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Categorical abstract algebraic logic: skywatching in semilattice systems*

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Abstract

Font and Moraschini established a bijective correspondence between congruences of semilattices with sectionally finite height and certain special subsets of their universes, called clouds. They provided a characterization of clouds and showed that the correspondence is given by the Leibniz operator of abstract algebraic logic. We extend the bijection to one between congruence systems on the semilattice systems of categorical abstract algebraic logic and what we call cloud families. In this context, the categorical analog of the Leibniz operator plays a similar role. In addition, we show that, even though the exact analogue of the Font–Moraschini condition fails in general, a more complex variant provides an analogous characterization of cloud families.

Keywords: Semilattices, Leibniz operator, Algebraic Systems, Congruence Systems, Rainbows, Clouds, Spectra.

1 Introduction

In universal algebra a *semilattice* is an algebra $\mathbf{A}=\langle A,\cdot\rangle$ of type $\langle 2\rangle$, whose operation is idempotent, commutative and associative. The accompanying partial ordering is defined by $a\leq b$ if and only if $a\cdot b=a$. As, with any partial ordering, the *covering relation* [2], denoted by $\prec$, is defined, for all $a,b\in A$ by

$$\begin{array}{lll} a\prec b & \text{iff} & a<b \text{ and, for all } c\in A,\\ & & a\leq c<b \text{ implies } a=c. \end{array}$$

The *principal down-set* $\downarrow a$ [2] (*generated by* $a\in A$), is defined by

$$\downarrow a=\{c\in A: c\leq a\}.$$

The remainder of this section contains some of the notions and the results presented in [4], which provided the motivation for the developments detailed in this work.

Denoting the subalgebra operator by S, we set $\mathcal{C}(\mathbf{A})=\{\mathbf{L}\in\mathsf{S}(\mathbf{A}):\mathbf{L}\text{ a chain}\}$ and define the **height** of $\mathbf{A}$ by $\mathcal{H}(\mathbf{A})=\max\{|\mathbf{L}|:\mathbf{L}\in\mathcal{C}(\mathbf{A})\}$, when this number is finite, in which case $\mathbf{A}$ is said to be of **finite height**.

A semilattice $\mathbf{A}$ is said to be of **sectionally finite height** if, for all $a\in A$, the principal downset $\downarrow a$ has finite height. The class of all semilattices of sectionally finite height, denoted FSL, does not form a variety because, even though it is closed under subalgebras and homomorphic images, it is (obviously) not closed under products.

Given a semilattice $\mathbf{A}$, the **height** of $a\in A$ is $\mathcal{H}(a)=\mathcal{H}(\downarrow a)$.

*To **Don Pigozzi** this work is dedicated on the occasion of his 80th Birthday.

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Given an algebra $\boldsymbol{A}$ and a subset $F \subseteq A$, the **Leibniz congruence of** F **in** $\boldsymbol{A}$ [1], denoted $\Omega^{\boldsymbol{A}}(F)$, is the largest congruence on $\boldsymbol{A}$ that is compatible with F in the sense that

$$\langle a,b\rangle \in \Omega^{\boldsymbol{A}}(F) \quad \text{and} \quad a \in F \quad \text{imply} \quad b \in F.$$

In Lemma 3.1 of [4], Font and Moraschini characterize Leibniz congruences in a semilattice $\boldsymbol{A} = \langle A, \cdot\rangle$ by showing that, for all $a,b \in A$, $\langle a,b\rangle \in \Omega^{\boldsymbol{A}}(F)$ if and only if,

$$a\cdot c \in F \quad \text{if and only if} \quad b\cdot c \in F, \quad \text{for all } c \in A.$$

By perceiving F as a palette assigning to its own elements a certain color and to its complement in A a different colour, Font and Moraschini [4] view $\Omega^{\boldsymbol{A}}(F)$ as identifying the identically coloured segments of $\boldsymbol{A}$ and conclude that the cardinality of the quotient $\boldsymbol{A}/\Omega^{\boldsymbol{A}}(F)$ counts the number of colour switches in ascending from bottom to top in a finite chain. This chromatic perspective motivates their definition of *rainbow*.

The **rainbow** $\mathcal{R}(\boldsymbol{A})$ of a semilattice $\boldsymbol{A}$ [4] is defined by

$$\mathcal{R}(\boldsymbol{A}) = \{a \in A : \mathcal{H}(a) \text{ is odd}\}.$$

Theorem 3.5 of [4] establishes that in a semilattice of sectionally finite height $\boldsymbol{A}$, $\Omega^{\boldsymbol{A}}(\mathcal{R}(\boldsymbol{A})) = \Delta_{\boldsymbol{A}}$, the identity congruence on $\boldsymbol{A}$.

A subset $F \subseteq A$ in a semilattice of sectionally finite height $\boldsymbol{A}$ is called a **cloud** (Definition 4.1 of [4]) if $F/\Omega^{\boldsymbol{A}}(F) = \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(F))$ and $\mathrm{Cl}(\boldsymbol{A})$ denotes the collection of all clouds of $\boldsymbol{A}$. In Theorem 4.2 of [4] it is shown that $\Omega^{\boldsymbol{A}} : \mathrm{Cl}(\boldsymbol{A}) \to \mathrm{Con}(\boldsymbol{A})$ establishes a bijection between the collection of clouds of a semilattice with sectionally finite height and the collection of its congruences.

To characterize clouds, Font and Moraschini introduce the **height** $\mathcal{H}_F(a)$ of an element a in a semilattice of sectionally finite height A **relative to** a subset $F \subseteq A$ by setting

$$\mathcal{H}_F(a) = \max\{|\boldsymbol{L}/\Omega^{\boldsymbol{L}}(F \cap L)| : \boldsymbol{L} \in \mathcal{C}(\downarrow a)\}.$$

Their characterization in Theorem 4.6 of [4] asserts that $F \in \mathrm{Cl}(\boldsymbol{A})$ if and only if $\bot \in F$ and $\mathcal{H}_F(a) = \mathcal{H}(a/\Omega^{\boldsymbol{A}}(F))$, for every $a \in A$.

In Section 2 of the present article, we start by recalling the definition of an algebraic system, of a sentence family of such a system and by briefly reminding the reader of the concept of the categorical Leibniz congruence system associated with a sentence family of a given algebraic system. We specialize this to semilattice systems and we provide a characterization of the Leibniz congruence system $\Omega^{\boldsymbol{A}}(T)$ associated with the sentence family T in a semilattice system $\boldsymbol{A}$ in a way analogous to the characterization in Lemma 3.1 of [4]. We also define *semilattice systems of sectionally finite height*. In Section 3, we introduce *rainbow systems* and *rainbow families* and show that, given a semilattice system $\boldsymbol{A}$ of sectionally finite height, its rainbow family $\mathcal{R}(\boldsymbol{A})$ satisfies $\Omega^{\boldsymbol{A}}(\mathcal{R}(\boldsymbol{A})) = \Delta^{\mathrm{SEN}}$, an analogue of Theorem 3.5 of [4]. A counterexample shows that this is not generally true for the rainbow system $\mathfrak{R}(\boldsymbol{A})$, which we take as a motivation for focusing on rainbow families rather than on rainbow systems for the remainder of our work. We define, next, in Section 4, a *cloud family* to be any sentence family whose quotient over its Leibniz congruence system coincides with the rainbow family of the quotient of the given semilattice system by the Leibniz congruence system of the sentence family. In Section 5, we introduce a Font–Moraschini type condition to characterize cloud families. A counterexample, however, shows that this condition, despite being necessary, does not suffice for our purposes. Therefore, we introduce in Section 6 the concept of a *spectrum* of an element in a semilattice system of sectionally finite height with respect to a given sentence family and use it to provide a characterization of cloud families in this more general context.

Motivated by the sky-inspired terminology of [4], we call the application of the Font–Moraschini Condition, which is successful in the trivial signature semilattice systems (corresponding to universal algebraic semilattices), *telescopy* and the more powerful method, based on the new condition, *spectroscopy*, which explains also the name *spectrum* for the concept defined in Section 6 to formalize our method in the categorical setting. At the beginning of Section 5, we offer a few more comments motivating this terminology related to astrophysical observation methods.

2 Semilattice systems

An **algebraic system** (see, e.g. [5] or the more recent [6]) is a triple $\boldsymbol{A}=\langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ consisting of:

- A category **Sign** of **signatures**;
- A functor $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ giving, for each signature $\Sigma\in|\mathbf{Sign}|$, the set $\mathrm{SEN}(\Sigma)$ of Σ**-sentences**;
- A category N of **natural transformations on** SEN; its objects are the finite powers $\mathrm{SEN}^k, k\in\omega$, and the arrows $\tau:\mathrm{SEN}^k\to\mathrm{SEN}^\ell$ are ℓ-tuples of natural transformations $\mathrm{SEN}^k\to\mathrm{SEN}$; the category is assumed to include all projection natural transformations and, also, to be 'closed under the formation of tuples', i.e. given a family $\{\tau^i:\mathrm{SEN}^k\to\mathrm{SEN}:i<\ell\}$ in N, the natural transformation $\langle\tau^i:i<\ell\rangle:\mathrm{SEN}^k\to\mathrm{SEN}^\ell$ must also be in N.

A **sentence family** (previously termed an **axiom family** in Categorical Abstract Algebraic Logic (CAAL)) is simply a collection $T=\{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of subsets $T_\Sigma\subseteq\mathrm{SEN}(\Sigma)$, for all $\Sigma\in|\mathbf{Sign}|$. We write $\mathrm{SenFam}(\boldsymbol{A})$ to denote the collection of all sentence families of $\boldsymbol{A}$, when $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$.

Let $\boldsymbol{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, termed the **base algebraic system** (see, e.g. Section 2 of [6]). An algebraic system $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ is called an $N^\flat$**-algebraic system** if there exists a surjective functor $N^\flat\to N$ that preserves all projection natural transformations and, therefore, preserves also the arities of all natural transformations in $N^\flat$. We write σ in N to indicate the image in N of a $\sigma^\flat$ in $N^\flat$ under the functor $N^\flat\to N$. We use similar conventions throughout, writing, e.g., σ' for the image of $\sigma^\flat$ under $N^\flat\to N'$, when N' is the category of natural transformations of an $N^\flat$-algebraic system $\boldsymbol{A}'$. Given two $N^\flat$-algebraic systems $\boldsymbol{A}=\langle\mathbf{Sign}',\mathrm{SEN}',N'\rangle$ and $\boldsymbol{B}=\langle\mathbf{Sign}'',\mathrm{SEN}'',N''\rangle$, an $N^\flat$-**(algebraic system) morphism** $\langle H,\gamma\rangle:\boldsymbol{A}\to\boldsymbol{B}$ consists of

- a functor $H:\mathbf{Sign}'\to\mathbf{Sign}''$ and
- a natural transformation $\gamma:\mathrm{SEN}'\to\mathrm{SEN}''\circ H$, such that, for all $\sigma:\mathrm{SEN}^k\to\mathrm{SEN}$ in $N^\flat$, all $\Sigma\in|\mathbf{Sign}'|$ and all $\varphi_0,\dots,\varphi_{k-1}\in\mathrm{SEN}'(\Sigma)$,

$$\gamma_\Sigma(\sigma'_\Sigma(\varphi_0,\dots,\varphi_{k-1}))=\sigma''_{H(\Sigma)}(\gamma_\Sigma(\varphi_0),\dots,\gamma_\Sigma(\varphi_{k-1})).$$

The $N^\flat$-morphism $\langle H,\gamma\rangle:\boldsymbol{A}\to\boldsymbol{B}$ is said to be **surjective** if both the functor H (on objects and on morphisms) and all components of the natural transformation γ are surjective.

Let $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be an algebraic system and $T\in\mathrm{SenFam}(\boldsymbol{A})$ a sentence family of $\boldsymbol{A}$. A congruence system $\theta\in\mathrm{ConSys}(\boldsymbol{A})$ is **compatible with** T if, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\langle\varphi,\psi\rangle\in\theta_\Sigma\quad\text{and}\quad\varphi\in T_\Sigma\quad\text{imply}\quad\psi\in T_\Sigma.$$

Given a sentence family T of $\boldsymbol{A}$, there always exists a largest congruence system on $\boldsymbol{A}$ that is compatible with T, called the **Leibniz congruence system** of T on $\boldsymbol{A}$ and denoted by $\Omega^{\boldsymbol{A}}(T)$ (see [5]).

The **Leibniz operator** is the map $\Omega^{\boldsymbol{A}}:\mathrm{SenFam}(\boldsymbol{A})\to\mathrm{ConSys}(\boldsymbol{A})$. Observe that, by the definition of compatibility, a congruence system $\theta\in\mathrm{ConSys}(\boldsymbol{A})$ is compatible with a sentence family T of $\boldsymbol{A}$ if and only if, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi\in\mathrm{SEN}(\Sigma)$,

$$\varphi\in T_\Sigma \quad \text{if and only if} \quad \varphi/\theta_\Sigma\in T_\Sigma/\theta_\Sigma. \tag{1}$$

As a particular case, one obtains that $\varphi\in T_\Sigma$ iff $\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)\in T_\Sigma/\Omega^{\boldsymbol{A}}_\Sigma(T)$.

Let $\langle H,\gamma\rangle:\boldsymbol{A}\to\boldsymbol{B}$ be a surjective $N^\flat$-morphism. Then it is well-known (see, e.g. Lemma 5.4 of [5]) that, for all $T'\in\mathrm{SenFam}(\boldsymbol{B})$,

$$\Omega^{\boldsymbol{A}}(\gamma^{-1}(T'))=\gamma^{-1}(\Omega^{\boldsymbol{B}}(T')). \tag{2}$$

Moreover, if $\theta\in\mathrm{ConSys}(\boldsymbol{A})$, such that $\theta\le\Omega^{\boldsymbol{A}}(T)$, then

$$\Omega^{\boldsymbol{A}/\theta}(T/\theta)=\Omega^{\boldsymbol{A}}(T)/\theta. \tag{3}$$

Property (3) follows directly from Property (2): Since $\theta\le\Omega^{\boldsymbol{A}}(T)$, θ is compatible with T, whence $T=\pi^{-1}(T/\theta)$, where $\langle I_{\mathbf{Sign}},\pi\rangle:\boldsymbol{A}\to\boldsymbol{A}/\theta$ is the projection morphism. Thus, since $\langle I_{\mathbf{Sign}},\pi\rangle$ is surjective, by Property (2), $\Omega^{\boldsymbol{A}}(T)=\Omega^{\boldsymbol{A}}(\pi^{-1}(T/\theta))=\pi^{-1}(\Omega^{\boldsymbol{A}/\theta}(T/\theta))$. Now, using again the surjectivity of $\langle I_{\mathbf{Sign}},\pi\rangle$, we get (3).

Let $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be an algebraic system, such that N is a category of natural transformations generated by a natural transformation $\bullet:\mathrm{SEN}^2\to\mathrm{SEN}$ satisfying, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi,\psi,\chi\in\mathrm{SEN}(\Sigma)$,

- **Idempotency**: $\bullet_\Sigma(\varphi,\varphi)=\varphi$;
- **Commutativity**: $\bullet_\Sigma(\varphi,\psi)=\bullet_\Sigma(\psi,\varphi)$;
- **Associativity**: $\bullet_\Sigma(\varphi,\bullet_\Sigma(\psi,\chi))=\bullet_\Sigma(\bullet_\Sigma(\varphi,\psi),\chi)$.

Such a natural transformation is called a **semilattice operation** and we usually write it in infix notation $\varphi\bullet_\Sigma\psi$, etc. We call $\boldsymbol{A}$ a **semilattice system**. Moreover, we define the relation family $\le=\{\le_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\boldsymbol{A}$ by setting, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\varphi\le_\Sigma\psi \quad \text{iff} \quad \varphi\bullet_\Sigma\psi=\varphi.$$

Lemma 1
Given a semilattice system $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, the relation family $\le$ is a partially order system (posystem) on $\boldsymbol{A}$.

Proof. By idempotency of $\bullet_\Sigma$, we get that $\le_\Sigma$ is reflexive. By commutativity of $\bullet_\Sigma$, we get that $\le_\Sigma$ is antisymmetric. Finally, using associativity of $\bullet_\Sigma$, we derive that $\le_\Sigma$ is also transitive. Thus, for all $\Sigma\in|\mathbf{Sign}|$, $\le_\Sigma$ is a partial ordering on $\mathrm{SEN}(\Sigma)$.

To see that $\le$ is a system, i.e., invariant under signature morphisms, assume that $\Sigma_1,\Sigma_2\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma_1,\Sigma_2)$ and $\varphi,\psi\in\mathrm{SEN}(\Sigma_1)$, such that $\varphi\le_{\Sigma_1}\psi$. Then, we get $\varphi\bullet_{\Sigma_1}\psi=\varphi$, whence $\mathrm{SEN}(f)(\varphi\bullet_{\Sigma_1}\psi)=\mathrm{SEN}(f)(\varphi)$. Since $\bullet:\mathrm{SEN}^2\to\mathrm{SEN}$ is a natural transformation, we obtain $\mathrm{SEN}(f)(\varphi)\bullet_{\Sigma_2}\mathrm{SEN}(f)(\psi)=\mathrm{SEN}(f)(\varphi)$ and, therefore, $\mathrm{SEN}(f)(\varphi)\le_{\Sigma_2}\mathrm{SEN}(f)(\psi)$. ■

Next, define the relation family $\prec=\{\prec_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\boldsymbol{A}$ by letting $\prec_\Sigma$ be the covering relation on $\mathrm{SEN}(\Sigma)$ with respect to the partial order $\le_\Sigma$, for all $\Sigma\in|\mathbf{Sign}|$. Recall from Section 1 that this means, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\begin{aligned}\varphi\prec_\Sigma\psi \quad \text{iff} \quad & \varphi<_\Sigma\psi \text{ and, for all } \chi\in\mathrm{SEN}(\Sigma),\\ & \varphi\le_\Sigma\chi<_\Sigma\psi \text{ implies } \varphi=\chi.\end{aligned}$$

In contrast to $\leq$, the relation family $\prec$ may fail to be invariant under **Sign**-morphisms, i.e. a relation *system*. That is, there may exist $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \prec_\Sigma \psi \quad \text{but } \mathrm{SEN}(f)(\phi) \not\prec \mathrm{SEN}(f)(\psi).$$

Example 2

We exhibit a semilattice system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ for which the relation family $\prec$ fails to satisfy the system property.

$$\Sigma \xrightarrow{f} \Sigma' \qquad\qquad \begin{tikzcd} 1 \arrow[dr, "\mathrm{SEN}(f)"] \arrow[dd, dash, "\prec_\Sigma"'] & \\ & \bot \\ 0 \arrow[ur, "\mathrm{SEN}(f)"'] & \end{tikzcd}$$

Consider the semilattice system with **Sign** as depicted on the left (omitting identity arrows) and $\mathrm{SEN}(\Sigma)$, $\mathrm{SEN}(\Sigma')$ and $\mathrm{SEN}(f)$ as on the right. The semilattice operation $\bullet$ is defined by setting $\phi \bullet_\Sigma \psi = \begin{cases} \phi, & \text{if } \phi = \psi \\ 0, & \text{if } \phi \neq \psi \end{cases}$, and similarly for Σ'. Thus, the result is assumed to be the minimum of the two arguments in the Hasse diagram depicting the orderings. We will adopt the same convention without explicit mention in all examples considered in the paper. The $\prec$ family fails to be a system: Indeed we have $0 \prec_\Sigma 1$, but $\mathrm{SEN}(f)(0) = \bot \not\prec_{\Sigma'} \bot = \mathrm{SEN}(f)(1)$.

Given a semilattice system $\boldsymbol{A}$, we denote by $\mathsf{S}(\boldsymbol{A})$ the class of all (simple) semilattice subsystems of $\boldsymbol{A}$. Simple here refers to the fact that they all have the same signature category as $\boldsymbol{A}$. Since, all subsystems we consider in this paper are simple, we will omit this qualifier in the sequel.

A semilattice system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is called a **chain system** if $\leq$ is a linear order system on SEN, i.e., for all $\Sigma \in |\mathbf{Sign}|$, $\leq_\Sigma$ is a linear ordering on $\mathrm{SEN}(\Sigma)$.

Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a semilattice system. We call the **Σ-component semilattice** and denote by $\boldsymbol{A}_\Sigma = \langle \mathrm{SEN}(\Sigma), \bullet_\Sigma \rangle$ the ordinary universal algebraic semilattice formed by restricting attention to the Σ-component of $\boldsymbol{A}$. A **Σ-chain** of $\boldsymbol{A}$ is a chain in $\boldsymbol{A}_\Sigma$. The **Σ-height** of $\boldsymbol{A}$ is $h_\Sigma(\boldsymbol{A}) = \mathcal{H}(\boldsymbol{A}_\Sigma)$, i.e. the maximum number of elements in any Σ-chain of $\boldsymbol{A}$, when this number is finite. The **height** $h(\boldsymbol{A})$ of $\boldsymbol{A}$ is defined by

$$h(\boldsymbol{A}) = \max\{h_\Sigma(\boldsymbol{A}) : \Sigma \in |\mathbf{Sign}|\},$$

when $h(\boldsymbol{A}) < \omega$. In this case, we say that the semilattice system $\boldsymbol{A}$ has **finite height**.

Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a semilattice system. For $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{SEN}(\Sigma)$, we set

$$\downarrow_\Sigma \varphi = \{\psi \in \mathrm{SEN}(\Sigma) : \psi \leq_\Sigma \varphi\}.$$

If $\downarrow_\Sigma \varphi$ has finite height, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, we say that $\boldsymbol{A}$ has **sectionally finite height** or is a **semilattice system of** (or **with**) **sectionally finite height** (**FSL**). We denote the class of all semilattice systems of sectionally finite height by FSL. For a semilattice system $\boldsymbol{A} \in \mathsf{FSL}$, $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, we define the **height** $h_\Sigma(\varphi)$ of φ as the height of the semilattice $\downarrow_\Sigma \varphi$: $h_\Sigma(\varphi) = \mathcal{H}(\downarrow_\Sigma \varphi)$.

3 Rainbow families

The next lemma provides a characterization of the Leibniz congruence system associated with a given sentence family in a semilattice system $\boldsymbol{A}$. It forms an analogue of Lemma 3.1 of [4] and has its roots, in the universal algebraic side, in [3] and, in the categorical side, in the characterization of the Leibniz congruence systems provided in [5].

Lemma 3

Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a semilattice system and let, also, $T \in \mathrm{SenFam}(\boldsymbol{A})$. For all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \varphi, \psi \rangle \in \Omega^{\boldsymbol{A}}_{\Sigma}(T)$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\chi \in \mathrm{SEN}(\Sigma')$,

$$\mathrm{SEN}(f)(\varphi) \bullet_{\Sigma'} \chi \in T_{\Sigma'} \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \bullet_{\Sigma'} \chi \in T_{\Sigma'}.$$

Proof. Let us set, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl} R_{\Sigma} & = & \{\langle \varphi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : \mathrm{SEN}(f)(\varphi) \bullet_{\Sigma'} \chi \in T_{\Sigma'} \text{ iff} \\ & & \qquad\qquad\qquad\qquad \mathrm{SEN}(f)(\psi) \bullet_{\Sigma'} \chi \in T_{\Sigma'}, \\ & & \quad \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma') \text{ and } \chi \in \mathrm{SEN}(\Sigma')\} \end{array}$$

and $R = \{R_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$. The goal is to show that R is a congruence system on $\boldsymbol{A}$ compatible with T and that it is the largest such.

- That R_{Σ} is an equivalence relation on $\mathrm{SEN}(\Sigma)$ is obvious.
- To see that R_{Σ} is a congruence system, we must show that, if $\langle \varphi, \varphi' \rangle \in R_{\Sigma}$ and $\langle \psi, \psi' \rangle \in R_{\Sigma}$, then $\langle \varphi \bullet_{\Sigma} \psi, \varphi' \bullet_{\Sigma} \psi' \rangle \in R_{\Sigma}$. This is easy to do using the naturality of $\bullet$, together with associativity and commutativity. The details are omitted.
- We now show that $R = \{R_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ is a system. Let $\langle \varphi, \psi \rangle \in R_{\Sigma}$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Then, for all $\Sigma'' \in |\mathbf{Sign}|$ all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ (see following diagram) and all $\chi \in \mathrm{SEN}(\Sigma'')$, we have

$$\Sigma \xrightarrow{\quad f \quad} \Sigma' \xrightarrow{\quad g \quad} \Sigma''$$

$$\begin{array}{ll} \mathrm{SEN}(g)(\mathrm{SEN}(f)(\varphi)) \bullet_{\Sigma''} \chi \in T_{\Sigma''} & \\ \qquad\qquad \text{iff} & \mathrm{SEN}(gf)(\varphi) \bullet_{\Sigma''} \chi \in T_{\Sigma''} \\ \qquad\qquad \text{iff} & \mathrm{SEN}(gf)(\psi) \bullet_{\Sigma''} \chi \in T_{\Sigma''} \\ \qquad\qquad \text{iff} & \mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)) \bullet_{\Sigma''} \chi \in T_{\Sigma''}, \end{array}$$

  whence $\langle \mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi) \rangle \in R_{\Sigma'}$.
- To see that R is compatible with T, let $\langle \varphi, \psi \rangle \in R_{\Sigma}$ and $\varphi \in T_{\Sigma}$. Then, we have

$$\begin{array}{rll} \varphi = \varphi \bullet_{\Sigma} \varphi \in T_{\Sigma} & \text{iff} & \psi \bullet_{\Sigma} \varphi \in T_{\Sigma} \\ & \text{iff} & \varphi \bullet_{\Sigma} \psi \in T_{\Sigma} \\ & \text{iff} & \psi = \psi \bullet_{\Sigma} \psi \in T_{\Sigma}. \end{array}$$

- Finally, to see that R is the largest congruence system compatible with T, assume that $\theta = \{\theta_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ is a congruence system compatible with T, $\Sigma \in |\mathbf{Sign}|$ and $\langle \varphi, \psi \rangle \in \theta_{\Sigma}$. Then, for all

$\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\chi \in \mathrm{SEN}(\Sigma')$,

$$\begin{array}{lll} \langle \varphi, \psi \rangle \in \theta_\Sigma & \text{implies} & \langle \mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi) \rangle \in \theta_{\Sigma'} \\ & \text{implies} & \langle \mathrm{SEN}(f)(\varphi) \bullet_{\Sigma'} \chi, \mathrm{SEN}(f)(\psi) \bullet_{\Sigma'} \chi \rangle \in \theta_{\Sigma'} \\ & \text{implies} & \mathrm{SEN}(f)(\varphi) \bullet_{\Sigma'} \chi \in T_{\Sigma'} \\ & & \qquad\qquad \text{iff } \mathrm{SEN}(f)(\psi) \bullet_{\Sigma'} \chi \in T_{\Sigma'} \\ & \text{implies} & \langle \varphi, \psi \rangle \in R_\Sigma. \end{array}$$

This proves that $\theta \leq R$.

■

Recall that, given a semilattice system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, we denote by $\boldsymbol{A}_\Sigma = \langle \mathrm{SEN}(\Sigma), \bullet_\Sigma \rangle$ the Σ-component semilattice of $\boldsymbol{A}$. Accordingly, given an $F \subseteq \mathrm{SEN}(\Sigma)$, $\Omega^{\boldsymbol{A}_\Sigma}(F)$ denotes the ordinary Leibniz congruence of F on $\boldsymbol{A}_\Sigma$ in the sense of AAL.

Corollary 4
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a semilattice system and consider $T \in \mathrm{SenFam}(\boldsymbol{A})$. Then, for all $\Sigma \in |\mathbf{Sign}|$,

$$\Omega^{\boldsymbol{A}}_\Sigma(T) \subseteq \Omega^{\boldsymbol{A}_\Sigma}(T_\Sigma).$$

Proof. By Lemma 3 we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \varphi, \psi \rangle \in \Omega^{\boldsymbol{A}}_\Sigma(T)$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\chi \in \mathrm{SEN}(\Sigma')$,

$$\mathrm{SEN}(f)(\varphi) \bullet_{\Sigma'} \chi \in T_{\Sigma'} \text{ iff } \mathrm{SEN}(f)(\psi) \bullet_{\Sigma'} \chi \in T_{\Sigma'}.$$

In particular, for all $\chi \in \mathrm{SEN}(\Sigma)$,

$$\varphi \bullet_\Sigma \chi \in T_\Sigma \text{ iff } \psi \bullet_\Sigma \chi \in T_\Sigma,$$

whence, by Lemma 3.1 of [4], $\langle \varphi, \psi \rangle \in \Omega^{\boldsymbol{A}_\Sigma}(T_\Sigma)$. ■

In the following example, it is shown that the inclusion of Corollary 4 may be a proper inclusion. Given an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, we denote by $\Delta^{\mathrm{SEN}} = \{\Delta^{\mathrm{SEN}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ the identity congruence system on $\boldsymbol{A}$, i.e., $\Delta^{\mathrm{SEN}}_\Sigma = \{\langle \varphi, \varphi \rangle : \varphi \in \mathrm{SEN}(\Sigma)\}$, for all $\Sigma \in |\mathbf{Sign}|$, and by $\nabla^{\mathrm{SEN}} = \{\nabla^{\mathrm{SEN}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ the all or nabla congruence system on $\boldsymbol{A}$, given by $\nabla^{\mathrm{SEN}}_\Sigma = \mathrm{SEN}(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}|$.

Example 5
The inclusion of Corollary 4 may be proper. Consider the category $\mathbf{Sign}$ shown on the left (again omitting identity arrows) and the functor SEN shown on the right.

$$\begin{tikzcd} \Sigma \arrow[r, "f"] & \Sigma' \end{tikzcd} \qquad\qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{SEN}(f)"] \arrow[d, dash, "\leq_\Sigma"'] & \top \arrow[d, dash, "\leq_{\Sigma'}"] \\ 0 \arrow[r, "\mathrm{SEN}(f)"'] & \bot \end{tikzcd}$$

Let $T_\Sigma = \{0, 1\}$ and $T_{\Sigma'} = \{\top\}$. Then, since $\mathrm{SEN}(f)(0) = \bot \notin T_{\Sigma'}$ whereas $\mathrm{SEN}(f)(1) = \top \in T_{\Sigma'}$, we have $\Omega^{\boldsymbol{A}}_\Sigma = \Delta^{\mathrm{SEN}}_\Sigma$. On the other hand, we can see that $\Omega^{\boldsymbol{A}_\Sigma}(T_\Sigma) = \nabla^{\mathrm{SEN}}_\Sigma$, being the largest congruence on $\boldsymbol{A}_\Sigma$ compatible with T_Σ. Thus, $\Omega^{\boldsymbol{A}}_\Sigma(T) \subsetneq \Omega^{\boldsymbol{A}_\Sigma}(T_\Sigma)$.

Next, we provide an analogue of Corollary 3.2 of [4] in the case of finite chain systems. This will also offer a flavor of the application of the characterization of $\Omega^{\mathcal{A}}(T)$ established in Lemma 3.

COROLLARY 6
Let $\mathcal{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a chain system, such that $\mathrm{SEN}(\Sigma)$ is finite, for all $\Sigma\in|\mathbf{Sign}|$. Let also $T\in\mathrm{SenFam}(\mathcal{A})$, $\Sigma\in|\mathbf{Sign}|$ and $\varphi,\psi\in\mathrm{SEN}(\Sigma)$, with $\varphi<_\Sigma\psi$. Then $\langle\varphi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)$ iff, for all $\Sigma'\in|\mathbf{Sign}|, f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$\begin{aligned}&[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]\subseteq T_{\Sigma'}\\ &\qquad\text{or}\quad[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]\subseteq\mathrm{SEN}(\Sigma')\backslash T_{\Sigma'}.\end{aligned}$$

PROOF. For the left-to-right implication, assume that $\langle\varphi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)$. Then either $\mathrm{SEN}(f)(\varphi)$, $\mathrm{SEN}(f)(\psi)\in T_{\Sigma'}$ or $\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)\notin T_{\Sigma'}$. Now let $\chi\in[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]$.

- If $\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)\in T_{\Sigma'}$, then

$$\chi=\mathrm{SEN}(f)(\psi)\bullet_{\Sigma'}\chi\ \ \Omega^{\mathbf{A}}_{\Sigma'}(T)\ \mathrm{SEN}(f)(\varphi)\bullet_{\Sigma'}\chi=\mathrm{SEN}(f)(\varphi),$$

  whence, since $\mathrm{SEN}(f)(\varphi)\in T_{\Sigma'}$, by compatibility, $\chi\in T_{\Sigma'}$.
- The case $\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)\notin T_{\Sigma'}$ is similar.

Suppose, conversely, that for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$\begin{aligned}&[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]\subseteq T_{\Sigma'}\\ &\qquad\text{or}\quad[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]\subseteq\mathrm{SEN}(\Sigma')\backslash T_{\Sigma'}.\end{aligned}$$

For a given $\chi\in\mathrm{SEN}(\Sigma')$, since $\mathcal{A}$ is a chain system, we have one of the following cases: $\chi<_{\Sigma'}\mathrm{SEN}(f)(\varphi)$ or $\chi\in[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]$ or $\mathrm{SEN}(f)(\psi)<_{\Sigma'}\chi$.

- In the first case, since $\mathrm{SEN}(f)(\varphi)\bullet_{\Sigma'}\chi=\chi=\mathrm{SEN}(f)(\psi)\bullet_{\Sigma'}\chi$, either both $\mathrm{SEN}(f)(\varphi)\bullet_{\Sigma'}\chi$ and $\mathrm{SEN}(f)(\psi)\bullet_{\Sigma'}\chi$ are in $T_{\Sigma'}$ or both outside.
- In the second case,

$$\begin{aligned}\mathrm{SEN}(f)(\varphi)\bullet_{\Sigma'}\chi\in T_{\Sigma'}\quad&\text{iff}\quad\mathrm{SEN}(f)(\varphi)\in T_{\Sigma'}\\ &\text{iff}\quad\chi\in T_{\Sigma'}\ \text{(by hypothesis)}\\ &\text{iff}\quad\mathrm{SEN}(f)(\psi)\bullet_{\Sigma'}\chi\in T_{\Sigma'}.\end{aligned}$$

- In the third case,

$$\begin{aligned}\mathrm{SEN}(f)(\varphi)\bullet_{\Sigma'}\chi\in T_{\Sigma'}\quad&\text{iff}\quad\mathrm{SEN}(f)(\varphi)\in T_{\Sigma'}\\ &\text{iff}\quad\mathrm{SEN}(f)(\psi)\in T_{\Sigma'}\ \text{(by hypothesis)}\\ &\text{iff}\quad\mathrm{SEN}(f)(\psi)\bullet_{\Sigma'}\chi\in T_{\Sigma'}.\end{aligned}$$

Thus, by Lemma 3, we get that $\langle\varphi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)$. ∎

COROLLARY 7
Let $\mathcal{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a chain system, such that, for all $\Sigma\in|\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma)$ is finite, and $T\in\mathrm{SenFam}(\mathcal{A})$. Then $\Omega^{\mathcal{A}}(T)=\Delta^{\mathrm{SEN}}$ iff, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\varphi\prec_\Sigma\psi$, there exists $\Sigma'\in|\mathbf{Sign}|$ and $f\in\mathbf{Sign}(\Sigma,\Sigma')$, such that, for some $\chi,\xi\in[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]$,

$$\chi\in T_{\Sigma'}\quad\text{iff}\quad\xi\notin T_{\Sigma'}.$$

Proof. Suppose that $\Omega^{\mathcal{A}}(T)=\Delta^{\mathrm{SEN}}$. Let $\Sigma\in|\mathbf{Sign}|$ and $\varphi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\varphi\prec_\Sigma\psi$. Clearly, $\langle\varphi,\psi\rangle\notin\Omega^{\mathcal{A}}_\Sigma(T)$. Therefore, by Corollary 6, there exists $\Sigma'\in|\mathbf{Sign}|$ and $f\in\mathbf{Sign}(\Sigma,\Sigma')$, such that $[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]\nsubseteq T_{\Sigma'}$ and $[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]\nsubseteq\mathrm{SEN}(T_{\Sigma'})\backslash T_{\Sigma'}$. But this is exactly the statement that there exist $\chi,\xi\in[\mathrm{SEN}(f)(\varphi),\mathrm{SEN}(f)(\psi)]$, such that $\chi\in T_{\Sigma'}$ iff $\xi\notin T_{\Sigma'}$. Conversely, if the postulated condition holds, then, by Corollary 6, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi,\psi\in\mathrm{SEN}(\Sigma)$, with $\varphi\prec_\Sigma\psi$, $\langle\varphi,\psi\rangle\notin\Omega^{\mathcal{A}}_\Sigma(T)$. This implies that $\Omega^{\mathcal{A}}(T)=\Delta^{\mathrm{SEN}}$. ∎

Let $\mathcal{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a semilattice system of sectionally finite height. The **rainbow system** of $\mathcal{A}$ is the family $\mathfrak{R}(\mathcal{A})=\{\mathfrak{R}_\Sigma(\mathcal{A})\}_{\Sigma\in|\mathbf{Sign}|}$, where

$$\begin{aligned}\mathfrak{R}_\Sigma(\mathcal{A}) \quad = \quad & \{\varphi\in\mathrm{SEN}(\Sigma): h_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \text{ is odd,}\\ & \qquad\qquad \text{for all } \Sigma'\in|\mathbf{Sign}|, f\in\mathbf{Sign}(\Sigma,\Sigma')\}.\end{aligned}$$

Lemma 8

Let $\mathcal{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a semilattice system with sectionally finite height. The rainbow system $\mathfrak{R}(\mathcal{A})$ of $\mathcal{A}$ is a system in the sense of CAAL, i.e. it is invariant under all signature morphisms.

Proof. Let $\Sigma\in|\mathbf{Sign}|$ and $\varphi\in\mathfrak{R}_\Sigma(\mathcal{A})$. Consider $\Sigma'\in|\mathbf{Sign}|$ and $f\in\mathbf{Sign}(\Sigma,\Sigma')$. The goal is to show that $\mathrm{SEN}(f)(\varphi)\in\mathfrak{R}_{\Sigma'}(\mathcal{A})$. Let $\Sigma''\in|\mathbf{Sign}|$ and $g\in\mathbf{Sign}(\Sigma',\Sigma'')$ as in the following diagram.

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

Since $\varphi\in\mathfrak{R}_\Sigma(\mathcal{A})$, $h_{\Sigma''}(\mathrm{SEN}(gf)(\varphi))$ is odd. Therefore, since SEN is a functor, $h_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\varphi)))$ is odd. Since g was arbitrary, this proves that $\mathrm{SEN}(f)(\varphi)\in\mathfrak{R}_{\Sigma'}(\mathcal{A})$. ∎

Example 9

We show that it is not necessarily the case that, given a semilattice system $\mathcal{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ of sectionally finite height, $\Omega^{\mathcal{A}}(\mathfrak{R}(\mathcal{A}))=\Delta^{\mathrm{SEN}}$.

Indeed, consider the following category **Sign** of signatures

$$\Sigma \xrightarrow{\ f\ } \Sigma' \qquad\qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{SEN}(f)"] \arrow[d, dash, "\prec_\Sigma"'] & \top \arrow[d, dash, "\prec_{\Sigma'}"] \\ 0 \arrow[ur, "\mathrm{SEN}(f)"'] & \bot \end{tikzcd}$$

and define SEN as shown on the right. Since

$$h_{\Sigma'}(\mathrm{SEN}(f)(0))=h_{\Sigma'}(\mathrm{SEN}(f)(1))=h_{\Sigma'}(\top)=2,$$

we have $0,1\notin\mathfrak{R}_\Sigma(\mathcal{A})$. Thus, $\mathfrak{R}_\Sigma(\mathcal{A})=\emptyset$. Since $h_{\Sigma'}(\bot)=1$ and the only outgoing arrow from Σ' is the identity, $\bot\in\mathfrak{R}_{\Sigma'}(\mathcal{A})$. Since $h_{\Sigma'}(\top)=2$, we get $\mathfrak{R}_{\Sigma'}(\mathcal{A})=\{\bot\}$.

Consider the congruence system $\theta=\{\theta_\Sigma,\theta_{\Sigma'}\}$, with $\theta_\Sigma=\nabla^{\mathrm{SEN}}_\Sigma$ and $\theta_{\Sigma'}=\Delta^{\mathrm{SEN}}_{\Sigma'}$. This congruence system on $\mathcal{A}$ is clearly compatible with $\mathfrak{R}(\mathcal{A})$, whence $\theta\le\Omega^{\mathcal{A}}(\mathfrak{R}(\mathcal{A}))$. On the other hand, it is the largest congruence system compatible with $\mathfrak{R}(\mathcal{A})$, since such a system must necessarily distinguish, by definition of compatibility, between $\bot\in\mathfrak{R}_{\Sigma'}(\mathcal{A})$ and $\top\notin\mathfrak{R}_{\Sigma'}(\mathcal{A})$. Hence $\Omega^{\mathcal{A}}(\mathfrak{R}(\mathcal{A}))=\theta$. Therefore, $\Omega^{\mathcal{A}}(\mathfrak{R}(\mathcal{A}))\neq\Delta^{\mathrm{SEN}}$.

The preceding example motivates turning attention to rainbow families in place of rainbow systems, i.e. relaxing the hypothesis of invariance under signature morphisms.

Let $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a semilattice system of sectionally finite height. The **rainbow family** of $\boldsymbol{A}$ is the family $\mathcal{R}(\boldsymbol{A})=\{\mathcal{R}_\Sigma(\boldsymbol{A})\}_{\Sigma\in|\mathbf{Sign}|}$, where

$$\mathcal{R}_\Sigma(\boldsymbol{A})=\{\varphi\in\mathrm{SEN}(\Sigma):h_\Sigma(\varphi)\text{ is odd}\}.$$

EXAMPLE 10
Note that in Example 9, we have $\mathcal{R}_\Sigma(\boldsymbol{A})=\{0\}$ and $\mathcal{R}_{\Sigma'}(\boldsymbol{A})=\{\bot\}$, whence, we actually obtain $\Omega^{\boldsymbol{A}}(\mathcal{R}(\boldsymbol{A}))=\Delta^{\mathrm{SEN}}$ is this case.

We next show that it is true in general that the Leibniz congruence system of a rainbow family is the identity, as illustrated in Example 10.

PROPOSITION 11
Let $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a semilattice system of sectionally finite height. Then $\Omega^{\boldsymbol{A}}(\mathcal{R}(\boldsymbol{A}))=\Delta^{\mathrm{SEN}}$.

PROOF. First, note that $\mathcal{R}_\Sigma(\boldsymbol{A})=\mathcal{R}(\boldsymbol{A}_\Sigma)$, where $\mathcal{R}(\boldsymbol{A}_\Sigma)$ denotes the rainbow of $\boldsymbol{A}_\Sigma=\langle\mathrm{SEN}(\Sigma),\bullet_\Sigma\rangle$ according to Definition 3.4 of [4]. Thus, by Theorem 3.5 of [4], we get that $\Omega^{\boldsymbol{A}_\Sigma}(\mathcal{R}_\Sigma(\boldsymbol{A}))=\Delta^{\mathrm{SEN}}_\Sigma$, for all $\Sigma\in|\mathbf{Sign}|$. By Corollary 4, we get $\Omega^{\boldsymbol{A}}_\Sigma(\mathcal{R}(\boldsymbol{A}))\subseteq\Omega^{\boldsymbol{A}_\Sigma}(\mathcal{R}_\Sigma(\boldsymbol{A}))$. Since this holds for all $\Sigma\in|\mathbf{Sign}|$, we, finally, get $\Omega^{\boldsymbol{A}}(\mathcal{R}(\boldsymbol{A}))=\Delta^{\mathrm{SEN}}$. ∎

4 Cloud families

Let $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a semilattice system with sectionally finite height and $T\in\mathrm{SenFam}(\boldsymbol{A})$. T is called a **cloud family** if $T/\Omega^{\boldsymbol{A}}(T)=\mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$. We set

$$\mathrm{ClFam}(\boldsymbol{A})=\{T\in\mathrm{SenFam}(\boldsymbol{A}):T\text{ is a cloud family}\}.$$

If $\langle I_{\mathbf{Sign}},\pi\rangle:\boldsymbol{A}\to\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T)$ is the projection N-morphism, then, by compatibility of $\Omega^{\boldsymbol{A}}(T)$ with T, we get

$$T=\pi^{-1}(T/\Omega^{\boldsymbol{A}}(T))=\pi^{-1}(\mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))),$$

That is cloud families are inverse images of rainbow families under the projection N-morphism of the semilattice system onto its quotient by the Leibniz congruence system of the cloud family.

LEMMA 12
Let $\boldsymbol{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be a semilattice system with sectionally finite height and $T\in\mathrm{ClFam}(\boldsymbol{A})$. Then, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi\in\mathrm{SEN}(\Sigma)$,

$$\varphi\in T_\Sigma\quad\text{iff}\quad h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))\text{ is odd}.$$

PROOF. Suppose $T\in\mathrm{ClFam}(\boldsymbol{A})$, $\Sigma\in|\mathbf{Sign}|$ and $\varphi\in\mathrm{SEN}(\Sigma)$. Then, we have

$$\begin{array}{lll}\varphi\in T_\Sigma & \text{iff} & \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)\in T_\Sigma/\Omega^{\boldsymbol{A}}_\Sigma(T)\quad\text{(by Equivalence (1))}\\ & \text{iff} & \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)\in\mathcal{R}_\Sigma(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))\quad(T\text{ a cloud family})\\ & \text{iff} & h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))\text{ is odd,}\end{array}$$

where the last equivalence follows by the definition of a rainbow family. ∎

Analogously with the universal algebraic case (Theorem 4.2 of [4]), we now obtain that the categorical Leibniz operator induces a bijection between the collection $\mathrm{ClFam}(\boldsymbol{A})$ of cloud families and the collection $\mathrm{ConSys}(\boldsymbol{A})$ of congruence systems of an FSL $\boldsymbol{A}$.

Theorem 13
Let $\boldsymbol{A}=\langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system with sectionally finite height. Then $\Omega^{\boldsymbol{A}}:\mathrm{ClFam}(\boldsymbol{A})\to \mathrm{ConSys}(\boldsymbol{A})$ is a bijection.

Proof. To show injectivity, suppose that $T, T'\in \mathrm{ClFam}(\boldsymbol{A})$. Then,

$$\begin{array}{rll}\Omega^{\boldsymbol{A}}(T)=\Omega^{\boldsymbol{A}}(T') & \text{iff} & \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))=\mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T')) \\ & \text{iff} & T/\Omega^{\boldsymbol{A}}(T)=T'/\Omega^{\boldsymbol{A}}(T') \\ & \text{iff} & T=T'.\end{array}$$

For surjectivity, let $\theta\in \mathrm{ConSys}(\boldsymbol{A})$. Then, we have $\boldsymbol{A}/\theta\in \mathsf{FSL}$, whence, by Proposition 11, $\Omega^{\boldsymbol{A}/\theta}(\mathcal{R}(\boldsymbol{A}/\theta))=\Delta^{\mathrm{SEN}/\theta}$. Set $T=\{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that, for all $\Sigma\in|\mathbf{Sign}|$,

$$T_\Sigma=\{\varphi\in \mathrm{SEN}(\Sigma):\varphi/\theta_\Sigma\in \mathcal{R}_\Sigma(\boldsymbol{A}/\theta)\}.$$

If we denote by $\langle I,\pi\rangle:=\langle I_{\mathbf{Sign}},\pi^\theta\rangle:\boldsymbol{A}\to\boldsymbol{A}/\theta$ the projection N-morphism, taking into account the commutativity of the Leibniz operator with inverse surjective N-morphisms, formulated in Equation (2), we get

$$\begin{array}{rcl}\Omega^{\boldsymbol{A}}(T) & = & \Omega^{\boldsymbol{A}}(\pi^{-1}(\mathcal{R}(\boldsymbol{A}/\theta))) \\ & = & \pi^{-1}(\Omega^{\boldsymbol{A}/\theta}(\mathcal{R}(\boldsymbol{A}/\theta))) \\ & = & \pi^{-1}(\Delta^{\mathrm{SEN}/\theta}) \\ & = & \theta.\end{array}$$

Hence, $\Omega^{\boldsymbol{A}}$ is also surjective and, therefore, a bijection. ∎

5 When telescopy is not sufficient...

In their Theorem 4.6, Font and Moraschini [4] characterize clouds in semilattices with sectionally finite height by providing a condition relating heights of elements relative to clouds with their 'absolute' heights. In this section, we show that the corresponding condition is necessary, but not sufficient, for cloud families of semilattice systems of sectionally finite height. Moreover, we illustrate, via example, that this shortcoming involves the inherent limitation of this condition to successfully detect and capture what happens in 'alien localities', i.e., in other 'local' semilattices when one 'observes' transformations under change of signatures. This shortcoming is mended in the following section by devising a necessary and sufficient condition that is able to 'sense' these 'remote signals' and, thus, to account for local effects of 'remote phenomena'. Based on both the sky-inspired terminology of [4] and on the aforementioned analogies between observing and capturing local versus remote features of the structures under consideration, we name, inspired by astrophysical methods of observation of incremental strength, the local conditions of [4] 'telescopic' and the global ones of the next section, that are powerful enough for the categorical context, 'spectroscopic'.

Let $\boldsymbol{A}=\langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system. Given a Σ-chain C_Σ, define $\boldsymbol{A}(C_\Sigma)$ to be the smallest (simple) semilattice subsystem of $\boldsymbol{A}$ including C_Σ, called the sub-semilattice of $\boldsymbol{A}$ **generated by** C_Σ.

Given an FSL $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $T \in \mathrm{SenFam}(\boldsymbol{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{SEN}(\Sigma)$, by analogy with Definition 4.3 of [4], the **Σ-height of φ relative to T** is defined by

$$h^T_\Sigma(\varphi) = \max\{|C_\Sigma / \Omega^{\boldsymbol{A}(C_\Sigma)}_\Sigma(T \cap \boldsymbol{A}(C_\Sigma))| : C_\Sigma \subseteq {\downarrow_\Sigma}\varphi\},$$

where ${\downarrow_\Sigma}\varphi$ refers to the principal downset of φ in $\boldsymbol{A}_\Sigma$.

We proceed to establish analogues of the properties shown to hold in the universal algebraic case in Lemma 4.4 of [4] in this more general context.

Lemma 14
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a semilattice system of sectionally finite height, $T \in \mathrm{SenFam}(\boldsymbol{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{SEN}(\Sigma)$. Then

$$h^T_\Sigma(\varphi) \leq h_\Sigma(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T)).$$

Proof. Suppose $C_\Sigma \subseteq {\downarrow_\Sigma}\varphi$ is such that $|C_\Sigma / \Omega^{\boldsymbol{A}(C_\Sigma)}_\Sigma(T \cap \boldsymbol{A}(C_\Sigma))| = h^T_\Sigma(\varphi)$. Note that $\Omega^{\boldsymbol{A}}(T) \cap \nabla^{\boldsymbol{A}(C_\Sigma)}$ is a congruence system that is compatible with $T \cap \boldsymbol{A}(C_\Sigma)$. Thus, we obtain $\Omega^{\boldsymbol{A}}(T) \cap \nabla^{\boldsymbol{A}(C_\Sigma)} \leq \Omega^{\boldsymbol{A}(C_\Sigma)}(T \cap \boldsymbol{A}(C_\Sigma))$. We now obtain

$$\begin{array}{rcl} h^T_\Sigma(\varphi) & = & |C_\Sigma / \Omega^{\boldsymbol{A}(C_\Sigma)}_\Sigma(T \cap \boldsymbol{A}(C_\Sigma))| \\ & \leq & |C_\Sigma / (\Omega^{\boldsymbol{A}}_\Sigma(T) \cap \nabla^{\boldsymbol{A}(C_\Sigma)})| \\ & \leq & h_\Sigma(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T)). \end{array}$$

■

We show, next, that the height of a Σ-sentence φ in an FSL coincides with its height relative to the rainbow family of the semilattice system.

Lemma 15
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a semilattice system with sectionally finite height, $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{SEN}(\Sigma)$. Then

$$h_\Sigma(\varphi) = h^{\mathcal{R}(\boldsymbol{A})}_\Sigma(\varphi).$$

Proof. Note that we have

$$\begin{array}{rcl} h^{\mathcal{R}(\boldsymbol{A})}_\Sigma(\varphi) & \stackrel{\text{Lemma 14}}{\leq} & h_\Sigma(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(\mathcal{R}(\boldsymbol{A}))) \\ & \stackrel{\text{Prop. 11}}{=} & h_\Sigma(\varphi / \Delta^{\mathrm{SEN}}_\Sigma) \\ & = & h_\Sigma(\varphi). \end{array}$$

For the reverse inequality, suppose that $h_\Sigma(\varphi) = k$ and consider a Σ-chain $C_\Sigma \subseteq {\downarrow_\Sigma}\varphi$, such that $|C_\Sigma| = k$, say

$$\bot_\Sigma = \varphi_1 \prec_\Sigma \varphi_2 \prec_\Sigma \cdots \prec_\Sigma \varphi_{k-1} \prec_\Sigma \varphi_k = \varphi.$$

Since $h_\Sigma(\varphi_i) = i$, for all $i = 1, \ldots, k$, we get that, for all $i = 1, \ldots, k-1$,

$$\varphi_i \in \mathcal{R}_\Sigma(\boldsymbol{A}) \quad \text{iff} \quad \varphi_{i+1} \notin \mathcal{R}_\Sigma(\boldsymbol{A}).$$

This shows that $|C_\Sigma/\Omega_\Sigma^{\boldsymbol{A}(C_\Sigma)}(\mathcal{R}(\boldsymbol{A})\cap A(C_\Sigma))|=k$. Now, we conclude

$$\begin{array}{rcl} h_\Sigma(\varphi) & = & |C_\Sigma| \\ & = & |C_\Sigma/\Omega_\Sigma^{\boldsymbol{A}(C_\Sigma)}(\mathcal{R}(\boldsymbol{A})\cap A(C_\Sigma))| \\ & \leq & h_\Sigma^{\mathcal{R}(\boldsymbol{A})}(\varphi) \quad (\text{definition of } h_\Sigma^{\mathcal{R}(\boldsymbol{A})}(\varphi)). \end{array}$$

∎

Lemma 16

Let $\boldsymbol{A}=\langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system with sectionally finite height, $T\in\mathrm{SenFam}(\boldsymbol{A})$, $\Sigma\in|\mathbf{Sign}|$ and $\varphi\in\mathrm{SEN}(\Sigma)$. Then

$$h_\Sigma^T(\varphi)=h_\Sigma^{T/\Omega^{\boldsymbol{A}}(T)}(\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T)).$$

Proof. Consider a Σ-chain $C_\Sigma\subseteq{\downarrow_\Sigma}\varphi$, such that $h_\Sigma^T(\varphi)=|C_\Sigma/\Omega_\Sigma^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))|$. We let $\theta=\Omega^{\boldsymbol{A}}(T)\cap\nabla^{\boldsymbol{A}(C_\Sigma)}$. Then, clearly, C_Σ/θ_Σ is a Σ-chain in $\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T)$, such that $C_\Sigma/\theta_\Sigma\subseteq{\downarrow_\Sigma}\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T)$.
Claim:

$$\boldsymbol{A}(C_\Sigma)/\Omega^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))\cong(\boldsymbol{A}(C_\Sigma)/\theta)/\Omega^{\boldsymbol{A}(C_\Sigma)/\theta}((T\cap A(C_\Sigma))/\theta).$$

Since $\theta\leq\Omega^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))$, by the Second Isomorphism Theorem for algebraic systems (see, e.g. Theorem 28 of [7]),

$$\boldsymbol{A}(C_\Sigma)/\Omega^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))\cong(\boldsymbol{A}(C_\Sigma)/\theta)/(\Omega^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))/\theta).$$

Moreover, based on the property expressed by Equation (3), we also have $\Omega^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))/\theta=\Omega^{\boldsymbol{A}(C_\Sigma)/\theta}((T\cap A(C_\Sigma))/\theta)$. This proves the claim.
Now we are able to conclude:

$$\begin{array}{rcl} h_\Sigma^T(\varphi) & = & |C_\Sigma/\Omega_\Sigma^{\boldsymbol{A}(C_\Sigma)}(T\cap A(C_\Sigma))| \\ & = & |(C_\Sigma/\theta_\Sigma)/\Omega_\Sigma^{\boldsymbol{A}(C_\Sigma)/\theta}((T\cap A(C_\Sigma))/\theta)| \\ & \leq & h_\Sigma^{T/\Omega^{\boldsymbol{A}}(T)}(\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T)) \\ & & (\text{by definition of } h_\Sigma^{T/\Omega^{\boldsymbol{A}}(T)}(\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T))). \end{array}$$

For the reverse inequality, assume that Q_Σ is a Σ-chain in $\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T)$, with $Q_\Sigma\subseteq{\downarrow_\Sigma}\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T)$, such that

$$h_\Sigma^{T/\Omega^{\boldsymbol{A}}(T)}(\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T))=|Q_\Sigma/\Omega_\Sigma^{(A/\Omega^{\boldsymbol{A}}(T))(Q_\Sigma)}(T/\Omega^{\boldsymbol{A}}(T)\cap(A/\Omega^{\boldsymbol{A}}(T))(Q_\Sigma))|.$$

Suppose that Q_Σ is

$$\varphi_1/\Omega_\Sigma^{\boldsymbol{A}}(T)\prec_\Sigma\varphi_2/\Omega_\Sigma^{\boldsymbol{A}}(T)\prec_\Sigma\cdots\prec_\Sigma\varphi_{k-1}/\Omega_\Sigma^{\boldsymbol{A}}(T)\prec_\Sigma\varphi/\Omega_\Sigma^{\boldsymbol{A}}(T).$$

Set $\psi_i=\varphi_i\bullet_\Sigma\varphi_{i+1}\bullet_\Sigma\cdots\bullet_\Sigma\varphi_{k-1}\bullet_\Sigma\varphi_k$. Consider $C_\Sigma=\{\psi_1,\psi_2,\ldots,\psi_{k-1},\varphi\}$. This is a Σ-chain in $\boldsymbol{A}$, such that $C_\Sigma\subseteq{\downarrow_\Sigma}\varphi$. Define $\langle I_{\mathbf{Sign}},\pi\rangle:\boldsymbol{A}(C_\Sigma)\to(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))(Q_\Sigma)$ as the restriction of $\langle I_{\mathbf{Sign}},\pi^{\Omega^{\boldsymbol{A}}(T)}\rangle:\boldsymbol{A}\to\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T)$ to $\boldsymbol{A}(C_\Sigma)$. Note that, for all $i=1,\ldots,k$, $\psi_i/\Omega_\Sigma^{\boldsymbol{A}}(T)=\varphi_i/\Omega_\Sigma^{\boldsymbol{A}}(T)$, i.e. $\langle\varphi_i,\psi_i\rangle\in\Omega_\Sigma^{\boldsymbol{A}}(T)$,

whence $\pi_\Sigma(\psi_i) = \varphi_i / \Omega^{\boldsymbol{A}}_\Sigma(T)$. Furthermore, by compatibility of $\Omega^{\boldsymbol{A}}(T)$ with T, for all $\Sigma' \in |\mathbf{Sign}|$, $\chi \in C_{\Sigma'}$, we have $\chi \in T_{\Sigma'}$ iff $\pi_{\Sigma'}(\chi) \in T_{\Sigma'} / \Omega^{\boldsymbol{A}}_{\Sigma'}(T)$.
Now we are able to conclude:

$$\begin{aligned} & h_\Sigma^{T/\Omega^{\boldsymbol{A}}(T)}(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T)) \\ & \quad = |Q_\Sigma / \Omega_\Sigma^{(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))(Q_\Sigma)}(T/\Omega^{\boldsymbol{A}}(T) \cap (\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))(Q_\Sigma))| \\ & \quad = |C_\Sigma / \Omega_\Sigma^{\boldsymbol{A}(C_\Sigma)}(T \cap \boldsymbol{A}(C_\Sigma))| \\ & \quad \leq h_\Sigma^T(\varphi) \quad \text{(by definition of } h_\Sigma^T(\varphi)). \end{aligned}$$

∎

Proposition 17
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be semilattice system with sectionally finite height and $T \in \mathrm{SenFam}(\boldsymbol{A})$. If $T \in \mathrm{ClFam}(\boldsymbol{A})$, then

- $\bot_\Sigma \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and
- $h_\Sigma^T(\varphi) = h_\Sigma(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T))$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$.

Proof. Suppose that $T \in \mathrm{ClFam}(\boldsymbol{A})$. Then $T/\Omega^{\boldsymbol{A}}(T) = \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$. Since $\bot_\Sigma / \Omega^{\boldsymbol{A}}_\Sigma(T) \in \mathcal{R}_\Sigma(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$, for all $\Sigma \in |\mathbf{Sign}|$, we get that $\bot_\Sigma \in T_\Sigma$. For the second condition, note that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned} h_\Sigma(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T)) &= h_\Sigma^{\mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))}(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T)) \quad \text{(by Lemma 15)} \\ &= h_\Sigma^{T/\Omega^{\boldsymbol{A}}(T)}(\varphi / \Omega^{\boldsymbol{A}}_\Sigma(T)) \quad (T \text{ a cloud family}) \\ &= h_\Sigma^T(T) \quad \text{(by Lemma 16)}. \end{aligned}$$

∎

In Part (a) of the following example we provide a counterexample to the converse implication of that of Proposition 17.

Example 18
We work with the following semilattice system $\boldsymbol{A}$:

$$\begin{tikzcd} \Sigma \arrow[r, "f"] & \Sigma' \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{SEN}(f)"] \arrow[d, dash, "\prec_\Sigma"'] & \top \arrow[d, dash, "\prec_{\Sigma'}"] \\ 0 \arrow[r, "\mathrm{SEN}(f)"'] & \bot \end{tikzcd}$$

(a) If $T_\Sigma = \{0, 1\}$ and $T_{\Sigma'} = \{\bot\}$, then, we get $\Omega^{\boldsymbol{A}}(T) = \Delta^{\mathrm{SEN}}$ (by compatibility, $\bot$ and $\top$ cannot be identified in $\Omega^{\boldsymbol{A}}_{\Sigma'}(T)$ and, then, because of the system property of $\Omega^{\boldsymbol{A}}(T)$, 0 and 1 cannot be identified in $\Omega^{\boldsymbol{A}}_\Sigma(T)$). Since, clearly, $1/\Omega^{\boldsymbol{A}}_\Sigma(T) \in T_\Sigma / \Omega^{\boldsymbol{A}}_\Sigma(T)$, but $1/\Omega^{\boldsymbol{A}}_\Sigma(T) \notin \mathcal{R}_\Sigma(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$, we get $T/\Omega^{\boldsymbol{A}}(T) \neq \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$. Thus, $T = \{T_\Sigma, T_{\Sigma'}\}$ is not a cloud family of $\boldsymbol{A}$.

(b) If $T_\Sigma = \{0\}$ and $T_{\Sigma'} = \{\bot\}$, then, we get $\Omega^{\boldsymbol{A}}(T) = \Delta^{\mathrm{SEN}}$ (by compatibility, no elements can be identified in $\Omega^{\boldsymbol{A}}(T)$ since the sentence family T has singleton components). Now we have $T/\Omega^{\boldsymbol{A}}(T) = \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$. Thus, T is a cloud family of $\boldsymbol{A}$.

(c) If $T_\Sigma = \{0\}$ and $T_{\Sigma'} = \{\perp, \top\}$, then, we get $\Omega^{\boldsymbol{A}}_\Sigma(T) = \Delta^{\mathrm{SEN}}_\Sigma$ and $\Omega^{\boldsymbol{A}}_{\Sigma'}(T) = \nabla^{\mathrm{SEN}}_{\Sigma'}$ (being the largest congruence system on $\boldsymbol{A}$ compatible with T). Still, we get $T/\Omega^{\boldsymbol{A}}(T) = \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$, whence T is a cloud family of $\boldsymbol{A}$.

(d) If $T_\Sigma = \{1\}$ and $T_{\Sigma'} = \{\perp, \top\}$, then, we get, as in (c), $\Omega^{\boldsymbol{A}}_\Sigma(T) = \Delta^{\mathrm{SEN}}_\Sigma$ and $\Omega^{\boldsymbol{A}}_{\Sigma'}(T) = \nabla^{\mathrm{SEN}}_{\Sigma'}$. However, $1/\Omega^{\boldsymbol{A}}_\Sigma(T) \in T_\Sigma/\Omega^{\boldsymbol{A}}_\Sigma(T)$, but $1/\Omega^{\boldsymbol{A}}_\Sigma(T) \notin \mathcal{R}_\Sigma(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$ and, hence, $T/\Omega^{\boldsymbol{A}}(T) \neq \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$. Thus, T is not a cloud family of $\boldsymbol{A}$.

Since the converse implication of that of Proposition 17 does not hold in general, the analogue of the characterization of clouds given in Theorem 4.6 of [4] using the categorical Leibniz operator fails in the case of cloud families. A more complex idea, that of a *spectrum of an element relative to a sentence family*, needs to be introduced. However, the following corollary of Theorem 4.6 of [4] does hold:

Corollary 19
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system of sectionally finite height, with **Sign** a trivial category, and $T \in \mathrm{SenFam}(\boldsymbol{A})$. Then, $T \in \mathrm{ClFam}(\boldsymbol{A})$, if and only if

- $\perp_\Sigma \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and
- $h^T_\Sigma(\varphi) = h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$.

Proof. The left to right implication is established without the triviality restriction on the signature in Proposition 17. The right to left implication, with the trivial category hypothesis in force, is Theorem 4.6 of [4]. ∎

6 ...Spectroscopy opens the skies

Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system and $T \in \mathrm{SenFam}(\boldsymbol{A})$. Define a family of binary functions $\ell^T = \{\ell^T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $\ell^T_\Sigma : \mathrm{SEN}(\Sigma)^2 \to \{0,1\}$ is given, for all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, by

$$\ell^T_\Sigma(\varphi,\psi) = \begin{cases} 1, & \text{if } \varphi \in T_\Sigma \text{ iff } \psi \in T_\Sigma \\ 0, & \text{otherwise} \end{cases}$$

We call ℓ^T the **local coherence with respect to** T or the T**-local coherence function**. Define, similarly, a family of binary functions $g^T = \{g^T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $g^T_\Sigma : \mathrm{SEN}(\Sigma)^2 \to \{0,1\}$ is given, for all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, by

$$g^T_\Sigma(\varphi,\psi) = \begin{cases} 1, & \text{if } \mathrm{SEN}(f)(\varphi) \bullet_{\Sigma'} \chi' \in T_{\Sigma'} \text{ iff } \mathrm{SEN}(f)(\psi) \bullet_{\Sigma'} \chi' \in T_{\Sigma'}, \\ & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'), \chi' \in \mathrm{SEN}(\Sigma') \\ 0, & \text{otherwise} \end{cases}$$

Note that, g^T is the indicator or characteristic function associated with the Leibniz congruence system of T in $\boldsymbol{A}$, i.e.

$$g^T_\Sigma(\varphi,\psi) = \begin{cases} 1, & \text{if } \langle \varphi,\psi\rangle \in \Omega^{\boldsymbol{A}}_\Sigma(T) \\ 0, & \text{otherwise} \end{cases}$$

We call g^T the **global coherence with respect to** T or the T**-global coherence function**.

Denote, as usual, by $\oplus$ the binary XOR operation. Define c^T, the **coherence with respect to** T or the T**-coherence**, as the family of binary functions $c^T = \{c^T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $c^T_\Sigma : \mathrm{SEN}(\Sigma)^2 \to \{0,1\}$ is given, for all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, by

$$c^T_\Sigma(\varphi,\psi) = \ell^T_\Sigma(\varphi,\psi) \oplus g^T_\Sigma(\varphi,\psi).$$

We have the following characterization of c^T:

Lemma 20
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system and consider $T \in \mathrm{SenFam}(\boldsymbol{A})$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$c^T_\Sigma(\varphi,\psi) = \begin{cases} 1, & \text{if } (\phi \in T_\Sigma \text{ iff } \psi \in T_\Sigma) \text{ and } \langle \varphi,\psi\rangle \notin \Omega^{\boldsymbol{A}}_\Sigma(T) \\ 0, & \text{otherwise} \end{cases}$$

Proof. Let $\Sigma \in |\mathbf{Sign}|$ and $\varphi, \psi \in \mathrm{SEN}(\Sigma)$. Note that, by the compatibility property of the Leibniz congruence system, $g^T_\Sigma(\varphi,\psi) = 1$ implies $\ell^T_\Sigma(\varphi,\psi) = 1$. Therefore, we have

$$\begin{array}{lll} c^T_\Sigma(\varphi,\psi) = 1 & \text{iff} & \ell^T_\Sigma(\varphi,\psi) \oplus g^T_\Sigma(\varphi,\psi) = 1 \\ & \text{iff} & \ell^T_\Sigma(\varphi,\psi) \neq g^T_\Sigma(\varphi,\psi) \\ & \text{iff} & \ell^T_\Sigma(\varphi,\psi) = 1 \text{ and } g^T_\Sigma(\varphi,\psi) = 0. \end{array}$$

This immediately yields the displayed characterization. ∎

Let $\Sigma \in |\mathbf{Sign}|$ and consider a finite Σ-chain

$$C: \quad \varphi_1 <_\Sigma \varphi_2 <_\Sigma \cdots <_\Sigma \varphi_{k-1} <_\Sigma \varphi_k$$

of length k. Define the **coherence vector of** C **with respect to** T or the T**-coherence vector** of C as the vector $c^T_\Sigma(C) = \langle c^T_\Sigma(C)_i : 1 \leq i < k\rangle$, where

$$c^T_\Sigma(C)_i = c^T_\Sigma(\varphi_i, \varphi_{i+1}), \quad i = 1, \ldots, k-1.$$

Let $\leq$ be the lexicographic binary ordering of binary vectors. We define, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, the **spectrum of** φ **with respect to** T or the T**-spectrum of** φ by

$$\mathsf{Spc}^T_\Sigma(\varphi) = \max\{c^T_\Sigma(C) : C/\Omega^{\boldsymbol{A}}_\Sigma(T) \subseteq {\downarrow_\Sigma}\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T) \text{ a } \Sigma\text{-chain}\}.$$

We conclude by using spectra of elements to provide a characterization of cloud families in FSLs in what may be viewed as an analogue of Theorem 4.6 of [4] in the categorical multi-signature context. For any given dimension, we denote by $\mathbf{0} = \langle 0, 0, \ldots, 0\rangle$, the zero vector of that dimension (which is left intentionally unspecified, but is determined by the context in which $\mathbf{0}$ is used).

Theorem 21
Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a semilattice system with sectionally finite height and $T \in \mathrm{SenFam}(\boldsymbol{A})$. Then $T \in \mathrm{ClFam}(\boldsymbol{A})$ if and only if

- $\bot_\Sigma \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and
- $\mathsf{Spc}^T_\Sigma(\varphi) = \mathbf{0}$, for all $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{SEN}(\Sigma) \backslash \{\bot_\Sigma\}$.

Proof. Suppose, first, that $T \notin \mathrm{ClFam}(\boldsymbol{A})$. Then $T/\Omega^{\boldsymbol{A}}(T) \neq \mathcal{R}(T/\Omega^{\boldsymbol{A}}(T))$. Thus, there exists $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, such that either $\varphi \in T_\Sigma$ and $h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))$ is even or $\varphi \notin T_\Sigma$ and $h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))$ is odd. Exploiting sectional finiteness, let us choose such a φ, necessarily $\varphi \neq \bot_\Sigma$, such that $h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))$ is minimum. Consider a maximal Σ-chain in $\downarrow_\Sigma \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)$:

$$\bot_\Sigma/\Omega^{\boldsymbol{A}}_\Sigma(T) = \varphi_1/\Omega^{\boldsymbol{A}}_\Sigma(T) \prec_\Sigma \cdots \prec_\Sigma \varphi_{k-1}/\Omega^{\boldsymbol{A}}_\Sigma(T) \prec_\Sigma \varphi_k/\Omega^{\boldsymbol{A}}_\Sigma(T) = \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T).$$

Then, we have $\langle \varphi_{k-1}, \varphi \rangle \notin \Omega^{\boldsymbol{A}}_\Sigma(T)$, whence $g^T_\Sigma(\varphi_{k-1}, \varphi) = 0$. On the other hand, since φ is of minimum height satisfying the property

$$\varphi \in T_\Sigma \quad \text{iff} \quad h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)) \text{ is even},$$

we get $\varphi_{k-1} \in T_\Sigma$ iff $h_\Sigma(\varphi_{k-1}/\Omega^{\boldsymbol{A}}_\Sigma(T))$ is odd. Now, if one of φ, φ_{k-1} was in T_Σ and the other outside T_Σ, then, by the preceding equivalences, the heights $h_\Sigma(\varphi/\Omega^{\boldsymbol{A}}_\Sigma(T))$ and $h_\Sigma(\varphi_{k-1}/\Omega^{\boldsymbol{A}}_\Sigma(T))$ would have the same parity, which would contradict the maximality of the Σ-chain chosen above in $\downarrow_\Sigma \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)$. Therefore, we must have $\varphi_{k-1} \in T_\Sigma$ iff $\varphi \in T_\Sigma$, i.e. $\ell^T_\Sigma(\varphi_{k-1}, \varphi) = 1$. This shows that $c^T_\Sigma(\varphi_{k-1}, \varphi) = 1$, whence $c^T_\Sigma(\varphi) \neq \mathbf{0}$. Since $C/\Omega^{\boldsymbol{A}}_\Sigma(T)$ is a maximal Σ-chain in $\downarrow_\Sigma \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)$, we get $\mathsf{Spc}^T_\Sigma(\varphi) \neq \mathbf{0}$.

Suppose, conversely, that there exists $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, such that $\mathsf{Spc}^T_\Sigma(\varphi) \neq \mathbf{0}$. Then, there exists a maximal Σ-chain $C/\Omega^{\boldsymbol{A}}_\Sigma(T) \subseteq \downarrow_\Sigma \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T)$, say

$$\bot_\Sigma/\Omega^{\boldsymbol{A}}_\Sigma(T) = \varphi_1/\Omega^{\boldsymbol{A}}_\Sigma(T) \prec_\Sigma \cdots \prec_\Sigma \varphi_{k-1}/\Omega^{\boldsymbol{A}}_\Sigma(T) \prec_\Sigma \varphi_k/\Omega^{\boldsymbol{A}}_\Sigma(T) = \varphi/\Omega^{\boldsymbol{A}}_\Sigma(T),$$

such that, for some $i < k$, $c^T_\Sigma(\varphi_i, \varphi_{i+1}) = 1$. By Lemma 20, we get $\varphi_i \in T_\Sigma$ iff $\varphi_{i+1} \in T_\Sigma$ and $\langle \varphi_i, \varphi_{i+1} \rangle \notin \Omega^{\boldsymbol{A}}_\Sigma(T)$. But, then, either $\psi = \varphi_i$ or $\psi = \varphi_{i+1}$ provides a counterexample for $\psi \in T_\Sigma$ iff $h(\psi/\Omega^{\boldsymbol{A}}_\Sigma(T))$ is odd, since, otherwise, the maximality of the chain would again be violated. Therefore, we get that $T/\Omega^{\boldsymbol{A}}(T) \neq \mathcal{R}(\boldsymbol{A}/\Omega^{\boldsymbol{A}}(T))$, proving that $T \notin \mathrm{ClFam}(\boldsymbol{A})$. ∎

Example 22 (18 Revisited)
Consider again the semilattice system $\boldsymbol{A}$ of Example 18,

$$\Sigma \xrightarrow{\;f\;} \Sigma' \qquad \begin{array}{ccc} 1 & \xrightarrow{\mathrm{SEN}(f)} & \top \\ {\scriptstyle \prec_\Sigma}\Big| & & \Big|{\scriptstyle \prec_{\Sigma'}} \\ 0 & \xrightarrow[\mathrm{SEN}(f)]{} & \bot \end{array}$$

with $T_\Sigma = \{0, 1\}$ and $T_{\Sigma'} = \{\bot\}$. Recall from Example 18 Part (a) that $\Omega^{\boldsymbol{A}}(T) = \Delta^{\mathrm{SEN}}$. Clearly, $\ell^T_\Sigma(0,1) = 1$, $g^T_\Sigma(0,1) = 0$, whence $c^T_\Sigma(0,1) = 1$ and, therefore, $\mathsf{Spc}^T_\Sigma(1) = \langle 1 \rangle \neq \mathbf{0}$. Thus, $T \notin \mathrm{ClFam}(\boldsymbol{A})$, as was shown by explicit calculation in Example 18.

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