle}))(\Theta) \text{ (by the term property)} \\
&= \alpha_{\Sigma_1'}(\mathrm{SEN}_1(f)(\phi)) \text{ (by definition of } \alpha_{\Sigma_1'}),
\end{aligned}$$

as required. Thus, $\langle F',\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is a translation, as was to be shown. ■

A characterization of interpretability follows.

LEMMA 2.30 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions. If there exists an interpretation $\langle F,\alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, then there exists a strongly monotonic, join-continuous, signature-respecting functor $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions.

Moreover, if $\mathbf{SIGN}_1 = \mathbf{SIGN}_2 = \mathbf{SIGN}$ and $F = I_{\mathbf{SIGN}}$, then $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ makes the following diagram commute

$$\begin{tikzcd}
\mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F'"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\
& \mathbf{SIGN} &
\end{tikzcd}$$

Proof:

Consider the functor $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that is given by Lemma 2.28. We show that it is strongly monotonic, join-continuous and commutes with substitutions. To this end, let $\langle\Sigma_1,T_1\rangle, \langle\Sigma_1,T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$. Then

$$\begin{aligned}
T_1 \subseteq T_1' \text{ iff } & \alpha_{\Sigma_1}(T_1)^c \subseteq \alpha_{\Sigma_1}(T_1')^c \text{ (since } \alpha \text{ is an interpretation)} \\
\text{iff } & \pi_2(F'(\langle\Sigma_1,T_1\rangle)) \subseteq \pi_2(F'(\langle\Sigma_1,T_1'\rangle)) \text{ (by the definitions of } F',\pi_2),
\end{aligned}$$

as required. To show that F' is join-continuous, let $\Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$. Then

$$\begin{aligned}
(\bigcup_{\phi\in\Phi}\pi_2(F'(\langle\Sigma_1,\{\phi\}^c\rangle)))^c &= (\bigcup_{\phi\in\Phi}\alpha_{\Sigma_1}(\{\phi\}^c)^c)^c \text{ (by the definition of } F' \text{ and } \pi_2)\\
&= (\bigcup_{\phi\in\Phi}\alpha_{\Sigma_1}(\phi)^c)^c \text{ (by Lemma 2.24)}\\
&= (\bigcup_{\phi\in\Phi}\alpha_{\Sigma_1}(\phi))^c\\
&= \alpha_{\Sigma_1}(\Phi)^c\\
&= \alpha_{\Sigma_1}(\Phi^c)^c \text{ (by Lemma 2.24)}\\
&= \pi_2(F'(\langle\Sigma_1,\Phi^c\rangle)) \text{ (by the definition of } F' \text{ and } \pi_2),
\end{aligned}$$

as required. Finally, for commutativity with substitutions, letting $f:\Sigma_1\to\Sigma_1'\in$ Mor($\mathbf{SIGN}_1$), we have

$$\begin{aligned}
\mathrm{SEN}_2(F(f))(\pi_2(F'(\langle\Sigma_1,T_1\rangle)))^c &= \mathrm{SEN}_2(F(f))(\alpha_{\Sigma_1}(T_1)^c)^c \text{ (by the def. of } F',\pi_2)\\
&= \mathrm{SEN}_2(F(f))(\alpha_{\Sigma_1}(T_1))^c \text{ (by Corollary 2.4)}\\
&= \alpha_{\Sigma_1'}(\mathrm{SEN}_1(f)(T_1))^c \text{ (since } \alpha \text{ is a nat.transf.)}\\
&= \alpha_{\Sigma_1'}(\mathrm{SEN}_1(f)(T_1)^c)^c \text{ (by Lemma 2.24)}\\
&= \pi_2(F'(\langle\Sigma_1',\mathrm{SEN}_1(f)(T_1)^c\rangle)) \text{ (by def. of } F',\pi_2).
\end{aligned}$$

as required. The second assertion follows by the last assertion of Lemma 2.28. ∎

THEOREM 2.31 *Let* $\mathcal{I}_1=\langle\mathbf{SIGN}_1,\mathrm{SEN}_1,\{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle$ *be a term* π*-institution and* $\mathcal{I}_2=\langle\mathbf{SIGN}_2,\mathrm{SEN}_2,\{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$ *be a* π*-institution.*

(i) There exists an interpretation $\langle F',\alpha\rangle:\mathcal{I}_1\to\mathcal{I}_2$ *if and only if there exists a strongly monotonic, join-continuous, signature-respecting functor* $F:\mathbf{TH}(\mathcal{I}_1)\to\mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

(ii) Moreover, in case $\mathbf{SIGN}_1=\mathbf{SIGN}_2=\mathbf{SIGN}$, *there exists an interpretation* $\langle I_{\mathbf{SIGN}},\alpha\rangle:\mathcal{I}_1\to\mathcal{I}_2$ *if and only if there exists a strongly monotonic, join-continuous functor* $F:\mathbf{TH}(\mathcal{I}_1)\to\mathbf{TH}(\mathcal{I}_2)$ *that makes the following diagram commute*

$$\begin{tikzcd}
\mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\
& \mathbf{SIGN} &
\end{tikzcd}$$

and commutes with substitutions.

Proof:

A stronger "only if" was proved in Lemma 2.30 without the requirement that $\mathcal{I}_1$ be term. For the "if" direction, let $F' : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2, \alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F'$ be the components of the translation given by Theorem 2.29. (ii) of 2.29 ensures that, if the given triangle commutes, then $F' = I_{\mathbf{SIGN}}$.

Note that

$$\pi_2(F(\langle \Sigma_1, \Phi^c \rangle)) = \alpha_{\Sigma_1}(\Phi)^c, \quad \text{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1). \tag{2.9}$$

In fact, we have

$$\begin{aligned}
\alpha_{\Sigma_1}(\Phi)^c &= (\textstyle\bigcup_{\phi\in\Phi} \alpha_{\Sigma_1}(\phi))^c \\
&= (\textstyle\bigcup_{\phi\in\Phi} \mathrm{SEN}_2(F'(f_{\langle\Sigma_1,\phi\rangle}))(\Theta))^c \text{ (by the definition of } \alpha_{\Sigma_1}) \\
&= (\textstyle\bigcup_{\phi\in\Phi} \mathrm{SEN}_2(F'(f_{\langle\Sigma_1,\phi\rangle}))(\pi_2(F(\langle A, \{p\}^c\rangle))))^c \text{ (by the definition of } \Theta) \\
&= (\textstyle\bigcup_{\phi\in\Phi} \mathrm{SEN}_2(F'(f_{\langle\Sigma_1,\phi\rangle}))(\pi_2(F(\langle A, \{p\}^c\rangle)))^c)^c \\
&= (\textstyle\bigcup_{\phi\in\Phi} \pi_2(F(\langle \Sigma_1, \mathrm{SEN}_1(f_{\langle\Sigma_1,\phi\rangle})(p)^c\rangle)))^c \text{ (by comm. with substit.)} \\
&= (\textstyle\bigcup_{\phi\in\Phi} \pi_2(F(\langle \Sigma_1, \{\phi\}^c\rangle)))^c \text{ (by the term property)} \\
&= \pi_2(F(\langle \Sigma_1, \Phi^c\rangle)) \text{ (by join-continuity)},
\end{aligned}$$

as required.

It only remains to show that $\langle F', \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an interpretation. To this end, let $\Sigma_1 \in |\mathbf{SIGN}_1|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$. Then

$$\begin{gathered}
\alpha_{\Sigma_1}(\phi) \subseteq \alpha_{\Sigma_1}(\Phi)^c \text{ iff} \\
\alpha_{\Sigma_1}(\phi)^c \subseteq \alpha_{\Sigma_1}(\Phi)^c \text{ iff, by Equation (2.9),} \\
\pi_2(F(\langle \Sigma_1, \{\phi\}^c\rangle)) \subseteq \pi_2(F(\langle \Sigma_1, \Phi^c\rangle)) \text{ iff, by strong monotonicity,} \\
\{\phi\}^c \subseteq \Phi^c \text{ iff} \\
\phi \in \Phi^c,
\end{gathered}$$

as required. ∎

Quasi-Equivalence

In this section the relation of quasi-equivalence between two term π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ is characterized. As a corollary, a characterization of strong quasi-equivalence is obtained. This also yields a characterization of deductive equivalence by looking at the special case where the adjunction between the signature categories happens to be an adjoint equivalence. However, in the main result of the next section, Theorem 2.41, it will be shown that in this special case, the additional requirement that the unit and counit of the adjunction be natural isomorphisms can simplify the conditions imposed significantly.

LEMMA 2.32 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions and $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ a signature-respecting adjunction. Then, for all $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|, \langle \Sigma_2, T_2\rangle, \langle \Sigma_2, T_2'\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$,

$$\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,T_1'\rangle}) \quad \textit{and} \quad \mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2\rangle}) = \mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2'\rangle}).$$

Proof:

We show that, for all $\Sigma_1 \in |\mathbf{SIGN}_1|, \langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,\emptyset^c\rangle}).$$

To this end, consider the theory morphism $i : \langle \Sigma_1, \emptyset^c\rangle \to \langle \Sigma_1, T_1\rangle$, that is the identity on signatures. This morphism agrees on signatures with the morphism $i_{\langle\Sigma_1,\emptyset^c\rangle} : \langle \Sigma_1, \emptyset^c\rangle \to \langle \Sigma_1, \emptyset^c\rangle$, that is also the identity on signatures, by definition. Thus, by signature-respectability,

$$\begin{aligned} \mathrm{SIG}(F(i)) &= \mathrm{SIG}(F(i_{\langle\Sigma_1,\emptyset^c\rangle})) \\ &= \mathrm{SIG}(i_{F(\langle\Sigma_1,\emptyset^c\rangle)}). \end{aligned}$$

Similarly, by signature-respectability, the above equation yields

$$\begin{aligned}\mathrm{SIG}(G(F(i))) &= \mathrm{SIG}(G(i_{F(\langle\Sigma_1,\emptyset^c\rangle)}))\\ &= \mathrm{SIG}(i_{G(F(\langle\Sigma_1,\emptyset^c\rangle))})\\ &= i_{\mathrm{SIG}(G(F(\langle\Sigma_1,\emptyset^c\rangle)))},\end{aligned}$$

and, therefore, the following diagram commutes, by the naturality of η :

$$\begin{tikzcd} \mathrm{SIG}(\langle\Sigma_1,\emptyset^c\rangle) \arrow[r, "\mathrm{SIG}(\eta_{\langle\Sigma_1,\emptyset^c\rangle})"] \arrow[d, "i_{\Sigma_1}"'] & \mathrm{SIG}(G(F(\langle\Sigma_1,\emptyset^c\rangle))) \arrow[d, "i_{\mathrm{SIG}(G(F(\langle\Sigma_1,\emptyset^c\rangle)))}"] \\ \mathrm{SIG}(\langle\Sigma_1,T_1\rangle) \arrow[r, "\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle})"'] & \mathrm{SIG}(G(F(\langle\Sigma_1,T_1\rangle))) \end{tikzcd}$$

This shows that $\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,\emptyset^c\rangle})$, as required. The corresponding relation for the counit ϵ can be proved similarly. ■

LEMMA 2.33 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions.

(i) If $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is a signature-respecting adjunction, then there exists an adjunction $\langle F',G',\eta',\epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$.

(ii) Moreover, if $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is a signature-respecting adjoint equivalence then $\langle F',G',\eta',\epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ is also an adjoint equivalence.

Proof:

By signature-respectability, there exist unique $F' : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ and $G' : \mathbf{SIGN}_2 \to \mathbf{SIGN}_1$, such that the following squares commute:

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{SIGN}_1 \arrow[r, "F'"'] & \mathbf{SIGN}_2 \end{tikzcd} \qquad \begin{tikzcd} \mathbf{TH}(\mathcal{I}_2) \arrow[r, "G"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_1) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{SIGN}_2 \arrow[r, "G'"'] & \mathbf{SIGN}_1 \end{tikzcd}$$

By Lemma 2.32, there exists unique $\eta' : I_{\mathbf{SIGN}_1} \to G'F'$, such that the following diagram commutes

$$\begin{tikzcd} I_{\mathbf{TH}(\mathcal{I}_1)} \arrow[r, "\eta"] \arrow[d, "\mathrm{SIG}"'] & GF \arrow[d, "\mathrm{SIG}"] \\ I_{\mathbf{SIGN}_1} \arrow[r, "\eta'"'] & G'F' \end{tikzcd}$$

Similarly, there exists unique $\epsilon' : F'G' \to I_{\mathbf{SIGN}_2}$, such that, the following diagram commutes

$$\begin{tikzcd} FG \arrow[r, "\epsilon"] \arrow[d, "\mathrm{SIG}"'] & I_{\mathbf{TH}(\mathcal{I}_2)} \arrow[d, "\mathrm{SIG}"] \\ F'G' \arrow[r, "\epsilon'"'] & I_{\mathbf{SIGN}_2} \end{tikzcd}$$

It is not difficult to check that $\langle F', G', \eta', \epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ is an adjunction and that, in case $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is an adjoint equivalence, $\langle F', G', \eta', \epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ is also an adjoint equivalence. ■

DEFINITION 2.34 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions. An adjunction $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be said to be **strong** *if the following hold*

(i) $\mathrm{SEN}_1(\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}))(T_1)^c = \pi_2(G(F(\langle\Sigma_1, T_1\rangle)))$, *for every* $\langle\Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, *and*

(ii) $\mathrm{SEN}_2(\mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2\rangle}))(\pi_2(F(G(\langle\Sigma_2, T_2\rangle))))^c = T_2$, *for every* $\langle\Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$.

LEMMA 2.35 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions.

(i) If $\mathcal{I}_1$ is left quasi-equivalent to $\mathcal{I}_2$ via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjunction $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, then there exists a strongly monotonic, join-continuous, signature-respecting adjunction $\langle F', G', \eta', \epsilon'\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions.

(ii) If $\mathcal{I}_1$ is strong left quasi-equivalent to $\mathcal{I}_2$ then the adjunction $\langle F', G', \eta', \epsilon'\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is strong.

(iii) If $\mathcal{I}_1$ is deductively equivalent to $\mathcal{I}_2$ then the adjunction $\langle F', G', \eta', \epsilon'\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is an adjoint equivalence.

(iv) If $\mathbf{SIGN}_1 = \mathbf{SIGN}_2 = \mathbf{SIGN}$, $F = G = I_{\mathbf{SIGN}}$ and η and ϵ are the identity natural transformations, then η' and ϵ' are the identities, i.e., F' and G' are inverse isomorphisms that make the following diagrams commute

$$
\begin{tikzcd}
\mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F'"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\
& \mathbf{SIGN} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{TH}(\mathcal{I}_2) \arrow[rr, "G'"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_1) \arrow[dl, "\mathrm{SIG}"] \\
& \mathbf{SIGN} &
\end{tikzcd}
$$

Proof:

(i) Let $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ be the two interpretations and $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ the adjunction witnessing the quasi-equivalence relation between $\mathcal{I}_1$ and $\mathcal{I}_2$. By Lemma 2.30 there exist strongly monotonic, join-continuous, signature-respecting functors $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2), G' : \mathbf{TH}(\mathcal{I}_2) \to \mathbf{TH}(\mathcal{I}_1)$, that commute with substitutions. Define $\eta' : I_{\mathbf{TH}(\mathcal{I}_1)} \to G'F'$ by

$$\eta'_{\langle\Sigma_1, T_1\rangle} : \langle\Sigma_1, T_1\rangle \to \langle G(F(\Sigma_1)), \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c\rangle,$$

with

$$\mathrm{SIG}(\eta'_{\langle\Sigma_1, T_1\rangle}) = \eta_{\Sigma_1}, \quad \text{for every} \quad \langle\Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|,$$

and $\epsilon' : F'G' \to I_{\mathbf{TH}(\mathcal{I}_2)}$ by $\epsilon'_{\langle\Sigma_2, T_2\rangle} : \langle F(G(\Sigma_2)), \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(T_2))^c\rangle \to \langle\Sigma_2, T_2\rangle$, with

$$\mathrm{SIG}(\epsilon'_{\langle\Sigma_2, T_2\rangle}) = \epsilon_{\Sigma_2}, \quad \text{for every} \quad \langle\Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|.$$

Since, by the definition of left quasi-equivalence and Lemma 2.25,

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(T_1)^c \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c$$

and

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(T_2)))^c \subseteq T_2,$$

both $\eta'_{\langle\Sigma_1,T_1\rangle}$ and $\epsilon'_{\langle\Sigma_2,T_2\rangle}$ are well-defined theory morphisms and it is clear that $\langle F', G', \eta', \epsilon'\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is an adjunction. Since both F' and G' are strongly monotonic, join-continuous, signature-respecting and commute with substitutions, $\langle F', G', \eta', \epsilon'\rangle$ is also strongly monotonic, join-continuous, signature-respecting and commutes with substitutions.

(ii) If $\mathcal{I}_1$ is strong left quasi-equivalent to $\mathcal{I}_2$ then, by Corollary 2.26,

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(\Phi)^c = \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi))^c \quad \text{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1) \quad \text{and}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Psi)))^c = \Psi^c \quad \text{for all} \quad \Sigma_2 \in |\mathbf{SIGN}_2|, \Psi \subseteq \mathrm{SEN}_2(\Sigma_2).$$

Thus, (i) and (ii) of Definition 2.34 hold and $\langle F', G', \eta', \epsilon'\rangle$ is a strong adjunction.

(iii) If $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent then $\langle F', G', \eta', \epsilon'\rangle$ is obviously an adjoint equivalence, since η' and ϵ' are isomorphisms.

(iv) This part is clear by Lemma 2.30 and the definition of η' and ϵ'. ■

THEOREM 2.36 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two term π-institutions.

(i) $\mathcal{I}_1$ is left quasi-equivalent to $\mathcal{I}_2$ via the interpretations $\langle F', \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G', \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjunction $\langle F', G', \eta', \epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, if and only if there exists a strongly monotonic, join-continuous, signature-respecting adjunction $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions.

(ii) $\mathcal{I}_1$ *is strong left quasi-equivalent to* $\mathcal{I}_2$ *via the interpretations* $\langle F', \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G', \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjunction* $\langle F', G', \eta', \epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, *if and only if there exists a strongly monotonic, join-continuous, signature-respecting strong adjunction* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

(iii) $\mathcal{I}_1$ *is deductively equivalent to* $\mathcal{I}_2$ *via the interpretations* $\langle F', \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G', \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjoint equivalence* $\langle F', G', \eta', \epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$, *if and only if there exists a strongly monotonic, join-continuous, signature-respecting adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

(iv) *If* $\mathbf{SIGN}_1 = \mathbf{SIGN}_2 = \mathbf{SIGN}$, *then* $\mathcal{I}_1$ *is deductively equivalent to* $\mathcal{I}_2$ *via the interpretations* $\langle I_{\mathbf{SIGN}}, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle I_{\mathbf{SIGN}}, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the identity adjoint equivalence if and only if there exist strongly monotonic, join-continuous inverse functors* $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *and* $G : \mathbf{TH}(\mathcal{I}_2) \to \mathbf{TH}(\mathcal{I}_1)$ *that make the following diagrams commute*

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{TH}(\mathcal{I}_2) \arrow[rr, "G"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_1) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

and commute with substitutions.

Proof:

A stronger "only if" was proved in Lemma 2.35 without the requirement that $\mathcal{I}_1, \mathcal{I}_2$ be term institutions. For the "if" direction construct the two interpretations $\langle F', \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G', \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, given by Theorem 2.31, and note that, since $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is signature-respecting, there exist, by Lemma 2.33, $\eta' : I_{\mathbf{SIGN}_1} \to G'F'$ and $\epsilon' : F'G' \to I_{\mathbf{SIGN}_2}$, such that $\langle F', G', \eta', \epsilon'\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$ is an adjunction and an adjoint equivalence in case $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is an adjoint equivalence. Thus, it only remains to show that

$$\mathrm{SEN}_1(\eta'_{\Sigma_1})(\phi)^c \subseteq \beta_{F'(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c, \quad \text{for all} \quad \Sigma_1 \in |\mathbf{SIGN}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1),$$

and

$$\mathrm{SEN}_2(\epsilon'_{\Sigma_2})(\alpha_{G'(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))^c \subseteq \{\psi\}^c, \quad \text{for all} \quad \Sigma_2 \in |\mathbf{SIGN}_2|, \psi \in \mathrm{SEN}_2(\Sigma_2)$$

with inequalities replaced by equalities in case of a strong quasi-equivalence or of an adjoint equivalence. We have

$$\begin{aligned}
\beta_{F'(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c &= \beta_{F'(\Sigma_1)}(\alpha_{\Sigma_1}(\phi)^c)^c \text{ (by Lemma 2.24)}\\
&= \beta_{F'(\Sigma_1)}(\alpha_{\Sigma_1}(\{\phi\}^c)^c)^c \text{ (by Lemma 2.24)}\\
&= \beta_{F'(\Sigma_1)}(\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c \text{ (by Equation (2.9))}\\
&= \pi_2(G(\langle F'(\Sigma_1),\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle))\rangle)) \text{ (by Equation (2.9))}
\end{aligned}$$

$$\begin{aligned}
&= \pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle)))\\
&\quad \text{(since } F(\langle\Sigma_1,\{\phi\}^c\rangle) = \langle F'(\Sigma_1),\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle))\rangle)\\
&\supseteq \mathrm{SEN}_1(\eta'_{\Sigma_1})(\{\phi\}^c)^c \text{ (by Lemma 2.32)}\\
&= \mathrm{SEN}_1(\eta'_{\Sigma_1})(\phi)^c,
\end{aligned}$$

as required. The remaining inclusion and the equalities in the cases of a strong quasi-equivalence and of an adjoint equivalence can be treated similarly. The last assertion also follows by Theorem 2.31(ii) and part (iii). ■

Deductive Equivalence

The notion of deductive equivalence was defined for π-institutions in the section on "Relating Institutions" and a characterization was obtained for the deductive equivalence of two term π-institutions in terms of their categories of theories in Theorem 2.36(iii) of the previous section, as a special case of a similar characterization for the more general notion of quasi-equivalence. In this section, we exploit the special additional features present in the case of a deductive equivalence, more precisely, the fact that units and counits of the adjunctions involved are natural isomorphisms, to obtain a refinement of part (iii) of Theorem 2.36.

LEMMA 2.37 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle,\quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions. A signature-respecting adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is monotonic.

Proof:

Suppose $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is signature-respecting and let $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, with $T_1 \subseteq T_1'$. Then, the identity on Σ_1 induces a theory morphism $i : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1, T_1'\rangle$. This morphism agrees on signatures with the identity $i_{\langle \Sigma_1, T_1\rangle} : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1, T_1\rangle$, whence, by signature-respectability,

$$\begin{aligned} \mathrm{SIG}(F(i)) &= \mathrm{SIG}(F(i_{\langle \Sigma_1, T_1\rangle})) \\ &= \mathrm{SIG}(i_{F(\langle \Sigma_1, T_1\rangle)}) \\ &= i_{\mathrm{SIG}(F(\langle \Sigma_1, T_1\rangle))}. \end{aligned}$$

Thus, $F(i) : F(\langle \Sigma_1, T_1\rangle) \to F(\langle \Sigma_1, T_1'\rangle)$ is the identity on signatures, showing that

$$\pi_2(F(\langle \Sigma_1, T_1\rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1'\rangle)),$$

as required. By symmetry, for all $\langle \Sigma_2, T_2\rangle, \langle \Sigma_2, T_2'\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$, with $T_2 \subseteq T_2'$,

$$\pi_2(G(\langle \Sigma_2, T_2\rangle)) \subseteq \pi_2(G(\langle \Sigma_2, T_2'\rangle)),$$

as required. ∎

LEMMA 2.38 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two π-institutions. A signature-respecting adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is injective on Σ_1-theories, i.e., for all $\Sigma_1 \in |\mathbf{SIGN}_1|, \langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$\langle \Sigma_1, T_1\rangle \neq \langle \Sigma_1, T_1'\rangle \quad \textit{implies} \quad F(\langle \Sigma_1, T_1\rangle) \neq F(\langle \Sigma_1, T_1'\rangle)$$

and the same holds for Σ_2-theories, for every $\Sigma_2 \in |\mathbf{SIGN}_2|$.

Proof:

Let $\langle\Sigma_1,T_1\rangle,\langle\Sigma_1,T_1'\rangle\in|\mathbf{TH}(\mathcal{I}_1)|$. If $F(\langle\Sigma_1,T_1\rangle)=F(\langle\Sigma_1,T_1'\rangle)$, then, by signature-respectability and Lemma 2.32,

$$\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,T_1\rangle}))(\pi_2(G(F(\langle\Sigma_1,T_1\rangle))))=\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,T_1'\rangle}))(\pi_2(G(F(\langle\Sigma_1,T_1'\rangle)))),$$

whence, by Lemma 2.16, $T_1=T_1'$, as required. An analogous argument can be used for G. ■

LEMMA 2.39 *Let*

$$\mathcal{I}_1=\langle\mathbf{SIGN}_1,\mathrm{SEN}_1,\{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle,\quad \mathcal{I}_2=\langle\mathbf{SIGN}_2,\mathrm{SEN}_2,\{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two π-institutions. A signature-respecting adjoint equivalence $\langle F,G,\eta,\epsilon\rangle:\mathbf{TH}(\mathcal{I}_1)\to\mathbf{TH}(\mathcal{I}_2)$ is join-continuous.

Proof:

Let $\Sigma_1\in|\mathbf{SIGN}_1|,\Phi\subseteq\mathrm{SEN}_1(\Sigma_1)$. Since, by Lemma 2.37, $\langle F,G,\eta,\epsilon\rangle$ is monotonic,

$$\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle))\subseteq\pi_2(F(\langle\Sigma_1,\Phi^c\rangle)),\quad\text{for every}\quad\phi\in\Phi,$$

whence

$$(\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\subseteq\pi_2(F(\langle\Sigma_1,\Phi^c\rangle)).$$

Suppose that the inclusion is proper, i.e., that

$$(\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\subset\pi_2(F(\langle\Sigma_1,\Phi^c\rangle)).$$

Then, by Lemmas 2.37 and 2.38, if $\Sigma_2=\mathrm{SIG}(F(\langle\Sigma_1,\Phi^c\rangle))$, we have

$$\begin{aligned}\pi_2(G(\langle\Sigma_2,(\textstyle\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle)) &\subset \pi_2(G(\langle\Sigma_2,\pi_2(F(\langle\Sigma_1,\Phi^c\rangle))\rangle))\\ &= \pi_2(G(F(\langle\Sigma_1,\Phi^c\rangle))),\end{aligned}$$

whence, since $\eta_{\langle\Sigma_1,\Phi^c\rangle}$ is an isomorphism,

$$\begin{aligned}&\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(\langle\Sigma_2,(\textstyle\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle)))\ \subset\\ &\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(F(\langle\Sigma_1,\Phi^c\rangle))))\end{aligned}$$

i.e., by Lemma 2.16,

$$\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(\langle\Sigma_2,(\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle))) \subset \Phi^c. \tag{2.10}$$

Now, note that

$$\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)) \subseteq (\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c,$$

for every $\phi \in \Phi$, whence, by Lemma 2.37,

$$\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle))) \subseteq \pi_2(G(\langle\Sigma_2,(\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle)),$$

and, hence,

$$(\bigcup_{\phi\in\Phi}\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle))))^c \subseteq \pi_2(G(\langle\Sigma_2,(\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle)).$$

Thus, by Lemma 2.32,

$$\begin{aligned}&\mathrm{SEN}_1(\eta'^{-1}_{\Sigma_1})((\textstyle\bigcup_{\phi\in\Phi}\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle))))^c) \subseteq \\ &\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(\langle\Sigma_2,(\textstyle\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle))),\end{aligned}$$

where $\eta'_{\Sigma_1} = \mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle})$, for every Σ_1-theory $\langle\Sigma_1,T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$. Therefore, by (2.10), and Corollaries 2.6 and 2.4, we have

$$\mathrm{SEN}_1(\eta'^{-1}_{\Sigma_1})(\bigcup_{\phi\in\Phi}\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle))))^c \subset \Phi^c,$$

i.e., by Lemma 2.32,

$$(\bigcup_{\phi\in\Phi}\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\{\phi\}^c\rangle}))(\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle)))))^c \subset \Phi^c,$$

whence, by Lemma 2.16,

$$(\bigcup_{\phi\in\Phi}\{\phi\}^c)^c \subset \Phi^c, \text{ i.e., } \Phi^c \subset \Phi^c,$$

a contradiction. ■

LEMMA 2.40 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|} \rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|} \rangle$$

be two π-institutions and $\langle F, G, \eta, \epsilon \rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ a signature-respecting adjoint equivalence. Then, for all $\langle \Sigma_1, T_1 \rangle, \langle \Sigma_1, T_1' \rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$T_1 \subseteq T_1' \quad \textit{iff} \quad \pi_2(F(\langle \Sigma_1, T_1 \rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1' \rangle)),$$

and, similarly, for G.

Proof:

The "only if" holds by Lemma 2.37.

For the "if" direction, assume that $\pi_2(F(\langle \Sigma_1, T_1 \rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1' \rangle))$. Then we must have, by Lemma 2.37,

$$\pi_2(G(F(\langle \Sigma_1, T_1 \rangle))) \subseteq \pi_2(G(F(\langle \Sigma_1, T_1' \rangle))),$$

and, therefore, by Lemma 2.32,

$$\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle \Sigma_1, T_1 \rangle}))(\pi_2(G(F(\langle \Sigma_1, T_1 \rangle)))) \subseteq \mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle \Sigma_1, T_1' \rangle}))(\pi_2(G(F(\langle \Sigma_1, T_1' \rangle)))),$$

i.e., by Lemma 2.16, $T_1 \subseteq T_1'$, as required. ■

THEOREM 2.41 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|} \rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|} \rangle$$

be two term π-institutions. $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions.

Proof:

A stronger "only if" was proved in part (iii) of Theorem 2.36.

For the "if" part, it suffices, by part (iii) of Theorem 2.36, to show that the signature-respecting adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions is also strongly monotonic and join-continuous. But this was shown in Lemmas 2.40 and 2.39, respectively. ■

Since the notions of deductive equivalence and the category of theories for institutions were defined in terms of the corresponding notions on the associated π-institutions. Theorem 2.41 can be reformulated to fit in the institution framework as follows:

COROLLARY 2.42 *Let* $\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \mathrm{MOD}_1, \models^1 \rangle$, $\mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \mathrm{MOD}_2, \models^2 \rangle$ *be two term institutions.* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence* $\langle F, G, \eta, \epsilon \rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

Deductive Auto-Equivalence

A special case of interest arises when we are considering two π-institutions $\mathcal{I}_1 = \langle \mathbf{SIGN}, \mathrm{SEN}_1, \{C^1_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$, $\mathcal{I}_2 = \langle \mathbf{SIGN}, \mathrm{SEN}_2, \{C^2_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$ with the same signature categories. On certain occasions we need to know when $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent via interpretations $\langle I_{\mathbf{SIGN}}, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle I_{\mathbf{SIGN}}, \beta \rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the identity adjoint equivalence. If this is the case we will say that $\mathcal{I}_1$ and $\mathcal{I}_2$ are **deductively auto-equivalent.** Part (iv) of Theorem 2.36 completely characterizes this particular case. In view of Lemmas 2.40 and 2.39 it assumes the following form

COROLLARY 2.43 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}, \mathrm{SEN}_1, \{C^1_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}, \mathrm{SEN}_2, \{C^2_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$$

be two term π-institutions with the same signature categories. $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are deductively auto-equivalent if and only if there exists an isomorphism* $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$, *such that the following diagrams commute*

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{TH}(\mathcal{I}_2) \arrow[rr, "F^{-1}"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_1) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

and both F and F^{-1} commute with substitutions.

Proof:

By (iv) of Theorem 2.36 and Lemmas 2.40 and 2.39. ∎

The following lemma will serve to simplify the conditions of Corollary 2.43.

LEMMA 2.44 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}, \mathrm{SEN}_1, \{C^1_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}, \mathrm{SEN}_2, \{C^2_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|}\rangle$$

be two term π-institutions with the same signature categories and $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ an isomorphism such that

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

commutes and F commutes with substitutions. Then $F^{-1} : \mathbf{TH}(\mathcal{I}_2) \to \mathbf{TH}(\mathcal{I}_1)$ makes the following diagram commute

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_2) \arrow[rr, "F^{-1}"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_1) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

and commutes with substitutions.

Proof:

If $\langle \Sigma, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$, then

$$\mathrm{SIG}(F^{-1}(\langle \Sigma, T_2\rangle)) = \mathrm{SIG}(F(F^{-1}(\langle \Sigma, T_2\rangle))) = \mathrm{SIG}(\langle \Sigma, T_2\rangle),$$

and, if $g \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_2))$, then

$$\mathrm{SIG}(F^{-1}(g)) = \mathrm{SIG}(F(F^{-1}(g))) = \mathrm{SIG}(g).$$

Hence, the diagram

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_2) \arrow[rr, "F^{-1}"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_1) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

also commutes. It suffices, thus, to show that, for all $f : \Sigma \to \Sigma' \in \mathrm{Mor}(\mathbf{SIGN})$, $\langle\Sigma,$ $T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$

$$\mathrm{SEN}_1(f)(\pi_2(F^{-1}(\langle\Sigma, T_2\rangle)))^c = \pi_2(F^{-1}(\langle\Sigma', \mathrm{SEN}_2(f)(T_2)^c\rangle)).$$

Since F is an isomorphism, it suffices to show that

$$F(\langle\Sigma', \mathrm{SEN}_1(f)(\pi_2(F^{-1}(\langle\Sigma, T_2\rangle)))^c\rangle) = \langle\Sigma', \mathrm{SEN}_2(f)(T_2)^c\rangle.$$

We have

$$\begin{aligned} & F(\langle\Sigma', \mathrm{SEN}_1(f)(\pi_2(F^{-1}(\langle\Sigma, T_2\rangle)))^c\rangle) = \\ = \ & \langle\Sigma', \mathrm{SEN}_2(f)(\pi_2(F(\langle\Sigma, \pi_2(F^{-1}(\langle\Sigma, T_2\rangle))\rangle)))^c\rangle \\ = \ & \langle\Sigma', \mathrm{SEN}_2(f)(\pi_2(F(F^{-1}(\langle\Sigma, T_2\rangle))))^c\rangle \\ = \ & \langle\Sigma', \mathrm{SEN}_2(f)(\pi_2(\langle\Sigma, T_2\rangle))^c\rangle \\ = \ & \langle\Sigma', \mathrm{SEN}_2(f)(T_2)^c\rangle, \end{aligned}$$

as required. ■

In view of Lemma 2.44, Corollary 2.43 takes the following simplified form

COROLLARY 2.45 *Let*

$$\mathcal{I}_1 = \langle\mathbf{SIGN}, \mathrm{SEN}_1, \{C^1_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle, \quad \mathcal{I}_2 = \langle\mathbf{SIGN}, \mathrm{SEN}_2, \{C^2_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$$

be two term π-institutions with the same signature categories. $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively auto-equivalent if and only if there exists an isomorphism $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that makes the following diagram commute

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_2) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

and commutes with substitutions.

Equivalence of Deductive Systems

In the Introduction to the thesis the notion of equivalence between k-deductive systems, introduced in [8], was described in some detail. It is now shown how this notion can be perceived as a special case of the notion of deductive equivalence of π-institutions that was introduced in this chapter.

Recall that, given a set X, by $\overline{X}$ is denoted a disjoint copy of X constructed in some canonical way. Given a finitary k-deductive system $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}}\rangle$ over $\mathcal{L}$, define a π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$, as follows:

(i) **SIGN** has $|\mathbf{SIGN}| = \{V\}$ and

$$\mathrm{Mor}(\mathbf{SIGN}) = \{h : V \to \mathrm{Tm}_{\mathcal{L}}(V) : h \text{ an assignment}\}.$$

Composition of $h_1, h_2 : V \to \mathrm{Tm}_{\mathcal{L}}(V)$ in **SIGN** is defined by $h_2 \circ h_1 = h_2^* h_1$, where $h_2^* : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{Tm}_{\mathcal{L}}(V)$ is the substitution extending h_2.

(ii) SEN : **SIGN** $\to$ **SET** is given by

$$\mathrm{SEN}(V) = \mathrm{Tm}_{\mathcal{L}}(V)^k,$$

and, for every $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$,

$$\mathrm{SEN}(h)(\phi) = h^*(\phi), \text{ for every } \phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k.$$

(iii)

$$C_V(\Gamma) = C_{\mathcal{S}}(\Gamma), \text{ for every } \Gamma \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k.$$

Since $\mathcal{S}$ is a (structural) deductive system, $\mathcal{I}_{\mathcal{S}}$ is clearly a π-institution.

For the proof of our main theorem, the following lemmas are needed:

LEMMA 2.46 *Let $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}}\rangle$ be a finitary k-deductive system. The corresponding π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ is a term π-institution.*

Proof:

The source signature is necessarily V and we choose the variable $\overline{v} = \langle \overline{v_0}, \ldots, \overline{v_{k-1}} \rangle \in \mathrm{Tm}_{\mathcal{L}}(V)^k$. Then there exists $f : \{\langle V, \phi\rangle : \phi \in \mathrm{Tm}_{\mathcal{L}}(V)^k\} \to |\mathbf{SIGN}(V, V)|$, given by $f_{\langle V,\phi\rangle} : V \to \mathrm{Tm}_{\mathcal{L}}(V)$, with $f_{\langle V,\phi\rangle}(v_i) = \phi_i, i < k$, and $f_{\langle V,\phi\rangle}(v_j) = \phi_0, j \geq k$, for every $\phi = \langle \phi_0, \ldots, \phi_{k-1}\rangle \in \mathrm{Tm}_{\mathcal{L}}(V)^k$. Then, for every $\phi = \langle \phi_0, \ldots, \phi_{k-1}\rangle \in \mathrm{Tm}_{\mathcal{L}}(V)^k$, $f^{\bullet}_{\langle V,\phi\rangle}(\overline{v}) = \phi$ and, for every $f : V \to \mathrm{Tm}_{\mathcal{L}}(V)$, $f^{\bullet} f_{\langle V,\phi\rangle} = f_{\langle V, \mathrm{SEN}(f)(\phi)\rangle}$. The last equality is true, since, for $i < k$, $f^{\bullet} f_{\langle V,\phi\rangle}(v_i) = f^{\bullet}(\phi_i) = f_{\langle V,\mathrm{SEN}(f)(\phi)\rangle}(v_i)$, and, for $j \geq k$, $f^{\bullet} f_{\langle V,\phi\rangle}(v_j) = f^{\bullet}(\phi_0) = f_{\langle V,\mathrm{SEN}(f)(\phi)\rangle}(v_j)$, as required. ■

LEMMA 2.47 *Let $\mathcal{S}_1 = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}_1}\rangle$ be a finitary k-deductive system and $\mathcal{S}_2 = \langle \mathbf{Tm}_{\mathcal{L}}(V)^l, \vdash_{\mathcal{S}_2}\rangle$ a finitary l-deductive system over the same signature $\mathcal{L}$, and $\mathcal{I}_{\mathcal{S}_1} = \langle \mathbf{SIGN}, \mathrm{SEN}_1, \{C^1_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$, $\mathcal{I}_{\mathcal{S}_2} = \langle \mathbf{SIGN}, \mathrm{SEN}_2, \{C^2_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ the corresponding π-institutions. The lattices $\mathbf{Th}_{\mathcal{S}_1}$ and $\mathbf{Th}_{\mathcal{S}_2}$ are isomorphic via an isomorphism that commutes with substitutions if and only if there exists an isomorphism $F : \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2})$ that makes the following diagram commute*

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1}) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2}) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

and commutes with substitutions.

Proof:

Suppose, first, that $\tau_{\mathcal{S}_2} : \mathbf{Th}_{\mathcal{S}_1} \to \mathbf{Th}_{\mathcal{S}_2}$ is a lattice isomorphism that commutes with substitutions in the sense that, for every $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$,

$$C_{\mathcal{S}_2}(h^{\bullet}(\tau_{\mathcal{S}_2}(T))) = \tau_{\mathcal{S}_2}(C_{\mathcal{S}_1}(h^{\bullet}(T))), \quad \text{for every} \quad T \in \mathrm{Th}_{\mathcal{S}_1}.$$

Define the functor $F : \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2})$ as follows:

$$F(\langle V, T_1\rangle) = \langle V, \tau_{\mathcal{S}_2}(T_1)\rangle, \quad \text{for every} \quad \langle V, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{S}_1})|,$$

and, given $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$, such that $C_{\mathcal{S}_1}(h^{\bullet}(T_1)) \subseteq T_1'$,

$$F(h) = h.$$

We have

$$\begin{aligned} C_{\mathcal{S}_2}(h^*(\tau_{\mathcal{S}_2}(T_1))) &= \tau_{\mathcal{S}_2}(C_{\mathcal{S}_1}(h^*(T_1))) \\ &\subseteq \tau_{\mathcal{S}_2}(T_1'), \end{aligned}$$

whence F is well defined on morphisms.

It is easy to verify that F is actually an isomorphism, with inverse $G : \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1})$ given by

$$G(\langle V, T_2\rangle) = \langle V, \tau_{\mathcal{S}_2}^{-1}(T_2)\rangle, \quad \text{for every} \quad \langle V, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{S}_2})|,$$

and, given $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$, such that $C_{\mathcal{S}_2}(h^*(T_2)) \subseteq T_2'$, $G(h) = h$.

Clearly, the triangle

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1}) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2}) \arrow[dl, "\mathrm{SIG}"] \\ & \mathbf{SIGN} & \end{tikzcd}$$

commutes and the fact that F commutes with substitutions is simply a restatement of the fact that the lattice isomorphism $\tau_{\mathcal{S}_2}$ commutes with substitutions.

Conversely, suppose that there exists an isomorphism $F : \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2})$ that commutes with substitutions and such that the triangle above commutes. Then F restricted to $|\mathbf{TH}(\mathcal{I}_{\mathcal{S}_1})|$ induces a lattice isomorphism $\tau_{\mathcal{S}_2} : \mathbf{Th}_{\mathcal{S}_1} \to \mathbf{Th}_{\mathcal{S}_2}$ and, if $T \in \mathrm{Th}_{\mathcal{S}_1}$, we have

$$\begin{aligned} C_{\mathcal{S}_2}(h^*(\tau_{\mathcal{S}_2}(T))) &= F(h)^*(\pi_2(F(\langle V, T\rangle)))^c \\ &= \pi_2(F(\pi_2(\langle V, h^*(T)^c\rangle))) \\ &= \tau_{\mathcal{S}_2}(C_{\mathcal{S}_1}(h^*(T))), \end{aligned}$$

as required. ■

Our main theorem for this section is the following:

THEOREM 2.48 *Let $\mathcal{S}_1 = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}_1}\rangle$ be a finitary k-deductive system and $\mathcal{S}_2 = \langle \mathbf{Tm}_{\mathcal{L}}(V)^l, \vdash_{\mathcal{S}_2}\rangle$ a finitary $l-$deductive system over the same signature $\mathcal{L}$, and $\mathcal{I}_{\mathcal{S}_1} = \langle \mathbf{SIGN}, \mathrm{SEN}_1, \{C^1_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$, $\mathcal{I}_{\mathcal{S}_2} = \langle \mathbf{SIGN}, \mathrm{SEN}_2, \{C^2_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ the corresponding π-institutions. The deductive systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent if and only if the π-institutions $\mathcal{I}_{\mathcal{S}_1}$ and $\mathcal{I}_{\mathcal{S}_2}$ are deductively auto-equivalent.*

Proof:

By Theorem 1.1, $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent if and only if there exists an isomorphism from $\mathbf{Th}_{\mathcal{S}_1}$ to $\mathbf{Th}_{\mathcal{S}_2}$ that commutes with substitutions. By Lemma 2.47, this is true if and only if there exists an isomorphism $F : \mathbf{TH}(\mathcal{I}_{\mathcal{S}_1}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{S}_2})$ that commutes with substitutions, such that $\mathrm{SIG} \circ F = \mathrm{SIG}$. And, finally, by Lemma 2.46 and Corollary 2.45, this is true if and only if the (term) π-institutions $\mathcal{I}_{\mathcal{S}_1}$ and $\mathcal{I}_{\mathcal{S}_2}$ are deductively auto-equivalent. ∎

3 ALGEBRAIZING INSTITUTIONS

" When a logic is algebraizable, the powerful methods of modern algebra can be used in its investigation, and this has had a profound influence on the development of these logics. " W.J. Blok and Don Pigozzi, Algebraizable logics, Memoirs of the A.M.S., Vol. 77, No. 396, (1989)

The notion of an algebraic institution is introduced and, through it and the use of the machinery developed in the previous chapters, the notion of an algebraizable institution is made precise. Some examples of algebraizable institutions are given that also serve to connect the present theory with the algebraizability of k-deductive systems.

Introduction

In 1974 Barwise [1] introduced and axiomatized abstract model theory, using elementary category theory, with the intention of generalizing basic results of classical model theory. In 1980 Burstall and Goguen [10], developing the semantics of the specification language CLEAR, introduced the notion of language. They reintroduced this same notion, together with some new concepts and improved notation, under the name of institution in 1984 [26]. They further elaborated on it in 1992 [27]. Meanwhile, in 1988, in a similar context, Fiadeiro and Sernadas [21] introduced the notion of π-institution. Rather than having semantical satisfaction as the basis for the formalism, the emphasis has now been shifted towards a syntactic consequence relation in the spirit of Tarski. Finally, in 1989, Meseguer [45] introduced general logics in an attempt to combine all previous approaches. He included axiomatizations of the notions of an entailment system and proof calculus as well as of that of an institution.

One hopes that the opening quote of Blok and Pigozzi concerning logics might be equally applicable to the case of institutions, i.e., that a possible algebraization of an institution will enable us to use the methods of universal algebra, or those of the theory of algebraic theories in the context of category theory, in its investigation. More ambitiously, one might even argue that successful application of the algebraic methods in the institution domain might influence the development of the latter notion itself and make it even more widely applicable to the solution of problems in the areas of logic, model theory and theoretical computer science.

The attempt at the algebraization of an institution that we make in the present work has as its starting influence the work of Blok and Pigozzi [6] on the algebraization of classical deductive systems. Roughly speaking, given a deductive system $\mathcal{S}$ an *algebraic semantics* for $\mathcal{S}$ is a class K of algebras such that the consequence relation $\vdash_{\mathcal{S}}$ of $\mathcal{S}$ can be interpreted in the semantical equational consequence relation $\models_K$ of K. An *equivalent algebraic semantics* for $\mathcal{S}$ is an algebraic semantics for $\mathcal{S}$ such that there is also an inverse interpretation of $\models_K$ in $\vdash_{\mathcal{S}}$. A deductive system $\mathcal{S}$ is then *algebraizable* in the sense of Blok and Pigozzi if it has an equivalent algebraic semantics K. In [6] it was proved that a class K of algebras is an equivalent algebraic semantics for a deductive system $\mathcal{S}$ if and only if there is an isomorphism between the theory lattice of $\mathcal{S}$ and the equational theory lattice of K that commutes with the substitution operators. Furthermore, given a theory T of $\mathcal{S}$, the *elementary Leibniz (equivalence) relation* associated with T, denoted ΩT, was defined and, based on the *Leibniz operator* Ω two intrinsic characterizations of algebraizability were obtained. In 1992 [7], the theory was generalized to include k-deductive systems and in 1995 [8], these algebraizability results were reformulated based on the notion of equivalence for two deductive systems.

The traces of the work of Blok and Pigozzi are more than apparent in the present work which owes much to it.

The second main influence for our work comes from an attempt by Diskin [18] to algebraize institutions. This attempt is heavily based on the work of the Zurich school on categorical algebra. The beginnings of these latter developments can be traced back to the works of Lawvere [33], Linton [36],[37] and Beck [3]. (For a more detailed account see Mac Lane [39], Manes [43] and Borceux [9].) In [18], starting from the notion of an algebraic theory, the notion of a *protodoctrine* was defined and it was shown how one can get an institution $IN(\mathcal{P})$ out of a given protodoctrine $\mathcal{P}$. This special kind of institution plays the role of the equivalent algebraic semantics in Diskin's development. An institution $\mathcal{I}$ was then said to be *algebraizable* if there exists a protodoctrine $\mathcal{P}$ and a suitably defined institution morphism $\alpha : \mathcal{I} \rightarrow IN(\mathcal{P})$, from the given institution $\mathcal{I}$ to the institution that arises from the protodoctrine $\mathcal{P}$, satisfying some additional conditions. Next, from an algebraizable institution $\langle\mathcal{I},\alpha\rangle$ the, so-called, *specification category* $\mathbf{SPEC}(\langle\mathcal{I},\alpha\rangle)$ was extracted, which, in turn, gave rise in a natural way to a *specification system* $SPSYS(\langle\mathcal{I},\alpha\rangle)$, and then, the notion of *regularity* for a protodoctrine was defined. In the main result of [18], it was shown that given an institution that is algebraizable through a regular protodoctrine, one can obtain an algebraization of the specification system $SPSYS(\langle\mathcal{I},\alpha\rangle)$, which has many desirable algebraic properties.

Following Diskin's ideas and elaborating on his notion of algebraizability, a modified version of the notion of an *algebraizable institution* is introduced in this chapter. Only basic notions and tools of category theory and some elements of the theory of algebraic theories are used. This precise notion will make it possible to answer more general questions pertaining to the algebraizability of institutions and π-institutions.

Inspired by [6, 8, 18], in the second chapter, the notion of *deductive equivalence* for two π-institutions was defined. Generally speaking, two π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ are *deductively equivalent* if the consequence relations between sentences of the first can be interpreted in the corresponding consequence relations of the second and vice versa. This notion of deductive equivalence generalizes the notion of equivalence for deductive systems introduced in [8]; see Theorem 2.48. The focus was then directed to a special, but yet wide, class of π-institutions, the, so-called, *term π-institutions*. Using the theory categories of π-institutions, necessary and sufficient conditions for the deductive equivalence of two term π-institutions were obtained. Namely, it was proved that two term π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent if and only if their categories of theories are naturally equivalent via an equivalence satisfying some additional, relatively simple and quite natural, conditions.

In this chapter, the notion of an algebraic institution is introduced. Algebraic theories over locally small categories with a terminal object 1, in which the coproduct $1 \sqcup 1$ exists are considered. An *algebraic institution* $\mathcal{I}_Q$ is one that is closely related to a prespecified subcategory $\mathcal{Q}$ of the Eilenberg-Moore category of algebras of such a theory. Based on this notion, the notion of an algebraic π-institution is then defined. If one specializes to **SET**, i.e., the category of small sets, algebraic theories over **SET** are obtained, whose Eilenberg-Moore categories of algebras, as is well-known, roughly correspond to universal algebraic varieties of algebras. An arbitrary π-institution $\mathcal{I}$ is then said to be *algebraizable* if it is deductively equivalent to some algebraic π-institution $\mathcal{I}_Q$. As a corollary of the main characterization theorem of the second chapter, a characterization of algebraizability for term π-institutions is obtained in terms of their categories of theories. This result generalizes a similar result in [6].

Two examples of algebraizable π-institutions are given next. The first inspired by the theory of algebraizable k-deductive systems and the second on the algebraizability of the equational institution, an institution that represents a version of equational logic.

Algebraic Institutions and Algebraizable π-Institutions

We now give an important example of an institution. Let $\mathcal{K}$ be a locally small category with a terminal object 1 and $\mathbf{T} = \langle T, \eta, \mu \rangle$ an algebraic theory in monoid form over $\mathcal{K}$. The Kleisli category of $\mathbf{T}$ in $\mathcal{K}$ is denoted, as usual, by $\mathcal{K}_{\mathbf{T}}$ and the Eilenberg-Moore category of $\mathbf{T}$-algebras over $\mathcal{K}$ by $\mathcal{K}^{\mathbf{T}}$.

DEFINITION 3.1 *Let $\mathcal{L}$ be an arbitrary full subcategory of $\mathcal{K}_{\mathbf{T}}$ and $\mathcal{Q}$ an arbitrary subcategory of $\mathcal{K}^{\mathbf{T}}$. Define $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}} = \langle \mathcal{L}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ as follows:*

(i) $\mathrm{EQ} : \mathcal{L} \to \mathbf{SET}$ *is defined by*

$$\mathrm{EQ}(L) = \mathcal{K}_{\mathbf{T}}(1, L)^2 = \mathcal{K}(1, T(L))^2, \quad \textit{for every} \quad L \in |\mathcal{L}|,$$

and, given $f : L \rightharpoonup K \in \mathrm{Mor}(\mathcal{L})$, $\mathrm{EQ}(f) : \mathcal{K}_{\mathbf{T}}(1, L)^2 \to \mathcal{K}_{\mathbf{T}}(1, K)^2$ is given by

$$\mathrm{EQ}(f)(\langle g_1, g_2 \rangle) = \langle f \circ g_1, f \circ g_2 \rangle, \quad \textit{for every} \quad \langle g_1, g_2 \rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2,$$

where $f \circ g_i = \mu_K T(f) g_i$ is the Kleisli composite of g_i and f, $i = 1, 2$.

(ii) $\mathrm{ALG} : \mathcal{L} \to \mathbf{CAT}^{op}$ *is defined as follows: For every $L \in |\mathcal{L}|$, $\mathrm{ALG}(L)$ is the category with objects all pairs of the form $\langle \langle K, \xi \rangle, f \rangle$, where $\langle K, \xi \rangle \in |\mathcal{Q}|$ and $f : L \rightharpoonup K \in \mathrm{Mor}(\mathcal{K}_{\mathbf{T}})$, and morphisms $h : \langle \langle K, \xi \rangle, f \rangle \to \langle \langle M, \zeta \rangle, g \rangle$, $\mathcal{Q}$-morphisms $h : \langle K, \xi \rangle \to \langle M, \zeta \rangle$, such that $g = T(h) f$.*

$$\begin{tikzcd}
 & T(K) \arrow[r, "\xi"] \arrow[dd, "T(h)"'] & K \arrow[dd, "h"] \\
L \arrow[ur, "f"] \arrow[dr, "g"'] & & \\
 & T(M) \arrow[r, "\zeta"'] & M
\end{tikzcd}$$

Given $k : L \rightharpoonup K \in \mathrm{Mor}(\mathcal{L})$ the functor $\mathrm{ALG}(k) : \mathrm{ALG}(K) \to \mathrm{ALG}(L)$ sends $\langle \langle M, \xi \rangle, f \rangle$ to $\langle \langle M, \xi \rangle, f \circ k \rangle$ and a morphism $h : \langle \langle M, \xi \rangle, f \rangle \to \langle \langle N, \zeta \rangle, g \rangle$ to the morphism $\mathrm{ALG}(k)(h) : \langle \langle M, \xi \rangle, f \circ k \rangle \to \langle \langle N, \zeta \rangle, g \circ k \rangle$ with $\mathrm{ALG}(k)(h) = h$.

$$K \xrightarrow{f} T(M) \xrightarrow{\xi} M$$

$\big\downarrow$ ALG(k):

$$L \xrightarrow{k} T(K) \xrightarrow{T(f)} T(T(M)) \xrightarrow{\mu_M} T(M) \xrightarrow{\xi} M$$

$$
\begin{tikzcd}
 & T(M) \arrow[r, "\xi"] \arrow[d, "T(h)"'] & M \arrow[d, "h"] \\
K \arrow[ur, "f"] \arrow[dr, "g"'] & T(N) \arrow[r, "\zeta"'] & N \\
 & T(N) \arrow[r, "\zeta"'] & N
\end{tikzcd}
$$

$\Big\downarrow$ ALG(k):

$$
\begin{tikzcd}
 & & T(T(M)) \arrow[r, "\mu_M"] \arrow[d, "T(T(h))"'] & T(M) \arrow[r, "\xi"] \arrow[d, "T(h)"'] & M \arrow[d, "h"] \\
L \arrow[r, "k"] & T(K) \arrow[ur, "T(f)"] \arrow[r, "T(g)"'] & T(T(N)) \arrow[r, "\mu_N"'] & T(N) \arrow[r, "\zeta"'] & N
\end{tikzcd}
$$

(iii) For all $L \in |\mathcal{L}|, \langle g_1, g_2 \rangle \in \mathrm{EQ}(L)$ and $\langle\langle K, \xi\rangle, f\rangle \in |\mathrm{ALG}(L)|$,

$$\langle\langle K, \xi\rangle, f\rangle \models_L \langle g_1, g_2\rangle \quad \textit{iff} \quad \xi\mu_K T(f) g_1 = \xi\mu_K T(f) g_2.$$

$$1 \underset{g_2}{\overset{g_1}{\rightrightarrows}} T(L) \xrightarrow{T(f)} T(T(K)) \xrightarrow{\mu_K} T(K) \xrightarrow{\xi} K$$

The next theorem states that the above construction gives an institution.

THEOREM 3.2 *Let $\mathcal{K}$ be a locally small category with a terminal object 1, $\mathbf{T} = \langle T, \eta, \mu\rangle$ an algebraic theory over $\mathcal{K}$, $\mathcal{L}$ a full subcategory of $\mathcal{K}_{\mathbf{T}}$ and $\mathcal{Q}$ a subcategory of $\mathcal{K}^{\mathbf{T}}$. Then $\mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} = \langle \mathcal{L}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$, as defined in 3.1, is an institution.*

Proof:

We only show that ALG is well-defined on morphisms and then verify that the satisfaction condition holds.

First, let $k : L \rightharpoonup K \in \mathrm{Mor}(\mathcal{L})$ and suppose that $h : \langle\langle M, \xi\rangle, f\rangle \to \langle\langle N, \zeta\rangle, g\rangle$ is a morphism in ALG(K). To see that h is a valid morphism $h : \langle\langle M, \xi\rangle, f \circ k\rangle \to \langle\langle N, \zeta\rangle, g \circ k\rangle$ in ALG(L) it suffices to show that $T(h)\mu_M T(f)k = \mu_N T(g)k$. We have

$$
\begin{tikzcd}
T(T(M)) \arrow[r, "\mu_M"] \arrow[d, "T(T(h))"'] & T(M) \arrow[d, "T(h)"] \\
T(T(N)) \arrow[r, "\mu_N"'] & T(N)
\end{tikzcd}
$$

$$\begin{aligned}
T(h)\mu_M T(f)k &= \mu_N T(T(h))T(f)k \text{ (since } \mu \text{ is a natural transf.; see diagram)}\\
&= \mu_N T(T(h)f)k \text{ (since } T \text{ is a functor)}\\
&= \mu_N T(g)k, \text{ (since } h \in \mathrm{Mor}(\mathrm{ALG}(K)))
\end{aligned}$$

as required.

Next, let $k : L \rightharpoonup K \in \mathrm{Mor}(\mathcal{L}), \langle g_1, g_2\rangle \in \mathrm{EQ}(L)$ and $\langle\langle M,\xi\rangle, f\rangle \in |\mathrm{ALG}(K)|$. Then

$\mathrm{ALG}(k)(\langle\langle M,\xi\rangle, f\rangle) \models_L \langle g_1, g_2\rangle$ iff, by definition of $\mathrm{ALG}(k)$,

$\langle\langle M,\xi\rangle, \mu_M T(f)k\rangle \models_L \langle g_1, g_2\rangle$ iff, by definition of $\models_L$,

$\xi\mu_M T(\mu_M T(f)k)g_1 = \xi\mu_M T(\mu_M T(f)k)g_2$ iff, since T is a functor,

$\xi\mu_M T(\mu_M)T(T(f))T(k)g_1 = \xi\mu_M T(\mu_M)T(T(f))T(k)g_2$ iff,

by commutativity of
$$\begin{tikzcd}
T(T(T(M))) \arrow[r, "\mu_{T(M)}"] \arrow[d, "T(\mu_M)"'] & T(T(M)) \arrow[d, "\mu_M"] \\
T(T(M)) \arrow[r, "\mu_M"'] & T(M)
\end{tikzcd}$$

$\xi\mu_M\mu_{T(M)}T(T(f))T(k)g_1 = \xi\mu_M\mu_{T(M)}T(T(f))T(k)g_2$ iff, since μ is a nat. transf.,

i.e., by commutativity of
$$\begin{tikzcd}
T(T(K)) \arrow[r, "\mu_K"] \arrow[d, "T(T(f))"'] & T(K) \arrow[d, "T(f)"] \\
T(T(T(M))) \arrow[r, "\mu_{T(M)}"'] & T(T(M))
\end{tikzcd}$$

$\xi\mu_M T(f)\mu_K T(k)g_1 = \xi\mu_M T(f)\mu_K T(k)g_2$ iff, by definition of $\models_K$,

$\langle\langle M,\xi\rangle, f\rangle \models_K \langle\mu_K T(k)g_1, \mu_K T(k)g_2\rangle$ iff, by definition of $\mathrm{EQ}(k)$,

$\langle\langle M,\xi\rangle, f\rangle \models_K \mathrm{EQ}(k)(\langle g_1, g_2\rangle)$,

as required. ∎

Now, suppose that $1 \sqcup 1$ exists in $\mathcal{K}$, i.e., for all $K \in \mathcal{K}, k_1, k_2 \in \mathcal{K}(1, K)$, there exists unique $k_1 \sqcup k_2 \in \mathcal{K}(1 \sqcup 1, K)$, such that the following diagram commutes, where

$q_1, q_2 : 1 \to 1 \sqcup 1$ are the coprojections,

$$\begin{tikzcd} 1 \arrow[r, "q_1"] \arrow[dr, "k_1"'] & 1 \sqcup 1 \arrow[d, dashed, "k_1 \sqcup k_2"] & 1 \arrow[l, "q_2"'] \arrow[dl, "k_2"] \\ & K' & \end{tikzcd}$$

Since left adjoints preserve colimits, $1 \sqcup 1$ is also a coproduct of 1 with itself in $\mathcal{K}_{\mathbf{T}}$. In fact, the following is a coproduct diagram in $\mathcal{K}_{\mathbf{T}}$, where $k_1, k_2 \in \mathcal{K}_{\mathbf{T}}(1, K')$,

$$\begin{tikzcd} 1 \arrow[r, "\eta_{1\sqcup 1} q_1"] \arrow[dr, "k_1"'] & 1 \sqcup 1 \arrow[d, dashed, "k_1 \sqcup k_2"] & 1 \arrow[l, "\eta_{1\sqcup 1} q_2"'] \arrow[dl, "k_2"] \\ & K' & \end{tikzcd}$$

Note that

$$\begin{aligned} (k_1 \sqcup k_2) \circ (\eta_{1\sqcup 1} q_1) &= \mu_{K'} T(k_1 \sqcup k_2) \eta_{1\sqcup 1} q_1 \\ &= \mu_{K'} \eta_{T(K')} (k_1 \sqcup k_2) q_1 \end{aligned}$$

by commutativity of

$$\begin{tikzcd} 1 \sqcup 1 \arrow[r, "\eta_{1\sqcup 1}"] \arrow[d, "k_1 \sqcup k_2"'] & T(1 \sqcup 1) \arrow[d, "T(k_1 \sqcup k_2)"] \\ T(K') \arrow[r, "\eta_{T(K')}"'] & T(T(K')) \end{tikzcd}$$

$$\begin{aligned} &= \mu_{K'} \eta_{T(K')} k_1 \\ &= k_1, \end{aligned}$$

and, similarly, $(k_1 \sqcup k_2) \circ (\eta_{1\sqcup 1} q_2) = k_2$.

In the sequel, we will denote $\eta_{1\sqcup 1} q_i$ by $p_i, i = 1, 2$.

In the next theorem, it is shown that, if $1 \sqcup 1$ exists in $\mathcal{K}$, the institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{K}_{\mathbf{T}}}$ is term, for every subcategory $\mathcal{Q}$ of $\mathcal{K}^{\mathbf{T}}$.

THEOREM 3.3 *Let* $\mathbf{T}$ *be an algebraic theory over a locally small category* $\mathcal{K}$ *with a terminal object 1, in which* $1 \sqcup 1$ *exists, and let* $\mathcal{Q}$ *be a subcategory of* $\mathcal{K}^{\mathbf{T}}$. *Then the institution* $\mathcal{I}_{\mathcal{Q}}^{\mathcal{K}_{\mathbf{T}}}$ *is term.*

Proof:

We show that $1 \sqcup 1$ is a source signature and $\langle p_1, p_2\rangle \in \mathrm{EQ}(1 \sqcup 1)$ a variable (see Definition 2.7). To this end, for all $K \in |\mathcal{K}_{\mathbf{T}}|$ and $\langle k_1, k_2\rangle \in \mathrm{EQ}(K) = \mathcal{K}_{\mathbf{T}}(1, K)^2 = \mathcal{K}(1, T(K))^2$, define $f_{\langle K, \langle k_1, k_2\rangle\rangle} = k_1 \sqcup k_2 \in \mathcal{K}_{\mathbf{T}}(1 \sqcup 1, K)$, where $k_1 \sqcup k_2$ is the coproduct of k_1, k_2 in $\mathcal{K}_{\mathbf{T}}$. Then, by the following coproduct diagram and the definition of $\mathrm{EQ}(f_{\langle K, \langle k_1, k_2\rangle\rangle})$, we have

$$\begin{tikzcd} 1 \arrow[r, "p_1"] \arrow[dr, "k_1"'] & 1 \sqcup 1 \arrow[d, dashed, "k_1 \sqcup k_2"] & 1 \arrow[l, "p_2"'] \arrow[dl, "k_2"] \\ & K & \end{tikzcd}$$

$$\mathrm{EQ}(f_{\langle K, \langle k_1, k_2\rangle\rangle})(\langle p_1, p_2\rangle) = \langle f_{\langle K, \langle k_1, k_2\rangle\rangle} \circ p_1, f_{\langle K, \langle k_1, k_2\rangle\rangle} \circ p_2\rangle = \langle k_1, k_2\rangle.$$

Moreover, if $f : K \rightharpoonup L \in \mathrm{Mor}(\mathcal{K}_{\mathbf{T}})$, $f \circ f_{\langle K, \langle k_1, k_2\rangle\rangle} \circ p_1 = f \circ k_1$ and $f \circ f_{\langle K, \langle k_1, k_2\rangle\rangle} \circ p_2 = f \circ k_2$. But we also have $f_{\langle L, \langle f \circ k_1, f \circ k_2\rangle\rangle} \circ p_1 = f \circ k_1$ and $f_{\langle L, \langle f \circ k_1, f \circ k_2\rangle\rangle} \circ p_2 = f \circ k_2$, whence, by uniqueness of coproduct,

$$f \circ f_{\langle K, \langle k_1, k_2\rangle\rangle} = f_{\langle L, \langle f \circ k_1, f \circ k_2\rangle\rangle} = f_{\langle L, \mathrm{EQ}(f)(\langle k_1, k_2\rangle)\rangle},$$

as required. ■

Next, suppose that $\mathcal{L}$ is a full subcategory of $\mathcal{K}_{\mathbf{T}}$, such that there exist $L \in |\mathcal{L}|, l_1, l_2 \in \mathcal{K}_{\mathbf{T}}(1, L)$, with the property that there exists a set function $f : \{\langle K, \langle k_1, k_2\rangle\rangle : K \in |\mathcal{K}_{\mathbf{T}}|, k_1, k_2 \in \mathcal{K}_{\mathbf{T}}(1, K)\} \to |(L \,|\, \mathcal{K}_{\mathbf{T}})|$, such that

$$f_{\langle K, \langle k_1, k_2\rangle\rangle} \in \mathcal{K}_{\mathbf{T}}(L, K), \text{ for all } K \in |\mathcal{K}_{\mathbf{T}}|, k_1, k_2 \in \mathcal{K}_{\mathbf{T}}(1, K), \text{ the following commutes} \tag{3.1}$$

$$\begin{tikzcd} 1 \arrow[r, "l_1"] \arrow[dr, "k_1"'] & L \arrow[d, dashed, "f_{\langle K, \langle k_1, k_2\rangle\rangle}"] & 1 \arrow[l, "l_2"'] \arrow[dl, "k_2"] \\ & K & \end{tikzcd}$$

and, for every $f \in \mathcal{K}_{\mathbf{T}}(K, K')$, $f \circ f_{\langle K, \langle k_1, k_2\rangle\rangle} = f_{\langle K', \langle f \circ k_1, f \circ k_2\rangle\rangle}$.

If (3.1) holds, we will refer to $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$ as the **$\mathcal{L}$-algebraic institution associated with $\mathcal{Q}$** and to its associated π-institution $\pi(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$ as the **$\mathcal{L}$-algebraic π-institution associated with $\mathcal{Q}$**. An **algebraic institution** or an **algebraic π-institution** is then an $\mathcal{L}$-algebraic institution associated with $\mathcal{Q}$ or an $\mathcal{L}$-algebraic π-institution associated with $\mathcal{Q}$, respectively, for some $\mathcal{L}$ and $\mathcal{Q}$. Note that, if $1 \sqcup 1$ exists in $\mathcal{K}$ and $1 \sqcup 1 \in |\mathcal{L}|$, then (3.1) is satisfied, with $L = 1 \sqcup 1, l_1 = p_1$ and $l_2 = p_2$.

From now on we will be following the convention of writing $f_{\langle k_1,k_2 \rangle}$ instead of the more cumbersome $f_{\langle K, \langle k_1,k_2 \rangle \rangle}$. The signature object K is usually clear from context and so there is no possibility of confusion.

DEFINITION 3.4 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$ be a π-institution. $\mathcal{I}$ is*

- **prealgebraizable** *if it is interpretable in some algebraic π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$*
- **quasi-algebraizable** *if it is left or right quasi-equivalent to an algebraic π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$*
- **strongly quasi-algebraizable** *if it left or right strongly quasi-equivalent to an algebraic π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$*
- **algebraizable** *if it is deductively equivalent to an algebraic π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$,*

where, as before, $\mathcal{L}$ is a full subcategory of $\mathcal{K}_{\mathbf{T}}$ and $\mathcal{Q}$ is a subcategory of $\mathcal{K}^{\mathbf{T}}$, for some algebraic theory $\mathbf{T}$ over a locally small category $\mathcal{K}$ with a terminal object 1, *in which* (3.1) *holds. In this case the algebraic π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$ will be referred to, respectively, as an* **algebraic,** *a* **quasi-algebraic,** *a* **strong quasi-algebraic** *and an* **equivalent algebraic semantics** *for $\mathcal{I}$.*

An institution $\mathcal{I}$ is **prealgebraizable, (strongly) quasi-algebraizable, algebraizable** *if its associated π-institution $\pi(\mathcal{I})$ is prealgebraizable, (strongly) quasi-algebraizable, algebraizable, respectively, in the previous sense.*

Since, by construction, every algebraic π-institution is term, we get as an immediate consequence of the characterization Theorems 2.36 and 2.41 of the previous chapter the following corollary.

COROLLARY 3.5 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a term π-institution.*

(i) $\mathcal{I}$ is prealgebraizable with algebraic π-institution semantics $\pi(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$, if and only if there exists a strongly monotonic, join-continuous, signature-respecting functor $F : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$ that commutes with substitutions.

(ii) $\mathcal{I}$ is (strongly) quasi-algebraizable with (strong) quasi-equivalent algebraic π-institution semantics $\pi(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$, if and only if there exists a strongly monotonic, join-continuous, signature-respecting (strong) adjunction

$$\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$$

that commutes with substitutions.

(iii) $\mathcal{I}$ is algebraizable with deductively equivalent algebraic π-institution $\pi(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$, if and only if there exists a signature-respecting adjoint equivalence $\langle F, G, \eta, \epsilon\rangle$: $\mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$ that commutes with substitutions.

An Application

In this section a collection of pairs of π-institutions is provided, that will be strongly quasi-equivalent but not deductively equivalent. This may serve as a motivation for the introduction of the notion of quasi-equivalence in the previous chapter. Given a locally small category $\mathcal{K}$ with a terminal object 1, and an algebraic theory $\mathbf{T}$ in $\mathcal{K}$, recall that by $\mathcal{K}_{\mathbf{T}}$ is denoted, as usual, the Kleisli category of $\mathbf{T}$ in $\mathcal{K}$, and by $\mathcal{K}^{\mathbf{T}}$ the Eilenberg-Moore category of $\mathbf{T}$-algebras over $\mathcal{K}$. The pairs of institutions, that are considered in this section, will consist of the institution $\mathcal{I}_{\mathcal{K}^{\mathbf{T}}}^{\mathcal{K}_{\mathbf{T}}}$, as constructed in Definition 3.1, and of another institution that results from this by slightly modifying its components. Namely, its signature category is the category $\mathcal{K}$ itself, instead of the Kleisli category of $\mathbf{T}$ in $\mathcal{K}$, its sentences are $\mathcal{K}$-morphisms instead of $\mathcal{K}_{\mathbf{T}}$-morphisms and similar modifications are introduced for the models and the satisfaction relations. Note that, despite these modifications, the two institutions in each pair can be thought of as having very closely related deductive mechanisms.

Let $\mathcal{K}$ be a locally small category with a terminal object 1 and $\mathbf{T} = \langle T, \eta, \mu \rangle$ an algebraic theory in monoid form over $\mathcal{K}$.

DEFINITION 3.6 *Define* $\mathcal{I} = \langle \mathcal{K}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ *as follows:*

(i) $\mathrm{SEN} : \mathcal{K} \to \mathbf{SET}$ *is defined by*

$$\mathrm{SEN}(K) = \mathcal{K}(1, K)^2, \quad \textit{for every} \quad K \in |\mathcal{K}|,$$

and, given $f : K \to L \in \mathrm{Mor}(\mathcal{K}), \mathrm{SEN}(f) : \mathcal{K}(1, K)^2 \to \mathcal{K}(1, L)^2$ *is given by*

$$\mathrm{SEN}(f)(\langle g_1, g_2 \rangle) = \langle f g_1, f g_2 \rangle, \quad \textit{for every} \quad \langle g_1, g_2 \rangle \in \mathcal{K}(1, K)^2.$$

(ii) $\mathrm{MOD} : \mathcal{K} \to \mathbf{CAT}^{op}$ *is defined as follows: For every* $K \in |\mathcal{K}|, \mathrm{MOD}(K)$ *is the category with objects all pairs of the form* $\langle \langle L, \xi \rangle, f \rangle$, *where* $\langle L, \xi \rangle \in |\mathcal{K}^{\mathbf{T}}|$ *and* $f : K \to T(L) \in \mathrm{Mor}(\mathcal{K})$, *and morphisms* $h : \langle \langle L, \xi \rangle, f \rangle \to \langle \langle M, \zeta \rangle, g \rangle, \mathcal{K}^{\mathbf{T}}$-*morphisms* $h : \langle L, \xi \rangle \to \langle M, \zeta \rangle$, *such that* $g = T(h) f$.

$$\begin{tikzcd}
 & T(L) \arrow[r, "\xi"] \arrow[dd, "T(h)"'] & L \arrow[dd, "h"] \\
K \arrow[ur, "f"] \arrow[dr, "g"'] & & \\
 & T(M) \arrow[r, "\zeta"'] & M
\end{tikzcd}$$

Given $k : K \to L \in \mathrm{Mor}(\mathcal{K})$ *the functor* $\mathrm{MOD}(k) : \mathrm{MOD}(L) \to \mathrm{MOD}(K)$ *sends* $\langle \langle M, \xi \rangle, f \rangle$ *to* $\langle \langle M, \xi \rangle, fk \rangle$ *and a morphism* $h : \langle \langle M, \xi \rangle, f \rangle \to \langle \langle N, \zeta \rangle, g \rangle$ *to the morphism* $\mathrm{MOD}(k)(h) : \langle \langle M, \xi \rangle, fk \rangle \to \langle \langle N, \zeta \rangle, gk \rangle$ *with* $\mathrm{MOD}(k)(h) = h$.

$$L \xrightarrow{f} T(M) \xrightarrow{\xi} M$$

$$\Big\downarrow \mathrm{MOD}(k):$$

$$K \xrightarrow{k} L \xrightarrow{f} T(M) \xrightarrow{\xi} M$$

$$\begin{tikzcd}
 & T(M) \arrow[r, "\xi"] \arrow[dd, "T(h)"'] & M \arrow[dd, "h"] \\
L \arrow[ur, "f"] \arrow[dr, "g"'] & & \\
 & T(N) \arrow[r, "\zeta"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T(M) \arrow[r, "\xi"] \arrow[dd, "T(h)"'] & M \arrow[dd, "h"] \\
K \arrow[r, "k"] & L \arrow[ur, "f"] \arrow[dr, "g"'] & & \\
 & & T(N) \arrow[r, "\zeta"'] & N
\end{tikzcd}$$

$$\xrightarrow{\mathrm{MOD}(k)}$$

(iii) For all $K \in |\mathcal{K}|, \langle g_1, g_2\rangle \in \mathrm{SEN}(K)$ and $\langle\langle L,\xi\rangle, f\rangle \in |\mathrm{MOD}(K)|$,

$$\langle\langle L,\xi\rangle,f\rangle \models_K \langle g_1,g_2\rangle \quad \textit{iff} \quad \xi f g_1 = \xi f g_2.$$

$$1 \overset{g_1}{\underset{g_2}{\rightrightarrows}} K \xrightarrow{f} T(L) \xrightarrow{\xi} L$$

The next theorem states that the above construction gives an institution.

THEOREM 3.7 *Let $\mathcal{K}$ be a locally small category with a terminal object* 1, *and $\mathbf{T} = \langle T, \eta, \mu\rangle$ an algebraic theory over $\mathcal{K}$. Then $\mathcal{I} = \langle \mathcal{K}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, as defined in* 3.6, *is an institution.*

Proof:

We only show that MOD is well-defined on morphisms and then verify that the satisfaction condition holds.

First, let $k : K \to L \in \mathrm{Mor}(\mathcal{K})$ and suppose that $h : \langle\langle M,\xi\rangle, f\rangle \to \langle\langle N,\zeta\rangle, g\rangle \in \mathrm{Mor}(\mathrm{MOD}(L))$. Then $T(h)f = g$, whence $T(h)fk = gk$. Thus, $h : \langle\langle M,\xi\rangle, fk\rangle \to \langle\langle N,\zeta\rangle, gk\rangle \in \mathrm{Mor}(\mathrm{MOD}(K))$, as required.

Next, let $k : K \to L \in \mathrm{Mor}(\mathcal{K}), \langle g_1, g_2\rangle \in \mathrm{SEN}(K)$ and $\langle\langle M,\xi\rangle, f\rangle \in |\mathrm{MOD}(L)|$. Then

$$\mathrm{MOD}(k)(\langle\langle M,\xi\rangle,f\rangle) \models_K \langle g_1,g_2\rangle \text{ iff, by definition of } \mathrm{MOD}(k),$$
$$\langle\langle M,\xi\rangle,fk\rangle \models_K \langle g_1,g_2\rangle \text{ iff, by definition of } \models_K,$$
$$\xi f k g_1 = \xi f k g_2 \text{ iff, by definition of } \models_L,$$
$$\langle\langle M,\xi\rangle,f\rangle \models_L \langle kg_1,kg_2\rangle \text{ iff, by definition of } \mathrm{SEN}(k),$$
$$\langle\langle M,\xi\rangle,f\rangle \models_L \mathrm{SEN}(k)(\langle g_1,g_2\rangle),$$

as required. ∎

Now let $\mathcal{I}_{\mathbf{T}} = \mathcal{I}_{\mathcal{K}^{\mathbf{T}}}^{\mathcal{K}_{\mathbf{T}}} = \langle \mathcal{K}_{\mathbf{T}}, \mathrm{EQ}, \mathrm{ALG}, \models^{\mathbf{T}}\rangle$ be the $\mathcal{K}_{\mathbf{T}}$-algebraic institution associated with $\mathcal{K}^{\mathbf{T}}$. We show that $\pi(\mathcal{I})$ is strong left quasi-equivalent to $\pi(\mathcal{I}_{\mathbf{T}})$. This result will provide many examples of pairs of π-institutions being strong quasi-equivalent but not deductively equivalent.

THEOREM 3.8 *Let $\mathcal{I}, \mathcal{I}_{\mathbf{T}} = \mathcal{I}^{\mathcal{K}_{\mathbf{T}}}_{\mathcal{K}^{\mathbf{T}}}$ be the institutions defined in 3.6, 3.1 and $\pi(\mathcal{I}), \pi(\mathcal{I}_{\mathbf{T}})$ the π-institutions associated with $\mathcal{I}, \mathcal{I}_{\mathbf{T}}$, respectively (see Chapter 2). $\pi(\mathcal{I})$ is left strong quasi-equivalent to $\pi(\mathcal{I}_{\mathbf{T}})$.*

Proof:

Let $\langle F_{\mathbf{T}}, U_{\mathbf{T}}, \eta_{\mathbf{T}}, \epsilon_{\mathbf{T}}\rangle : \mathcal{K} \to \mathcal{K}_{\mathbf{T}}$ be the Kleisli adjunction (see Chapter 1). Define $\alpha : \mathrm{SEN} \to \mathcal{P}\,\mathrm{EQ}\,F_{\mathbf{T}}$ by $\alpha_K : \mathrm{SEN}(K) \to \mathcal{P}(\mathrm{EQ}(K))$, with

$$\alpha_K(\langle g_1, g_2\rangle) = \{\langle \eta_K g_1, \eta_K g_2\rangle\}, \quad \text{for every} \quad \langle g_1, g_2\rangle \in \mathrm{SEN}(K),$$

and $\beta : \mathrm{EQ} \to \mathcal{P}\,\mathrm{SEN}\,U_{\mathbf{T}}$ by $\beta_K : \mathrm{EQ}(K) \to \mathcal{P}(\mathrm{SEN}(T(K)))$, with

$$\beta_K(\langle g_1, g_2\rangle) = \{\langle g_1, g_2\rangle\}, \quad \text{for every} \quad \langle g_1, g_2\rangle \in \mathrm{EQ}(K).$$

We first show that α and β are natural transformations. To this end, let $f : K \to L$ $\in \mathrm{Mor}(\mathcal{K})$. We need to show that the following diagram commutes. For every $\langle g_1, g_2\rangle \in \mathrm{SEN}(K)$, we have

$$\begin{tikzcd}
\mathrm{SEN}(K) \arrow[r, "\alpha_K"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathcal{P}(\mathrm{EQ}(K)) \arrow[d, "\mathcal{P}\mathrm{EQ}(\eta_L f)"] \\
\mathrm{SEN}(L) \arrow[r, "\alpha_L"'] & \mathcal{P}(\mathrm{EQ}(L))
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{P}\mathrm{EQ}(F_{\mathbf{T}}(f))(\alpha_K(\langle g_1, g_2\rangle)) &= \mathrm{EQ}(\eta_L f)(\langle \eta_K g_1, \eta_K g_2\rangle) \\
&= \langle \mu_L T(\eta_L f)\eta_K g_1, \mu_L T(\eta_L f)\eta_K g_2\rangle \\
&= \langle \mu_L T(\eta_L) T(f)\eta_K g_1, \mu_L T(\eta_L) T(f)\eta_K g_2\rangle \\
&= \langle T(f)\eta_K g_1, T(f)\eta_K g_2\rangle
\end{aligned}$$

by commutativity of

$$\begin{tikzcd}
T(L) \arrow[r, "T(\eta_L)"] \arrow[dr, "i_{T(L)}"'] & T(T(L)) \arrow[d, "\mu_L"] \\
 & T(L)
\end{tikzcd}$$

$$= \langle \eta_L f g_1, \eta_L f g_2 \rangle$$

by commutativity of
$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[d, "f"'] & T(K) \arrow[d, "T(f)"] \\ L \arrow[r, "\eta_L"'] & T(L) \end{tikzcd}$$

$$\begin{aligned} &= \alpha_L(\langle f g_1, f g_2 \rangle) \\ &= \alpha_L(\mathrm{SEN}(f)(\langle g_1, g_2 \rangle)), \end{aligned}$$

as required. For β, let $f : K \rightarrow L \in \mathrm{Mor}(\mathcal{K}_{\mathbf{T}})$. We need to show that the following diagram commutes. For every $\langle g_1, g_2 \rangle \in \mathrm{EQ}(K)$, we have

$$\begin{tikzcd} \mathrm{EQ}(K) \arrow[r, "\beta_K"] \arrow[d, "\mathrm{EQ}(f)"'] & \mathcal{P}(\mathrm{SEN}(T(K))) \arrow[d, "\mathcal{P}\mathrm{SEN}(\mu_L T(f))"] \\ \mathrm{EQ}(L) \arrow[r, "\beta_L"'] & \mathcal{P}(\mathrm{SEN}(T(L))) \end{tikzcd}$$

$$\begin{aligned} \mathcal{P}\mathrm{SEN}(\mu_L T(f))(\beta_K(\langle g_1, g_2 \rangle)) &= \mathrm{SEN}(\mu_L T(f))(\langle g_1, g_2 \rangle) \\ &= \langle \mu_L T(f) g_1, \mu_L T(f) g_2 \rangle \\ &= \beta_L(\langle \mu_L T(f) g_1, \mu_L T(f) g_2 \rangle) \\ &= \beta_L(\mathrm{EQ}(f)(\langle g_1, g_2 \rangle)), \end{aligned}$$

as required.

Next, we show that $\langle F_{\mathbf{T}}, \alpha \rangle : \pi(\mathcal{I}) \rightarrow \pi(\mathcal{I}_{\mathbf{T}}), \langle U_{\mathbf{T}}, \beta \rangle : \pi(\mathcal{I}_{\mathbf{T}}) \rightarrow \pi(\mathcal{I})$ are interpretations.

For $\langle F_{\mathbf{T}}, \alpha \rangle$, let $K \in |\mathcal{K}|, \Theta \cup \{\langle g_1, g_2 \rangle\} \subseteq \mathrm{SEN}(K)$. Then

$$\langle g_1, g_2 \rangle \in \Theta^c$$

iff, for every $\langle \langle L, \xi \rangle, f \rangle \in |\mathrm{MOD}(K)|$,

$$\langle \langle L, \xi \rangle, f \rangle \models_K \langle \theta_1, \theta_2 \rangle, \text{ for every } \langle \theta_1, \theta_2 \rangle \in \Theta, \text{ implies } \langle \langle L, \xi \rangle, f \rangle \models_K \langle g_1, g_2 \rangle,$$

iff, by the definition of $\models_{K'}$, for every $\langle\langle L, \xi\rangle, f\rangle \in |\mathrm{MOD}(K')|$,

$$\xi f\theta_1 = \xi f\theta_2, \quad \text{for every} \quad \langle\theta_1, \theta_2\rangle \in \Theta, \quad \text{implies} \quad \xi f g_1 = \xi f g_2,$$

iff, by commutativity of

$$\begin{tikzcd} T(L) \arrow[r, "\eta_{T(L)}"] \arrow[dr, "i_{T(L)}"'] & T(T(L)) \arrow[d, "\mu_L"] \\ & T(L) \end{tikzcd}$$

for every $\langle\langle L, \xi\rangle, f\rangle \in |\mathrm{MOD}(K')|$,

$$\xi\mu_L\eta_{T(L)}f\theta_1 = \xi\mu_L\eta_{T(L)}f\theta_2, \quad \text{for every} \quad \langle\theta_1, \theta_2\rangle \in \Theta, \quad \text{implies}$$

$$\xi\mu_L\eta_{T(L)}fg_1 = \xi\mu_L\eta_{T(L)}fg_2,$$

iff, by commutativity of

$$\begin{tikzcd} K' \arrow[r, "\eta_{K'}"] \arrow[d, "f"'] & T(K') \arrow[d, "T(f)"] \\ T(L) \arrow[r, "\eta_{T(L)}"'] & T(T(L)) \end{tikzcd}$$

for every $\langle\langle L, \xi\rangle, f\rangle \in |\mathrm{MOD}(K')|$,

$$\xi\mu_L T(f)\eta_{K'}\theta_1 = \xi\mu_L T(f)\eta_{K'}\theta_2, \quad \text{for every} \quad \langle\theta_1, \theta_2\rangle \in \Theta, \quad \text{implies}$$

$$\xi\mu_L T(f)\eta_{K'}g_1 = \xi\mu_L T(f)\eta_{K'}g_2,$$

iff, by the definition of $\models^{\mathbf{T}}_{K'}$, for every $\langle\langle L, \xi\rangle, f\rangle \in |\mathrm{ALG}(K')|$,

$\langle\langle L, \xi\rangle, f\rangle \models^{\mathbf{T}}_{K'} \langle\eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle$, for every $\langle\theta_1, \theta_2\rangle \in \Theta$, implies $\langle\langle L, \xi\rangle, f\rangle \models^{\mathbf{T}}_{K'} \langle\eta_{K'}g_1, \eta_{K'}g_2\rangle$,

$$\text{iff} \quad \langle\eta_{K'}g_1, \eta_{K'}g_2\rangle \in \{\langle\eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle : \langle\theta_1, \theta_2\rangle \in \Theta\}^c,$$

$$\text{iff} \quad \alpha_{K'}(\langle g_1, g_2\rangle) \in \alpha_{K'}(\Theta)^c,$$

as required.

The proof for $\langle U_{\mathbf{T}}, \beta\rangle$ is more complicated. Let $K' \in |\mathcal{K}_{\mathbf{T}}|, \Theta \cup \{\langle g_1, g_2\rangle\} \subseteq \mathrm{EQ}(K')$. We will first show that, if $\langle g_1, g_2\rangle \in \Theta^c$, then $\beta_{K'}(\langle g_1, g_2\rangle) \in \beta_{K'}(\Theta)^c$ and then that, if $\beta_{K'}(\langle g_1, g_2\rangle) \in \beta_{K'}(\Theta)^c$, then $\langle g_1, g_2\rangle \in \Theta^c$.

Suppose that $\langle g_1, g_2\rangle \in \Theta^c$. Then, for every $\langle\langle L,\xi\rangle, f\rangle \in |\mathrm{ALG}(K)|$,

$$\langle\langle L,\xi\rangle, f\rangle \models^{\mathbf{T}}_{K} \langle\theta_1,\theta_2\rangle, \text{ for every } \langle\theta_1,\theta_2\rangle \in \Theta, \text{ implies } \langle\langle L,\xi\rangle, f\rangle \models^{\mathbf{T}}_{K} \langle g_1, g_2\rangle. \tag{3.2}$$

Now, assume that $\langle\langle L,\xi\rangle, f\rangle \in |\mathrm{MOD}(T(K))|$ is such that $\langle\langle L,\xi\rangle, f\rangle \models_{T(K)} \langle\theta_1,\theta_2\rangle$, for every $\langle\theta_1,\theta_2\rangle \in \beta_K(\Theta)$. Then $\xi f\theta_1 = \xi f\theta_2$, whence $\xi f\mu_K T(\eta_K)\theta_1 = \xi f\mu_K T(\eta_K)\theta_2$, and, therefore, $\xi\mu_L T(f)T(\eta_K)\theta_1 = \xi\mu_L T(f)T(\eta_K)\theta_2$. Thus, $\langle\langle L,\xi\rangle, f\eta_K\rangle \models^{\mathbf{T}}_{K} \langle\theta_1,\theta_2\rangle$ and , by (3.2), $\langle\langle L,\xi\rangle, f\eta_K\rangle \models^{\mathbf{T}}_{K} \langle g_1, g_2\rangle$. Following the same steps backwards, we conclude that $\langle\langle L,\xi\rangle, f\rangle \models_{T(K)} \langle g_1, g_2\rangle$. Thus, $\beta_K(\langle g_1, g_2\rangle) \in \beta_K(\Theta)^c$, as required.

Conversely, suppose that $\beta_K(\langle g_1, g_2\rangle) \in \beta_K(\Theta)^c$. Then,

$$\text{for every } \langle\langle L,\xi\rangle, f\rangle \in |\mathrm{MOD}(T(K))|,$$

$$\langle\langle L,\xi\rangle, f\rangle \models_{T(K)} \langle\theta_1,\theta_2\rangle, \text{ for every } \langle\theta_1,\theta_2\rangle \in \Theta, \text{ implies } \langle\langle L,\xi\rangle, f\rangle \models_{T(K)} \langle g_1, g_2\rangle. \tag{3.3}$$

Now, assume that $\langle\langle L,\xi\rangle, f\rangle \in |\mathrm{ALG}(K)|$ is such that $\langle\langle L,\xi\rangle, f\rangle \models^{\mathbf{T}}_{K} \langle\theta_1,\theta_2\rangle$, for every $\langle\theta_1,\theta_2\rangle \in \Theta$. Then $\xi\mu_L T(f)\theta_1 = \xi\mu_L T(f)\theta_2$, i.e., $\langle\langle L,\xi\rangle, \mu_L T(f)\rangle \models_{T(K)} \langle\theta_1,\theta_2\rangle$ and , by (3.3), $\langle\langle L,\xi\rangle, \mu_L T(f)\rangle \models_{T(K)} \langle g_1, g_2\rangle$. Following the same steps backwards, we conclude that $\langle\langle L,\xi\rangle, f\rangle \models^{\mathbf{T}}_{K} \langle g_1, g_2\rangle$. Thus, $\langle g_1, g_2\rangle \in \Theta^c$, as required.

Finally, we need to show that, for all $K \in |\mathcal{K}|, \langle g_1, g_2\rangle \in \mathrm{SEN}(K)$,

$$\mathrm{SEN}(\eta_K)(\langle g_1, g_2\rangle)^c = \beta_K(\alpha_K(\langle g_1, g_2\rangle))^c,$$

and, for all $K \in |\mathcal{K}_{\mathbf{T}}|, \langle g_1, g_2\rangle \in \mathrm{EQ}(K)$,

$$\mathrm{EQ}(\epsilon_K)(\alpha_{T(K)}(\beta_K(\langle g_1, g_2\rangle)))^c = \{\langle g_1, g_2\rangle\}^c.$$

We have

$$\begin{aligned} \beta_K(\alpha_K(\langle g_1, g_2\rangle))^c &= \alpha_K(\langle g_1, g_2\rangle)^c \\ &= \{\langle \eta_K g_1, \eta_K g_2\rangle\}^c \\ &= \mathrm{SEN}(\eta_K)(\langle g_1, g_2\rangle)^c, \end{aligned}$$

as required, and

$$\begin{aligned}
\mathrm{EQ}(\epsilon_{K})(\alpha_{T(K)}(\beta_{K}(\langle g_1, g_2\rangle)))^c &= \mathrm{EQ}(i_{T(K)})(\alpha_{T(K)}(\langle g_1, g_2\rangle))^c \\
&= \mathrm{EQ}(i_{T(K)})(\langle \eta_{T(K)} g_1, \eta_{T(K)} g_2\rangle)^c \\
&= \{\langle \mu_K T(i_{T(K)})\eta_{T(K)} g_1, \mu_K T(i_{T(K)})\eta_{T(K)} g_2\rangle\}^c \\
&= \{\langle \mu_K \eta_{T(K)} g_1, \mu_K \eta_{T(K)} g_2\rangle\}^c \\
&= \{\langle g_1, g_2\rangle\}^c, \quad \text{as required.}
\end{aligned}$$

■

Deductive π-Institutions

In this section, part (iii) of Corollary 3.5 is applied to the, so-called, deductive π-institutions, that naturally arise from deductive systems, and a theorem is proved that provides a relationship between the algebraizability of a deductive system and the algebraizability of the corresponding deductive π-institution.

Recall from the last section of Chapter 2 that, given a language type $\mathcal{L}$ and a finitary k-deductive system $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}}\rangle$ over $\mathcal{L}$, we can define a π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ by letting **SIGN** be the one-element category with the single object V and morphisms all assignments $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$. $\mathrm{SEN} : \mathbf{SIGN} \to \mathbf{SET}$ is given by $\mathrm{SEN}(V) = \mathrm{Tm}_{\mathcal{L}}(V)^k$ and $\mathrm{SEN}(h)(\phi) = h^*(\phi)$, for every $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$, where h^* denotes the unique endomorphism on the $\mathcal{L}$–term algebra extending the assignment h. Finally, $C_V(\Gamma) = C_{\mathcal{S}}(\Gamma)$, for every $\Gamma \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^k$. We call the π-institution, thus obtained, the **deductive π-institution associated with $\mathcal{S}$**. In Lemma 2.46 it was proved that it is a term π-institution.

Given a language type $\mathcal{L}$, we can construct an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in **SET**, whose Eilenberg-Moore category of **T**-algebras $\mathbf{SET}^{\mathbf{T}}$ is isomorphic to the category of the variety of all $\mathcal{L}$-algebras. Recall that, given $X \in |\mathbf{SET}|$, we denote by $\overline{X}$ a disjoint copy of X, constructed in some canonical way, and by $\mathrm{Tm}_{\mathcal{L}}(X)$ the set of all $\mathcal{L}$-terms over $\overline{X}$. Briefly, we have

$$T(X) = \mathrm{Tm}_{\mathcal{L}}(X), \quad \text{for every} \quad X \in |\mathbf{SET}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{SET})$, $T(f) : \mathrm{Tm}_{\mathcal{L}}(X) \to \mathrm{Tm}_{\mathcal{L}}(Y)$ is the unique extension of f to $\mathcal{L}$-terms. It is formally defined by recursion on the structure of $\mathcal{L}$−terms as follows:

- $T(f)(\overline{x}) = \overline{f(x)}$, for every $x \in X$, and
- $T(f)(\lambda(t_0, \dots, t_{\rho(\lambda)-1})) = \lambda(T(f)(t_0), \dots, T(f)(t_{\rho(\lambda)-1}))$, for all $\lambda \in \Lambda, t_0, \dots, t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(X)$.

Moreover, $\eta_X : X \to \mathrm{Tm}_{\mathcal{L}}(X)$ is the map given by $\eta_X(x) = \overline{x}$, for every $x \in X$, and $\mu_X : \mathrm{Tm}_{\mathcal{L}}(\mathrm{Tm}_{\mathcal{L}}(X)) \to \mathrm{Tm}_{\mathcal{L}}(X)$ combines $\mathcal{L}$-terms over $\mathcal{L}$-terms to simple $\mathcal{L}$-terms and is defined formally by recursion on the structure of $\mathcal{L}$-terms over $\mathrm{Tm}_{\mathcal{L}}(X)$ as follows

- $\mu_X(\overline{t}) = t$, for every $t \in \mathrm{Tm}_{\mathcal{L}}(X)$, and
- $\mu_X(\lambda(t_0, \dots, t_{\rho(\lambda)-1})) = \lambda(\mu_X(t_0), \dots, \mu_X(t_{\rho(\lambda)-1}))$, for all $\lambda \in \Lambda, t_0, \dots, t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(\mathrm{Tm}_{\mathcal{L}}(X))$.

Note that $\mathbf{SIGN}$ is the full subcategory of $\mathbf{SET_T}$ with the single object V.

Given an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$, the corresponding $\mathbf{T}$-algebra under the above isomorphism is $\mathbf{A}^* = \langle A, \xi_{\mathbf{A}} \rangle$, where the structure map $\xi_{\mathbf{A}} : \mathrm{Tm}_{\mathcal{L}}(A) \to A$ is defined by recursion on the structure of $\mathcal{L}$-terms as follows

- $\xi_{\mathbf{A}}(\overline{a}) = a$, for every $a \in A$, and
- $\xi_{\mathbf{A}}(\lambda(t_0, \dots, t_{\rho(\lambda)-1})) = \lambda^{\mathbf{A}}(\xi_{\mathbf{A}}(t_0), \dots, \xi_{\mathbf{A}}(t_{\rho(\lambda)-1}))$, for all $\lambda \in \Lambda, t_0, \dots, t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(A)$.

Given an $\mathcal{L}$-term t over V, let us denote by $f_t \in \mathbf{SET_T}(1, V)$ the set map from the singleton $1 = \{\emptyset\}$ to $\mathrm{Tm}_{\mathcal{L}}(V)$, with $f_t(\emptyset) = t$ and, given a class K of $\mathcal{L}$-algebras we denote by K^* the full subcategory of $\mathbf{SET^T}$ with objects $\{\mathbf{A}^* : \mathbf{A} \in K\}$.

LEMMA 3.9 *Let* $t \in \mathrm{Tm}_{\mathcal{L}}(V), \mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ *an* $\mathcal{L}$*-algebra and* $\vec{a} : V \to A$*. Then, if* $\mathbf{A}^* = \langle A, \xi_{\mathbf{A}} \rangle$,

$$t^{\mathbf{A}}(\vec{a}) = \xi_{\mathbf{A}} \mu_A T(\eta_A \vec{a}) f_t(\emptyset).$$

$$\{\emptyset\} \xrightarrow{f_t} \mathrm{Tm}_{\mathcal{L}}(V) \xrightarrow{T(\vec{a})} \mathrm{Tm}_{\mathcal{L}}(A) \xrightarrow{T(\eta_A)}$$
$$\mathrm{Tm}_{\mathcal{L}}(\mathrm{Tm}_{\mathcal{L}}(A)) \xrightarrow{\mu_A} \mathrm{Tm}_{\mathcal{L}}(A) \xrightarrow{\xi_{\mathbf{A}}} A$$

Proof:

First, note that

$$\begin{aligned}
\xi_{\mathbf{A}} \mu_A T(\eta_A \vec{a}) f_t(\emptyset) &= \xi_{\mathbf{A}} \mu_A T(\eta_A) T(\vec{a}) f_t(\emptyset) \quad \text{(since } T \text{ is a functor)} \\
&= \xi_{\mathbf{A}} T(\vec{a}) f_t(\emptyset) \quad \text{(since } \mu_A T(\eta_A) = 1_{T(A)}\text{)} \\
&= \xi_{\mathbf{A}} T(\vec{a})(t). \quad \text{(since } f_t(\emptyset) = t\text{)}
\end{aligned}$$

We now work by recursion on the structure of an $\mathcal{L}$-term. If $t = \overline{v} \in \overline{V}$, then

$$\xi_{\mathbf{A}}(T(\vec{a})(\overline{v})) = \xi_{\mathbf{A}}(\overline{\vec{a}(v)}) = \vec{a}(v) = \overline{v}^{\mathbf{A}}(\vec{a}),$$

as required. Next, let $t = \lambda(t_0, \ldots, t_{\rho(\lambda)-1})$, for some $\lambda \in \Lambda, t_0, \ldots, t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(V)$, and suppose that $\xi_{\mathbf{A}}(T(\vec{a})(t_i)) = t_i^{\mathbf{A}}(\vec{a})$, for every $i < \rho(\lambda)$. Then

$$\begin{aligned}
\xi_{\mathbf{A}}(T(\vec{a})(t)) &= \xi_{\mathbf{A}}(T(\vec{a})(\lambda(t_0, \ldots, t_{\rho(\lambda)-1}))) \\
&= \xi_{\mathbf{A}}(\lambda(T(\vec{a})(t_0), \ldots, T(\vec{a})(t_{\rho(\lambda)-1}))) \text{ (by the definition of } T(\vec{a})\text{)} \\
&= \lambda^{\mathbf{A}}(\xi_{\mathbf{A}}(T(\vec{a})(t_0)), \ldots, \xi_{\mathbf{A}}(T(\vec{a})(t_{\rho(\lambda)-1}))) \text{ (by the definition of } \xi_{\mathbf{A}}\text{)} \\
&= \lambda^{\mathbf{A}}(t_0^{\mathbf{A}}(\vec{a}), \ldots, t_{\rho(\lambda)-1}^{\mathbf{A}}(\vec{a})) \text{ (by the induction hypothesis)} \\
&= t^{\mathbf{A}}(\vec{a}),
\end{aligned}$$

as required. ■

LEMMA 3.10 *Let* $t \in \mathrm{Tm}_{\mathcal{L}}(V), \mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ *an* $\mathcal{L}$*-algebra and* $\vec{a} : V \to T(A)$*. Then, if* $\mathbf{A}^* = \langle A, \xi_{\mathbf{A}} \rangle$,

$$\xi_{\mathbf{A}} \mu_A T(\vec{a}) f_t(\emptyset) = t^{\mathbf{A}}(\xi_{\mathbf{A}} \vec{a}).$$

$$\{\emptyset\} \xrightarrow{f_t} \mathrm{Tm}_{\mathcal{L}}(V) \xrightarrow{T(\vec{a})} \mathrm{Tm}_{\mathcal{L}}(\mathrm{Tm}_{\mathcal{L}}(A)) \xrightarrow{\mu_A} \mathrm{Tm}_{\mathcal{L}}(A) \xrightarrow{\xi_{\mathbf{A}}} A$$

Proof:

Note, first, that $\xi_{\mathbf{A}}\mu_A T(\vec{a})f_t(\emptyset) = \xi_{\mathbf{A}}\mu_A T(\vec{a})(t)$. We apply again recursion on the structure of $\mathcal{L}$-terms. For $t = \overline{v} \in \overline{V}$, we have

$$\xi_{\mathbf{A}}\mu_A T(\vec{a})(\overline{v}) = \xi_{\mathbf{A}}\mu_A(\overline{\vec{a}(v)}) = \xi_{\mathbf{A}}(\vec{a}(v)) = \overline{v}^{\mathbf{A}}(\xi_{\mathbf{A}}\vec{a}),$$

as required. Next, let $t = \lambda(t_0, \ldots, t_{\rho(\lambda)-1})$, for some $\lambda \in \Lambda, t_0, \ldots, t_{\rho(\lambda)-1} \in \mathrm{Tm}_{\mathcal{L}}(V)$, and suppose that $\xi_{\mathbf{A}}\mu_A T(\vec{a})(t_i) = t_i^{\mathbf{A}}(\xi_{\mathbf{A}}\vec{a})$, for every $i < \rho(\lambda)$. Then

$$\begin{aligned}
\xi_{\mathbf{A}}\mu_A T(\vec{a})(t) &= \xi_{\mathbf{A}}\mu_A T(\vec{a})(\lambda(t_0, \ldots, t_{\rho(\lambda)-1}))\\
&= \xi_{\mathbf{A}}\mu_A(\lambda(T(\vec{a})(t_0) \ldots, T(\vec{a})(t_{\rho(\lambda)-1}))) \text{ (by the definition of } T(\vec{a}))\\
&= \xi_{\mathbf{A}}(\lambda(\mu_A(T(\vec{a})(t_0)), \ldots, \mu_A(T(\vec{a})(t_{\rho(\lambda)-1})))) \text{ (by the defin. of } \mu_A)\\
&= \lambda^{\mathbf{A}}(\xi_{\mathbf{A}}(\mu_A(T(\vec{a})(t_0))), \ldots, \xi_{\mathbf{A}}(\mu_A(T(\vec{a})(t_{\rho(\lambda)-1})))) \text{ (by defin. of } \xi_{\mathbf{A}})\\
&= \lambda^{\mathbf{A}}(t_0^{\mathbf{A}}(\xi_{\mathbf{A}}\vec{a}), \ldots, t_{\rho(\lambda)-1}^{\mathbf{A}}(\xi_{\mathbf{A}}\vec{a})) \text{ (by the induction hypothesis)}\\
&= t^{\mathbf{A}}(\xi_{\mathbf{A}}\vec{a}), \text{ (by the definition of } t^{\mathbf{A}})
\end{aligned}$$

as required. ∎

THEOREM 3.11 *Let $\mathcal{L}$ be a language type, K a class of $\mathcal{L}$-algebras, $\mathcal{S}_K = \langle \mathbf{Tm}_{\mathcal{L}}(V)^2, \models_K\rangle$ the equational 2-deductive system of K and $\mathbf{T} = \langle T, \eta, \mu\rangle$ the algebraic theory that corresponds to the variety of all $\mathcal{L}$-algebras. Then $\mathcal{I}_{\mathcal{S}_K}$ and $\pi(\mathcal{I}_{K^\bullet}^{\mathbf{SIGN}})$ are deductively auto-equivalent π-institutions.*

Proof:

Clearly, the two given π-institutions have the same signature categories. So it suffices to exhibit natural transformations $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$ and $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$, such that $\langle I_{\mathbf{SIGN}}, \alpha\rangle : \mathcal{I}_{\mathcal{S}_K} \to \pi(\mathcal{I}_{K^\bullet}^{\mathbf{SIGN}}), \langle I_{\mathbf{SIGN}}, \beta\rangle : \pi(\mathcal{I}_{K^\bullet}^{\mathbf{SIGN}}) \to \mathcal{I}_{\mathcal{S}_K}$ are inverse interpretations.

Define $\alpha_V : \mathrm{Tm}_{\mathcal{L}}(V)^2 \to \mathcal{P}(\mathbf{SET_T}(1, V)^2)$ by

$$\alpha_V(\langle t_0, t_1\rangle) = \{\langle f_{t_0}, f_{t_1}\rangle\}, \quad \text{for all} \quad t_0, t_1 \in \mathrm{Tm}_{\mathcal{L}}(V),$$

and $\beta_V : \mathbf{SET_T}(1, V)^2 \to \mathcal{P}(\mathrm{Tm}_{\mathcal{L}}(V)^2)$ by

$$\beta_V(\langle f_0, f_1\rangle) = \{\langle f_0(\emptyset), f_1(\emptyset)\rangle\}, \quad \text{for all} \quad f_0, f_1 \in \mathbf{SET_T}(1, V).$$

We first show that α and β are natural transformations. To this end, let $h : V \to \mathrm{Tm}_{\mathcal{L}}(V) \in \mathrm{Mor}(\mathbf{SIGN})$. We need to show that the following diagram commutes. If $\langle t_0, t_1\rangle \in \mathrm{Tm}_{\mathcal{L}}(V)^2$, we have

$$\begin{array}{ccc}
\mathrm{Tm}_{\mathcal{L}}(V)^2 & \xrightarrow{\alpha_V} & \mathcal{P}(\mathbf{SET_T}(1, V)^2) \\
{\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h} \\
\mathrm{Tm}_{\mathcal{L}}(V)^2 & \xrightarrow[\alpha_V]{} & \mathcal{P}(\mathbf{SET_T}(1, V)^2)
\end{array}$$

$$\begin{aligned}
h(\alpha_V(\langle t_0, t_1\rangle)) &= h(\langle f_{t_0}, f_{t_1}\rangle) \\
&= \langle h \circ f_{t_0}, h \circ f_{t_1}\rangle \\
&= \langle h^* f_{t_0}, h^* f_{t_1}\rangle \\
&= \langle f_{h^*(t_0)}, f_{h^*(t_1)}\rangle \\
&= \alpha_V(h^*(\langle t_0, t_1\rangle)),
\end{aligned}$$

as required. The proof for β is similar.

Now, we show that $\langle I_{\mathbf{SIGN}}, \alpha\rangle : \mathcal{I}_{S_K} \to \pi(\mathcal{I}_{K^*}^{\mathbf{SIGN}})$ is an interpretation. To this end, let $E \cup \{\langle t_0, t_1\rangle\} \subseteq \mathrm{Tm}_{\mathcal{L}}(V)^2$. We first show that, if $\langle t_0, t_1\rangle \in E^c$, then $\langle f_{t_0}, f_{t_1}\rangle \in \{\langle f_{e_0}, f_{e_1}\rangle : \langle e_0, e_1\rangle \in E\}^c$.

If $\langle t_0, t_1\rangle \in E^c$, then, for every $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle \in K, \vec{a} : V \to A$,

$$e_0^{\mathbf{A}}(\vec{a}) = e_1^{\mathbf{A}}(\vec{a}), \quad \text{for every} \quad \langle e_0, e_1\rangle \in E, \quad \text{implies} \quad t_0^{\mathbf{A}}(\vec{a}) = t_1^{\mathbf{A}}(\vec{a}). \tag{3.4}$$

Now, suppose that $\langle\langle A, \xi_{\mathbf{A}}\rangle, f\rangle \in |\mathrm{ALG}(V)|$, such that $\langle\langle A, \xi_{\mathbf{A}}\rangle, f\rangle \models_V \langle f_{e_0}, f_{e_1}\rangle$, for every $\langle e_0, e_1\rangle \in E$. Then $\xi_{\mathbf{A}}\mu_A T(f) f_{e_0} = \xi_{\mathbf{A}}\mu_A T(f) f_{e_1}$, for every $\langle e_0, e_1\rangle \in E$, whence, by Lemma 3.10, $e_0^{\mathbf{A}}(\xi_{\mathbf{A}} f) = e_1^{\mathbf{A}}(\xi_{\mathbf{A}} f)$, for every $\langle e_0, e_1\rangle \in E$. Therefore, by (3.4), $t_0^{\mathbf{A}}(\xi_{\mathbf{A}} f) = t_1^{\mathbf{A}}(\xi_{\mathbf{A}} f)$. Thus, by Lemma 3.10 again, we obtain $\xi_{\mathbf{A}}\mu_A T(f) f_{t_0} = \xi_{\mathbf{A}}\mu_A T(f) f_{t_1}$, i.e., $\langle\langle A, \xi_{\mathbf{A}}\rangle, f\rangle \models_V \langle f_{t_0}, f_{t_1}\rangle$, as required.

Suppose, conversely, that $\langle f_{t_0}, f_{t_1}\rangle \in \{\langle f_{e_0}, f_{e_1}\rangle : \langle e_0, e_1\rangle \in E\}^c$. Then, for every $\langle\langle A, \xi_{\mathbf{A}}\rangle, f\rangle \in |\mathrm{ALG}(V)|$,

$$\xi_{\mathbf{A}}\mu_A T(f) f_{e_0} = \xi_{\mathbf{A}}\mu_A T(f) f_{e_1}, \quad \text{for every} \quad \langle e_0, e_1\rangle \in E,$$

$$\text{implies} \quad \xi_{\mathbf{A}}\mu_A T(f)f_{t_0} = \xi_{\mathbf{A}}\mu_A T(f)f_{t_1}. \tag{3.5}$$

Let $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle \in K, \vec{a} : V \to A$ be such that $e_0^{\mathbf{A}}(\vec{a}) = e_1^{\mathbf{A}}(\vec{a})$, for every $\langle e_0, e_1 \rangle \in E$. Then, by Lemma 3.9, $\xi_{\mathbf{A}}\mu_A T(\eta_A\vec{a})f_{e_0} = \xi_{\mathbf{A}}\mu_A T(\eta_A\vec{a})f_{e_1}$, for every $\langle e_0, e_1 \rangle \in E$. Hence, by (3.5), $\xi_{\mathbf{A}}\mu_A T(\eta_A\vec{a})f_{t_0} = \xi_{\mathbf{A}}\mu_A T(\eta_A\vec{a})f_{t_1}$, whence, by Lemma 3.9 again, $t_0^{\mathbf{A}}(\vec{a}) = t_1^{\mathbf{A}}(\vec{a})$. Therefore $\langle t_0, t_1 \rangle \in E^c$, as required.

The proof that $\langle I_{\mathbf{SIGN}}, \beta \rangle : \pi(\mathcal{I}_{K}^{\mathbf{SIGN}}) \to \mathcal{I}_{\mathcal{S}_K}$ is an interpretation is similar.

Finally, for every $\langle t_0, t_1 \rangle \in \mathrm{Tm}_{\mathcal{L}}(V)^2$,

$$\begin{aligned} \beta_V(\alpha_V(\langle t_0, t_1 \rangle))^c &= \beta_V(\langle f_{t_0}, f_{t_1} \rangle) \\ &= \{\langle f_{t_0}(\emptyset), f_{t_1}(\emptyset) \rangle\}^c \\ &= \{\langle t_0, t_1 \rangle\}^c, \end{aligned}$$

and, for every $\langle f_0, f_1 \rangle \in \mathbf{SET_T}(1, V)^2$,

$$\begin{aligned} \alpha_V(\beta_V(\langle f_0, f_1 \rangle))^c &= \alpha_V(\langle f_0(\emptyset), f_1(\emptyset) \rangle)^c \\ &= \{\langle f_{f_0(\emptyset)}, f_{f_1(\emptyset)} \rangle\}^c \\ &= \{\langle f_0, f_1 \rangle\}^c, \end{aligned}$$

whence $\langle I_{\mathbf{SIGN}}, \alpha \rangle, \langle I_{\mathbf{SIGN}}, \beta \rangle$ are in fact inverse interpretations, as required. ■

Now, we are ready to prove the main theorem of the present section giving the relationship between algebraizability of a k-deductive system $\mathcal{S}$ and algebraizability of its associated π-institution $\mathcal{I}_{\mathcal{S}}$.

THEOREM 3.12 *Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}} \rangle$ a finitary k-deductive system over $\mathcal{L}$. If $\mathcal{S}$ is algebraizable then $\mathcal{I}_{\mathcal{S}}$ is algebraizable.*

Proof:

Suppose that $\mathcal{S}$ is algebraizable with equivalent algebraic semantics (see [6]) the class K of $\mathcal{L}$-algebras. This means that $\mathcal{S}$ is equivalent to the 2-deductive system $\mathcal{S}_K = \langle \mathbf{Tm}_{\mathcal{L}}(V)^2, \models_K \rangle$, having as its consequence relation the semantical equational consequence relation $\models_K$ of K. By Theorem 2.48, this implies that the π-institutions

$\mathcal{I}_{\mathcal{S}}$ and $\mathcal{I}_{\mathcal{S}_K}$ are deductively auto-equivalent. By Theorem 3.11, $\mathcal{I}_{\mathcal{S}_K}$ is deductively auto-equivalent to the algebraic π-institution $\mathcal{I}_{K}^{\mathbf{SIGN}}$. Therefore $\mathcal{S}$ is deductively auto-equivalent to $\mathcal{I}_{K}^{\mathbf{SIGN}}$ and, hence, algebraizable. ■

Actually, the proof of Theorem 3.12 gives the stronger result that, if $\mathcal{S}$ is algebraizable then $\mathcal{I}_{\mathcal{S}}$ is auto-algebraizable in a sense that will be made precise in the following chapter.

Algebraizing the Equational Institution

In this section, the, so-called, equational institution, an institution that naturally represents a version of equational logic, is constructed. In this version the operation symbols of each equational signature have no fixed arity. Instead, the arity of each symbol varies over different models of the same signature. Then, an algebraic institution $\mathcal{I}_Q = \mathcal{I}_Q^{\mathbf{SET_T}}$ is used to algebraize the equational institution. The theory $\mathbf{T}$ in $\mathbf{SET}$ over which this algebraic institution is constructed is discussed briefly in the second subsection, but is presented in detail in Chapter 6 of the thesis.

The Equational Institution

In this subsection, the basic construction of the equational institution is provided. It represents a version of equational logic in which the operation symbols of each language type do not have fixed arities. More precisely, for every language type $\mathcal{L}$, other than the empty type, there exist algebras in which the operations corresponding to the same operation symbol of the type have different arities. These arities must be finite but vary.

A countably infinite set V, called **set of variables,** is fixed in advance and well-ordered and, as usual, the category of all small sets is denoted by $\mathbf{SET}$. The definition of a term is given first.

DEFINITION 3.13 *Let* $X \in |\mathbf{SET}|$. *We define the* **set of** X**-terms** $\mathrm{Tm}_X(V) \in |\mathbf{SET}|$, *to be the smallest set with*

(i) $V \subseteq \mathrm{Tm}_X(V)$ *and*

(ii) If $x \in X, n \in \omega$ *and* $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)$, *with* $t_{n-1} \neq v_{n-1}$, *then*

$$x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V).$$

Next, the definition of an algebra is provided.

DEFINITION 3.14 *Let* $X \in |\mathbf{SET}|$ *and* $\rho : X \to \omega$ *be a rank function. By an* (X,ρ)-**algebra** *we mean an* $\mathcal{L}$*-algebra* $\mathbf{A} = \langle A, \alpha\rangle$, *where* $\mathcal{L} = \langle X, \rho\rangle$, *i.e., a set* A *together with a mapping* $\alpha : X \to \mathrm{Cl}(A)$, *where* $\mathrm{Cl}(A) = \bigcup_{k=0}^{\infty} A^{A^k}$, *such that* $\alpha(x) \in A^{A^{\rho(x)}}$, *for every* $x \in X$. *By an* X-**algebra** *we mean an* (X,ρ)*-algebra for some rank function* ρ *on* X.

Given $\vec{a} : V \to A$, we denote by a_i the element $\vec{a}(v_i), i < \omega$. If $\mathbf{A} = \langle A, \alpha\rangle$ is an (X,ρ)-algebra, $x \in X$ and $\alpha(x) \in A^{A^{\rho(x)}}$, then we use the notation $\alpha(x)(\vec{a}), \vec{a} : V \to A$, to denote the element $\alpha(x)(a_0, \ldots, a_{\rho(x)-1}) \in A$.

Next, the notion of homomorphism is defined.

DEFINITION 3.15 *Given two* X*-algebras* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *with corresponding rank functions* $\rho_{\mathbf{A}}, \rho_{\mathbf{B}}$, *an* X-**homomorphism** $h : \mathbf{A} \to \mathbf{B}$ *is a map* $h : A \to B$ *such that, for every* $x \in X$,

$$h(\alpha(x)(\vec{a})) = \beta(x)(h(\vec{a})), \quad \textit{for every} \quad \vec{a} : V \to A,$$

where $h(\vec{a})_i = h(a_i)$, *for every* $i \in \omega$.

The collection of all X-algebras together with all X-homomorphisms between them forms a category, called the **category of** X**-algebras** and denoted by $\mathbf{ALG}_X$.

Finally, before the definition of the equational institution, the formal definitions of the **evaluation** of an X-term in an X-algebra and that of the **extension** of a given set map $f : X \to \mathrm{Tm}_Y(V)$ to a map $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ must be given.

DEFINITION 3.16 *Let* $X \in |\mathbf{SET}|$ *and* $\mathbf{A} = \langle A, \alpha\rangle$ *an* X*-algebra with rank function* $\rho_{\mathbf{A}} : X \to \omega$. *Define* $e^{\mathbf{A}} : \mathrm{Tm}_X(V) \times A^V \to A$ *by recursion on the structure of* X*-terms, as follows:*

(i) $e^{\mathbf{A}}(v, \vec{a}) = \vec{a}(v)$, *for all* $v \in V, \vec{a} : V \to A$,

(ii) $e^{\mathbf{A}}(x(t_0,\dots,t_{n-1}),\vec{a}) = \alpha(x)((\langle e^{\mathbf{A}}(t_0,\vec{a}),\dots,e^{\mathbf{A}}(t_{n-1},\vec{a}),a_n,a_{n+1},\dots\rangle)$, *for all* $x \in X, n \in \omega, t_0,\dots,t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}, \vec{a} \in A^V$.

Define $\alpha^* : \mathrm{Tm}_X(V) \to \mathrm{Cl}(A)$ *as follows: The rank* $\rho(\alpha^*(t))$ *is defined by*

- $\rho(\alpha^*(v_i)) = i+1$, *for every* $v_i \in V$,
- $\rho(\alpha^*(x(t_0,\dots,t_{n-1}))) = \max\{\rho_{\mathbf{A}}(x), \rho(t_0),\dots,\rho(t_{\min\{n,\rho_{\mathbf{A}}(x)\}-1})\}$, *for all* $x \in X$, $t_0,\dots,t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$

and

$$\alpha^*(t)(\vec{a}) = e^{\mathbf{A}}(t,\vec{a}), \quad \textit{for all} \quad t \in \mathrm{Tm}_X(V), \vec{a} \in A^V.$$

DEFINITION 3.17 *Let* $X \in |\mathbf{SET}|$, *as before. Define a function*

$$R_X : \mathrm{Tm}_X(V) \times \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)^k \to \mathrm{Tm}_X(V)$$

by recursion on the structure of X-terms as follows:

(i)

$$R_X(v_i,\langle s_0,\dots,s_{m-1}\rangle) = \begin{cases} s_i, & i < m \\ v_i, & i \geq m \end{cases}$$

for all $m \in \omega, s_0,\dots,s_{m-1} \in \mathrm{Tm}_X(V)$,

(ii)

$$R_X(x(t_0,\dots,t_{n-1}),\vec{s}) = \begin{cases} x(R_X(t_0,\vec{s}),\dots,R_X(t_{k-1},\vec{s}), & \textit{if } m \leq n \textit{ or } n < m \\ & \textit{and } s_i = v_i\ \forall i \geq m \\ x(R_X(t_0,\vec{s}),\dots,R_X(t_{n-1},\vec{s}),s_n,\dots,s_{k-1}), & \textit{if } n < m \end{cases}$$

for all $x \in X, n \in \omega, t_0,\dots,t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, *and all* $m \in \omega, \vec{s} \in \mathrm{Tm}_X(V)^m$, *where, in the first branch,* $k = \max\{l : R_X(t_l,\vec{s}) \neq v_l\}$, *and, in the second branch,* $k = \max\{l : s_l \neq v_l\}$.

In other words, it is understood that the last, say k-th, term inside the parenthesis on the right, i.e., $R_X(t_{k-1},\vec{s}), 0 \leq k < n$, *if* $m \leq n$, *and either* $R_X(t_{k-1},\vec{s})$ *or* $s_{k-1}, 0 \leq k < m$, *if* $n < m$, *must be the last term that is not equal to the variable* v_{k-1}.

DEFINITION 3.18 *Let $X, Y \in |\mathbf{SET}|$ and $f : X \to \mathrm{Tm}_Y(V)$. Define $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ by recursion on the structure of X-terms as follows:*

(i) $f^*(v) = v$, *for every* $v \in V$,

(ii) $f^*(x(t_0, \ldots, t_{n-1})) = R_Y(f(x), \langle f^*(t_0), \ldots, f^*(t_{n-1})\rangle)$, *for all* $x \in X, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$.

In the sequel, we write $f : X \rightharpoondown Y$ to denote a **SET**-map $f : X \to \mathrm{Tm}_Y(V)$, as above. Given two such maps $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$, their *composition* $g \circ f : X \rightharpoondown Z$ is defined to be

$$g \circ f = g^* f.$$

We denote by **SIGN** the category having as collection of objects $|\mathbf{SET}|$ and as its collections of morphisms

$$\mathbf{SIGN}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{SET}(X, \mathrm{Tm}_Y(V))\},$$

for all $X, Y \in |\mathbf{SET}|$. This category, which is denoted by **FACA** in chapter 6 of the thesis (see Theorem 6.7), has as its composition the composition $\circ$ as defined above and its identity arrows $j_X : X \rightharpoondown X$ are the set maps $j_X : X \to \mathrm{Tm}_X(V)$, with

$$j_X(x) = x(), \quad \text{for every} \quad x \in X.$$

The definition of the equational institution follows.

DEFINITION 3.19 *Define $\mathcal{EQ} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ by letting*

(i) **SIGN** *be the category just defined.*

(ii) $\mathrm{SEN} : \mathbf{SIGN} \to \mathbf{SET}$ *sends an object* $X \in |\mathbf{SIGN}|$ *to the set* $\mathrm{Tm}_X(V)^2$ *and a morphism* $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{SIGN})$ *to the set map* $\mathrm{SEN}(f) = (f^*)^2 : \mathrm{SEN}(X) \to \mathrm{SEN}(Y)$.

We usually denote $\langle t_0, t_1\rangle \in \mathrm{Tm}_X(V)^2$ *by* $t_0 \approx t_1$.

(iii) MOD : **SIGN** $\rightarrow$ **CAT**^{op} *sends an object* $X \in |\mathbf{SIGN}|$ *to the category* $\mathbf{ALG}_X$ *and a morphism* $f : X \rightarrow Y \in$ Mor(**SIGN**) *to the functor* MOD(f) : $\mathbf{ALG}_Y \rightarrow \mathbf{ALG}_X$ *sending* $\langle A, \alpha\rangle$ *to* $\langle A, \alpha^* f\rangle$ *and a morphism* $h : \langle A, \alpha\rangle \rightarrow \langle B, \beta\rangle$ *to* MOD(f) $(h) : \langle A, \alpha^* f\rangle \rightarrow \langle B, \beta^* f\rangle$ *defined by* MOD$(f)(h) = h$.

(iv) *For every* $X \in |\mathbf{SIGN}|, \models_X \subseteq |\mathrm{MOD}(X)| \times \mathrm{SEN}(X)$ *is given by*

$$\langle A, \alpha\rangle \models_X t_1 \approx t_2 \quad \textit{if and only if} \quad e^{\mathbf{A}}(t_1, \vec{a}) = e^{\mathbf{A}}(t_2, \vec{a}), \quad \textit{for every} \quad \vec{a} \in A^V,$$

for all $\langle A, \alpha\rangle \in |\mathrm{MOD}(X)|, t_1 \approx t_2 \in \mathrm{SEN}(X)$.

To prove that $\mathcal{EQ}$ is an institution, we first need to prove three lemmas.

LEMMA 3.20 *Let* $X \in |\mathbf{SET}|, \mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be* X*-algebras,* $h : \mathbf{A} \rightarrow \mathbf{B}$ *be an* X*-homomorphism and* $t \in \mathrm{Tm}_X(V)$ *an* X*-term. Then*

$$h(\alpha^*(t)(\vec{a})) = \beta^*(t)(h(\vec{a})).$$

Proof:

The proof is by induction on the structure of the X-term t.

If $t = v_i \in V$, then $h(\alpha^*(v_i)(\vec{a})) = h(\vec{a}(v_i)) = h(a_i) = h(\vec{a})_i = \beta^*(v_i(h(\vec{a})))$.

If $x \in X, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, with

$$h(\alpha^*(t_i)(\vec{a})) = \beta^*(t_i)(h(\vec{a})), \quad \text{for every} \quad i < n,$$

$$\begin{aligned}
h(\alpha^*(x(t_0, \ldots, t_{n-1}))(\vec{a})) &= h(e^{\mathbf{A}}(x(t_0, \ldots, t_{n-1}), \vec{a})) \\
&= h(\alpha(x)(\langle e^{\mathbf{A}}(t_0, \vec{a}), \ldots, e^{\mathbf{A}}(t_{n-1}, \vec{a}), a_n, a_{n+1}, \ldots\rangle)) \\
&\quad \text{(by the definition of } e^{\mathbf{A}}) \\
&= \beta(x)(\langle h(e^{\mathbf{A}}(t_0, \vec{a})), \ldots, h(e^{\mathbf{A}}(t_{n-1}, \vec{a})), h(a_n), \ldots\rangle) \\
&\quad \text{(since } h : \mathbf{A} \rightarrow \mathbf{B} \in \mathrm{Mor}(\mathbf{ALG}_X)) \\
&= \beta(x)(\langle h(\alpha^*(t_0)(\vec{a})), \ldots, h(\alpha^*(t_{n-1})(\vec{a})), h(a_n), \ldots\rangle) \\
&\quad \text{(by the definition of } \alpha^*) \\
&= \beta(x)(\langle \beta^*(t_0)(h(\vec{a})), \ldots, \beta^*(t_{n-1})(h(\vec{a})), h(a_n), \ldots\rangle) \\
&\quad \text{(by the induction hypothesis)} \\
&= \beta^*(x(t_0, \ldots, t_{n-1}))(h(\vec{a})), \ \text{(by the definition of } \beta^*)
\end{aligned}$$

as required. ■

LEMMA 3.21 *Let $X \in |\mathbf{SET}|, \mathbf{A} = \langle A, \alpha\rangle$ be an X-algebra with rank function $\rho : X \to \omega$ $t \in \mathrm{Tm}_X(V)$, $\vec{s} \in \mathrm{Tm}_X(V)^m$ and $\vec{a} \in A^V$. Then*

$$e^{\mathbf{A}}(R_X(t, \vec{s}), \vec{a}) = e^{\mathbf{A}}(t, \langle e^{\mathbf{A}}(s_0, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, a_{m+1}, \ldots\rangle).$$

Proof:

The proof is by recursion on the structure of t.

If $t = v_i \in V$, then

$$e^{\mathbf{A}}(R_X(v_i, \vec{s}), \vec{a}) = \left\{ \begin{array}{ll} e^{\mathbf{A}}(s_i, \vec{a}), & i < m \\ e^{\mathbf{A}}(v_i, \vec{a}), & i \geq m \end{array} \right\} = \left\{ \begin{array}{ll} e^{\mathbf{A}}(s_i, \vec{a}), & i < m \\ a_i, & i \geq m \end{array} \right\} =$$

$$= e^{\mathbf{A}}(v_i, \langle e^{\mathbf{A}}(s_0, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, a_{m+1}, \ldots\rangle),$$

as required. Now suppose that $x \in X, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, and

$$e^{\mathbf{A}}(R_X(t_i, \vec{s}), \vec{a}) = e^{\mathbf{A}}(t_i, \langle e^{\mathbf{A}}(s_0, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, a_{m+1}, \ldots\rangle), \quad \text{for every} \quad i < n.$$

Then

$$\begin{aligned}
e^{\mathbf{A}}(R_X(x(t_0, \ldots, t_{n-1}), \vec{s}), \vec{a}) &= e^{\mathbf{A}}(x(R_X(t_0, \vec{s}), \ldots, R_X(t_{n-1}, \vec{s}), s_n, \ldots, s_{m-1}), \vec{a}) \\
&\quad \text{(by the definition of } R_X) \\
&= \alpha(x)(e^{\mathbf{A}}(R_X(t_0, \vec{s}), \vec{a}), \ldots, e^{\mathbf{A}}(R_X(t_{n-1}, \vec{s}), \vec{a}), \\
&\quad e^{\mathbf{A}}(s_n, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, a_{m+1}, \ldots) \\
&\quad \text{(by the definition of } e^{\mathbf{A}}) \\
&= \alpha(x)(e^{\mathbf{A}}(t_0, \langle e^{\mathbf{A}}(s_0, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, \ldots\rangle), \\
&\quad \ldots, e^{\mathbf{A}}(t_{n-1}, \langle e^{\mathbf{A}}(s_0, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, \ldots\rangle), \\
&\quad e^{\mathbf{A}}(s_n, \vec{a}), \ldots, e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, a_{m+1}, \ldots) \\
&\quad \text{(by the induction hypothesis)} \\
&= e^{\mathbf{A}}(x(t_0, \ldots, t_{n-1}), \langle e^{\mathbf{A}}(s_0, \vec{a}), \ldots, \\
&\quad e^{\mathbf{A}}(s_{m-1}, \vec{a}), a_m, a_{m+1}, \ldots\rangle), \text{ (by definition of } e^{\mathbf{A}})
\end{aligned}$$

as required. ■

LEMMA 3.22 *Let* $X, Y \in |\mathbf{SIGN}|, f : X \to Y \in \mathrm{Mor}(\mathbf{SIGN}), \langle A, \alpha\rangle \in |\mathrm{MOD}(Y)|$ *and* $t \in \mathrm{Tm}_X(V)$. *Then*

$$e^{\langle A,\alpha^{\bullet}f\rangle}(t,\vec{a}) = e^{\langle A,\alpha\rangle}(f^{\bullet}(t),\vec{a}), \quad \textit{for every} \quad \vec{a} \in A^V.$$

Proof:

The proof is by recursion on the structure of t.

If $t = v_i \in V$, then $e^{\langle A,\alpha^{\bullet}f\rangle}(v_i,\vec{a}) = a_i = e^{\langle A,\alpha\rangle}(v_i,\vec{a}) = e^{\langle A,\alpha\rangle}(f^{\bullet}(v_i),\vec{a})$, as required.

If $x \in X, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, with

$$e^{\langle A,\alpha^{\bullet}f\rangle}(t_i,\vec{a}) = e^{\langle A,\alpha\rangle}(f^{\bullet}(t_i),\vec{a}). \quad \text{for every} \quad i < n.$$

then

$$\begin{aligned}
e^{\langle A,\alpha^{\bullet}f\rangle}(x(t_0,\ldots,t_{n-1}),\vec{a}) &= \alpha^{\bullet}(f(x))(e^{\langle A,\alpha^{\bullet}f\rangle}(t_0,\vec{a}),\ldots,e^{\langle A,\alpha^{\bullet}f\rangle}(t_{n-1},\vec{a}),\\
&\quad a_n, a_{n+1},\ldots) \text{ (by the definition of } e^{\langle A,\alpha^{\bullet}f\rangle})\\
&= \alpha^{\bullet}(f(x))(e^{\langle A,\alpha\rangle}(f^{\bullet}(t_0),\vec{a}),\ldots,e^{\langle A,\alpha\rangle}(f^{\bullet}(t_{n-1}),\vec{a}),\\
&\quad a_n, a_{n+1},\ldots) \text{ (by the induction hypothesis)}\\
&= e^{\langle A,\alpha\rangle}(f(x),\langle e^{\langle A,\alpha\rangle}(f^{\bullet}(t_0),\vec{a}),\ldots,\\
&\quad e^{\langle A,\alpha\rangle}(f^{\bullet}(t_{n-1}),\vec{a}),a_n,\ldots\rangle) \text{ (by the definition of } \alpha^{\bullet})\\
&= e^{\langle A,\alpha\rangle}(R_Y(f(x),\langle f^{\bullet}(t_0),\ldots,f^{\bullet}(t_{n-1})\rangle),\vec{a})\\
&\quad \text{(by Lemma 3.21)}\\
&= e^{\langle A,\alpha\rangle}(f^{\bullet}(x(t_0,\ldots,t_{n-1})),\vec{a}), \text{ (by the definition of } f^{\bullet})
\end{aligned}$$

as required. ■

THEOREM 3.23 $\mathcal{EQ} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ *as defined in* 3.19 *is an institution.*

Proof:

We show that MOD is well-defined on morphisms and then verify the satisfaction condition.

To this end, let $f : X \to Y \in \mathrm{Mor}(\mathbf{SIGN})$ and $h : \langle A, \alpha\rangle \to \langle B, \beta\rangle \in \mathrm{Mor}(\mathbf{ALG}_Y)$. Then $h(\alpha(y)(\vec{a})) = \beta(y)(h(\vec{a}))$, for every $\vec{a} : V \to A$. We need to show that

$$h(\alpha^{*}(f(x))(\vec{a})) = \beta^{*}(f(x))(h(\vec{a})), \quad \text{for all} \quad x \in X, \vec{a} : V \to A.$$

This, however, was proved in Lemma 3.20.

For the satisfaction condition, let $f : X \to Y \in \mathrm{Mor}(\mathbf{SIGN}), t_0 \approx t_1 \in \mathrm{SEN}(X)$ and $\langle A, \alpha\rangle \in |\mathrm{MOD}(Y)|$. Then

$$\mathrm{MOD}(f)(\langle A, \alpha\rangle) \models_X t_0 \approx t_1 \quad \text{if and only if} \quad \langle A, \alpha^{*}f\rangle \models_X t_0 \approx t_1 \quad \text{if and only if}$$

$$e^{\langle A,\alpha^{*}f\rangle}(t_0, \vec{a}) = e^{\langle A,\alpha^{*}f\rangle}(t_1, \vec{a}), \quad \text{for every} \quad \vec{a} \in A^V, \quad \text{if and only if, by Lemma 3.22,}$$

$$e^{\langle A,\alpha\rangle}(f^{*}(t_0), \vec{a}) = e^{\langle A,\alpha\rangle}(f^{*}(t_1), \vec{a}), \quad \text{for every} \quad \vec{a} \in A^V, \quad \text{if and only if}$$

$$\langle A, \alpha\rangle \models_Y f^{*}(t_0) \approx f^{*}(t_1) \quad \text{if and only if} \quad \langle A, \alpha\rangle \models_Y \mathrm{SEN}(f)(t_0 \approx t_1),$$

as required. ∎

We refer to $\mathcal{EQ}$ as the **equational institution.**

The Algebraic Counterpart

In this subsection, the construction of the algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ that will serve as the basis for the algebraic institution algebraizing the equational institution $\mathcal{EQ}$ is overviewed. Details are omitted, since the entire construction is carefully developed in Chapter 6 of the thesis.

The functor $T : \mathbf{SET} \to \mathbf{SET}$ is defined by

$$T(X) = \mathrm{Tm}_X(V), \quad \text{for every} \quad X \in |\mathbf{SET}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{SET})$, $T(f) : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ is defined by

$$T(f) = (j_Y f)^{*}.$$

The natural transformation $\eta : I_{\mathbf{SET}} \to T$ is given by $\eta_X : X \to T(X)$, with

$$\eta_X = j_X, \quad \text{for every} \quad X \in |\mathbf{SET}|.$$

Finally, the natural transformation $\mu : TT \to T$ is defined by $\mu_X : T(T(X)) \to T(X)$, with

$$\mu_X = i_{T(X)} \circ i_{T(T(X))} = i^{*}_{T(X)}, \quad \text{for every} \quad X \in |\mathbf{SET}|.$$

Given a set A, define $\mathbf{A}^{\#} = \langle \mathrm{Cl}(A), \xi_A \rangle$ as follows:

$\mathrm{Cl}(A)$ is the full clone of operations on the set A, i.e.,

$$\mathrm{Cl}(A) = \bigcup_{k=0}^{\infty} A^{A^k}.$$

$\xi_A : \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \to \mathrm{Cl}(A)$ is defined by induction on the structure of $\mathrm{Cl}(A)$-terms over V, as follows

- $\xi_A(v_i) = p_i$, for every $i \in \omega$, where $p_i : A^{i+1} \to A$ is the i-th projection map.
- For all $f \in \mathrm{Cl}(A), n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V), t_{n-1} \neq v_{n-1}$,

$$\rho(\xi_A(f(t_0, \ldots, t_{n-1})) = max\{\rho(f), \rho(\xi_A(t_0)), \ldots, \rho(\xi_A(t_{\min\{n, \rho(f)\}-1}))\}, \quad \text{and}$$

$$\xi_A(f(t_0, \ldots, t_{n-1}))(\vec{a}) = f(\langle \xi_A(t_0)(\vec{a}), \ldots . \xi_A(t_{n-1})(\vec{a}), a_n, a_{n+1}, \ldots \rangle).$$

LEMMA 3.24 *Let A be a set, $t, s_0, \ldots, s_{m-1} \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V)$. Then*

$$\xi_A(R_{\mathrm{Cl}(A)}(t, \vec{s})) = \xi_A(t)(\xi_A(s_0), \ldots, \xi_A(s_{m-1})).$$

Proof:

The proof is by induction on the structure of the $\mathrm{Cl}(A)$-term t.

If $t = v_i \in V$, then

$$\xi_A(R_{\mathrm{Cl}(A)}(v_i, \vec{s})) = \left\{ \begin{array}{ll} \xi_A(s_i), & i < m \\ \xi_A(v_i), & i \geq m \end{array} \right\} =$$

$$= p_i(\xi_A(s_0), \dots, \xi_A(s_{m-1})) = \xi_A(v_i)(\xi_A(s_0), \dots, \xi_A(s_{m-1})),$$

as required. Next, if $f \in \mathrm{Cl}(A), t_0, \dots, t_{n-1} \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V), t_{n-1} \neq v_{n-1}$, such that

$$\xi_A(R_{\mathrm{Cl}(A)}(t_i, \vec{s})) = \xi_A(t_i)(\xi_{\mathbf{A}}(s_0), \dots, \xi_A(s_{m-1})), \quad \text{for every} \quad i < n,$$

we have

$$\begin{aligned}
\xi_A(R_{\mathrm{Cl}(A)}(f(t_0, \dots, t_{n-1}), \vec{s})) &= \xi_A(f(R_{\mathrm{Cl}(A)}(t_0, \vec{s}), \dots, R_{\mathrm{Cl}(A)}(t_{n-1}, \vec{s}), \\
&\quad s_n, \dots, s_{m-1})) \text{ (by the definition of } R_{\mathrm{Cl}(A)}) \\
&= f(\xi_A(R_{\mathrm{Cl}(A)}(t_0, \vec{s}), \dots, \xi_A(R_{\mathrm{Cl}(A)}(t_{n-1}, \vec{s})), \\
&\quad \xi_A(s_n), \dots, \xi_A(s_{m-1})) \text{ (by the definition of } \xi_A) \\
&= f(\xi_A(t_0)(\xi_A(\vec{s})), \dots, \xi_A(t_{n-1})(\xi_A(\vec{s})), \\
&\quad \xi_A(s_n), \dots, \xi_A(s_{m-1})) \text{ (by the ind. hypothesis)} \\
&= f(\xi_A(t_0), \dots, \xi_A(t_{n-1}))(\xi_A(\vec{s})) \\
&\quad \text{(by associativity of the clone composition)} \\
&= \xi_A(f(t_0, \dots, t_{n-1}))(\xi_A(\vec{s})) \text{ (by the definition of } \xi_A)
\end{aligned}$$

as required. ■

In the next lemma it is proved that $\mathbf{A}^\#$ is a $\mathbf{T}$-algebra.

LEMMA 3.25 *$\mathbf{A}^\# = \langle \mathrm{Cl}(A), \xi_A \rangle$ is a $\mathbf{T}$-algebra.*

Proof:

By definition, we need to check commutativity of the following diagrams:

$$\begin{tikzcd}
\mathrm{Cl}(A) \arrow[r, "\eta_{\mathrm{Cl}(A)}"] \arrow[dr, "i_{\mathrm{Cl}(A)}"'] & \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \arrow[d, "\xi_A"] \\
& \mathrm{Cl}(A)
\end{tikzcd}$$

To this end, let $f \in \mathrm{Cl}(A)$. We have

$$\xi_A(\eta_{\mathrm{Cl}(A)}(f)) = \xi_A(f()) = f = i_{\mathrm{Cl}(A)}(f).$$

$$\begin{tikzcd}
\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V) \arrow[r, "T(\xi_A)"] \arrow[d, "\mu_{\mathrm{Cl}(A)}"'] & \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \arrow[d, "\xi_A"] \\
\mathrm{Tm}_{\mathrm{Cl}(A)}(V) \arrow[r, "\xi_A"'] & \mathrm{Cl}(A)
\end{tikzcd}$$

For the commutativity of the rectangle we work by induction on the structure of a $\mathrm{Tm}_{\mathrm{Cl}(A)}(V)$–term t.

For $t = v_i \in V$,

$$\begin{aligned}
\xi_A(T(\xi_A)(v_i)) &= \xi_A((\eta_{\mathrm{Cl}(A)}\xi_A)^{*}(v_i)) \\
&= \xi_A(v_i) \\
&= \xi_A(i^{*}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)} i_{\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)}(v_i)) \\
&= \xi_A \mu_{\mathrm{Cl}(A)}(v_i),
\end{aligned}$$

as required. Next, if $t \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V), n \in \omega, s_0, \ldots, s_{m-1} \in \mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V), s_{m-1} \neq v_{m-1}$, such that $\xi_A(T(\xi_A)(s_i)) = \xi_A(\mu_{\mathrm{Cl}(A)}(s_i))$, for every $i < n$, then

$$\xi_A(T(\xi_A)(t(s_0, \ldots, s_{n-1}))) = \xi_A((\eta_{\mathrm{Cl}(A)}\xi_A)^{*}(t(s_0, \ldots, s_{n-1}))) \text{ (by defin. of } T)$$

$$\begin{aligned}
&= \xi_A(R_{\mathrm{Cl}(A)}((\eta_{\mathrm{Cl}(A)}\xi_A)(t), \langle(\eta_{\mathrm{Cl}(A)}\xi_A)^{*}(s_0), \ldots, \\
&\quad (\eta_{\mathrm{Cl}(A)}\xi_A)^{*}(s_{n-1})\rangle)) \text{ (by the definition of } ^{*}) \\
&= \xi_A((\eta_{\mathrm{Cl}(A)}\xi_A)(t))(\xi_A((\eta_{\mathrm{Cl}(A)}\xi_A)^{*}(s_0)), \ldots, \\
&\quad \xi_A((\eta_{\mathrm{Cl}(A)}\xi_A)^{*}(s_{n-1}))) \text{ (by Lemma 3.24)} \\
&= \xi_A(t)(\xi_A(\mu_{\mathrm{Cl}(A)}(s_0)), \ldots, \xi_A(\mu_{\mathrm{Cl}(A)}(s_{n-1}))) \\
&\quad \text{(by commut. of triangle and induction hyp.)} \\
&= \xi_A(R_{\mathrm{Cl}(A)}(t, \langle\mu_{\mathrm{Cl}(A)}(s_0), \ldots, \mu_{\mathrm{Cl}(A)}(s_{n-1})\rangle)) \text{ (by Lemma 3.24)} \\
&= \xi_A(R_{\mathrm{Cl}(A)}(i_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(t), \langle i^{*}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)} i_{\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)}(s_0), \ldots, \\
&\quad i^{*}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)} i_{\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)}(s_{n-1})\rangle)) \text{ (by definition of } \mu_{\mathrm{Cl}(A)})
\end{aligned}$$

$$
\begin{aligned}
&= \xi_A(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(t(i_{\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)}(s_0),\ldots, \\
&\quad i_{\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)}(s_{n-1})))) \text{ (by definition of }^*) \\
&= \xi_A(i^*_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)} i_{\mathrm{Tm}_{\mathrm{Tm}_{\mathrm{Cl}(A)}(V)}(V)}(t(s_0,\ldots,s_{n-1}))) \\
&= \xi_A(\mu_{\mathrm{Cl}(A)}(t(s_0,\ldots,s_{n-1}))),
\end{aligned}
$$

as required. ■

The Algebraization

In this subsection, the algebraization of the equational institution $\mathcal{EQ}$, that was constructed in the first subsection, is presented. For the algebraization, an algebraic institution that is based on the algebraic theory $\mathbf{T}$ over $\mathbf{SET}$, that was constructed in the preceding subsection, is used.

Let $\mathcal{Q}$ be the full subcategory of $\mathbf{SET}^{\mathbf{T}}$ with objects

$$\{\mathbf{A}^{\#} = \langle \mathrm{Cl}(A), \xi_A\rangle : A \in |\mathbf{SET}|\}.$$

Set $\mathcal{I}_{\mathcal{Q}} = \mathcal{I}_{\mathcal{Q}}^{\mathbf{SET}_{\mathbf{T}}}$. It will be shown that $\mathcal{I}_{\mathcal{Q}}$ is deductively equivalent to $\mathcal{EQ}$ and, therefore, that $\mathcal{EQ}$ is algebraizable.

First, we need to prove the following lemma:

LEMMA 3.26 *Let $X \in |\mathbf{SET}|, \mathbf{A} = \langle A, \alpha\rangle \in |\mathrm{ALG}(X)|$, $t \in \mathrm{Tm}_X(V)$ and $\vec{a} \in A^V$. Then*

$$e^{\mathbf{A}}(t, \vec{a}) = \xi_A T(\alpha)(t)(\vec{a}).$$

Proof:

By induction on the structure of t.

If $t = v_i \in V$, then

$$\xi_A(T(\alpha)(v_i))(\vec{a}) = \xi_A((\eta_{\mathrm{Cl}(A)}\alpha)^*(v_i))(\vec{a}) = \xi_A(v_i)(\vec{a}) =$$

$$= p_i(\vec{a}) = a_i = e^{\mathbf{A}}(v_i, \vec{a}).$$

If $x \in X, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, such that

$$e^{\mathbf{A}}(t_i, \vec{a}) = \xi_A T(\alpha)(t_i)(\vec{a}), \quad \text{for every} \quad i < n,$$

then

$$\begin{aligned}
\xi_A(T(\alpha)(x(t_0, \ldots, t_{n-1})))(\vec{a}) &= \xi_A((\eta_{\mathrm{Cl}(A)}\alpha)^{*}(x(t_0, \ldots, t_{n-1})))(\vec{a}) \text{ (by defin. of } T) \\
&= \xi_A(R_{\mathrm{Cl}(A)}(\eta_{\mathrm{Cl}(A)}\alpha(x), \langle T(\alpha)(t_0), \ldots, \\
&\quad T(\alpha)(t_{n-1})\rangle))(\vec{a}) \text{ (by the definition of }{}^{*}) \\
&= \xi_A(\eta_{\mathrm{Cl}(A)}(\alpha(x))(\xi_A(T(\alpha)(t_0)), \ldots, \\
&\quad \xi_A(T(\alpha)(t_{n-1}))))(\vec{a}) \text{ (by Lemma 3.24)} \\
&= \alpha(x)(\xi_A(T(\alpha)(t_0)), \ldots, \xi_A(T(\alpha)(t_{n-1})))(\vec{a}) \\
&\quad \text{(by commutativity of triangle)} \\
&= \alpha(x)(\xi_A(T(\alpha)(t_0))(\vec{a}), \ldots, \xi_A(T(\alpha)(t_{n-1}))(\vec{a}), \\
&\quad a_n, a_{n+1}, \ldots) \text{ (by the definition of clone)} \\
&= \alpha(x)(e^{\mathbf{A}}(t_0, \vec{a}), \ldots, e^{\mathbf{A}}(t_{n-1}, \vec{a}), a_n, \ldots) \\
&\quad \text{(by the induction hypothesis)} \\
&= e^{\mathbf{A}}(x(t_0, \ldots, t_{n-1}), \vec{a}), \text{ (by definition of } e^{\mathbf{A}})
\end{aligned}$$

as required. ■

And now for the main theorem of this subsection.

THEOREM 3.27 $\mathcal{EQ} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ *and* $\mathcal{I}_Q = \langle \mathbf{SET_T}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ *are deductively equivalent institutions.*

Proof:

First, note that $\mathbf{SIGN} = \mathbf{SET_T}$, whence it is legal to take the identity functor as the signature component of the interpretations $\langle I_{\mathbf{SIGN}}, \alpha\rangle : \mathcal{EQ} \to \mathcal{I}_Q$ and $\langle I_{\mathbf{SET_T}}, \beta\rangle : \mathcal{I}_Q \to \mathcal{EQ}$. Define $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$ by $\alpha_X : \mathrm{SEN}(X) \to \mathcal{P}(\mathrm{EQ}(X))$, with

$$\alpha_X(t_0 \approx t_1) = \{\langle f_{t_0}, f_{t_1}\rangle\}, \quad \text{for every} \quad t_0 \approx t_1 \in \mathrm{SEN}(X),$$

where, as before, given $t \in \mathrm{Tm}_X(V)$, we denote by $f_t : \{\emptyset\} \to \mathrm{Tm}_X(V)$ the map that sends $\emptyset$ to the X-term t. Similarly, define $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$ by $\beta_X : \mathrm{EQ}(X) \to \mathcal{P}(\mathrm{SEN}(X))$, with

$$\beta_X(\langle f_0, f_1 \rangle) = \{f_0(\emptyset) \approx f_1(\emptyset)\}, \quad \text{for every} \quad \langle f_0, f_1 \rangle \in \mathrm{EQ}(X).$$

We first show that $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$, $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$ are natural transformations. To this end, let $f : X \to Y \in \mathrm{Mor}(\mathbf{SIGN})$. We need to show that the following diagram commutes. If $t_0 \approx t_1 \in \mathrm{SEN}(X)$, we have

$$\begin{tikzcd} \mathrm{SEN}(X) \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathcal{P}(\mathrm{EQ}(X)) \arrow[d, "\mathcal{P}\mathrm{EQ}(f)"] \\ \mathrm{SEN}(Y) \arrow[r, "\alpha_Y"'] & \mathcal{P}(\mathrm{EQ}(Y)) \end{tikzcd}$$

$$\begin{aligned} \mathcal{P}\mathrm{EQ}(f)(\alpha_X(t_0 \approx t_1)) &= \mathrm{EQ}(f)(\langle f_{t_0}, f_{t_1} \rangle) \\ &= \langle f \circ f_{t_0}, f \circ f_{t_1} \rangle \\ &= \langle f^{*} f_{t_0}, f^{*} f_{t_1} \rangle \\ &= \langle f_{f^{*}(t_0)}, f_{f^{*}(t_1)} \rangle \\ &= \alpha_Y(f^{*}(t_0) \approx f^{*}(t_1)) \\ &= \alpha_Y(\mathrm{SEN}(f)(t_0 \approx t_1)), \end{aligned}$$

as required. The proof for β is similar. Next, we show that $\langle I_{\mathbf{SIGN}}, \alpha \rangle : \mathcal{EQ} \to \mathcal{I}_{\mathcal{Q}}$ is an interpretation. To this end, let $X \in |\mathbf{SIGN}|, E \cup \{t_0 \approx t_1\} \subseteq \mathrm{SEN}(X)$. We need to show that

$$t_0 \approx t_1 \in E^c \quad \text{iff} \quad \langle f_{t_0}, f_{t_1} \rangle \in \{\langle f_{e_0}, f_{e_1} \rangle : e_0 \approx e_1 \in E\}^c.$$

We first show that, if $t_0 \approx t_1 \in E^c$, then $\langle f_{t_0}, f_{t_1} \rangle \in \{\langle f_{e_0}, f_{e_1} \rangle : e_0 \approx e_1 \in E\}^c$. Suppose that $t_0 \approx t_1 \in E^c$. Then, for every $\mathbf{A} = \langle A, \alpha \rangle \in |\mathrm{MOD}(X)|$,

$$e^{\mathbf{A}}(e_0, \vec{a}) = e^{\mathbf{A}}(e_1, \vec{a}), \text{ for every } e_0 \approx e_1 \in E, \vec{a} : V \to A, \text{ implies } e^{\mathbf{A}}(t_0, \vec{a}) = e^{\mathbf{A}}(t_1, \vec{a}). \tag{3.6}$$

Now assume that $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle \in |\mathbf{ALG}_X|$, such that $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle \models_X \langle f_{e_0}, f_{e_1}\rangle$, for every $e_0 \approx e_1 \in E$. Then $\xi_A\mu_{\mathrm{Cl}(A)}T(f)f_{e_0} = \xi_A\mu_{\mathrm{Cl}(A)}T(f)f_{e_1}$, for every $e_0 \approx e_1 \in E$, whence $\xi_A\mu_{\mathrm{Cl}(A)}\eta_{T(\mathrm{Cl}(A))}f^{\bullet}f_{e_0} = \xi_A\mu_{\mathrm{Cl}(A)}\eta_{T(\mathrm{Cl}(A))}f^{\bullet}f_{e_1}$, for every $e_0 \approx e_1 \in E$, and, therefore $\xi_A f^{\bullet} f_{e_0} = \xi_A f^{\bullet} f_{e_1}$, for every $e_0 \approx e_1 \in E$. Thus, $\langle \mathrm{Cl}(A), \xi_A f\rangle \models_X e_0 \approx e_1$, for every $e_0 \approx e_1 \in E$. By (3.6), then, $\langle \mathrm{Cl}(A), \xi_A f\rangle \models_X t_0 \approx t_1$, and, reversing the steps in the above deduction, $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle \models_X \langle f_{t_0}, f_{t_1}\rangle$. Hence $\langle f_{t_0}, f_{t_1}\rangle \in \{\langle f_{e_0}, f_{e_1}\rangle : e_0 \approx e_1 \in E\}^c$, as was to be shown.

Suppose, conversely, that $\langle f_{t_0}, f_{t_1}\rangle \in \{\langle f_{e_0}, f_{e_1}\rangle : e_0 \approx e_1 \in E\}^c$. Then, for every $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle \in |\mathrm{ALG}(X)|$,

$$\langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle \models_X \langle f_{e_0}, f_{e_1}\rangle, \quad \text{for every} \quad e_0 \approx e_1 \in E,$$

$$\text{implies} \quad \langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle \models_X \langle f_{t_0}, f_{t_1}\rangle. \tag{3.7}$$

Now assume that $\langle A, \alpha\rangle \in |\mathrm{MOD}(X)|$, such that $e^{\mathbf{A}}(e_0, \vec{a}) = e^{\mathbf{A}}(e_1, \vec{a})$, for every $e_0 \approx e_1 \in E, \vec{a} : V \to A$. Then, by Lemma 3.26, we have $\xi_A T(\alpha)(e_0) = \xi_A T(\alpha)(e_1)$, for every $e_0 \approx e_1 \in E$, i.e., $\xi_A\mu_{\mathrm{Cl}(A)}T(\eta_{\mathrm{Cl}(A)})T(\alpha)(e_0) = \xi_A\mu_{\mathrm{Cl}(A)}T(\eta_{\mathrm{Cl}(A)})T(\alpha)(e_1)$, for every $e_0 \approx e_1 \in E$. Thus, $\xi_A\mu_{\mathrm{Cl}(A)}T(\eta_{\mathrm{Cl}(A)}\alpha)(e_0) = \xi_A\mu_{\mathrm{Cl}(A)}T(\eta_{\mathrm{Cl}(A)}\alpha)(e_1)$, for every $e_0 \approx e_1 \in E$, and, therefore, $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, \eta_{\mathrm{Cl}(A)}\alpha\rangle \models \langle f_{e_0}, f_{e_1}\rangle$, for every $e_0 \approx e_1 \in E$. Thus, by (3.7), we have $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, \eta_{\mathrm{Cl}(A)}\alpha\rangle \models \langle f_{t_0}, f_{t_1}\rangle$. Reversing the steps in the above deduction, we get $e^{\mathbf{A}}(t_0, \vec{a}) = e^{\mathbf{A}}(t_1, \vec{a})$, for every $\vec{a} : V \to A$. Hence $t_0 \approx t_1 \in E^c$, as required.

The proof that $\langle I_{\mathbf{SET_T}}, \beta\rangle : \mathcal{I}_Q \to \mathcal{EQ}$ is an interpretation is similar.

It only remains to show that $\langle I_{\mathbf{SIGN}}, \alpha\rangle : \mathcal{EQ} \to \mathcal{I}_Q$ and $\langle I_{\mathbf{SET_T}}, \beta\rangle : \mathcal{I}_Q \to \mathcal{EQ}$ are inverse interpretations. To this end. let $X \in |\mathbf{SIGN}|, t_0 \approx t_1 \in \mathrm{SEN}(X)$. We have

$$\begin{aligned} \beta_X(\alpha_X(t_0 \approx t_1))^c &= \beta_X(\langle f_{t_0}, f_{t_1}\rangle)^c \\ &= \{f_{t_0}(\emptyset) \approx f_{t_1}(\emptyset)\}^c \\ &= \{t_0 \approx t_1\}^c, \end{aligned}$$

as required, and if $X \in |\mathbf{SET_T}|$, $\langle f_0, f_1\rangle \in \mathrm{EQ}(X)$,

$$\begin{aligned} \alpha_X(\beta_X(\langle f_0, f_1\rangle))^c &= \alpha_X(f_0(\emptyset) \approx f_1(\emptyset))^c \\ &= \{\langle f_{f_0(\emptyset)}, f_{f_1(\emptyset)}\rangle\}^c \\ &= \{\langle f_0, f_1\rangle\}^c, \end{aligned}$$

as required. ∎

4 AUTO-ALGEBRAIZABLE THEORY INSTITUTIONS

A very special subclass of term π-institutions, the so-called theory institutions, is introduced and the notion of auto-algebraizability is defined for theory institutions. The notion of Leibniz operator, introduced in [6], is extended and an intrinsic characterization of auto-algebraizability for a subclass of theory institutions is then obtained along the lines of [6]. An example of an auto-algebraizable theory institution is provided and the relation between auto-algebraizability of theory institutions and classical algebraizability of deductive systems is explored.

Introduction

In [6], Blok and Pigozzi developed a general framework for the algebraization of deductive systems in the sense of Tarski. They dealt with propositional-like logics over a fixed signature $\mathcal{L}$. In this framework, the algebraization of logics dealing with varying signatures, like equational or first-order logic, requires first the transformation of the logic to a propositional-like structural counterpart. For example (see appendix in [6]) the algebraization of first-order logic presupposes its "cylindrification". This initial ad-hoc step makes the process cumbersome and clumsy and seems artificial and unsatisfactory.

In a different context, Goguen and Burstall [26, 27] introduced the notion of institution in order to exploit some nice features of equational and other logics in the area of specification of programming languages. The institution structure has proved to be very appropriate for handling logics with varying signatures.

Inspired by a later work of Blok and Pigozzi [8] on the equivalence of deductive systems, a generalization, using the institution structure, of the theory of algebraization of [6] was developed in Chapters 2 and 3 of the thesis. This more general framework incorporates nicely the algebraization of logics with varying signatures (see Chapter 3). The notion of deductive equivalence was defined for institutions and necessary and sufficient conditions for the deductive equivalence of two term institutions were given. Further, the notion of algebraizable institution was introduced and, based on the characterization result on deductive equivalence, a characterization of the algebraizability of term institutions was provided.

In this chapter a very special subclass of term institutions, the so-called *theory institutions*, is introduced. Roughly speaking, theory institutions are π-institutions whose syntax has the specially desirable feature that it is already algebraic in nature. To algebraize them, therefore, it is only necessary to interpret their closure systems in algebraic closure systems and vice-versa. It is in this sense, that the class of theory institutions may be seen to be the natural first generalization of the class of classical deductive systems, whose syntax component is essentially an absolutely free algebra over some pre-specified signature.

Auto-algebraizability of theory institutions results from applying the general techniques developed in Chapters 2 and 3 of the thesis to this special class of term π-institutions, by imposing the additional restriction that the syntax component must remain invariant. See the section on deductive auto-equivalence in Chapter 2 for more details on the idea of invariance.

The nice algebraic-like structure of the syntax of theory institutions makes it possible to extend the definition of the Leibniz operator of [6] to this more general context. Again the generalized congruence, thus obtained, may be thought of, as the largest congruence identifying elements with the "same behaviour" with respect to the system under consideration. The introduction of this notion enables one to get an intrinsic characterization of auto-algebraizability for a special subclass of theory institutions along the lines of [6]. Finally, the relation of auto-algebraizability of theory institutions with the classical notion of algebraizability for deductive systems of [6] is also explored.

Theory Institutions and Algebraic Institutions

From now on we will be considering only categories $\mathcal{K}$ with the following properties:

1. $\mathcal{K}$ is locally small.
2. $\mathcal{K}$ has a terminal object 1 and
3. in $\mathcal{K}$ the coproduct $1 \sqcup 1$ exists.

Let $\mathbf{T} = \langle T, \eta, \mu \rangle$ be an algebraic theory in monoid form over a category $\mathcal{K}$, as above, and, denote, as usual, by $\mathcal{K}_{\mathbf{T}}$ the Kleisli category of $\mathbf{T}$ in $\mathcal{K}$, and by $\mathcal{K}^{\mathbf{T}}$ the Eilenberg-Moore category of $\mathbf{T}$-algebras over $\mathcal{K}$. Moreover, let $\mathcal{L}$ be a full subcategory of $\mathcal{K}_{\mathbf{T}}$, satisfying the following condition (see also the section on algebraic institutions in Chapter 3):

There exists $L_0 \in |\mathcal{L}|, l_0, l_1 \in \mathcal{K}_{\mathbf{T}}(1, L_0)$, with the property that there exists $f : \{\langle K, \langle k_0, k_1 \rangle \rangle : K \in |\mathcal{K}_{\mathbf{T}}|, k_0, k_1 \in \mathcal{K}_{\mathbf{T}}(1, K)\} \to |(L_0|\mathcal{K}_{\mathbf{T}})|$, such that

$f_{\langle K, \langle k_0, k_1 \rangle \rangle} \in \mathcal{K}_{\mathbf{T}}(L_0, K)$, for all $K \in |\mathcal{K}_{\mathbf{T}}|, k_0, k_1 \in \mathcal{K}_{\mathbf{T}}(1, K)$ the following commutes

$$\begin{tikzcd} 1 \arrow[r, "l_0"] \arrow[dr, "k_0"'] & L_0 \arrow[d, dashed, "f_{\langle K, \langle k_0, k_1 \rangle \rangle}"] & 1 \arrow[l, "l_1"'] \arrow[dl, "k_1"] \\ & K & \end{tikzcd} \tag{4.1}$$

and, for every $g \in \mathcal{K}_{\mathbf{T}}(K, K'), g \circ f_{\langle K, \langle k_0, k_1 \rangle \rangle} = f_{\langle K', \langle g \circ k_0, g \circ k_1 \rangle \rangle}$.

DEFINITION 4.1 *An* $\mathcal{L}$-**theory institution** *is a* π*-institution* $\mathcal{I}$ *of the form*

$$\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L \in |\mathcal{L}|} \rangle,$$

for some algebraic theory $\mathbf{T}$ *over* $\mathcal{K}$ *and some full subcategory* $\mathcal{L}$ *of* $\mathcal{K}_{\mathbf{T}}$, *satisfying* (4.1), *where* $\mathcal{K}_{\mathbf{T}}(1,-) : \mathcal{K}_{\mathbf{T}} \to \mathbf{SET}$ *is the* **representable covariant functor.**

Next, let $\mathbf{T} = \langle T, \eta, \mu \rangle$ be an algebraic theory in monoid form over a category $\mathcal{K}$, satisfying 1-3, $\mathcal{L}$ a full subcategory of $\mathcal{K}_{\mathbf{T}}$, satisfying (4.1), and $\mathcal{Q}$ a subcategory of $\mathcal{K}^{\mathbf{T}}$. Recall from Chapter 3 that the $\mathcal{L}$-algebraic π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$ associated with $\mathcal{Q}$ is the π-institution $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-)^2, \{E_L\}_{L \in |\mathcal{L}|} \rangle$, where $E_L : \mathcal{P}(\mathcal{K}_{\mathbf{T}}(1,L)^2) \to \mathcal{P}(\mathcal{K}_{\mathbf{T}}(1,L)^2)$ is given, for every $\Delta \subseteq \mathcal{K}_{\mathbf{T}}(1,L)^2$, by

$$\begin{aligned} E_L(\Delta) \;=\; & \{\langle t_1, t_2 \rangle \in \mathcal{K}_{\mathbf{T}}(1,L)^2 : \forall \langle K, \xi \rangle \in |\mathcal{Q}| \ \forall f \in \mathcal{K}_{\mathbf{T}}(L,K) \\ & (\xi \mu_K T(f) \delta_1 = \xi \mu_K T(f) \delta_2, \quad \text{for every } \langle \delta_1, \delta_2 \rangle \in \Delta, \\ & \text{implies} \quad \xi \mu_K T(f) t_1 = \xi \mu_K T(f) t_2)\}. \end{aligned}$$

Note that, in the present context, $\mathcal{L}$-algebraic π-institutions and $\mathcal{L}$-theory institutions are in the same relation that equational deductive systems and 1-deductive systems are in. So it is natural to also consider k-**theory institutions** that are obtained by taking the sentence functor to be $\mathcal{K}_{\mathbf{T}}(1,-)^k$. This entails a slight complication because (4.1) is no longer sufficient to ensure that the k-theory institution is a term institution. Rather, one has to postulate the existence of $l_0, \ldots, l_{k-1} \in \mathcal{K}_{\mathbf{T}}(1, L_0)$, satisfying an analogous condition. Moreover, the introduction of k-tuples overloads the notation. So, from this point on we will only be considering 1-theory institutions.

Now, the following result may be derived as a special case of Corollary 3.5(iii).

COROLLARY 4.2 *Let* $\mathcal{K}_1, \mathcal{K}_2$ *be categories satisfying* 1-3, $\mathbf{T}_1, \mathbf{T}_2$ *algebraic theories in* $\mathcal{K}_1$, $\mathcal{K}_2$, *respectively,* $\mathcal{L}_1, \mathcal{L}_2$ *full subcategories of* $\mathcal{K}_{1\mathbf{T}_1}, \mathcal{K}_{2\mathbf{T}_2}$, *respectively, satisfying* (4.1), *and* $\mathcal{Q}$ *a subcategory of* $\mathcal{K}_2^{\mathbf{T}_2}$. *A theory institution* $\mathcal{I} = \langle \mathcal{L}_1, \mathcal{K}_{1\mathbf{T}_1}(1,-), \{C_{L_1}\}_{L_1 \in |\mathcal{L}_1|} \rangle$ *is algebraizable, with deductively equivalent algebraic institution* $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}_2}$, *if and only if there exists a signature respecting adjoint equivalence* $\langle F, G, \eta, \epsilon \rangle : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}_2})$, *that commutes with substitutions.*

This characterization of algebraizability for theory institutions is not intrinsic, in the sense that it requires a priori knowledge of the equivalent algebraic institution semantics

$\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}_2}$. A general result giving such an intrinsic characterization is not known. In the sequel, we define the notion of **auto-algebraizability** for a theory institution and prove a partial result, characterizing intrinsically this more restricted notion for a special class of theory institutions.

DEFINITION 4.3 *A theory institution* $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ *will be called* **auto-algebraizable** *(***a.a.***, for short,) if there exists a subcategory* $\mathcal{Q}$ *of* $\mathcal{K}^{\mathbf{T}}$*, such that* $\mathcal{I}$ *is deductively equivalent to* $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$ *via interpretations* $\langle I_{\mathcal{L}}, \alpha\rangle : \mathcal{I} \to \mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$ *and* $\langle I_{\mathcal{L}}, \beta\rangle : \mathcal{I}_{\mathcal{Q}}^{\mathcal{L}} \to \mathcal{I}$.

Clearly, auto-algebraizability trivially implies algebraizability as defined in 3.4. Corollary 2.45 yields the following corollary referring to autoalgebraizability.

COROLLARY 4.4 *A theory institution* $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ *is autoalgebraizable, with deductively equivalent algebraic institution* $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$*, if and only if there exists an isomorphism* $F : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})$*, that makes the following diagram commute*

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}) \arrow[rr, "F"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}) \arrow[dl, "\mathrm{SIG}"] \\ & \mathcal{L} & \end{tikzcd}$$

and commutes with substitutions.

The Leibniz Operator

Let $\mathcal{K}$ be a category satisfying conditions 1-3 of the previous section and $\mathbf{T} = \langle T, \eta, \mu\rangle$ an algebraic theory in monoid form over $\mathcal{K}$. Recall that $\mathcal{K}(1, U^{\mathbf{T}}) : \mathcal{K}^{\mathbf{T}} \to \mathbf{SET}$ is the functor from the Eilenberg-Moore category $\mathcal{K}^{\mathbf{T}}$ of $\mathbf{T}$ in $\mathcal{K}$ into $\mathbf{SET}$, that sends a $\mathbf{T}$-algebra $\langle K, \xi\rangle$ to the set $\mathcal{K}(1, K)$ and a $\mathbf{T}$-algebra homomorphism $h : \langle K, \xi\rangle \to \langle L, \zeta\rangle$

$$\begin{tikzcd} T(K) \arrow[r, "T(h)"] \arrow[d, "\xi"'] & T(L) \arrow[d, "\zeta"] \\ K \arrow[r, "h"'] & L \end{tikzcd}$$

to the set map $\mathcal{K}(1,h) : \mathcal{K}(1,K) \to \mathcal{K}(1,L)$, with

$$\begin{tikzcd} K \arrow[rr, "h"] & & L \\ & 1 \arrow[ul, "f"] \arrow[ur, "hf"'] & \end{tikzcd}$$

$$\mathcal{K}(1,h)(f) = hf, \quad \text{for every} \quad f \in \mathcal{K}(1,K).$$

Now, let $\langle K,\xi\rangle$ be a $\mathbf{T}$-algebra, as above. A **congruence** Θ of $\langle K,\xi\rangle$ is an equivalence relation on $\mathcal{K}(1,K)$, such that, for all $n \in \omega$, all natural transformations $\tau : \mathcal{K}(1,U^{\mathbf{T}})^n \to \mathcal{K}(1,U^{\mathbf{T}})$ and all $t_i, s_i \in \mathcal{K}(1,K), i < n$,

$$\langle t_i, s_i\rangle \in \Theta, i < n, \quad \text{implies} \quad \langle \tau_{\langle K,\xi\rangle}(\vec{t}), \tau_{\langle K,\xi\rangle}(\vec{s})\rangle \in \Theta.$$

LEMMA 4.5 *Let* $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-)^2, \{E_L\}_{L\in|\mathcal{L}|}\rangle$ *be an* $\mathcal{L}$*-algebraic institution. For every theory* $\langle L,\Theta\rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})|$, Θ *is a congruence of* $\langle T(L), \mu_L\rangle$.

Proof:

Let $\langle L,\Theta\rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})|$ and denote by $\langle L,\Theta\rangle^*$ the class of all pairs $\langle\langle K,\xi\rangle, f\rangle$, with $\langle K,\xi\rangle \in |\mathcal{Q}|$ and $f \in \mathcal{K}_{\mathbf{T}}(L,K)$, such that $\xi\mu_K T(f)\theta_1 = \xi\mu_K T(f)\theta_2$, for every $\langle\theta_1,\theta_2\rangle \in \Theta$.

$$1 \xrightarrow{\theta_1} T(L) \xrightarrow{T(f)} T(T(K)) \xrightarrow{\mu_K} T(K) \xrightarrow{\xi} K$$

Clearly, by the definition of satisfaction in $\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}}$, we have

$$\Theta = \{\langle\theta_1,\theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1,L)^2 : \xi\mu_K T(f)\theta_1 = \xi\mu_K T(f)\theta_2, \quad \text{for every } \langle\langle K,\xi\rangle, f\rangle \in \langle L,\Theta\rangle^*\} \tag{4.2}$$

It is easy to see from (4.2) that Θ is an equivalence relation on $\mathcal{K}(1,T(L))$. To show that it is a congruence, let $n \in \omega$, $\tau : \mathcal{K}(1,U^{\mathbf{T}})^n \to \mathcal{K}(1,U^{\mathbf{T}})$ a natural transformation and $t_i, s_i \in \mathcal{K}(1,T(L)), i < n$. Then $\langle t_i, s_i\rangle \in \Theta, i < n$, implies

$$\xi\mu_K T(f)t_i = \xi\mu_K T(f)s_i, i < n, \quad \text{for every} \quad \langle\langle K,\xi\rangle, f\rangle \in \langle L,\Theta\rangle^*,$$

whence $\tau_{\langle K,\xi\rangle}(\xi\mu_K T(f)\vec{t}) = \tau_{\langle K,\xi\rangle}(\xi\mu_K T(f)\vec{s})$ and therefore, since $\xi\mu_K T(f) : \langle T(L), \mu_L\rangle \to \langle K, \xi\rangle \in \mathrm{Mor}(\mathcal{K}^{\mathbf{T}})$,

$$\begin{tikzcd}
\mathcal{K}(1, T(L))^{\kappa} \arrow[r, "\tau_{\langle T(L),\mu_L\rangle}"] \arrow[d, "\mathcal{K}(1,\xi\mu_K T(f))^{\kappa}"'] & \mathcal{K}(1, T(L)) \arrow[d, "\mathcal{K}(1,\xi\mu_K T(f))"] \\
\mathcal{K}(1, K)^{\kappa} \arrow[r, "\tau_{\langle K,\xi\rangle}"'] & \mathcal{K}(1, K)
\end{tikzcd}$$

$$\xi\mu_K T(f)\tau_{\langle T(L),\mu_L\rangle}(\vec{t}) = \xi\mu_K T(f)\tau_{\langle T(L),\mu_L\rangle}(\vec{s}).$$

i.e., $\langle \tau_{\langle T(L),\mu_L\rangle}(\vec{t}), \tau_{\langle T(L),\mu_L\rangle}(\vec{s})\rangle \in \Theta$, as required. ■

A congruence Θ on $\langle K, \xi\rangle$ is said to be **compatible with** a subset $A \subseteq \mathcal{K}(1, K)$ if, for all $\theta_1, \theta_2 \in \mathcal{K}(1, K)$,

$$\langle \theta_1, \theta_2\rangle \in \Theta \quad \text{and} \quad \theta_1 \in A \quad \text{imply} \quad \theta_2 \in A.$$

A binary relation Θ on $\mathcal{K}(1, K)$ is said to be **explicitly definable over** a **T**-algebra $\langle K, \xi\rangle$ and a subset $A \subseteq \mathcal{K}(1, K)$ if there exists $n \in \omega$, an I-indexed family of natural transformations $\tau^i : \mathcal{K}(1, U^{\mathbf{T}})^{n+2} \to \mathcal{K}(1, U^{\mathbf{T}})$ and $r_i \in \mathcal{K}(1, K), i < n$, such that

$$\langle \theta_1, \theta_2\rangle \in \Theta \quad \text{iff} \quad \tau^i_{\langle K,\xi\rangle}(\theta_1, \theta_2, \vec{r}) \in A, \text{ for all } i \in I.$$

DEFINITION 4.6 *Let $\langle K, \xi\rangle \in |\mathcal{K}^{\mathbf{T}}|$ and $A \subseteq \mathcal{K}(1, K)$. Define*

$$\begin{array}{rcl}
\Omega_{\langle K,\xi\rangle}(A) & = & \{\langle \theta_1, \theta_2\rangle \in \mathcal{K}(1, K)^2 : \quad \textit{for all} \quad n \in \omega, \quad \textit{natural transformations} \\
& & \tau : \mathcal{K}(1, U^{\mathbf{T}})^{n+1} \to \mathcal{K}(1, U^{\mathbf{T}}) \quad \textit{and} \quad r_i \in \mathcal{K}(1, K), i < n, \\
& & \tau_{\langle K,\xi\rangle}(\theta_1, \vec{r}) \in A \quad \textit{if and only if} \quad \tau_{\langle K,\xi\rangle}(\theta_2, \vec{r}) \in A\}
\end{array}$$

$\Omega_{\langle K,\xi\rangle}(A)$ *is called the* **Leibniz $\langle K, \xi\rangle$-equivalence over** A *and*

$$\Omega_{\langle K,\xi\rangle} : \mathcal{P}(\mathcal{K}(1, K)) \to \mathcal{P}(\mathcal{K}(1, K)^2)$$

is called the **Leibniz operator** *on $\langle K, \xi\rangle$. By Ω_K we will sometimes denote the Leibniz operator on $\langle T(K), \mu_K\rangle$.*

THEOREM 4.7 *Let $\langle K, \xi\rangle \in |\mathcal{K}^{\mathbf{T}}|, A \subseteq \mathcal{K}(1, K)$. $\Omega_{\langle K,\xi\rangle}(A)$ is the largest congruence on $\langle K, \xi\rangle$ compatible with A.*

Proof:

It is clear that $\Omega_{\langle K,\xi\rangle}(A)$ is an equivalence on $\mathcal{K}(1, K)$. We show that it is a congruence. To this end, let $n \in \omega$, $\tau : \mathcal{K}(1, U^{\mathbf{T}})^n \to \mathcal{K}(1, U^{\mathbf{T}})$ a natural transformation and $\langle s_i, t_i\rangle \in \Omega_{\langle K,\xi\rangle}(A), i < n$. Then, if $m \in \omega$, $\sigma : \mathcal{K}(1, U^{\mathbf{T}})^{m+1} \to \mathcal{K}(1, U^{\mathbf{T}})$ a natural transformation and $r_j \in \mathcal{K}(1, K), j < m$, we have

$$\sigma_{\langle K,\xi\rangle}(\tau_{\langle K,\xi\rangle}(t_0, \ldots, t_{n-1}), \vec{r}) \in A \quad \text{iff}$$

$$\sigma_{\langle K,\xi\rangle}(\tau_{\langle K,\xi\rangle}(s_0, t_1, \ldots, t_{n-1}), \vec{r}) \in A \quad \text{iff}$$

$$\sigma_{\langle K,\xi\rangle}(\tau_{\langle K,\xi\rangle}(s_0, s_1, t_2, \ldots, t_{n-1}), \vec{r}) \in A \quad \text{iff}$$

$$\ldots$$

$$\sigma_{\langle K,\xi\rangle}(\tau_{\langle K,\xi\rangle}(\vec{s}), \vec{r}) \in A.$$

Thus, $\langle \tau_{\langle K,\xi\rangle}(\vec{t}), \tau_{\langle K,\xi\rangle}(\vec{s})\rangle \in \Omega_{\langle K,\xi\rangle}(A)$, as required.

Compatibility of $\Omega_{\langle K,\xi\rangle}(A)$ with A is obvious if one considers the identity natural transformation $i : \mathcal{K}(1, U^{\mathbf{T}}) \to \mathcal{K}(1, U^{\mathbf{T}})$.

Finally, let Θ be a congruence on $\langle K, \xi\rangle$, compatible with A and $\langle \theta_1, \theta_2\rangle \in \Theta$. Then, for all $n < \omega$, natural transformation $\tau : \mathcal{K}(1, U^{\mathbf{T}})^{n+1} \to \mathcal{K}(1, U^{\mathbf{T}})$ and $r_i \in \mathcal{K}(1, K), i < n$, $\langle \tau_{\langle K,\xi\rangle}(\theta_1, \vec{r}), \tau_{\langle K,\xi\rangle}(\theta_2, \vec{r})\rangle \in \Theta$, whence, by compatibility with A,

$$\tau_{\langle K,\xi\rangle}(\theta_1, \vec{r}) \in A \quad \text{iff} \quad \tau_{\langle K,\xi\rangle}(\theta_2, \vec{r}) \in A,$$

i.e., $\langle \theta_1, \theta_2\rangle \in \Omega_{\langle K,\xi\rangle}(A)$, as required. ∎

THEOREM 4.8 *Let $\langle K, \xi\rangle \in |\mathcal{K}^{\mathbf{T}}|, A \subseteq \mathcal{K}(1, K)$ and $\Theta \subseteq \mathcal{K}(1, K)^2$ explicitly definable over $\langle K, \xi\rangle$ and A.*

(i) If Θ is reflexive, then $\Omega_{\langle K,\xi\rangle}(A) \subseteq \Theta$.

(ii) If, in addition, Θ is a congruence compatible with A, then $\Omega_{\langle K,\xi\rangle}(A) = \Theta$.

Proof:

(i) Suppose

$$\Theta = \{\langle \theta_1, \theta_2 \rangle \in \mathcal{K}(1, K)^2 : \tau^i_{\langle K,\xi \rangle}(\theta_1, \theta_2, \vec{r}) \in A, i \in I\}$$

and $\langle \theta_1, \theta_2 \rangle \in \Omega_{\langle K,\xi \rangle}(A)$. By reflexivity of Θ, $\tau^i_{\langle K,\xi \rangle}(\theta_2, \theta_2, \vec{r}) \in A, i \in I$. Therefore, since $\langle \theta_1, \theta_2 \rangle \in \Omega_{\langle K,\xi \rangle}(A)$, $\tau^i_{\langle K,\xi \rangle}(\theta_1, \theta_2, \vec{r}) \in A, i \in I$, and, hence, $\langle \theta_1, \theta_2 \rangle \in \Theta$. Thus $\Omega_{\langle K,\xi \rangle}(A) \subseteq \Theta$, as required.

(ii) Obvious by (i) and Theorem 4.7. ∎

Uniqueness of Autoalgebraizability

Let $\mathcal{K}$ be a category satisfying conditions 1-3, $\mathbf{T} = \langle T, \eta, \mu \rangle$ an algebraic theory in monoid form over $\mathcal{K}$ and $\mathcal{L}$ a full subcategory of $\mathcal{K}_{\mathbf{T}}$ satisfying (4.1).

LEMMA 4.9 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L \in |\mathcal{L}|} \rangle$ be an autoalgebraizable theory institution via the interpretations $\langle I_{\mathcal{L}}, \alpha \rangle : \mathcal{I} \to \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}}$ and $\langle I_{\mathcal{L}}, \beta \rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$. Then, for all $L \in |\mathcal{L}|$ and all $t_0, t_1, t_2 \in \mathcal{K}_{\mathbf{T}}(1, L)$,*

(i) $\beta_L(\langle t_0, t_0 \rangle) \subseteq \emptyset^c$

(ii) $\beta_L(\langle t_1, t_0 \rangle) \subseteq \beta_L(\langle t_0, t_1 \rangle)^c$

(iii) $\beta_L(\langle t_0, t_2 \rangle) \subseteq \beta_L(\{\langle t_0, t_1 \rangle, \langle t_1, t_2 \rangle\})^c$.

Furthermore, for all $n \in \omega$, natural transformation $\tau : \mathcal{K}(1, U^{\mathbf{T}})^{n+1} \to \mathcal{K}(1, U^{\mathbf{T}})$ and $r_i \in \mathcal{K}_{\mathbf{T}}(1, L), i < n$,

(iv) $\beta_L(\langle \tau_{\langle T(L), \mu_L \rangle}(t_0, \vec{r}), \tau_{\langle T(L), \mu_L \rangle}(t_1, \vec{r}) \rangle) \subseteq \beta_L(\langle t_0, t_1 \rangle)^c$.

Proof:

(i) We have $\langle t_0, t_0 \rangle \in \emptyset^c$, whence, since $\langle I_{\mathcal{L}}, \beta \rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$ is an interpretation,

$$\beta_L(\langle t_0, t_0 \rangle) \subseteq \beta_L(\emptyset)^c = \emptyset^c,$$

as required.

(ii) and (iii) can be proved similarly.

(iv) Suppose that for some $\mathbf{T}$–algebra $\langle K,\zeta\rangle \in |\mathcal{Q}|$ and $f \in \mathcal{K}_{\mathbf{T}}(L,K)$, $\langle t_0,t_1\rangle \in \langle\langle K,\zeta\rangle,f\rangle^{*}$. Then $\zeta\mu_K T(f)t_0 = \zeta\mu_K T(f)t_1$, whence

$$\tau_{\langle K,\zeta\rangle}(\zeta\mu_K T(f)t_0,\zeta\mu_K T(f)\vec{r}) = \tau_{\langle K,\zeta\rangle}(\zeta\mu_K T(f)t_1,\zeta\mu_K T(f)\vec{r})$$

and therefore, by commutativity of the diagram

$$\begin{tikzcd}
\mathcal{K}(1,T(L))^{n+1} \arrow[r, "\tau_{\langle T(L),\mu_L\rangle}"] \arrow[d, "\zeta\mu_K T(f)^{n+1}"'] & \mathcal{K}(1,T(L)) \arrow[d, "\zeta\mu_K T(f)"] \\
\mathcal{K}(1,K)^{n+1} \arrow[r, "\tau_{\langle K,\zeta\rangle}"'] & \mathcal{K}(1,K)
\end{tikzcd}$$

$\zeta\mu_K T(f)\tau_{\langle T(L),\mu_L\rangle}(t_0,\vec{r}) = \zeta\mu_K T(f)\tau_{\langle T(L),\mu_L\rangle}(t_1,\vec{r})$. Thus

$$\langle\tau_{\langle T(L),\mu_L\rangle}(t_0,\vec{r}),\tau_{\langle T(L),\mu_L\rangle}(t_1,\vec{r})\rangle \in \langle\langle K,\zeta\rangle,f\rangle^{*}.$$

This shows that $\langle\tau_{\langle T(L),\mu_L\rangle}(t_0,\vec{r}),\tau_{\langle T(L),\mu_L\rangle}(t_1,\vec{r})\rangle \in \{\langle t_0,t_1\rangle\}^c$, whence, since $\langle I_{\mathcal{L}},\beta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$ is an interpretation, $\beta_L(\langle\tau_{\langle T(L),\mu_L\rangle}(t_0,\vec{r}),\tau_{\langle T(L),\mu_L\rangle}(t_1,\vec{r})\rangle) \subseteq \beta_L(\langle t_0,t_1\rangle)^c$, as required. ∎

LEMMA 4.10 *Let $\mathcal{I} = \langle\mathcal{L},\mathcal{K}_{\mathbf{T}}(1,-),\{C_L\}_{L\in|\mathcal{L}|}\rangle$ be an a.a. theory institution via the interpretations $\langle I_{\mathcal{L}},\alpha\rangle : \mathcal{I} \to \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}}$ and $\langle I_{\mathcal{L}},\beta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$. Then,*

(i) there exist an index set J and a J-indexed family of natural transformations $\alpha^j : \mathcal{K}(1,U^{\mathbf{T}}) \to \mathcal{K}(1,U^{\mathbf{T}})^2$, such that

$$\alpha_L(s) = \{\alpha^j_{\langle T(L),\mu_L\rangle}(s) : j\in J\}, \quad \textit{for all} \quad L\in|\mathcal{L}|, s\in\mathcal{K}_{\mathbf{T}}(1,L),$$

(ii) there exist an index set I and an I-indexed family of natural transformations $\beta^i : \mathcal{K}(1,U^{\mathbf{T}})^2 \to \mathcal{K}(1,U^{\mathbf{T}})$, such that

$$\beta_L(\langle s,t\rangle) = \{\beta^i_{\langle T(L),\mu_L\rangle}(\langle s,t\rangle) : i\in I\}, \quad \textit{for all} \quad L\in|\mathcal{L}|, s,t\in\mathcal{K}_{\mathbf{T}}(1,L).$$

Proof:

(i) Let $l_0, l_1, f_{\langle s,s\rangle}$ be as in (4.1).

$$\begin{tikzcd} 1 \arrow[r, "l_0"] \arrow[dr, "s"'] & L_0 \arrow[d, "f_{\langle s,s\rangle}"] & 1 \arrow[l, "l_1"'] \arrow[dl, "s"] \\ & L & \end{tikzcd}$$

Then

$$\begin{aligned} \alpha_L(s) &= \alpha_L(\mathcal{K}_{\mathbf{T}}(1, f_{\langle s,s\rangle})(l_0)) \\ &= \mathcal{K}_{\mathbf{T}}(1, f_{\langle s,s\rangle})^2(\alpha_{L_0}(l_0)), \end{aligned}$$

by commutativity of

$$\begin{tikzcd} \mathcal{K}_{\mathbf{T}}(1, L_0) \arrow[r, "\alpha_{L_0}"] \arrow[d, "\mathcal{K}_{\mathbf{T}}(1, f_{\langle s,s\rangle})"'] & \mathcal{P}(\mathcal{K}_{\mathbf{T}}(1, L_0)^2) \arrow[d, "\mathcal{P}\mathcal{K}_{\mathbf{T}}(1, f_{\langle s,s\rangle})^2"] \\ \mathcal{K}_{\mathbf{T}}(1, L) \arrow[r, "\alpha_L"'] & \mathcal{P}(\mathcal{K}_{\mathbf{T}}(1, L)^2) \end{tikzcd}$$

Let $\alpha_{L_0}(l_0) = \{\langle \phi_j, \psi_j\rangle : j \in J\}$. Then, for every $\langle K', \xi\rangle \in |\mathcal{K}^{\mathbf{T}}|, j \in J$, define $\alpha^j_{\langle K',\xi\rangle} : \mathcal{K}(1, K') \to \mathcal{K}(1, K')^2$ by

$$\alpha^j_{\langle K',\xi\rangle}(\theta) = \langle \xi\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\phi_j, \xi\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\psi_j\rangle, \quad \text{for every} \quad \theta \in \mathcal{K}(1, K').$$

We first check that $\alpha^j : \mathcal{K}(1, U^{\mathbf{T}}) \to \mathcal{K}(1, U^{\mathbf{T}})^2$ is a natural transformation. Let $\langle K', \xi\rangle, \langle L, \zeta\rangle \in |\mathcal{K}^{\mathbf{T}}|, h : \langle K', \xi\rangle \to \langle L, \zeta\rangle \in \mathrm{Mor}(\mathcal{K}^{\mathbf{T}})$. We need to show that the following diagram commutes. If $\theta \in \mathcal{K}(1, K')$, we have

$$\begin{tikzcd} \mathcal{K}(1, K') \arrow[r, "\alpha^j_{\langle K',\xi\rangle}"] \arrow[d, "\mathcal{K}(1,h)"'] & \mathcal{K}(1, K')^2 \arrow[d, "\mathcal{K}(1,h)^2"] \\ \mathcal{K}(1, L) \arrow[r, "\alpha^j_{\langle L,\zeta\rangle}"'] & \mathcal{K}(1, L)^2 \end{tikzcd}$$

$$\begin{aligned} \mathcal{K}(1, h)^2(\alpha^j_{\langle K',\xi\rangle}(\theta)) &= \mathcal{K}(1, h)^2(\langle \xi\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\phi_j, \xi\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\psi_j\rangle) \\ &= \langle h\xi\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\phi_j, h\xi\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\psi_j\rangle \\ &= \langle \zeta T(h)\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\phi_j, \zeta T(h)\mu_{K'}T(f_{\langle \eta_{K'}\theta, \eta_{K'}\theta\rangle})\psi_j\rangle \end{aligned}$$

by commutativity of
$$\begin{tikzcd} T(K') \arrow[r, "T(h)"] \arrow[d, "\xi"'] & T(L) \arrow[d, "\zeta"] \\ K' \arrow[r, "h"'] & L \end{tikzcd}$$

$$= \langle \zeta\mu_L T(T(h))T(f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(T(h))T(f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\psi_j\rangle$$

by commutativity of
$$\begin{tikzcd} T(T(K')) \arrow[r, "\mu_{K'}"] \arrow[d, "T(T(h))"'] & T(K') \arrow[d, "T(h)"] \\ T(T(L)) \arrow[r, "\mu_L"'] & T(L) \end{tikzcd}$$

$$\begin{aligned}
&= \langle \zeta\mu_L T(T(h)f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(T(h)f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(T(h)\mu_{K'}T(\eta_{K'})f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(T(h)\mu_{K'}T(\eta_{K'})f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\psi_j\rangle
\end{aligned}$$

by commutativity of
$$\begin{tikzcd} T(K') \arrow[r, "T(\eta_{K'})"] \arrow[dr, "i_{T(K')}"'] & T(T(K')) \arrow[d, "\mu_{K'}"] \\ & T(K') \end{tikzcd}$$

$$\begin{aligned}
&= \langle \zeta\mu_L T(\mu_L T(T(h))T(\eta_{K'})f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(\mu_L T(T(h))T(\eta_{K'})f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(\mu_L T(T(h)\eta_{K'})f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(\mu_L T(T(h)\eta_{K'})f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(T(h)\eta_{K'} \circ f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(T(h)\eta_{K'} \circ f_{\langle \eta_{K'}\theta,\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(f_{\langle T(h)\eta_{K'}\circ\eta_{K'}\theta, T(h)\eta_{K'}\circ\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(f_{\langle T(h)\eta_{K'}\circ\eta_{K'}\theta, T(h)\eta_{K'}\circ\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(f_{\langle \mu_L T(T(h)\eta_{K'})\eta_{K'}\theta, \mu_L T(T(h)\eta_{K'})\eta_{K'}\theta\rangle})\phi_j, \\
&\quad \zeta\mu_L T(f_{\langle \mu_L T(T(h)\eta_{K'})\eta_{K'}\theta, \mu_L T(T(h)\eta_{K'})\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(f_{\langle \mu_L T(T(h))T(\eta_{K'})\eta_{K'}\theta, \mu_L T(T(h))T(\eta_{K'})\eta_{K'}\theta\rangle})\phi_j, \\
&\quad \zeta\mu_L T(f_{\langle \mu_L T(T(h))T(\eta_{K'})\eta_{K'}\theta, \mu_L T(T(h))T(\eta_{K'})\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(f_{\langle T(h)\mu_{K'}T(\eta_{K'})\eta_{K'}\theta, T(h)\mu_{K'}T(\eta_{K'})\eta_{K'}\theta\rangle})\phi_j, \\
&\quad \zeta\mu_L T(f_{\langle T(h)\mu_{K'}T(\eta_{K'})\eta_{K'}\theta, T(h)\mu_{K'}T(\eta_{K'})\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(f_{\langle T(h)\eta_{K'}\theta, T(h)\eta_{K'}\theta\rangle})\phi_j, \zeta\mu_L T(f_{\langle T(h)\eta_{K'}\theta, T(h)\eta_{K'}\theta\rangle})\psi_j\rangle \\
&= \langle \zeta\mu_L T(f_{\langle \eta_L h\theta, \eta_L h\theta\rangle})\phi_j, \zeta\mu_L T(f_{\langle \eta_L h\theta, \eta_L h\theta\rangle})\psi_j\rangle
\end{aligned}$$

by commutativity of
$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[d, "h"'] & T(K) \arrow[d, "T(h)"] \\ L \arrow[r, "\eta_L"'] & T(L) \end{tikzcd}$$

$$\begin{aligned} &= \alpha^j_{\langle L,\zeta\rangle}(h\theta) \\ &= \alpha^j_{\langle L,\zeta\rangle}(\mathcal{K}(1,h)(\theta)), \end{aligned}$$

as required. Finally, we have

$$\begin{aligned}
\alpha_L(s) &= \mathcal{K}_{\mathbf{T}}(1, f_{\langle s,s\rangle})^2(\{\langle \phi_j, \psi_j\rangle : j \in J\}) \\
&= \{\langle f_{\langle s,s\rangle} \circ \phi_j, f_{\langle s,s\rangle} \circ \psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L T(f_{\langle s,s\rangle})\phi_j, \mu_L T(f_{\langle s,s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(\eta_{T(L)})T(f_{\langle s,s\rangle})\phi_j, \mu_L \mu_{T(L)} T(\eta_{T(L)})T(f_{\langle s,s\rangle})\psi_j : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(\eta_{T(L)} f_{\langle s,s\rangle})\phi_j, \mu_L \mu_{T(L)} T(\eta_{T(L)} f_{\langle s,s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(\eta_{T(L)} \mu_L T(\eta_L) f_{\langle s,s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(\eta_{T(L)} \mu_L T(\eta_L) f_{\langle s,s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(\mu_{T(L)} T(\eta_{T(L)}) T(\eta_L) f_{\langle s,s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(\mu_{T(L)} T(\eta_{T(L)}) T(\eta_L) f_{\langle s,s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(\mu_{T(L)} T(\eta_{T(L)} \eta_L) f_{\langle s,s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(\mu_{T(L)} T(\eta_{T(L)} \eta_L) f_{\langle s,s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(\eta_{T(L)} \eta_L \circ f_{\langle s,s\rangle})\phi_j, \mu_L \mu_{T(L)} T(\eta_{T(L)} \eta_L \circ f_{\langle s,s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)}\eta_L \circ s, \eta_{T(L)}\eta_L \circ s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)}\eta_L \circ s, \eta_{T(L)}\eta_L \circ s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(f_{\langle \mu_{T(L)} T(\eta_{T(L)}\eta_L)s, \mu_{T(L)} T(\eta_{T(L)}\eta_L)s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(f_{\langle \mu_{T(L)} T(\eta_{T(L)}\eta_L)s, \mu_{T(L)} T(\eta_{T(L)}\eta_L)s\rangle})\psi_j\rangle : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(f_{\langle \mu_{T(L)} T(\eta_{T(L)})T(\eta_L)s, \mu_{T(L)} T(\eta_{T(L)})T(\eta_L)s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(f_{\langle \mu_{T(L)} T(\eta_{T(L)})T(\eta_L)s, \mu_{T(L)} T(\eta_{T(L)})T(\eta_L)s\rangle})\psi_j : j \in J\} \\
&= \{\langle \mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)} \mu_L T(\eta_L)s, \eta_{T(L)} \mu_L T(\eta_L)s\rangle})\phi_j, \\
&\qquad \mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)} \mu_L T(\eta_L)s, \eta_{T(L)} \mu_L T(\eta_L)s\rangle})\psi_j\rangle : j \in J\}
\end{aligned}$$

$$\begin{aligned} &= \{\langle \mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)} s, \eta_{T(L)} s\rangle})\phi_j, \mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)} s, \eta_{T(L)} s\rangle})\psi_j\rangle : j \in J\} \\ &= \{\alpha^j_{\langle T(L), \mu_L\rangle}(s) : j \in J\} \end{aligned}$$

as required.

(ii) Let $l_0, l_1, f_{\langle s,t\rangle}$ be as in (4.1).

$$\begin{tikzcd} 1 \arrow[r, no head, "l_0"] \arrow[dr, no head, "s"'] & L_0 \arrow[d, "f_{\langle s,t\rangle}"] & 1 \arrow[l, no head, "l_1"'] \arrow[dl, "t"] \\ & L & \end{tikzcd}$$

Then

$$\begin{aligned} \beta_L(\langle s, t\rangle) &= \beta_L(\mathcal{K}_{\mathbf{T}}(1, f_{\langle s,t\rangle})^2(\langle l_0, l_1\rangle)) \\ &= \mathcal{K}_{\mathbf{T}}(1, f_{\langle s,t\rangle})(\beta_{L_0}(\langle l_0, l_1\rangle)), \end{aligned}$$

by commutativity of
$$\begin{tikzcd} \mathcal{K}_{\mathbf{T}}(1, L_0)^2 \arrow[r, "\beta_{L_0}"] \arrow[d, "\mathcal{K}_{\mathbf{T}}(1, f_{\langle s,t\rangle})^2"'] & \mathcal{P}(\mathcal{K}_{\mathbf{T}}(1, L_0)) \arrow[d, "\mathcal{P}\mathcal{K}_{\mathbf{T}}(1, f_{\langle s,t\rangle})"] \\ \mathcal{K}_{\mathbf{T}}(1, L)^2 \arrow[r, "\beta_L"'] & \mathcal{P}(\mathcal{K}_{\mathbf{T}}(1, L)) \end{tikzcd}$$

Let $\beta_{L_0}(\langle l_0, l_1\rangle) = \{\phi_i : i \in I\}$. Then, for every $\langle K, \xi\rangle \in |\mathcal{K}^{\mathbf{T}}|, i \in I$, define $\beta^i_{\langle K, \xi\rangle} : \mathcal{K}(1, K)^2 \to \mathcal{K}(1, K)$ by

$$\beta^i_{\langle K,\xi\rangle}(\langle \theta_1, \theta_2\rangle) = \xi \mu_K T(f_{\langle \eta_K \theta_1, \eta_K \theta_2\rangle})\phi_i, \quad \text{for every} \quad \theta_1, \theta_2 \in \mathcal{K}(1, K).$$

We first check that $\beta^i : \mathcal{K}(1, U^{\mathbf{T}})^2 \to \mathcal{K}(1, U^{\mathbf{T}})$ is a natural transformation. Let $\langle K, \xi\rangle, \langle L, \zeta\rangle \in |\mathcal{K}^{\mathbf{T}}|$ and $h : \langle K, \xi\rangle \to \langle L, \zeta\rangle \in \mathrm{Mor}(\mathcal{K}^{\mathbf{T}})$. We need to show that the following diagram commutes. If $\theta_1, \theta_2 \in \mathcal{K}(1, K)$, we have

$$\begin{tikzcd} \mathcal{K}(1, K)^2 \arrow[r, "\beta^i_{\langle K,\xi\rangle}"] \arrow[d, "\mathcal{K}(1,h)^2"'] & \mathcal{K}(1, K) \arrow[d, "\mathcal{K}(1,h)"] \\ \mathcal{K}(1, L)^2 \arrow[r, "\beta^i_{\langle L,\zeta\rangle}"'] & \mathcal{K}(1, L) \end{tikzcd}$$

$$\begin{aligned} \mathcal{K}(1, h)(\beta^i_{\langle K,\xi\rangle}(\langle \theta_1, \theta_2\rangle)) &= \mathcal{K}(1, h)(\xi \mu_K T(f_{\langle \eta_K \theta_1, \eta_K \theta_2\rangle})\phi_i \\ &= h \xi \mu_K T(f_{\langle \eta_K \theta_1, \eta_K \theta_2\rangle})\phi_i \\ &= \zeta T(h) \mu_K T(f_{\langle \eta_K \theta_1, \eta_K \theta_2\rangle})\phi_i \end{aligned}$$

by commutativity of
$$\begin{tikzcd} T(K') \arrow[r, "T(h)"] \arrow[d, "\xi"'] & T(L) \arrow[d, "\zeta"] \\ K' \arrow[r, "h"'] & L \end{tikzcd}$$

$$= \zeta\mu_L T(T(h))T(f_{\langle \eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle})\phi_i$$

by commutativity of
$$\begin{tikzcd} T(T(K')) \arrow[r, "\mu_{K'}"] \arrow[d, "T(T(h))"'] & T(K') \arrow[d, "T(h)"] \\ T(T(L)) \arrow[r, "\mu_L"'] & T(L) \end{tikzcd}$$

$$= \zeta\mu_L T(T(h)f_{\langle \eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle})\phi_i$$

$$= \zeta\mu_L T(T(h)\mu_{K'}T(\eta_{K'})f_{\langle \eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle})\phi_i$$

by commutativity of
$$\begin{tikzcd} T(K') \arrow[r, "T(\eta_{K'})"] \arrow[dr, "i_{T(K')}"'] & T(T(K')) \arrow[d, "\mu_{K'}"] \\ & T(K') \end{tikzcd}$$

$$\begin{aligned}
&= \zeta\mu_L T(\mu_L T(T(h))T(\eta_{K'})f_{\langle \eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(\mu_L T(T(h)\eta_{K'})f_{\langle \eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(T(h)\eta_{K'} \circ f_{\langle \eta_{K'}\theta_1, \eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(f_{\langle T(h)\eta_{K'}\circ\eta_{K'}\theta_1, T(h)\eta_{K'}\circ\eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(f_{\langle \mu_L T(T(h)\eta_{K'})\eta_{K'}\theta_1, \mu_L T(T(h)\eta_{K'})\eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(f_{\langle \mu_L T(T(h))T(\eta_{K'})\eta_{K'}\theta_1, \mu_L T(T(h))T(\eta_{K'})\eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(f_{\langle T(h)\mu_{K'}T(\eta_{K'})\eta_{K'}\theta_1, T(h)\mu_{K'}T(\eta_{K'})\eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(f_{\langle T(h)\eta_{K'}\theta_1, T(h)\eta_{K'}\theta_2\rangle})\phi_i \\
&= \zeta\mu_L T(f_{\langle \eta_L h\theta_1, \eta_L h\theta_2\rangle})\phi_i
\end{aligned}$$

by commutativity of
$$\begin{tikzcd} K' \arrow[r, "\eta_{K'}"] \arrow[d, "h"'] & T(K') \arrow[d, "T(h)"] \\ L \arrow[r, "\eta_L"'] & T(L) \end{tikzcd}$$

$$= \beta^i_{\langle L,\zeta\rangle}(\langle h\theta_1, h\theta_2\rangle)$$
$$= \beta^i_{\langle L,\zeta\rangle}(\mathcal{K}(1,h)^2(\langle\theta_1,\theta_2\rangle)),$$

as required. Finally, we have

$$\begin{aligned}
\beta_L(\langle s,t\rangle) &= \mathcal{K}_{\mathbf{T}}(1, f_{\langle s,t\rangle})(\{\phi_i : i \in I\}) \\
&= \{f_{\langle s,t\rangle} \circ \phi_i : i \in I\} \\
&= \{\mu_L T(f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(\eta_{T(L)}) T(f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(\eta_{T(L)} f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(\eta_{T(L)} \mu_L T(\eta_L) f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(\mu_{T(L)} T(\eta_{T(L)}) T(\eta_L) f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(\mu_{T(L)} T(\eta_{T(L)}\eta_L) f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(\eta_{T(L)}\eta_L \circ f_{\langle s,t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)}\eta_L \circ s, \eta_{T(L)}\eta_L \circ t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(f_{\langle \mu_{T(L)} T(\eta_{T(L)}\eta_L) s, \mu_{T(L)} T(\eta_{T(L)}\eta_L) t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(f_{\langle \mu_{T(L)} T(\eta_{T(L)}) T(\eta_L) s, \mu_{T(L)} T(\eta_{T(L)}) T(\eta_L) t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)} \mu_L T(\eta_L) s, \eta_{T(L)} \mu_L T(\eta_L) t\rangle})\phi_i : i \in I\} \\
&= \{\mu_L \mu_{T(L)} T(f_{\langle \eta_{T(L)} s, \eta_{T(L)} t\rangle})\phi_i : i \in I\} \\
&= \{\beta^i_{\langle T(L),\mu_L\rangle}(\langle s,t\rangle) : i \in I\}
\end{aligned}$$

as required. ■

LEMMA 4.11 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ be an a.a. theory institution via the interpretations $\langle I_{\mathcal{L}}, \alpha\rangle : \mathcal{I} \to \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}}$ and $\langle I_{\mathcal{L}}, \beta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$. Then, for all $L \in |\mathcal{L}|, s, t \in \mathcal{K}_{\mathbf{T}}(1, L)$,*

$$t \in (\{s\} \cup \beta_L(\langle s,t\rangle))^c.$$

Proof:

Let $\langle K, \zeta\rangle \in |\mathcal{Q}|$ and $f \in \mathcal{K}_{\mathbf{T}}(L, K)$, such that $\alpha_L(s) \subseteq \langle\langle K,\zeta\rangle, f\rangle^*$ and $\langle s,t\rangle \in \langle\langle K,\zeta\rangle, f\rangle^*$. Then $(\zeta\mu_K T(f))^2\alpha_L(s) \subseteq \Delta_{\mathcal{K}(1,K)}$ and $\zeta\mu_K T(f)s = \zeta\mu_K T(f)t$, where, by

$\Delta_{\mathcal{K}(1,K)}$, we denote the identity binary relation on $\mathcal{K}(1,K)$. Then, by Lemma 4.10, we get, $(\zeta\mu_K T(f))^2\alpha^j_{\langle T(L),\mu_L\rangle}(s) \subseteq \Delta_{\mathcal{K}(1,K)}$, for every $j \in J$, and $\zeta\mu_K T(f)s = \zeta\mu_K T(f)t$. But $\zeta\mu_K T(f) : \langle T(L), \mu_L\rangle \to \langle K, \zeta\rangle \in \mathrm{Mor}(\mathcal{K}^{\mathbf{T}})$, whence, since $\alpha^j : \mathcal{K}(1, U^{\mathbf{T}}) \to \mathcal{K}(1, U^{\mathbf{T}})^2$ is a natural transformation, we have

$$\begin{array}{ccc} \mathcal{K}(1,T(L)) & \xrightarrow{\alpha^j_{\langle T(L),\mu_L\rangle}} & \mathcal{K}(1,T(L))^2 \\ \Big\downarrow{\scriptstyle \mathcal{K}(1,\zeta\mu_K T(f))} & & \Big\downarrow{\scriptstyle \mathcal{K}(1,\zeta\mu_K T(f))^2} \\ \mathcal{K}(1,K) & \xrightarrow[\alpha^j_{\langle K,\zeta\rangle}]{} & \mathcal{K}(1,K)^2) \end{array}$$

$\alpha^j_{\langle K,\zeta\rangle}(\zeta\mu_K T(f)s) \subseteq \Delta_{\mathcal{K}(1,K)}$ and, therefore, $\alpha^j_{\langle K,\zeta\rangle}(\zeta\mu_K T(f)t) \subseteq \Delta_{\mathcal{K}(1,K)}$, for every $j \in J$. Thus, $(\zeta\mu_K T(f))^2\alpha^j_{\langle T(L),\mu_L\rangle}(t) \subseteq \Delta_{\mathcal{K}(1,K)}$, for every $j \in J$, whence $(\zeta\mu_K T(f))^2\alpha_L(t) \subseteq \Delta_{\mathcal{K}(1,K)}$. Hence $\alpha_L(t) \subseteq \langle\langle K,\zeta\rangle, f\rangle^*$. Therefore $\alpha_L(t) \subseteq (\{\langle s,t\rangle\} \cup \alpha_L(s))^c$ and, since $\langle I_{\mathcal{L}}, \beta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$ is an interpretation,

$$\beta_L(\alpha_L(t))^c \subseteq (\beta_L(\langle s,t\rangle) \cup \beta_L(\alpha_L(s))^c)^c,$$

and therefore

$$t \in (\{s\} \cup \beta_L(\langle s,t\rangle))^c,$$

as required. ∎

THEOREM 4.12 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ be an a.a. theory institution and suppose there exist interpretations $\langle I_{\mathcal{L}}, \alpha\rangle : \mathcal{I} \to \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}}$, $\langle I_{\mathcal{L}}, \beta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$ and $\langle I_{\mathcal{L}}, \gamma\rangle : \mathcal{I} \to \mathcal{I}^{\mathcal{L}}_{\mathcal{P}}$, $\langle I_{\mathcal{L}}, \delta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{P}} \to \mathcal{I}$ that both autoalgebraize $\mathcal{I}$. Then, for all $L \in |\mathcal{L}|, s,t \in \mathcal{K}_{\mathbf{T}}(1,L)$,*

$$\alpha_L(s)^{c_{\mathcal{Q}}} = \gamma_L(s)^{c_{\mathcal{P}}} \quad \textit{and} \quad \beta_L(\langle s,t\rangle)^c = \delta_L(\langle s,t\rangle)^c.$$

Proof:

Because of symmetry, it suffices to show that $\alpha_L(s)^{c_{\mathcal{Q}}} = \gamma_L(s)^{c_{\mathcal{P}}}$ and $\delta_L(\langle s,t\rangle) \subseteq \beta_L(\langle s,t\rangle)^c$, for all $L \in |\mathcal{L}|, s,t \in \mathcal{K}_{\mathbf{T}}(1,L)$.

Let $\delta^i : \mathcal{K}(1,U^{\mathbf{T}})^2 \to \mathcal{K}(1,U^{\mathbf{T}})$ be the natural transformations associated with δ via Lemma 4.10. By Lemma 4.9(iv),

$$\beta_L(\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,s_0\rangle), \delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,t_0\rangle)) \subseteq \beta_L(\langle s_0,t_0\rangle)^c, i \in I. \qquad (4.3)$$

By Lemmas 4.11 and 4.9(i),

$$\begin{aligned}\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,t_0\rangle) &\in (\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,s_0\rangle)\cup \\ &\qquad \beta_L(\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,s_0\rangle),\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,t_0\rangle)))^c \\ &\subseteq \beta_L(\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,s_0\rangle),\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,t_0\rangle))^c.\end{aligned}$$

Thus, by (4.3),

$$\delta^i_{\langle T(L),\mu_L\rangle}(\langle s_0,t_0\rangle) \subseteq \beta_L(\langle s_0,t_0\rangle)^c, i \in I,$$

and therefore

$$\delta_L(\langle s_0,t_0\rangle) \subseteq \beta_L(\langle s_0,t_0\rangle)^c,$$

as required.

For the first equality we get

$$\gamma_L(s)^{c\mathcal{Q}} = \alpha_L(s)^{c\mathcal{P}} \quad \text{iff}$$

$$\beta_L(\gamma_L(s))^c = \beta_L(\alpha_L(s))^c \quad \text{iff, by the first part,}$$

$$\delta_L(\gamma_L(s))^c = \beta_L(\alpha_L(s))^c \quad \text{iff}$$

$$\{s\}^c = \{s\}^c,$$

as required. ■

This theorem shows that for an a.a. theory institution the adjoint equivalence between the theory of categories investigated in the previous chapters of the thesis must be unique.

THEOREM 4.13 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ be an a.a. theory institution via the interpretations $\langle I_{\mathcal{L}},\alpha\rangle : \mathcal{I} \to \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}}$ and $\langle I_{\mathcal{L}},\beta\rangle : \mathcal{I}^{\mathcal{L}}_{\mathcal{Q}} \to \mathcal{I}$. Then, for all $L \in |\mathcal{L}|, \langle L,T\rangle \in |\mathbf{TH}(\mathcal{I})|$,*

$$\alpha_L(T)^c = \Omega_{\langle T(L),\mu_L\rangle}(T).$$

Proof:

By Theorem 4.8, it suffices to show that $\alpha_L(T)^c$ is a congruence of $\langle T(L), \mu_L\rangle$ that is compatible with T and explicitly definable over $\langle T(L), \mu_L\rangle$ and T.

By Lemma 4.5, $\alpha_L(T)^c$ is a congruence of $\langle T(L), \mu_L\rangle$. It is explicitly definable over $\langle T(L), \mu_L\rangle$ and T, since

$$\begin{aligned}\alpha_L(T)^c &= \{\langle\theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \beta_L(\langle\theta_1, \theta_2\rangle) \subseteq T\}\\ &= \{\langle\theta_1, \theta_2\rangle \in \mathcal{K}(1, T(L))^2 : \beta^i_{\langle T(L), \mu_L\rangle}(\langle\theta_1, \theta_2\rangle) \in T, i \in I\},\end{aligned}$$

where $\beta^i : \mathcal{K}(1, U^{\mathbf{T}})^2 \to \mathcal{K}(1, U^{\mathbf{T}}), i \in I$, are the natural transformations associated with β via Lemma 4.10.

For compatibility with T, let $\langle\theta_1, \theta_2\rangle \in \alpha_L(T)^c$ and $\theta_1 \in T$. Then $\beta_L(\langle\theta_1, \theta_2\rangle) \subseteq T$ and $\theta_1 \in T$. But, by Lemma 4.11,

$$\theta_2 \in (\{\theta_1\} \cup \beta_L(\langle\theta_1, \theta_2\rangle))^c \subseteq T^c = T,$$

as required. ∎

Properties of the Leibniz Operator

In this section, several properties of the Leibniz operator are introduced, that will be used in the next section to obtain a partial characterization result for autoalgebraizability for a special class of theory institutions. The notion of the Leibniz functor, which is, essentially, an extension of the Leibniz operator to theory morphisms is also introduced.

Recall that, given a theory institution $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L\in|\mathcal{L}|}\rangle$, $L \in |\mathcal{L}|$ and $\langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|$,

$$\begin{aligned}\Omega_L(T) = \{\langle\theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : &\quad \text{for all } n \in \omega, \quad \text{natural transformation}\\ &\tau : \mathcal{K}(1, U^{\mathbf{T}})^{n+1} \to \mathcal{K}(1, U^{\mathbf{T}}) \quad \text{and} \quad r_0, \ldots, r_{n-1} \in \mathcal{K}_{\mathbf{T}}(1, L),\\ &\tau_{\langle T(L), \mu_L\rangle}(\theta_1, \vec{r}) \in T \quad \text{if and only if} \quad \tau_{\langle T(L), \mu_L\rangle}(\theta_2, \vec{r}) \in T\}\end{aligned}$$

Ω will be said to be **join-continuous** if, for all $L \in |\mathcal{L}|, \Phi \subseteq \mathcal{K}_{\mathbf{T}}(1, L)$,

$$(\bigcup_{\phi\in\Phi} \Omega_L(\{\phi\}^c))^c = \Omega_L(\Phi^c).$$

It is said to be **injective** if, for all $L \in |\mathcal{L}|, \langle L, T_1\rangle, \langle L, T_2\rangle \in |\mathbf{TH}(\mathcal{I})|$,

$$\langle L, T_1\rangle \neq \langle L, T_2\rangle \quad \text{implies} \quad \Omega_L(T_1) \neq \Omega_L(T_2).$$

Finally, it is said to **commute with substitutions** if, for all $L \in |\mathcal{L}|, \langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|$ and $f \in \mathcal{L}(L, K)$,

$$\mathcal{K}_{\mathbf{T}}(1, f)^2(\Omega_L(T))^c = \Omega_K(\mathcal{K}_{\mathbf{T}}(1, f)(T)^c).$$

LEMMA 4.14 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L \in |\mathcal{L}|}\rangle$ be a theory institution. If Ω is join-continuous, then for all $L \in |\mathcal{L}|, \langle L, T_1\rangle, \langle L, T_2\rangle \in |\mathbf{TH}(\mathcal{I})|$,*

$$T_1 \subseteq T_2 \quad \textit{implies} \quad \Omega_L(T_1) \subseteq \Omega_L(T_2).$$

Proof:

$$\Omega_L(T_1) = (\bigcup_{t \in T_1} \Omega_L(\{t\}^c))^c \subseteq (\bigcup_{t \in T_2} \Omega_L(\{t\}^c))^c = \Omega_L(T_2),$$

as required. ∎

When the Leibniz operator satisfies the conclusion of the above Lemma it will be said to be **monotonic**.

LEMMA 4.15 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L \in |\mathcal{L}|}\rangle$ be an a.a. theory institution. Then the Leibniz operator is injective, join-continuous and commutes with substitutions.*

Proof:

All properties are direct consequences of Corollary 4.4 and Theorem 4.13. ∎

Next, suppose that $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L \in |\mathcal{L}|}\rangle$ is a theory institution, such that, for all $L \in |\mathcal{L}|, \langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|$, $\langle L, \Omega_L(T)\rangle \in |\mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{K}\mathbf{T}})|$, and that the Leibniz operator is injective, join-continuous and commutes with substitutions. We define the **Leibniz functor** $\Omega : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{K}\mathbf{T}})$, as follows:

$$\Omega(\langle L, T\rangle) = \langle L, \Omega_L(T)\rangle, \quad \text{for every} \quad \langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|,$$

and

$$\Omega(f) = f, \quad \text{for every} \quad f : \langle L, T\rangle \to \langle L', T'\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I})).$$

To see that Ω is well defined at the morphism level, note that

$$\begin{aligned}
\mathcal{K}_{\mathbf{T}}(1, f)^2(\Omega_L(T))^c &= \Omega_{L'}(\mathcal{K}_{\mathbf{T}}(1, f)(T)^c), \quad \text{by commutativity with substitutions,} \\
&\subseteq \Omega_{L'}(T'), \quad \text{by Lemma 4.14, since } f \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I})),
\end{aligned}$$

as required.

Since Ω is the identity on morphisms, it is clearly a functor.

An Intrinsic Characterization

As before, let $\mathcal{K}$ be a locally small category with a terminal object 1, such that $1 \sqcup 1$ exists in $\mathcal{K}$, $\mathbf{T} = \langle T, \eta, \mu\rangle$ an algebraic theory in monoid form over $\mathcal{K}$ and $\mathcal{L}$ a full subcategory of the Kleisli category $\mathcal{K}_{\mathbf{T}}$ of $\mathbf{T}$ in $\mathcal{K}$, satisfying (4.1). Recall also that $\mathcal{K}^{\mathbf{T}}$ is the Eilenberg-Moore category of $\mathbf{T}$-algebras over $\mathcal{K}$.

LEMMA 4.16 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1, -), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ be a theory institution, such that the Leibniz operator Ω is join-continuous. Then, for every collection $\{\langle L, T_i\rangle : i \in I\}$ of L-theories,*

$$(\bigcup_{i\in I} \Omega_L(T_i))^c = \Omega_L((\bigcup_{i\in I} T_i)^c).$$

Proof:

We have

$$\begin{aligned}
\Omega_L((\textstyle\bigcup_{i\in I} T_i)^c) &= (\textstyle\bigcup_{\phi\in\bigcup_{i\in I} T_i} \Omega_L(\{\phi\}^c))^c, \quad \text{by join-continuity,} \\
&= (\textstyle\bigcup_{i\in I} \bigcup_{\phi\in T_i} \Omega_L(\{\phi\}^c))^c \\
&= (\textstyle\bigcup_{i\in I} (\bigcup_{\phi\in T_i} \Omega_L(\{\phi\}^c))^c)^c \\
&= (\textstyle\bigcup_{i\in I} \Omega_L(T_i^c))^c, \quad \text{by join-continuity,} \\
&= (\textstyle\bigcup_{i\in I} \Omega_L(T_i))^c,
\end{aligned}$$

as required. ∎

LEMMA 4.17 *Let $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ be a theory institution, such that the Leibniz operator Ω is monotonic. Then,*

(i) For every collection $\{\langle L, T_i\rangle : i \in I\} \subseteq |\mathbf{TH}(\mathcal{I})|$,

$$\Omega_L(\bigcap_{i\in I} T_i) = \bigcap_{i\in I} \Omega_L(T_i).$$

(ii) For all $\langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|$, $f \in \mathcal{L}(L, K)$, such that $\mathcal{K}_{\mathbf{T}}(1, f)$ is onto,

$$\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(\Omega_K(T)) = \Omega_L(\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T)).$$

Proof:

(i) Since Ω is monotonic, $\Omega_L(\bigcap_{i\in I} T_i) \subseteq \bigcap_{i\in I} \Omega_L(T_i)$. For the reverse inclusion, it suffices, by Theorem 4.7, to show that $\bigcap_{i\in I} \Omega_L(T_i)$ is compatible with $\bigcap_{i\in I} T_i$. But, if $\langle \theta_1, \theta_2\rangle \in \bigcap_{i\in I} \Omega_L(T_i)$ and $\theta_1 \in \bigcap_{i\in I} T_i$, we have $\langle \theta_1, \theta_2\rangle \in \Omega_L(T_i)$ and $\theta_1 \in T_i$, for every $i \in I$, whence, by the compatibility of $\Omega_L(T_i)$ with T_i, $\theta_2 \in T_i$, for every $i \in I$, i.e., $\theta_2 \in \bigcap_{i\in I} T_i$, as required.

(ii) We first show that $\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(\Omega_K(T))$ is compatible with $\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T)$ and, hence, that $\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(\Omega_K(T)) \subseteq \Omega_L(\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T))$.

Let $\langle \theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1,f)^{-1}(\Omega_K(T))$ and $\theta_1 \in \mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T)$. Then $\langle f\circ\theta_1, f\circ\theta_2\rangle \in \Omega_K(T)$ and $f\circ\theta_1 \in T$, whence $f\circ\theta_2 \in T$, and therefore $\theta_2 \in \mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T)$, as required.

For the reverse inclusion, suppose that $\langle \theta_1, \theta_2\rangle \in \Omega_L(\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T))$ and assume, to the contrary, that $\langle \theta_1, \theta_2\rangle \notin \mathcal{K}_{\mathbf{T}}(1,f)^{-1}(\Omega_K(T))$. Then, there exists $n \in \omega$, a natural transformation $\tau : \mathcal{K}(1, U^{\mathbf{T}})^{n+1} \to \mathcal{K}(1, U^{\mathbf{T}})$ and $r_0, \ldots, r_{n-1} \in \mathcal{K}_{\mathbf{T}}(1, K)$, such that

$$\tau_{\langle T(K),\mu_K\rangle}(f\circ\theta_1, \vec{r}) \in T \quad \text{but} \quad \tau_{\langle T(K),\mu_K\rangle}(f\circ\theta_2, \vec{r}) \notin T \quad \text{or vice-versa.}$$

By surjectivity of $\mathcal{K}_{\mathbf{T}}(1,f)$, there exist $s_0, \ldots, s_{n-1} \in \mathcal{K}_{\mathbf{T}}(1, L)$, such that $f\circ s_i = r_i, i < n$, whence

$$\tau_{\langle T(K),\mu_K\rangle}(f\circ\theta_1, f\circ\vec{s}) \in T \quad \text{but} \quad \tau_{\langle T(K),\mu_K\rangle}(f\circ\theta_2, f\circ\vec{s}) \notin T,$$

$$\text{i.e.,} \quad \tau_{\langle T(K),\mu_K\rangle}(\mu_K T(f)\theta_1, \mu_K T(f)\vec{s}) \in T \quad \text{but} \quad \tau_{\langle T(K),\mu_K\rangle}(\mu_K T(f)\theta_2, \mu_K T(f)\vec{s}) \notin T,$$

and, thus, by commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{K}(1,T(L))^{n+1} \arrow[r, "\tau_{\langle T(L),\mu_L\rangle}"] \arrow[d, "(\mu_K T(f))^{n+1}"'] & \mathcal{K}(1,T(L)) \arrow[d, "\mu_K T(f)"] \\
\mathcal{K}(1,T(K))^{n+1} \arrow[r, "\tau_{\langle T(K),\mu_K\rangle}"'] & \mathcal{K}(1,T(K))
\end{tikzcd}$$

$$\mu_K T(f)\tau_{\langle T(L),\mu_L\rangle}(\theta_1,\vec{s}) \in T \quad \text{but} \quad \mu_K T(f)\tau_{\langle T(L),\mu_L\rangle}(\theta_2,\vec{s}) \notin T,$$

$$\text{i.e.,} \quad \mathcal{K}_{\mathbf{T}}(1,f)(\tau_{\langle T(L),\mu_L\rangle}(\theta_1,\vec{s})) \in T \quad \text{but} \quad \mathcal{K}_{\mathbf{T}}(1,f)(\tau_{\langle T(L),\mu_L\rangle}(\theta_2,\vec{s})) \notin T,$$

whence $\langle\theta_1,\theta_2\rangle \notin \Omega_L(\mathcal{K}_{\mathbf{T}}(1,f)^{-1}(T))$, contradicting our hypothesis. ∎

DEFINITION 4.18 *1. Let* $\langle K',\xi\rangle \in |\mathcal{K}^{\mathbf{T}}|$. *A morphism* $f \in \mathcal{K}_{\mathbf{T}}(K,K')$ *will be said to be* **special with respect to** $\mathcal{L}$ *if, for every* $L \in |\mathcal{L}|$, $\theta_1,\theta_2 \in \mathcal{K}_{\mathbf{T}}(1,L), g \in \mathcal{K}_{\mathbf{T}}(L,K')$, *such that* $\xi\mu_{K'}T(g)\theta_1 = \xi\mu_{K'}T(g)\theta_2$,

$$\begin{tikzcd}
K \arrow[rr, "f"] & & K' \\
& L \arrow[ul, dashed, "h"] \arrow[ur, "g"'] & \\
1 \arrow[ur, shift left, "\theta_1"] \arrow[ur, shift right, "\theta_2"'] & &
\end{tikzcd}$$

there exists $h \in \mathcal{K}_{\mathbf{T}}(L,K)$, *such that*

- $\xi\mu_{K'}T(g)\theta_i = \xi\mu_{K'}T(f\circ h)\theta_i, i = 1,2,$
- $\mathcal{K}_{\mathbf{T}}(1,h)$ *is surjective.*

2. **T** *will be said to* **simply create theories** *for some theory institution* $\mathcal{I} = \langle\mathcal{L},\mathcal{K}_{\mathbf{T}}(1,-),\{C_L\}_{L\in|\mathcal{L}|}\rangle$ *if, for every* $\langle L,T\rangle \in |\mathbf{TH}(\mathcal{I})|$, *there exists* $\langle K,\xi\rangle \in |\mathcal{K}^{\mathbf{T}}|, f \in \mathcal{K}_{\mathbf{T}}(L,K)$, *such that*

$$\Omega_L(T) = \{\langle\theta_1,\theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1,L)^2 : \xi\mu_K T(f)\theta_1 = \xi\mu_K T(f)\theta_2\}.$$

3. **T** *will be said to* **specially create theories** *for some theory institution* $\mathcal{I} = \langle\mathcal{L},\mathcal{K}_{\mathbf{T}}(1,-),\{C_L\}_{L\in|\mathcal{L}|}\rangle$ *if it simply creates theories for* $\mathcal{I}$ *and, moreover, the* f*'s in (2) can be chosen to be special with respect to* $\mathcal{L}$.

4. *Finally, a theory institution* $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ *is said to be* **Blok-Pigozzi** *if* $\mathbf{T}$ *specially creates theories for* $\mathcal{I}$.

THEOREM 4.19 *A Blok-Pigozzi theory institution* $\mathcal{I} = \langle \mathcal{L}, \mathcal{K}_{\mathbf{T}}(1,-), \{C_L\}_{L\in|\mathcal{L}|}\rangle$ *is auto-algebraizable iff the Leibniz operator is injective, join-continuous and commutes with substitutions.*

Proof:

A stronger "only if" was proved in Lemma 4.15 without the requirement that $\mathcal{I}$ be Blok-Pigozzi.

By Corollary 4.4, for the "if" part, it suffices to show that there exists a subcategory $\mathcal{Q}$ of $\mathcal{K}^{\mathbf{T}}$ and an isomorphism $\Omega : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{Q}})$ that makes the following diagram commute

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}) \arrow[rr, "\Omega"] \arrow[dr, "\mathrm{SIG}"'] & & \mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{Q}}) \arrow[dl, "\mathrm{SIG}"] \\ & \mathcal{L} & \end{tikzcd}$$

and commutes with substitutions. Since $\mathcal{I}$ is Blok-Pigozzi and the Leibniz operator is injective, join-continuous and commutes with substitutions, we can define the Leibniz functor $\Omega : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{K}^{\mathbf{T}}})$. Since $\mathcal{I}$ is Blok-Pigozzi, $\mathbf{T}$ specially creates theories. Thus, for every $\langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|$, there exists $\mathbf{K}_{\langle L,T\rangle} = \langle K, \xi\rangle \in |\mathcal{K}^{\mathbf{T}}|, f \in \mathcal{K}_{\mathbf{T}}(L, K)$, such that

$$\Omega_L(T) = \{\langle \theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi\mu_K T(f)\theta_1 = \xi\mu_K T(f)\theta_2\}$$

and f is special with respect to $\mathcal{L}$.

Let $\mathcal{Q}$ be the full subcategory of $\mathcal{K}^{\mathbf{T}}$ with objects

$$|\mathcal{Q}| = \{\mathbf{K}_{\langle L,T\rangle} : \langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|\}.$$

Clearly, $\Omega(|\mathbf{TH}(\mathcal{I})|) \subseteq |\mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{Q}})|$. We first show that $\Omega : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}^{\mathcal{L}}_{\mathcal{Q}})$ is an isomorphism and, then, that it commutes with substitutions, since commutativity of the triangle is straightforward.

Since Ω is the identity on morphisms, it is full and faithfull. So it suffices to show that it induces a bijection $\Omega : |\mathbf{TH}(\mathcal{I})| \to |\mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})|$. Injectivity is guaranteed by our assumption on Ω. So it suffices to show that Ω is surjective.

Let $\langle L, \Theta\rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\mathcal{L}})|$. then, there exists a collection $\langle L_i, T_i\rangle \in |\mathbf{TH}(\mathcal{I})|, i \in I$, and $g_i \in \mathcal{K}_{\mathbf{T}}(L, K_i)$, such that

$$\Theta = \{\langle\theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi_i\mu_{K_i}T(g_i)\theta_1 = \xi_i\mu_{K_i}T(g_i)\theta_2 \quad \forall i \in I\},$$

where $\langle K_i, \xi_i\rangle = \mathbf{K}_{\langle L_i, T_i\rangle} \in |\mathcal{Q}|$. Let $f_i \in \mathcal{K}_{\mathbf{T}}(L_i, K_i)$ be the special morphism associated with $\langle L_i, T_i\rangle$. Clearly,

$$\Theta = \bigcap_{i\in I}\{\langle\theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi_i\mu_{K_i}T(g_i)\theta_1 = \xi_i\mu_{K_i}T(g_i)\theta_2\},$$

whence, by Lemma 4.17(i), it suffices to show that

$$\Theta_i = \{\langle\theta_1, \theta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi_i\mu_{K_i}T(g_i)\theta_1 = \xi_i\mu_{K_i}T(g_i)\theta_2\} \in \Omega(|\mathbf{TH}(\mathcal{I})|).$$

Since f_i is special with respect to $\mathcal{L}$, there exists $h_{\langle\theta_1, \theta_2\rangle} \in \mathcal{K}_{\mathbf{T}}(L, L_i)$, such that

- $\xi_i\mu_{K_i}T(g_i)\theta_j = \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\theta_j, j = 1, 2,$
- $\mathcal{K}_{\mathbf{T}}(1, h_{\langle\theta_1,\theta_2\rangle})$ is surjective.

We claim that

$$\Theta_i = (\bigcup_{\langle\theta_1,\theta_2\rangle\in\Theta_i} \{\langle\delta_1, \delta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\delta_2\})^c.$$

Left to right inclusion is clear. For right to left, note that, for every $\langle\theta_1, \theta_2\rangle \in \Theta_i$,

$$\{\langle\delta_1, \delta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\delta_2\} \subseteq \{\langle\theta_1, \theta_2\rangle\}^c,$$

whence

$$(\bigcup_{\langle\theta_1,\theta_2\rangle\in\Theta_i} \{\langle\delta_1, \delta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1, L)^2 : \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i \circ h_{\langle\theta_1,\theta_2\rangle})\delta_2\})^c \subseteq$$

$$\subseteq \Big(\bigcup_{\langle\theta_1,\theta_2\rangle\in\Theta_i} \{\langle\theta_1,\theta_2\rangle\}^c\Big)^c = \Theta_i^c = \Theta_i,$$

as claimed.

Hence, by Lemma 4.16, it suffices to show that

$$\{\langle\delta_1,\delta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1,L)^2 : \xi_i\mu_{K_i}T(f_i\circ h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i\circ h_{\langle\theta_1,\theta_2\rangle})\delta_2\} \in \Omega(|\mathbf{TH}(\mathcal{I})|).$$

We have

$$\xi_i\mu_{K_i}T(f_i\circ h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i\circ h_{\langle\theta_1,\theta_2\rangle})\delta_2 \quad \text{iff, by the def. of Kleisli comp.,}$$

$$\xi_i\mu_{K_i}T(\mu_{K_i}T(f_i)h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(\mu_{K_i}T(f_i)h_{\langle\theta_1,\theta_2\rangle})\delta_2 \quad \text{iff, since } T \text{ is a functor,}$$

$$\xi_i\mu_{K_i}T(\mu_{K_i})T(T(f_i))T(h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(\mu_{K_i})T(T(f_i))T(h_{\langle\theta_1,\theta_2\rangle})\delta_2 \quad \text{iff}$$

by commutativity of
$$\begin{tikzcd} TTT(K_i) \arrow[r, "T(\mu_{K_i})"] \arrow[d, "\mu_{T(K_i)}"'] & TT(K_i) \arrow[d, "\mu_{K_i}"] \\ TT(K_i) \arrow[r, "\mu_{K_i}"'] & T(K_i) \end{tikzcd}$$

$$\xi_i\mu_{K_i}\mu_{T(K_i)}T(T(f_i))T(h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}\mu_{T(K_i)}T(T(f_i))T(h_{\langle\theta_1,\theta_2\rangle})\delta_2 \quad \text{iff}$$

by commutativity of
$$\begin{tikzcd} TT(L_i) \arrow[r, "TT(f_i)"] \arrow[d, "\mu_{L_i}"'] & TTT(K_i) \arrow[d, "\mu_{T(K_i)}"] \\ T(L_i) \arrow[r, "T(f_i)"'] & TT(K_i) \end{tikzcd}$$

$$\xi_i\mu_{K_i}T(f_i)\mu_{L_i}T(h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i)\mu_{L_i}T(h_{\langle\theta_1,\theta_2\rangle})\delta_2 \quad \text{iff}$$

$$\xi_i\mu_{K_i}T(f_i)(h_{\langle\theta_1,\theta_2\rangle}\circ\delta_1) = \xi_i\mu_{K_i}T(f_i)(h_{\langle\theta_1,\theta_2\rangle}\circ\delta_2).$$

Therefore

$$\{\langle\delta_1,\delta_2\rangle \in \mathcal{K}_{\mathbf{T}}(1,L)^2 : \xi_i\mu_{K_i}T(f_i\circ h_{\langle\theta_1,\theta_2\rangle})\delta_1 = \xi_i\mu_{K_i}T(f_i\circ h_{\langle\theta_1,\theta_2\rangle})\delta_2\} =$$

$$= \mathcal{K}_{\mathbf{T}}(1, h_{\langle\theta_1,\theta_2\rangle})^{-1}(\{\langle\epsilon_1,\epsilon_2\rangle \in \mathcal{K}_{\mathbf{T}}(1,L)^2 : \xi_i\mu_{K_i}T(f_i)\epsilon_1 = \xi_i\mu_{K_i}T(f_i)\epsilon_2\}) =$$

$$= \mathcal{K}_{\mathbf{T}}(1, h_{\langle\theta_1,\theta_2\rangle})^{-1}(\Omega_{L_i}(T_i)),$$

which is in $\Omega(|\mathbf{TH}(\mathcal{I})|)$, by Lemma 4.17(ii), as required.

Finally, it remains to show that $\Omega : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{Q}}^{\xi})$ commutes with substitutions. We have, for every $\langle L, T\rangle \in |\mathbf{TH}(\mathcal{I})|$, $f \in \mathcal{K}_{\mathbf{T}}(L, K)$,

$$\Omega_K(\mathcal{K}_{\mathbf{T}}(1,f)(T)^c) = \mathcal{K}_{\mathbf{T}}(1,f)^2(\Omega_L(T))^c,$$

by our assumption, as required. ∎

Deductive π-Institutions Revisited

In this section, the general theory of auto-algebraizability is applied to the class of deductive π-institutions, that were studied in Chapter 3. Recall from Chapter 3 that, given a language type $\mathcal{L}$ and a finitary k-deductive system $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}}\rangle$ over $\mathcal{L}$, the deductive π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$, associated with $\mathcal{S}$, has as its signature category $\mathbf{SIGN}$ the category with the single object V and morphisms all assignments $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$, as its sentence functor $\mathrm{SEN} : \mathbf{SIGN} \to \mathbf{SET}$ the functor sending V to $\mathrm{Tm}_{\mathcal{L}}(V)^k$ and the assignment $h : V \to \mathrm{Tm}_{\mathcal{L}}(V)$ to $\mathrm{SEN}(h) : \mathrm{Tm}_{\mathcal{L}}(V)^k \to \mathrm{Tm}_{\mathcal{L}}(V)^k$ with $\mathrm{SEN}(h)(\phi) = h^*(\phi)$, and as its closure $C_V : \mathcal{P}(\mathrm{Tm}_{\mathcal{L}}(V)^k) \to \mathcal{P}(\mathrm{Tm}_{\mathcal{L}}(V)^k)$ the closure $C_{\mathcal{S}}$ of the given k-deductive system $\mathcal{S}$. Recall, also, that, given a language type $\mathcal{L}$, we can construct an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in $\mathbf{SET}$, whose Eilenberg-Moore category of $\mathbf{T}$-algebras, $\mathbf{SET}^{\mathbf{T}}$, is isomorphic to the category of the variety of all $\mathcal{L}$-algebras. More details can be found in Chapter 3.

In the comments following Theorem 3.12, it was mentioned that the proof was providing the following stronger result that can now be stated explicitly as follows.

COROLLARY 4.20 *Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V)^k, \vdash_{\mathcal{S}}\rangle$ a finitary k-deductive system over $\mathcal{L}$. If $\mathcal{S}$ is algebraizable then $\mathcal{I}_{\mathcal{S}}$ is auto-algebraizable.*

Next, Theorems 1.3 and 4.19 will be used to show that the converse of this corollary holds. Namely, restricting attention to 1-deductive systems, it is shown that, given a finitary 1-deductive system $\mathcal{S}$, if $\mathcal{I}_{\mathcal{S}}$ is auto-algebraizable, then $\mathcal{S}$ is algebraizable. The following lemma is needed first.

LEMMA 4.21 *Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V), \vdash_{\mathcal{S}} \rangle$ a finitary deductive system over $\mathcal{L}$. $\mathcal{I}_{\mathcal{S}}$ is a Blok-Pigozzi theory institution.*

Proof:

$\mathcal{I}_{\mathcal{S}} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_{\Sigma}\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$ is a theory institution, since $\mathbf{SET}$ is locally small, has a terminal object $\{\emptyset\}$ and $\{\emptyset\} \sqcup \{\emptyset\}$ exists in $\mathbf{SET}$, $\mathbf{SIGN}$ is a full subcategory of $\mathbf{SET_T}$, satisfying (4.1), and SEN can be taken to be $\mathbf{SET_T}(\{\emptyset\}, -)$ by identifying $\phi \in Tm_{\mathcal{L}}(V)$ with $f_{\phi} : \{\emptyset\} \to \mathrm{Tm}_{\mathcal{L}}(V)$, sending $\emptyset$ to ϕ.

Next, suppose that $\langle V, T \rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{S}})|$. Note that, under the identification just made, the generalized Leibniz congruence $\Omega_V(T)$, defined in this chapter, coincides with the Leibniz congruence $\Omega(T)$ of [6]. We can, thus, consider the $\mathbf{T}$-algebra $\langle \mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T), \xi \rangle$ corresponding to the $\mathcal{L}$-algebra $\mathbf{Tm}_{\mathcal{L}}(V)/\Omega(T)$ and let $f \in \mathbf{SET_T}(V, \mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T))$ be the map $\eta_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)} q$, where $q : V \to \mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)$ sends $v \in V$ to $v/\Omega(T)$. We then have

$$\Omega_V(T) = \{\langle f_{\phi_1}, f_{\phi_2} \rangle \in \mathbf{SET_T}(\{\emptyset\}, V)^2 : \xi \mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)} T(f) f_{\phi_1} = \xi \mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)} T(f) f_{\phi_2}\},$$

i.e., $\mathbf{T}$ simply creates theories for $\mathcal{I}_{\mathcal{S}}$.

To see that f is special with respect to $\mathbf{SIGN}$, let $f_{\phi_1}, f_{\phi_2} \in \mathbf{SET_T}(\{\emptyset\}, V), g \in \mathbf{SET_T}(V, \mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T))$, such that

$$\xi \mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)} T(g) f_{\phi_1} = \xi \mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)} T(g) f_{\phi_2}.$$

Then the $h \in \mathbf{SET_T}(V, V)$ which is such that

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, dashed, "h"'] & & \mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T) \\
& V \arrow[ur, "g"'] & \\
1 \arrow[ur, shift left, "f_{\phi_1}"] \arrow[ur, shift right, "f_{\phi_2}"'] & &
\end{tikzcd}$$

- $\xi\mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(g)f_{\phi_i} = \xi\mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(f \circ h)f_{\phi_i}, i = 1, 2$

- $\mathbf{SET_T}(\{\emptyset\}, h)$ is surjective

can be constructed using the following argument, borrowed from the proof of Lemma 4.5 of [6].

First, note that, since ϕ_1, ϕ_2 contain only finitely many variables of V, there exists a $g' \in \mathbf{SET_T}(V, \mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T))$, such that, each element of $\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)$ is the image of an infinite number of variables and $T(g)f_{\phi_i} = T(g')f_{\phi_i}, i = 1, 2$. Next, let $h \in \mathbf{SET_T}(V, V)$ be such that $h(v_i) \in \xi g'(v_i), i \in \omega$, and v_i is the image under h of some v_j;such an h exists because of the assumption that each element of $\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)$ is the image of an infinite number of variables. Then h and, hence, $\mathbf{SET_T}(\{\emptyset\}, h)$ is surjective and $\xi g'(v_i) = h(v_i)/\Omega(T), i \in \omega$. Therefore

$$\begin{aligned}
\xi(f \circ h)(v_i) &= \xi\mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(f)h(v_i) \\
&= \xi\mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(\eta_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)}q)h(v_i) \\
&= \xi\mu_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(\eta_{\mathrm{Tm}_{\mathcal{L}}(V)/\Omega(T)})T(q)h(v_i) \\
&= \xi T(q)h(v_i) \\
&= h(v_i)/\Omega(T) \\
&= \xi g'(v_i).
\end{aligned}$$

Thus,

$$\begin{aligned}
\xi\mu_{\mathbf{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(f\circ h)(v_i) &= \xi T(\xi)T(f\circ h)(v_i)\\
&= \xi T(\xi(f\circ h))(v_i)\\
&= \xi T(\xi g')(v_i)\\
&= \xi T(\xi)T(g')(v_i)\\
&= \xi\mu_{\mathbf{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(g')(v_i),
\end{aligned}$$

i.e., $\xi\mu_{\mathbf{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(f\circ h) = \xi\mu_{\mathbf{Tm}_{\mathcal{L}}(V)/\Omega(T)}T(g')$, and, since $T(g)$ agrees with $T(g')$ on ϕ_1, ϕ_2, the conclusion follows. ∎

THEOREM 4.22 *Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V), \vdash_{\mathcal{S}}\rangle$ a finitary deductive system over $\mathcal{L}$. If $\mathcal{I}_{\mathcal{S}}$ is auto-algebraizable then $\mathcal{S}$ is algebraizable.*

Proof:

Suppose that $\mathcal{I}_{\mathcal{S}}$ is auto-algebraizable. Then, by Theorem 4.19 and Lemma 4.21, the Leibniz operator is injective, join-continuous and commutes with substitutions. In particular, it preserves unions of directed subsets of theories. Hence, by Theorem 1.3, $\mathcal{S}$ is algebraizable. ∎

THEOREM 4.23 *Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathbf{Tm}_{\mathcal{L}}(V), \vdash_{\mathcal{S}}\rangle$ a finitary deductive system over $\mathcal{L}$. $\mathcal{S}$ is algebraizable if and only if $\mathcal{I}_{\mathcal{S}}$ is auto-algebraizable.*

Proof:

By Theorems 3.12 and 4.22. ∎

5 METALOGICAL PROPERTIES

Metalogical properties that have traditionally been studied in the deductive system context (see [23]) and transfered later in the institution context [50], are here formulated in the π-institution context. Preservation under deductive equivalence of π-institutions is investigated.

Introduction

Two have been the main directions of development of abstract algebraic logic. One is the study of the *algebraization process* itself and the other is the extent to which *metalogical properties* are related to algebraic properties via algebraizability, or, more generally, whether they are preserved or not under equivalence of deductive systems. [8, 14, 15, 23], e.g., study in detail the deduction-detachment property for deductive systems. It is only natural that these two directions will be the main focus of categorical abstract algebraic logic as well, its starting point being relations between π-institutions or institutions like the ones introduced in Chapter 2 of the thesis. In Chapters 3 and 4, the first direction has been pursued further. The study of the algebraization process has begun. This chapter is a contribution to the second direction of research. Various metalogical properties of institutions have already been defined in [50]. We reformulate some of those, in a somewhat nonstandard way, in the π-institution framework and define some new ones. Then we study the effect that deductive equivalence has on these properties.

Deduction-Detachment Property

The Deduction-Detachment property for a π-institution is now introduced and it is shown, as an application of the notion of deductive equivalence, that it is invariant under this equivalence.

DEFINITION 5.1 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|}\rangle$ be a π-institution. A natural transformation $E : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ will be called a* **Deduction-Detachment transformation (DDT,** *for short.) for $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{SIGN}|, \Gamma \cup \Delta \cup \Phi \subseteq \mathrm{SEN}(\Sigma)$,*

$$\Phi \subseteq (\Gamma \cup \Delta)^c \quad \textit{iff} \quad E_\Sigma(\Delta, \Phi) \subseteq \Gamma^c.$$

$\mathcal{I}$ will be said to **have the Deduction-Detachment property (DDP,** *for short.) if there exists a Deduction-Detachment transformation for $\mathcal{I}$.*

THEOREM 5.2 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π-institutions. Then $\mathcal{I}_1$ has the DDP *if and only if $\mathcal{I}_2$ has the* DDP.

Proof:

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Suppose $\mathcal{I}_1$ has the DDP with Deduction-Detachment transformation $E : \mathcal{P}\mathrm{SEN}_1^2 \to \mathcal{P}\mathrm{SEN}_1$. Then, for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \Gamma \cup \Delta \cup \Phi \subseteq \mathrm{SEN}_2(\Sigma_2)$,

$$\Phi \subseteq (\Gamma \cup \Delta)^c \quad \text{iff, since } \langle G, \beta\rangle \text{ is an intrepretation,}$$

$$\beta_{\Sigma_2}(\Phi) \subseteq \beta_{\Sigma_2}(\Gamma \cup \Delta)^c \quad \text{iff}$$

$$\beta_{\Sigma_2}(\Phi) \subseteq (\beta_{\Sigma_2}(\Gamma) \cup \beta_{\Sigma_2}(\Delta))^c \quad \text{iff, since } E \text{ is a DDT for } \mathcal{I}_1,$$

$$E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi)) \subseteq \beta_{\Sigma_2}(\Gamma)^c \quad \text{iff, since } \langle F, \alpha\rangle \text{ is an interpretation,}$$

$\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi))) \subseteq \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c$ iff, since ϵ_{Σ_2} is an isomorphism,

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi)))) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c)$$

iff, by Lemma 2.26, $\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi)))) \subseteq \Gamma^c$.

Let $E' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ be defined by

$$E'_{\Sigma_2}(\Delta, \Phi) = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta^2_{\Sigma_2}(\Delta, \Phi)))),$$

for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \Delta, \Phi \subseteq \mathrm{SEN}_2(\Sigma_2)$. Note that $E' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ is a natural transformation since it is the composition of the natural transformations $\beta^2 : \mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_1^2 G$, $E_G : \mathcal{P}\mathrm{SEN}_1^2 G \to \mathcal{P}\mathrm{SEN}_1 G$, $\alpha_G : \mathrm{SEN}_1 G \to \mathcal{P}\mathrm{SEN}_2 FG$ and $\mathrm{SEN}_2\epsilon : SEN_2FG \to SEN_2$. Thus, it follows from what was just shown that E' is a DDT for $\mathcal{I}_2$ and, thus $\mathcal{I}_2$ has the DDP, as required.

The converse follows by symmetry. ■

Disjunction Property

The abstract property of disjunction for deductive systems in the context of abstract algebraic logic has been studied in [22] and taken up again in [23]. The property of conjunction for institutions has been introduced in [50]. Modifying this definition appropriately, an institution $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ is said to *have disjunction* if, for every signature Σ and finite set $\Phi \subseteq \mathrm{SEN}(\Sigma)$, there exists $\bigvee \Phi \in \mathrm{SEN}(\Sigma)$, such that, for every $M \in |\mathrm{MOD}(\Sigma)|$, $M \models_\Sigma \bigvee \Phi$ if and only if $M \models_\Sigma \phi$, for some $\phi \in \Phi$.

A somewhat nonstandard formulation of the conjunction property for a π-institution will now be given and it will be shown that it is preserved under deductive equivalence of π-institutions.

DEFINITION 5.3 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|} \rangle$ *be a* π*-institution. A natural transformation* $\bigvee : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ *will be called a* **disjunction for** $\mathcal{I}$ *if, for all* $\Sigma \in |\mathbf{SIGN}|, \Phi, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$(\Phi \cup \bigvee_\Sigma(\Gamma, \Delta))^c = (\Phi \cup \Gamma)^c \cap (\Phi \cup \Delta)^c.$$

$\mathcal{I}$ will be said to **have disjunction** *if there exists a disjunction for $\mathcal{I}$.*

A lemma is needed for the proof of our main result.

LEMMA 5.4 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Then, for all $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, $\alpha_{\Sigma_1}(T_1)^c \cap \alpha_{\Sigma_1}(T_1')^c = \alpha_{\Sigma_1}(T_1 \cap T_1')^c$.

Proof:

First, note that, for all $\Sigma_1 \in |\mathbf{SIGN}_1|, \langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, we have

$$\alpha_{\Sigma_1}(T_1)^c = \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1)\}.$$

In fact,

$$\begin{array}{c} \psi \in \alpha_{\Sigma_1}(T_1)^c \quad \text{iff, since } \langle G, \beta\rangle \text{ is an interpretation,} \\ \beta_{F(\Sigma_1)}(\psi) \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c \quad \text{iff, by Lemma 2.26,} \\ \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1), \end{array}$$

as required. Thus, we have

$$\begin{aligned} \alpha_{\Sigma_1}(T_1)^c \cap \alpha_{\Sigma_1}(T_1')^c &= \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1)\} \\ &\quad \cap \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1')\} \\ &= \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \\ &\quad \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1) \cap \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1')\} \\ &= \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1 \cap T_1')\} \\ &\quad \text{(since } \eta_{\Sigma_1} \text{ is an isomorphism)} \\ &= \alpha_{\Sigma_1}(T_1 \cap T_1')^c, \end{aligned}$$

as required. ∎

THEOREM 5.5 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|} \rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|} \rangle$$

be two deductively equivalent π-institutions. $\mathcal{I}_1$ has disjunction if and only if $\mathcal{I}_2$ has disjunction.

Proof:

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta \rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Suppose that $\mathcal{I}_1$ has disjunction and let $\bigvee : \mathcal{P}\mathrm{SEN}_1^2 \to \mathcal{P}\mathrm{SEN}_1$ be a disjunction for $\mathcal{I}_1$. Then, for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \Phi, \Gamma, \Delta \subseteq \mathrm{SEN}_2(\Sigma_2)$,

$$(\Phi \cup \Gamma)^c \cap (\Phi \cup \Delta)^c =$$

$$\begin{aligned}
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi \cup \Gamma))^c) \cap \\
&\quad \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi \cup \Delta))^c) \text{ (by Lemma 2.26)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi \cup \Gamma))^c \cap \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi \cup \Delta))^c) \\
&\quad \text{(since } \epsilon_{\Sigma_2} \text{ is an isomorphism)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi) \cup \beta_{\Sigma_2}(\Gamma))^c \cap \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi) \cup \beta_{\Sigma_2}(\Delta))^c) \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi) \cup \beta_{\Sigma_2}(\Gamma))^c)^c \cap \alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi) \cup \beta_{\Sigma_2}(\Delta))^c)^c) \\
&\quad \text{(by Lemma 2.24)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi) \cup \beta_{\Sigma_2}(\Gamma))^c \cap (\beta_{\Sigma_2}(\Phi) \cup \beta_{\Sigma_2}(\Delta))^c)^c) \\
&\quad \text{(by Lemma 5.4)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi) \cup \textstyle\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta)))^c)^c) \\
&\quad \text{(since } \textstyle\bigvee \text{ is a disjunction for } \mathcal{I}_1) \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi) \cup \textstyle\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta)))^c) \\
&\quad \text{(by Lemma 2.24)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})((\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi)) \cup \alpha_{G(\Sigma_2)}(\textstyle\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta))))^c) \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})((\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\textstyle\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta))))^c)
\end{aligned}$$

$$= \quad \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta))))^c$$

(by Corollaries 2.6 and 2.4)

$$= \quad (\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c) \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta)))))^c$$

$$= \quad (\Phi^c \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta)))))^c \text{ (by Lemma 2.26)}$$

$$= \quad (\Phi \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma), \beta_{\Sigma_2}(\Delta)))))^c.$$

Let $\bigvee' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ be defined by

$$\bigvee_{\Sigma_2}'(\Gamma, \Delta) = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\bigvee_{G(\Sigma_2)}(\beta_{\Sigma_2}^2(\Gamma, \Delta)))).$$

for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \Gamma, \Delta \subseteq \mathrm{SEN}_2(\Sigma_2)$. $\bigvee' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ is a natural transformation, since it is the composite of the natural transformations $\beta^2 : \mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_1^2 G, \bigvee_G : \mathcal{P}\mathrm{SEN}_1^2 G \to \mathcal{P}\mathrm{SEN}_1 G, \alpha_G : \mathrm{SEN}_1 G \to \mathcal{P}\mathrm{SEN}_2 GF$ and $\mathrm{SEN}_2\epsilon : \mathrm{SEN}_2 GF \to \mathrm{SEN}_2$. Since, from what was just shown, we have

$$(\Phi \cup \bigvee_{\Sigma_2}'(\Gamma, \Delta))^c = (\Phi \cup \Gamma)^c \cap (\Phi \cup \Delta)^c,$$

$\bigvee'$ is a disjunction for $\mathcal{I}_2$, as required.

The converse follows by symmetry. ∎

A Note on Conjunction

By analogy with the previous section, one may attempt to define conjunction for π-institutions as follows

DEFINITION 5.6 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|}\rangle$ *be a* π*-institution. A natural transformation* $\bigwedge : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ *will be called a* **conjunction for** $\mathcal{I}$ *if, for all* $\Sigma \in |\mathbf{SIGN}|, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$(\Gamma \cup \Delta)^c = \bigwedge_{\Sigma}(\Gamma, \Delta)^c.$$

$\mathcal{I}$ *will be said to* **have conjunction** *if there exists a conjunction for* $\mathcal{I}$.

The property of conjunction will now be shown to be an intrinsic property of all π-institutions, owing to the fact that their sentence functor is postulated to map into **SET**.

LEMMA 5.7 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$, *be a π-institution.* $\bigwedge : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ *with*

$$\bigwedge_\Sigma(\Gamma, \Delta) = \Gamma \cup \Delta, \quad \textit{for all} \quad \Sigma \in |\mathbf{SIGN}|, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma),$$

is a natural transformation.

Proof:

Let $f : \Sigma \to \Sigma' \in \mathrm{Mor}(\mathbf{SIGN})$. We need to show that the following diagram commutes. If $\Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$, then

$$\begin{tikzcd}
\mathcal{P}\mathrm{SEN}^2(\Sigma) \arrow[r, "\bigwedge_\Sigma"] \arrow[d, "\mathcal{P}\mathrm{SEN}^2(f)"'] & \mathcal{P}\mathrm{SEN}(\Sigma) \arrow[d, "\mathcal{P}\mathrm{SEN}(f)"] \\
\mathcal{P}\mathrm{SEN}^2(\Sigma') \arrow[r, "\bigwedge_{\Sigma'}"'] & \mathcal{P}\mathrm{SEN}(\Sigma')
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{P}\mathrm{SEN}(f)(\bigwedge_\Sigma(\Gamma, \Delta)) &= \mathcal{P}\mathrm{SEN}(f)(\Gamma \cup \Delta) \\
&= \mathcal{P}\mathrm{SEN}(f)(\Gamma) \cup \mathcal{P}\mathrm{SEN}(f)(\Delta) \\
&= \bigwedge_{\Sigma'}(\mathcal{P}\mathrm{SEN}(f)(\Gamma), \mathcal{P}\mathrm{SEN}(f)(\Delta)) \\
&= \bigwedge_{\Sigma'}(\mathcal{P}\mathrm{SEN}^2(f)(\Gamma, \Delta)),
\end{aligned}$$

as required. ■

THEOREM 5.8 *Every π-institution has conjunction.*

Proof:

By Lemma 5.7. ■

Negation

Following the same line of thought that was followed in the previous sections, the property of negation for π-institutions is now introduced.

DEFINITION 5.9 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ *be a* π*-institution. A natural transformation* $\neg : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ *will be called a* **negation for** $\mathcal{I}$ *if, for all* $\Sigma \in |\mathbf{SIGN}|, \Phi, \Gamma \subseteq \mathrm{SEN}(\Sigma)$,

$$\Gamma \subseteq \Phi^c \quad \textit{iff} \quad (\Phi \cup \neg_\Sigma \Gamma)^c = \mathrm{SEN}(\Sigma).$$

$\mathcal{I}$ *will be said to* **have negation** *if there exists a negation for* $\mathcal{I}$.

For the proof of the main theorem a lemma is needed first.

LEMMA 5.10 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π*-institutions via the interpretations* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$*. Then, for every* $\Sigma_1 \in |\mathbf{SIGN}_1|$, $\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c = \mathrm{SEN}_2(F(\Sigma_1))$ *and, similarly, for every* $\Sigma_2 \in |\mathbf{SIGN}_2|$, $\beta_{\Sigma_2}(\mathrm{SEN}_2(\Sigma_2))^c = \mathrm{SEN}_1(G(\Sigma_2))$.

Proof:

Obviously, $\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c \subseteq \mathrm{SEN}_2(F(\Sigma_1))$. Suppose that

$$\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c \subset \mathrm{SEN}_2(F(\Sigma_1)).$$

Then, by Theorem 2.41 and Lemmas 2.37 and 2.38, we have

$$\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1)))^c \subset \beta_{F(\Sigma_1)}(\mathrm{SEN}_2(F(\Sigma_1)))^c,$$

whence, since η_{Σ_1} is an isomorphism,

$$\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1)))^c) \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\mathrm{SEN}_2(F(\Sigma_1)))^c),$$

i.e., by Lemma 2.26,

$$\mathrm{SEN}_1(\Sigma_1) \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\mathrm{SEN}_2(F(\Sigma_1)))^c),$$

which is absurd. ■

THEOREM 5.11 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|} \rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|} \rangle$$

be two deductively equivalent π-institutions. $\mathcal{I}_1$ has negation if and only if $\mathcal{I}_2$ has negation.

Proof:

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta \rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Suppose that $\mathcal{I}_1$ has negation and let $\neg : \mathcal{P}\mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_1$ be a negation for $\mathcal{I}_1$. Then, for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \Gamma \cup \Phi \subseteq \mathrm{SEN}_2(\Sigma_2)$,

$$\Gamma \subseteq \Phi^c \quad \text{iff}$$

$$\Gamma^c \subseteq \Phi^c \quad \text{iff}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c), \quad \text{by Lemma 2.26, iff}$$

$$\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c \subseteq \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c, \quad \text{since } \epsilon_{\Sigma_2} \text{ is an iso, iff}$$

$$\beta_{\Sigma_2}(\Gamma)^c \subseteq \beta_{\Sigma_2}(\Phi)^c, \quad \text{since } \langle F, \alpha \rangle \text{ is an interpretation, iff}$$

$$(\beta_{\Sigma_2}(\Phi) \cup \neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))^c = \mathrm{SEN}_1(G(\Sigma_2)), \quad \text{since } \neg \text{ is a negation for } \mathcal{I}_1, \text{ iff}$$

$$\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi) \cup \neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))^c = \alpha_{G(\Sigma_2)}(\mathrm{SEN}_1(G(\Sigma_2)))^c, \quad \text{since } \langle F, \alpha \rangle \text{ is an int., iff}$$

$$(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi)) \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c = \mathrm{SEN}_2(F(G(\Sigma_2))), \quad \text{by Lemma 5.10, iff}$$

$$(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c = \mathrm{SEN}_2(F(G(\Sigma_2))), \quad \text{iff}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})((\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c) = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\mathrm{SEN}_2(F(G(\Sigma_2)))),$$

since ϵ_{Σ_2} is an iso, iff

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\mathrm{SEN}_2(F(G(\Sigma_2)))),$$

by Corollaries 2.6 and 2.4, iff

$$(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c) \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))))^c = \mathrm{SEN}_2(\Sigma_2),$$

since ϵ_{Σ_2} is an iso, iff

$$(\Phi^c \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))))^c = \mathrm{SEN}_2(\Sigma_2), \quad \text{by Lemma 2.26, iff}$$

$$(\Phi \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))))^c = \mathrm{SEN}_2(\Sigma_2).$$

Let $\neg' : \mathcal{P}\mathrm{SEN}_2 \to \mathcal{P}\mathrm{SEN}_2$ be defined by

$$\neg'_{\Sigma_2}\Gamma = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))),$$

for all $\Sigma_2 \in |\mathbf{SIGN}_2|, \Gamma \subseteq \mathrm{SEN}_2(\Sigma_2)$. $\neg' : \mathcal{P}\mathrm{SEN}_2 \to \mathcal{P}\mathrm{SEN}_2$ is a natural transformation, since it is the composite of natural transformations. Thus, from what was just shown, we have

$$\Gamma \subseteq \Phi^c \quad \text{iff} \quad (\Phi \cup \neg'_{\Sigma_2}\Gamma)^c = \mathrm{SEN}_2(\Sigma_2),$$

i.e., $\neg'$ is a negation for $\mathcal{I}_2$, as required.

The converse follows by symmetry. ■

Craig Interpolation

Tarlecki [50] introduced and studied the Craig Interpolation Theorem for institutions. Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ be an institution and the following

$$\begin{tikzcd}
 & \Sigma''' & \\
\Sigma' \arrow[ur, "g'"] & & \Sigma'' \arrow[ul, "g''"'] \\
 & \Sigma \arrow[ul, "f'"] \arrow[ur, "f''"'] &
\end{tikzcd}$$

a pushout diagram in **SIGN**. According to [50], $\mathcal{I}$ is said to satisfy the *Craig Interpolation Theorem* if, for all $\phi' \in \mathrm{SEN}(\Sigma'), \phi'' \in \mathrm{SEN}(\Sigma'')$, with $\mathrm{SEN}(g')(\phi') \models \mathrm{SEN}(g'')(\phi'')$, there exists $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi' \models \mathrm{SEN}(f')(\phi)$ and $\mathrm{SEN}(f'')(\phi) \models \phi''$.

Modifying slightly Tarlecki's definition the following is obtained.

DEFINITION 5.12 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}|}\rangle$ *be a* π*-institution.* $\mathcal{I}$ *is said to have the* **Craig Interpolation Property** (**CIP**, *for short,*) *if, for all* $\Sigma, \Sigma', \Sigma'' \in |\mathbf{SIGN}|$ *and pushout diagram*

$$\begin{tikzcd} & \Sigma''' & \\ \Sigma' \arrow[ur, "g'"] & & \Sigma'' \arrow[ul, "g''"'] \\ & \Sigma \arrow[ul, "f'"] \arrow[ur, "f''"'] & \end{tikzcd}$$

we have that, for all $\Phi' \subseteq \mathrm{SEN}(\Sigma'), \Phi'' \subseteq \mathrm{SEN}(\Sigma'')$, *with*

$$\mathrm{SEN}(g'')(\Phi'') \subseteq \mathrm{SEN}(g')(\Phi')^c,$$

there exists $\Phi \subseteq \mathrm{SEN}(\Sigma)$, *such that* $\mathrm{SEN}(f')(\Phi) \subseteq \Phi'^c$ *and* $\Phi'' \subseteq \mathrm{SEN}(f'')(\Phi)^c$.

THEOREM 5.13 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π*-institutions.* $\mathcal{I}_1$ *has the CIP if and only if* $\mathcal{I}_2$ *has the CIP.*

Proof:

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Suppose that $\mathcal{I}_1$ has the CIP and assume that

$$\begin{tikzcd} & \Sigma_2''' & \\ \Sigma_2' \arrow[ur, "g_2'"] & & \Sigma_2'' \arrow[ul, "g_2''"'] \\ & \Sigma_2 \arrow[ul, "f_2'"] \arrow[ur, "f_2''"'] & \end{tikzcd}$$

is a pushout diagram in $\mathbf{SIGN}_2$ and $\Phi_2' \subseteq \mathrm{SEN}_2(\Sigma_2'), \Phi_2'' \subseteq \mathrm{SEN}_2(\Sigma_2'')$, with

$$\mathrm{SEN}_2(g_2'')(\Phi_2'') \subseteq \mathrm{SEN}_2(g_2')(\Phi_2')^c.$$

Since left adjoints preserve colimits, the following is, then, a pushout diagram in $\mathbf{SIGN}_1$.

$$\begin{tikzcd} & G(\Sigma_2''') & \\ G(\Sigma_2') \arrow[ur, "G(g_2')"] & & G(\Sigma_2'') \arrow[ul, "G(g_2'')"'] \\ & G(\Sigma_2) \arrow[ul, "G(f_2')"] \arrow[ur, "G(f_2'')"'] & \end{tikzcd}$$

Moreover, since $\langle G, \beta\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an interpretation, we have

$$\beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2'')(\Phi_2'')) \subseteq \beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(\Phi_2'))^c.$$

Since β is a natural transformation,

$$\begin{tikzcd} \mathrm{SEN}_2(\Sigma_2') \arrow[r, "\beta_{\Sigma_2'}"] \arrow[d, "\mathrm{SEN}_2(g_2')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2'))"] \\ \mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2''')) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{SEN}_2(\Sigma_2'') \arrow[r, "\beta_{\Sigma_2''}"] \arrow[d, "\mathrm{SEN}_2(g_2'')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2'')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2''))"] \\ \mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2''')) \end{tikzcd}$$

we obtain

$$\mathrm{SEN}_1(G(g_2''))(\beta_{\Sigma_2''}(\Phi_2'')) \subseteq \mathrm{SEN}_1(G(g_2'))(\beta_{\Sigma_2'}(\Phi_2'))^c.$$

Since $\mathcal{I}_1$ has the CIP, there exists $\Phi_1 \subseteq \mathrm{SEN}_1(G(\Sigma_2))$, such that

$$\mathrm{SEN}_1(G(f_2'))(\Phi_1) \subseteq \beta_{\Sigma_2'}(\Phi_2')^c \quad \text{and} \quad \beta_{\Sigma_2''}(\Phi_2'') \subseteq \mathrm{SEN}_1(G(f_2''))(\Phi_1)^c.$$

Thus, since $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an interpretation,

$$\alpha_{G(\Sigma_2')}(\mathrm{SEN}_1(G(f_2'))(\Phi_1)) \subseteq \alpha_{G(\Sigma_2')}(\beta_{\Sigma_2'}(\Phi_2'))^c$$

and

$$\alpha_{G(\Sigma_2'')}(\beta_{\Sigma_2''}(\Phi_2'')) \subseteq \alpha_{G(\Sigma_2'')}(\mathrm{SEN}_1(G(f_2''))(\Phi_1))^c$$

and, since α is a natural transformation,

$$\begin{tikzcd}
\mathrm{SEN}_1(G(\Sigma_2)) \arrow[r, "\alpha_{G(\Sigma_2)}"] \arrow[d, "\mathrm{SEN}_1(G(f_2'))"'] & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2))) \arrow[d, "\mathcal{P}\mathrm{SEN}_2(F(G(f_2')))"] \\
\mathrm{SEN}_1(G(\Sigma_2')) \arrow[r, "\alpha_{G(\Sigma_2')}"'] & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2')))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{SEN}_1(G(\Sigma_2)) \arrow[r, "\alpha_{G(\Sigma_2)}"] \arrow[d, "\mathrm{SEN}_1(G(f_2''))"'] & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2))) \arrow[d, "\mathcal{P}\mathrm{SEN}_2(F(G(f_2'')))"] \\
\mathrm{SEN}_1(G(\Sigma_2'')) \arrow[r, "\alpha_{G(\Sigma_2'')}"'] & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2'')))
\end{tikzcd}$$

we obtain

$$\mathrm{SEN}_2(F(G(f_2')))(\alpha_{G(\Sigma_2)}(\Phi_1)) \subseteq \alpha_{G(\Sigma_2')}(\beta_{\Sigma_2'}(\Phi_2'))^c \quad \text{and}$$

$$\alpha_{G(\Sigma_2'')}(\beta_{\Sigma_2''}(\Phi_2'')) \subseteq \mathrm{SEN}_2(F(G(f_2'')))(\alpha_{G(\Sigma_2)}(\Phi_1))^c. \quad \text{Hence,}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2'})(\mathrm{SEN}_2(F(G(f_2')))(\alpha_{G(\Sigma_2)}(\Phi_1))) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2'})(\alpha_{G(\Sigma_2')}(\beta_{\Sigma_2'}(\Phi_2'))^c) \quad \text{and}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2''})(\alpha_{G(\Sigma_2'')}(\beta_{\Sigma_2''}(\Phi_2''))^c) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2''})(\mathrm{SEN}_2(F(G(f_2'')))(\alpha_{G(\Sigma_2)}(\Phi_1))^c),$$

i.e., by Lemma 2.26, $\mathrm{SEN}_2(\epsilon_{\Sigma_2'})(\mathrm{SEN}_2(F(G(f_2')))(\alpha_{G(\Sigma_2)}(\Phi_1))) \subseteq \Phi_2'^c$ and

$$\Phi_2'' \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2''})(\mathrm{SEN}_2(F(G(f_2'')))(\alpha_{G(\Sigma_2)}(\Phi_1))^c).$$

Thus,

$$
\begin{tikzcd}
F(G(\Sigma_2)) \arrow[r, "\epsilon_{\Sigma_2}"] \arrow[d, "F(G(f_2'))"'] & \Sigma_2 \arrow[d, "f_2'"] \\
F(G(\Sigma_2')) \arrow[r, "\epsilon_{\Sigma_2'}"'] & \Sigma_2'
\end{tikzcd}
\qquad
\begin{tikzcd}
F(G(\Sigma_2)) \arrow[r, "\epsilon_{\Sigma_2}"] \arrow[d, "F(G(f_2''))"'] & \Sigma_2 \arrow[d, "f_2''"] \\
F(G(\Sigma_2'')) \arrow[r, "\epsilon_{\Sigma_2''}"'] & \Sigma_2''
\end{tikzcd}
$$

$$\mathrm{SEN}_2(f_2'\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1)) \subseteq \Phi_2'^c \quad \text{and} \quad \Phi_2'' \subseteq \mathrm{SEN}_2(f_2''\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1))^c$$

and, therefore,

$$\mathrm{SEN}_2(f_2')(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1))) \subseteq \Phi_2'^c$$

$$\text{and} \quad \Phi_2'' \subseteq \mathrm{SEN}_2(f_2'')(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1)))^c.$$

Thus, $\mathcal{I}_2$ has the CIP, as required.

The converse follows by symmetry. ∎

Robinson Consistency

Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a π-institution and $\Sigma \in |\mathbf{SIGN}|$. Recall that a theory $\langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|$ is said to be **consistent** if $T \neq \mathrm{SEN}(\Sigma)$ and **complete** if, for every $\langle \Sigma, T'\rangle \in |\mathbf{TH}(\mathcal{I})|, T \subset T'$ implies $T' = \mathrm{SEN}(\Sigma)$.

DEFINITION 5.14 *Let $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ be a π-institution. $\mathcal{I}$ will be said to* **have the Robinson Consistency Property (RCP,** *for short,*) *if, for every consistent complete theory $\langle \Sigma, T\rangle$ and consistent theories $\langle \Sigma', T'\rangle, \langle \Sigma'', T''\rangle$, such that $f' : \langle \Sigma, T\rangle \to \langle \Sigma', T'\rangle, f'' : \langle \Sigma, T\rangle \to \langle \Sigma'', T''\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, the theory*

$$\langle \Sigma''', (\mathrm{SEN}(g')(T') \cup \mathrm{SEN}(g'')(T''))^c\rangle$$

is consistent, where as before, the following diagram

$$
\begin{tikzcd}
 & \Sigma''' & \\
\Sigma' \arrow[ur, "g'"] & & \Sigma'' \arrow[ul, "g''"'] \\
 & \Sigma \arrow[ul, "f'"] \arrow[ur, "f''"'] &
\end{tikzcd}
$$

is a pushout in **SIGN**.

Before presenting our main result, a lemma is needed.

LEMMA 5.15 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle$: $\mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Then, for every $\langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, if $\langle \Sigma_1, T_1\rangle$ is consistent, then so is $\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle$ and if $\langle \Sigma_1, T_1\rangle$ is complete, then so is $\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle$.

Proof:

Suppose that $\langle \Sigma_1, T_1\rangle$ is consistent, i.e., that $T_1 \neq \mathrm{SEN}_1(\Sigma_1)$ and assume, to the contrary, that $\alpha_{\Sigma_1}(T_1)^c = \mathrm{SEN}_2(F(\Sigma_1))$. By Lemma 5.10, $\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c = \mathrm{SEN}_2(F(\Sigma_1))$, whence $\alpha_{\Sigma_1}(T_1)^c = \alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c$, which contradicts Theorem 2.41 and Lemma 2.38.

Next, suppose that $\langle \Sigma_1, T_1\rangle$ is complete, i.e., that, for every $\langle \Sigma_1, T_1'\rangle$, with $T_1 \subset T_1'$, we have $T_1' = \mathrm{SEN}_1(\Sigma_1)$. Suppose to the contrary, that $\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle$ is not complete, i.e., that there exists $\langle F(\Sigma_1), T_2\rangle$, such that $\alpha_{\Sigma_1}(T_1)^c \subset T_2$, but $T_2 \neq \mathrm{SEN}_2(F(\Sigma_1))$. Then

$$\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c) \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(T_2)^c), \quad \text{i.e.,}$$

$$T_1 \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(T_2)^c),$$

with $\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(T_2)^c) \neq \mathrm{SEN}_1(\Sigma_1)$, which contradicts our hypothesis. ■

THEOREM 5.16 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π-institutions. $\mathcal{I}_1$ has the RCP if and only if $\mathcal{I}_2$ has the RCP.

Proof:

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Suppose that $\mathcal{I}_1$ has the RCP and assume that

$$\begin{tikzcd} & \Sigma_2''' & \\ \Sigma_2' \arrow[ur, "g_2'"] & & \Sigma_2'' \arrow[ul, "g_2''"'] \\ & \Sigma_2 \arrow[ul, "f_2'"] \arrow[ur, "f_2''"'] & \end{tikzcd}$$

is a pushout diagram in $\mathbf{SIGN}_2$ and that $\langle\Sigma_2, T_2\rangle$ is a consistent complete theory and $\langle\Sigma_2', T_2'\rangle, \langle\Sigma_2'', T_2''\rangle$ are consistent theories in $|\mathbf{TH}(\mathcal{I}_2)|$, such that $f_2' : \langle\Sigma_2, T_2\rangle \to \langle\Sigma_2', T_2'\rangle, f_2'' : \langle\Sigma_2, T_2\rangle \to \langle\Sigma_2'', T_2''\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_2))$. Since left adjoints preserve colimits, the following diagram

$$\begin{tikzcd} & G(\Sigma_2''') & \\ G(\Sigma_2') \arrow[ur, "G(g_2')"] & & G(\Sigma_2'') \arrow[ul, "G(g_2'')"'] \\ & G(\Sigma_2) \arrow[ul, "G(f_2')"] \arrow[ur, "G(f_2'')"'] & \end{tikzcd}$$

is a pushout diagram in $\mathbf{SIGN}_1$.

Consider the theories $\langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle, \langle G(\Sigma_2'), \beta_{\Sigma_2'}(T_2')^c\rangle$ and $\langle G(\Sigma_2''), \beta_{\Sigma_2''}(T_2'')^c\rangle$ in $\mathbf{TH}(\mathcal{I}_1)$. By Lemma 5.15, $\langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle$ is consistent and complete and

$$\langle G(\Sigma_2'), \beta_{\Sigma_2'}(T_2')^c\rangle, \langle G(\Sigma_2''), \beta_{\Sigma_2''}(T_2'')^c\rangle$$

are consistent. Moreover $G(f_2') : \langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle \to \langle G(\Sigma_2'), \beta_{\Sigma_2'}(T_2')^c\rangle$ and $G(f_2'') : \langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle \to \langle G(\Sigma_2''), \beta_{\Sigma_2''}(T_2'')^c\rangle$ are theory morphisms. Hence, since $\mathcal{I}_1$ has the RCP, the theory

$$\langle G(\Sigma_2'''), (\mathrm{SEN}_1(G(g_2'))(\beta_{\Sigma_2'}(T_2')) \cup \mathrm{SEN}_1(G(g_2''))(\beta_{\Sigma_2''}(T_2'')))^c\rangle$$

is a consistent theory in $\mathbf{TH}(\mathcal{I}_1)$. This theory is the same as

$$\begin{tikzcd}
\mathrm{SEN}_2(\Sigma_2') \arrow[r, "\beta_{\Sigma_2'}"] \arrow[d, "\mathrm{SEN}_2(g_2')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2'))"] \\
\mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2'''))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{SEN}_2(\Sigma_2'') \arrow[r, "\beta_{\Sigma_2''}"] \arrow[d, "\mathrm{SEN}_2(g_2'')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2'')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2''))"] \\
\mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2'''))
\end{tikzcd}$$

$$\langle G(\Sigma_2'''), (\beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(T_2')) \cup \beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2'')(T_2'')))^c\rangle$$

$$\text{i.e.,}\quad \langle G(\Sigma_2'''), \beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(T_2') \cup \mathrm{SEN}_2(g_2'')(T_2''))^c\rangle.$$

Consistency of this theory implies, by Lemma 5.15, consistency of

$$\langle F(G(\Sigma_2''')), \alpha_{G(\Sigma_2''')}(\beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(T_2') \cup \mathrm{SEN}_2(g_2'')(T_2'')))^c\rangle$$

and, therefore, since $\epsilon_{\Sigma_2'''}$ is an isomorphism, of

$$\langle \Sigma_2''', (\mathrm{SEN}_2(g_2')(T_2') \cup \mathrm{SEN}_2(g_2'')(T_2''))^c\rangle.$$

Thus, $\mathcal{I}_2$ has the RCP, as required.

The converse follows by symmetry. ∎

The Lindenbaum Property

DEFINITION 5.17 *Let* $\mathcal{I} = \langle \mathbf{SIGN}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{SIGN}|}\rangle$ *be a* π*-institution.* $\mathcal{I}$ *will be said to* **have the Lindenbaum Property** **(LP,** *for short,) if, for all* $\Sigma \in |\mathbf{SIGN}|$, $\langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|$, *if* $\langle \Sigma, T\rangle$ *is consistent, then there exists a consistent, complete theory* $\langle \Sigma, T'\rangle$, *such that* $T \subseteq T'$.

Theorem 5.18 *Let*

$$\mathcal{I}_1 = \langle \mathbf{SIGN}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_1|}\rangle, \quad \mathcal{I}_2 = \langle \mathbf{SIGN}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{SIGN}_2|}\rangle$$

be two deductively equivalent π-institutions. $\mathcal{I}_1$ has the LP if and only if $\mathcal{I}_2$ has the LP.

Proof:

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{SIGN}_1 \to \mathbf{SIGN}_2$. Suppose that $\mathcal{I}_1$ has the LP and let $\Sigma_2 \in |\mathbf{SIGN}_2|, \langle \Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$ a consistent theory. By Lemma 5.15, $\langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle$ is a consistent theory in $\mathbf{TH}(\mathcal{I}_1)$. Thus, since $\mathcal{I}_1$ has the LP, there exists a consistent, complete theory $\langle G(\Sigma_2), T_1\rangle$, such that $\beta_{\Sigma_2}(T_2)^c \subseteq T_1$. But then, by Lemma 5.15, $\langle F(G(\Sigma_2)), \alpha_{G(\Sigma_2)}(T_1)^c\rangle$ is a consistent, complete theory of $\mathcal{I}_2$, such that $\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(T_2))^c \subseteq \alpha_{G(\Sigma_2)}(T_1)^c$, whence $\langle \Sigma_2, \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(T_1))^c\rangle$ is a consistent, complete theory of $\mathbf{TH}(\mathcal{I}_2)$, such that $T_2 \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(T_1))^c$. Hence, $\mathcal{I}_2$ has the LP, as required.

The converse follows by symmetry. ■

6 ABSTRACT CLONE ALGEBRAS

The category **FACA** of free abstract clone algebras with a designated set of generators together with an adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{SET} \to \mathbf{FACA}$ is constructed. This gives rise to an algebraic theory $\mathbf{T}$ over $\mathbf{SET}$. A variety $\mathcal{ACA}$ of algebras is, then, equationally defined. It is shown that the Eilenberg-Moore category of $\mathbf{T}$-algebras is isomorphic to the category $\mathbf{ACA} = \vec{\mathcal{ACA}}$ corresponding to the variety $\mathcal{ACA}$.

Introduction

In algebraic logic one studies the classes of algebras that form the so-called *algebraic semantics* of deductive systems ([6, 7]). Along these lines several attempts have been made to define algebras that would be appropriate for algebraizing equational logic. Some of these attempts focused on ordinary, single-sorted, algebras, whereas others used many-sorted algebras. The general theory of this latter type of algebras has been developed independently in [41, 42],[30] and [4]. Some of these attempts are P. Hall's notion of clone (see [12]), which gives a partial single-sorted algebra, B.H. Neumann and E.C. Wiegold's representation of varieties in terms of semigroups [47], W.D. Neumann's substitution algebras [46], having infinitary substitution operations, Lawvere's algebraic theories [33, 34] (see also [36, 48]), W. Taylor's heterogeneous variety of substitution algebras [52] and, finally, N. Feldman's polynomial substitution algebras [20] (see also [11]). In a similar direction Czelakowski and Pigozzi [17] view equational logic as a 2-deductive system in the sense of [7] and propose its algebraization via another 2-deductive system, based on [20], which they call *hyperequational logic*.

In Chapter 3, a general framework for the algebraization of institutions was introduced. The attempt to algebraize the equational institution in this framework (see Chapter 3) leads naturally to the construction of an adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{SET} \to \mathbf{FACA}$. This adjunction gives, in turn, rise to an algebraic theory $\mathbf{T}$ in monoid form over $\mathbf{SET}$ (see Chapter 1). Based on [52], a variety $\mathcal{ACA}$ of single-sorted algebras is also constructed, that corresponds to clones of algebras with operations of arbitrary finite arities. It is then shown that the Eilenberg-Moore category of $\mathbf{T}$-algebras of the theory $\mathbf{T}$ is isomorphic to the category $\mathbf{ACA} = \vec{\mathcal{ACA}}$ of the variety $\mathcal{ACA}$.

Basic Constructions

A countably infinite set V, called **set of variables,** is fixed in advance and well-ordered and, as usual, the category of all small sets is denoted by $\mathbf{SET}$. The definition of a term, which has already been given in Chapter 3, is repeated below.

DEFINITION 6.1 *Let* $X \in |\mathbf{SET}|$. *We define the* **set of X-terms** $\mathrm{Tm}_X(V) \in |\mathbf{SET}|$, *to be the smallest set with*

(i) $V \subseteq \mathrm{Tm}_X(V)$ *and*

(ii) If $x \in X, n \in \omega$ *and* $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)$, *with* $t_{n-1} \neq v_{n-1}$, *then*

$$x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V).$$

The definitions of simultaneous substitution of terms for variables in a term and that of the extension of a given set map $f : X \to \mathrm{Tm}_Y(V)$ to a map $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ are also repeated below.

DEFINITION 6.2 *Let* $X \in |\mathbf{SET}|$, *as before. Define a function*

$$R_X : \mathrm{Tm}_X(V) \times \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)^k \to \mathrm{Tm}_X(V)$$

by recursion on the structure of X-terms as follows:

(i)

$$R_X(v_i, \langle s_0, \dots, s_{m-1}\rangle) = \begin{cases} s_i, & i < m \\ v_i, & i \geq m \end{cases}$$

for all $m \in \omega, s_0, \dots, s_{m-1} \in \mathrm{Tm}_X(V)$,

(ii)

$$R_X(x(t_0, \dots, t_{n-1}), \vec{s}) = \begin{cases} x(R_X(t_0, \vec{s}), \dots, R_X(t_{k-1}, \vec{s})), & \text{if } m \leq n, \text{ or } n < m \\ & \text{and } s_i = v_i\ \forall i \geq m \\ x(R_X(t_0, \vec{s}), \dots, R_X(t_{n-1}, \vec{s}), s_n, \dots, s_{k-1}), & \text{if } n < m \end{cases}$$

for all $x \in X, n \in \omega, t_0, \dots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, *and all* $m \in \omega$ *and* $\vec{s} \in \mathrm{Tm}_X(V)^m$, *where, in the first alternative,* $k = \max\{l : R_X(t_l, \vec{s}) \neq v_l\}$, *and, in the second,* $k = \max\{l : s_l \neq v_l\}$.

In other words, it is understood that the last, say k-th, term inside the parenthesis on the right, i.e., $R_X(t_{k-1}, \vec{s}), 0 \leq k < n$, *if* $m \leq n$, *and either* $R_X(t_{k-1}, \vec{s})$ *or* $s_{k-1}, 0 \leq k < m$, *if* $n < m$, *must be the last term that is not equal to the variable* v_{k-1}.

In what follows the second alternative in Definition 6.2 will be used as shorthand for both alternatives. If the first actually holds, then the trailing s's inside the parenthesis on the right hand side should be disregarded.

DEFINITION 6.3 *Let* $X, Y \in |\mathbf{SET}|$ *and* $f : X \to \mathrm{Tm}_Y(V)$. *Define* $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ *by recursion on the structure of X-terms as follows:*

(i) $f^*(v) = v$, *for every* $v \in V$,

(ii) $f^*(x(t_0, \dots, t_{n-1})) = R_Y(f(x), \langle f^*(t_0), \dots, f^*(t_{n-1})\rangle)$, *for every* $x \in X, n \in \omega, t_0, \dots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$.

In the sequel, we write $f : X \rightharpoondown Y$ to denote a **SET**−map $f : X \to \mathrm{Tm}_Y(V)$, as above. Given two such maps $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$, their **composition** $g \circ f : X \rightharpoondown Z$ is defined to be

$$g \circ f = g^* f.$$

LEMMA 6.4 *Let* $f : X \rightharpoondown Y, k, m \in \omega, t \in \mathrm{Tm}_X(V), \vec{u} \in \mathrm{Tm}_X(V)^k$ *and* $\vec{s} \in \mathrm{Tm}_X(V)^m$. *Then*

$$R_X(R_X(t, \vec{u}), \vec{s}) = R_X(t, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle).$$

Proof:

By recursion on the structure of t.

If $t = v_i \in V$,

$$R_X(R_X(v_i, \vec{u}), \vec{s}) = \left\{ \begin{array}{ll} R_X(u_i, \vec{s}), & i < k \\ R_X(v_i, \vec{s}), & i \geq k \end{array} \right\} = \left\{ \begin{array}{ll} R_X(u_i, \vec{s}), & i < k \\ s_i, & k \leq i < m \\ v_i, & m \leq i \end{array} \right\} =$$

$$= R_X(v_i, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle),$$

as required.

Next, if $x \in X, n \in \omega$ and $t_0, \dots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$,

$$R_X(R_X(x(t_0, \dots, t_{n-1}), \vec{u}), \vec{s}) =$$

$$\begin{aligned}
&= R_X(x(R_X(t_0, \vec{u}), \dots, R_X(t_{n-1}, \vec{u}), u_n, \dots, u_{k-1}), \vec{s}) \text{ (by definition of } R_X)\\
&= x(R_X(R_X(t_0, \vec{u}), \vec{s}), \dots, R_X(R_X(t_{n-1}, \vec{u}), \vec{s}),\\
&\quad R_X(u_n, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}) \text{ (by definition of } R_X)\\
&= x(R_X(t_0, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle), \dots,\\
&\quad R_X(t_{n-1}, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle), R_X(u_n, \vec{s}), \dots,\\
&\quad R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}) \text{ (by the induction hypothesis)}\\
&= R_X(x(t_0, \dots, t_{n-1}), \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle),\\
&\quad \text{(by definition of } R_X)
\end{aligned}$$

as required. ∎

LEMMA 6.5 *Let* $f : X \rightharpoondown Y, m \in \omega, t \in \mathrm{Tm}_X(V), \vec{s} \in \mathrm{Tm}_X(V)^m$. *Then*

$$f^{*}(R_X(t, \vec{s})) = R_Y(f^{*}(t), f^{*}(\vec{s})).$$

Proof:

By induction on the structure of t. If $t = v_i \in V$,

$$f^*(R_X(v_i, \langle s_0, \dots, s_{m-1}\rangle)) = \left\{ \begin{array}{ll} f^*(s_i), & i < m \\ f^*(v_i), & i \geq m \end{array} \right\} = \left\{ \begin{array}{ll} f^*(s_i), & i < m \\ v_i, & i \geq m \end{array} \right\} =$$

$$= R_Y(v_i, \langle f^*(s_0), \dots, f^*(s_{m-1})\rangle) = R_Y(f^*(v_i), \langle f^*(s_0), \dots, f^*(s_{m-1})\rangle),$$

as required.

Next, if $x \in X, n \in \omega$ and $t_0, \dots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$,

$$f^*(R_X(x(t_0, \dots, t_{n-1}), \langle s_0, \dots, s_{m-1}\rangle)) =$$

$$\begin{aligned}
&= f^*(x(R_X(t_0, \vec{s}), \dots, R_X(t_{n-1}, \vec{s}), s_n, \dots, s_{m-1})) \text{ (by definition of } R_X) \\
&= R_Y(f(x), \langle f^*(R_X(t_0, \vec{s})), \dots, f^*(R_X(t_{n-1}, \vec{s})), f^*(s_n), \dots, f^*(s_{m-1})\rangle) \\
&\quad \text{(by definition of } f^*) \\
&= R_Y(f(x), \langle R_Y(f^*(t_0), f^*(\vec{s})), \dots, R_Y(f^*(t_{n-1}), f^*(\vec{s})), f^*(s_n), \dots, f^*(s_{n-1})\rangle) \\
&\quad \text{(by the induction hypothesis)} \\
&= R_Y(R_Y(f(x), \langle f^*(t_0), \dots, f^*(t_{n-1})\rangle), f^*(\vec{s})) \quad \text{(by Lemma 6.4)} \\
&= R_Y(f^*(x(t_0, \dots, t_{n-1})), f^*(\vec{s})), \text{ (by definition of } f^*)
\end{aligned}$$

as required. ■

LEMMA 6.6 *Let* $f : X \to Y, g : Y \to Z$. *Then*

$$(g \circ f)^* = g^* f^*.$$

Proof:

By induction on the structure of $t \in \mathrm{Tm}_X(V)$.

If $t = v_i \in V$, $(g \circ f)^*(v_i) = v_i = g^*(f^*(v_i))$, as required.

Next, if $x \in X, n \in \omega$ and $t_0, \dots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$,

$$\begin{aligned}
(g \circ f)^*(x(t_0, \dots, t_{n-1})) &= R_Z((g \circ f)(x), \langle (g \circ f)^*(t_0), \dots, (g \circ f)^*(t_{n-1}) \rangle) \\
&\quad \text{(by definition of } (g \circ f)^*) \\
&= R_Z(g^*(f(x)), \langle g^*(f^*(t_0)), \dots, g^*(f^*(t_{n-1})) \rangle) \\
&\quad \text{(by definition of } g \circ f \text{ and the induction hypothesis)} \\
&= g^*(R_Y(f(x), \langle f^*(t_0), \dots, f^*(t_{n-1}) \rangle)) \quad \text{(by Lemma 6.5)} \\
&= g^*(f^*(x(t_0, \dots, t_{n-1}))). \text{ (by definition of } f^*)
\end{aligned}$$

as required. ■

Define $|\mathbf{FACA}| = |\mathbf{SET}|$ and, for all $X, Y \in |\mathbf{SET}|$,

$$\mathbf{FACA}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{SET}(X, \mathrm{Tm}_Y(V))\}.$$

Then the following holds

THEOREM 6.7 **FACA** *is a category with objects* $|\mathbf{FACA}|$, *morphisms* $\mathbf{FACA}(X, Y)$, *for all* $X, Y \in |\mathbf{FACA}|$, *morphism composition* $\circ$ *and identity arrows* $j_X : X \rightharpoondown X$ *the set maps* $j_X : X \to \mathrm{Tm}_X(V)$, *with* $j_X(x) = x()$, *for every* $x \in X$.

Proof:

We show that $\circ$ is associative. Ti this end, let $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ and $h : Z \rightharpoondown W$ be **FACA**-morphisms. Then

$$\begin{aligned}
h \circ (g \circ f) &= h^*(g \circ f) \quad \text{(by definition of } \circ) \\
&= h^*(g^* f) \quad \text{(by definition of } \circ) \\
&= (h^* g^*) f \quad \text{(by associativity of composition)} \\
&= (h \circ g)^* f \quad \text{(by Lemma 6.6)} \\
&= (h \circ g) \circ f \quad \text{(by definition of } \circ)
\end{aligned}$$

■

It will turn out that **FACA** is the Kleisli category of the algebraic theory **T** to be constructed later. Moreover, the adjunction that will be constructed in the next section will turn out to be the associated Kleisli adjunction.

The Adjunction

We are now ready to proceed with the construction of the promised adjunction

$$\langle F, U, \eta, \epsilon \rangle : \mathbf{SET} \to \mathbf{FACA}.$$

First, define a functor $F : \mathbf{SET} \to \mathbf{FACA}$ by

$$F(X) = X, \quad \text{for every} \quad X \in |\mathbf{SET}|,$$

and, if $f : X \to Y \in \mathrm{Mor}(\mathbf{SET})$,

$$F(f) = j_Y f : X \rightharpoondown Y.$$

If $f : X \to Y, g : Y \to Z \in \mathrm{Mor}(\mathbf{SET})$, then

$$F(gf) = j_Z(gf) = (j_Z g)^{*}(j_Y f) = F(g)^{*}F(f) = F(g) \circ F(f),$$

i.e., F is a functor, as required.

Now define a functor $U : \mathbf{FACA} \to \mathbf{SET}$ by

$$U(X) = \mathrm{Tm}_X(V), \quad \text{for every} \quad X \in |\mathbf{FACA}|,$$

and, if $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{FACA})$,

$$U(f) = f^{*} : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V).$$

Then, if $f : X \rightharpoondown Y, g : Y \rightharpoondown Z \in \mathrm{Mor}(\mathbf{FACA})$, we have

$$\begin{aligned} U(g \circ f) &= (g \circ f)^{*} \\ &= g^{*} f^{*} \quad \text{(by Lemma 6.6)} \\ &= U(g)U(f), \end{aligned}$$

i.e., U is also a functor, as required.

Finally, define natural transformations $\eta : I_{\mathbf{SET}} \to UF$ by $\eta_X : X \to \mathrm{Tm}_X(V)$ with $\eta_X = j_X$, for every $X \in |\mathbf{SET}|$, and $\epsilon : FU \to I_{\mathbf{FACA}}$ by $\epsilon_X : \mathrm{Tm}_X(V) \rightharpoondown X$ with

$\epsilon_X = i_{\mathrm{Tm}_X(V)}$, for every $X \in |\mathbf{FACA}|$. We now show that η and ϵ are indeed natural transformations.

To this end, let $f : X \to Y \in \mathrm{Mor}(\mathbf{SET})$. Then, for every $x \in X$,

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & U(F(X)) \arrow[d, "U(F(f))"] \\ Y \arrow[r, "\eta_Y"'] & U(F(Y)) \end{tikzcd}$$

$$U(F(f))(\eta_X(x)) = U(F(f))(x()) = (j_Y f)^{*}(x()) = f(x)() = j_Y f(x) = \eta_Y(f(x)),$$

as required.

Next, let $f : X \to Y \in \mathrm{Mor}(\mathbf{FACA})$. Then, for every $t \in \mathrm{Tm}_X(V)$,

$$\begin{tikzcd} F(U(X)) \arrow[r, "\epsilon_X"] \arrow[d, "F(U(f))"'] & X \arrow[d, "f"] \\ F(U(Y)) \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

$$(f \circ \epsilon_X)(t) = f^{*}(\epsilon_X(t)) = f^{*}(t) = \epsilon_Y^{*}(j_{\mathrm{Tm}_Y(V)}(f^{*}(t))) = (\epsilon_Y \circ F(U(f)))(t),$$

as required.

THEOREM 6.8 $\langle F, U, \eta, \epsilon \rangle : \mathbf{SET} \to \mathbf{FACA}$ *is an adjunction.*

Proof:

By the preceding discussion η and ϵ are natural transformations, whence it suffices to show that the following triangles commute, for every $X \in |\mathbf{FACA}|, Y \in |\mathbf{SET}|$,

$$\begin{tikzcd} U(X) \arrow[r, "\eta_{U(X)}"] \arrow[dr, "i_{U(X)}"'] & U(F(U(X))) \arrow[d, "U(\epsilon_X)"] \\ & U(X) \end{tikzcd} \qquad \begin{tikzcd} F(Y) \arrow[r, "F(\eta_Y)"] \arrow[dr, "\eta_{F(Y)}"'] & F(U(F(Y))) \arrow[d, "\epsilon_{F(Y)}"] \\ & F(Y) \end{tikzcd}$$

These are

$$
\begin{tikzcd}
\mathrm{Tm}_X(V) \arrow[r, "\eta_{\mathrm{Tm}_X(V)}"] \arrow[dr, "i_{\mathrm{Tm}_X(V)}"'] & \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[d, "i^{*}_{\mathrm{Tm}_X(V)}"] \\
& \mathrm{Tm}_X(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\eta_{\mathrm{Tm}_Y(V)}\eta_Y"] \arrow[dr, "\eta_Y"'] & \mathrm{Tm}_Y(V) \arrow[d, "i_{\mathrm{Tm}_Y(V)}"] \\
& Y
\end{tikzcd}
$$

If $t \in \mathrm{Tm}_X(V)$, then

$$i^{*}_{\mathrm{Tm}_X(V)}(\eta_{\mathrm{Tm}_X(V)}(t)) = i^{*}_{\mathrm{Tm}_X(V)}(t()) = t,$$

and, if $y \in Y$,

$$i^{*}_{\mathrm{Tm}_X(V)}(\eta_{\mathrm{Tm}_Y(V)}(\eta_Y(y))) = i^{*}_{\mathrm{Tm}_X(V)}(y()()) = y() = \eta_Y(y),$$

as required. ■

The Theory of the Adjunction

It is well-known ([39, 43, 9], see also Chapter 1) that the adjunction $\langle F, U, \eta, \epsilon\rangle$: $\mathbf{SET} \to \mathbf{FACA}$ gives rise to an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in monoid form over $\mathbf{SET}$, with $T = UF$ and $\mu = U\epsilon_F$. Moreover there exists a unique functor $K : \mathbf{SET_T} \to \mathbf{FACA}$ from the Kleisli category of the theory to $\mathbf{FACA}$, called the *Kleisli comparison functor of the adjunction*, that makes the F- and U-paths of the following diagrams commute.

$$
\begin{tikzcd}
\mathbf{SET_T} \arrow[rr, "K"] & & \mathbf{FACA} \\
& \mathbf{SET} \arrow[ul, "F_{\mathbf{T}}"] \arrow[ur, "F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{SET_T} \arrow[rr, "K"] \arrow[dr, "U_{\mathbf{T}}"'] & & \mathbf{FACA} \arrow[dl, "U"] \\
& \mathbf{SET} &
\end{tikzcd}
$$

Given such an adjunction, the Kleisli category $\mathbf{SET_T}$ of $\mathbf{T}$ in $\mathbf{SET}$ has as objects $|\mathbf{SET}|$ and as morphisms $\mathbf{SET_T}(X, Y) = \mathbf{SET}(X, U(F(Y)))$, for all $X, Y \in |\mathbf{SET}|$. Moreover, it is easy to verify that the Kleisli composition coincides with the composition $\circ$ in $\mathbf{FACA}$. Thus, in this case $\mathbf{SET_T} = \mathbf{FACA}$ and $K = I_{\mathbf{FACA}}$. Therefore $\mathbf{FACA}$ is the category of all free algebras of the algebraic theory $\mathbf{T}$ over $\mathbf{SET}$.

Also recall that a $\mathbf{T}$-algebra $\langle X, \xi\rangle$ consists of a set X together with a map $\xi : T(X) \to X$, i.e., $\xi : \mathrm{Tm}_X(V) \to X$, such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "i_X"'] & T(X) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T(T(X)) \arrow[r, "T(\xi)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "\xi"] \\ T(X) \arrow[r, "\xi"'] & X \end{tikzcd}$$

These take the form

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[dr, "i_X"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[r, "(j_X\xi)^\bullet"] \arrow[d, "i^\bullet_{\mathrm{Tm}_X(V)}"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ \mathrm{Tm}_X(V) \arrow[r, "\xi"'] & X \end{tikzcd}$$

Abstract Clone Algebras

In this section we equationally define a variety of algebras $\mathcal{ACA}$, whose members we call **abstract clone algebras.** In the next section, it will be shown that the category $\mathbf{ACA} = \vec{\mathcal{ACA}}$ of this variety is isomorphic to the Eilenberg-Moore category $\mathbf{SET}^{\mathbf{T}}$ of the algebraic theory $\mathbf{T}$ in $\mathbf{SET}$, that was constructed in the previous section.

Let $\mathcal{L} = \langle \Lambda, \rho\rangle$ be the language type defined as follows.

$$\Lambda = \{\mathrm{v}_i, \mathrm{C}_i : i \in \omega\}, \quad \text{with} \quad \rho(\mathrm{v}_i) = 0, \rho(\mathrm{C}_i) = i+1.$$

DEFINITION 6.9 *An* **abstract clone algebra A** *is an $\mathcal{L}$-algebra that satisfies the following identities, for all $n, m \in \omega$,*

- $\mathrm{C}_0(x) = x$
- $\mathrm{C}_n(x, y_0 \ldots, y_{n-2}, \mathrm{v}_{i-1}) = \mathrm{C}_{n-1}(x, y_0 \ldots, y_{n-2})$
- $$\mathrm{C}_n(\mathrm{v}_m, x_0, \ldots, x_{n-1}) = \begin{cases} x_m, & \textit{if } m < n \\ \mathrm{v}_m, & \textit{otherwise} \end{cases}$$
- $\mathrm{C}_n(z, \mathrm{C}_n(y_0, \vec{x}), \ldots, \mathrm{C}_n(y_{m-1}, \vec{x}), x_m, \ldots, x_{n-1}) = \mathrm{C}_n(C_m(z, \vec{y}), \vec{x})$

Let $\mathcal{ACA}$ be the variety of all abstract clone algebras and denote by $\mathbf{ACA} = \vec{\mathcal{ACA}}$ *the category associated with $\mathcal{ACA}$.*

Let $\mathbf{A} = \langle A, \mathcal{C}^{\mathbf{A}} \rangle$ be an abstract clone algebra. Define $\mathbf{A}^{\bullet} = \langle A, \xi_{\mathbf{A}^{\bullet}} \rangle$ as follows. $\xi_{\mathbf{A}^{\bullet}} : \mathrm{Tm}_A(V) \to A$ is defined by recursion on the structure of A-terms, by

- $\xi_{\mathbf{A}^{\bullet}}(v_i) = \mathrm{v}_i^{\mathbf{A}}$, for every $i \in \omega$.
- If $a \in A, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_A(V), t_{n-1} \neq v_{n-1}$,

$$\xi_{\mathbf{A}^{\bullet}}(a(t_0, \ldots, t_{n-1})) = \mathrm{C}_n^{\mathbf{A}}(a, \xi_{\mathbf{A}^{\bullet}}(t_0), \ldots, \xi_{\mathbf{A}^{\bullet}}(t_{n-1})).$$

LEMMA 6.10 *Let* $\mathbf{A} \in \mathcal{ACA}, \mathbf{A}^{\bullet} = \langle A, \xi_{\mathbf{A}^{\bullet}} \rangle$. *Then, for every* $t \in \mathrm{Tm}_A(V), m \in \omega, \vec{s} \in \mathrm{Tm}_A(V)^m$,

$$\xi_{\mathbf{A}^{\bullet}}(R_A(t, \vec{s})) = \mathrm{C}_m^{\mathbf{A}}(\xi_{\mathbf{A}^{\bullet}}(t), \xi_{\mathbf{A}^{\bullet}}(\vec{s})).$$

Proof:

By induction on the structure of t.

If $t = v_i \in V$, then

$$\begin{aligned}
\xi_{\mathbf{A}^{\bullet}}(R_A(v_i, \vec{s})) &= \left\{ \begin{array}{ll} \xi_{\mathbf{A}^{\bullet}}(s_i), & \text{if } i < m \\ \xi_{\mathbf{A}^{\bullet}}(v_i), & \text{if } i \geq m \end{array} \right\} = \left\{ \begin{array}{ll} \xi_{\mathbf{A}^{\bullet}}(s_i), & \text{if } i < m \\ \mathrm{v}_i^{\mathbf{A}}, & \text{if } i \geq m \end{array} \right\} \\
&= \mathrm{C}_m^{\mathbf{A}}(\mathrm{v}_i^{\mathbf{A}}, \xi_{\mathbf{A}^{\bullet}}(\vec{s})) \qquad \text{(by the third axiom)} \\
&= \mathrm{C}_m^{\mathbf{A}}(\xi_{\mathbf{A}^{\bullet}}(v_i), \xi_{\mathbf{A}^{\bullet}}(\vec{s})),
\end{aligned}$$

as required.

If $a \in A, n \in \omega, \vec{t} \in \mathrm{Tm}_A(V)^n, t_{n-1} \neq v_{n-1}$, then

$$\begin{aligned}
\xi_{\mathbf{A}^{\bullet}}(R_A(a(\vec{t}), \vec{s})) &= \xi_{\mathbf{A}^{\bullet}}(a(R_A(t_0, \vec{s}), \ldots, R_A(t_{n-1}, \vec{s}), s_n, \ldots, s_{m-1})) \\
&\quad \text{(by definition of } R_A) \\
&= \mathrm{C}_m^{\mathbf{A}}(a, \xi_{\mathbf{A}^{\bullet}}(R_A(t_0, \vec{s})), \ldots, \xi_{\mathbf{A}^{\bullet}}(R_A(t_{n-1}, \vec{s})), \\
&\quad \xi_{\mathbf{A}^{\bullet}}(s_n), \ldots, \xi_{\mathbf{A}^{\bullet}}(s_{m-1})) \text{ (by definition of } \xi_{\mathbf{A}^{\bullet}})
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{C}^{\mathbf{A}}_m(a, \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^\bullet}(t_0), \xi_{\mathbf{A}^\bullet}(\vec{s})), \dots, \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^\bullet}(t_{n-1}), \xi_{\mathbf{A}^\bullet}(\vec{s})), \\
&\qquad \xi_{\mathbf{A}^\bullet}(s_n), \dots, \xi_{\mathbf{A}^\bullet}(s_{m-1})) \text{ (by the induction hypothesis)} \\
&= \mathrm{C}^{\mathbf{A}}_m(\mathrm{C}^{\mathbf{A}}_n(a, \xi_{\mathbf{A}^\bullet}(\vec{t})), \xi_{\mathbf{A}^\bullet}(\vec{s})) \text{ (by the fourth axiom)} \\
&= \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^\bullet}(a(\vec{t})), \xi_{\mathbf{A}^\bullet}(\vec{s})), \text{ (by definition of } \xi_{\mathbf{A}^\bullet})
\end{aligned}
$$

as required. ■

LEMMA 6.11 *Let* $\mathbf{A} \in \mathcal{ACA}$. *Then* $\mathbf{A}^\bullet = \langle A, \xi_{\mathbf{A}^\bullet} \rangle \in |\mathbf{SET}^{\mathbf{T}}|$.

Proof:

We need to show that the following diagrams commute

$$
\begin{tikzcd}
A \arrow[r, "j_A"] \arrow[dr, "i_A"'] & \mathrm{Tm}_A(V) \arrow[d, "\xi_{\mathbf{A}^\bullet}"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Tm}_{\mathrm{Tm}_A(V)}(V) \arrow[r, "(j_A\xi_{\mathbf{A}^\bullet})^\bullet"] \arrow[d, "i^\bullet_{\mathrm{Tm}_A(V)}"'] & \mathrm{Tm}_A(V) \arrow[d, "\xi_{\mathbf{A}^\bullet}"] \\
\mathrm{Tm}_A(V) \arrow[r, "\xi_{\mathbf{A}^\bullet}"'] & A
\end{tikzcd}
$$

For the triangle, we have, for every $a \in A$,

$$
\begin{aligned}
\xi_{\mathbf{A}^\bullet}(j_A(a)) &= \xi_{\mathbf{A}^\bullet}(a()) \text{ (by definition of } j_A) \\
&= \mathrm{C}^{\mathbf{A}}_0(a) \text{ (by definition of } \xi_{\mathbf{A}^\bullet}) \\
&= a \text{ (by the first axiom)} \\
&= i_A(a), \quad \text{as required.}
\end{aligned}
$$

For the rectangle, we proceed by induction on the structure of a $\mathrm{Tm}_A(V)$-term t. If $t = v_i \in V$, then

$$\xi_{\mathbf{A}^\bullet}((j_A\xi_{\mathbf{A}^\bullet})^\bullet(v_i)) = \xi_{\mathbf{A}^\bullet}(v_i) = \xi_{\mathbf{A}^\bullet}(i^\bullet_{Tm_A(V)}(v_i)),$$

as required.

If $s \in \mathrm{Tm}_A(V), n \in \omega, \vec{t} \in \mathrm{Tm}_{\mathrm{Tm}_A(V)}(V)^n, t_{n-1} \neq v_{n-1}$, then

$$
\begin{aligned}
\xi_{\mathbf{A}^\bullet}((j_A\xi_{\mathbf{A}^\bullet})^\bullet(s(\vec{t}))) &= \xi_{\mathbf{A}^\bullet}(R_A((j_A\xi_{\mathbf{A}^\bullet})(s), (j_A\xi_{\mathbf{A}^\bullet})^\bullet(\vec{t}))) \text{ (by definition of } (j_A\xi_{\mathbf{A}^\bullet})^\bullet) \\
&= \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^\bullet}(j_A(\xi_{\mathbf{A}^\bullet}(s))), \xi_{\mathbf{A}^\bullet}((j_A\xi_{\mathbf{A}^\bullet})^\bullet(\vec{t}))) \text{ (by Lemma 6.10)} \\
&= \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^\bullet}(s), \xi_{\mathbf{A}^\bullet}(i^\bullet_{\mathrm{Tm}_A(V)}(\vec{t}))) \\
&\qquad \text{(by commutativity of triangle and the induction hypothesis)}
\end{aligned}
$$

$$= \xi_{\mathbf{A}^{\bullet}}(R_A(s, i^{*}_{\mathrm{Tm}_A(V)}(\vec{t}))) \text{ (by Lemma 6.10)}$$
$$= \xi_{\mathbf{A}^{\bullet}}(i^{*}_{\mathrm{Tm}_A(V)}(s(\vec{t}))), \text{ (by definition of } i^{*}_{\mathrm{Tm}_A(V)})$$

as required. ■

Next suppose that $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle, \mathbf{B} = \langle B, \mathcal{L}^{\mathbf{B}} \rangle \in \mathcal{ACA}$ and $h : \mathbf{A} \to \mathbf{B} \in \mathbf{ACA}(\mathbf{A}, \mathbf{B})$. We show that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Tm}_A(V) \arrow[r, "(j_B h)^{*}"] \arrow[d, "\xi_{\mathbf{A}^{\bullet}}"'] & \mathrm{Tm}_B(V) \arrow[d, "\xi_{B^{\bullet}}"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

i.e., that $h \in \mathbf{SET}^{\mathbf{T}}(\mathbf{A}^{*}, \mathbf{B}^{*})$.

We work by induction on the structure of an A-term t.

If $t = v_i \in V$, then

$$\begin{aligned}
\xi_{\mathbf{B}^{\bullet}}((j_B h)^{*}(v_i)) &= \xi_{\mathbf{B}^{\bullet}}(v_i) \text{ (by definition of } (j_B h)^{*}) \\
&= \mathrm{v}_i^{\mathbf{B}} \text{ (by definition of } \xi_{\mathbf{B}^{\bullet}}) \\
&= h(\mathrm{v}_i^{\mathbf{A}}) \text{ (since } h \in \mathbf{ACA}(\mathbf{A}, \mathbf{B})) \\
&= h(\xi_{\mathbf{A}^{\bullet}}(v_i)), \text{ (by definition of } \xi_{\mathbf{A}^{\bullet}})
\end{aligned}$$

as required.

If $a \in A, n \in \omega, \vec{t} \in \mathrm{Tm}_A(V)^n, t_{n-1} \neq v_{n-1}$,

$$\begin{aligned}
\xi_{\mathbf{B}^{\bullet}}((j_B h)^{*}(a(\vec{t}))) &= \xi_{\mathbf{B}^{\bullet}}(R_B((j_B h)(a), (j_B h)^{*}(\vec{t}))) \text{ (by definition of } (j_B h)^{*}) \\
&= \mathrm{C}_n^{\mathbf{B}}(\xi_{\mathbf{B}^{\bullet}}(j_B(h(a))), \xi_{\mathbf{B}^{\bullet}}((j_B h)^{*}(\vec{t}))) \text{ (by Lemma 6.10)} \\
&= \mathrm{C}_n^{\mathbf{B}}(h(a), h(\xi_{\mathbf{A}^{\bullet}}(\vec{t}))) \text{ (by comm. of triangle and the ind. hyp.)} \\
&= h(\mathrm{C}_n^{\mathbf{A}}(a, \xi_{\mathbf{A}^{\bullet}}(\vec{t}))) \text{ (since } h \in \mathbf{ACA}(\mathbf{A}, \mathbf{B})) \\
&= h(\xi_{\mathbf{A}^{\bullet}}(a(\vec{t}))), \text{ (by definition of } \xi_{\mathbf{A}^{\bullet}})
\end{aligned}$$

as required.

Thus, it is possible to define the functor $P : \mathbf{ACA} \to \mathbf{SET}^{\mathbf{T}}$ by

$$P(\mathbf{A}) = \mathbf{A}^{*}, \quad \text{for every} \quad \mathbf{A} \in \mathcal{ACA},$$

and, given $h \in \mathbf{ACA}(\mathbf{A}, \mathbf{B})$, $P(h) \in \mathbf{SET}^{\mathbf{T}}(\mathbf{A}^{\cdot}, \mathbf{B}^{\cdot})$, by

$$P(h) = h.$$

The Equivalence

In this section, a functor $Q : \mathbf{SET}^{\mathbf{T}} \to \mathbf{ACA}$ in the opposite direction is defined and it is shown that P and Q are inverses of each other. Therefore the two categories $\mathbf{SET}^{\mathbf{T}}$ and $\mathbf{ACA}$ are isomorphic categories.

Let $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle$ be a $\mathbf{T}$-algebra. Define an $\mathcal{L}$-algebra $\mathbf{A}^{\#} = \langle A, \mathcal{L}^{\mathbf{A}^{\#}} \rangle$ as follows:

- $\mathrm{v}_i^{\mathbf{A}^{\#}} = \xi_{\mathbf{A}}(v_i)$, for every $i \in \omega$,
- $\mathrm{C}_n^{\mathbf{A}^{\#}}(a, a_0, \ldots, a_{n-1}) = \xi_{\mathbf{A}}(R_A(j_A(a), j_A(a_0), \ldots, j_A(a_{n-1})))$, for every $n \in \omega$, $a, a_0, \ldots, a_{n-1} \in A$.

LEMMA 6.12 *Let* $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle \in |\mathbf{SET}^{\mathbf{T}}|$. *Then* $j_A\xi_{\mathbf{A}} = (j_A\xi_{\mathbf{A}})^{\cdot} j_{\mathrm{Tm}_A(V)}$.

Proof:

Let $t \in \mathrm{Tm}_A(V)$. Then

$$\begin{aligned} (j_A\xi_{\mathbf{A}})^{\cdot}(j_{\mathrm{Tm}_A(V)}(t)) &= (j_A\xi_{\mathbf{A}})^{\cdot}(t()) \text{ (by definition of } j_{\mathrm{Tm}_A(V)}) \\ &= j_A\xi_{\mathbf{A}}(t), \text{ (by definition of } (j_A\xi_{\mathbf{A}})^{\cdot}) \end{aligned}$$

as required. ∎

LEMMA 6.13 *Let* $\mathbf{A} \in |\mathbf{SET}^{\mathbf{T}}|$. *Then* $\mathbf{A}^{\#} \in \mathcal{ACA}$.

Proof:

We need to verify that the identities of Definition 6.9 hold. For the first one,

$$\mathrm{C}_0^{\mathbf{A}^{\#}}(a) = \xi_{\mathbf{A}}(R_A(j_A(a), \langle\rangle)) = \xi_{\mathbf{A}} j_A(a) = a,$$

as required.

For the second, we have

$$
\begin{aligned}
C_n^{\mathbf{A}\#}(a, b_0, \ldots, b_{n-2}, v_{n-1}^{\mathbf{A}\#}) &= C_n^{\mathbf{A}\#}(a, \vec{b}, \xi_{\mathbf{A}}(v_{n-1})) \text{ (by definition of } v_{n-1}^{\mathbf{A}\#}) \\
&= \xi_{\mathbf{A}}(R_A(j_A(a), j_A(\vec{b}), j_A(\xi_{\mathbf{A}}(v_{n-1})))) \text{ (by defin. of } C_n^{\mathbf{A}\#}) \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(j_A(a))), j_A(\xi_{\mathbf{A}}(j_A(\vec{b}))), j_A(\xi_{\mathbf{A}}(v_{n-1})))) \\
&\quad \text{(by } \xi_{\mathbf{A}} j_A = i_A) \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(a())), j_A(\xi_{\mathbf{A}}(\vec{b}())), j_A(\xi_{\mathbf{A}}(v_{n-1})))) \\
&\quad \text{(by definition of } j_A) \\
&= \xi_{\mathbf{A}}(R_A((j_A\xi_{\mathbf{A}})(a()), (j_A\xi_{\mathbf{A}})^{*}(\vec{b}()()), (j_A\xi_{\mathbf{A}})^{*}(v_{n-1}()))) \\
&\quad \text{(by Lemma 6.12)} \\
&= \xi_{\mathbf{A}}((j_A\xi_{\mathbf{A}})^{*}(a()(\vec{b}()(), v_{n-1}()))) \text{ (by defin. of } (j_A\xi_{\mathbf{A}})^{*}) \\
&= \xi_{\mathbf{A}}(i^{*}_{\mathrm{Tm}_A(V)}(a()(\vec{b}()(), v_{n-1}()))) \\
&\quad \text{(since } \xi_{\mathbf{A}}(j_A\xi_{\mathbf{A}})^{*} = \xi_{\mathbf{A}} i^{*}_{\mathrm{Tm}_A(V)}) \\
&= \xi_{\mathbf{A}}(R_A(a(), \vec{b}(), v_{n-1})) \text{ (by definition of } i^{*}_{\mathrm{Tm}_A(V)}) \\
&= \xi_{\mathbf{A}}(R_A(a(), \vec{b}())). \text{ (by definition of } R_A) \\
&= \ldots \text{ (reverse all the steps in the deduction above)} \\
&= C_{n-1}^{\mathbf{A}\#}(a, \vec{b}),
\end{aligned}
$$

as required.

For the third identity, we have

$$
\begin{aligned}
C_n^{\mathbf{A}\#}(v_m^{\mathbf{A}\#}, \vec{a}) &= C_n^{\mathbf{A}\#}(\xi_{\mathbf{A}}(v_m), \vec{a}) \text{ (by definition of } v_m^{\mathbf{A}\#}) \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(v_m)), j_A(\vec{a}))) \text{ (by definition of } C_n^{\mathbf{A}\#}) \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(v_m)), j_A(\xi_{\mathbf{A}}(j_A(\vec{a}))))) \text{ (since } \xi_{\mathbf{A}} j_A = i_A) \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(v_m)), j_A(\xi_{\mathbf{A}}(\vec{a}())))) \text{ (by the definition of } j_A) \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(v_m)), (j_A\xi_{\mathbf{A}})^{*}(\vec{a}()()))) \text{ (by Lemma 6.12)} \\
&= \xi_{\mathbf{A}}((j_A\xi_{\mathbf{A}})^{*}(v_m(\vec{a}()()))) \text{ (by definition of } (j_A\xi_{\mathbf{A}})^{*}) \\
&= \xi_{\mathbf{A}}(i^{*}_{\mathrm{Tm}_A(V)}(v_m(\vec{a}()()))) \text{ (since } \xi_{\mathbf{A}}(j_A\xi_{\mathbf{A}})^{*} = \xi_{\mathbf{A}} i^{*}_{\mathrm{Tm}_A(V)}) \\
&= \xi_{\mathbf{A}}(R_A(v_m, \vec{a}())) \text{ (by definition of } i^{*}_{\mathrm{Tm}_A(V)})
\end{aligned}
$$

$$= \left\{ \begin{array}{ll} \xi_{\mathbf{A}}(a_m()), & \text{if } m < n \\ \xi_{\mathbf{A}}(v_m), & \text{if } m \geq n \end{array} \right\} \text{ (by definition of } R_A\text{)}$$

$$= \left\{ \begin{array}{ll} a_m, & \text{if } m < n \\ \mathrm{v}_m^{\mathbf{A}^\#}, & \text{if } m \geq n \end{array} \right\}, \text{ (since } \xi_{\mathbf{A}} j_A = i_A \text{ and by defin. of } \mathrm{v}_m^{\mathbf{A}^\#}\text{)}$$

as required.

For the fourth identity, we have

$$\mathrm{C}_n^{\mathbf{A}^\#}(a, \mathrm{C}_n^{\mathbf{A}^\#}(b_0, \vec{c}), \ldots, \mathrm{C}_n^{\mathbf{A}^\#}(b_{m-1}, \vec{c}), c_m, \ldots, c_{n-1}) =$$

$$\begin{aligned}
&= \mathrm{C}_n^{\mathbf{A}^\#}(a, \xi_{\mathbf{A}}(R_A(j_A(b_0), j_A(\vec{c}))), \ldots, \xi_{\mathbf{A}}(R_A(j_A(b_{m-1}), j_A(\vec{c}))), c_m, \ldots, c_{n-1}) \\
&\quad \text{(by definition of } \mathrm{C}_n^{\mathbf{A}^\#}\text{)} \\
&= \xi_{\mathbf{A}}(R_A(j_A(a), j_A(\xi_{\mathbf{A}}(R_A(j_A(b_0), j_A(\vec{c})))), \ldots, j_A(\xi_{\mathbf{A}}(R_A(j_A(b_{m-1}), j_A(\vec{c})))), \\
&\quad j_A(c_m), \ldots, j_A(c_{n-1}))) \text{ (by definition of } \mathrm{C}_n^{\mathbf{A}^\#}\text{)} \\
&= \xi_{\mathbf{A}}(R_A((j_A\xi_{\mathbf{A}})(j_A(a)), (j_A\xi_{\mathbf{A}})^{*}(R_A(j_A(b_0), j_A(\vec{c}))()), \ldots, \\
&\quad (j_A\xi_{\mathbf{A}})^{*}(R_A(j_A(b_{m-1}), j_A(\vec{c}))()), \\
&\quad (j_A\xi_{\mathbf{A}})^{*}(c_m()()), \ldots, (j_A\xi_{\mathbf{A}})^{*}(c_{n-1}()()))) \text{ (since } \xi_{\mathbf{A}} j_A = i_A \text{ and by Lemma 6.12)} \\
&= \xi_{\mathbf{A}}((j_A\xi_{\mathbf{A}})^{*}(j_A(a)(R_A(j_A(b_0), j_A(\vec{c}))(), \ldots, R_A(j_A(b_{m-1}), j_A(\vec{c}))(), \\
&\quad c_m()(), \ldots, c_{n-1}()()))) \text{ (by definition of } (j_A\xi_{\mathbf{A}})^{*}\text{)} \\
&= \xi_{\mathbf{A}}(i^{*}_{\mathrm{Tm}_A(V)}(j_A(a)(R_A(j_A(b_0), j_A(\vec{c}))(), \ldots, R_A(j_A(b_{m-1}), j_A(\vec{c}))(), \\
&\quad c_m()(), \ldots, c_{n-1}()()))) \text{ (since } \xi_{\mathbf{A}}(j_A\xi_{\mathbf{A}})^{*} = \xi_{\mathbf{A}} i^{*}_{\mathrm{Tm}_A(V)}\text{)} \\
&= \xi_{\mathbf{A}}(R_A(j_A(a), R_A(j_A(b_0), j_A(\vec{c})), \ldots, R_A(j_A(b_{m-1}), j_A(\vec{c})), c_m(), \ldots, c_{n-1}())) \\
&\quad \text{(by definition of } i^{*}_{\mathrm{Tm}_A(V)}\text{)} \\
&= \xi_{\mathbf{A}}(R_A(R_A(j_A(a), j_A(\vec{b})), j_A(\vec{c}))) \\
&\quad \text{(by Lemma 6.4)} \\
&= \ldots \text{ (by reversing the steps in the deduction above)} \\
&= \mathrm{C}_n^{\mathbf{A}^\#}(\mathrm{C}_m^{\mathbf{A}^\#}(a, \vec{b}), \vec{c}),
\end{aligned}$$

as required. ∎

Next, let $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle, \mathbf{B} = \langle B, \xi_{\mathbf{B}} \rangle \in |\mathbf{SET^T}|$ and $h \in \mathbf{SET^T}(\mathbf{A}, \mathbf{B})$, i.e., $h \in \mathbf{SET}(A, B)$ and the following diagram commutes

$$\begin{tikzcd}
\mathrm{Tm}_A(V) \arrow[r, "(j_B h)^*"] \arrow[d, "\xi_{\mathbf{A}}"'] & \mathrm{Tm}_B(V) \arrow[d, "\xi_{\mathbf{B}}"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

We show that $h \in \mathbf{ACA}(\mathbf{A}^{\#}, \mathbf{B}^{\#})$. To this end, we need to verify the following two equations

- $h(\mathrm{v}_i^{\mathbf{A}^{\#}}) = \mathrm{v}_i^{\mathbf{B}^{\#}}$, for every $i \in \omega$, and
- $h(\mathrm{C}_n^{\mathbf{A}^{\#}}(a, a_0, \ldots, a_{n-1})) = \mathrm{C}_n^{\mathbf{B}^{\#}}(h(a), h(a_0), \ldots, h(a_{n-1}))$, for every $n \in \omega, a, a_0, \ldots, a_{n-1} \in A$.

We have

$$\begin{aligned}
h(\mathrm{v}_i^{\mathbf{A}^{\#}}) &= h(\xi_{\mathbf{A}}(v_i)) \text{ (by definition of } \mathrm{v}_i^{\mathbf{A}^{\#}}) \\
&= \xi_{\mathbf{B}}((j_B h)^*(v_i)) \text{ (by commutativity of rectangle)} \\
&= \xi_{\mathbf{B}}(v_i) \text{ (by definition of } (j_B h)^*) \\
&= \mathrm{v}_i^{\mathbf{B}^{\#}}, \text{ (by definition of } \mathrm{v}_i^{\mathbf{B}^{\#}})
\end{aligned}$$

as required and

$$\begin{aligned}
h(\mathrm{C}_n^{\mathbf{A}^{\#}}(a, \vec{a})) &= h(\xi_{\mathbf{A}}(R_A(j_A(a), j_A(\vec{a})))) \text{ (by definition of } \mathrm{C}_n^{\mathbf{A}^{\#}}) \\
&= \xi_{\mathbf{B}}((j_B h)^*(R_A(j_A(a), j_A(\vec{a})))) \text{ (by commut. of rectangle)} \\
&= \xi_{\mathbf{B}}(R_B((j_B h)^*(j_A(a)), (j_B h)^*(j_A(\vec{a})))) \text{ (by Lemma 6.5)} \\
&= \xi_{\mathbf{B}}(R_B((j_B h)(a), (j_B h)(\vec{a}))) \\
&= \mathrm{C}_n^{\mathbf{B}^{\#}}(h(a), h(\vec{a})), \text{ (by definition of } \mathrm{C}_n^{\mathbf{B}^{\#}})
\end{aligned}$$

as required.

Therefore, we can define a functor $Q : \mathbf{SET^T} \to \vec{\mathcal{V}}$, by

$$Q(\mathbf{A}) = \mathbf{A}^{\#}, \quad \text{for every} \quad \mathbf{A} \in |\mathbf{SET^T}|,$$

and, given $h \in \mathbf{SET}^{\mathbf{T}}(\mathbf{A}, \mathbf{B}), Q(h) \in \mathbf{ACA}(\mathbf{A}^{\#}, \mathbf{B}^{\#})$, by

$$Q(h) = h.$$

We finally proceed to show that $QP = I_{\mathbf{ACA}}$ and $PQ = I_{\mathbf{SET}^{\mathbf{T}}}$. To this end, let $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle \in \mathcal{ACA}$. We have

$$\mathrm{v}_i^{\mathbf{A}^{\bullet\#}} = \xi_{\mathbf{A}^{\bullet}}(v_i) = \mathrm{v}_i^{\mathbf{A}}$$

and, for every $n \in \omega, a, a_0, \ldots, a_{n-1} \in A$,

$$\begin{aligned}
\mathrm{C}_n^{\mathbf{A}^{\bullet\#}}(a, \vec{a}) &= \xi_{\mathbf{A}^{\bullet}}(R_A(j_A(a), j_A(\vec{a}))) \text{ (by definition of } \mathrm{C}_n^{\mathbf{A}^{\bullet\#}}) \\
&= \xi_{\mathbf{A}^{\bullet}}(a(j_A(\vec{a}))) \text{ (by definition of } R_A) \\
&= \mathrm{C}_n^{\mathbf{A}}(a, \xi_{\mathbf{A}^{\bullet}}(j_A(\vec{a}))) \text{ (by definition of } \xi_{\mathbf{A}^{\bullet}}) \\
&= \mathrm{C}_n^{\mathbf{A}}(a, \vec{a}), \text{ (by } \xi_{\mathbf{A}^{\bullet}} j_A = i_A)
\end{aligned}$$

as required.

Finally, let $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle \in |\mathbf{SET}^{\mathbf{T}}|$. We have

$$\xi_{\mathbf{A}^{\#\bullet}}(v_i) = \mathrm{v}_i^{\mathbf{A}^{\#}} = \xi_{\mathbf{A}}(v_i)$$

and, for every $a \in A, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_A(V), t_{n-1} \neq v_{n-1}$,

$$\begin{aligned}
\xi_{\mathbf{A}^{\#\bullet}}(a(\vec{t})) &= \mathrm{C}_n^{\mathbf{A}^{\#}}(a, \xi_{\mathbf{A}^{\#\bullet}}(\vec{t})) \text{ (by definition of } \xi_{\mathbf{A}^{\#\bullet}}) \\
&= \xi_{\mathbf{A}}(R_A(j_A(a), j_A(\xi_{\mathbf{A}}(\vec{t})))) \text{ (by definition of } \mathrm{C}_n^{\mathbf{A}^{\#}} \text{ and the ind. hyp.)} \\
&= \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(j_A(a))), (j_A\xi_{\mathbf{A}})^{\bullet}(\vec{t}()))) \text{ (by } \xi_{\mathbf{A}} j_A = i_A \text{ and Lemma 6.12)} \\
&= \xi_{\mathbf{A}}((j_A\xi_{\mathbf{A}})^{\bullet}(a()(\vec{t}()))) \text{ (by definition of } R_A) \\
&= \xi_{\mathbf{A}}(i^{\bullet}_{\mathrm{Tm}_A(V)}(a()(\vec{t}()))) \text{ (since } \xi_{\mathbf{A}}(j_A\xi_{\mathbf{A}})^{\bullet} = \xi_{\mathbf{A}} i^{\bullet}_{\mathrm{Tm}_A(V)}) \\
&= \xi_{\mathbf{A}}(R_A(a(), \vec{t})) \text{ (by definition of } i^{\bullet}_{\mathrm{Tm}_A(V)}) \\
&= \xi_{\mathbf{A}}(a(\vec{t})). \text{ (by definition of } R_A)
\end{aligned}$$

Thus, the following theorem holds

THEOREM 6.14 $\mathbf{ACA} \cong \mathbf{SET}^{\mathbf{T}}$.

A Related Result

In this section we build on the universe $\mathrm{Tm}_X(V)$ a substitution algebra $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ in the sense of Feldman [20]. We use the universal mapping property of free algebras in a variety to show that $\mathbf{Tm}_X(V)$ is the free algebra in the variety of representable substitution algebras, i.e., the variety generated by the class of all locally finite polynomial substitution algberas of [20]. This result, first proved in [49], is of interest because it shows that an algebraic theory $\mathbf{T}'$ in **SET** can be built, whose Eilenberg-Moore category of algebras is isomorphic with the category corresponding to the variety of Feldman's representable substitution algebras, in such a way that the functor $T' : \mathbf{SET} \to \mathbf{SET}$ is defined on objects by $T'(X) = \mathrm{Tm}_X(V)$.

First, we have to recall some definitions from [20].

A **substitution algebra A** is an algebra of type $\mathcal{L} = \{\Lambda, \rho\}$, where $\Lambda = \{\mathbf{S}^n, v_n : n \in \omega\}$ and $\rho : L \to \omega$ a rank function on Λ, defined by $\rho(\mathbf{S}^n) = 2$ and $\rho(v_n) = 0$, that satisfies the following axioms, for every $n, m, l < \omega, s, t, u \in A$,

(SA1) $\mathbf{S}^n_{v_n}(t) \approx t$

(SA2) $\mathbf{S}^n_t(v_n) \approx t$

(SA3) $\mathbf{S}^n_t(v_m) \approx v_m$ if $n \neq m$

(SA4) $\mathbf{S}^n_s(\mathbf{S}^n_u(t)) \approx \mathbf{S}^n_{\mathbf{S}^n_s(u)}(t)$

(SA5) $\mathbf{S}^n_{s(m/l)}(\mathbf{S}^m_u(t)) \approx \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}(\mathbf{S}^n_{s(m/l)}(t))$, where $s(m/l) = \mathbf{S}^m_{v_l}(s)$ and m, n and l are all distinct.

If **A** is a substitution algebra and $a \in A$, by the **dimension set** $D(a)$ of the element $a \in A$ we mean the set

$$D(a) = \{n \in \omega : \mathbf{S}^n_t(a) \neq a, \text{ for some } t \in A\}.$$

A substitution algebra **A** is called **locally finite** if, for every $a \in A, |D(a)| < \omega$. The variety generated by the class of all locally finite substitution algebras will be called the variety of **representable substitution algebras.** By a representation result of Feldman [20], this is the variety generated by a class of polynomial substitution algebras.

Let $X \in |\mathbf{SET}|$ and define on $\mathrm{Tm}_X(V)$ nullary operations $v_k : \mathrm{Tm}_X(V)^0 \to \mathrm{Tm}_X(V)$ with $v_k^{\mathbf{Tm}_X(V)} = v_k$, for every $k \in \omega$, and binary operations $\mathbf{S}^k : \mathrm{Tm}_X(V)^2 \to \mathrm{Tm}_X(V)$ by

$$\mathbf{S}^k_s(t) = R_X(t, \langle v_0, \dots, v_{k-1}, s\rangle), \quad \text{for every} \quad s, t \in \mathrm{Tm}_X(V), k \in \omega.$$

Our goal is to show that $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ is a free representable substitution algebra over X.

LEMMA 6.15 $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ *is a substitution algebra.*

Proof:

We verify that $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ satisfies the identities (SA1)-(SA5). For (SA1) we apply induction on the structure of $t \in \mathrm{Tm}_X(V)$. If $t = v_i \in V$, then

$$\mathbf{S}^n_{v_n}(v_i) = R_X(v_i, \langle v_0, \dots, v_{n-1}, v_n\rangle) = v_i.$$

If $x \in X, m \in \omega, t_0, \dots, t_{m-1} \in \mathrm{Tm}_X(V)$, then

$$\begin{aligned}
\mathbf{S}^n_{v_n}(x(t_0, \dots, t_{m-1})) &= R_X(x(t_0, \dots, t_{m-1}), \langle v_0, \dots, v_{n-1}, v_n\rangle)\\
&= x(R_X(t_0, \langle v_0, \dots, v_{n-1}, v_n\rangle), \dots,\\
&\quad R_X(t_{m-1}, \langle v_0, \dots, v_{n-1}, v_n\rangle))\\
&= x(t_0, \dots, t_{m-1}), \quad \text{as required.}
\end{aligned}$$

Next, for (SA2), $\mathbf{S}^n_t(v_n) = R_X(v_n, \langle v_0, \dots, v_{n-1}, t\rangle) = t$.

Now, for (SA3), $\mathbf{S}^n_t(v_m) = R_X(v_m, \langle v_0, \dots, v_{n-1}, t\rangle) = v_m$.

Next, for (SA4), we need to apply induction on the structure of t. First, if $t = v_n$, then

$$\begin{aligned}
\mathbf{S}^n_s\mathbf{S}^n_u(v_n) &= R_X(R_X(v_n, \langle v_0, \dots, v_{n-1}, u\rangle), \langle v_0, \dots, v_{n-1}, s\rangle)\\
&= R_X(u, \langle v_0, \dots, v_{n-1}, s\rangle)
\end{aligned}$$

$$
\begin{aligned}
&= R_X(v_n, \langle v_0, \ldots, v_{n-1}, R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)\rangle) \\
&= \mathbf{S}^n_{\mathbf{S}^n_s(u)}(v_n), \quad \text{as required.}
\end{aligned}
$$

If $t = v_m \neq v_n$, then

$$
\begin{aligned}
\mathbf{S}^n_s\mathbf{S}^n_u(v_m) &= R_X(R_X(v_m, \langle v_0, \ldots, v_{n-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s\rangle) \\
&= R_X(v_m, \langle v_0, \ldots, v_{n-1}, s\rangle) \\
&= v_m \\
&= R_X(v_m, \langle v_0, \ldots, v_{n-1}, R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)\rangle) \\
&= \mathbf{S}^n_{\mathbf{S}^n_s(u)}(v_m),
\end{aligned}
$$

as required. If $x \in X, k \in \omega, t_0, \ldots, t_{k-1} \in \mathrm{Tm}_X(V), t_{k-1} \neq v_{k-1}$, then

$$
\begin{aligned}
\mathbf{S}^n_s\mathbf{S}^n_u(x(t_0, \ldots, t_{k-1})) &= R_X(R_X(x(t_0, \ldots, t_{k-1}), \langle v_0, \ldots, v_{n-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s\rangle) \\
&= R_X(x(R_X(t_0, \langle v_0, \ldots, v_{n-1}, u\rangle), \ldots, R_X(t_{k-1}, \langle v_0, \ldots, v_{n-1}, \\
&\quad u\rangle), v_k, \ldots, v_{n-1}, u), \langle v_0, \ldots, v_{n-1}, s\rangle) \\
&= x(R_X(R_X(t_0, \langle v_0, \ldots, v_{n-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s\rangle), \ldots, \\
&\quad R_X(R_X(t_{k-1}, \langle v_0, \ldots, v_{n-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s\rangle), \\
&\quad R_X(v_k, \langle v_0, \ldots, v_{n-1}, s\rangle), \ldots, R_X(v_{n-1}, \langle v_0, \ldots, v_{n-1}, s\rangle), \\
&\quad R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)) \\
&= x(R_X(t_0, \langle v_0, \ldots, v_{n-1}, R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)\rangle), \ldots, \\
&\quad R_X(t_{k-1}, \langle v_0, \ldots, v_{n-1}, R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)\rangle), \\
&\quad v_k, \ldots, v_{n-1}, R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)) \\
&= R_X(x(t_0, \ldots, t_{k-1}), \langle v_0, \ldots, v_{n-1}, R_X(u, \langle v_0, \ldots, v_{n-1}, s\rangle)\rangle) \\
&= \mathbf{S}^n_{\mathbf{S}^n_s(u)}(x(t_0, \ldots, t_{k-1})).
\end{aligned}
$$

Finally, for (SA5), we need again to apply induction on the structure of t. If $t = v_n$, then

$$
\mathbf{S}^n_{s(m/l)}\mathbf{S}^m_u(v_n) = \mathbf{S}^n_{s(m/l)}(v_n) = s(m/l) = \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}(s(m/l)) = \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}(\mathbf{S}^n_{s(m/l)}(v_n)),
$$

as required. If $t = v_m$, then

$$
\mathbf{S}^n_{s(m/l)}\mathbf{S}^m_u(v_m) = \mathbf{S}^n_{s(m/l)}(u) = \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}(v_m) = \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}\mathbf{S}^n_{s(m/l)}(v_m),
$$

as required. If $t = v_k \neq v_n, v_m$, then

$$\mathbf{S}^n_{s(m/l)}\mathbf{S}^m_u(v_k) = v_k = \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}\mathbf{S}^n_{s(m/l)}(v_k).$$

Finally, if $x \in X, k \in \omega, t_0, \ldots, t_{k-1} \in \mathrm{Tm}_X(V), t_{k-1} \neq v_{k-1}$, then

$$\mathbf{S}^n_{s(m/l)}\mathbf{S}^m_u(x(t_0, \ldots, t_{n-1})) =$$

$$\begin{aligned}
&= \mathbf{S}^n_{s(m/l)}(R_X(x(t_0, \ldots, t_{k-1}), \langle v_0, \ldots, v_{m-1}, u\rangle))\\
&= R_X(R_X(x(t_0, \ldots, t_{k-1}), \langle v_0, \ldots, v_{m-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle)\\
&= R_X(x(R_X(t_0, \langle v_0, \ldots, v_{m-1}, u\rangle), \ldots, R_X(t_{k-1}, \langle v_0, \ldots, v_{m-1}, u\rangle),\\
&\quad v_k, \ldots, v_{m-1}, u), \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle)\\
&= x(R_X(R_X(t_0, \langle v_0, \ldots, v_{m-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), \ldots,\\
&\quad R_X(R_X(t_{k-1}, \langle v_0, \ldots, v_{m-1}, u\rangle), \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle),\\
&\quad R_X(v_k, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), \ldots, R_X(v_{m-1}, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle),\\
&\quad R_X(u, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), v_{m+1}, \ldots, v_{n-1}, s(m/l))\\
&= x(R_X(R_X(t_0, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), \langle v_0, \ldots, v_{m-1}, \mathbf{S}^n_{s(m/l)}(u)\rangle), \ldots\\
&\quad R_X(R_X(t_{k-1}, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), \langle v_0, \ldots, v_{m-1}, \mathbf{S}^n_{s(m/l)}(u)\rangle), v_k, \ldots, v_{m-1},\\
&\quad \mathbf{S}^n_{s(m/l)}(u), v_{m+1}, \ldots, v_{n-1}, s(m/l))\\
&= R_X(x(R_X(t_0, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), \ldots, R_X(t_{k-1}, \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle),\\
&\quad v_k, \ldots, v_{n-1}, s(m/l)), \langle v_0, \ldots, v_{m-1}, \mathbf{S}^n_{s(m/l)}(u)\rangle)\\
&= R_X(R_X(x(t_0, \ldots, t_{k-1}), \langle v_0, \ldots, v_{n-1}, s(m/l)\rangle), \langle v_0, \ldots, v_{m-1}, \mathbf{S}^n_{s(m/l)}(u)\rangle)\\
&= \mathbf{S}^m_{\mathbf{S}^n_{s(m/l)}(u)}\mathbf{S}^n_{s(m/l)}(x(t_0, \ldots, t_{k-1})),
\end{aligned}$$

as required. ∎

LEMMA 6.16 $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ *possesses the universal mapping property over X with respect to the class of all locally finite substitution algebras.*

Proof:

Let $\mathbf{A} = \langle A, \mathbf{S}^k, v_k\rangle_{k\in\omega}$ be a locally finite substitution algebra and $f : X \to A$ a set map. We define a substitution algebra homomorphism $f^\# : \mathbf{Tm}_X(V) \to \mathbf{A}$ that agrees with f on X, by recursion on the structure of X-terms as follows:

$$f^{\#}(v_i) = v_i^{\mathbf{A}}, \quad \text{for every} \quad v_i \in V,$$

and, if $x \in X, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, then, if

$$k_0, \ldots, k_{n-1} \notin D(f(x)) \cup \bigcup_{i=0}^{n-1} D(f^{\#}(t_i)),$$

$$f^{\#}(x(t_0, \ldots, t_{n-1})) = \mathbf{S}^{k_{n-1}}_{f^{\#}(t_{n-1})} \cdots \mathbf{S}^{k_0}_{f^{\#}(t_0)} \mathbf{S}^{n-1}_{v^{\mathbf{A}}_{k_{n-1}}} \cdots \mathbf{S}^{0}_{v^{\mathbf{A}}_{k_0}}(f(x)).$$

This element of A is independent of the choice of $k_0, \ldots, k_{n-1}$ and hence well-defined.

We first show that, if $s, t \in \mathrm{Tm}_X(V), m \in \omega$, then

$$f^{\#}(R_X(t, \langle v_0, \ldots, v_{m-1}, s \rangle)) = \mathbf{S}^m_{f^{\#}(s)}(f^{\#}(t)).$$

We do this by induction on the structure of t. If $t = v_m$, then

$$f^{\#}(R_X(v_m, \langle v_0, \ldots, v_{m-1}, s \rangle)) = f^{\#}(s) = \mathbf{S}^m_{f^{\#}(s)}(v^{\mathbf{A}}_m) = \mathbf{S}^m_{f^{\#}(s)}(f^{\#}(v_m)).$$

If $t = v_n \neq v_m$, then

$$f^{\#}(R_X(v_n, \langle v_0, \ldots, v_{m-1}, s \rangle)) = f^{\#}(v_n) = v^{\mathbf{A}}_n = \mathbf{S}^m_{f^{\#}(s)}(v^{\mathbf{A}}_n) = \mathbf{S}^m_{f^{\#}(s)}(f^{\#}(v_n)).$$

Finally, if $x \in X, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, then

$$f^{\#}(R_X(x(t_0, \ldots, t_{n-1}), \langle v_0, \ldots, v_{m-1}, s \rangle)) =$$

$$\begin{aligned}
&= f^{\#}(x(R_X(t_0, \langle v_0, \ldots, v_{m-1}, s \rangle), \ldots, R_X(t_{n-1}, \langle v_0, \ldots, v_{m-1}, s \rangle), v_n, \ldots, v_{m-1}, s)) \\
&= \mathbf{S}^{k_m}_{f^{\#}(s)} \mathbf{S}^{k_{m-1}}_{f^{\#}(v_{m-1})} \cdots \mathbf{S}^{k_n}_{f^{\#}(v_n)} \mathbf{S}^{k_{n-1}}_{f^{\#}(R_X(t_{n-1}, \langle v_0, \ldots, v_{m-1}, s \rangle))} \cdots \\
&\quad\ \mathbf{S}^{k_0}_{f^{\#}(R_X(t_0, \langle v_0, \ldots, v_{m-1}, s \rangle))} \mathbf{S}^m_{v^{\mathbf{A}}_{k_m}} \cdots \mathbf{S}^n_{v^{\mathbf{A}}_{k_n}} \mathbf{S}^{n-1}_{v^{\mathbf{A}}_{k_{n-1}}} \cdots \mathbf{S}^0_{v^{\mathbf{A}}_{k_0}}(f(x)) \\
&= \mathbf{S}^{k_m}_{f^{\#}(s)} \mathbf{S}^{k_{m-1}}_{v^{\mathbf{A}}_{m-1}} \cdots \mathbf{S}^{k_n}_{v^{\mathbf{A}}_n} \mathbf{S}^{k_{n-1}}_{\mathbf{S}^m_{f^{\#}(s)}(f^{\#}(t_{n-1}))} \cdots \mathbf{S}^{k_0}_{\mathbf{S}^m_{f^{\#}(s)}(f^{\#}(t_0))} \mathbf{S}^m_{f^{\#}(s)} \mathbf{S}^m_{v^{\mathbf{A}}_{k_m}} \cdots \mathbf{S}^0_{v^{\mathbf{A}}_{k_0}}(f(x)) \\
&= \mathbf{S}^{k_m}_{f^{\#}(s)} \mathbf{S}^{k_{m-1}}_{v^{\mathbf{A}}_{m-1}} \cdots \mathbf{S}^{k_n}_{v^{\mathbf{A}}_n} \mathbf{S}^{k_{n-1}}_{\mathbf{S}^m_{f^{\#}(s)}(f^{\#}(t_{n-1}))} \cdots \mathbf{S}^{k_1}_{\mathbf{S}^m_{f^{\#}(s)}(f^{\#}(t_1))} \mathbf{S}^m_{f^{\#}(s)} \mathbf{S}^{k_0}_{f^{\#}(t_0)} \mathbf{S}^m_{v^{\mathbf{A}}_{k_m}} \cdots \mathbf{S}^0_{v^{\mathbf{A}}_{k_0}}(f(x)) \\
&= \cdots
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{S}^{k_m}_{f^\#(s)} \mathbf{S}^{k_{m-1}}_{v^{\mathbf{A}}_{m-1}} \cdots \mathbf{S}^{k_n}_{v^{\mathbf{A}}_n} \mathbf{S}^{m}_{f^\#(s)} \mathbf{S}^{k_{n-1}}_{f^\#(t_{n-1})} \cdots \mathbf{S}^{k_0}_{f^\#(t_0)} \mathbf{S}^{m}_{v^{\mathbf{A}}_{k_m}} \cdots \mathbf{S}^{0}_{v^{\mathbf{A}}_{k_0}} (f(x)) \\
&= \mathbf{S}^{k_m}_{f^\#(s)} \mathbf{S}^{m}_{f^\#(s)} \mathbf{S}^{k_{n-1}}_{f^\#(t_{n-1})} \cdots \mathbf{S}^{k_0}_{f^\#(t_0)} \mathbf{S}^{m}_{v^{\mathbf{A}}_{k_m}} \mathbf{S}^{n-1}_{v^{\mathbf{A}}_{k_{n-1}}} \cdots \mathbf{S}^{0}_{v^{\mathbf{A}}_{k_0}} (f(x)) \\
&= \mathbf{S}^{m}_{f^\#(s)} \mathbf{S}^{k_{n-1}}_{f^\#(t_{n-1})} \cdots \mathbf{S}^{k_0}_{f^\#(t_0)} \mathbf{S}^{n-1}_{v^{\mathbf{A}}_{k_{n-1}}} \cdots \mathbf{S}^{0}_{v^{\mathbf{A}}_{k_0}} (f(x)) \\
&= \mathbf{S}^{m}_{f^\#(s)} (f^\#(x(t_0, \ldots, t_{n-1})))).
\end{aligned}
$$

as required.

Finally, we show that $f^\#$ is a substitution algebra homomorphism. If $n \in \omega$, then

$$f^\#(v_i^{\mathbf{Tm}_X(V)}) = f^\#(v_i) = v_i^{\mathbf{A}},$$

and, if $m \in \omega, s, t \in \mathrm{Tm}_X(V)$, then

$$f^\#(\mathbf{S}^m_s(t)) = f^\#(R_X(t, \langle v_0, \ldots, v_{m-1}, s\rangle)) = \mathbf{S}^m_{f^\#(s)}(f^\#(t)),$$

as required. ■

COROLLARY 6.17 $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ *possesses the universal mapping property over X with respect to the variety of representable substitution algebras.*

Proof:

Follows directly from lemma 6.16, since the variety of representable substitution algebras is the variety generated by the class of all locally finite substitution algebras.

■

LEMMA 6.18 $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ *is a representable substitution algebra, i.e., belongs to the variety generated by the class of locally finite substitution algebras.*

Proof:

We show that $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k\rangle_{k\in\omega}$ belongs to the variety generated by the class of locally finite substitution algebras by proving that $\mathbf{Tm}_X(V)$ is a subdirect product of locally finite substitution algebras.

Let $\rho : X \to \omega$ be a rank function on X. Construct a binary relation $\Theta_\rho \subseteq \mathrm{Tm}_X(V)^2$ by recursion on the structure of X-terms as follows: $(v_i, v_j) \in \Theta_\rho$ if and only if $i = j$

and given $x, y \in X, m, n \in \omega, t_0, \dots, t_{n-1}, s_0, \dots, s_{m-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}, s_{m-1} \neq v_{m-1}$,

$$(x(t_0, \dots, t_{n-1}), y(s_0, \dots, s_{m-1})) \in \Theta_\rho \quad \text{if and only if} \quad x = y \quad \text{and} \quad (t_i, s_i) \in \Theta_\rho,$$

for every $0 \leq i < \rho(x)$, where we set $t_j = v_j, s_k = v_k$, for every $j \geq n, k \geq m$. Θ_ρ is clearly an equivalence relation on $\mathrm{Tm}_X(V)$. We show that Θ_ρ is a congruence on $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k \rangle_{k \in \omega}$.

First, we show by induction on the structure of the X-term t that, if $(t, t'), (s, s') \in \Theta_\rho, m \in \omega$, then

$$(R_X(t, \langle v_0, \dots, v_{m-1}, s \rangle), R_X(t', \langle v_0, \dots, v_{m-1}, s' \rangle)) \in \Theta_\rho.$$

If $t = v_m$, then $t' = v_m$, whence

$$(R_X(t, \langle v_0, \dots, v_{m-1}, s \rangle), R_X(t', \langle v_0, \dots, v_{m-1}, s' \rangle)) = (s, s') \in \Theta_\rho.$$

If $t = v_n \neq v_m$, then $t' = v_n$, whence

$$(R_X(t, \langle v_0, \dots, v_{m-1}, s \rangle), R_X(t', \langle v_0, \dots, v_{m-1}, s' \rangle)) = (v_n, v_n) \in \Theta_\rho.$$

If $x \in X, n \in \omega, t_0, \dots, t_{n-1} \in Tm_X(V), t_{n-1} \neq v_{n-1}$, then, if $t = x(t_0, \dots, t_{n-1})$, we must have $t' = x(t'_0, \dots, t'_{k-1})$, with $(t_i, t'_i) \in \Theta_\rho, 0 \leq i < \rho(x), t_j = v_j, t'_l = v_l, j \geq n, l \geq k$. Thus,

$$(R_X(x(t_0, \dots, t_{n-1}), \langle v_0, \dots, v_{m-1}, s \rangle), R_X(x(t'_0, \dots, t'_{k-1}), \langle v_0, \dots, v_{m-1}, s' \rangle)) =$$

$$= (x(R_X(t_0, \langle v_0, \dots, v_{m-1}, s \rangle), \dots, R_X(t_{n-1}, \langle v_0, \dots, v_{m-1}, s \rangle), v_n, \dots, v_{m-1}, s),$$

$$x(R_X(t'_0, \langle v_0, \dots, v_{m-1}, s' \rangle), \dots, R_X(t'_{k-1}, \langle v_0, \dots, v_{m-1}, s' \rangle), v_k, \dots, v_{m-1}, s')) \in \Theta_\rho,$$

by the definition of Θ_ρ and the induction hypothesis.

Now, if $n \in \omega$ and $(s, s'), (t, t') \in \Theta_\rho$, then

$$\mathbf{S}^n_s(t) = R_X(t, \langle v_0, \dots, v_{n-1}, s \rangle) \equiv_{\Theta_\rho} R_X(t', \langle v_0, \dots, v_{n-1}, s' \rangle) = \mathbf{S}^n_{s'}(t'),$$

and hence $\Theta_\rho \in Co(\mathbf{Tm}_X(V))$, as required.

Next, consider the quotient $\mathbf{Tm}_X(V)/\Theta_\rho$. We show that it is a locally finite substitution algebra. To this end, we show that if $x \in X, n \in \omega$ and $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, then

$$D(x(t_0, \ldots, t_{n-1})/\Theta_\rho) \subseteq \{0, 1, \ldots, \rho(x) - 1\} \cup \bigcup_{i=0}^{n-1} D(t_i/\Theta_\rho).$$

Suppose that $m \notin \{0, 1, \ldots, \rho(x) - 1\} \cup \bigcup_{i=0}^{n-1} D(t_i/\Theta_\rho)$. Then, for every

$$s \in \mathrm{Tm}_X(V), k_0, \ldots, k_{n-1} \notin \bigcup_{i=0}^{n-1} D(t_i/\Theta_\rho),$$

$$\mathbf{S}^m_{s/\Theta_\rho}(x(t_0, \ldots, t_{n-1})/\Theta_\rho) = \mathbf{S}^m_s(x(t_0, \ldots, t_{n-1}))/\Theta_\rho =$$

$$= R_X(x(t_0, \ldots, t_{n-1}), \langle v_0, \ldots, v_{m-1}, s\rangle)/\Theta_\rho =$$

$$= x(R_X(t_0, \langle v_0, \ldots, v_{m-1}, s\rangle), \ldots, R_X(t_{n-1}, \langle v_0, \ldots, v_{m-1}, s\rangle), v_n, \ldots, v_{m-1}, s)/\Theta_\rho =$$

$$= x(t_0, \ldots, t_{n-1})/\Theta_\rho,$$

since, for every $0 \leq i < n$,

$$R_X(t_i, \langle v_0, \ldots, v_{m-1}, s\rangle)/\Theta_\rho = \mathbf{S}^m_s(t_i)/\Theta_\rho = \mathbf{S}^m_{s/\Theta_\rho}(t_i/\Theta_\rho) = t_i/\Theta_\rho$$

and $m \notin \{0, 1, \ldots, \rho(x) - 1\}$. Hence $m \notin D(x(t_0, \ldots, t_{n-1})/\Theta_\rho)$, as was to be shown.

Finally, we show that if $R = \omega^X$ is the collection of all rank functions $\rho : X \to \omega$, then $\bigcap_{\rho \in R} \Theta_\rho = \Delta_{\mathrm{Tm}_X(V)}$. To this end, suppose that $(s, t) \in \mathrm{Tm}_X(V)^2$, with $s \neq t$. If $t = v_i \in V$, then $(s, t) \notin \Theta_\rho$, for every $\rho \in R$. So suppose that $s = y(s_0, \ldots, s_{m-1})$ and $t = x(t_0, \ldots, t_{n-1})$, for some $m, n \in \omega$, $s_0, \ldots, s_{m-1}$, $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), s_{m-1} \neq v_{m-1}, t_{n-1} \neq v_{n-1}$. If $x \neq y$, then $(s, t) \notin \Theta_\rho$, for every $\rho \in R$. If $x = y$ then, for some $i < max\{m, n\}, t_i \neq s_i$. Thus by the induction hypothesis, there exists $\rho \in R$, such that $(s_i, t_i) \notin \Theta_\rho$. Let $\rho' \in R$ be defined by

$$\rho'(z) = \begin{cases} max\{\rho(x), i+1\} & \text{if } z = x \\ \rho(z) & \text{otherwise} \end{cases}$$

We show that, since $\rho'(z) \geq \rho(z)$, for every $z \in X$, $\Theta_{\rho'} \subseteq \Theta_\rho$. This being the case $(s,t) \notin \Theta_{\rho'}$ and hence $(s,t) \notin \bigcap_{\rho\in R} \Theta_\rho$, as was to be shown. In fact, if

$$(x(t_0,\ldots,t_{n-1}), y(s_0,\ldots,s_{m-1})) \notin \Theta_\rho$$

then, either $x \neq y$, in which case $(x(t_0,\ldots,t_{n-1}), y(s_0,\ldots,s_{m-1})) \notin \Theta_{\rho'}$, or $x = y$ and $(t_i, s_i) \notin \Theta_\rho$, for some $i \leq \rho(x) \leq \rho'(x)$. In this case, applying the induction hypothesis, we get that $(t_i, s_i) \notin \Theta_{\rho'}$, whence $(x(t_0,\ldots,t_{n-1}), y(s_0,\ldots,s_{m-1})) \notin \Theta_{\rho'}$, by the definition of $\Theta_{\rho'}$, as required.

This concludes the proof that $\mathbf{Tm}_X(V)$ is subdirectly representable by means of the product $\prod_{\rho\in R} \mathbf{Tm}_X(V)/\Theta_\rho$, with all $\mathbf{Tm}_X(V)/\Theta_\rho$ locally finite, and therefore is representable. ∎

THEOREM 6.19 $\mathbf{Tm}_X(V) = \langle \mathrm{Tm}_X(V), \mathbf{S}^k, v_k \rangle_{k\in\omega}$ *is the free representable substitution algebra over* X.

Proof:

This is a direct consequence of Corollary 6.17 and Lemma 6.18. ∎

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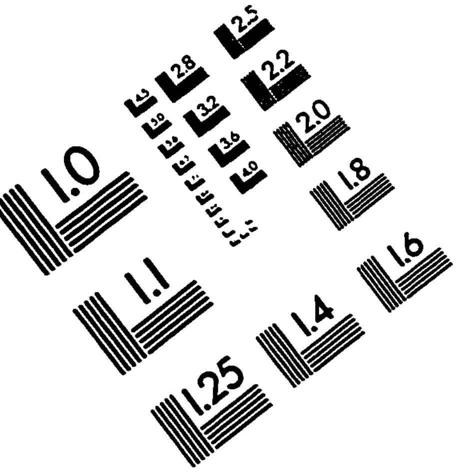
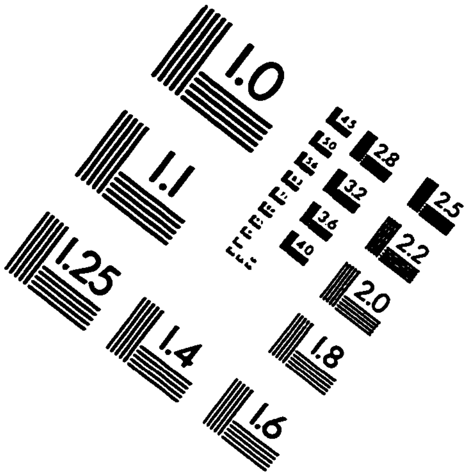
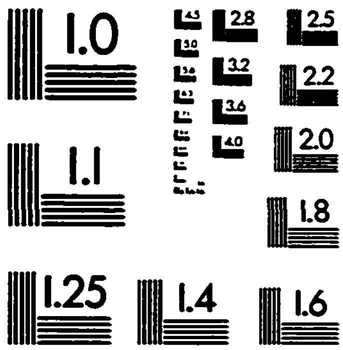
IMAGE EVALUATION TEST TARGET (QA-3)







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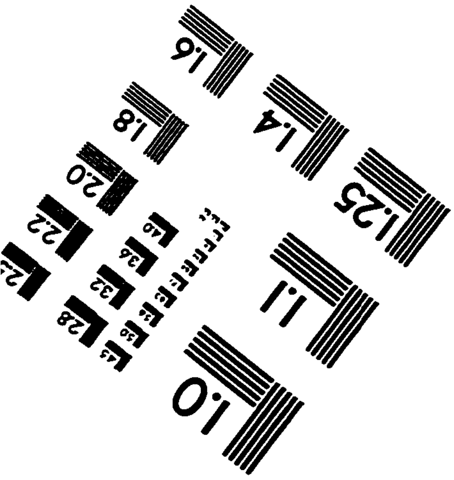

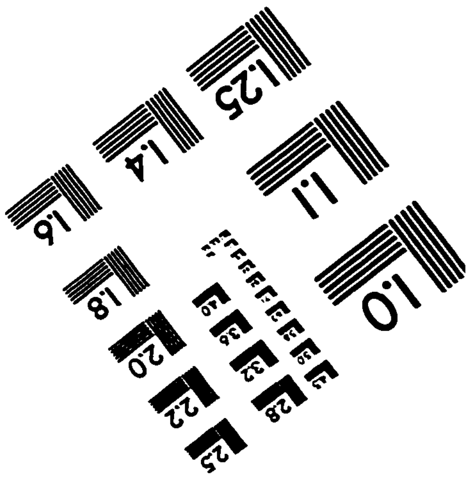

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