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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC WEAKLY REFERENTIAL π-INSTITUTIONS

A b s t r a c t. Wójcicki introduced in the late 1970s the concept of a referential semantics for propositional logics. Referential semantics incorporate features of the Kripke possible world semantics for modal logics into the realm of algebraic and matrix semantics of arbitrary sentential logics. A well-known theorem of Wójcicki asserts that a logic has a referential semantics if and only if it is selfextensional. A second theorem of Wójcicki asserts that a logic has a weakly referential semantics if and only if it is weakly self-extensional. We formulate and prove an analog of this theorem in the categorical setting. We show that a π-institution has a weakly referential semantics if and only if it is weakly self-extensional.

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1. Introduction

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a logical signature/algebraic type, i.e., a set of logical connectives/operation symbols Λ with attached finite arities given by the function $\rho : \Lambda \to \omega$. Let, also, V be a countably infinite set of propositional variables and T a set of reference/base points. Wójcicki [5] defines a **referential algebra $\boldsymbol{A}$** to be an $\mathcal{L}$-algebra with universe $A \subseteq \{0,1\}^T$. Such an algebra determines the consequence operation $C^{\boldsymbol{A}}$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ by setting, for all $X \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\alpha \in C^{\boldsymbol{A}}(X)$ iff, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A}$ and all $t \in T$,

$$h(\beta)(t) = 1, \text{ for all } \beta \in X, \quad \text{implies} \quad h(\alpha)(t) = 1.$$

Moreover, Wójcicki calls a propositional logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where $C = C^{\boldsymbol{A}}$, for a referential algebra $\boldsymbol{A}$, a **referential** (or **referentially truth-functional**) **propositional logic**.

Wójcicki shows in [5] that, given a class K of referential algebras, there exists a single referential algebra $\boldsymbol{A}$, such that $C^{\mathsf{K}} := \bigcap_{\boldsymbol{K} \in \mathsf{K}} C^{\boldsymbol{K}} = C^{\boldsymbol{A}}$. Thence follows that a propositional logic is referential if and only if it is defined by a class of referential algebras.

Given a propositional logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$, the **Frege** or **interderivability relation** of $\mathcal{S}$ (see, e.g., Definition 2.37 of [3]), denoted $\Lambda(\mathcal{S})$, is the equivalence relation on $\mathrm{Fm}_{\mathcal{L}}(V)$, defined, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\langle \alpha, \beta \rangle \in \Lambda(\mathcal{S}) \quad \text{iff} \quad C(\alpha) = C(\beta).$$

The **Tarski congruence** $\widetilde{\Omega}(\mathcal{S})$ of $\mathcal{S}$ (see [3]) is the largest congruence relation on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with all theories of $\mathcal{S}$. The Tarski congruence is a special case of the **Suszko congruence** $\widetilde{\Omega}^{\mathcal{S}}(T)$ associated with a given theory T of $\mathcal{S}$, which is defined as the largest congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with all theories of $\mathcal{S}$ that contain the given theory T (see [2]). In fact, by definition, $\widetilde{\Omega}(\mathcal{S}) = \widetilde{\Omega}^{\mathcal{S}}(C(\varnothing))$, i.e., the Tarski congruence of $\mathcal{S}$ is the Suszko congruence associated with the set of theorems of the logic $\mathcal{S}$. Font and Jansana (see p.17 of [3]), extending Blok and Pigozzi's [1] well-known characterization of the *Leibniz congruence* $\Omega(T)$ associated with a theory T of a sentential logic, have shown that, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\langle \alpha, \beta \rangle \in \widetilde{\Omega}(\mathcal{S}) \quad \text{iff} \quad \text{for all } \varphi(p, \vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \; C(\varphi(\alpha, \vec{q})) = C(\varphi(\beta, \vec{q})).$$

Whereas $\widetilde{\Omega}(\mathcal{S}) \subseteq \Lambda(\mathcal{S})$, for every propositional logic $\mathcal{S}$, the reverse inclusion does not hold in general. A propositional logic is called **selfextensional** in [5] if $\Lambda(\mathcal{S}) \subseteq \widetilde{\Omega}(\mathcal{S})$. In fact, Wójcicki shows in what has become a fundamental theorem in the theory of referential semantics, Theorem 2 of [5], that a propositional logic is referential if and only if it is self-extensional.

Wójcicki in [6] revisited the equivalence between referentiality and self-extensionality, proving a "weak version" by replacing the entirety of theories (equivalently, the closure operator C) by the set of theorems. More precisely, Wójcicki considers in [6] (see the Theorem in [6]) propositional logics $\mathcal{S} = \langle \mathcal{L}, C\rangle$, where $C(\varnothing) = C^{\boldsymbol{A}}(\varnothing)$, for a referential algebra $\boldsymbol{A}$. We call such logics **weakly referential logics**.

Given a propositional logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$, the **Leibniz congruence** $\Omega(T)$ of a theory T of $\mathcal{S}$ (see [1]) is the largest congruence relation on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with T. Blok and Pigozzi's well-known characterization of the Leibniz congruence $\Omega(T)$ (see p. 11 of [1]) asserts that, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{aligned} \langle \alpha, \beta\rangle \in \Omega(T) \quad \text{iff} \quad & \text{for all } \varphi(p, \vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \\ & \varphi(\alpha, \vec{q}) \in T \quad \text{iff} \quad \varphi(\beta, \vec{q}) \in T. \end{aligned}$$

A propositional logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$ is called **weakly selfextensional** in [6] if, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\alpha, \beta \in C(\varnothing) \quad \text{implies} \quad \langle \alpha, \beta\rangle \in \Omega(C(\varnothing)).$$

In the Theorem of [6], Wójcicki shows that a propositional logic is weakly referential if and only if it is weakly self-extensional.

2. π-Institutions and Closure Systems

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a **Set**-valued functor. The **clone of all natural transformations on** SEN (see Section 2 of [8]) is the category U with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition of $\langle \tau_i : i < \beta\rangle : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ with $\langle \sigma_j : j < \gamma\rangle : \mathrm{SEN}^\beta \to \mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory of this category with objects all objects of the form SEN^k, $k < \omega$, and such that:

- it contains all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}$ given by
$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$
- for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, $\langle \tau_i : i < l \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N,

is referred to as a **category of natural transformations on** SEN.

Consider an **algebraic system** $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, i.e., a triple consisting of

- a category **Sign**, called the **category of signatures**;
- a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, called the **sentence functor**;
- a category of natural transformations N on SEN.

A π-**institution based on** $\boldsymbol{F}$ is a pair $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$, where $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is a **closure system on** SEN, i.e., a $|\mathbf{Sign}|$-indexed collection of closure operators $C_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$, such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$,

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\Phi)).$$

This condition is sometimes referred to as **structurality**. In this context, $\boldsymbol{F}$ is also referred to as the **base algebraic system**. Given a π-institution $\mathcal{I}$, a **theory family** $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is a $|\mathbf{Sign}|$-indexed collection of subsets $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, closed under C_Σ, i.e., such that $C_\Sigma(T_\Sigma) = T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$. The collection of all theory families of $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$. Ordered by signature-wise inclusion, it forms a complete lattice $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$.

Note, also, that, given a base algebraic system $\boldsymbol{F}$, the collection of all closure systems based on $\boldsymbol{F}$ is closed under signature-wise intersections and, hence, forms a complete lattice under the signature-wise ordering $\leq$:

$$C^1 \leq C^2 \quad \text{iff} \quad \text{for all } \Sigma \in |\mathbf{Sign}| \text{ and all } \Phi \subseteq \mathrm{SEN}(\Sigma), \; C^1_\Sigma(\Phi) \subseteq C^2_\Sigma(\Phi).$$

3. Referential π-Institutions

We assume a base algebraic system $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Consider also an N-algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, i.e., one such that there exists a surjective functor $' : N \to N'$, preserving all projection natural transformations and, as a consequence, all arities of the natural transformations involved. We denote by $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ the natural transformation in N' that is the image of $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N under $'$.

More specifically, we want to focus on N-algebraic systems $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}'_s, N'\rangle$, where SEN'_s is a simple subfunctor (having the same domain) of the inverse powerset functor $\overleftarrow{\mathcal{P}}\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ of a contravariant functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}^{\mathsf{op}}$. For $\Sigma \in |\mathbf{Sign}'|$, the elements of $\mathrm{SEN}'(\Sigma)$ in this context are referred to as **Σ-reference** or **Σ-base points** (see, e.g., [9]). An N-morphism $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$ will be viewed as a valuation of sentences of SEN in the following way: For all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\varphi \in \mathrm{SEN}(\Sigma)$ is **true at** $p \in \mathrm{SEN}'(F(\Sigma))$ **under** $\langle F, \alpha\rangle$ iff $p \in \alpha_\Sigma(\varphi)$.

An N-algebraic system of this special form is called a **referential N-algebraic system**. By slightly abusing terminology, we use the same term to refer to an (**interpreted**) **referential N-algebraic system**, which is a pair $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, with $\langle F, \alpha\rangle : \boldsymbol{F} \to \boldsymbol{A}$ an algebraic system morphism, also referred to as an N-morphism. We sometimes drop the subscript $_s$ when referring to the subfunctor to make notation less cumbersome, provided that this is unlikely to cause any confusion.

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ an interpreted referential N-algebraic system. Then $\mathcal{A}$ determines a closure system $C^{\mathcal{A}}$ on SEN (or on $\boldsymbol{F}$) according to the following definition:

For all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $\varphi \in C^{\mathcal{A}}_\Sigma(\Phi)$ iff, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))$$

(ϕ and φ, here, are intentionally different).

Proposition 1 (Proposition 1 of [11]). *Suppose* $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *is a base algebraic system and* $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ *an interpreted referential N-algebraic system. Then $C^{\mathcal{A}}$ is a closure system on* $\boldsymbol{F}$.

Since $C^{\mathcal{A}}$ is a closure system on $\boldsymbol{F}$, the pair $\mathcal{I}^{\mathcal{A}} = \langle \boldsymbol{F}, C^{\mathcal{A}} \rangle$ is a π-institution. We call an institution having this form a **referential π-institution**. Such π-institutions correspond in the theory of categorical abstract algebraic logic (CAAL) to the referential propositional logics of Wójcicki [5].

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ a π-institution based on $\boldsymbol{F}$. We define the **Frege equivalence system** $\Lambda(\mathcal{I})$ of $\mathcal{I}$ (see p. 37 of [7]), also known as the **interderivability equivalence system**, by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{if and only if} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

The **Tarski congruence system** $\widetilde{\Omega}(\mathcal{I})$ of $\mathcal{I}$ ([3] for the universal algebraic notion and [10] for its categorical extension) is the largest N-congruence system on SEN that is compatible with every theory family $T \in \mathrm{ThFam}(\mathcal{I})$.

Clearly, it is always the case that $\widetilde{\Omega}(\mathcal{I}) \le \Lambda(\mathcal{I})$. We call the π-institution $\mathcal{I}$ **self-extensional** if $\Lambda(\mathcal{I}) \le \widetilde{\Omega}(\mathcal{I})$. In view of the preceding remark, $\mathcal{I}$ is self-extensional if and only if $\Lambda(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$.

A generalization to π-institutions of Wójcicki's Theorem (see Theorem 2 of [5], but, also, Theorem 2.2 of [4] for a complete proof) provides a characterization of referential π-institutions

Theorem 2 (Theorem 8 of [9])**.** *A π-institution $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ is referential if and only if it is self-extensional.*

4. Weakly Referential π-Institutions

We assume a base algebraic system $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$. Recall that for any (interpreted) referential N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$, the pair $\mathcal{I}^{\mathcal{A}} = \langle \boldsymbol{F}, C^{\mathcal{A}} \rangle$ is a referential π-institution. We call a π-institution $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ a **weakly referential π-institution** if, for all $\Sigma \in |\mathbf{Sign}|$,

$$C_\Sigma(\varnothing) = C_\Sigma^{\mathcal{A}}(\varnothing),$$

for some referential π-institution $\mathcal{I}^{\mathcal{A}}$. Such π-institutions correspond in the theory of CAAL to the weakly referential propositional logics of Wójcicki [6].

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ a π-institution based on $\boldsymbol{F}$. Let, also $T \in \mathrm{ThFam}(\mathcal{I})$. The **Leibniz congruence system** $\Omega(T)$ of T ([1] for the universal algebraic notion and p. 223 of [8] for its categorical extension) is the largest N-congruence system on SEN that is compatible with the theory family T. We denote by $\mathrm{Thm} = \{\mathrm{Thm}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ the **theorem family** of $\mathcal{I}$, i.e., $\mathrm{Thm}_\Sigma = C_\Sigma(\varnothing)$, for all $\Sigma \in |\mathbf{Sign}|$.

We call the π-institution $\mathcal{I}$ **weakly self-extensional** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\varphi, \psi \in \mathrm{Thm}_\Sigma \quad \text{implies} \quad \langle \varphi, \psi\rangle \in \Omega_\Sigma(\mathrm{Thm}_\Sigma).$$

A generalization to π-institutions of Wójcicki's Theorem (see the Theorem of [6]) provides a characterization of weakly referential π-institutions. This is the main result of the present work, formulated in Theorem 9. The value rests in both furnishing a more detailed proof based on the sketch provided in [6], and, also, in extending the scope of the result to encompass logics formalized as π-institutions. We start with the easy direction.

Proposition 3. *If a π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is weakly referential, then it is weakly self-extensional.*

Proof. Suppose that $\mathcal{I}$ is weakly referential. Thus, there exists a referential N-algebraic system $\mathcal{A}$, such that $C_\Sigma(\varnothing) = C^{\mathcal{A}}_\Sigma(\varnothing)$, for all $\Sigma \in |\mathbf{Sign}|$. Let $\Sigma \in |\mathbf{Sign}|$ and $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\varphi, \psi \in C_\Sigma(\varnothing) = C^{\mathcal{A}}_\Sigma(\varnothing)$. This implies that $C^{\mathcal{A}}_\Sigma(\varphi) = C^{\mathcal{A}}_\Sigma(\psi)$, i.e., that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}^{\mathcal{A}})$. Since $\mathcal{I}^{\mathcal{A}}$ is referential, it is self-extensional by Theorem 2. Thus, we get $\langle \phi, \psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}^{\mathcal{A}})$. Therefore, by the characterization theorem of the Tarski Operator in CAAL, Theorem 4 of [10], for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$,

$$C^{\mathcal{A}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})) = C^{\mathcal{A}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})).$$

Thus, we obtain, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}) \in \mathrm{Thm}_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in \mathrm{Thm}_{\Sigma'}.$$

This shows that $\langle \varphi, \psi\rangle \in \Omega_\Sigma(\mathrm{Thm})$. □

Let $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ be a weakly self-extensional π-institution, with theorem family Thm. Define the family $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ by setting

$$R_\Sigma = \left\{\frac{\sigma_\Sigma(\varphi,\vec{\chi})}{\sigma_\Sigma(\psi,\vec{\chi})} : \sigma \text{ in } N, \vec{\chi} \in \mathrm{SEN}(\Sigma)^k, \varphi,\psi \in \mathrm{Thm}_\Sigma\right\},$$

where, following a common convention in CAAL, when we write $\frac{\sigma_\Sigma(\varphi,\vec{\chi})}{\sigma_\Sigma(\psi,\vec{\chi})}$, we mean that φ,ψ may occupy any position in σ and not just the first, as long as they occupy the same position in both the antecedent and the consequent of the rule.

Define on $\boldsymbol{F}$ the operator family $C^{\mathrm{Thm},R} = \{C^{\mathrm{Thm},R}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $C^{\mathrm{Thm},R}_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$ is given, for all $\Phi\cup\{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\varphi \in C^{\mathrm{Thm},R}_\Sigma(\Phi) \quad \text{iff} \quad \varphi \text{ is } R_\Sigma\text{-provable from } \Phi \cup \mathrm{Thm}_\Sigma.$$

Then, we can show that $C^{\mathrm{Thm},R}$ is a closure system on $\boldsymbol{F}$:

Lemma 4. *Let $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ be a weakly self-extensional π-institution, with theorem family* Thm. *Then $C^{\mathrm{Thm},R}$ is a closure system on $\boldsymbol{F}$.*

Proof. By classical proof-theoretic arguments, one shows that $C^{\mathrm{Thm},R}_\Sigma$ is a closure operator on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. So it suffices to show that $C^{\mathrm{Thm},R}$ is structural. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C^{\mathrm{Thm},R}_\Sigma(\Phi)$. This means that there exists an R_Σ-proof

$$\varphi_0, \varphi_1, \dots, \varphi_n = \varphi$$

of φ from $\Phi\cup\mathrm{Thm}_\Sigma$. We must show that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f)(\varphi) \in C^{\mathrm{Thm},R}_{\Sigma'}(\mathrm{SEN}(f)(\Phi))$. Consider the sequence of Σ'-sentences

$$\mathrm{SEN}(f)(\varphi_0), \mathrm{SEN}(f)(\varphi_1), \dots, \mathrm{SEN}(f)(\varphi_n) = \mathrm{SEN}(f)(\varphi).$$

It suffices to show that this is a valid $R_{\Sigma'}$-proof of $\mathrm{SEN}(f)(\varphi)$ from hypotheses $\mathrm{SEN}(f)(\Phi) \cup \mathrm{Thm}_{\Sigma'}$. This is accomplished by induction on $0 \le k \le n$:

Base: If $k = 0$, then φ_0 must be a Σ-sentence in $\Phi \cup \mathrm{Thm}_\Sigma$. But then, since the theorem family of any π-institution is a theory system, we get that $\mathrm{SEN}(f)(\varphi_0)$ is in $\mathrm{SEN}(f)(\Phi) \cup \mathrm{Thm}_{\Sigma'}$.

Hypothesis: Suppose, for all $i < k \leq n$, $\mathrm{SEN}(f)(\varphi_i)$ is either in $\mathrm{SEN}(f)(\Phi) \cup \mathrm{Thm}_{\Sigma'}$ or follows from previous sentences in the sequence by a single application of an $R_{\Sigma'}$-rule.

Step: If φ_k is in $\Phi \cup \mathrm{Thm}_\Sigma$, then, as in the Base, it follows that $\mathrm{SEN}(f)(\varphi_k)$ is in $\mathrm{SEN}(f)(\Phi) \cup \mathrm{Thm}_{\Sigma'}$. Suppose, finally, that φ_k follows from $\varphi_i, i < k$, by a single application of an R_Σ-rule, i.e., there exists σ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^p$, such that $\varphi_i = \sigma_\Sigma(\varphi, \vec{\chi})$ and $\varphi_k = \sigma_\Sigma(\psi, \vec{\chi})$, for some $\varphi, \psi \in \mathrm{Thm}_\Sigma$. But, then, for the same σ in N and $\mathrm{SEN}(f)(\vec{\chi}) \in \mathrm{SEN}(\Sigma')^p$, we have that $\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi) \in \mathrm{Thm}_{\Sigma'}$ and

$$\begin{aligned}\mathrm{SEN}(f)(\varphi_i) &= \sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)^p(\vec{\chi})),\\ \mathrm{SEN}(f)(\varphi_k) &= \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)^p(\vec{\chi})).\end{aligned}$$

Thus, $\mathrm{SEN}(f)(\varphi_k)$ follows from $\mathrm{SEN}(f)(\varphi_i)$ by an application of the $R_{\Sigma'}$-rule $\dfrac{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\vec{\chi}))}{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\vec{\chi}))}$.

This concludes the proof of structurality of $C^{\mathrm{Thm},R}$. □

Thus, $\mathcal{I}^{\mathrm{Thm},R} = \langle \boldsymbol{F}, C^{\mathrm{Thm},R} \rangle$ is a π-institution. Let us denote by $\mathrm{Thm}^R = \{\mathrm{Thm}^R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ the theorem system of $\mathcal{I}^{\mathrm{Thm},R}$. It turns out that the theorem system Thm^R coincides with the theorem system Thm of $\mathcal{I}$:

Lemma 5. *Let $\mathcal{I} = \langle \boldsymbol{F}, C \rangle$ be a weakly self-extensional π-institution, with theorem family* Thm. *Then* $\mathrm{Thm} = \mathrm{Thm}^R$.

Proof. Clearly, by the definition of $C^{\mathrm{Thm},R}$, $\mathrm{Thm} \leq \mathrm{Thm}^R$.

For the converse, suppose that $\Sigma \in |\mathbf{Sign}|$ and $\varphi \in \mathrm{Thm}^R_\Sigma$. Thus, $\varphi \in C^{\mathrm{Thm},R}_\Sigma(\varnothing)$. This means that there exists an R_Σ-proof

$$\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$$

of ϕ from Thm_Σ. We show by induction on $k \leq n$ that $\varphi_k \in \mathrm{Thm}_\Sigma$.

Base: If $k = 0$, then φ_0 must be in Thm_Σ by hypothesis.

Hypothesis: Suppose that, for all $i < k \leq n$, $\varphi_i \in \mathrm{Thm}_\Sigma$.

Step: If $\varphi_k \in \mathrm{Thm}_\Sigma$, then there is nothing to prove. Otherwise, φ_k follows from φ_i, $i < k$, by an application of an R_Σ-rule. Thus, for some σ in N, some $\vec{\chi} \in \mathrm{SEN}(\Sigma)^p$ and some $\varphi, \psi \in \mathrm{Thm}_\Sigma$,

$$\varphi_i = \sigma_\Sigma(\varphi, \vec{\chi}), \quad \varphi_k = \sigma_\Sigma(\psi, \vec{\chi}).$$

By weak selfextensionality of $\mathcal{I}$, we get $\langle\varphi,\psi\rangle \in \Omega_\Sigma(\mathrm{Thm})$. Thus, since $\Omega(\mathrm{Thm})$ is a congruence system, $\langle\varphi_i,\varphi_k\rangle \in \Omega_\Sigma(\mathrm{Thm})$. Since, by the Induction Hypothesis, $\varphi_i \in \mathrm{Thm}_\Sigma$, by the compatibility of the Leibniz congruence system, we get $\varphi_k \in \mathrm{Thm}_\Sigma$.

This shows that $\varphi \in \mathrm{Thm}_\Sigma$. Therefore $\mathrm{Thm}^R \leq \mathrm{Thm}$. □

The next result shows that $\mathcal{I}^{\mathrm{Thm},R}$ is a self-extensional π-institution. Intuitively speaking, this feature is instilled to the π-institution by virtue of its definition.

Lemma 6. *Let $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ be a weakly self-extensional π-institution, with theorem family* Thm. *Then $\mathcal{I}^{\mathrm{Thm},R}$ is a selfextensional π-institution.*

Proof. Suppose $\Sigma \in |\mathbf{Sign}|$ and $\varphi,\psi \in \mathrm{SEN}(\Sigma)$ are such that

$$C_\Sigma^{\mathrm{Thm},R}(\varphi) = C_\Sigma^{\mathrm{Thm},R}(\psi).$$

Then $\varphi \in C_\Sigma^{\mathrm{Thm},R}(\psi)$. Let $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$ be fixed but arbitrary. Our goal is to show that $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi),\vec{\chi}) \in C_{\Sigma'}^{\mathrm{Thm},R}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}))$. By symmetry, it then follows

$$C_{\Sigma'}^{\mathrm{Thm},R}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi),\vec{\chi})) = C_{\Sigma'}^{\mathrm{Thm},R}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})),$$

i.e., that $\mathcal{I}^{\mathrm{Thm},R}$ is self-extensional.

Suppose first that $\varphi \in \mathrm{Thm}_\Sigma$. Then, $\psi \in \mathrm{Thm}_\Sigma$ also. Hence $\mathrm{SEN}(f)(\varphi)$ and $\mathrm{SEN}(f)(\psi)$ are in $\mathrm{Thm}_{\Sigma'}$. Therefore, $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi),\vec{\chi})$ follows by an application of a rule in $R_{\Sigma'}$ from $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})$. This proves that $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi),\vec{\chi}) \in C_{\Sigma'}^{\mathrm{Thm},R}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}))$.

Now we turn to the case where $\varphi \notin \mathrm{Thm}_\Sigma$. Since $\varphi \in C_\Sigma^{\mathrm{Thm},R}(\psi)$, there exists an R_Σ-proof

$$\varphi_0,\varphi_1,\ldots,\varphi_n = \varphi$$

of φ from premises $\{\psi\} \cup \mathrm{Thm}_\Sigma$. Consider the sequence

$$\varphi'_0,\varphi'_1,\ldots,\varphi'_n,$$

defined by induction on $k \leq n$ as follows:

- If $\varphi_k = \psi$, then $\varphi'_k = \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})$.

- If $\varphi_k \in \mathrm{Thm}_\Sigma$, then $\varphi'_k = \mathrm{SEN}(f)(\varphi_k)$.
- If φ_k follows from φ_i, $i < k$, by an application of the R_Σ-rule $\frac{\tau_\Sigma(\zeta,\vec{\eta})}{\tau_\Sigma(\xi,\vec{\eta})}$, we set:
  - $\varphi'_k = \mathrm{SEN}(f)(\phi_k)$, if $\varphi'_i = \mathrm{SEN}(f)(\varphi_i)$;
  - $\varphi'_k = \sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi_k), \vec{\chi})$, if $\varphi'_i = \sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi_i), \vec{\chi})$.

Our goal is to show that this is a valid $R_{\Sigma'}$-proof of $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})$ from premises $\{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\} \cup \mathrm{Thm}_{\Sigma'}$. We do this by employing induction on $k \leq n$ to show that the sequence

$$\varphi'_0, \varphi'_1, \dots, \varphi'_k$$

is an $R_{\Sigma'}$-proof of φ'_k from premises $\{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\} \cup \mathrm{Thm}_{\Sigma'}$.

Base: If $k = 0$, we have two cases:

- If $\varphi_0 = \psi$, then $\varphi'_0 = \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})$ follows by hypothesis.
- If $\varphi_0 \in \mathrm{Thm}_\Sigma$, then $\varphi'_0 = \mathrm{SEN}(f)(\varphi_0) \in \mathrm{Thm}_{\Sigma'}$ also follows by hypothesis.

Hypothesis: Assume that, for all $i < k \leq n$,

$$\varphi'_0, \varphi'_1, \dots, \varphi'_i$$

is a valid $R_{\Sigma'}$-proof of φ'_i from premises $\{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\} \cup \mathrm{Thm}_{\Sigma'}$.

Step: If $\varphi_k = \psi$ or $\varphi_k \in \mathrm{Thm}_\Sigma$, then we replicate the reasoning in the Base.

Suppose that φ_k follows from φ_i, $i < k$, by an application of the R_Σ-rule $\frac{\tau_\Sigma(\zeta,\vec{\eta})}{\tau_\Sigma(\xi,\vec{\eta})}$, where $\zeta, \eta \in \mathrm{Thm}_\Sigma$.

- If $\varphi'_i = \mathrm{SEN}(f)(\varphi_i)$, then $\varphi'_k = \mathrm{SEN}(f)(\varphi_k)$. Since $\zeta, \xi \in \mathrm{Thm}_\Sigma$, $\mathrm{SEN}(f)(\zeta), \mathrm{SEN}(f)(\xi) \in \mathrm{Thm}_{\Sigma'}$. Thus, this step in the proof is justified by the fact that

$$\frac{\varphi'_i}{\varphi'_k} = \frac{\mathrm{SEN}(f)(\varphi_i)}{\mathrm{SEN}(f)(\varphi_k)} = \frac{\tau_{\Sigma'}(\mathrm{SEN}(f)(\zeta), \mathrm{SEN}(f)^p(\vec{\eta}))}{\tau_{\Sigma'}(\mathrm{SEN}(f)(\xi), \mathrm{SEN}(f)^p(\vec{\eta}))}$$

is a valid $R_{\Sigma'}$-rule.

– If $\varphi'_i = \sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi_i),\vec{\chi})$, then $\varphi'_k = \sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi_k),\vec{\chi})$. Once more, since $\zeta,\xi \in \mathrm{Thm}_\Sigma$, we get $\mathrm{SEN}(f)(\zeta),\mathrm{SEN}(f)(\xi) \in \mathrm{Thm}_{\Sigma'}$. Thus, this step in the proof is justified by the fact that

$$\frac{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi_i),\vec{\chi})}{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi_k),\vec{\chi})} = \frac{\sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\zeta),\mathrm{SEN}(f)^p(\vec{\eta})),\vec{\chi})}{\sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\xi),\mathrm{SEN}(f)^p(\vec{\eta})),\vec{\chi})}$$

is a valid $R_{\Sigma'}$-rule.

By symmetry, interchanging the roles of φ,ψ in the preceding reasoning, we get that, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$,

$$C^{\mathrm{Thm},R}_{\Sigma'}(\mathrm{SEN}(f)(\varphi),\vec{\chi}) = C^{\mathrm{Thm},R}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}).$$

By the CAAL characterization theorem of the Tarski congruence system of a π-institution (Theorem 4 of [10]), we get that $\langle\varphi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}^{\mathrm{Thm},R})$. This proves that $\mathcal{I}^{\mathrm{Thm},R}$ is a selfextensional π-institution. □

Corollary 7. *Let $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ be a weakly self-extensional π-institution, with theorem family* Thm. *Then $\mathcal{I}^{\mathrm{Thm},R}$ is a referential π-institution.*

Proof. By Lemma 6 and Theorem 2 (Theorem 8 of [9]). □

Proposition 8. *If a π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is weakly self-extensional, then it is weakly referential.*

Proof. Let $\mathcal{I}$ be weakly self-extensional. Denote by Thm its theorem family. Construct the π-institution $\mathcal{I}^{\mathrm{Thm},R}$ and denote by Thm^R its theorem family. By Corollary 7, $\mathcal{I}^{\mathrm{Thm},R}$ is referential and, by Lemma 5, $\mathrm{Thm} = \mathrm{Thm}^R$. Therefore, $\mathcal{I}$ is weakly referential. □

Theorem 9. *A π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ is weakly referential if and only if it is weakly self-extensional.*

Proof. The left-to-right implication is Proposition 3. The right-to-left implication is Proposition 8. □

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