Advances in Computer Science and Engineering
Volume 5, Number 2, 2010, Pages 131-172
This paper is available online at http://pphmj.com/journals/acse.htm

PROBABILISTIC FEDERATED $\mathcal{ALCI}$

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Abstract

In previous work, we have introduced a fully contextualized federated ontology language F-$\mathcal{ALCI}$, based on the well-known description logic $\mathcal{ALCI}$. Inspired by the work of Lukasiewicz on expressive probabilistic logics, we augment that work by considering a probabilistic extension of F-$\mathcal{ALCI}$, termed PF-$\mathcal{ALCI}$. Although its modules employ a less expressive description logic than the rich $\mathcal{SHIF}(\mathbf{D})$ or $\mathcal{SHOIN}(\mathbf{D})$ of Lukasiewicz and, in particular, do not provide support for concrete domains, PF-$\mathcal{ALCI}$ is the first ontology language in the literature to offer modularity and contextualization of all logical connectives combined with the ability to express probabilistic terminological and default knowledge.

1. Introduction

The large amount of data and services that have become available on the world wide web have led to the *semantic web initiative* [4, 15], which aims at making information machine-interpretable and services machine-operable so that data

Keywords and phrases: description logics, modular description logics, modular ontologies, context, probabilistic description logics, probabilistic knowledge, default knowledge.

Communicated by Tarek M. Sobh

Received September 20, 2009

discovery, integration and navigation can be enhanced. The precision in the definition and the meaning of the terms representing the available information, required for succeeding in this goal, is provided by organizing them into ontologies. Ontologies are knowledge bases that typically cover a specific domain of expertise. Different ontologies may cover related domains with partially overlapping, or interdependent, information, but are typically developed independently of each other. One of the most widely used languages for ontology construction is OWL [39]. Ontology languages are based on description logics [1], which, typically, are decidable fragments of first-order logic or various other decidable extensions with additional constructs that are used to enrich expressivity without compromising decidability [7]. The basic building blocks of a description logic are concepts and roles. Concepts represent classes of individuals in the domain of discourse and roles represent relationships between individuals. The most elementary statements that are encoded in a description logic knowledge base and on which we will focus in this paper are subsumption relationships between concepts.

In various applications of description logics in the semantic web, the need arises to express uncertain or imprecise information and to reason about it. In this direction a body of work has focused on integrating fuzzy representation and reasoning into ontology languages (see, e.g., [40-42]). On the other hand, an alternative approach is to use probabilistic methods to represent and reason about uncertain information on the web. This has been explored extensively in the area of logic programs (see, e.g., [32]) and various researchers have advocated and treated the introduction of probabilistic features in knowledge representation [26-29, 34] and ontology engineering [13, 14, 48, 37, 16, 35, 10, 11].

Typically, development of ontologies in the semantic web is occurring autonomously by independent contributors, each of whom addresses a different area of expertise. But the ontology modules that are constructed in this federated fashion are not entirely disjoint. They may cover related or partially overlapping domains, e.g., biology, medicine, pharmacology. In order to avoid reconstructing the same terminology and repeating parts of an already existing ontology, tools have been developed that allow an ontology developer to reuse concepts and definitions from other ontology modules. The theoretical study on the foundations of ontology languages that allow this feature has led to the development of several possible platforms that may be used for selectively reusing parts of other ontology modules in the development of a new ontology. These modular ontology languages include distributive description logics [5, 19], $\mathcal{E}$-connections [24], semantic importing [36], semantic binding [49] and package-based description logics [2, 3]. A slightly different approach that also has as its main goal partial reuse of available knowledge is based on the notion of conservative extensions [20, 23, 22]. Of particular interest to us, since it will form the foundation for our studies in this paper, is the framework of federated, fully-contextualized description logics that was introduced recently in [47, 44]. Apart from enabling the user to partially reuse information by importing concepts and roles from different modules, it also recognizes the need to contextualize information. This need arises because imported terms from other modules may be interpreted differently depending on the context in which they are being reused. Context as a key concept in reasoning in AI has been studied before in [8, 9] and, more specifically, in the area of ontology languages in [6, 17, 18]. The additional recognition of the need to reason with imprecise or fuzzy information in this federated setting has recently led to the formulation of a federated reasoning framework [46], where instead of a two-valued semantics, an arbitrary certainty lattice may be used, as was done previously in the single-module setting in [42].

In the present work, we introduce probabilistic terminological axioms in the federated fully-contextualized description logic F-$\mathcal{ALCI}$ to obtain the probabilistic federated description logic PF-$\mathcal{ALCI}$. We follow in this endeavor the leads from the pioneering work of Lukasiewicz [34], where probabilistic analogs of the very expressive description logics $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$ were introduced and studied in detail. Because this is, to the best of our knowledge, the first attempt at the creation of a relatively expressive *modular* description logic with probabilistic features, we opted for a rather simplified version of the description logic used, as compared with the powerful logics used in [34]. More expressive DLs and a more general framework will be studied in future work. Our framework has the following three limitations when comparing the underlying language used with those of Lukasiewicsz: First, $\mathcal{ALCI}$ is significantly less expressive than either $\mathcal{SHIF}(\mathbf{D})$ or $\mathcal{SHOIN}(\mathbf{D})$. Second, we do not treat concrete domains as does Lukasiewicz. Finally, we restrict our attention only to terminological axioms. Despite these simplifications our innovation relies on several features that are introduced *collectively* for the first time in an ontology language. First, our language is *modular*. That is, its semantics handles readily interactions between various modules that are developed independently on the web. Second, in each of these modules, all logical connectives are *contextualized*. Each logical connective has a local meaning that is transferred across modules via image domain relations. Finally, several of the nice *probabilistic features* of Lukasiewicz's approach pertaining to default and probabilistic terminological axioms still hold in the distributed context, despite the limited expressivity of the underlying description logic.

2. A Quick Review of $\mathcal{ALCI}$ and F-$\mathcal{ALCI}$

2.1. $\mathcal{ALCI}$ basics

Recall, e.g., from [1], that the description logic $\mathcal{ALCI}$ consists of role expressions and concept expressions that are built starting from two disjoint collections of concept names $\mathcal{C}$ and role names $\mathcal{R}$, using the top and bottom concepts, negation, conjunction, disjunction, value and existential restriction (for concepts) and inverse roles. More precisely, if A is a concept name and C, D are concept expressions, then

$$\top,\ \bot,\ A,\ \neg C,\ C \sqcap D,\ C \sqcup D,\ \forall R.C \text{ and } \exists R.C$$

are concept expressions, where R is a role expression, i.e., a role name R or of the form R^-, with R a role name. The set of role expressions is denoted by $\hat{\mathcal{R}}$ and the set of concept expressions by $\hat{\mathcal{C}}$. A subsumption in $\mathcal{ALCI}$ is a formula of the form $C \sqsubseteq D$, where $C,\ D \in \hat{\mathcal{C}}$. An ontology T (also known as knowledge base (KB) or TBox) is a finite set of subsumptions. This language is provided a formal semantics as follows: An interpretation for T is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a nonempty set, called the *domain of the interpretation*, and $\cdot^{\mathcal{I}}$ is a function, that assigns to each concept name C a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to each role name R a set $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, such that $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $\bot^{\mathcal{I}} = \varnothing$. One uses the recursive nature of the concept expressions to extend the function $\cdot^{\mathcal{I}}$ over all role and concept expressions as follows: Let $(R^-)^{\mathcal{I}} = (R^{\mathcal{I}})^-$, the inverse relation of $R^{\mathcal{I}}$, for every $R \in \mathcal{R}$, and, for all concept expressions C, D and role expressions R,

$-(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$;

$-(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \bigcap D^{\mathcal{I}}$;

$-(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \bigcup D^{\mathcal{I}}$;

$-(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : (\forall y \in \Delta^{\mathcal{I}})((x, y) \in R^{\mathcal{I}} \text{ implies } y \in C^{\mathcal{I}})\}$;

$-(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : (\exists y \in C^{\mathcal{I}})((x, y) \in R^{\mathcal{I}})\}$.

The interpretation $\mathcal{I}$ **satisfies** the subsumption $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a **model** of the KB *T* if it satisfies every subsumption in *T*. A KB *T* is said to be **consistent** or **satisfiable** if it has a model, whereas a concept expression $C \in \hat{\mathcal{C}}$ is said to be **satisfiable with respect to**, or **relative to**, *T* if *T* has a model $\mathcal{I}$, such that $C^{\mathcal{I}} \neq \emptyset$.

2.2. F-$\mathcal{ALCI}$ basics

In this section, we revisit the basic definitions concerning the syntax and semantics of the modular ontology language F-$\mathcal{ALCI}$ that was introduced in [47]. This language will constitute one of the basic underlying components of the probabilistic counterpart that will be presented in the following sections.

A directed acyclic graph $G = \langle V, E \rangle$ is given, whose vertices represent modules of a federated ontology and whose edges correspond to direct importing relations between the modules. In other words, if $(i, j) \in E$, then module *j* may import concept names, role names and logical connectives from module *i*. Note that F-$\mathcal{ALCI}$ is the first modular ontology language that supports contextualization of all logical connectives, rather than just logical negation, as was done in previous proposals [5, 25, 3]. The language of the *i*-th module in F-$\mathcal{ALCI}$ consists of a set of role expressions $\hat{\mathcal{R}}_i$ and concept expressions $\hat{\mathcal{C}}_i$, that are built starting from disjoint collections of concept names $\mathcal{C}_i$ and role names $\mathcal{R}_i$, for each module $i \in V$. The *i*-th role expressions are of the form R or R^-, where $R \in \mathcal{R}_j$, $(j, i) \in E$. The *i*-th concept expressions are built recursively by

$$\top_j, \bot_j, A, \neg_j C, C \sqcap_j D, C \sqcup_j D, \forall_j R.C \text{ and } \exists_j R.C,$$

where $A \in \mathcal{C}_j$, $C, D \in \hat{\mathcal{C}}_i \cap \hat{\mathcal{C}}_j$ and $R \in \hat{\mathcal{R}}_i \cap \hat{\mathcal{R}}_j$, for $(j, i) \in E$. An *i*-subsumption in F-$\mathcal{ALCI}$ is a formula of the form $C \sqsubseteq D$, where $C, D \in \hat{\mathcal{C}}_i$. An ontology $T = \{T_i\}_{i \in V}$ (also known as knowledge base (KB) or TBox) is a *V*-indexed collection of finite sets T_i of *i*-subsumptions. The language is provided a formal semantics as follows: An interpretation for T is a pair $\mathcal{I} = \langle \{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i, j) \in E} \rangle$, where $\mathcal{I}_i = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ is a local interpretation and $r_{ij} \subseteq \Delta^i \times \Delta^j$ is an image-domain relation. The local interpretations $\mathcal{I}_i = \langle \Delta^i, \cdot^i \rangle$ consist of a nonempty local domain Δ^i and an interpretation function $\cdot^i$, that assigns to each i-concept name C a set $C^i \subseteq \Delta^i$ and to each i-role name R a set $R^i \subseteq \Delta^i \times \Delta^i$, such that $\top_i^i = \Delta^i$ and $\bot_i^i = \emptyset$. One uses the recursive nature of the concept expressions to extend the functions $\cdot^i$ over all i-role and i-concept expressions as follows:

First, we introduce some notation. For a binary relation $r \subseteq \Delta^i \times \Delta^j$, $X \subseteq \Delta^i$ and $S \subseteq \Delta^i \times \Delta^i$, we set

$$r(X) := \{y \in \Delta^j : (\exists x \in X)((x, y) \in r)\},$$

$$r(S) := \{(z, w) \in \Delta^j \times \Delta^j : (\exists (x, y) \in S)((x, z), (y, w) \in r)\}$$

to denote the images of X and S under the binary relation r.

Let $(R^-)^i = (R^i)^-$, for every $R \in \mathcal{R}_j$, $(j, i) \in E$. Then, for every $A \in \mathcal{C}_j$ and all concept expressions $C, D \in \hat{\mathcal{C}}_i \cap \hat{\mathcal{C}}_j$ and role expressions $R \in \hat{\mathcal{R}}_i \cap \hat{\mathcal{R}}_j$, with $(j, i) \in E$,

– $\top_j^i = r_{ji}(\Delta^j)$ and $\bot_j^i = \emptyset$;

– $A^i = r_{ji}(A^j)$;

– $(\neg_j C)^i = r_{ji}(\Delta^j \setminus C^j)$;

– $(C \sqcap_j D)^i = r_{ji}(C^j \cap D^j)$;

– $(C \sqcup_j D)^i = r_{ji}(C^j \cup D^j)$;

– $(\forall_j R.C)^i = r_{ji}(\{x \in \Delta^j : (\forall y \in \Delta^j)((x, y) \in R^j \text{ implies } y \in C^j)\})$;

– $(\exists_j R.C)^i = r_{ji}(\{x \in \Delta^j : (\exists y \in C^j)((x, y) \in R^j)\})$.

The interpretation $\mathcal{I}$ **satisfies** the *i*-subsumption $C \sqsubseteq D$ **as witnessed by** module *i* iff $C^i \subseteq D^i$. An interpretation $\mathcal{I}$ **satisfies**, or is a **model of**, the KB $T = \{T_i\}_{i \in V}$ if it satisfies every *i*-subsumption in T_i as witnessed by *i*, for all $i \in V$. A KB *T* is said to be **consistent** or **satisfiable** if it has a model $\mathcal{I}$. On the other hand, an *i*-concept expression $C \in \hat{\mathcal{C}}_i$ is said to be **satisfiable as witnessed by** *i* **with respect to**, or **relative to**, T if T has a model $\mathcal{I}$, such that $C^i \neq \emptyset$. Finally, a collection $\mathcal{E} = \{\mathcal{E}_i\}_{i \in V}$, where $\mathcal{E}_i \subseteq \hat{\mathcal{C}}_i$, is **satisfiable relative to** (or **with respect to**) *T* if, there exists an interpretation $\mathcal{I}$ (which is a model of T), such that $\bigcap \{C^i : C \in \mathcal{E}_i\} \neq \emptyset$, for all $i \in V$. When *T* is empty, we say that $\mathcal{E}$ is **satisfiable** omitting the reference relative to the empty TBox. Note that, because we are assuming that all local domains of every model are nonempty, satisfiability of $\{\mathcal{E}_i\}_{i \in V}$ relative to *T* is equivalent to the satisfiability of $\{\mathcal{E}_i \cup \top_i\}_{i \in V}$, relative to T.

3. The Probabilistic Extension of F-$\mathcal{ALCI}$

To introduce the syntax of PF-$\mathcal{ALCI}$, we define first the concept of conditional constraint. It was given in [31] and forms a cornerstone in the definitions of both P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$ in [34]. To define the semantics of the new probabilistic language involving conditional constraints, the notion of lexicographic entailment, introduced by Lehmann in [30] in the context of default reasoning from conditional knowledge bases, will be employed. This type of reasoning has also been employed in the context of probabilistic default reasoning in [33, 32] and in the definition of the semantics of both P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$ in [34].

3.1. Syntax

Roughly speaking, a PF-$\mathcal{ALCI}$ knowledge base or PF-$\mathcal{ALCI}$ ontology is a collection of PTboxes $\{PT_i\}_{i \in V}$, each of which is an ordinary module of an F-$\mathcal{ALCI}$ knowledge base along with axioms for terminological probabilistic knowledge and default knowledge. An example of an ordinary F-$\mathcal{ALCI}$-ontology follows.

Example 1. Consider an ontology *T* with two modules. Module T_1 consists of information regarding the insurance status of the personnel at a given university. It has a concept corresponding to insured personnel and also one corresponding to fully insured and one corresponding to partially insured personnel:

$$\texttt{FullyInsured} \sqsubseteq \texttt{Insured}$$

$$\texttt{PartiallyInsured} \sqsubseteq \texttt{Insured}$$

$$\texttt{PartiallyInsured} \sqsubseteq \neg_1 \texttt{FullyInsured}$$

Module T_2, on the other hand, consists of information about the titles of the personnel, i.e., their role in the university structure. It contains concepts for lecturers, faculty, male and female lectures and imports the concept Insured from T_1 (equality between two concept expressions stands for subsumption in both directions):

$$\texttt{MaleLecturer} \sqsubseteq \texttt{Lecturer}$$

$$\texttt{FemaleLecturer = Lecturer} \sqcap_2 \neg_2 \texttt{ MaleLecturer}$$

$$\texttt{Lecturer} \sqsubseteq \texttt{Faculty}$$

$$\texttt{Faculty} \sqsubseteq \texttt{Insured}$$

Let $\mathcal{B}_i$ be a finite nonempty set of **basic classification *i*-concepts**, which are *i*-concept expressions *C* in F-$\mathcal{ALCI}$, i.e., $\mathcal{B}_i \subseteq \hat{\mathcal{C}}_i$. These are the concepts that will be used in conditional constraints to define terminological probabilistic relationships. They will also be used in the semantics of PF-$\mathcal{ALCI}$ to obtain finite sets of worlds. A **classification *i*-concept** is defined by recursion starting from basic classification *i*-concepts as follows:

– Every basic classification *i*-concept $\phi \in \mathcal{B}_i$ is a classification *i*-concept.

– If ϕ, ψ are classification *i*-concepts, then $\neg_i\phi, \phi \sqcap_i \psi, \phi \sqcup_i \psi$, are also classification *i*-concepts.

The collection of all classification *i*-concepts is denoted by $\hat{\mathcal{B}}_i$.

An ***i*-conditional constraint** is an expression of the form $(\psi \mid \phi)[l, u]$, where ϕ, ψ are classification *i*-concepts and l, $u \in [0, 1]$ are reals in the unit interval. This constraint formally expresses the statement that the conditional probability of ψ given ϕ lies between l and u.

As Lukasiewicz observes in [34], the use of classification concepts rather than of only basic classification concepts adds flexibility, reduces the number of worlds that need to be considered in the semantics and brings the framework closer to probabilistic lexicographic entailment in probabilistic default reasoning [33, 32].

Example 2. Assume that all three 1-concept names that we have seen in Example 1, together with another one (that we have not used yet) `HasDental`, with the intended meaning that member employees have dental insurance, are basic classification 1-concepts. The terminological probabilistic knowledge "generally, insured personnel are fully insured with probability at least 0.8", i.e., "typically, a randomly chosen insured employee is fully insured with probability of at least 0.8" can be expressed by the conditional constraint

$$(\texttt{FullyInsured}|\texttt{Insured})[0.8, 1].$$

On the other hand, the terminological default knowledge "generally, insured personnel have dental insurance" can be expressed by

$$(\texttt{HasDental}|\texttt{Insured})[1, 1]$$

and the default knowledge "generally, partially insured personnel do not have dental insurance" by

$$(\neg_1 \texttt{HasDental}|\texttt{PartiallyInsured})[1, 1].$$

This is different from the strict terminological knowledge "all insured employees have dental insurance", which is expressed by the concept subsumption $\texttt{Insured} \sqsubseteq \texttt{HasDental}$. The difference lies in the way these two assertions are handled when used to draw conclusions. More details on this point will come later.

To illustrate our modular approach, we consider also some basic classification 2-concepts and some 2-conditional constraints. Alongside `Faculty` and `Lecturer`, the imported 1-concept name `FullyInsured` and another 2-concept name, that we have not met yet, `DoesResearch`, are basic-classification 2-concepts. Here we have the default knowledge

$$(\texttt{DoesResearch}|\texttt{Faculty})\ [1, 1]$$

$$(\neg_2 \texttt{DoesResearch}|\texttt{Lecturer})\ [1, 1]$$

$$(\texttt{FullyInsured}|\texttt{Faculty})\ [1, 1]$$

and the terminological probabilistic knowledge

$$(\texttt{FullyInsured}|\texttt{Lecturer})[0.7, 1].$$ □

A PF-$\mathcal{ALCI}$-**knowledge base** or PF-$\mathcal{ALCI}$-**ontology** $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$ is a collection of **PTBoxes** $\mathrm{PT}_i = \langle T_i, P_i \rangle$, where T_i is the *i*-**TBox** of an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$ and P_i is a finite set of *i*-conditional constraints. P_i encodes both probabilistic terminological knowledge and terminological default knowledge. In particular, a specific *i*-conditional constraint $(\psi \mid \phi)[l, u]$ has the intended meaning that "generally, if $\phi(a)$ holds, then $\psi(a)$ holds with probability at least *l* and at most *u*", for every randomly chosen individual (a) in the domain of discourse.

3.2. Semantics

In this section, the key concepts of consistency and lexicographic entailment for a PF-$\mathcal{ALCI}$ knowledge base will be introduced. The inspiration comes from the work of Lehmann [30] on lexicographic entailment in default reasoning from conditional knowledge bases. Lukasiewicz used this notion to define lexicographic entailment in probabilistic default reasoning in [33, 32] and, more recently, in [34] to obtain lexicographic entailment for his probabilistic description logics. We rely on his latest work to develop the semantics for our framework.

Our goal in reasoning with PF-$\mathcal{ALCI}$ is to define new terminological probabilistic knowledge from a given PF-$\mathcal{ALCI}$ knowledge base $\mathrm{PT} = \{\langle T_i, P_i \rangle\}_{i \in V}$. To perform this reasoning, contextual inconsistencies inside each PTBox $\mathrm{PT}_i = \langle T_i, P_i \rangle$ have to be resolved. For instance, if the PTBox PT_2 includes the probabilistic default statements

$$(\texttt{DoesResearch} \mid \texttt{Faculty})[1, 1]$$

$$(\neg_2 \texttt{DoesResearch} \mid \texttt{Lecturer})[1, 1], \qquad (1)$$

then an inconsistency is created, given the strict terminological knowledge axiom $\texttt{Lecturer} \sqsubseteq \texttt{Faculty}$. Following [34], we use the maximum specificity rule to resolve such inconsistencies. This rule stipulates that more specific information is preferred over less specific one. Since "lecturers do not generally conduct research" is more specific than "faculty do in general conduct research", the first probability statement in (1) will be ignored to resolve this inconsistency.

More generally, the specificity of each conditional constraint in each probabilistic box P_i is analyzed and this analysis leads to establishing a preference relation between all subsets of P_i, which extends to a preference relation between all probabilistic interpretations. This relation is the one that will be used to resolve inconsistencies and draw conclusions whenever possible, i.e., in all cases when the knowledge base is consistent.

World models and probabilistic models. Given a collection $\mathcal{E} \subseteq \hat{\mathcal{C}}_i$, denote by $\neg_i\mathcal{E}$ the set $\neg_i\mathcal{E} = \{\neg_i\phi : \phi \in \mathcal{E}\}$. Let $I = \{I_i\}_{i \in V}$ be a collection of sets of basic classification *i*-concepts, such that $\{I_i \cup \neg_i(\mathcal{B}_i \setminus I_i)\}_{i \in V}$ is satisfiable.[1] *I* is called a **world relative to** $\mathcal{B} = \{\mathcal{B}_i\}_{i \in V}$. The set of all worlds relative to $\mathcal{B}$ will be denoted by $\mathcal{I}_{\mathcal{B}}$. Since, for all $i \in V$, $|\mathcal{B}_i| < \omega$, we also have that $|\mathcal{I}_{\mathcal{B}}| < \omega$.

Example 3. Consider the knowledge base *K* that was discussed in the previous examples. We have that

$$\mathcal{B}_1 = \{\texttt{Insured}, \texttt{PartiallyInsured}, \texttt{FullyInsured}, \texttt{HasDental}\},$$

$$\mathcal{B}_2 = \{\texttt{Faculty}, \texttt{Lecturer}, \texttt{FullyInsured}, \texttt{DoesResearch}\}.$$

Clearly, every $I = \{I_1, I_2\}$, with $I_1 \subseteq \mathcal{B}_1$, $I_2 \subseteq \mathcal{B}_2$, yields a world relative to $\mathcal{B}$. Thus, in this example, there are $2^4 \cdot 2^4$ worlds relative to $\mathcal{B}$. □

Given a world $I = \{I_i\}_{i \in V}$ and an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$, *I* **satisfies** *T* or *I* is a **model** of *T*, written $I \vDash T$, if $\{I_i \cup \neg_i(\mathcal{B}_i \setminus I_i)\}_{i \in V}$ is satisfiable *relative* to *T*. *I* **satisfies** a basic classification *i*-concept $\phi \in \mathcal{B}_i$ or *I* is a **model** of ϕ, denoted by $I \vDash_i \phi$, if $\phi \in I_i$. Satisfaction of classification *i*-concepts by worlds is defined by extending the definition inductively over Boolean connectives in the usual way.

The following proposition is an analog of Proposition 4.8 of [34] and shows that an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$ is satisfiable iff it has a world model.

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[1]Recall that this means that there exists an interpretation $\mathcal{I} = \langle\{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i,j) \in E}\rangle$, such that $\bigcap\{C^i : C \in I_i \cup \neg_i(\mathcal{B}_i \setminus I_i)\} \neq \emptyset$, for all $i \in V$.

Proposition 1. *Let* $\mathcal{B} = \{\mathcal{B}_i\}_{i \in V}$, $\mathcal{B}_i \neq \emptyset$, *be a family of finite sets of basic classification i-concepts and* $T = \{T_i\}_{i \in V}$ *an* F-$\mathcal{ALCI}$ *knowledge base. T has a model* $\mathcal{I} = \langle \{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i, j) \in E} \rangle$, *with* $\mathcal{I}_i = \langle \Delta^i, \cdot^i \rangle$, $i \in V$, *iff T has a world model* $I = \{I_i\}_{i \in V}$ *relative to* $\mathcal{B}$.

Proof. Suppose, first, that $\mathcal{I} = \langle \{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i, j) \in E} \rangle$ is a model of $T = \{T_i\}_{i \in V}$, with $\mathcal{I}_i = \langle \Delta^i, \cdot^i \rangle$, $i \in V$. Recall that we are assuming that $\Delta^i \neq \emptyset$, for all $i \in V$. Let, for each $i \in V$, $a_i \in \Delta^i$. Define $I_i = \{\phi \in \mathcal{B}_i : a_i \in \phi^i\}$, $i \in V$. Then $I = \{I_i\}_{i \in V}$ is a world relative to $\mathcal{B}$ that is also a model of *T*. If, conversely, *T* has a world model $I = \{I_i\}_{i \in V} \in \mathcal{I}_{\mathcal{B}}$, then $\{I_i \cup \neg_i(\mathcal{B}_i \setminus I_i)\}_{i \in V}$ is satisfiable relative to *T*, whence *T* is a fortiori satisfiable.

A **probabilistic interpretation** Pr is a probability function on the set of all worlds $\mathcal{I}_{\mathcal{B}}$ over the set $\mathcal{B}$ of basic classification concepts, i.e., a mapping $\Pr : \mathcal{I}_{\mathcal{B}} \to [0, 1]$, such that $\sum_{I \in \mathcal{I}_{\mathcal{B}}} \Pr(I) = 1$. Pr **satisfies** an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$ or Pr is a **model** of *T*, denoted $\Pr \vDash T$, if, for all $I \in \mathcal{I}_{\mathcal{B}}$, such that $\Pr(I) > 0$, $I \vDash T$. As far as satisfaction of conditional constraints goes, we set it up as follows: The **probability** of a classification *i*-concept ϕ in a probabilistic interpretation Pr, denoted $\Pr_i(\phi)$, is defined by

$$\Pr_i(\phi) = \sum \{\Pr(I) : I \vDash_i \phi\}.$$

Furthermore, for all classification *i*-concepts ϕ and ψ, such that $\Pr_i(\phi) > 0$, we set

$$\Pr_i(\psi \mid \phi) = \frac{\Pr_i(\phi \sqcap_i \psi)}{\Pr_i(\phi)}.$$

Pr **satisfies** an *i*-conditional constraint $(\psi \mid \phi)[l, u]$ or Pr is a **model** of $(\psi \mid \phi)[l, u]$, denoted $\Pr \vDash_i (\psi \mid \phi)[l, u]$, if $\Pr_i(\phi) = 0$ or $\Pr_i(\psi \mid \phi) \in [l, u]$. Pr satisfies a set of *i*-conditional constraints $\mathcal{F}_i$ or Pr is a model of $\mathcal{F}_i$, written $\Pr \vDash_i \mathcal{F}_i$, if $\Pr \vDash_i F$, for all $F \in \mathcal{F}_i$. Finally, if $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ is a collection of sets of *i*-conditional constraints for $i \in V$, we write $\Pr \vDash \mathcal{F}$ to signify that $\Pr \vDash_i \mathcal{F}_i$, for all $i \in V$.

In Proposition 2, an analog of Proposition 4.9 of [34] in the federated setting, it is shown that an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$ is satisfiable if and only if it has a probabilistic model.

Proposition 2. *Let* $\mathcal{B} = \{\mathcal{B}_i\}_{i \in V}$ *be a collection of nonempty sets of basic classification concepts and* $T = \{T_i\}_{i \in V}$ *be an* F-$\mathcal{ALCI}$ *knowledge base. T has a model* $\mathcal{I} = \langle \{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i,j) \in E} \rangle$, *with* $\mathcal{I}_i = \langle \Delta^i, \cdot^i \rangle$, $i \in V$, *if and only if it has a probabilistic model* Pr *on* $\mathcal{I}_{\mathcal{B}}$.

Proof. Suppose that T has a model $\mathcal{I} = \langle \{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i,j) \in E} \rangle$. By Proposition 1, T has a world model $I = \{I_i\}_{i \in V}$. Define $\Pr(I) = 1$ and $\Pr(I') = 0$, for all $I' \in \mathcal{I}_{\mathcal{B}}$, with $I' \neq I$. Then Pr is a probabilistic model of T on $\mathcal{I}_{\mathcal{B}}$. Suppose, conversely, that T has a probabilistic model Pr on $\mathcal{I}_{\mathcal{B}}$. Then, there exists $I \in \mathcal{I}_{\mathcal{B}}$, such that $\Pr(I) > 0$. Since $\Pr \vDash T$, this implies that $I \vDash T$. Hence, again by Proposition 1, T has a model $\mathcal{I} = \langle \{\mathcal{I}_i\}_{i \in V}, \{r_{ij}\}_{(i,j) \in E} \rangle$.

***z*-partitions and consistency.** *A* probabilistic interpretation Pr **verifies** an i-conditional constraint $(\psi \mid \phi)[l, u]$ if $\Pr_i(\phi) = 1$ and $\Pr_i(\psi) \in [l, u]$ (see also [33, 32]). On the other hand, Pr **falsifies** $(\psi \mid \phi)[l, u]$ if $\Pr_i(\phi) = 1$ and $\Pr \nvDash_i (\psi \mid \phi)[l, u]$. A collection $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ of sets of conditional constraints **tolerates** an i-conditional constraint $(\psi \mid \phi)[l, u]$ under an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$, or $(\psi \mid \phi)[l, u]$ **is tolerated under** T **by** $\mathcal{F}$, if $T \cup \mathcal{F}$ has a model that verifies $(\psi \mid \phi)[l, u]$.

Example 4. We illustrate, using our previous examples, ways in which tolerance of an i-conditional constraint by a collection of sets of conditional constraints may fail. We consider again the F-$\mathcal{ALCI}$ TBox $T = \{T_1, T_2\}$ of Example 1 and we set

$$\mathcal{F}_1 = \{(\texttt{FullyInsured} \mid \texttt{Insured})[0.8, 1], (\texttt{HasDental} \mid \texttt{Insured})[1, 1]\},$$

$$\mathcal{F}_2 = \{(\texttt{DoesResearch} \mid \texttt{Faculty})[1, 1], (\texttt{FullyInsured} \mid \texttt{Faculty})[1, 1]\}$$

and $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2\}$. Assume, for the sake of obtaining a contradiction that $F_1 = (\neg_1 \texttt{HasDental} \mid \texttt{PartiallyInsured})\ [1, 1]$ is tolerated under T by $\mathcal{F}$. Let Pr be a probabilistic model of $T \cup \mathcal{F}$ that verifies F_1. Then $\Pr_1(\texttt{PartiallyInsured})$

$= 1$ and $\Pr_1(\neg_1 \texttt{HasDental}) = 1$. But, since

$$\texttt{PartiallyInsured} \sqsubseteq \texttt{Insured} \in T_1,$$

$\Pr_1(\texttt{PartiallyInsured}) = 1$ implies that $\Pr_1(\texttt{Insured}) = 1$, whence, since $(\texttt{HasDental}|\texttt{Insured})[1, 1] \in \mathcal{F}_1$ and Pr is a model of $\mathcal{F}$, $\Pr_1(\texttt{HasDental}) = 1$. This clearly contradicts $\Pr_1(\neg_1 \texttt{HasDental}) = 1$. Therefore, F_1 is not tolerated under T by $\mathcal{F}$.

Similarly, it is easily seen that $F_2 = (\neg_2 \texttt{DoesResearch}|\texttt{Lecturer})[1, 1]$ is not tolerated under T by $\mathcal{F}$ because, if one assumes that a person is a lecturer, then they are clearly faculty, who are assumed, in general, to perform research and this would contradict the information that lecturers, in general, do not do research. If, on the other hand, our sets of conditional constraints included the more specific information represented by F_1 and F_2 as opposed to the facts that insured people, in general, have dental insurance and faculty, in general, do research, respectively, then the more general information would be tolerated by the more specific pieces of information. □

Concerning tolerance, it is not difficult to see that the following proposition, relating tolerance with the existence of a probabilistic model, which is due essentially to Lukasiewicz [34], holds:

Proposition 3. *An i-conditional constraint* $(\psi|\phi)[l, u]$ *is tolerated under an* F-$\mathcal{ALCI}$ *knowledge base* $T = \{T_i\}_{i\in V}$ *by a collection* $\mathcal{F} = \{\mathcal{F}_i\}_{i\in V}$ *of sets of conditional constraints iff there exists a probabilistic model Pr of* $T \cup \mathcal{F}'$, *where* $\mathcal{F}' = \{\mathcal{F}'_i\}_{i\in V}$, *with*

$$\mathcal{F}'_j = \begin{cases} \mathcal{F}_i \cup \{(\psi|\phi)[l, u], (\phi|\top_i)[1, 1]\}, & \text{if } j = i, \\ \mathcal{F}_j, & \text{otherwise}. \end{cases}$$

Proof. Suppose, first, that $(\psi|\phi)[l, u]$ is tolerated under T by $\mathcal{F}$. Thus, by definition, $T \cup \mathcal{F}$ has a model Pr verifying $(\psi|\phi)[l, u]$. Hence $\Pr_i(\phi) = 1$ and $\Pr_i(\psi) \in [l, u]$. But then $\Pr_i(\phi|\top_i) = \frac{\Pr_i(\phi \sqcap_i \top_i)}{\Pr_i(\top_i)} = \Pr_i(\phi) = 1$ and also $\Pr_i(\psi|\phi) = \frac{\Pr_i(\phi \sqcap_i \psi)}{\Pr_i(\phi)} = \Pr_i(\phi \sqcap_i \psi) = \Pr_i(\psi) \in [l, u]$. Therefore, Pr is a probabilistic model of $T \cup \mathcal{F}'$. For the converse we follow the reverse steps.

A PF-$\mathcal{ALCI}$ knowledge base $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$, with $\mathrm{PT}_i = \langle T_i, P_i \rangle$, is **consistent** if

(i) $T = \{T_i\}_{i \in V}$ is satisfiable;

(ii) There exist $k_i \geq 0$, $i \in V$, and ordered partitions $(P_0^i, ..., P_{k_i}^i)$ of P_i, such that each P_j^i, $j = 0, ..., k_i$, is the set of all $F \in P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)$ that are tolerated under *T* by $P \setminus (P_0 \cup \cdots \cup P_{j-1})$, where

$$P \setminus (P_0 \cup \cdots \cup P_{j-1}) = \{P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)\}_{i \in V}.$$

(In case, for some $i \in V$, $j > k_i$, we set $P_j^i = \varnothing$.)

The ordered partitions $(P_0^i, ..., P_{k_i}^i)$ are unique if they exist. Taking after [34], the term *z*-**partition** of $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$ will be used to refer to these partitions. Intuitively speaking, the *z*-partition enables one to resolve contextual inconsistencies by selecting more specific conditional constraints over less specific ones, as was demonstrated in Example 4.

Example 5. Consider again the TBox $T = \{T_1, T_2\}$ of Example 1 and the PBox $P = \{P_1, P_2\}$ discussed in Example 2. It is not difficult to see that *T* is satisfiable and that there exists a unique *z*-partition $P_1 = (P_0^1, P_1^1)$, $P_2 = (P_0^2, P_1^2)$, given by

$$P_0^1 = \{(\psi \mid \phi)[l, u] \in P_1 : \phi = \mathtt{Insured}\},$$

$$P_1^1 = \{(\neg_1 \mathtt{HasDental} \mid \mathtt{PartiallyInsured})[1, 1]\}$$

and

$$P_0^2 = P_2 \setminus \{(\neg_2 \mathtt{DoesResearch} \mid \mathtt{Lecturer})[1, 1]\},$$

$$P_1^2 = \{(\neg_2 \mathtt{DoesResearch} \mid \mathtt{Lecturer})[1, 1]\}.$$

Thus, the knowledge base $\mathrm{PT} = \{\mathrm{PT}_1, \mathrm{PT}_2\}$ is consistent. Note that, in the construction of the *z*-partition of P_2, the 2-conditional constraint

$$(\mathtt{FullyInsured} \mid \mathtt{Lecturer})[0.7, 1]$$

is in the first block; it is tolerated under *T* by *P*, even though P_2 contains the "more general" piece of information $(\mathtt{FullyInsured} \mid \mathtt{Faculty})[1, 1]$. □

Lexicographic entailment. Let $\mathrm{PT} = \{\langle T_i, P_i\rangle\}_{i\in V}$ be a consistent PF-$\mathcal{ALCI}$ knowledge base, with z-partition $\{(P_0^i, \ldots, P_{k_i}^i) : i \in V\}$. We follow [34] in first defining a lexicographic preference relation on probabilistic interpretations and, then, a lexicographic entailment for sets of conditional constraints under PTBoxes.

Given two probabilistic interpretations Pr and Pr′, Pr is said to be **lexicographically preferable** or **lex-preferable** to Pr′ if, for all $i \in V$, there exists $j_i = \{0, 1, \ldots, k_i\}$, such that

$|\{F \in P_{j_i}^i : \mathrm{Pr} \vDash_i F\}| > |\{F \in P_{j_i}^i : \mathrm{Pr}' \vDash_i F\}|$ and

$|\{F \in P_l^i : \mathrm{Pr} \vDash_i F\}| = |\{F \in P_l^i : \mathrm{Pr}' \vDash_i F\}|$, for all $j_i < l \leq k_i$.

The lex-preference relation implements the idea of preferring more specific sets of conditional constraints to less specific ones. This is used to resolve contextual inconsistencies when drawing conclusions.

A probabilistic interpretation Pr, satisfying an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i\in V}$ and a collection $\mathcal{F} = \{\mathcal{F}_i\}_{i\in V}$ of sets of conditional constraints is a **lexicographically minimal** or **lex-minimal model** of $T \cup \mathcal{F}$ if no model of $T \cup \mathcal{F}$ is lex-preferable to Pr.

An i-conditional constraint $(\psi\,|\,\phi)[l, u]$ is a **lexicographic-consequence** or **lex-consequence of** a collection of sets of conditional constraints $\mathcal{F} = \{\mathcal{F}_i\}_{i\in V}$ **under** a PTBox PT, written $\mathcal{F} \Vdash_i^{\mathrm{lex}} (\psi\,|\,\phi)[l, u]$ **under** PT if $\mathrm{Pr}_i(\psi) \in [l, u]$, for every lex-minimal model Pr of $T \cup \mathcal{F}'$, where $\mathcal{F}' = \{\mathcal{F}'_i\}_{i\in V}$, with

$$\mathcal{F}'_j = \begin{cases} \mathcal{F}_i \cup \{(\phi\,|\,\top_i)[1, 1]\}, & \text{if } j = i, \\ \mathcal{F}_j, & \text{otherwise.} \end{cases} \tag{2}$$

An i-conditional constraint $(\psi\,|\,\phi)[l, u]$ is a **tight lexicographic-consequence** or **tight lex-consequence of** $\mathcal{F} = \{\mathcal{F}_i\}_{i\in V}$ **under** PT, written $\mathcal{F} \Vdash_i^{\mathrm{tlex}} (\psi\,|\,\phi)[l, u]$ **under** PT if l and u are, respectively, the infimum and supremum of $\mathrm{Pr}_i(\psi)$ subject to all lex-minimal models Pr of $T \cup \mathcal{F}'$, with $\mathcal{F}'$ as in Equation (2).

Note that $[l, u] = [1, 0]$ when no such models exist. Moreover, we stipulate that

$\mathcal{F} \Vdash_i^{\text{lex}} (\psi|\phi)[l, u]$ and $\mathcal{F} \Vdash_i^{\text{tlex}} (\psi|\phi)[0, 1]$ under PT, for all $\mathcal{F}$ and all $(\psi|\phi)[l, u]$ in case PT is an inconsistent PTBox.

Example 6. Consider again the knowledge base PT with TBox introduced in Example 1 and PBox introduced in Example 2. The analysis performed in Examples 4 and 5, together with the definition of lex-preference and lex-consequence, show that, for any three probabilistic interpretations Pr, Pr' and Pr'', such that

(a) $\text{Pr} \vDash P$,

(b) $\text{Pr}' \vDash P \setminus \{(\texttt{HasDental}|\texttt{Insured})[1,1], (\texttt{DoesResearch}|\texttt{Faculty})[1,1]\}$,

(c) $\text{Pr}'' \vDash P \setminus \{(\neg_1\texttt{HasDental}|\texttt{PartiallyInsured})[1,1],$

$(\neg_2\texttt{DoesResearch}|\texttt{Lecturer})[1,1]\}$.

Pr is lex-preferable to both Pr' and Pr'' and Pr' is lex-preferable to Pr''. Thus, according to lex-preference, any probabilistic interpretation Pr, as above, will be preferred to Pr', resulting in the more specific 1-conditional constraint

$$(\neg_1\texttt{HasDental}|\texttt{PartiallyInsured})[1,1] \in P_1^1$$

to be preferred in reasoning over the more general 1-conditional constraint $(\texttt{HasDental|Insured})\ [1, 1] \in P_0^1$. Similar comments apply in comparing the more specific 2-conditional constraint

$$(\neg_2\ \texttt{DoesResearch}|\texttt{Lecturer})[1, 1] \in P_1^2$$

with the more general 2-conditional constraint $(\texttt{DoesResearch}|\texttt{Faculty})$ $[1, 1] \in P_0^2$. These choices will be used when deriving lex-consequences to resolve conflicts involving default information. □

An *i*-conditional constraint F is a **lex-consequence** of a PTBox PT, denoted $\text{PT} \Vdash_i^{\text{lex}} F$, if $\varnothing \Vdash_i^{\text{lex}} F$ under PT. F is a **tight lex-consequence** of PT, denoted $\text{PT} \Vdash_i^{\text{tlex}} F$, if $\varnothing \Vdash_i^{\text{tlex}} F$ under PT.

An analog of Theorem 4.18 of [34] provides a characterization of lexicographic entailment for a set of conditional constraints under a PTBox in terms of satisfiability and logical entailment of conditional constraints under an F-$\mathcal{ALCI}$ knowledge base. We introduce some additional definitions to prepare the groundwork for formulating this analog.

Given an F-$\mathcal{ALCI}$ knowledge base $T = \{T_i\}_{i \in V}$ and a V-indexed collection $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ of sets of conditional constraints, $T \cup \mathcal{F}$ is satisfiable if there exists a model of $T \cup \mathcal{F}$. An i-conditional constraint $(\psi \mid \phi)[l, u]$ is a **logical consequence** of $T \cup \mathcal{F}$, denoted $T \cup \mathcal{F} \vDash_i (\psi \mid \phi)[l, u]$, if each model of $T \cup \mathcal{F}$ is also a model of $(\psi \mid \phi)[l, u]$. $(\psi \mid \phi)[l, u]$ is a **tight logical consequence** of $T \cup \mathcal{F}$, denoted $T \cup \mathcal{F} \vDash_i^t (\psi \mid \phi)[l, u]$, if l and u are, respectively, the infimum and supremum of $\mathrm{Pr}_i(\psi \mid \phi)$ subject to all models Pr of $T \cup \mathcal{F}$, with $\mathrm{Pr}_i(\phi) > 0$.

Let $P = \{P_i\}_{i \in V}$ be the PBox of a PF-$\mathcal{ALCI}$ knowledge base $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$. An indexed subfamily $Q = \{Q_i\}_{i \in V}$ with $Q_i \subseteq P_i$, $i \in V$, denoted $Q \leq P$, is **lexicographically-preferable** or **lex-preferable** to $Q' = \{Q'_i\}_{i \in V}$, with $Q' \leq P$, if, for all $i \in V$, there exists $j_i \in \{0, 1, ..., k_i\}$, such that

– $| Q_i \cap P^i_{j_i} | > | Q'_i \cap P^i_{j_i} |$ and

– $| Q_i \cap P^i_l | = | Q'_i \cap P^i_l |$, for all $j_i < l \leq k_i$,

where $\{(P^i_0, ..., P^i_{k_i}) : i \in V\}$ is the z-partition of PT. Q is **lexicographically-minimal** or **lex-minimal** in $S = \{S^k\}_{k \in I}$, $S^k = \{S^k_i\}_{i \in V}$, $S^k \leq P$, if $Q \in S$ and no $Q' \in S$ is lex-preferable to Q.

Theorem 1. *Let* $\mathrm{PT} = \{\langle T_i, P_i \rangle\}_{i \in V}$ *be a consistent* PTBox, $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ *be a family of sets of conditional constraints and* ϕ, ψ *two i-classification concepts. Consider the collection* $\mathcal{Q}$ *of all lex-minimal elements in the set of all* $Q \leq P$, *such that* $T \cup Q \cup \mathcal{F}'$ *is satisfiable, where* $\mathcal{F}' = \{\mathcal{F}'_i\}_{i \in V}$ *is given by Equation* (2).

1. *If* $\mathcal{Q} = \varnothing$, *then* $\mathcal{F} \Vdash_i^{\mathrm{tlex}} (\psi \mid \phi)[1, 0]$ *under* PT.

2. *If* $\mathcal{Q} \neq \varnothing$, *then* $\mathcal{F} \Vdash_i^{\mathrm{tlex}} (\psi \mid \phi)[l, u]$ *under* PT, *where* $l = \min l'$ *and* $u = \max u'$, *subject to* $T \cup Q \cup \mathcal{F}' \vDash_i^t (\psi \mid \top_i)[l', u']$ *and* $Q \in \mathcal{Q}$.

Proof. The proof is very similar to the proof of Theorem 4.18 of [34] (see page 878).

1. If $\mathcal{Q} = \emptyset$, then $T \cup Q \cup \mathcal{F}'$ is not satisfiable, for any $Q \leq P$. Thus, $T \cup \mathcal{F}'$ is not satisfiable. This shows that $\mathcal{F} \Vdash_i^{\text{tlex}} (\psi \mid \phi)[1, 0]$ under PT.

2. Let $\mathcal{Q} \neq \emptyset$. Then Pr is a lex-minimal model of $T \cup \mathcal{F}'$ iff (i) Pr is a model of $T \cup \mathcal{F}'$ and (ii) $\{\{F_i \in P_i : \Pr \vDash_i F_i\}\}_{i \in V}$ is a lex minimal element in the set of all $Q \leq P$, such that $T \cup Q \cup \mathcal{F}'$ is satisfiable. This shows that Pr is a model of $T \cup Q \cup \mathcal{F}'$, for some $Q \in \mathcal{Q}$.

4. A Modified P-$\mathcal{ALCI}$

In the remainder of the paper, we formulate some inference problems concerning PF-$\mathcal{ALCI}$ knowledge bases. Our goal is to reduce these problems to corresponding problems for non-federated probabilistic knowledge bases and, then, use already known procedures from [34] (or slightly modified versions) in order to solve them. Apart from the obvious algorithmic advantage, we also get the side-benefit of being able to pinpoint the algorithmic complexity of the federated problems, based on the complexities of the unimodule versions. The unfortunate fact is that the probabilistic description logic that results by restricting the logic P-$\mathcal{SHIF}(\mathbf{D})$, as presented in [34], by adopting as its underlying description logic $\mathcal{ALCI}$ and by disregarding its assertional part, does not serve exactly our goal. Intuitively, this happens because the world models in the semantics of P-$\mathcal{SHIF}(\mathbf{D})$ are based in a single set of basic classification concepts, whereas, our reduction will necessitate the existence of multiple sets of basic classification concepts. For this reason, in this section, we present a single module probabilistic language P-$\mathcal{ALCI}$ by slightly generalizing the semantics of the probabilistic terminological and terminological default knowledge given in [34]. This modified version will be appropriate for accommodating the sound and complete reductions of the inference problems for PF-$\mathcal{ALCI}$. Furthermore, we introduce the main problems that we will consider in P-$\mathcal{ALCI}$. They are the same as those of Lukasiewicz [34], but refer to the modified semantics. Finally, we argue that these problems may be solved with algorithms virtually identical to the ones provided by Lukasiewicz and, as a result, maintain the same computational complexities.

4.1. Syntax and semantics

Since in the federated language PF-$\mathcal{ALCI}$ we deal only with PTBoxes, we restrict our attention in this section to P-$\mathcal{ALCI}$ knowledge bases with only PTBoxes of the form $\mathrm{PT} = \langle T, P \rangle$. We assume that among the $\mathcal{ALCI}$ concept names, there are $|V|$ names $\top_i$, $i \in V$, whose extensions, intuitively, will represent the parts of the local domains of a PF-$\mathcal{ALCI}$ interpretation corresponding to the various modules, when combined into a single large domain. The TBox T is an ordinary $\mathcal{ALCI}$ TBox and the semantics concerning the P-$\mathcal{ALCI}$ roles, concepts and TBox axioms is exactly the ordinary semantics of $\mathcal{ALCI}$, with the only exception that an $\mathcal{ALCI}$ model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ must satisfy $\top_i^{\mathcal{I}} \neq \emptyset$, for all $i \in V$. To construct the PBox, fix a collection $\{\mathcal{C}_i\}_{i \in V}$ of nonempty sets of **basic classification concepts** or **basic** c**-concepts**, which are concepts in $\mathcal{ALCI}$, such that $C \sqsubseteq \top_i \in T$, for every $C \in \mathcal{C}_i$, $i \in V$. These will be the sets of relevant concepts for defining probabilistic relationships. This is the main deviation from the languages in [34], which have one set of basic classification concepts. We adopt this modification, as explained previously, because we would like to use P-$\mathcal{ALCI}$ to simulate PF-$\mathcal{ALCI}$, which employs multiple sets of basic classification concepts each of which is used to classify probabilistic relationships between concepts appearing in a corresponding module of a federated ontology. Accordingly, we also obtain $|V|$ sets of **classification concepts** or c**-concepts**. These are defined recursively by taking negations, conjunctions and disjunctions starting from the corresponding set of basic c-concepts. We also construct conditional constraints of $|V|$ types, each using corresponding c-concepts. Thus, a **conditional constraint of type** i is one of the form $(\psi \mid \phi)[l, u]$, where ϕ, ψ are c-concepts from the i-th set and l, $u \in [0, 1]$. We denote by P a finite set of conditional constraints (possibly of many types), called a **PBox**. A **PTBox** $\mathrm{PT} = \langle T, P \rangle$ consists of an $\mathcal{ALCI}$ TBox T together with a PBox P. For our purposes a **probabilistic knowledge base** and a PTBox coincide, since we do not consider individuals and, as a result we do not consider either classical ABoxes or PABoxes in the sense of [34].

To accommodate the multiplicity of the sets of basic classification concepts, the notion of a world in the semantics must be modified as compared to the standard one of [34]. Namely, a **world** is a collection of sets $I = \{I_i\}_{i \in V}$, with $I_i \subseteq \mathcal{C}_i$, such that the concept $\sqcap_{C \in I_i} C \sqcap \sqcap_{C \in \mathcal{C}_i \setminus I_i} \neg C$ is satisfiable, for all $i \in V$. The set of all worlds is denoted by $\mathcal{I}_{\mathcal{C}}$. A world I is said to **satisfy** an $\mathcal{ALCI}$ TBox T or to be a

model of *T*, written $I \vDash T$, if there exists a model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of *T*, such that $\bigcap_{C \in I_i} C^{\mathcal{I}} \cap \bigcap_{C \in \mathcal{C}_i \setminus I_i} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \neq \varnothing$, for all $i \in V$. A world $I = \{I_i\}_{i \in V}$ **satisfies** a basic classification concept $\phi \in \mathcal{C}_i$ if $\phi \in I_i$. This notion can be extended in the usual way over classification concepts of type *i*.

It can be shown, now, using the modified definitions that were introduced above that an $\mathcal{ALCI}$ TBox T has a model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ iff it has a world model $I = \{I_i\}_{i \in V}$.

A **probabilistic interpretation** Pr is a probability function on $\mathcal{I}_{\mathcal{C}}$, as in the ordinary case [34]. It **satisfies** an $\mathcal{ALCI}$ TBox *T* or is a **model** of *T* if $I \vDash T$, for all *I*, such that $\Pr(I) > 0$. Moreover, for a classification concept ϕ of type *i*, we have $\Pr(\phi) = \sum \{\Pr(I) : I \vDash \phi\}$ and, for two classification concepts ϕ, ψ of type *i*, such that $\Pr(\phi) > 0$, we define $\Pr(\psi \mid \phi) = \dfrac{\Pr(\phi \sqcap \psi)}{\Pr(\phi)}$. Then Pr **satisfies** or is a **model** of a conditional constraint $(\psi \mid \phi)[l, u]$ of type *i*, written $\Pr \vDash (\psi \mid \phi)[l, u]$, if $\Pr(\phi) = 0$ or $\Pr(\psi \mid \phi) \in [l, u]$. This notion extends to satisfiability of a set of conditional constraints, possibly consisting of constraints of more than one types.

A probabilistic interpretation is said to **verify** a conditional constraint $(\psi \mid \phi)[l, u]$ if $\Pr(\phi) = 1$ and $\Pr(\psi) \in [l, u]$. A set $\mathcal{F}$ of conditional constraints (possibly of various types) is said to **tolerate** a conditional constraint $(\psi \mid \phi)[l, u]$ **under** an $\mathcal{ALCI}$ TBox *T* if $T \cup \mathcal{F}$ has a model that verifies $(\psi \mid \phi)[l, u]$. These concepts help in defining the notion of consistency for a P-$\mathcal{ALCI}$ PTBox, which is a slightly modified version of the one given in Section 4.2.3 of [34]. A PT-Box $\mathrm{PT} = \langle T, P \rangle$ is **consistent** if *T* is satisfiable (i.e., has a model; recall that the $\top_i$, $i \in V$, must have nonempty extensions in the model) and there exists an ordered partition $(P_0, \ldots, P_k)$ of *P*, such that each P_i is the set of all $F \in P \setminus \bigcup_{j=0}^{i-1} P_j$ that are tolerated by $P \setminus \bigcup_{j=0}^{i-1} P_j$ under *T*. The ordered partition $(P_0, \ldots, P_k)$ is unique, if it exists, and, following [34], we call it the z-**partition** of PT and define $P_j^i = \{F \in P_j : F \text{ is of type } i\}$, for all $j = 1, \ldots, k$, $i \in V$.

To define lexicographic entailment, we fix a consistent P-$\mathcal{ALCCI}$ PTBox $\mathrm{PT} = \langle T, P \rangle$. Thus, there exists a z-partition $(P_0, \ldots, P_k)$ of PT. A probabilistic interpretation Pr is said to be **lexicographically preferable** or **lex-preferable** to a probabilistic interpretation Pr' if, for every $i \in V$, there exists j_i, $0 \leq j_i \leq k$, such that $|\{F \in P^i_{j_i} : \mathrm{Pr} \vDash F\}| > |\{F \in P^i_{j_i} : \mathrm{Pr}' \vDash F\}|$ and $|\{F \in P^i_l : \mathrm{Pr} \vDash F\}| = |\{F \in P^i_l : \mathrm{Pr}' \vDash F\}|$, for all $j_i < l \leq k$. A probabilistic interpretation Pr, that satisfies T and a set $\mathcal{F}$ of conditional constraints (of possibly various types) is a **lexicographically minimal** or **lex-minimal model** of $T \cup \mathcal{F}$ if no model of $T \cup \mathcal{F}$ is lex-preferable to Pr. A conditional constraint $(\psi | \phi)[l, u]$ of type i is a **lexicographic consequence** or **lex-consequence** of a set $\mathcal{F}$ of conditional constraints under a PTBox PT, denoted $\mathcal{F} \Vdash^{\mathrm{lex}} (\psi | \phi)[l, u]$ **under** PT, if $\mathrm{Pr}(\psi) \in [l, u]$ for every lex-minimal model Pr of $T \cup \mathcal{F} \cup \{(\phi | \top)[1, 1]\}$. The conditional constraint $(\psi | \phi)[l, u]$ of type i is a **tight lexicographic consequence** or **tight lex-consequence** of a set $\mathcal{F}$ of conditional constraints under a PTBox PT, denoted $\mathcal{F} \Vdash^{\mathrm{lex}}_{\mathrm{tight}} (\psi | \phi)[l, u]$ **under** PT, if l and u are, respectively, the infimum and supremum of $\mathrm{Pr}(\psi)$ subject to all lex-minimal models Pr of $T \cup \mathcal{F} \cup \{(\phi | \top)[1, 1]\}$. For inconsistent PTBoxes PT, we define $\mathcal{F} \Vdash^{\mathrm{lex}} (\psi | \phi)[l, u]$ and $\mathcal{F} \Vdash^{\mathrm{lex}}_{\mathrm{tight}} (\psi | \phi)[1, 0]$ under PT, for all sets of conditional constraints $\mathcal{F}$ and all conditional constraints $(\psi | \phi)[l, u]$. Finally, a conditional constraint F is a **lex-consequence** of a PTBox PT, denoted $\mathrm{PT} \Vdash^{\mathrm{lex}} F$, if $\varnothing \Vdash^{\mathrm{lex}} F$ under PT and F is a **tight lex-consequence** of PT, denoted $\mathrm{PT} \Vdash^{\mathrm{lex}}_{\mathrm{tight}} F$, if $\varnothing \Vdash^{\mathrm{lex}}_{\mathrm{tight}} F$ under PT.

Along the lines of the characterization of lexicographic entailment for a set of conditional constraints under a PTBox in terms of satisfiability and logical entailment for a set of conditional constraints under a classical knowledge base that was provided in [34] (Theorem 4.18), one may give a characterization for the modified language P-$\mathcal{ALCCI}$, presented in this section. Given an $\mathcal{ALCCI}$ TBox T and a set of conditional constraints $\mathcal{F}$ of possibly various types, $T \cup \mathcal{F}$ is **satisfiable** if a model of $T \cup \mathcal{F}$ exists. Again, such a model in the present context is assumed to assign nonempty extensions to all $\top_i$, $i \in V$. A conditional constraint $(\psi | \phi)[l, u]$

of type i is a **logical consequence** of $T \cup \mathcal{F}$, denoted $T \cup \mathcal{F} \vDash (\psi \mid \phi)[l, u]$, if each model of $T \cup \mathcal{F}$ is also a model of $(\psi \mid \phi)[l, u]$. The conditional constraint $(\psi \mid \phi)[l, u]$ is a **tight logical consequence** of $T \cup \mathcal{F}$, denoted $T \cup \mathcal{F} \vDash_{\text{tight}} (\psi \mid \phi)[l, u]$, if l and u are, respectively, the infimum and the supremum of $\Pr(\psi \mid \phi)$ subject to all models Pr of $T \cup \mathcal{F}$, with $\Pr(\phi) > 0$. Let $\text{PT} = \langle T, P \rangle$ be a consistent P-$\mathcal{ALCI}$ PTBox, where $(P_0, \ldots, P_k)$ denotes the z-partition of PT and P_j^i, $i \in V$, $j = 1, \ldots, k$ are as before. A subset $Q \subseteq P$ is **lexicographically preferable** or **lex-preferable** to $Q' \subseteq P$ if, for every $i \in V$, there exists a $j_i \in \{0, \ldots, k\}$, such that $| Q \cap P_{j_i}^i | > | Q' \cap P_{j_i}^i |$ and $| Q \cap P_l^i | = | Q' \cap P_l^i |$, for all $j_i < l \leq k$. Q is **lexicographically minimal** or **lex-minimal** in $\mathcal{S} \subseteq \mathcal{P}(P)$ if $Q \in \mathcal{S}$ and no $Q' \in \mathcal{S}$ is lex-preferable to Q. Analogously to Theorem 4.18 of [34], we obtain for our modified language P-$\mathcal{ALCI}$ that for a consistent PTBox $\text{PT} = \langle T, P \rangle$, a set $\mathcal{F}$ of conditional constraints of possibly various types and two c-concepts ϕ, ψ of type i, if $\mathcal{Q}$ is the set of all lex-minimal elements in the set of all $Q \subseteq P$, such that $T \cup Q \cup \mathcal{F} \cup \{(\phi \mid \top)[1, 1]\}$ is satisfiable, then

– if $\mathcal{Q} = \emptyset$, $\mathcal{F} \Vdash_{\text{tight}}^{\text{lex}} (\psi \mid \phi)[1, 0]$ under PT;

– if $\mathcal{Q} \neq \emptyset$, then $\mathcal{F} \Vdash_{\text{tight}}^{\text{lex}} (\psi \mid \phi)[l, u]$ under PT, where $l = \min l'$ and $u = \max u'$, respectively, subject to $T \cup Q \cup \mathcal{F} \cup \{(\phi \mid \top)[1, 1]\} \vDash_{\text{tight}} (\psi \mid \top)[l', u']$ and $Q \in \mathcal{Q}$.

4.2. P-$\mathcal{ALCI}$ problems of interest

Concerning the modified language P-$\mathcal{ALCI}$, whose syntax and semantics we presented in the previous subsection, we would like to revisit the following two computational problems, assuming that all reals in [0, 1] considered are taken to be rational and denoting the set of rational numbers by $\mathbb{Q}$:

PTBox Consistency (PTCon): Decide whether a given PTBox $\text{PT} = \langle T, P \rangle$, is consistent.

Tight Lexicographic Entailment (TLexEnt): Given a PTBox $\text{PT} = \langle T, P \rangle$, a finite set $\mathcal{F}$ of conditional constraints (of possibly various types) and two *c*-concepts ϕ and ψ of type *i*, compute $l,\ u \in [0, 1] \cap \mathbb{Q}$, such that $\mathcal{F} \Vdash^{\text{lex}}_{\text{tight}} (\psi \mid \phi)[l, u]$ under PT.

Lukasiewicz [34] shows that the problems PTCon and TLEXENT for his version of the language P-$\mathcal{ALCI}$ (in fact for his languages P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$, based on the (non-probabilistic) description logics $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$, respectively, which are significantly more expressive than $\mathcal{ALCI}$) can be reduced to the following two problems:

Satisfiability (SAT): Given an $\mathcal{ALCI}$ knowledge base *T* and a finite set $\mathcal{F}$ of conditional constraints, decide whether $T \cup \mathcal{F}$ is satisfiable.

Tight Logical Entailment (TLOGENT): Given an $\mathcal{ALCI}$ knowledge base *T*, a finite set $\mathcal{F}$ of conditional constraints and a *c*-concept ψ, compute $l, u \in [0, 1] \cap \mathbb{Q}$, such that $T \cup \mathcal{F} \vDash_{\text{tight}} (\psi \mid \top)[l, u]$.

In fact, Lukasiewicz presents in Subsection 5.2 of [34] algorithms that reduce both PTCON and TLEXENT for his description logics P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$ to algorithms for SAT and TLOGENT for P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$, respectively. Some of his ideas are borrowed from [33] and [21]. His algorithms apply also to the modified language P-$\mathcal{ALCI}$, that we presented in the previous subsection, that contains neither ABoxes nor PABoxes. We emphasize, however, that satisfiability for the modified P-$\mathcal{ALCI}$ language refers to the existence of a model that satisfies the additional stipulation concerning satisfiability of all $\top_i$, $i \in V$, and the modified semantics. The following result allows us to estimate the number of instances of SAT and TLOGENT that one has to solve in order to obtain solutions of given instances of PTCON and TLEXENT.

Theorem 2 (Theorem 5.4 (a) and (c) of [34]).

(a) *An algorithm that solves* PTCON *uses* $O(|P|^2)$ *instances of* SAT.

(b) *An algorithm that solves* TLEXENT *uses* $O(2^{|P|})$ *instances of* SAT *and* TLOGENT.

SAT and TLOGENT, on the other hand, can be handled by reductions to deciding

TBOX satisfiability in $\mathcal{ALCI}$, deciding the solvability of systems of linear constraints and computing the optimal value of linear programs exactly as is the case for SAT and TLOGENT for P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$ in [34].

As far as SAT is concerned, it is reducible to deciding TBOX satisfiability in $\mathcal{ALCI}$ and whether a system of linear constraints in solvable. First, the set of possible worlds $R = \{I \in \mathcal{I}_{\mathcal{C}} : I \vDash T\}$ is computed, using satisfiability $|\mathcal{I}_{\mathcal{C}}|$ times to decide whether there exists a model $\mathcal{I}$ of T, such that $\bigcap_{C \in I_i} C^{\mathcal{I}} \cap \bigcap_{C \in \mathcal{C}_i \setminus I_i} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \neq \varnothing$, for all $i \in V$. Then the following result, an analog of Theorem 5.5 of [34] for our modified language, applies:

Theorem 3. *Let T be an $\mathcal{ALCI}$ TBox and $\mathcal{F}$ be a finite set of conditional constraints of possibly various types. Let $R = \{I \in \mathcal{I}_{\mathcal{C}} : I \vDash T\}$. Then $T \cup \mathcal{F}$ is satisfiable iff the system of linear constraints*

$$\begin{cases} \sum_{r \in R,\, r \vDash \neg\psi \sqcap \phi} -l y_r + \sum_{r \in R,\, r \vDash \psi \sqcap \phi} (1 - l) y_r \geq 0, & (\psi|\phi)[l, u] \in \mathcal{F},\, l > 0, \\ \sum_{r \in R,\, r \vDash \neg\psi \sqcap \phi} u y_r + \sum_{r \in R,\, r \vDash \psi \sqcap \phi} (u - 1) y_r \geq 0, & (\psi|\phi)[l, u] \in \mathcal{F},\, u < 1, \\ \sum_{r \in R} y_r = 1, \quad y_r \geq 0,\ r \in R \end{cases} \tag{3}$$

over the variables y_r, $r \in R$, is solvable.

As far as TLOGENT is concerned, it is reducible to deciding TBOX satisfiability in $\mathcal{ALCI}$ and computing the optimal values of two linear programs. The following result, an analog of Theorem 5.7 of [34], details the situation.

Theorem 4. *Let T be an $\mathcal{ALCI}$ TBox, $\mathcal{F}$ be a finite set of conditional constraints of possibly various types and ψ be a c-concept of type i. Assume that $T \cup \mathcal{F}$ is satisfiable and let $R = \{I \in \mathcal{I}_{\mathcal{C}} : I \vDash T\}$. Then l and u, such that $T \cup \mathcal{F} \vDash_{\text{tight}} (\psi|\top)[l, u]$ are given, respectively, by the optimal values of the following linear programs over the variables y_r, $r \in R$:*

$$\textit{minimize } \sum_{r \in R,\, r \vDash \psi} y_r \ \textit{ subject to the linear constraints } (3)$$

$$\textit{maximize } \sum_{r \in R,\, r \vDash \psi} y_r \ \textit{ subject to the linear constraints } (3).$$

It should be fairly obvious from the work accomplished in [34] (see Theorems 6.3 and 6.4) that the following result applies concerning the complexity of the problems SAT, PTCON, on the one hand, and TLOGENT, TLEXENT, on the other:

Theorem 5. (a) SAT *and* PTCON *are in* EXP *when* $T \cup \mathcal{F}$ *and* PT, *respectively*, *are defined in* P-$\mathcal{ALCI}$;

(b) TLOGENT *and* TLEXENT *are in* FEXP, *when* $T \cup \mathcal{F}$ *and* $\mathrm{PT} \cup \mathcal{F}$, *respectively*, *are defined in* P-$\mathcal{ALCI}$.

5. Reduction from Federated to Unimodule Problems

We start this section by introducing the computational problems that are of interest when reasoning with a PF-$\mathcal{ALCI}$ knowledge base. We assume, once more that all reals in [0, 1] considered are rational. Following [34], we would like to study the following decision and computation problems in the framework of PF-$\mathcal{ALCI}$ knowledge bases:

Federated PTBOX Consistency (FPTCON): Decide whether a given PT-BOX $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$, with $\mathrm{PT}_i = \langle T_i, P_i \rangle$, is consistent.

Federated Tight Lexicographic Entailment (FTLEXENT): Given a PT-BOX $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$, with $\mathrm{PT}_i = \langle T_i, P_i \rangle$, a finite collection $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ of sets of conditional constraints and two i-classification concepts ϕ and ψ, *compute* l, $u \in [0, 1] \cap \mathbb{Q}$, such that $\mathcal{F} \Vdash_i^{\mathrm{tlex}} (\psi \mid \phi)[l, u]$ under PT.

Our strategy for solving these two problems is to reduce them to the corresponding unimodule problems PTCON and TLEXENT for the language P-$\mathcal{ALCI}$, that were introduced in the previous section. Since algorithms for solving these problems can be easily extracted as modifications of the algorithms for the corresponding problems for the languages P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$, presented in [34], our reduction will provide a solution for the federated case. Moreover, based on the complexities of the unimodule problems, we can provide estimates for the corresponding complexities of the federated versions.

More precisely, we provide a reduction of a given instance of the federated PTBOX consistency or the federated tight lexicographic entailment problem to an instance of the corresponding unimodule problem. Then, we apply the method of

Lukasiewicz, using the algorithms involving SAT and TLOGENT, to obtain a solution for the original problem. Our task is to show how, given an instance α of either the FPTCON or the FTLEXENT problem, we can obtain an instance α^s of the corresponding problem for P-$\mathcal{ALCI}$, such that

– for FPTCON, α is consistent if and only if α^s is consistent and

– for FTLEXENT, l, $u \in [0, 1] \cap \mathbb{Q}$ are solutions of α if and only if l, u are solutions of α^s.

This will prove the decidability of FPTCON and FTLEXENT and will allow us to draw conclusions on their complexities based on the complexities of the corresponding problems for P-$\mathcal{ALCI}$.

Let $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$ be a PF-$\mathcal{ALCI}$ PTBOX, with $\mathrm{PT}_i = \langle T_i, P_i \rangle$. Taking after a similar construction, presented in [47], we construct a unimodule PTBOX $\mathrm{PT}^s = \langle T^s, P^s \rangle$ as follows:

The signature of PT^s is the union of the local signatures of the modules together with a global top $\top$, a global bottom $\bot$, local top concepts $\top_i$, for all $i \in V$, and, finally, a collection of new role names $\{R_{ij}\}_{(i, j) \in E}$, whose extensions in P-$\mathcal{ALCI}$ interpretations will be used, roughly speaking, to simulate the image-domain relations of the federated interpretations. Formally,

$$\mathrm{Sig}(\mathrm{PT}^s) = \bigcup_i (\mathcal{C}_i \cup \mathcal{R}_i) \cup \{\top, \bot\} \cup \{\top_i, 1 \le i \le n\} \cup \{R_{ij} : (i, j) \in E\}.$$

To construct the unimodule TBOX T^s and the unimodule PBOX P^s, given the federated PTBOX PT, we first introduce a mapping $\#_i$, which translates i-concept expressions C of the federated instance to concept expressions $\#_i(C)$ of the unimodule counterpart, and serves to maintain the compatibility of the concept domains. It is defined by induction on the structure of $C \in \hat{\mathcal{C}}_i$ (for $i = j$, R_{ij} is assumed to be interpreted as the identity on Δ^i in any interpretation and, as a result, may be omitted from the following translation):

– $\#_i(C) = C$, if $C \in \mathcal{C}_i$;

– $\#_i(C) = \exists R_{ji}^{-}.\#_j(C)$, if $C \in \mathcal{C}_j \cap \hat{\mathcal{C}}_i$;

– $\#_i(\neg_j D) = \exists R_{ji}^{-}.(\neg \#_j(D) \sqcap \top_j)$;

– $\#_i(D \boxplus_j E) = \exists R_{ji}^{-}.(\#_j(D) \boxplus \#_j(E))$, where $\boxplus = \sqcap$ or $\boxplus = \sqcup$;

– $\#_i(\exists_j R.D) = \exists R_{ji}^{-}.(\exists R_{kj}^{-}.(\exists R.(\exists R_{kj}.\#_j(D))))$, for $R \in \mathcal{R}_k \cup \mathcal{R}_k^{-}$;

– $\#_i(\forall_j R.D) = \exists R_{ji}^{-}.(\forall R_{kj}^{-}.(\forall R.(\forall R_{kj}.\#_j(D))))$, for $R \in \mathcal{R}_k \cup \mathcal{R}_k^{-}$.

Having defined $\#_i$, we show how various axioms derived from the structure of PT are added to T^s :

– For each $C \in \mathcal{C}_i$, $C \sqsubseteq \top_i$ is added to T^s.

– For each $R \in \mathcal{R}_i$, $\top_i$ is stipulated to be the domain and range of *R*, i.e., $\top \sqsubseteq \forall R^{-}.\top_i$ and $\top \sqsubseteq \forall R.\top_i$ are added to T^s.

– For each new role name R_{ij}, $\top_i$ is stipulated to be its domain and $\top_j$ to be its range, i.e., $\top \sqsubseteq \forall R_{ij}^{-}.\top_i$ and $\top \sqsubseteq \forall R_{ij}.\top_j$ are added to T^s.

– For each $C \sqsubseteq D \in T_i$, $\#_i(C) \sqsubseteq \#_i(D)$ is added to T^s.

Finally, various axioms derived from the conditional constraints in PT, using the transformations $\#_i$, $i \in V$, are added to P^s:

– For every basic *i*-classification concept ϕ, the concept expression $\#_i(\phi)$ is added as a basic classification concept of type *i* of PT^s. Moreover, $\top_i$ is declared to be a basic classification concept of type *i* of PT^s. This defines the collection $\mathcal{B}^s = \{\mathcal{B}_i^s\}_{i \in V}$ of sets $\mathcal{B}_i^s$ of basic classification concepts of type *i* of PT^s.

– For each *i*-conditional constraint $(\psi | \phi)[l, u]$ in P_i, the conditional constraint $(\#_i(\psi) | \#_i(\phi))[l, u]$ is added to P^s.

Applying these definitions, we may obtain an instance of PTCON, given an instance of FPTCON. On the other hand, considering instances of the problem

FTLEXENT, the collection $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ of sets of conditional constraints is translated to the collection $\mathcal{F}^s$ by including, for every *i*-conditional constraint $(\psi | \phi)[l, u]$ in $\mathcal{F}_i$, the conditional constraint $(\#_i(\psi) | \#_i(\phi))[l, u]$ of type *i* in $\mathcal{F}^s$. We also translate the additional *i*-conditional constraint $(\psi | \phi)[l, u]$, that is given in the instance of the problem, to the conditional constraint $(\#_i(\psi) | \#_i(\phi))[l, u]$ of type *i*.

Example 7. In this example, we illustrate the reduction defined in this section by transforming the PTBOX PT $= \langle \mathrm{PT}_1, \mathrm{PT}_2 \rangle$, defined in Examples 1 and 2 using the language PF-$\mathcal{ALCI}$, to the corresponding PTBOX PT$^s = \langle T^s, P^s \rangle$ over the language P-$\mathcal{ALCI}$.

The general axioms included in T^s are:

Insured $\sqsubseteq \top_1$	Faculty $\sqsubseteq \top_2$
FullyInsured $\sqsubseteq \top_1$	Lecturer $\sqsubseteq \top_2$
PartiallyInsured $\sqsubseteq \top_1$	MaleLecturer $\sqsubseteq \top_2$
	FemaleLecturer $\sqsubseteq \top_2$
$\top \sqsubseteq \forall R_{12}^{-}.\top_1$	$\top \sqsubseteq \forall R_{12}.\top_2$

The axioms in T^s that are induced by the axioms in T_1 and T_2 are, respectively,

FullyInsured $\sqsubseteq$ Insured

PartiallyInsured $\sqsubseteq$ Insured

PartiallyInsured $\sqsubseteq$ ¬FullyInsured $\sqcap \top_1$

MaleLecturer $\sqsubseteq$ Lecturer

FemaleLecturer = Lecturer $\sqcap$ (¬MaleLecturer $\sqcap \top_2$)

Lecturer $\sqsubseteq$ Faculty

Faculty $\sqsubseteq \exists R_{12}^{-}$.Insured

Finally, the axioms in P^s that are induced by the axioms in P_1 and P_2 are, respectively,

$(\texttt{FullyInsured}|\texttt{Insured})[0.8, 1]$

$(\texttt{HasDental}|\texttt{Insured})[1, 1]$

$(\neg\texttt{HasDental} \sqcap \top_1 | \texttt{PartiallyInsured})[1, 1]$

$(\texttt{DoesResearch}|\texttt{Faculty})[1, 1]$

$(\neg\texttt{DoesResearch} \sqcap \top_2 |\texttt{Lecturer})[1, 1]$

$(\exists R_{12}^{-}.\texttt{FullyInsured}|\texttt{Faculty})[1, 1]$

$(\exists R_{12}^{-}.\texttt{FullyInsured}|\texttt{Lecturer})\ [0.7, 1]$ □

In the next section we show that the reduction $\mathfrak{R}$ is sound and complete for both FPTCON and FTLEXENT. For FPTCON, this means that PT is consistent if and only if PT^s is consistent. On the other hand, for FTLEXENT, it means that $l,\ u \in [0, 1] \cap \mathbb{Q}$, are such that $\mathcal{F} \Vdash_i^{\text{tlex}} (\psi | \phi)[l, u]$ under PT if and only if they are such that $\mathcal{F}^s \Vdash_{\text{tight}}^{\text{lex}} (\#_i(\psi) | \#_i(\phi))[l, u]$ under PT^s.

6. Soundness and Completeness

6.1. Soundness and completeness for FPTCON

In this section, we present the soundness and completeness proofs of the translations from the federated problem FPTCON to the corresponding unimodule problem PTCON for P-$\mathcal{ALCI}$, the modified version of the problem studied for P-$\mathcal{SHOIN}(\mathbf{D})$ and P-$\mathcal{SHIF}(\mathbf{D})$ in [34]. More precisely, we aim to show that $\text{PT} = \{\text{PT}_i\}_{i \in V}$, with $\text{PT}_i = \langle T_i, P_i \rangle$, is consistent if and only if $\text{PT}^s = \langle T^s, P^s \rangle$ is consistent. To some degree, we rely on the proofs of soundness and completeness of a reduction from F-$\mathcal{ALCI}$ to $\mathcal{ALCI}$, that were presented in [47] (see also [44]).

Recall that a P-$\mathcal{ALCI}$ knowledge base $\langle T, P \rangle$ is consistent if

(i) T is satisfiable and

(ii) there exists an ordered partition $(P_0, ..., P_k)$ of P, such that each P_i with $i \in \{0, 1, ..., k\}$ is the set of all $F \in P \setminus (P_0 \cup \cdots \cup P_{i-1})$ that are tolerated under T by $P \setminus (P_0 \cup \cdots \cup P_{i-1})$.

On the other hand, as defined before, $\mathrm{PT} = \{\langle T_i, P_i \rangle\}_{i \in V}$ is *consistent* if

(i) $T = \{T_i\}_{i \in V}$ is consistent;

(ii) There exist $k_i \geq 0$, $i \in V$, and ordered partitions $(P_0^i, ..., P_{k_i}^i)$ of P_i, such that each P_j^i, $j = 0, ..., k_i$, is the set of all $F \in P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)$ that are tolerated under T by $P \setminus (P_0 \cup \cdots \cup P_{j-1})$, where

$$P \setminus (P_0 \cup \cdots \cup P_{j-1}) = \{P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)\}_{i \in V}.$$

(In case, for some $i \in V$, $j > k_i$, we set $P_j^i = \varnothing$.)

Suppose that $T = \{T_i\}_{i \in V}$ is a F-$\mathcal{ALCI}$ knowledge base, $\mathcal{B} = \{\mathcal{B}_i\}_{i \in V}$ a collection of sets of basic classification concepts, and $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ a collection of sets of conditional constraints. Consider the $\mathcal{ALCI}$ knowledge base T^s, the collection $\mathcal{B}^s = \{\mathcal{B}_i^s\}_{i \in V}$ of sets of basic classification concepts and the set of conditional constraints P^s.

To each world $I = \{I_i\}_{i \in V}$ in $\mathcal{I}_{\mathcal{B}}$ there corresponds a world $I^s = \{I_i^s\}_{i \in V}$, with $I_i^s = \{\#_i(\phi) : \phi \in I_i\}$, in $\mathcal{I}_{\mathcal{B}^s}$. Conversely, to every world $I = \{I_i\}_{i \in V}$ in $\mathcal{I}_{\mathcal{B}^s}$, there corresponds a world $I^d = \{I_i^d\}_{i \in I}$, with $I_i^d = \{\phi \in \mathcal{B}_i : \#_i(\phi) \in I_i\}$, in $\mathcal{I}_{\mathcal{B}}$. Obviously, by definition, $I^{ds} = I$, for every $I \in \mathcal{I}_{\mathcal{B}^s}$, and $I^{sd} = I$, for every $I \in \mathcal{I}_{\mathcal{B}}$.

Let $\mathrm{Pr} : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ be a probabilistic interpretation on $\mathcal{I}_{\mathcal{B}}$. Define $\mathrm{Pr}^s : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ by setting $\mathrm{Pr}^s(I^s) = \mathrm{Pr}(I)$, for all $I \in \mathcal{I}_{\mathcal{B}}$. Clearly, Pr^s is a probabilistic interpretation on $\mathcal{I}_{\mathcal{B}^s}$.

Lemma 1. *If* $\mathrm{Pr} : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ *is a probabilistic model of* $T \cup \mathcal{F}$, *then* $\mathrm{Pr}^s : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ *is a probabilistic model of* $T^s \cup \mathcal{F}^s$. *Moreover, for every basic i-classification concept* $\phi \in \mathcal{B}_i$, *we have* $\mathrm{Pr}_i(\phi) = \mathrm{Pr}^s(\#_i(\phi))$.

Proof. The fact that $I \vDash T$ iff $I^s \vDash T^s$ follows from the Soundness and Completeness Theorem of [45]. Furthermore, we have

$$\begin{aligned}\Pr_i(\phi) &= \sum \{\Pr(I) : I \vDash_i \phi\} \\ &= \sum \{\Pr^s(I^s) : I^s \vDash \#_i(\phi)\} \\ &= \Pr^s(\#_i(\phi)).\end{aligned}$$

This also shows that, if I is a model of $\mathcal{F}$, then I^s is a model of $\mathcal{F}^s$.

Let $\Pr : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ be a probabilistic interpretation on $\mathcal{I}_{\mathcal{B}^s}$. Define $\Pr^d : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ by setting $\Pr^d(I^d) = \Pr(I)$, for all $I \in \mathcal{I}_{\mathcal{B}^s}$. Clearly, $\Pr^d$ is a probabilistic interpretation on $\mathcal{I}_{\mathcal{B}}$.

Lemma 2. *If* $\Pr : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ *is a probabilistic model of* $T^s \cup \mathcal{F}^s$, *then* $\Pr^d : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ *is a probabilistic model of* $T \cup \mathcal{F}$. *Moreover, for every basic i-classification concept* $\phi \in \mathcal{B}_i$, *we have* $\Pr_i^d(\phi) = \Pr(\#_i(\phi))$.

Proof. Very similar to the proof of Lemma 1.

We continue the process of proving the soundness and completeness of the reduction $\mathrm{PT} \to \mathrm{PT}^s$ for FPTCON by showing that the notion of tolerance in the federated case and that in the unimodule case are very tightly related. The following lemma expresses this connection precisely.

Lemma 3. *A collection* $\mathcal{F} = \{\mathcal{F}_i\}_{i\in V}$ *of sets of conditional constraints tolerates an i-conditional constraint* $(\psi \mid \phi)[l, u]$ *under an F-$\mathcal{ALCI}$ knowledge base* $T = \{T_i\}_{i\in V}$ *if and only if* $\mathcal{F}^s$ *tolerates* $(\#_i(\psi) \mid \#_i(\phi))[l, u]$ *under* T^s.

Proof. Suppose, first, that $\mathcal{F} = \{\mathcal{F}_i\}_{i\in V}$ tolerates $(\psi \mid \phi)[l, u]$ under $T = \{T_i\}_{i\in V}$. Then $T \cup \mathcal{F}$ has a model $\Pr : \mathcal{I}_{\mathcal{B}} \to [0, 1]$, that verifies $(\psi \mid \phi)[l, u]$, i.e., such that $\Pr_i(\phi) = 1$ and $\Pr_i(\psi) \in [l, u]$. Let $\mathcal{B}^s = \{\mathcal{B}_i^s\}_{i\in V}$ be the collection of sets of basic classification concepts of PT^s, on which $\mathcal{F}^s$ is based. Then, by Lemma 1, the function $\Pr^s : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$, defined by $\Pr^s(I^s) = \Pr(I)$, is a probabilistic model of $T^s \cup \mathcal{F}^s$. Under this model we have $\Pr^s(\#_i(\phi)) = \Pr_i(\phi) = 1$ and $\Pr^s(\#_i(\psi)) = \Pr_i(\psi) \in [l, u]$. Thus, $\mathcal{F}^s$ tolerates $(\#_i(\psi)|\#_i(\phi))[l, u]$ under T^s.

Suppose, conversely, that $\mathcal{F}^s$ tolerates the conditional constraint of type i $(\#_i(\psi)|\#_i(\phi))[l, u]$ under T^s. Then there exists a probabilistic model $\Pr : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ of $T^s \cup \mathcal{F}^s$, that verifies $(\#_i(\psi)|\#_i(\phi))[l, u]$. Then, by Lemma 2, the function $\Pr^d : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ is a probabilistic model of $T \cup \mathcal{F}$, such that $\Pr_i^d(\phi) = \Pr(\#_i(\phi)) = 1$ and $\Pr_i^d(\psi) = \Pr(\#_i(\psi)) \in [l, u]$. Hence, $\mathcal{F}$ tolerates $(\psi|\phi)[l, u]$ under *T*.

Taking into account the definitions concerning the relevant notions for P-$\mathcal{ALCI}$ and PF-$\mathcal{ALCI}$, to show that PT is consistent if and only if PT^s is consistent, it suffices to show the following:

Lemma 4. *There exist* $k_i \geq 0$, $i \in V$, *and ordered partitions* $(P_0^i, \ldots, P_{k_i}^i)$ *of* P_i, *such that each* P_j^i, $j = 0, \ldots, k_i$, *is the set of all* $F \in P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)$ *that are tolerated under T by* $P \setminus (P_0 \cup \cdots \cup P_{j-1})$, *where*

$$P \setminus (P_0 \cup \cdots \cup P_{j-1}) = \{P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)\}_{i \in V},$$

if and only if there exists an ordered partition $(P_0^s, \ldots, P_k^s)$ *of* P^s, *such that each* P_i^s *with* $i \in \{0, 1, \ldots, k\}$ *is the set of all* $F \in P^s \setminus (P_0^s \cup \cdots \cup P_{i-1}^s)$ *that are tolerated under* T^s *by* $P^s \setminus (P_0^s \cup \cdots \cup P_{i-1}^s)$.

Proof. Suppose, first, that there exist $k_i \geq 0$, $i \in V$, and ordered partitions $(P_0^i, \ldots, P_{k_i}^i)$ of P_i, such that each P_j^i, $j = 0, \ldots, k_i$, is the set of all $F \in P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)$ that are tolerated under T by $P \setminus (P_0 \cup \cdots \cup P_{j-1})$, where $P \setminus (P_0 \cup \cdots \cup P_{j-1}) = \{P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)\}_{i \in V}$. Let $k := \max_{i \in V} k_i$ and set $P_j^s = \bigcup_{i \in V} \{(\#_i(\psi)|\#_i(\phi))[l, u] : (\psi|\phi)[l, u] \in P_j^i\}$, for $j = 0, \ldots, k$, where $P_j^i = \varnothing$, for all $j > k_i$. We must show that each P_i^s, $i \in \{0, 1, ..., k\}$, is the set of all $F \in P^s \setminus (P_0^s \cup \cdots \cup P_{i-1}^s)$ that are tolerated under T^s by $P^s \setminus (P_0^s \cup \cdots \cup P_{i-1}^s)$. This follows from Lemma 3.

Conversely, assume there exists an ordered partition $(P_0^s, ..., P_k^s)$ of P^s, such that each P_i^s with $i \in \{0, 1, ..., k\}$ is the set of all $F \in P^s \setminus (P_0^s \cup \cdots \cup P_{i-1}^s)$ that are tolerated under T^s by $P^s \setminus (P_0^s \cup \cdots \cup P_{i-1}^s)$. Then, set, for all $i \in V$ and all $j = 0, 1, ..., k$,

$$P_j^i = \{(\psi \mid \phi)[l, u] \in P_i : (\#_i(\psi) \mid \#_i(\phi))[l, u] \in P_j^s\}.$$

Moreover, let $k_i = \max\{j : P_j^i \neq \emptyset\}$. Then, by Lemma 3, the ordered partitions $(P_0^i, ..., P_{k_i}^i)$ of P_i are such that each P_j^i, $j = 0, ..., k_i$, is the set of all $F \in P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)$ that are tolerated under T by $P \setminus (P_0 \cup \cdots \cup P_{j-1})$, where $P \setminus (P_0 \cup \cdots \cup P_{j-1}) = \{P_i \setminus (P_0^i \cup \cdots \cup P_{j-1}^i)\}_{i \in V}$.

Lemma 4 immediately yields

Theorem 6 (Soundness and Completeness of $\mathfrak{R}$ for PTCON). *Let* PT $= \{\text{PT}_i\}_{i \in V}$ *be a* PF-$\mathcal{ALCI}$ *knowledge base*. PT *is consistent if and only if* PT^s *is consistent*.

6.2. Soundness and completeness for FTLEXENT

In this subsection, we prove the soundness and completeness of the translation from the federated FTLEXENT problem to the corresponding problem for P-$\mathcal{ALCI}$. More precisely, we show that, given a PTBOX PT $= \{\langle T_i, P_i \rangle\}_{i \in V}$, a finite collection $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ of sets of conditional constraints and two i-classification concepts ϕ and ψ, the set of rational numbers $l, u \in [0, 1] \cap \mathbb{Q}$, that satisfy $\mathcal{F} \Vdash_i^{\text{tlex}} (\psi \mid \phi)[l, u]$ under PT, is the same with that of the rational numbers $l, u \in [0, 1] \cap \mathbb{Q}$, satisfying $\mathcal{F}^s \Vdash_{\text{tight}}^{\text{lex}} (\#_i(\psi) \mid \#_i(\phi))[l, u]$ under PT^s. The fact that these two subsets of $\mathbb{Q}$ coincide shows that the algorithm developed in [34] for solving TLexEnt in the context of P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$, appropriately adjusted to solve

TLEXENT for the modified language P-$\mathcal{ALCI}$, that we presented in Section 4, may be used to also solve an instance of FTLEXENT. It also helps in providing a complexity estimate for the federated problem based on the corresponding complexity for the unimodule problem.

Given a finite collection $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ of sets of conditional constraints and an i-classification concept ϕ, let us define the collection $\mathcal{F}(\phi) = \{\mathcal{F}_i(\phi)\}_{i \in V}$, by

$$\mathcal{F}_j(\phi) = \begin{cases} \mathcal{F}_j, & \text{if } j \neq i, \\ \mathcal{F}_i \cup \{(\phi | \top_i)[1, 1]\}, & \text{if } j = i. \end{cases}$$

Moreover, in the unimodule setting, given a finite collection $\mathcal{F}$ of conditional constraints, of possibly various types, and a classification concept ϕ of type i, let (see also [34]) $\mathcal{F}(\phi) = \mathcal{F} \cup \{(\phi | \top)[1, 1]\}$.

Note that $\mathcal{F}^s(\#_i(\phi)) = \mathcal{F}^s \cup \{(\#_i(\phi) | \top)[1, 1]\}$ whereas $\mathcal{F}(\phi)^s = \mathcal{F}^s \cup \{(\#_i(\phi) | \top_i)[1, 1]\}$. Because of the added stipulation adopted in our semantics that every interpretation of a federated ontology must have nonempty local domains and the fact that the extension of $\#_i(\phi)$ in any interpretation is a subset of the i-th domain, the interpretation of the two conditional constraints appearing in the singleton sets above coincide in any probabilistic model. More formally, in the following technical lemma it is shown that these two sets of conditional constraints have identical probabilistic models.

Lemma 5. *Let* $\mathrm{PT} = \{\mathrm{PT}_i\}_{i \in V}$ *be a* PF-$\mathcal{ALCI}$ *knowledge base,* $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ *be a collection of sets of conditional constraints and* ϕ *be an i-classification concept. A function* $\Pr : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ *is a probabilistic model of* $\mathcal{F}^s(\#_i(\phi))$ *iff it is a probabilistic model of* $\mathcal{F}(\phi)^s$.

Proof. Taking into account the forms of the two sets of conditional constraints, the following string of equalities proves the lemma:

$$\Pr(\#_i(\phi) | \top) = \frac{\Pr(\#_i(\phi) \sqcap \top)}{\Pr(\top)}$$

$$= \Pr(\#_i(\phi))$$

$$= \frac{\Pr(\#_i(\phi) \sqcap \top_i)}{\Pr(\top_i)}$$

$$= \Pr(\#_i(\phi) | \top_i).$$

The third equality follows from the fact that the interpretation of $\#_i(\phi)$ is stipulated to be a subset of the interpretation of $\top_i$ in any model of the translation of PT and from the fact that our models are assumed to have nonempty local domains.

In Lemmas 6 and 7 it is shown that a probabilistic interpretation is a lex-minimal model of a federated knowledge base iff its unimodule counterpart is a lex-minimal model of the reduction of the federated knowledge base. This is the last auxiliary step on the road to proving Theorems 7 and 8, which establish the soundeness and completeness of the given reduction from the problem FTLEXENT for a PF-$\mathcal{ALCI}$ PTBOX to the problem TLEXENT for a P-$\mathcal{ALCI}$ PTBOX.

Lemma 6. *Suppose that* $T = \{T_i\}_{i \in V}$ *is an* F-$\mathcal{ALCI}$ *knowledge base and* $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ *is a collection of finite sets of conditional constraints. If* $\Pr : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ *is a lex-min model of* $T \cup \mathcal{F}$, *then* $\Pr^s : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ *is a lex-min model of* $T^s \cup \mathcal{F}^s$.

Proof. We prove the statement by contraposition. Suppose that $\Pr^s : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ is not a lex-min model of $T^s \cup \mathcal{F}^s$. Thus, there exists a model $\Pr' : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ of $T^s \cup \mathcal{F}^s$ that is lex-preferable to $\Pr^s$. But then, by Lemma 1 and the definitions of lex-preference, $\Pr'^d : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ is a model of $T \cup \mathcal{F}$, which is lex-preferable to $\Pr^{sd} = \Pr$. This shows that $\Pr : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ is not a lex-min model of $T \cup \mathcal{F}$.

Lemma 7. *Suppose that* $T = \{T_i\}_{i \in V}$ *is an* F-$\mathcal{ALCI}$ *knowledge base and* $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ *is a collection of finite sets of conditional constraints. If* $\Pr : \mathcal{I}_{\mathcal{B}^s} \to [0, 1]$ *is a lex-min model of* $T^s \cup \mathcal{F}^s$, *then* $\Pr^d : \mathcal{I}_{\mathcal{B}} \to [0, 1]$ *is a lex-min model of* $T \cup \mathcal{F}$.

Proof. Very similar to the proof of Lemma 6.

Theorem 7. *Suppose that* $\mathrm{PT} = \{\langle T_i, P_i \rangle\}_{i \in V}$ *is a* PF-$\mathcal{ALCI}$ *knowledge base,*

$\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ is a collection of finite sets of conditional constraints and ϕ, ψ are two i-classification concepts. Then $\mathcal{F} \Vdash_i^{\text{lex}} (\psi \mid \phi)[l, u]$ under PT *if and only if $\mathcal{F}^s \Vdash^{\text{lex}} (\#_i(\psi) \mid \#_i(\phi))[l, u]$ under* PT^s.

Proof. Suppose, first, that $\mathcal{F} \Vdash_i^{\text{lex}} (\psi \mid \phi)[l, u]$ under PT. Then $\text{Pr}_i(\psi) \in [l, u]$, for every lex-min model Pr of $T \cup \mathcal{F}(\phi)$. Assume, for the sake of obtaining a contradiction, that $\mathcal{F}^s \nVdash^{\text{lex}} (\#_i(\psi) \mid \#_i(\phi))[l, u]$ under PT^s. Thus, there exists a lex-min model Pr' of $T^s \cup \mathcal{F}(\phi)^s$, such that $\text{Pr}'(\#_i(\psi)) \notin [l, u]$. By Lemmas 2, 5 and 7, Pr'^d is a lex-min model of $T \cup \mathcal{F}(\phi)$, such that $\text{Pr}_i'^d(\psi) = \text{Pr}'^{ds}(\#_i(\psi)) = \text{Pr}'(\#_i(\psi)) \notin [l, u]$. But this is a contradiction.

The proof of the converse statement is very similar, but uses Lemmas 1 and 6 in place of Lemmas 2 and 7, respectively.

Theorem 8. *Let $\text{PT} = \{\langle T_i, P_i \rangle\}_{i \in V}$ be a PF-$\mathcal{ALCI}$ PTBox, $\mathcal{F} = \{\mathcal{F}_i\}_{i \in V}$ be a collection of finite sets of conditional constraints and ϕ, ψ be two i-classification concepts. Then it is the case that $\mathcal{F} \Vdash_i^{\text{tlex}} (\psi \mid \phi)[l, u]$ under* PT *if and only if $\mathcal{F}^s \Vdash_{\text{tight}}^{\text{lex}} (\#_i(\psi) \mid \#_i(\phi))[l, u]$ under* PT^s.

Proof. Obvious from Theorem 7.

6.3. Algorithmic significance and complexities

In this subsection, we examine the significance of the reductions from the federated problems FPTCON and FTLEXENT to the corresponding unimodule problems, that were presented in the previous subsections. In Section 4, it was shown, based on the algorithms provided by Lukasiewicz [34] for the expressive probabilistic description logics P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$, that both PTCON and TLEXENT are decidable for our version of the language P-$\mathcal{ALCI}$. Thus, our reductions show that both problems FPTCON and FTLEXENT for PF-$\mathcal{ALCI}$ are also decidable. In fact, it is not much more difficult, based on Theorems 6 and 8, to show, using Theorems 6.3 and 6.4 of [34], that FPTCON is complete for exponential time and that FTLEXENT is complete for FEXP, the class corresponding to exponential time for problems that output a value. We formally state these results in the next theorem:

Theorem 9. FPTCON *is complete for* EXP *and* FTLEXENT *is complete for* FEXP.

Proof. Since FPTCON and FTLEXENT were reduced in Theorems 6 and 8, respectively, to the corresponding unimodule problems in polynomial time, and, by Theorem 5 (an adaptation of Theorems 6.3 and 6.4 of [34]), these problems are in EXP and FEXP, respectively, we know that FPTCON and FTLEXENT are in EXP and FEXP, respectively. Hardness for both problems is inherited from the fact that deciding the satisfiability of a knowledge base with arbitrary TBoxes in $\mathcal{ALCI}$ is complete for EXP (see [38, 43]).

Summary

In this paper, inspired by the work of Lukasiewicz on expressive probabilistic description logics [34], we have introduced the federated probabilistic description language PF-$\mathcal{ALCI}$. This language is, to the best of our knowledge, the first language presented in the literature that combines three desirable features:

– modularity, so as to support autonomous but interrelated ontology development in the semantic web;

– contextualization of all logical connectives so that meaning depends on the module where a definition is provided;

– probabilistic features that support probabilistic terminological and default terminological reasoning.

Probabilistic treatment of both terminological and assertional knowledge was presented for the description logics P-$\mathcal{SHIF}(\mathbf{D})$ and P-$\mathcal{SHOIN}(\mathbf{D})$ in [34]. On the other hand, a federated description logic F-$\mathcal{ALCI}$, based on $\mathcal{ALCI}$, supporting contextual connectives, was introduced in [47]. Since in the present work, we make a first attempt at integrating all these features into a single language, we opted to keep the language rather simple. Instead of a more expressive description logic, we based our language on $\mathcal{ALCI}$ and chose to deal only with TBoxes and probabilistic terminological and terminological default knowledge rather than incorporating also ABoxes and assertional probabilistic statements.

These extensions, which are very desirable for obvious reasons, are left as goals for future work.

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