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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: SUBDIRECT REPRESENTATION FOR CLASSES OF STRUCTURE SYSTEMS

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Abstract

The notion of subdirect irreducibility in the context of languages without equality, as presented by Elgueta, is extended in order to obtain subdirect representation theorems for abstract and reduced classes of structure systems. Structure systems serve as models of first-order theories but, rather than having universal algebras as their algebraic reducts, they have algebraic systems in the sense of Categorical Abstract Algebraic Logic. The subdirect representation theory for partially ordered functors, presented in previous work by the author, becomes a special case of the theory presented here.

1. Introduction

This paper expands on the effort to adapt notions and results concerning the ordinary model theory of equality-free first-order logic, based on the notion of a first-order structure, as developed by Elgueta, Dellunde and others in the context of Abstract Algebraic Logic (AAL), to the equality-free first-order logic model theory that is based on the more general notion of an $\mathcal{L}$-structure system or, more simply, an $\mathcal{L}$-system. The concept of an $\mathcal{L}$-system was introduced in [25] as a vehicle for transporting results from the well-developed theory of first-order structures to structures whose underlying algebraic component is an algebraic system rather than an

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ordinary universal algebra. Algebraic systems, in turn, appeared first in the context of Categorical Abstract Algebraic Logic (CAAL) in [20], where sufficient evidence was provided to the effect that, in that context, they are the "right" algebraic entities to consider in place of ordinary universal algebras that have been at the focus from the very beginning in AAL.

In the present installment of these ongoing investigations, the focus is on revisiting and adapting the last part of [9], concerning subdirect irreducibility of first-order structures and subdirect representability of full and of reduced classes of structures, to the framework of $\mathcal{L}$-systems. As a consequence, the present work depends, to a large degree, on the work introducing and studying protoalgebraic classes of $\mathcal{L}$-systems [28], which was inspired by and based on the first part of [9]. In previous studies by the author on the same topic [21, 22, 25, 26, 27], the basic notions were introduced and the basic results of the theory developed, also inspired by and based on work of Elgueta [8, 9, 10, 11] and Elgueta and his collaborators [7, 12].

More specifically, in [9], Elgueta, based on his previous work [8], sets out to develop a theory of subdirect representability for full and for reduced classes of first-order structures defined without equality. His motivation, which is very similar to ours, is two-fold. On the one hand, it consists of abstracting results known previously in the context of the model theory of universal Horn logic without equality and with one unary predicate symbol, i.e., in the context of logical matrix models of sentential logics, to the level of arbitrary equality-free first-order structures. On the other hand, by obtaining such general results, he aims at expanding the scope of AAL by bringing under its wings various aspects of the theory that were previously thought to be outside its realm. Elgueta's journey brings him, first, to the land of filter congruences and protoalgebraic structures. A filter congruence on a structure is a pair consisting of a filter extension of the structure and of a congruence on that filter extension. Elgueta provides elegant and convincing arguments to the effect that filter congruences are the appropriate notions to replace ordinary universal algebraic congruences in the context of structures, since they help establish analogs of the well-known homomorphism theorems of Universal Algebra in that context. Analogs of these theorems, with the exception of the First Isomorphism Theorem, have also filter versions. In his quest to find a necessary and sufficient condition to establish a filter version of the First Isomorphism Theorem and inspired by the work of Blok and

Pigozzi [2], Elgueta introduces the notion of a protoalgebraic class of $\mathcal{L}$-structures and provides several interesting characterizations of protoalgebraic classes, that also serve to tie the theory of structures to various previous results known for the special case of logical matrices.

Both the notion of a filter congruence and that of a protoalgebraic class of structures were generalized by the author to cover $\mathcal{L}$-systems in [28]. In the present work the journey is continued with the treatment of the subdirect representation theory. The contents of the paper are briefly discussed in the remainder of this Introduction.

First, a necessary and sufficient condition is established for the subdirect representability of an $\mathcal{L}$-system in a given class of $\mathcal{L}$-systems. This condition generalizes both Corollary 3.3 of [9], which gives a necessary and sufficient condition for the subdirect representability of a structure into a class of structures, and the subdirect representation theorem for partially ordered functors of [24]. Then, two notions of subdirect irreducibility are formulated for $\mathcal{L}$-systems inside a given class, taking after corresponding notions of [9] from the theory of first-order structures. One of the two notions, that of subdirect irreducibility, uses reductive $\mathcal{L}$-morphisms, whereas the second, that of complete subdirect irreducibility, uses reductive $\mathcal{L}$-morphisms with isomorphic natural transformation components. It is shown that the two notions coincide when subdirect irreducibility in reduced classes is considered. Complete subdirect irreducibility and subdirect irreducibility are characterized in terms of meet irreducibility of filter congruence systems in various partially ordered structures of filter congruence systems. To make this more precise, recall that, given two $\mathcal{L}$-systems $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{B}}\rangle$ over the same underlying $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$, $\mathfrak{B}$ is said to be a *filter extension* of $\mathfrak{A}$, denoted $\mathfrak{A} \sqsubseteq \mathfrak{B}$, if, for every relation symbol $r \in R$, we have that $r^{\mathfrak{A}} \leq r^{\mathfrak{B}}$. Also recall that, given an $\mathcal{L}$-system $\mathfrak{A}$, as before, by a *congruence system on* $\mathfrak{A}$ is meant an $\mathbf{N}$-congruence system θ on SEN, that is compatible with the relation system $R^{\mathfrak{A}}$. Moreover, by a *filter congruence system on* $\mathfrak{A}$ is meant a pair $\langle \mathfrak{B}, \theta\rangle$, where $\mathfrak{A} \sqsubseteq \mathfrak{B}$ and θ is a congruence system on $\mathfrak{B}$. The trivial filter congruence system of an $\mathcal{L}$-system $\mathfrak{A}$ is the pair $\langle \mathfrak{A}, \Delta^{\mathrm{SEN}^{\mathfrak{A}}} \rangle$. With these definitions in mind, it is shown that an $\mathcal{L}$-system $\mathfrak{A}$ is subdirectly irreducible in a full class K if its trivial filter congruence system is meet irreducible in the poset $\mathbf{Fc}^{\Delta}_{\mathsf{K}}(\mathfrak{A}) = \langle \mathrm{Fc}^{\Delta}_{\mathsf{K}}(\mathfrak{A}), \sqsubseteq \rangle$, where

$$\mathrm{Fc}^{\Delta}_{\mathsf{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \Delta^{\mathrm{SEN}^{\mathfrak{A}}} \rangle : \mathfrak{A} \sqsubseteq \mathfrak{B} \in \mathsf{K}\},$$

whereas an $\mathcal{L}$-system $\mathfrak{A}$ is completely subdirectly irreducible in a full class K if its trivial filter congruence system is meet irreducible in the poset $\mathbf{Fc}_{\mathsf{K}}(\mathfrak{A}) = \langle \mathrm{Fc}_{\mathsf{K}}(\mathfrak{A}), \sqsubseteq \rangle$, where

$$\mathrm{Fc}_{\mathsf{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \theta \rangle : \mathfrak{A} \sqsubseteq \mathfrak{B} \in \mathsf{K} \text{ and } \theta \in \mathrm{Con}(\mathfrak{B})\}.$$

Furthermore, recalling that by $\Omega(\mathfrak{A})$ is denoted the Leibniz congruence system of an $\mathcal{L}$-system $\mathfrak{A}$, it is also shown that an $\mathcal{L}$-system $\mathfrak{A}$ is subdirectly irreducible in the reduction K^* of a full class K if its trivial filter congruence system is meet irreducible in the poset $\mathbf{Fc}^{l}_{\mathsf{K}}(\mathfrak{A}) = \langle \mathrm{Fc}^{l}_{\mathsf{K}}(\mathfrak{A}), \sqsubseteq \rangle$, where

$$\mathrm{Fc}^{l}_{\mathsf{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \theta \rangle \in \mathrm{Fc}_{\mathsf{K}}(\mathfrak{A}) : \theta = \Omega(\mathfrak{B})\}.$$

Finally, after elaborating on some of the connections between the different kinds of subdirectly irreducible members of a given class of $\mathcal{L}$-systems and its reduction, the task of formulating and proving subdirect representation theorems for $\mathcal{L}$-systems is undertaken. More precisely, given a full class K of $\mathcal{L}$-systems and an $\mathcal{L}$-algebraic system $\mathbf{A}$, it is shown that, if $\mathsf{K}_{\mathbf{A}}$, the class of all $\mathcal{L}$-systems in K with underlying $\mathcal{L}$-algebraic system $\mathbf{A}$, is closed under unions of $\sqsubseteq$-chains, then every $\mathcal{L}$-system in K is isomorphic to a subdirect product of subdirectly irreducible members of K. This theorem forms an analog of the Subdirect Representation Theorem for Full Classes of equality-free first-order structures of Elgueta (Theorem 3.14 of [9]). Finally, it is shown that, under the same conditions, every $\mathcal{L}$-system in the reduced class K^* is isomorphic to the reduction of a subdirect product of members of the class $\mathsf{K}_{\mathrm{RSI}} \cap \mathsf{K}^*$, i.e., the class of all reduced K-subdirectly irreducible $\mathcal{L}$-systems.

For all unexplained categorical definitions and notation, see, e.g., [1, 4, 16]. Two standard references on model theory are the books [5, 15]. For an overview of the present state of affairs in Abstract Algebraic Logic the reader is referred to the review article [14], the monograph [13] and the book [6]. For recent developments on the categorical side of the theory, see [18-20, 23].

2. Subdirect Representability

Let **Sign** be a category and SEN : **Sign** → **Set** a functor. Recall from [23] that the **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition is defined by

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i<\beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j<\gamma\rangle} \mathrm{SEN}^\gamma$$

$$\langle \sigma_j : j < \gamma\rangle \circ \langle \tau_i : i < \beta\rangle = \langle \sigma_j(\langle \tau_i : i < \beta\rangle) : j < \gamma\rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$, $i < k$, $k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

Given a functor SEN : **Sign** → **Set**, an n-**ary relation family** $r = \{r_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN is a $|\mathbf{Sign}|$-indexed collection of n-ary relations $r_\Sigma \subseteq \mathrm{SEN}(\Sigma)^n$, for all $\Sigma \in |\mathbf{Sign}|$. The relation family r is called a **relation system** if, in addition, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\mathrm{SEN}(f)^n(r_{\Sigma_1}) \subseteq r_{\Sigma_2}.$$

Recall, also, from [25] that a **(structure system) language** $\mathcal{L} = \langle \mathbf{F}, R, \rho\rangle$ consists of a category $\mathbf{F}$ of natural transformations on a given set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a nonempty collection R of relation symbols and an arity function $\rho : R \to \omega$, giving the arity of a relation symbol in R. An $\mathcal{L}$-**(structure) system** $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ consists of

- a functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Set}$,
- a category $\mathbf{N}^{\mathfrak{A}}$ of natural transformations on $\mathrm{SEN}^{\mathfrak{A}}$,
- a surjective functor $F^{\mathfrak{A}} : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}}$, that preserves all projections and, as result, preserves also the arity of all natural transformations and
- a collection $R^{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$ of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ indexed by R, such that $r^{\mathfrak{A}}$ is an n-ary relation system if $\rho(r) = n$.

Given two $\mathcal{L}$-systems $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$, an $\mathcal{L}$-**(system) morphism** $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathcal{L}$-algebraic morphism

$$\langle F, \alpha\rangle : \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle \to \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle,$$

such that $\alpha_\Sigma(r_\Sigma^{\mathfrak{A}}) \subseteq r_{F(\Sigma)}^{\mathfrak{B}}$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $r \in R$. $\langle F, \alpha\rangle$ is **surjective**, denoted $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, if F is surjective and $\alpha_\Sigma : \mathrm{SEN}^{\mathfrak{A}}(\Sigma) \to \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. Moreover, $\langle F, \alpha\rangle$ is called **strict**, denoted $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$, if $\alpha_\Sigma(r_\Sigma^{\mathfrak{A}}) = r_{F(\Sigma)}^{\mathfrak{B}}$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $r \in R$. Given two $\mathcal{L}$-systems $\mathfrak{A}$ and $\mathfrak{B}$, as above, $\mathfrak{A}$ is said to be an **expansion** of $\mathfrak{B}$ and $\mathfrak{B}$ a **contraction** of $\mathfrak{A}$ if there exists a strict surjective $\mathcal{L}$-morphism, also known as a **reductive $\mathcal{L}$-morphism**, $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$.

Now recall, from, e.g., [28], that a class $\mathtt{K}$ of $\mathcal{L}$-systems is said to be a **full class** whenever it is closed under expansions and contains an $\mathcal{L}$-system with at least one nonempty relation system. $\mathtt{K}$ is said to be an **abstract class** if it is full and closed under contractions. On the other hand, K is called a **reduced class** if it contains some nontrivial $\mathcal{L}$-system and all its members are (Leibniz) reduced $\mathcal{L}$-systems.

Let $\mathcal{L} = \langle \mathbf{F}, R, \rho\rangle$ be a system language, K be a class of $\mathcal{L}$-systems and $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ an $\mathcal{L}$-system not necessarily in K. $\mathfrak{A}$ is said to be **subdirectly representable in** K if it can be subdirectly embedded into a direct product of members of K. The two lemmas that follow from analogs of Lemmas 3.1 and 3.2 of [9] and provide a characterization of subdirect representability of an $\mathcal{L}$-system in a class of $\mathcal{L}$-systems. Given a subdirect embedding $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{B}_i$, we will denote by $\langle F^i, \alpha^i\rangle : \mathfrak{A} \to \mathfrak{B}_i$ the composition $\langle F^i, \alpha^i\rangle = \langle P^i, \pi^i\rangle \circ \langle F, \alpha\rangle$,

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "{\langle F,\alpha\rangle}"] \arrow[dr, "{\langle F^i,\alpha^i\rangle}"'] & \prod_{i\in I} \mathfrak{B}_i \arrow[d, "{\langle P^i,\pi^i\rangle}"] \\ & \mathfrak{B}_i \end{tikzcd}$$

where $\langle P^i, \pi^i\rangle : \prod_{i \in I} \mathfrak{B}_i \to \mathfrak{B}_i$ is the projection $\mathcal{L}$-morphism, for all $i \in I$. Of course $\langle F^i, \alpha^i\rangle$ is a surjective $\mathcal{L}$-morphism, for all $i \in I$, by the definition of a subdirect embedding.

Given an $\mathcal{L}$-structure system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, an $\mathbf{N}^{\mathfrak{A}}$-congruence system $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ on $\mathrm{SEN}^{\mathfrak{A}}$ is said to be a **congruence system of** $\mathfrak{A}$ if, for all $r \in R$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi}, \vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r_\Sigma^{\mathfrak{A}} \text{ and } \vec{\phi}\ \theta_\Sigma^n\ \vec{\varphi} \text{ imply } \vec{\varphi} \in r_\Sigma^{\mathfrak{A}}.$$

An $\mathcal{L}$-system $\mathfrak{B}$ is said to be a **filter extension** of an $\mathcal{L}$-system $\mathfrak{A}$, denoted $\mathfrak{A} \sqsubseteq \mathfrak{B}$, if, for all $r \in R$, $r^{\mathfrak{A}} \le r^{\mathfrak{B}}$. A **filter congruence system** of $\mathfrak{A}$ is a pair $\langle \mathfrak{B}, \theta\rangle$, where $\mathfrak{B}$ is a filter extension of $\mathfrak{A}$ and θ is a congruence system of $\mathfrak{B}$. Given a class K of $\mathcal{L}$-systems, by $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$ is denoted the class of all

$\mathbb{K}$-**filter congruence systems** of $\mathfrak{A}$, i.e., all those filter congruence systems $\langle \mathfrak{B}, \theta \rangle$ of $\mathfrak{A}$, such that $\mathfrak{B} \in \mathbb{K}$, and by $\mathrm{Fc}^{l}_{\mathbb{K}}(\mathfrak{A})$ the class of all $\mathbb{K}$-filter congruence systems $\langle \mathfrak{B}, \theta \rangle$ of $\mathfrak{A}$, such that $\theta = \Omega(\mathfrak{B})$, the Leibniz congruence of $\mathfrak{B}$, i.e., its largest congruence system.

Given an $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$, by $\mathrm{Ker}(\langle F, \alpha \rangle)$ is denoted the *kernel* of $\langle F, \alpha \rangle$, i.e., the collection $\mathrm{Ker}\,(\langle F, \alpha \rangle) = \{\mathrm{Ker}_{\Sigma}(\langle F, \alpha \rangle)\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$, with

$$\mathrm{Ker}_{\Sigma}(\langle F, \alpha \rangle) = \{\langle \phi, \varphi \rangle \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^2 : \alpha_{\Sigma}(\phi) = \alpha_{\Sigma}(\varphi)\}.$$

If $\langle F, \alpha \rangle$ is strict, then $\mathrm{Ker}(\langle F, \alpha \rangle)$ is a congruence system of $\mathfrak{A}$. Moreover, by $\mathrm{FKer}(\langle F, \alpha \rangle)$ is denoted the **filter kernel** of $\langle F, \alpha \rangle$, i.e., the pair $\langle \alpha^{-1}(\mathfrak{B}), \mathrm{Ker}(\langle F, \alpha \rangle) \rangle$, which is a filter congruence system of $\mathfrak{A}$.

Before proceeding to Lemma 2.1, recall from [28] that, given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, by $\Theta^{\mathfrak{A}}$ is denoted the filter congruence system $\Theta^{\mathfrak{A}} = \langle \mathfrak{A}, \Delta^{\mathrm{SEN}^{\mathfrak{A}}} \rangle$ of $\mathfrak{A}$. Lemma 2.1 provides necessary conditions for an embedding of an $\mathcal{L}$-system $\mathfrak{A}$ into the direct product $\prod_{i \in I} \mathfrak{B}_i$ of $\mathcal{L}$-systems $\mathfrak{B}_i$, $i \in I$, in a full class $\mathbb{K}$ to be a subdirect representation of $\mathfrak{A}$ in $\mathbb{K}$. A collection of functors $F^i : \mathbf{C} \to \mathbf{D}_i, i \in I$, is said to be **collectively mono** if, for all $\Sigma_1, \Sigma_2 \in |\mathbf{C}|$, $F^i(\Sigma_1) = F^i(\Sigma_2)$, for all $i \in I$, implies that $\Sigma_1 = \Sigma_2$, and similarly for morphisms.

Lemma 2.1. *Let* $\mathbb{K}$ *be a full class of* $\mathcal{L}$*-systems,* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ *an* $\mathcal{L}$*-system and* $\mathfrak{B}_i = \langle \mathrm{SEN}^{\mathfrak{B}_i}, \langle \mathbf{N}^{\mathfrak{B}_i}, F^{\mathfrak{B}_i} \rangle, R^{\mathfrak{B}_i} \rangle \in \mathbb{K}$, *for all* $i \in I$. *If* $\langle F, \alpha \rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{B}_i$ *is a subdirect embedding, then* $\mathrm{FKer}(\langle F^i, \alpha^i \rangle) \in \mathrm{Fc}_{\mathbb{K}}(\mathfrak{A})$, *for all* $i \in I$, *and, moreover,*

1. $\bigcap_{i \in I} \mathrm{FKer}(\langle F^i, \alpha^i \rangle) = \Theta^{\mathfrak{A}}$ *and*

2. *the pair* $\langle H^i, \gamma^i \rangle$, *given by* $H^i = F^i$ *and* $\gamma^i : \mathrm{SEN}^{\mathfrak{A}} / \mathrm{FKer}(\langle F^i, \alpha^i \rangle) \to \mathrm{SEN}^{\mathfrak{B}_i} \circ F^i$, *defined, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,

by

$$\gamma^i_\Sigma(\phi / \mathrm{FKer}_\Sigma(\langle F^i, \alpha^i\rangle)) = \alpha^i_\Sigma(\phi),$$

is a reductive $\mathcal{L}$-morphism $\langle H^i, \gamma^i\rangle : \mathfrak{A} / \mathrm{FKer}(\langle F^i, \alpha^i\rangle) \twoheadrightarrow_s \mathfrak{B}_i$, such that

(a) $\{H^i : i \in I\}$ *are collectively mono and*

(b) γ^i_Σ *is a bijection, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

Proof. Let $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{B}_i$, with $\mathfrak{B}_i \in \mathbb{K}$, for all $i \in I$, be a subdirect embedding and set $\langle F^i, \alpha^i\rangle = \langle P^i, \pi^i\rangle \circ \langle F, \alpha\rangle$, for all $i \in I$. First, let us see that $\bigcap_{i\in I} \mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$. Suppose that $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then, if $\alpha^i_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}_i}_{F^i(\Sigma)}$, for all $i \in I$, we obtain that $\alpha_\Sigma(\vec{\phi}) \in r^{\prod_{i\in I}\mathfrak{B}_i}_{F(\Sigma)}$, whence $\vec{\phi} \in r^{\mathfrak{B}}_\Sigma$, since $\langle F, \alpha\rangle$ is an embedding. Thus, we get that $\bigcap_{i\in I} \alpha^{i^{-1}}(\mathfrak{B}_{\mathrm{i}}) = \mathfrak{A}$. Also, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\phi, \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, if $\langle \phi, \varphi\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle F^i, \alpha^i\rangle)$, then $\alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\varphi)$, for all $i \in I$, whence $\alpha_\Sigma(\phi) = \alpha_\Sigma(\varphi)$, showing that $\phi = \varphi$, since $\langle F, \alpha\rangle$ is injective. Therefore, $\bigcap_{i\in I} \mathrm{Ker}(\langle F^i, \alpha^i\rangle) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$, which concludes the proof that $\bigcap_{i\in I} \mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$.

Since $\langle F^i, \alpha^i\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_i$ is a surjective $\mathcal{L}$-morphism, by Corollary 7 of [28], the pair $\langle F^i, \gamma^i\rangle : \mathfrak{A} / \mathrm{FKer}(\langle F^i, \alpha^i\rangle) \twoheadrightarrow_s \mathfrak{B}_i$ is a strict surjective $\mathcal{L}$-morphism with γ^i_Σ a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $i \in I$. This also shows that $\mathrm{FKer}(\langle F^i, \alpha^i\rangle) \in \mathrm{Fc}_{\mathbb{K}}(\mathfrak{A})$, since $\mathbb{K}$ is a full class and $\mathfrak{B}_i \in \mathbb{K}$ for all $i \in I$. To see that $\{F^i : i \in I\}$ are collectively mono, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|$, such that $F^i(\Sigma_1) = F^i(\Sigma_2)$, for all $i \in I$. Then $F(\Sigma_1) = F(\Sigma_2)$, whence $\Sigma_1 = \Sigma_2$, since $\langle F, \alpha\rangle$ is injective, and similarly for morphisms.

Lemma 2.2 complements Lemma 2.1 by providing sufficient conditions for an $\mathcal{L}$-system $\mathfrak{A}$ to be subdirectly embeddable into a direct product $\prod_{i\in I} \mathfrak{B}_i$

of a collection $\mathfrak{B}_i$, $i \in I$, of $\mathcal{L}$-systems. It also provides sufficient conditions for $\mathfrak{A}$ to be subdirectly representable in a given class $\mathbb{K}$ of $\mathcal{L}$-systems that is closed under contractions.

Lemma 2.2. *Let* $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$ *be an* $\mathcal{L}$*-system,* $\Theta^i = \langle \mathfrak{A}_i, \theta^i\rangle$, $i \in I$, *filter congruence systems on* $\mathfrak{A}$ *and* $\mathfrak{B}_i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i\rangle, R^i\rangle$, $i \in I$, *a family of* $\mathcal{L}$*-systems. If* $\bigcap_{i\in I} \Theta^i = \Theta^{\mathfrak{A}}$, *and there exists a family of reductive* $\mathcal{L}$*-morphisms* $\langle H^i, \gamma^i\rangle : \mathfrak{A}/\Theta^i \twoheadrightarrow_s \mathfrak{B}_i$, $i \in I$, *such that*

1. $\{H^i : i \in I\}$ *are collectively mono and*
2. γ^i_Σ *is a bijection, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $i \in I$,

then $\mathfrak{A}$ *is subdirectly embeddable in the direct product* $\prod_{i\in I} \mathfrak{B}_i$.

If, in addition, $\Theta^i \in \mathrm{Fc}_{\mathbb{K}}(\mathfrak{A})$, *for all* $i \in I$, *where* $\mathbb{K}$ *is a class closed under contractions, then* $\mathfrak{A}$ *is subdirectly representable in* $\mathbb{K}$.

Proof. Consider, for all $i \in I$, the projection $\mathcal{L}$-morphisms $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta^i}\rangle : \mathfrak{A} \to \mathfrak{A}/\Theta^i$. Compose with $\langle H^i, \gamma^i\rangle : \mathfrak{A}/\Theta^i \twoheadrightarrow_s \mathfrak{B}_i$ to obtain the $\mathcal{L}$-morphisms $\langle H^i, \gamma'^i\rangle : \mathfrak{A} \to \mathfrak{B}_i$, for all $i \in I$. Finally, set $\langle K, \kappa\rangle := \prod_{i\in I}\langle H^i, \gamma'^i\rangle : \mathfrak{A} \to \prod_{i\in I} \mathfrak{B}_i$.

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta_i}\rangle"] \arrow[d, "\langle K, \kappa\rangle"'] \arrow[dr, "\langle H^i, \gamma'^i\rangle"] & \mathfrak{A}/\Theta^i \arrow[d, "\langle H^i, \gamma^i\rangle"] \\
\prod_{i\in I} \mathfrak{B}_i \arrow[r, "\langle P^i, \pi^i\rangle"'] & \mathfrak{B}_i
\end{tikzcd}$$

As shown in the proof of Proposition 1 of [24], $\langle K, \kappa\rangle : \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle \to \prod_{i\in I}\langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i\rangle\rangle$ is a subdirect embedding of the underlying $\mathcal{L}$-algebraic systems. So it suffices to show that $\kappa(\mathfrak{A})$ is a subsystem of $\prod_{i\in I} \mathfrak{B}_i$. To this end, suppose that $r \in R$, with $\rho(r) = n$, and $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$. Then we have

$\vec{\phi} \in r_\Sigma^{\mathfrak{A}}$ iff $(\forall i \in I)\, (\vec{\phi} / \Theta_\Sigma^i \in r_\Sigma^{\mathfrak{A}/\Theta^i})$ (since $\bigcap_{i \in I} \Theta^i = \Theta^{\mathfrak{A}}$)

iff $(\forall i \in I)\, (\gamma_\Sigma^i(\vec{\phi} / \Theta_\Sigma^i) \in r_{H^i(\Sigma)}^{\mathfrak{B}_i})$ (since $\langle H^i, \gamma^i \rangle : \mathfrak{A} / \Theta^i \twoheadrightarrow_s \mathfrak{B}_i$)

iff $(\forall i \in I)\, (\gamma'^i_\Sigma(\vec{\phi}) \in r_{H^i(\Sigma)}^{\mathfrak{B}_i})$ (by the definition of $\langle H^i, \gamma'^i \rangle$)

iff $\kappa_\Sigma(\vec{\phi}) \in r_{K(\Sigma)}^{\prod_{i \in I} \mathfrak{B}_i}$. (by the definition of $\langle K, \kappa \rangle$)

Thus $\kappa(\mathfrak{A}) \subseteq_{\mathrm{sd}} \prod_{i \in I} \mathfrak{B}_i$.

If, now, $\Theta^i \in \mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$, for all $i \in I$, and K is closed under contractions, then $\mathfrak{A} / \Theta^i \in \mathrm{K}$, for all $i \in I$, whence $\mathfrak{B}_i \in \mathrm{K}$ and $\mathfrak{A}$ is subdirectly representable in K.

If Lemma 2.1 and Lemma 2.2 are combined, then the following corollary may be formulated. It gives a very simple and elegant characterization of the subdirect representability of an $\mathcal{L}$-system $\mathfrak{A}$ in an abstract class K of $\mathcal{L}$-systems in terms of the existence of a collection of K-filter congruence systems on $\mathfrak{A}$ whose meet is the trivial filter congruence system on $\mathfrak{A}$.

Corollary 2.3. *Let K be an abstract class of $\mathcal{L}$-systems and $\mathfrak{A}$ be an $\mathcal{L}$-system. Then $\mathfrak{A}$ is subdirectly representable in K if and only if, there exist $\Theta^i \in \mathrm{Fc}_{\mathrm{K}}(\mathfrak{A}), i \in I$, such that $\bigcap_{i \in I} \Theta^i = \Theta^{\mathfrak{A}}$.*

Corollary 2.3 has the following version when applied to reduced classes of $\mathcal{L}$-systems. Corollary 2.4 is an analog in the present framework of Corollary 3.4 of [9]. It characterizes the subdirect representability of a reduced $\mathcal{L}$-system $\mathfrak{A}$ into the reduction K^* of an abstract class K of $\mathcal{L}$-systems in terms of the existence of a collection of K-filter congruence systems on $\mathfrak{A}$ in the subset $\mathrm{Fc}_{\mathrm{K}}^l(\mathfrak{A})$ of $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$, whose meet is the trivial filter congruence system on $\mathfrak{A}$.

Corollary 2.4. *Let K be an abstract class of $\mathcal{L}$-systems and $\mathfrak{A}$ be a reduced $\mathcal{L}$-system. Then $\mathfrak{A}$ is subdirectly representable in K^* if and only if, there exist $\Theta^i \in \mathrm{Fc}_{\mathrm{K}}^l(\mathfrak{A}), i \in I$, such that $\bigcap_{i \in I} \Theta^i = \Theta^{\mathfrak{A}}$.*

Proof. Suppose that $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{B}_i$ with $\mathfrak{B}_i \in \mathrm{K}^*$, for all $i \in I$. Take $\Theta^i = \mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \langle {\alpha^i}^{-1}(\mathfrak{B}_i), \mathrm{Ker}(\langle F^i, \alpha^i\rangle)\rangle$. By Lemma 2.1, we have that the mapping $\langle H^i, \gamma^i\rangle : \mathfrak{A} / \mathrm{FKer}(\langle F^i, \alpha^i\rangle) \twoheadrightarrow_s \mathfrak{B}_i$ is a reductive $\mathcal{L}$-morphism. Therefore, by Theorem 5 of [21], we get that

$$\begin{aligned}
\Omega({\gamma^i}^{-1}(\mathfrak{B}_i)) &= {\gamma^i}^{-1}(\Omega(\mathfrak{B}_i)) \\
&= {\gamma^i}^{-1}(\Delta^{\mathrm{SEN}^{\mathfrak{B}_i}}) \\
&= \mathrm{Ker}(\langle F^i, \alpha^i\rangle) / \mathrm{FKer}\,(\langle F^i, \alpha^i\rangle) \\
&= \Delta^{\mathrm{SEN}^{\mathfrak{A}}/\mathrm{Ker}(\langle F^i, \alpha^i\rangle)}.
\end{aligned}$$

Thus, denoting by $\langle \mathrm{I}, \pi\rangle : {\alpha^i}^{-1}(\mathfrak{B}_i) \twoheadrightarrow_s {\alpha^i}^{-1}(\mathfrak{B}_i)/\mathrm{Ker}(\langle F^i, \alpha^i\rangle)$ the natural projection $\mathcal{L}$-morphism, we have

$$\begin{aligned}
\Omega({\alpha^i}^{-1}(\mathfrak{B}_i)) &= \Omega(\pi^{-1}({\gamma^i}^{-1}(\mathfrak{B}_i))) \\
&= \pi^{-1}(\Omega({\gamma^i}^{-1}(\mathfrak{B}_i))) \\
&= \pi^{-1}(\Delta^{\mathrm{SEN}^{\mathfrak{A}}/\mathrm{Ker}(\langle F^i, \alpha^i\rangle)}) \\
&= \mathrm{Ker}(\langle F^i, \alpha^i\rangle).
\end{aligned}$$

Therefore, $\Theta^i \in \mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$. Since, by Lemma 2.1,

$$\bigcap_{i\in I} \Theta^i = \bigcap_{i\in I} \mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}},$$

the left-to-right implication is complete.

Suppose, conversely, that $\Theta^i = \langle \mathfrak{A}_i, \Omega(\mathfrak{A}_i)\rangle \in \mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A}), i \in I$. Since the relation system $\bigcap_{i\in I} \Omega(\mathfrak{A}_i)$ is a congruence system of $\mathfrak{A}$ and $\mathfrak{A}$ is reduced, we get that $\bigcap_{i\in I} \Omega(\mathfrak{A}_i) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$. Now apply Lemma 2.2.

As is the case when one passes from algebras to first-order structures, when one passes from $\mathcal{L}$-algebraic systems to $\mathcal{L}$-systems the role of congruence systems is assumed by the filter congruence systems.

3. Notions of Subdirect Irreducibility

In [9], Elgueta explains that, in the context of equality-free languages, strict epimorphisms of structures behave, to a large extent, like isomorphisms. This motivates him to consider simultaneously two notions of subdirect irreducibility; one based on strict epimorphisms and one on isomorphisms, as is usually done in the context of universal algebras.

In extending the scope of Elgueta's work, we take into consideration, besides his approach, some of the guiding principles provided by the subdirect representation theory for partially ordered functors of [24], which show that, at the functor level, it is reasonable to only ask for surjectivity rather than a full isomorphism. The combination of these two approaches suggests adopting the following definitions as suitable analogs of those of Elgueta in the present context.

Let K be a class of $\mathcal{L}$-systems and $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle \in \mathrm{K}$ a nontrivial $\mathcal{L}$-system, i.e., such that $r^{\mathfrak{A}} \neq (\mathrm{SEN}^{\mathfrak{A}})^{\rho(r)}$, for some $r \in R$. $\mathfrak{A}$ is said to be (**finitely**) **subdirectly irreducible in** K, or simply (**finitely**) K-**subdirectly irreducible** if

$$\mathfrak{A} \overset{\langle P, \pi \rangle}{\twoheadrightarrow} \mathfrak{B} \overset{\langle F, \alpha \rangle}{\rightarrowtail}_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A}_i,$$

with P an isomorphism and $\mathfrak{A}_i \in \mathrm{K}$, for all $i \in I$ (I finite), implies $\langle F^i, \alpha^i \rangle \circ \langle P, \pi \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$ for some $i \in I$.

The $\mathcal{L}$-system $\mathfrak{A}$, on the other hand, is **completely subdirectly irreducible in** K or K-**completely subdirectly irreducible** if $\langle F, \alpha \rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A}_i$, with $\mathfrak{A}_i \in \mathrm{K}$, for all $i \in I$, implies $\langle F^i, \alpha^i \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$, with α^i_Σ a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, for some $i \in I$.

We write $\mathrm{K}_{\mathrm{RSI}}$ ($\mathrm{K}_{\mathrm{RFSI}}$) for the class of all (finitely) K-subdirectly irreducible $\mathcal{L}$-systems and $\mathrm{K}_{\mathrm{RCSI}}$ for the class of all K-completely subdirectly irreducible $\mathcal{L}$-systems. As a result, $\mathrm{K}^*_{\mathrm{RSI}}$, $\mathrm{K}^*_{\mathrm{RCSI}}$ denote, respectively, the classes of K^*-subdirectly irreducible and K^*-completely subdirectly irreducible $\mathcal{L}$-systems.

The next proposition shows that, for any class $\mathbb{K}$ of $\mathcal{L}$-systems, the class of all $\mathbb{K}^*$-completely subdirectly irreducible $\mathcal{L}$-systems and the class of all $\mathbb{K}^*$-subdirectly irreducible $\mathcal{L}$-systems coincide, i.e., that $\mathbb{K}^*_{\mathrm{RCSI}} = \mathbb{K}^*_{\mathrm{RSI}}$, for any class $\mathbb{K}$.

Proposition 3.1. *For every class* $\mathbb{K}$ *of* $\mathcal{L}$*-systems*, $\mathbb{K}^*_{\mathrm{RSI}} = \mathbb{K}^*_{\mathrm{RCSI}}$.

Proof. Suppose that $\mathfrak{A} \in \mathbb{K}^*_{\mathrm{RSI}}$ and that $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A}_i$, with $\mathfrak{A}_i \in \mathbb{K}^*$, $i \in I$. Then $\mathfrak{A} \overset{\langle I, \iota\rangle}{\twoheadrightarrow_s} \mathfrak{A} \overset{\langle F, \alpha\rangle}{\rightarrowtail_{\mathrm{sd}}} \prod_{i \in I} \mathfrak{A}_i$, satisfies the hypothesis of the condition for membership in $\mathbb{K}^*_{\mathrm{RSI}}$, whence, by the hypothesis, there exists $i \in I$, such that $\langle F^i, \alpha^i\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$. Since $\mathfrak{A}_i$ is reduced, by the Filter Homomorphism Theorem 2 of [28], we get that α^i_Σ is an isomorphism, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, whence $\mathfrak{A} \in \mathbb{K}^*_{\mathrm{RCSI}}$.

If, conversely, $\mathfrak{A} \in \mathbb{K}^*_{\mathrm{RCSI}}$ and $\mathfrak{A} \overset{\langle P, \pi\rangle}{\twoheadrightarrow_s} \mathfrak{B} \overset{\langle F, \alpha\rangle}{\rightarrowtail_{\mathrm{sd}}} \prod_{i \in I} \mathfrak{A}_i$, with P an isomorphism and $\mathfrak{A}_i \in \mathbb{K}^*$, then, since both $\mathfrak{A}$ and $\mathfrak{B}$ are reduced, we get, once more by Theorem 2 of [28], that $\langle P, \pi\rangle : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism. (Note here the assumption that P is an isomorphism.) Therefore $\mathfrak{A} \overset{\langle F, \alpha\rangle\langle P, \pi\rangle}{\rightarrowtail_{\mathrm{sd}}} \prod_{i \in I} \mathfrak{A}_i$ satisfies the hypothesis of the condition for membership in $\mathbb{K}^*_{\mathrm{RCSI}}$, which yields that, there exists $i \in I$, such that $\langle F^i, \alpha^i\rangle \circ \langle P, \pi\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$, with α^i_Σ a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. Therefore $\mathfrak{A} \in \mathbb{K}^*_{\mathrm{RSI}}$.

On page 234 of [9], a counterexample is provided for the claim $\mathbb{K}_{\mathrm{RSI}} = \mathbb{K}_{\mathrm{RCSI}}$, where, of course, $\mathbb{K}$ is a nonreduced class of $\mathcal{L}$-structures. The counterexample takes as $\mathbb{K}$ the class of all lattice quasi-ordered sets.

4. Characterizations of Subdirect Irreducibility

Recall from [28] that, given an $\mathcal{L}$-system $\mathfrak{A}$ and a class $\mathbb{K}$ of $\mathcal{L}$-systems, by $\mathrm{Fe}_{\mathbb{K}}(\mathfrak{A})$ is denoted the collection of all filter extensions of $\mathfrak{A}$ that are in $\mathbb{K}$, i.e.,

$$\mathrm{Fe}_{\mathrm{K}}(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{A} \sqsubseteq \mathfrak{B} \in \mathrm{K}\}.$$

Moreover, $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$ is denoted the collection of all K-filter congruence systems of $\mathfrak{A}$, i.e., all pairs $\langle \mathfrak{B}, \theta \rangle$ such that $\mathfrak{B}$ is a K-filter extension of $\mathfrak{A}$ and θ is a congruence system on $\mathfrak{B}$. Formally,

$$\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \theta \rangle : \mathfrak{A} \sqsubseteq \mathfrak{B} \in \mathrm{K} \text{ and } \theta \in Con(\mathfrak{B})\}.$$

Let K be a class of $\mathcal{L}$-systems, $\mathfrak{A}$ an $\mathcal{L}$-system and $X \subseteq \mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$. An element $\Theta \in \mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$ is said to be **meet-irreducible in** *X,* if, for all $\Theta^i \in X, i \in I, \bigwedge_{i \in I} \Theta^i = \Theta$ implies that $\Theta^i = \Theta$, for some $i \in I$. We provide in this section some results that characterize subdirect irreducibility of $\mathcal{L}$-systems in terms of meet irreducibility of filter congruence systems in appropriate partially ordered structures of filter congruence systems. These results generalize corresponding results of Elgueta [9], which, in turn, abstract well-known characterizations of subdirectly irreducible universal algebras via properties of their lattices of congruences.

Theorem 4.1 is an analog of Theorem 3.5 of [9] and characterizes K-subdirect irreducibility in terms of trivial K-filter congruence systems, i.e., in terms of K-filter extensions. Recall, once more from [28] that, given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle, R \rangle$, by $\mathrm{Fc}_{\mathrm{K}}^{\Delta}(\mathfrak{A})$ is denoted the Δ^{SEN}-section of $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$, i.e.,

$$\mathrm{Fc}_{\mathrm{K}}^{\Delta}(\mathfrak{A}) = \{\langle \mathfrak{B}, \Delta^{\mathrm{SEN}} \rangle : \mathfrak{A} \sqsubseteq \mathfrak{B} \in \mathrm{K}\}.$$

Clearly, this collection forms an isomorphic partially ordered set under $\sqsubseteq$ to the one formed by the collection $\mathrm{Fe}_{\mathrm{K}}(\mathfrak{A})$.

Theorem 4.1 *Let* K *be a full class of* $\mathcal{L}$*-systems and* $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle, R^{\mathfrak{A}} \rangle \in \mathrm{K}$ *a nontrivial* $\mathcal{L}$*-system. Then the following statements are equivalent*:

1. $\mathfrak{A} \in \mathrm{K}_{\mathrm{RSI}}$.

2. *For all* $\mathfrak{A}_i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i \rangle, R^i \rangle \in \mathrm{K}, i \in I, \langle F, \alpha \rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A}_i$ *implies* $\langle F^i, \alpha^i \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$, *for some* $i \in I$.

3. $\Theta^{\mathfrak{A}}$ *is meet irreducible in* $\mathrm{Fc}_{\mathbb{K}}^{\Delta}(\mathfrak{A})$.

4. *There exists* $r \in R$, *with* $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, *such that* $\vec{\phi} \notin r_{\Sigma}^{\mathfrak{A}}$ *but* $\vec{\phi} \in r_{\Sigma}^{\mathfrak{B}}$, *for all* $\mathfrak{B} \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{A}) - \{\mathfrak{A}\}$.

Proof. $1 \to 2$ This implication is obvious from the definition of $\mathbb{K}$-subdirect irreducibility.

$2 \to 3$ Let $\Theta^i = \langle \mathfrak{A}_i, \Delta^{\mathrm{SEN}} \rangle \in \mathrm{Fc}_{\mathbb{K}}^{\Delta}(\mathfrak{A})$, for all $i \in I$, and assume that $\Theta^{\mathfrak{A}} = \bigwedge_{i \in I} \Theta^i$. Then, by Lemma 2.2, $\langle F, \alpha \rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A} / \Theta^i$, where $F(\Sigma) = \prod_{i \in I} \Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and similarly for morphisms, and $\alpha_{\Sigma}(\phi) = \prod_{i \in I} \phi / \Theta_{\Sigma}^i$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$. Thus, by the hypothesis, $\langle F^i, \alpha^i \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A} / \Theta^i$, for some $i \in I$, whence $\Theta^{\mathfrak{A}} = \Theta^i$ and $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}_{\mathbb{K}}^{\Delta}(\mathfrak{A})$.

$3 \to 4$ Suppose, now, that $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}_{\mathbb{K}}^{\Delta}(\mathfrak{A})$ and let $\mathfrak{A}_0 = \bigwedge \{\mathfrak{B} \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{A}) : \mathfrak{B} \neq \mathfrak{A}\}$. Then $\mathfrak{A}_0 \neq \mathfrak{A}$, whence, there exists $r \in R$, such that $r^{\mathfrak{A}_0} \neq r^{\mathfrak{A}}$. Thus, since $\mathfrak{A}_0 \sqsubseteq \mathfrak{B}$, for all $\mathfrak{B} \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{A}) - \{\mathfrak{A}\}$, there exist $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, such that $\vec{\phi} \in r_{\Sigma}^{\mathfrak{B}}$, for all $\mathfrak{B} \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{A}) - \{\mathfrak{A}\}$, but $\vec{\phi} \notin r_{\Sigma}^{\mathfrak{A}}$.

$4 \to 1$ Suppose, finally, that there exist $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, such that $\vec{\phi} \in r_{\Sigma}^{\mathfrak{B}}$, for all $\mathfrak{B} \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{A}) - \{\mathfrak{A}\}$, and $\vec{\phi} \notin r_{\Sigma}^{\mathfrak{A}}$. Assume that

$$\mathfrak{A} \overset{\langle P, \pi \rangle}{\twoheadrightarrow_s} \mathfrak{B} \overset{\langle F, \alpha \rangle}{\rightarrowtail_{\mathrm{sd}}} \prod_{i \in I} \mathfrak{A}_i,$$

with P an isomorphism and $\mathfrak{A}_i \in \mathbb{K}$, for all $i \in I$. Fix $i \in I$. Then $\langle F^i, \alpha^i \rangle \circ \langle P, \pi \rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{A}_i$ is a surjective $\mathcal{L}$-morphism. Since $\mathbb{K}$ is full, $\mathrm{FKer}(\langle F^i, \alpha^i \rangle \circ \langle P, \pi \rangle) \in \mathrm{Fc}_{\mathbb{K}}(\mathfrak{A})$. We show that $\bigwedge_{i \in I} \mathrm{FKer}(\langle F^i, \alpha^i \rangle \circ \langle P, \pi \rangle) = \langle \mathfrak{A}, \mathrm{Ker}(\langle P, \pi \rangle) \rangle$. Since $\langle F, \alpha \rangle : \mathfrak{B} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A}_i$, we get, by Lemma 2.1, that $\bigwedge_{i\in I}\mathrm{FKer}\,(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{B}}$. But, by definition,

$$\mathrm{FKer}(\langle F^i, \alpha^i\rangle \circ \langle P, \pi\rangle) = \pi^{-1}(\mathrm{FKer}(\langle F^i, \alpha^i\rangle)).$$

This yields

$$\begin{aligned}\bigwedge_{i\in I}\mathrm{FKer}(\langle F^i, \alpha^i\rangle \circ \langle P, \pi\rangle) &= \bigwedge_{i\in I}\pi^{-1}(\mathrm{FKer}(\langle F^i, \alpha^i\rangle))\\ &= \pi^{-1}(\bigwedge_{i\in I}\mathrm{FKer}(\langle F^i, \alpha^i\rangle))\\ &= \pi^{-1}(\Theta^{\mathfrak{B}})\\ &= \langle \mathfrak{A},\ \mathrm{Ker}(\langle P,\ \pi\rangle)\rangle.\end{aligned}$$

Now we get $\mathfrak{A} = \bigcap_{i\in I}(\alpha^i \circ \pi)^{-1}(\mathfrak{A}_i)$, whence, by the hypothesis, there exists $i \in I$, such that $\mathfrak{A} = (\alpha^i \circ \pi)^{-1}(\mathfrak{A}_i)$. Therefore $\langle F^i, \alpha^i\rangle \circ \langle P, \pi\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$ is a reductive $\mathcal{L}$-morphism and $\mathfrak{A} \in \mathrm{K}_{\mathrm{RSI}}$.

Theorem 4.2 is an analog of Theorem 3.6 of [9] and characterizes K-complete subdirect irreducibility in terms of K-filter congruence systems. More precisely, it states that for any full class K of $\mathcal{L}$-systems and any nontrivial $\mathcal{L}$-system $\mathfrak{A}$ in K, $\mathfrak{A}$ is completely subdirectly irreducible in K if and only if the trivial filter congruence system $\Theta^{\mathfrak{A}}$ is meet irreducible in the collection $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$ of all K-filter congruence systems on $\mathfrak{A}$.

Theorem 4.2. *Let K be a full class of $\mathcal{L}$-systems and $\mathfrak{A} \in \mathrm{K}$ a nontrivial $\mathcal{L}$-system. Then $\mathfrak{A} \in \mathrm{K}_{\mathrm{RCSI}}$ if and only if $\Theta^{\mathfrak{A}}$ is meet irreducible in* $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$.

Proof. Suppose, first, that $\mathfrak{A} \in \mathrm{K}_{\mathrm{RCSI}}$ and let $\Theta^i = \langle \mathfrak{A}_i, \theta^i\rangle \in \mathrm{Fc}_{\mathrm{K}}(\mathfrak{A}), i \in I$, such that $\bigwedge_{i\in I}\Theta^i = \Theta^{\mathfrak{A}}$. Then, by Lemma 2.2, $\langle F,\ \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I}\mathfrak{A}/\Theta^i$, whence, since $\mathfrak{A} \in \mathrm{K}_{\mathrm{RCSI}}$ there exists $i \in I$, such that $\langle F^i, \alpha^i\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\Theta^i$, with α^i_Σ a bijection, for all $\Sigma \in |\,\mathbf{Sign}^{\mathfrak{A}}\,|$. Hence, we have that $\Theta^i = \Theta^{\mathfrak{A}}$ and $\Theta^{\mathfrak{A}}$ is meet-irreducible in $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$.

Suppose, conversely, that $\Theta^{\mathfrak{A}}$ is meet-irreducible in $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$ and that $\langle F,\ \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I}\mathfrak{A}_i$. Then, by Lemma 2.1, $\bigwedge_{i\in I}\mathrm{FKer}\,(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$.

Hence, by the meet-irreducibility of $\Theta^{\mathfrak{A}}$ in $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$, FKer $(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$, for some $i \in I$. Thus, there exists a reductive homomorphism $\langle G, \beta\rangle : \mathfrak{A} / \mathrm{FKer}(\langle F^i, \alpha^i\rangle) \twoheadrightarrow_s \mathfrak{A}_i$, such that β_Σ is a bijection, for all $\Sigma \in |\,\mathbf{Sign}^{\mathfrak{A}}\,|$. Since $\mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$, this proves that $\mathfrak{A} \in \mathrm{K}_{\mathrm{RCSI}}$.

For reduced classes of $\mathcal{L}$-systems, we have the following analog of Theorem 3.7 of [9] that uses the set of all Leibniz K-filter congruence systems. According to Theorem 4.3, a nontrivial reduced $\mathcal{L}$-system $\mathfrak{A}$ is subdirectly irreducible in the reduction K^* of a full class K of $\mathcal{L}$-systems if and only if its trivial filter congruence system $\Theta^{\mathfrak{A}}$ is meet irreducible in the collection $\mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$ of all filter congruence systems on $\mathfrak{A}$ of the form $\langle \mathfrak{B}, \Omega(\mathfrak{B})\rangle$.

For the proof, recall from Proposition 3.1 that, for every class K of $\mathcal{L}$-systems, $\mathrm{K}^*_{\mathrm{RSI}} = \mathrm{K}^*_{\mathrm{RCSI}}$.

Theorem 4.3. *Let* K *be a full class of* $\mathcal{L}$*-systems and let* $\mathfrak{A} \in \mathrm{K}^*$ *a nontrivial reduced* $\mathcal{L}$*-system. Then* $\mathfrak{A} \in \mathrm{K}^*_{\mathrm{RSI}}$ *if and only if* $\Theta^{\mathfrak{A}}$ *is meet irreducible in* $\mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$.

Proof. Suppose, first, that $\mathfrak{A} \in \mathrm{K}^*_{\mathrm{RSI}}$ and $\Theta^i = \langle \mathfrak{A}_i, \Omega(\mathfrak{A}_i)\rangle \in \mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A}), i \in I$, such that $\bigwedge_{i\in I}\Theta^i = \Theta^{\mathfrak{A}}$. Thus, by Lemma 2.2, there exists $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{A} / \Theta^i$. Therefore, since $\mathfrak{A} \in \mathrm{K}^*_{\mathrm{RSI}}$, there exists $i \in I$, such that $\langle F^i, \alpha^i\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A} / \Theta^i$, which shows that $\Theta^i = \Theta^{\mathfrak{A}}$ and $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$.

If, conversely, $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$ and $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{A}_i$, with $\mathfrak{A}_i \in \mathrm{K}^*$, for all $i \in I$, then we get, on the one hand, by Lemma 2.1, that $\bigwedge_{i\in I} \mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$ and, on the other, since $\langle F^i, \alpha^i\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{A}_i$ and $\mathfrak{A}_i$ is reduced, that $\mathrm{FKer}(\langle F^i, \alpha^i\rangle) \in \mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$. Therefore, by the hypothesis, there exists $i \in I$, such that $\mathrm{FKer}(\langle F^i, \alpha^i\rangle) = \Theta^{\mathfrak{A}}$. This shows that $\langle F^i, \alpha^i\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$ is a reductive $\mathcal{L}$-morphism and $\mathfrak{A} \in \mathrm{K}^*_{\mathrm{RSI}}$.

Finally, the section is concluded with an analog of Theorem 3.8 of [9] that provides a necessary condition for an $\mathcal{L}$-system to be completely subdirectly irreducible in terms of the congruence systems of the system rather than its filter congruence systems.

Theorem 4.4. *Let* K *be a class of* $\mathcal{L}$*-systems closed under contractions and* $\mathfrak{A} \in \mathrm{K}$ *a nontrivial* $\mathcal{L}$*-system. If* $\mathfrak{A} \in \mathrm{K}_{\mathrm{RCSI}}$, *then* $Con(\mathfrak{A})$ *has a monolith* (*i.e., for all* $\theta^i \in Con(\mathfrak{A}), i \in I$, *if* $\bigcap_{i \in I} \theta^i = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$, *then, there exists* $i \in I$, *such that* $\theta^i = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$).

Proof. Suppose that $Con(\mathfrak{A})$ has no monolith. Then

$$\bigcap (Con(\mathfrak{A}) - \{\Delta^{\mathrm{SEN}^{\mathfrak{A}}}\}) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}.$$

Hence, by Lemma 2.2, the pair $\langle F, \alpha \rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{\theta \neq \Delta^{\mathrm{SEN}^{\mathfrak{A}}}} \mathfrak{A} / \theta$, with $F(\Sigma) = \prod \Sigma$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and similarly for morphisms, and

$$\alpha_\Sigma(\phi) = \prod_{\theta \neq \Delta^{\mathrm{SEN}^{\mathfrak{A}}}} \phi/\theta_\Sigma,$$

for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, is a subdirect embedding. None of the projection $\mathcal{L}$-morphisms $\langle \mathrm{I}, \pi^\theta \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A} / \theta$, however, has injective natural transformation components, since $\theta \neq \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$, for all θ. Moreover, since K is closed under contractions, $\mathfrak{A} / \theta \in \mathrm{K}$, for all θ. Therefore, $\mathfrak{A}$ is not K-completely subdirectly irreducible.

5. Relating the Notions of Subdirect Irreducibility

Recall from [28], that a Lyndon class is a class of $\mathcal{L}$-systems which is full and closed under subdirect products. Theorem 6 of [27] states that, if K is a full class, then K is closed under subdirect products if and only if, for every $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle \rangle$, the collection $\mathrm{K}_{\mathbf{A}}$, of all members of K with underlying $\mathcal{L}$-algebraic system $\mathbf{A}$, is closed under arbitrary meets. Recall also from [28] that a class K of $\mathcal{L}$-systems is called **protoalgebraic**, if

Ω is $\sqsubseteq$- monotone in $\mathbb{K}$, i.e., if, for all $\mathcal{L}$-systems $\mathfrak{A}, \mathfrak{B} \in \mathbb{K}$,

$$\text{if } \mathfrak{A} \sqsubseteq \mathfrak{B}, \text{ then } \Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B}).$$

By Proposition 3.1, we know that, for every class $\mathbb{K}$ of $\mathcal{L}$-systems, $\mathbb{K}^*_{\mathrm{RSI}} = \mathbb{K}^*_{\mathrm{RCSI}}$. Moreover, by Theorem 4.1, it follows that $\mathbb{K}_{\mathrm{RCSI}} \subseteq \mathbb{K}_{\mathrm{RSI}}$.

In the following lemma, an analog of Lemma 3.9 of [9], it is shown that, if $\mathbb{K}$ is full, the class $\mathbb{K}_{\mathrm{RSI}}$ is closed under contractions via reductive $\mathcal{L}$-morphisms with isomorphic functor components, and that, if $\mathbb{K}$ is a protoalgebraic Lyndon class, it is closed under expansions via reductive $\mathcal{L}$-morphisms with isomorphic functor components.

Lemma 5.1. *Suppose that $\mathbb{K}$ is a full class of $\mathcal{L}$-systems and $\mathfrak{A}, \mathfrak{B}$ two $\mathcal{L}$-systems. If $\mathfrak{B}$ is a contraction of $\mathfrak{A}$ via a reductive $\mathcal{L}$-morphism with an isomorphic functor component, then $\mathfrak{A} \in \mathbb{K}_{\mathrm{RSI}}$ implies that $\mathfrak{B} \in \mathbb{K}_{\mathrm{RSI}}$. If, in addition, $\mathbb{K}$ is a protoalgebraic Lyndon class, then the converse also holds.*

Proof. Suppose, first, that $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathcal{L}$-morphism, with F an isomorphism, and $\mathfrak{A} \in \mathbb{K}_{\mathrm{RSI}}$. Let $\mathfrak{B}_i \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{B})$, $i \in I$, such that $\mathfrak{B} = \bigcap_{i \in I} \mathfrak{B}_i$. Since $\mathbb{K}$ is full, $\alpha^{-1}(\mathfrak{B}_i) \in \mathrm{Fe}_{\mathbb{K}}(\mathfrak{A})$, for all $i \in I$. Therefore, since $\langle F, \alpha \rangle$ is strict, we get that $\mathfrak{A} = \alpha^{-1}(\mathfrak{B}) = \alpha^{-1}(\bigcap_{i \in I} \mathfrak{B}_i) = \bigcap_{i \in I} \alpha^{-1}(\mathfrak{B}_i)$. Hence, since $\mathfrak{A} \in \mathbb{K}_{\mathrm{RSI}}$, there exists an $i \in I$, such that $\mathfrak{A} = \alpha^{-1}(\mathfrak{B}_i)$ and, therefore, using the surjectivity of $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_i$, we obtain $\mathfrak{B} = \alpha(\mathfrak{A}) = \alpha(\alpha^{-1}(\mathfrak{B}_i)) = \mathfrak{B}_i$. Thus, by Theorem 4.1, we get that $\mathfrak{B} \in \mathbb{K}_{\mathrm{RSI}}$.

Suppose, conversely, that $\mathbb{K}$ is a protoalgebraic Lyndon class and $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ a reductive $\mathcal{L}$-morphism, with F an isomorphism, and $\mathfrak{B} \in \mathbb{K}_{\mathrm{RSI}}$. By Theorem 4.1, an $\mathcal{L}$-system in $\mathbb{K}$ is $\mathbb{K}$-subdirectly irreducible if and only if the lattice of all its filter extensions has a monolith. Thus, we have $\mathbf{Fe}_{\mathbb{K}}(\mathfrak{B})$ has a monolith, whence, since, by the Filter Correspondence Property (Theorem 15 of [28]), $\mathbf{Fe}_{\mathbb{K}}(\mathfrak{A}) \cong \mathbf{Fe}_{\mathbb{K}}(\mathfrak{B})$, $\mathbf{Fe}_{\mathbb{K}}(\mathfrak{A})$ also has a monolith, and, therefore, $\mathfrak{A} \in \mathbb{K}_{\mathrm{RSI}}$.

Once more, if K is not a protoalgebraic Lyndon class, then the last statement of Lemma 5.1 does not hold in general. Elgueta provides on page 235 of [9] a counterexample to illustrate this.

The following corollary of Lemma 5.1 shows that the class of all reducts of K-subdirectly irreducible $\mathcal{L}$-systems consists of exactly the reduced $\mathcal{L}$-systems in $\mathrm{K}_{\mathrm{RSI}}$.

Corollary 5.2. *If* K *is a full class of* $\mathcal{L}$*-systems, then* $(\mathrm{K}_{\mathrm{RSI}})^* = \mathrm{K}_{\mathrm{RSI}} \cap \mathrm{K}^*$.

Proof. By Lemma 5.1, $(\mathrm{K}_{\mathrm{RSI}})^* \subseteq \mathrm{K}_{\mathrm{RSI}}$, whence $(\mathrm{K}_{\mathrm{RSI}})^* \subseteq \mathrm{K}_{\mathrm{RSI}} \cap \mathrm{K}^*$. The reverse inclusion is obvious.

Lemma 5.3, an analog of Lemma 3.11 of [9], shows that for an abstract class K of $\mathcal{L}$-systems, the K^*-completely subdirectly irreducible members form a subclass of all the K-completely subdirectly irreducible members.

Lemma 5.3. *If* K *is an abstract class of* $\mathcal{L}$*-systems, then* $\mathrm{K}^*_{\mathrm{RCSI}} \subseteq \mathrm{K}_{\mathrm{RCSI}}$.

Proof. Suppose that $\mathfrak{A} \in \mathrm{K}^*_{\mathrm{RCSI}}$. By Proposition 3.1 and Theorem 4.3, $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$. Since K is abstract, we get $\mathfrak{A} \in \mathrm{K}$. Hence, by Theorem 4.2, it suffices to show that $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$. Suppose, to this end, that $\Theta^{\mathfrak{A}} = \bigwedge_{i \in I} \Theta^i$, for some collection $\Theta^i = \langle \mathfrak{A}_i, \theta^i \rangle, i \in I$, of K-filter congruence systems of $\mathfrak{A}$. Define $\Theta^{i,\Omega} = \langle \mathfrak{A}_i, \Omega(\mathfrak{A}_i) \rangle, i \in I$. We have that $\Theta^{\mathfrak{A}} = \bigwedge_{i \in I} \Theta^{i,\Omega}$ because $\mathfrak{A} = \bigcap_{i \in I} \mathfrak{A}_i$ and $\bigcap_{i \in I} \Omega(\mathfrak{A}_i)$ is a congruence system on $\mathfrak{A}$, so that, since $\mathfrak{A}$ is reduced, $\bigcap_{i \in I} \Omega(\mathfrak{A}_i) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$. Thus, since $\Theta^{\mathfrak{A}}$ is meet irreducible in $\mathrm{Fc}^l_{\mathrm{K}}(\mathfrak{A})$, there exists $i \in I$, such that $\Theta^{\mathfrak{A}} = \Theta^{i,\Omega}$. Hence, since $\Theta^i \le \Theta^{i,\Omega}$, we get that $\Theta^{\mathfrak{A}} = \Theta^i$.

The next lemma shows that all completely subdirectly irreducible members in a full class that contains all trivial $\mathcal{L}$-systems are reduced.

Lemma 5.4. *Let* K *be a full class of* $\mathcal{L}$*-systems. If* K *contains all trivial* $\mathcal{L}$*-systems, then every member of* $\mathrm{K}_{\mathrm{RCSI}}$ *is reduced.*

Proof. Suppose that $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle \in \mathrm{K}_{\mathrm{RCSI}}$. Then $\Theta^{\mathfrak{A}} = \Theta^{\mathfrak{C}_{\mathbf{A}}} \wedge \Theta^{\mathfrak{A}, \Omega}$, where, of course, by $\Theta^{\mathfrak{C}_{\mathbf{A}}}$ is denoted the trivial filter congruence system $\Theta^{\mathfrak{C}_{\mathbf{A}}} := \langle \mathfrak{C}_{\mathbf{A}}, \Delta^{\mathrm{SEN}}\rangle$ on the $\mathcal{L}$-system $\mathfrak{C}_{\mathbf{A}}$, the trivial $\mathcal{L}$-system on the $\mathcal{L}$-algebraic system reduct $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ of $\mathfrak{A}$. But, by hypothesis, $\Theta^{\mathfrak{C}_{\mathfrak{A}}} \in \mathrm{Fc}_{\mathrm{K}}(\mathfrak{A})$, whence, by Theorem 4.2, $\Theta^{\mathfrak{A}} = \Theta^{\mathfrak{A}, \Omega}$, i.e., $\Omega(\mathfrak{A}) = \Delta^{\mathrm{SEN}}$ and $\mathfrak{A}$ is reduced.

Finally, combining Lemmas 5.3 and 5.4, we obtain the following analog of Theorem 3.13 of [9]. Its first part states that, for an abstract class K of $\mathcal{L}$-systems, the K^*-subdirectly irreducible $\mathcal{L}$-systems coincide with the reduced members of the class of all K-subdirectly irreducible $\mathcal{L}$-systems. Its second part adds the hypothesis that K contains all trivial $\mathcal{L}$-systems to conclude, based on Lemmas 5.3 and 5.4, that the class of all K-completely subdirectly irreducible members coincides with the subclass of all reduced members of the class of all K-subdirectly irreducible $\mathcal{L}$-systems.

Theorem 5.5. *The following holds, for all classes* K *of* $\mathcal{L}$*-systems*:

1. *If* K *is abstract, then* $\mathrm{K}^*_{\mathrm{RSI}} = \mathrm{K}_{\mathrm{RSI}} \cap \mathrm{K}^*$.

2. *If* K *is abstract and contains all trivial* $\mathcal{L}$*-systems, then* $\mathrm{K}_{\mathrm{RCSI}} = \mathrm{K}_{\mathrm{RSI}} \cap \mathrm{K}^*$.

Proof. 1. By Proposition 3.1, $\mathrm{K}^*_{\mathrm{RSI}} = \mathrm{K}^*_{\mathrm{RCSI}}$. Thus, by Lemma 5.3, $\mathrm{K}^*_{\mathrm{RSI}} \subseteq \mathrm{K}_{\mathrm{RCSI}}$. Hence, since $\mathrm{K}_{\mathrm{RCSI}} \subseteq \mathrm{K}_{\mathrm{RSI}}$, we obtain $\mathrm{K}^*_{\mathrm{RSI}} \subseteq \mathrm{K}_{\mathrm{RSI}} \cap \mathrm{K}^*$. The reverse inclusion is obvious.

2. By Part 1, Proposition 3.1 and Lemmas 5.3 and 5.4, we get that

$$\begin{aligned} \mathrm{K}_{\mathrm{RCSI}} &= \mathrm{K}^*_{\mathrm{RCSI}} \text{ (by Lemmas 5.3 and 5.4)} \\ &= \mathrm{K}^*_{\mathrm{RSI}} \text{ (by Proposition 3.1)} \\ &= \mathrm{K}_{\mathrm{RSI}} \cap \mathrm{K}^*. \text{ (by Part 1)} \end{aligned}$$

6. Subdirect Representation Theorems

In this section, analogs are provided of Theorems 3.14 and 3.15 of [9] for the subdirect representability of an $\mathcal{L}$-system in full and in reduced classes of $\mathcal{L}$-systems, respectively.

Recall that, given a class K of $\mathcal{L}$-systems and an $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle\rangle$, by $\mathrm{K}_{\mathbf{A}}$ is denoted the class of all $\mathcal{L}$-systems in K with underlying $\mathcal{L}$-algebraic system $\mathbf{A}$.

Theorem 6.1 provides a sufficient condition for the subdirect representability of a member of a full class K inside the class in terms of the closure of $\mathrm{K}_{\mathbf{A}}$ under unions of $\sqsubseteq$-chains, for every $\mathcal{L}$-algebraic system $\mathbf{A}$. More precisely, it is shown that, if K is a full class of $\mathcal{L}$-systems, such that $\mathrm{K}_{\mathbf{A}}$ is closed under unions of $\sqsubseteq$-chains, for all $\mathcal{L}$-algebraic systems $\mathbf{A}$, then every $\mathcal{L}$-system in K is isomorphic to a subdirect product of K-subdirectly irreducible members. As pointed out in [9], this condition is due essentially to Mal'cev (see Theorem 3 of the paper "*Subdirect Products of Models*" in [17]).

Theorem 6.1 (Subdirect Representation for Full Classes)**.** *Suppose that* K *is a full class of* $\mathcal{L}$*-systems such that* $\mathrm{K}_{\mathbf{A}}$ *is closed under unions of* $\sqsubseteq$*-chains, for all* $\mathcal{L}$*-algebraic systems* $\mathbf{A}$. *Then* $\mathrm{K} \subseteq \mathbf{P}_{\mathrm{sd}}(\mathrm{K}_{\mathrm{RSI}})$, *i.e., every* $\mathcal{L}$*-system in* K *is isomorphic to a subdirect product of members of* $\mathrm{K}_{\mathrm{RSI}}$.

Proof. Let $\mathfrak{A} \in \mathrm{K}$ and

$$I = \{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle : r \in R \text{ with } \rho(r) = n,\ \vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n - r_{\Sigma}^{\mathfrak{A}}\}.$$

For every $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$, choose $\mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \in \mathrm{Fe}_{\mathrm{K}}(\mathfrak{A})$, maximal with respect to the condition $\vec{\phi} \notin r_{\Sigma}^{\mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}}$. Such a maximal K-filter extension of $\mathfrak{A}$ exists, by Zorn's Lemma, since $\mathrm{K}_{\mathbf{A}}$ is closed under unions of $\sqsubseteq$-chains. Then we have that $\bigcap_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} = \mathfrak{A}$. Thus, by Corollary 2.3, $\mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$. Moreover, by definition and by Theorem 4.1, $\mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$ is K-subdirectly irreducible, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$. Hence we obtain that $\mathfrak{A} \in \mathbf{P}_{\mathrm{sd}}(\mathrm{K}_{\mathrm{RSI}})$.

Finally, Theorem 6.2, an analog of Theorem 3.15 of [9], is presented, that provides a sufficient condition for the subdirect representability of a member of the reduction $\mathbb{K}^*$ of a full class $\mathbb{K}$ inside $\mathbb{K}^*$, once more, in terms of the closure of $\mathbb{K}_{\mathbf{A}}$ under unions of $\sqsubseteq$-chains, for every $\mathcal{L}$-algebraic system $\mathbf{A}$. Theorem 6.2, shows that, if $\mathbb{K}$ is a full class of $\mathcal{L}$-systems, such that $\mathbb{K}_{\mathbf{A}}$ is closed under unions of $\sqsubseteq$-chains, for all $\mathcal{L}$-algebraic systems $\mathbf{A}$, then every $\mathcal{L}$-system in $\mathbb{K}^*$ is isomorphic to the reduction of a subdirect product of reduced $\mathbb{K}$-subdirectly irreducible members.

Before introducing the statement of the theorem, recall that, given an operator $\mathbf{O}$ on classes of $\mathcal{L}$-systems, the notation $\mathbf{O}^*$ is used to denote the operator $\mathbf{LO}$, where $\mathbf{L}$ is the operator that takes a class of $\mathcal{L}$-systems and maps it to the class of all isomorphic copies of the Leibniz reductions of its members.

Theorem 6.2 (Subdirect Repres. for Reduced Classes)**.** *Let $\mathbb{K}$ be a full class of $\mathcal{L}$-systems, such that $\mathbb{K}_{\mathbf{A}}$ is closed under unions of $\sqsubseteq$-chains, for every $\mathcal{L}$-algebraic system $\mathbf{A}$. Then $\mathbb{K}^* \subseteq \mathbf{P}^*_{\rm sd}(\mathbb{K}_{\rm RSI} \cap \mathbb{K}^*)$, i.e., every $\mathcal{L}$-system of $\mathbb{K}^*$ is isomorphic to the reduction of a subdirect product of members of $\mathbb{K}_{\rm RSI} \cap \mathbb{K}^*$.*

Proof. By the Filter Homomorphism Theorem (Theorem 2 of [28]), it suffices to show that, for all $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle \in \mathbb{K}$, some contraction of $\mathfrak{A}$, via a reductive $\mathcal{L}$-morphism with an isomorphic functor component, is subdirectly embeddable into a direct product of $\mathcal{L}$-systems in the class $\mathbb{K}_{\rm RSI} \cap \mathbb{K}^*$. Consider, again, the subdirect embedding $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\rm sd} \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$, where

$$I = \{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle : r \in R \text{ with } \rho(r) = n,\ \vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n - r^{\mathfrak{A}}_{\Sigma}\},$$

of the proof of Theorem 6.1, which is such that $\mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \in \mathbb{K}_{\rm RSI}$, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$. Consider also the strict $\mathcal{L}$-morphism

$$\langle K, \kappa\rangle : \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \to_s \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle},$$

given by $K = \mathrm{I}$, where I is the identity functor on $\prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathbf{Sign}^{\mathfrak{A}}$, and, for all $\Sigma_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\varphi_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle})$, $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$, by

$$\kappa_{\prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \Sigma_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}}(\vec{\varphi}) = \langle \varphi^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} : \langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I\rangle,$$

where, of course, $\varphi^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$ denotes the reduction of $\varphi_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$ in $\mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$. Since both $\langle F, \alpha\rangle$ and $\langle K, \kappa\rangle$ are strict $\mathcal{L}$-morphisms, so is their composition $\langle K, \kappa\rangle \circ \langle F, \alpha\rangle$, whence, by Lemma 2 of [21], $\mathrm{Ker}(\langle K, \kappa\rangle \circ \langle F, \alpha\rangle) \in Con(\mathfrak{A})$. Moreover, the quotient $\mathfrak{A}/\mathrm{Ker}(\langle K, \kappa\rangle \circ \langle F, \alpha\rangle)$ is embeddable into $\prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$ by the induced $\mathcal{L}$-morphism

$$\langle G, \beta\rangle : \mathfrak{A}/\mathrm{Ker}(\langle K, \kappa\rangle \circ \langle F, \alpha\rangle) \to \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle},$$

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "{\langle F, \alpha\rangle}"] \arrow[dr, "{\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\mathrm{Ker}(\langle K,\kappa\rangle \circ \langle F,\alpha\rangle)}\rangle}"'] & \prod_{\langle r,\langle \Sigma,\vec{\phi}\rangle\rangle \in I} \mathfrak{A}_{\langle r,\langle \Sigma,\vec{\phi}\rangle\rangle} \arrow[r, "{\langle K, \kappa\rangle}"] & \prod_{\langle r,\langle \Sigma,\vec{\phi}\rangle\rangle \in I} \mathfrak{A}^*_{\langle r,\langle \Sigma,\vec{\phi}\rangle\rangle} \\
& \mathfrak{A}/\mathrm{Ker}(\langle K, \kappa\rangle \circ \langle F, \alpha\rangle) \arrow[ur, "{\langle G, \beta\rangle}"'] &
\end{tikzcd}$$

in such a way that $\langle \mathrm{I}, \pi^{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}\rangle \circ \langle G, \beta\rangle : \mathfrak{A}/\mathrm{Ker}(\langle K, \kappa\rangle \circ \langle F, \alpha\rangle) \to \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$ is surjective, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$, where by

$$\langle \mathrm{I}, \pi^{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}\rangle : \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \to \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$$

is denoted the natural projection $\mathcal{L}$-morphism, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$. Therefore $\langle G, \beta\rangle : \mathfrak{A}/\mathrm{Ker}(\langle K, \kappa\rangle \circ \langle F, \alpha\rangle) \rightarrowtail_{\mathrm{sd}} \prod_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I} \mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle}$.

To finish up the proof, note, now, that, since $\mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \in (\mathbb{K}_{\mathrm{RSI}})^*$, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$, Corollary 5.2 gives that $\mathfrak{A}^*_{\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle} \in \mathbb{K}_{\mathrm{RSI}} \cap \mathbb{K}^*$, for all $\langle r, \langle \Sigma, \vec{\phi}\rangle\rangle \in I$, which completes the proof.

The following corollary is now easily derivable from Theorem 6.2.

Corollary 6.3. *Let* $\mathbb{K}$ *be an abstract class of* $\mathcal{L}$*-systems, such that* $\mathbb{K}_{\mathbf{A}}$ *is closed under unions of* $\sqsubseteq$*-chains, for every* $\mathcal{L}$*-algebraic system* $\mathbf{A}$. *Then the following hold*:

1. *If* $\mathbb{K}$ *contains all trivial* $\mathcal{L}$*-systems, then* $\mathbb{K}^* \subseteq \mathbf{P}^*_{\rm sd}(\mathbb{K}_{\rm RCSI})$.

2. *If, in addition,* $\mathbb{K}$ *is protoalgebraic, then* $\mathbb{K}^* \subseteq \mathbf{P}_{\rm sd}(\mathbb{K}_{\rm RCSI})$.

Proof. Part 1 follows by combining Theorem 6.2 and Lemma 5.5. Part 2 also follows easily by applying Theorem 17 of [28] to Part 1, since all $\mathcal{L}$-systems in $\mathbb{K}_{\rm RCSI}$ are reduced.

Finally, let us point out, in closing, that Elgueta gives an example on page 239 of [9] that shows that the subdirect representation theory of first-order structures gives different results for lattices considered as lattice-ordered sets from the ones usually obtained in universal algebra, when lattices are viewed as algebras. This, Elgueta remarks, is simply due to the fact that the analysis of the structure theory in the context of first-order structures critically depends on the language that is being used.

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