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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: REFERENTIAL π-INSTITUTIONS*

To **Don Pigozzi** this work is dedicated
on the occasion of his 80th Birthday.

Abstract

Wójcicki introduced in the late 1970s the concept of a referential semantics for propositional logics. Referential semantics incorporate features of the Kripke possible world semantics for modal logics into the realm of algebraic and matrix semantics of arbitrary sentential logics. A well-known theorem of Wójcicki asserts that a logic has a referential semantics if and only if it is selfextensional. Referential semantics was subsequently studied in detail by Malinowski and the concept of selfextensionality has played, more recently, an important role in the field of abstract algebraic logic in connection with the operator approach to algebraizability. We introduce and review some of the basic definitions and results pertaining to the referential semantics of π-institutions, abstracting corresponding results from the realm of propositional logics.

Keywords: Referential Logics, Selfextensional Logics, Leibniz operator, Tarski operator, Suszko operator, π-institutions.

1. Introduction

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a logical signature/algebraic type, i.e., a set of logical connectives/operation symbols Λ with attached finite arities given by the

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function $\rho : \Lambda \to \omega$. Let, also, V be a countably infinite set of propositional variables and T a set of reference/base points. Wójcicki [20] defines a **referential algebra $\boldsymbol{A}$** to be an $\mathcal{L}$-algebra with universe $A \subseteq \{0,1\}^T$. Such an algebra determines the consequence operation $C^{\boldsymbol{A}}$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ by setting, for all $X \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\alpha \in C^{\boldsymbol{A}}(X)$ iff, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A}$ and all $t \in T$,

$$h(\beta)(t) = 1, \text{ for all } \beta \in X, \quad \text{implies} \quad h(\alpha)(t) = 1.$$

Moreover, Wójcicki calls a propositional logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where $C = C^{\boldsymbol{A}}$, for a referential algebra $\boldsymbol{A}$, a **referential** (or **referentially truth-functional**) **propositional logic**.

Wójcicki shows in [20] that, given a class K of referential algebras, there exists a single referential algebra $\boldsymbol{A}$, such that $C^{\mathsf{K}} := \bigcap_{\boldsymbol{K} \in \mathsf{K}} C^{\boldsymbol{K}} = C^{\boldsymbol{A}}$. Thence follows that a propositional logic is referential if and only if it is defined by a class of referential algebras.

Given a propositional logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$, the **Frege** or **interderivability relation** of $\mathcal{S}$, denoted $\Lambda(\mathcal{S})$, is the equivalence relation on $\mathrm{Fm}_{\mathcal{L}}(V)$, defined, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\langle \alpha, \beta \rangle \in \Lambda(\mathcal{S}) \quad \text{iff} \quad C(\alpha) = C(\beta).$$

The **Tarski congruence** $\widetilde{\Omega}(\mathcal{S})$ of $\mathcal{S}$ (see [8]) is the largest congruence relation on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with all theories of $\mathcal{S}$. The Tarski congruence is a special case of the **Suszko congruence** $\widetilde{\Omega}^{\mathcal{S}}(T)$ associated with a given theory T of $\mathcal{S}$, which is defined as the largest congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with all theories of $\mathcal{S}$ that contain the given theory T (see [3]). In fact, by definition, $\widetilde{\Omega}(\mathcal{S}) = \widetilde{\Omega}^{\mathcal{S}}(C(\varnothing))$, i.e., the Tarski congruence of $\mathcal{S}$ is the Suszko congruence associated with the set of theorems of the logic $\mathcal{S}$. Font and Jansana [8], extending Blok and Pigozzi's [1] well-known characterization of the *Leibniz congruence* $\Omega(T)$ associated with a theory T of a sentential logic, have shown that, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\langle \alpha, \beta \rangle \in \widetilde{\Omega}(\mathcal{S}) \quad \text{iff} \quad \begin{array}{l} \text{for all } \varphi(p, \vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \\ C(\varphi(\alpha, \vec{q})) = C(\varphi(\beta, \vec{q})). \end{array}$$

Whereas $\widetilde{\Omega}(\mathcal{S}) \subseteq \Lambda(\mathcal{S})$, for every propositional logic $\mathcal{S}$, the reverse inclusion does not hold in general. A propositional logic is called **selfextensional** in [20] if $\Lambda(\mathcal{S}) \subseteq \widetilde{\Omega}(\mathcal{S})$. In fact, Wójcicki shows in what has become a fundamental theorem in the theory of referential semantics, Theorem 2 of [20], that a propositional logic is referential if and only if it is selfextensional.

Several authors, inspired by this pioneering work, have subsequently contributed to the development of referential semantics. Various related variants and extensions of this type of semantics have also been introduced. We provide below a very brief survey, as well as an overview of some influences exerted by this work on the field of *abstract algebraic logic*.

Wójcicki in [21], reasserting the importance of referentiality, revisited the equivalence between referentiality and selfextensionality, proving a "weak version" by replacing the entirety of theories (equivalently, the closure operator C) by the set of theorems. Malinowski [13] uses a counterexample crafted by Dziobiak to show that not every propositional logic admits a matrix semantics consisting only of matrices with one-element filters. According to Malinowski [13], Wójcicki conjectured a characterization of such logics at the Autumn School on Strongly Finite Sentential Calculi held in Międzygórze in 1977, to the effect that this happens for a logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ iff, for all $X \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, if $\alpha \in C(X)$, then $C(X) = \{\beta \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \alpha, \beta \rangle \in \widetilde{\Omega}^{\mathcal{S}}(C(X))\}$. This means that the theory generated by a set X of formulas coincides with a class of its Suszko congruence. Besides this interesting early connection between referentiality and the seeds of abstract algebraic logic [2], this result was shown to have some important ramifications in the study of the algebraic semantics of referential logics in [16].

Malinowski [14], in a paper dedicated to Wójcicki, introduced the so-called *pseudo-referential matrix semantics* for propositional logics in an attempt to maintain some of the desirable features of referentiality, but also to amend the limitation of its applicability as a strongly adequate semantics exclusively to selfextensional logics. He showed that all propositional logics have a strongly adequate pseudo-referential semantics. His work was continued by Marek [17], who constructed, for any generalized matrix an equivalent *discrete* pseudo-referential matrix, thus, proving that every sentential logic has a strongly adequate discrete pseudo-referential matrix.

Referential semantics has given rise to two other interesting extensions: First, Tokarz [19] introduced and studied *pragmatic matrix semantics*, in which the universe of the underlying algebras of the pragmatic matrices consists of functions from situations to situations, with some situations singled out as facts and deciding the filter of the matrix. Second, Malinowski [15] introduced and studied *many-valued referential semantics*, influenced both by ordinary referential semantics, but, also fusing elements from both modal logic and many-valued logic techniques. For an expert view of these developments, we refer the reader to Malinowski's more recent work [16]. It is also worth mentioning that Chapter 5 of Wójcicki's monograph [22] is devoted to an exposition of referential semantics of propositional logics.

Related to this work, but in a rather different direction, selfextensionality itself, together with its equivalence with referentiality, have given rise to a substantial body of work in the field of abstract algebraic logic.

This line of work was initiated by Pigozzi [18]. During the years straddling the transition to the new millennium, two groups worked tidally, independently, but also in close contact and with mutual cross-fertilization, to advance these ideas. On the one hand, Czelakowski and Pigozzi developed the theory of *Fregean logics*, obtaining both general [6] and more specialized results, pertaining, for example, to the multi-term deduction-detachment theorem [7] and the amalgamation property [5, 4]. On the other hand, Font and Jansana, starting with their seminal monograph on the "general algebraic semantics of sentential logics" [8], paved the way for Jansana's subsequent extensive study of *selfextensionality* in [10, 11], a "brief survey" of which can be found in [9]. In the categorical context, the author followed closely the developments of both groups with work presented in [24] (see, also, the unpublished [27, 28]). In closing this summary, one should mention the more recent work on relating referentiality with the theory of *duality*, both in the universal algebraic [12] and the categorical [25] framework.

In the present work, we return to the nascent steps of the study of referentiality and introduce some basic concepts and ideas in the *categorical abstract algebraic logic* framework of logics formalized as π-institutions, paralleling the pioneering work of Wójcicki and the Polish School.

2. π-Institutions and Closure Systems

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a **Set**-valued functor. The **clone of all natural transformations on** SEN is the category U with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition of $\langle \tau_i : i < \beta \rangle : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ with $\langle \sigma_j : j < \gamma \rangle : \mathrm{SEN}^\beta \to \mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma\rangle \circ \langle \tau_i : i < \beta\rangle = \langle \sigma_j(\langle \tau_i : i < \beta\rangle) : j < \gamma\rangle.$$

A subcategory of this category with objects all objects of the form SEN^k : $k < \omega$, and such that:

- it contains all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}$ given by
$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$
- for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, $\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N,

is referred to as a **category of natural transformations on** SEN.

Consider an **algebraic system** $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, i.e., a triple consisting of

- a category **Sign**, called the **category of signatures**;
- a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, called the **sentence functor**;
- a category of natural transformations N on SEN.

A π-**institution based on** $\boldsymbol{F}$ is a pair $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$, where $C = \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a **closure system on** SEN, i.e., a $|\mathbf{Sign}|$-indexed collection of closure operators $C_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$, such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$,

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\Phi)).$$

This condition is sometimes referred to as **structurality**. In this context, $\boldsymbol{F}$ is also referred to as the **base algebraic system**. Given a π-institution $\mathcal{I}$, a **theory family** $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a $|\mathbf{Sign}|$-indexed collection of subsets $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, closed under C_Σ, i.e., such that $C_\Sigma(T_\Sigma) = T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$. The collection of all theory families of $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$ and it is well-known that, ordered by signature-wise inclusion, it forms a complete lattice $\mathbf{ThFam}(\mathcal{I})$.

Note, also, that, given a base algebraic system $\boldsymbol{F}$, the collection of all closure systems based on $\boldsymbol{F}$ is closed under signature-wise intersections and, hence, forms a complete lattice under the signature-wise ordering $\leq$:

$$C^1 \leq C^2 \quad \text{iff} \quad \text{for all } \Sigma \in |\mathbf{Sign}| \text{ and all } \Phi \subseteq \mathrm{SEN}(\Sigma), \; C^1_\Sigma(\Phi) \subseteq C^2_\Sigma(\Phi).$$

3. Referential π-Institutions: Algebraic Systems

We assume a base algebraic system $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Consider also an N-algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, i.e., one such that there exists a surjective functor $' : N \to N'$, preserving all projection natural transformations and, as a consequence, all arities of the natural transformations involved. We denote by $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ the natural transformation in N' that is the image of $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N under $'$.

More specifically, we want to focus on N-algebraic systems $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}'_s, N'\rangle$, where SEN'_s is a simple subfunctor (having the same domain) of the inverse powerset functor $\overleftarrow{\mathcal{P}}\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ of a contravariant functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}^{\mathsf{op}}$. For $\Sigma \in |\mathbf{Sign}'|$, the elements of $\mathrm{SEN}'(\Sigma)$ in this context are referred to as **Σ-reference** or **Σ-base points** (see, e.g., [25]). An N-morphism $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$ will be viewed as a valuation of sentences of SEN in the following way: For all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\varphi \in \mathrm{SEN}(\Sigma)$ is **true at** $p \in \mathrm{SEN}'(F(\Sigma))$ **under** $\langle F, \alpha\rangle$ iff $p \in \alpha_\Sigma(\varphi)$.

An N-algebraic system of this special form is called a **referential N-algebraic system**. By slightly abusing terminology, we use the same term to refer to an (**interpreted**) **referential N-algebraic system**, which is a pair $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, with $\langle F, \alpha\rangle : \boldsymbol{F} \to \boldsymbol{A}$ an N-morphism. We sometimes drop the subscript $_s$ when referring to the subfunctor to make notation less cumbersome, provided that this is unlikely to cause any confusion.

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ an interpreted referential N-algebraic system. Then $\mathcal{A}$ determines a closure system $C^{\mathcal{A}}$ on SEN (or on $\boldsymbol{F}$) according to the following definition:

For all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $\varphi \in C^{\mathcal{A}}_\Sigma(\Phi)$ iff, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$$

(ϕ and φ, here, are intentionally different).

Proposition 1. *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ an interpreted referential N-algebraic system. Then $C^{\mathcal{A}}$ is a closure system on* SEN.

Proof: One needs to check that, for all $\Sigma \in |\mathbf{Sign}|$, $C^{\mathcal{A}}_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$ is a closure operator and, also, that the structurality condition holds. Reflexivity of $C^{\mathcal{A}}_{\Sigma}$ is obvious. Monotonicity follows from the fact that, if $\Phi \subseteq \Psi \subseteq \mathrm{SEN}(\Sigma)$, then $\bigcap_{\psi \in \Psi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \subseteq \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$. For transitivity, to see that $C^{\mathcal{A}}_{\Sigma}(C^{\mathcal{A}}_{\Sigma}(\Phi)) \subseteq C^{\mathcal{A}}_{\Sigma}(\Phi)$, note that the following hold, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\varphi \in C^{\mathcal{A}}_{\Sigma}(C^{\mathcal{A}}_{\Sigma}(\Phi))$: $\bigcap_{\psi \in C^{\mathcal{A}}_{\Sigma}(\Phi)} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$ and, also, for all $\psi \in C^{\mathcal{A}}_{\Sigma}(\Phi)$, $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, whence

$$\begin{array}{rcl} \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & \subseteq & \bigcap_{\psi \in C^{\mathcal{A}}_{\Sigma}} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \\ & \subseteq & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)). \end{array}$$

This proves $\varphi \in C^{\mathcal{A}}_{\Sigma}(\Phi)$. Finally, for structurality we must show that for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$, $\mathrm{SEN}(f)(C^{\mathcal{A}}_{\Sigma_1}(\Phi)) \subseteq C^{\mathcal{A}}_{\Sigma_2}(\mathrm{SEN}(f)(\Phi))$. Suppose $\varphi' \in \mathrm{SEN}(f)(C^{\mathcal{A}}_{\Sigma_1}(\Phi))$. Then, there exists $\varphi \in C^{\mathcal{A}}_{\Sigma_1}(\Phi)$, such that $\varphi' = \mathrm{SEN}(f)(\varphi)$. Thus, for all $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$, $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(g)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(g)(\varphi))$.

$$\begin{tikzcd} \Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "g'"] \\ & \Sigma' & \end{tikzcd}$$

But then, for all $g' \in \mathbf{Sign}(\Sigma_2, \Sigma')$, we get $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(g')\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(g')\mathrm{SEN}(f)(\varphi))$ or

$$\bigcap_{\phi' \in \mathrm{SEN}(f)(\Phi)} \alpha_{\Sigma'}(\mathrm{SEN}(g')(\phi')) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(g')(\varphi')).$$

This proves that $\varphi' \in C^{\mathcal{A}}_{\Sigma_2}(\mathrm{SEN}(f)(\Phi))$. □

Since $C^{\mathcal{A}}$ is a closure system on $\boldsymbol{F}$, the pair $\mathcal{I}^{\mathcal{A}} = \langle \boldsymbol{F}, C^{\mathcal{A}} \rangle$ is a π-institution. We call an institution having this form a **referential π-institution**. Such π-institutions correspond in the theory of categorical abstract algebraic logic to the referential propositional logics of Wójcicki [20].

4. Referentiality via Classes of Algebraic Systems

Consider, again, a base algebraic system $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$.

Given categories $\mathbf{Sign}'^{i}$, $i \in I$, we denote by $\prod_{i \in I} \mathbf{Sign}'^{i}$ the product category of the categories $\{\mathbf{Sign}'^{i} : i \in I\}$.

Suppose $\mathrm{SEN}^i : \mathbf{Sign}'^i \to \mathbf{Set}^{\mathsf{op}}$, $i \in I$, is a collection of functors and $\mathrm{SEN}'^i : \mathbf{Sign}'^i \to \mathbf{Set}$ corresponding simple subfunctors of $\overleftarrow{\mathcal{P}}\mathrm{SEN}^i : \mathbf{Sign}'^i \to \mathbf{Set}$, for all $i \in I$. (Here, we use SEN'^i instead of SEN^i_s to distinguish the two functors.) Consider the functor $\bigsqcup \mathrm{SEN}^i : \prod_{i\in I} \mathbf{Sign}'^i \to \mathbf{Set}^{\mathsf{op}}$, defined, for all $i \in I$ and all $\Sigma_i \in |\mathbf{Sign}'^i|$, by

$$\bigsqcup \mathrm{SEN}^i(\langle \Sigma_i : i \in I\rangle) = \biguplus_{i\in I} \mathrm{SEN}^i(\Sigma_i),$$

where $\biguplus_{i\in I} \mathrm{SEN}^i(\Sigma_i)$ denotes the disjoint union of sets, and, for all $i \in I$, $\Sigma^1_i, \Sigma^2_i \in |\mathbf{Sign}'^i|$ and $f_i \in \mathbf{Sign}'^i(\Sigma^1_i, \Sigma^2_i)$, $\varphi \in \bigsqcup \mathrm{SEN}^i(\langle \Sigma^1_i : i \in I\rangle)$,

$$\bigsqcup \mathrm{SEN}^i(\langle f_i : i \in I\rangle)(\varphi) = \mathrm{SEN}^i(f_i)(\varphi), \quad \text{if } \varphi \in \mathrm{SEN}^i(\Sigma^1_i).$$

Let $\mathrm{SEN}' : \prod_{i\in I} \mathbf{Sign}'^i \to \mathbf{Set}$ denote the simple subfunctor of $\overleftarrow{\mathcal{P}} \bigsqcup \mathrm{SEN}^i : \prod_{i\in I} \mathbf{Sign}'^i \to \mathbf{Set}$ specified by setting, for all $i \in I$ and all $\Sigma_i \in |\mathbf{Sign}'^i|$,

$$\begin{array}{rcl}\mathrm{SEN}'(\langle \Sigma_i : i \in I\rangle) & = & \{\Phi \subseteq \bigsqcup \mathrm{SEN}^i(\langle \Sigma_i : i \in I\rangle) : \\ & & \qquad \Phi \cap \mathrm{SEN}^i(\Sigma_i) \in \mathrm{SEN}'^i(\Sigma_i), i \in I\}.\end{array}$$

Proposition 2. *The definition of* $\mathrm{SEN}' : \prod_{i\in I} \mathbf{Sign}'^i \to \mathbf{Set}$ *is sound, i.e., for all* $f_i : \Sigma^1_i \to \Sigma^2_i, i \in I$, *and all* $\Phi \in \mathrm{SEN}'(\langle \Sigma^1_i : i \in I\rangle)$,

$$\bigsqcup \mathrm{SEN}^i(\langle f_i : i \in I\rangle)(\Phi) \in \mathrm{SEN}'(\langle \Sigma^2_i : i \in I\rangle).$$

Proof: By definition,

$$\bigsqcup \mathrm{SEN}^i(\langle f_i : i \in I\rangle)(\Phi) = \biguplus_{i\in I} \mathrm{SEN}^i(f_i)^{-1}(\Phi \cap \mathrm{SEN}^i(\Sigma^1_i)).$$

Since $\Phi\cap\mathrm{SEN}^i(\Sigma^1_i) \in \mathrm{SEN}^i(\Sigma^1_i), i \in I$, we have $\mathrm{SEN}^i(f_i)^{-1}(\Phi\cap\mathrm{SEN}^i(\Sigma^1_i)) \in \mathrm{SEN}'^i(\Sigma^2_i)$. Now, noting that $\mathrm{SEN}'(\langle f_i : i \in I\rangle)(\Phi)\cap\mathrm{SEN}^i(\Sigma^2_i) \in \mathrm{SEN}'^i(\Sigma^2_i)$, we conclude that $\mathrm{SEN}'(\langle f_i : i \in I\rangle)(\Phi) \in \mathrm{SEN}'(\langle \Sigma^2_i : i \in I\rangle)$, as required. □

Corollary 3. *For all* $i \in I$, *all* $f_i : \Sigma^1_i \to \Sigma^2_i$ *and all* $\Phi \in \mathrm{SEN}'(\langle \Sigma^1_i : i \in I\rangle)$,

$$\mathrm{SEN}'^i(f_i)(\Phi \cap \mathrm{SEN}^i(\Sigma^1_i)) = \mathrm{SEN}'(\langle f_i : i \in I\rangle)(\Phi) \cap \mathrm{SEN}^i(\Sigma^2_i).$$

Proof: By the proof of Proposition 2. □

Next, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, let $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ be defined, for all $\Sigma_i \in |\mathbf{Sign}'^i|, i \in I$, and all $\Phi_0, \dots, \Phi_{k-1} \in \mathrm{SEN}'^k(\langle \Sigma_i : i \in I\rangle)$, by

$$\sigma'_{\langle\Sigma_i:i\in I\rangle}(\Phi_0,\dots,\Phi_{k-1}) = \biguplus_{i\in I}\sigma'^i_{\Sigma_i}(\Phi_0\cap\mathrm{SEN}^i(\Sigma_i),\dots,\Phi_{k-1}\cap\mathrm{SEN}^i(\Sigma_i)).$$

We show that this is a bona fide natural transformation: Consider $f_i:\Sigma^1_i\to\Sigma^2_i$ in $\mathbf{Sign}'^i$, $i\in I$, and let $\Phi_0,\dots,\Phi_{k-1}\in\mathrm{SEN}'^k(\langle\Sigma^1_i:i\in I\rangle)$. Then:

$$\begin{tikzcd}
\mathrm{SEN}'^k(\langle\Sigma^1_i:i\in I\rangle) \arrow[r, "\sigma'_{\langle\Sigma^1_i:i\in I\rangle}"] \arrow[d, "\mathrm{SEN}'^k(\langle f_i:i\in I\rangle)"'] & \mathrm{SEN}'(\langle\Sigma^1_i:i\in I\rangle) \arrow[d, "\mathrm{SEN}'(\langle f_i:i\in I\rangle)"] \\
\mathrm{SEN}'^k(\langle\Sigma^2_i:i\in I\rangle) \arrow[r, "\sigma'_{\langle\Sigma^2_i:i\in I\rangle}"'] & \mathrm{SEN}'(\langle\Sigma^2_i:i\in I\rangle)
\end{tikzcd}$$

$$\begin{aligned}
&\mathrm{SEN}'(\langle f_i:i\in I\rangle)(\sigma'_{\langle\Sigma^1_i:i\in I\rangle}(\Phi_0,\dots,\Phi_{k-1}))\\
&= \mathrm{SEN}'(\langle f_i:i\in I\rangle)(\uplus_{i\in I}\sigma'^i_{\Sigma^1_i}(\Phi_0\cap\mathrm{SEN}^i(\Sigma^1_i),\dots,\Phi_{k-1}\cap\mathrm{SEN}^i(\Sigma^1_i)))\\
&= \uplus_{i\in I}\mathrm{SEN}'^i(f_i)(\sigma'^i_{\Sigma^1_i}(\Phi_0\cap\mathrm{SEN}^i(\Sigma^1_i),\dots,\Phi_{k-1}\cap\mathrm{SEN}^i(\Sigma^1_i)))\\
&= \uplus_{i\in I}\sigma'^i_{\Sigma^2_i}(\mathrm{SEN}'^i(f_i)(\Phi_0\cap\mathrm{SEN}^i(\Sigma^1_i)),\dots,\mathrm{SEN}'^i(f_i)(\Phi_{k-1}\cap\mathrm{SEN}^i(\Sigma^1_i)))\\
&= \uplus_{i\in I}\sigma'^i_{\Sigma^2_i}(\mathrm{SEN}'(\langle f_i:i\in I\rangle)(\Phi_0)\cap\mathrm{SEN}^i(\Sigma^2_i),\dots,\\
&\qquad\qquad \mathrm{SEN}'(\langle f_i:i\in I\rangle)(\Phi_{k-1})\cap\mathrm{SEN}^i(\Sigma^2_i))\\
&= \sigma'_{\langle\Sigma^2_i:i\in I\rangle}(\mathrm{SEN}'(\langle f_i:i\in I\rangle)(\Phi_0),\dots,\mathrm{SEN}'(\langle f_i:i\in I\rangle)(\Phi_{k-1})).
\end{aligned}$$

Let N' be the category of natural transformations on SEN' determined by these natural transformations. The triple

$$\boldsymbol{A}' = \langle\prod_{i\in I}\mathbf{Sign}'^i,\mathrm{SEN}',N'\rangle$$

is a referential N-algebraic system, called the **pasting** of the referential N-algebraic systems $\boldsymbol{A}'^i=\langle\mathbf{Sign}'^i,\mathrm{SEN}'^i,N'^i\rangle$, $i\in I$.

This construction may be further extended to interpreted referential N-algebraic systems. Suppose, maintaining the notation used previously, that $\mathcal{A}'^i=\langle\boldsymbol{A}'^i,\langle F^i,\alpha^i\rangle\rangle$, $i\in I$, is a collection of interpreted referential N-algebraic systems, with $\boldsymbol{A}'^i=\langle\mathbf{Sign}'^i,\mathrm{SEN}'^i,N'^i\rangle, i\in I$. Then, we construct the pasting $\boldsymbol{A}'=\langle\prod_{i\in I}\mathbf{Sign}'^i,\mathrm{SEN}',N'\rangle$, as above, and define the pair $\langle F,\alpha\rangle$ as follows:

- $F:\mathbf{Sign}\to\prod_{i\in I}\mathbf{Sign}'^i$ is given by $F(\Sigma)=\langle F^i(\Sigma):i\in I\rangle$, for all $\Sigma\in|\mathbf{Sign}|$, and $F(f)=\langle F^i(f):i\in I\rangle$, for all $f\in\mathbf{Sign}(\Sigma_1,\Sigma_2)$.

- $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$ is the natural transformation, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(\langle F^i(\Sigma) : i \in I\rangle)$ is determined, for all $\varphi \in \mathrm{SEN}(\Sigma)$, by

$$\alpha_\Sigma(\varphi) = \biguplus_{i\in I} \alpha^i_\Sigma(\varphi), \text{ for all } \varphi \in \mathrm{SEN}(\Sigma).$$

We check that the latter is a natural transformation. Let $f : \Sigma_1 \to \Sigma_2$ be a morphism in **Sign** and $\varphi \in \mathrm{SEN}(\Sigma_1)$. Then

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma_1) \arrow[r, "\alpha_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'(\langle F^i(\Sigma_1) : i \in I\rangle) \arrow[d, "\mathrm{SEN}'(\langle F^i(f) : i \in I\rangle)"] \\
\mathrm{SEN}(\Sigma_2) \arrow[r, "\alpha_{\Sigma_2}"'] & \mathrm{SEN}'(\langle F^i(\Sigma_2) : i \in I\rangle)
\end{tikzcd}$$

$$\begin{array}{rcl}
\alpha_{\Sigma_2}(\mathrm{SEN}(f)(\varphi)) & = & \biguplus_{i\in I} \alpha^i_{\Sigma_2}(\mathrm{SEN}(f)(\varphi)) \\
& = & \biguplus_{i\in I} \mathrm{SEN}(F^i(f))(\alpha^i_{\Sigma_1}(\varphi)) \\
& = & \mathrm{SEN}'(\langle F^i(f) : i \in I\rangle)(\biguplus_{i\in I} \alpha^i_{\Sigma_1}(\varphi)) \\
& = & \mathrm{SEN}'(\langle F^i(f) : i \in I\rangle)(\alpha_{\Sigma_1}(\varphi)).
\end{array}$$

We also call the pair $\mathcal{A}' = \langle \boldsymbol{A}', \langle F, \alpha\rangle\rangle$ the **pasting referential N-algebraic system** of the collection of interpreted referential N-algebraic systems $\mathcal{A}'^i = \langle \boldsymbol{A}'^i, \langle F^i, \alpha^i\rangle\rangle, i \in I$.

We have now paved the way for proving an analog of Theorem 1 of [20]. In the proof of Theorem 4, we maintain the notation introduced in the preceding definitions.

Theorem 4. *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base N-algebraic system. Let $\mathsf{A} = \{\mathcal{A}'^i = \langle \boldsymbol{A}'^i, \langle F^i, \alpha^i\rangle\rangle : i \in I\}$, be a collection of interpreted referential N-algebraic systems and $\mathcal{A}' = \langle \boldsymbol{A}', \langle F, \alpha\rangle\rangle$ their pasting. Then $C^{\mathcal{A}'} = C^{\mathsf{A}}$.*

Proof: Suppose, first, that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C^{\mathsf{A}}_\Sigma(\Phi)$. This happens iff, for all $i \in I$, $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\bigcap_{\phi\in\Phi} \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)).$$

Suppose, next, that, for some $p \in \mathrm{SEN}'(F(\Sigma'))$, $p \in \bigcap_{\phi\in\Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Thus, keeping in mind the disjointness of the unions in the construction of the pasting, there exists $i \in I$, such that $p \in \bigcap_{\phi \in \Phi} \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Then, by hypothesis, $p \in \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$. Therefore, $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$, i.e., $\varphi \in C^{\mathcal{A}'}_{\Sigma}(\Phi)$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C^{\mathcal{A}'}_{\Sigma}(\Phi)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)).$$

Next, let $i \in I$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $p \in \mathrm{SEN}'^i(F^i(\Sigma'))$, such that $p \in \bigcap_{\phi \in \Phi} \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Then, since, by definition, $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = \biguplus_{i \in I} \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, we have $p \in \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Thus, by hypothesis, $p \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \biguplus_{i \in I} \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$. Since $p \in \mathrm{SEN}'^i(F^i(\Sigma'))$, we conclude that $p \in \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$. We have, therefore, shown that $\bigcap_{\phi \in \Phi} \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$. Since this holds for all $i \in I$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get $\varphi \in C^{\mathsf{A}}_{\Sigma}(\Phi)$. $\square$

We conclude that, even if multiple referential algebraic systems are used in the definition of its closure system, a π-institution retains its referential character, an analog of the corollary following Theorem 1 of [20]:

Corollary 5. *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a base algebraic system. A π-institution $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ is referential iff it is defined by a class $\{\mathcal{A}'^i\}_{i \in I}$ of interpreted referential N-algebraic systems.*

5. Referential π-Institutions: Operators

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ a π-institution based on $\boldsymbol{F}$. We define the **Frege equivalence system** $\Lambda(\mathcal{I})$ of $\mathcal{I}$, also known as the **interderivability equivalence system**, by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \Lambda_{\Sigma}(\mathcal{I}) \quad \text{if and only if} \quad C_{\Sigma}(\varphi) = C_{\Sigma}(\psi).$$

The **Tarski congruence system** $\widetilde{\Omega}(\mathcal{I})$ of $\mathcal{I}$ ([8] for the universal algebraic notion and [26] for its categorical extension) is the largest N-congruence system on SEN that is compatible with every theory family $T \in \mathrm{ThFam}(\mathcal{I})$.

Clearly, it is always the case that $\widetilde{\Omega}(\mathcal{I}) \leq \Lambda(\mathcal{I})$. We call the π-institution $\mathcal{I}$ **self-extensional** if $\Lambda(\mathcal{I}) \leq \widetilde{\Omega}(\mathcal{I})$. In view of the preceding remark, $\mathcal{I}$ is self-extensional if and only if $\Lambda(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$.

We prove, next, a generalization to π-institutions of Wójcicki's Theorem (see Theorem 2 of [20], but, also, Theorem 2.2 of [12] for a complete proof), providing a characterization of referential sentential logics. We split the proof into two lemmas.

Lemma 6. *Every referential π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is self-extensional.*

Proof: Suppose that $\mathcal{I}$ is referential, i.e., that $C = C^{\mathcal{A}}$, for some interpreted referential N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$. Let $\Sigma \in |\mathbf{Sign}|, \varphi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \varphi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I})$. Then, by definition, $C_\Sigma(\varphi) = C_\Sigma(\psi)$, whence, since $C = C^{\mathcal{A}}$, we obtain $C^{\mathcal{A}}_\Sigma(\varphi) = C^{\mathcal{A}}_\Sigma(\psi)$. Thus, by the definition of $C^{\mathcal{A}}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)).$$

Thus, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma'' \in |\mathbf{Sign}|$, all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$, we get

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

$$\begin{aligned}
&\sigma'_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}(gf)(\varphi)), \alpha_{\Sigma''}(\mathrm{SEN}(g)(\vec{\chi})))\\
&\qquad = \sigma'_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}(gf)(\psi)), \alpha_{\Sigma''}(\mathrm{SEN}(g)(\vec{\chi})))\\
\Rightarrow\ &\alpha_{\Sigma''}(\sigma_{\Sigma''}(\mathrm{SEN}(gf)(\varphi), \mathrm{SEN}(g)(\vec{\chi})))\\
&\qquad = \alpha_{\Sigma''}(\sigma_{\Sigma''}(\mathrm{SEN}(gf)(\psi), \mathrm{SEN}(g)(\vec{\chi})))\\
\Rightarrow\ &\alpha_{\Sigma''}(\mathrm{SEN}(g)(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})))\\
&\qquad = \alpha_{\Sigma''}(\mathrm{SEN}(g)(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})))\\
\Rightarrow\ &C^{\mathcal{A}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})) = C^{\mathcal{A}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}))\\
\Rightarrow\ &C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})) = C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}))\\
\Rightarrow\ &\langle \varphi, \psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}),
\end{aligned}$$

where the last implication follows by a well-known characterization of the Tarski congruence system of a π-institution, Theorem 4 of [26]. □

The following lemma proves the converse:

Lemma 7. *Every self-extensional π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is referential.*

Proof: Suppose that $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is selfextensional, i.e., that $\Lambda(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$. We construct the following **canonical referential N-algebraic system** $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ as follows:

- $\mathbf{Sign}' = \mathbf{Sign}$;
- Take as the set of Σ-points the collection of all Σ-theories $\mathrm{Th}_\Sigma(\mathcal{I})$ of $\mathcal{I}$. Thus, for $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}'(\Sigma) = \mathrm{Th}_\Sigma(\mathcal{I})$ and, for $f : \Sigma_1 \to \Sigma_2$ in $\mathbf{Sign}$, $\mathrm{SEN}'(f) : \mathrm{Th}_{\Sigma_2}(\mathcal{I}) \to \mathrm{Th}_{\Sigma_1}(\mathcal{I})$ is given by $\mathrm{SEN}'(f)(T) = \mathrm{SEN}(f)^{-1}(T)$, for all $T \in \mathrm{Th}_{\Sigma_2}(\mathcal{I})$.
- We now define SEN'_s as a subfunctor of $\overleftarrow{\mathcal{P}}\mathrm{SEN}' : \mathcal{P}(\mathrm{SEN}'(\Sigma_1)) \to \mathcal{P}(\mathrm{SEN}'(\Sigma_2))$, but we keep the same symbol SEN' instead of writing SEN'_s. For $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{SEN}(\Sigma)$, let $\mathrm{Th}_\Sigma(\varphi) = \{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : \varphi \in T\}$ and set
$$\mathrm{SEN}'(\Sigma) = \{\mathrm{Th}_\Sigma(\varphi) : \varphi \in \mathrm{SEN}(\Sigma)\}.$$
- Finally, define, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \dots, \varphi_{k-1} \in \mathrm{SEN}(\Sigma)$,
$$\sigma'_\Sigma(\mathrm{Th}_\Sigma(\varphi_0), \dots, \mathrm{Th}_\Sigma(\varphi_{k-1})) = \mathrm{Th}_\Sigma(\sigma_\Sigma(\varphi_0, \dots, \varphi_{k-1})).$$
  Let N' be the category of natural transformations consisting of all the $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$, for $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N.

It is not difficult to see that SEN' is a subfunctor of the inverse powerset functor $\overleftarrow{\mathcal{P}}\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}^{\mathsf{op}}$ defined above. To check this, it suffices to show that for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, and $\varphi \in \mathrm{SEN}(\Sigma_1)$,
$$\mathrm{SEN}'(f)(\mathrm{Th}_{\Sigma_1}(\varphi)) = \mathrm{Th}_{\Sigma_2}(\mathrm{SEN}(f)(\varphi)).$$
In fact, we have
$$\begin{array}{lll} T \in \mathrm{Th}_{\Sigma_2}(\mathrm{SEN}(f)(\varphi)) & \text{iff} & \mathrm{SEN}(f)(\varphi) \in T \\ & \text{iff} & \phi \in \mathrm{SEN}(f)^{-1}(T) \\ & \text{iff} & \mathrm{SEN}(f)^{-1}(T) \in \mathrm{Th}_{\Sigma_1}(\varphi) \\ & \text{iff} & T \in \mathrm{SEN}'(f)(\mathrm{Th}_{\Sigma_1}(\varphi)). \end{array}$$
Moreover, the "natural" transformations are well-defined because of the assumption of selfextensionality of the π-institution $\mathcal{I}$. More precisely, if $\Sigma \in |\mathbf{Sign}|$, $\varphi_0, \dots, \varphi_{k-1}, \psi_0, \dots, \psi_{k-1} \in \mathrm{SEN}(\Sigma)$, such that $\mathrm{Th}_\Sigma(\varphi_i) = \mathrm{Th}_\Sigma(\psi_i)$, $i < k$, then, $\langle \varphi_i, \psi_i \rangle \in \Lambda_\Sigma(\mathcal{I})$, whence $\langle \varphi_i, \psi_i \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$, by selfextensionality. Since the latter is an N-congruence system, it follows that
$$\langle \sigma_\Sigma(\varphi_0, \dots, \varphi_{k-1}), \sigma_\Sigma(\psi_0, \dots, \psi_{k-1}) \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) = \Lambda_\Sigma(\mathcal{I}).$$
Hence, we obtain

$$\begin{array}{rcl}\sigma'_\Sigma(\mathrm{Th}_\Sigma(\varphi_0),\dots,\mathrm{Th}_\Sigma(\varphi_{k-1})) & = & \mathrm{Th}_\Sigma(\sigma_\Sigma(\varphi_0,\dots,\varphi_{k-1}))\\ & = & \mathrm{Th}_\Sigma(\sigma_\Sigma(\psi_0,\dots,\psi_{k-1}))\\ & = & \sigma'_\Sigma(\mathrm{Th}_\Sigma(\psi_0),\dots,\mathrm{Th}_\Sigma(\psi_{k-1})).\end{array}$$

And, moreover, they are indeed natural:

$$\begin{tikzcd} \mathrm{SEN}'^k(\Sigma_1) \arrow[r, "\sigma'_1"] \arrow[d, "\mathrm{SEN}'(f)^k"'] & \mathrm{SEN}'(\Sigma_1) \arrow[d, "\mathrm{SEN}'(f)"] \\ \mathrm{SEN}'^k(\Sigma_2) \arrow[r, "\sigma'_2"'] & \mathrm{SEN}'(\Sigma_2) \end{tikzcd}$$

$$\begin{array}{l}\mathrm{SEN}'(f)(\sigma'_{\Sigma_1}(\mathrm{Th}_{\Sigma_1}(\varphi_0),\dots,\mathrm{Th}_{\Sigma_1}(\varphi_{k-1}))\\ \quad = \mathrm{SEN}'(f)(\mathrm{Th}_{\Sigma_1}(\sigma_{\Sigma_1}(\varphi_0,\dots,\varphi_{k-1})))\\ \quad = \mathrm{Th}_{\Sigma_2}(\mathrm{SEN}(f)(\sigma_{\Sigma_1}(\varphi_0,\dots,\varphi_{k-1})))\\ \quad = \mathrm{Th}_{\Sigma_2}(\sigma_{\Sigma_2}(\mathrm{SEN}(f)(\varphi_0),\dots\mathrm{SEN}(f)(\varphi_{k-1})))\\ \quad = \sigma'_{\Sigma_2}(\mathrm{Th}_{\Sigma_2}(\mathrm{SEN}(f)(\varphi_0)),\dots,\mathrm{Th}_{\Sigma_2}(\mathrm{SEN}(f)(\varphi_{k-1})))\\ \quad = \sigma'_{\Sigma_2}(\mathrm{SEN}'(f)(\mathrm{Th}_{\Sigma_1}(\varphi_0)),\dots,\mathrm{SEN}'(f)(\mathrm{Th}_{\Sigma_1}(\varphi_{k-1}))).\end{array}$$

The next step is to create an interpreted referential N-algebraic system based on $\boldsymbol{A}$ by defining an interpretation $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \boldsymbol{F} \to \boldsymbol{A}$, where $\alpha : \mathrm{SEN} \to \mathrm{SEN}'$ is the natural transformation, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\varphi) = \mathrm{Th}_\Sigma(\varphi).$$

Having laid the groundwork, it is now easy to verify the this is a natural transformation and that it respects all natural transformations in N, i.e., $\langle F, \alpha\rangle$ is an N-morphism.

It only remains to see that $C^{\mathcal{A}} = C$, where $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$. We have, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $\varphi \in C^{\mathcal{A}}_\Sigma(\Phi)$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\bigcap_{\phi\in\Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\bigcap_{\phi\in\Phi} \mathrm{Th}_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \mathrm{Th}_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$ if and only if $\varphi \in C_\Sigma(\Phi)$. □

THEOREM 8. *A π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is referential if and only if it is self-extensional.*

PROOF: The left-to-right implication is given in Lemma 6 and the right-to-left implication in Lemma 7. □

6. Referential π-Institutions: Matrix Systems

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system. Consider an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, where $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$. A **sentence family** of $\mathcal{A}$ (or of $\boldsymbol{A}$ or of SEN') is a $|\mathbf{Sign}'|$-indexed collection $T' = \{T'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$, such that $T'_\Sigma \subseteq \mathrm{SEN}'(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'|$. The pair $\mathfrak{A} = \langle \mathcal{A}, T'\rangle$ is called an N-**matrix system**.

Given an N-matrix system $\mathfrak{A}$, the closure system $C^{\mathfrak{A}}$ on $\boldsymbol{F}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, by $\varphi \in C^{\mathfrak{A}}_\Sigma(\Phi)$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}.$$

Given a π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$, the N-matrix system $\mathfrak{A} = \langle \mathcal{A}, T'\rangle$ is called an $\mathcal{I}$-**matrix system** or a **matrix system model** of $\mathcal{I}$ in case $C \leq C^{\mathfrak{A}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \quad \text{implies} \quad \varphi \in C^{\mathfrak{A}}_\Sigma(\Phi).$$

Using the same notation, given an N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, we may consider a collection $\mathcal{T}' = \{T^i\}_{i\in I}$ of sentence families of $\mathcal{A}$ and, in this way, we form an N-**generalized matrix system**, or N-**gmatrix system** for short, $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$ based on $\mathcal{A}$. The closure system $C^{\mathbb{A}}$ on $\boldsymbol{F}$ defined by this N-gmatrix system is given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, by $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$ iff, for all $T' \in \mathcal{T}'$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}.$$

Note that this is equivalent to saying that

$$\varphi \in C^{\mathbb{A}}_\Sigma(\Phi) \quad \text{iff} \quad \varphi \in C^{\mathfrak{A}}_\Sigma(\Phi), \text{ for all } \mathfrak{A} = \langle \mathcal{A}, T'\rangle, T' \in \mathcal{T}',$$

or, more succinctly, $C^{\mathbb{A}} = \bigcap_{\mathfrak{A}\in\mathbb{A}} C^{\mathfrak{A}}$, where we write $\mathfrak{A} = \langle \mathcal{A}, T'\rangle \in \langle \mathcal{A}, \mathcal{T}'\rangle = \mathbb{A}$ as an alias for $T' \in \mathcal{T}'$ and $\bigcap$ is meant to represent signature-wise intersection. Given a π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ based on $\boldsymbol{F}$, we say that $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$ is a **generalized matrix system model** of $\mathcal{I}$ or an $\mathcal{I}$-**gmatrix system** if $C \leq C^{\mathbb{A}}$. Note that, since $C^{\mathbb{A}}$ is a closure system, the pair $\mathcal{I}^{\mathbb{A}} = \langle \boldsymbol{F}, C^{\mathbb{A}}\rangle$ is a π-institution. If $\mathbb{A}$ is an $\mathcal{I}$-gmatrix system, then $\mathcal{I}^{\mathbb{A}}$ is a *π-institution model* of $\mathcal{I}$, according to the definitions in [23], where those models are studied in some detail.

The following is a version of the well-known Completeness Theorem for sentential logics lifted to the level of π-institutions:

PROPOSITION 9. *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ a π-institution based on $\boldsymbol{F}$. Then, there exists an N-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$, such that $C = C^{\mathbb{A}}$.*

PROOF: Set $\boldsymbol{A} = \boldsymbol{F}$ and $\langle F, \alpha\rangle : \boldsymbol{F} \to \boldsymbol{A}$ be the identity N-morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle$. In addition, set $\mathcal{T} = \mathrm{ThFam}(\mathcal{I})$, the collection of all theory families of the π-institution $\mathcal{I}$. Define $\mathbb{A} = \langle\langle \boldsymbol{F}, \langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle\rangle, \mathrm{ThFam}(\mathcal{I})\rangle$. Then, by definition, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $\varphi \in C^{\mathbb{A}}_{\Sigma}(\Phi)$ if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\Phi) \subseteq T_{\Sigma'}$ implies $\mathrm{SEN}(f)(\varphi) \in T_{\Sigma'}$, which is equivalent to $\varphi \in C_{\Sigma}(\Phi)$. □

We consider, next, the special case in which the N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, is a referential N-algebraic system. We construct the **canonical referential N-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$ associated with $\mathcal{A}$**. This construction abstracts to the categorical level one introduced by Wójcicki in [20] in the context of referential propositional logics.

Let us assume, for simplicity, that sets of base points over different signatures are disjoint. Let $\Sigma \in |\mathbf{Sign}'|$ and let p be a Σ-reference or Σ-base point of $\mathrm{SEN}'(\Sigma)$. We write $p \flat \mathrm{SEN}'(\Sigma)$ to denote that p is a Σ-reference point of SEN'. (This helps ameliorate the overloading of notation adopted when SEN' was supposed to denote both the contravariant base functor and the covariant subfunctor of its inverse powerset functor, which should have been formally denoted by SEN' and SEN'_s, respectively.)

Given $\Sigma^* \in |\mathbf{Sign}'|$ and $p \flat \mathrm{SEN}'(\Sigma^*)$, define the sentence family $T^p = \{T^p_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ of SEN' by setting

$$T^p_\Sigma = \begin{cases} \{\varphi' \in \mathrm{SEN}'(\Sigma) : p \in \varphi'\}, & \text{if } \Sigma = \Sigma^* \\ \mathrm{SEN}(\Sigma), & \text{if } \Sigma \neq \Sigma^* \end{cases}$$

Finally, define

$$\mathcal{T}' = \{T^p : p \flat \mathrm{SEN}'(\Sigma^*), \Sigma^* \in |\mathbf{Sign}'|\},$$

and let the canonical referential N-gmatrix system associated with the referential N-algebraic system $\mathcal{A}$ be the N-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$.

PROPOSITION 10. *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ a referential N-algebraic system, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$.*

Then, $C^{\mathcal{A}} = C^{\mathbb{A}}$, where $\mathbb{A}$ is the canonical referential N-gmatrix system associated with $\mathcal{A}$.

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, we have the following chain of equivalences: $\varphi \in C^{\mathbb{A}}_{\Sigma}(\Phi)$ iff, for all $\Sigma^* \in |\mathbf{Sign}'|$, all $p \flat \mathrm{SEN}'(\Sigma^*)$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T^*_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T^*_{F(\Sigma')},$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $p \flat \mathrm{SEN}'(F(\Sigma'))$,

$$p \in \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \quad \text{implies} \quad p \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)),$$

iff $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$ iff $\varphi \in C^{\mathcal{A}}_{\Sigma}(\Phi)$. □

Corollary 11. *Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ a referential π-institution based on $\boldsymbol{F}$. Then, there exists an N-gmatrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}' \rangle$, such that $C = C^{\mathbb{A}}$.*

Proof: Directly from Proposition 10. □

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