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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC

WEAKLY REFERENTIAL $\pi$-INSTITUTIONS

Abstract. Wójicki introduced in the late 1970s the concept of a referential semantics for propositional logics. Referential semantics incorporate features of the Kripke possible world semantics for modal logics into the realm of algebraic and matrix semantics of arbitrary sentential logics. A well-known theorem of Wójicki asserts that a logic has a referential semantics if and only if it is selfextensional. A second theorem of Wójicki asserts that a logic has a weakly referential semantics if and only if it is weakly self-extensional. We formulate and prove an analog of this theorem in the categorical setting. We show that a $\pi$-institution has a weakly referential semantics if and only if it is weakly self-extensional.

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To Don Pigozzi this work is dedicated on the occasion of his 80th Birthday.

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1. Introduction

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a logical signature/algebraic type, i.e., a set of logical connectives/operation symbols $\Lambda$ with attached finite arities given by the function $\rho : \Lambda \to \omega$. Let, also, $V$ be a countably infinite set of propositional variables and $T$ a set of reference/base points. Wójcicki [5] defines a referential algebra $A$ to be an $\mathcal{L}$-algebra with universe $A \subseteq \{0, 1\}^T$. Such an algebra determines the consequence operation $C^A$ on $\text{Fm}_\mathcal{L}(V)$ by setting, for all $X \cup \{\alpha\} \subseteq \text{Fm}_\mathcal{L}(V)$, $\alpha \in C^A(X)$ iff, for all $h : \text{Fm}_\mathcal{L}(V) \to A$ and all $t \in T$,

$$h(\beta)(t) = 1, \text{ for all } \beta \in X, \text{ implies } h(\alpha)(t) = 1.$$ 

Moreover, Wójcicki calls a propositional logic $S = \langle \mathcal{L}, C \rangle$, where $C = C^A$, for a referential algebra $A$, a referential (or referentially truth-functional) propositional logic.

Wójcicki shows in [5] that, given a class $K$ of referential algebras, there exists a single referential algebra $A$, such that $C^K := \cap_{A \in K} C^K = C^A$. Thence follows that a propositional logic is referential if and only if it is defined by a class of referential algebras.

Given a propositional logic $S = \langle \mathcal{L}, C \rangle$, the Frege or interderivability relation of $S$ (see, e.g., Definition 2.37 of [3]), denoted $\Lambda(S)$, is the equivalence relation on $\text{Fm}_\mathcal{L}(V)$, defined, for all $\alpha, \beta \in \text{Fm}_\mathcal{L}(V)$, by

$$\langle \alpha, \beta \rangle \in \Lambda(S) \iff C(\alpha) = C(\beta).$$

The Tarski congruence $\tilde{\Omega}(S)$ of $S$ (see [3]) is the largest congruence relation on $\text{Fm}_\mathcal{L}(V)$ that is compatible with all theories of $S$. The Tarski congruence is a special case of the Suszko congruence $\tilde{\Omega}^S(T)$ associated with a given theory $T$ of $S$, which is defined as the largest congruence on $\text{Fm}_\mathcal{L}(V)$ that is compatible with all theories of $S$ that contain the given theory $T$ (see [2]). In fact, by definition, $\tilde{\Omega}(S) = \tilde{\Omega}^S(C(\emptyset))$, i.e., the Tarski congruence of $S$ is the Suszko congruence associated with the set of theorems of the logic $S$. Font and Jansana (see p.17 of [3]), extending Blok and Pigozzi’s [1] well-known characterization of the Leibniz congruence $\Omega(T)$ associated with a theory $T$ of a sentential logic, have shown that, for all $\alpha, \beta \in \text{Fm}_\mathcal{L}(V)$,

$$\langle \alpha, \beta \rangle \in \tilde{\Omega}(S) \iff \text{for all } \varphi(p, \bar{q}) \in \text{Fm}_\mathcal{L}(V),
C(\varphi(\alpha, \bar{q})) = C(\varphi(\beta, \bar{q})).$$
Whereas $\tilde{\Omega}(S) \subseteq \Lambda(S)$, for every propositional logic $S$, the reverse inclusion does not hold in general. A propositional logic is called **selfextensional** in [5] if $\Lambda(S) \subseteq \tilde{\Omega}(S)$. In fact, Wójcicki shows in what has become a fundamental theorem in the theory of referential semantics, Theorem 2 of [5], that a propositional logic is referential if and only if it is self-extensional.

Wójcicki in [6] revisited the equivalence between referentiality and self-extensionality, proving a “weak version” by replacing the entirety of theories (equivalently, the closure operator $C$) by the set of theorems. More precisely, Wójcicki considers in [6] (see the Theorem in [6]) propositional logics $S = \langle L, C \rangle$, where $C(\emptyset) = C^A(\emptyset)$, for a referential algebra $A$. We call such logics **weakly referential logics**.

Given a propositional logic $S = \langle L, C \rangle$, the **Leibniz congruence** $\Omega(T)$ of a theory $T$ of $S$ (see [1]) is the largest congruence relation on $\text{Fm}_L(V)$ that is compatible with $T$. Blok and Pigozzi’s well-known characterization of the Leibniz congruence $\Omega(T)$ (see p. 11 of [1]) asserts that, for all $\alpha, \beta \in \text{Fm}_L(V),$

$$\langle \alpha, \beta \rangle \in \Omega(T) \; \text{iff} \; \text{for all } \varphi(p, \bar{q}) \in \text{Fm}_L(V), \varphi(\alpha, \bar{q}) \in T \; \text{iff} \; \varphi(\beta, \bar{q}) \in T.$$  

A propositional logic $S = \langle L, C \rangle$ is called **weakly selfextensional** in [6] if, for all $\alpha, \beta \in \text{Fm}_L(V),$

$$\alpha, \beta \in C(\emptyset) \; \text{implies} \; \langle \alpha, \beta \rangle \in \Omega(C(\emptyset)).$$  

In the Theorem of [6], Wójcicki shows that a propositional logic is weakly referential if and only if it is weakly self-extensional.

### 2. $\pi$-Institutions and Closure Systems

Let $\text{Sign}$ be a category and $\text{SEN} : \text{Sign} \to \text{Set}$ a $\text{Set}$-valued functor. The **clone of all natural transformations on** $\text{SEN}$ (see Section 2 of [8]) is the category $U$ with collection of objects $\{\text{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \text{SEN}^\alpha \to \text{SEN}^\beta \; \beta$-sequences of natural transformations $\tau : \text{SEN}^\alpha \to \text{SEN}$. Composition of $\langle \tau_i : i < \beta \rangle : \text{SEN}^\alpha \to \text{SEN}^\beta$ with $\langle \sigma_j : j < \gamma \rangle : \text{SEN}^\beta \to \text{SEN}^\gamma$

$$\text{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \text{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \text{SEN}^\gamma$$
is defined by
\[
\{ \sigma_j : j < \gamma \} \circ \{ \tau_i : i < \beta \} = \{ \sigma_j(\{ \tau_i : i < \beta \}) : j < \gamma \}.
\]
A subcategory of this category with objects all objects of the form SEN<sup>k</sup>,
k < \omega, and such that:

- it contains all projection morphisms \( p^{k,i} : \text{SEN}<sup>k</sup> \to \text{SEN}, i < k, k < \omega, \)
  with \( p^{k,i}_\Sigma : \text{SEN}(\Sigma)<sup>k</sup> \to \text{SEN} \)
  given by
  \[
  p^{k,i}_\Sigma(\tilde{\phi}) = \phi_i, \text{ for all } \tilde{\phi} \in \text{SEN}(\Sigma)<sup>k</sup>,
  \]
- for every family \( \{ \tau_i : \text{SEN}<sup>k</sup> \to \text{SEN} : i < l \} \) of natural transformations
  in \( N, \{ \tau_i : i < l \} : \text{SEN}<sup>k</sup> \to \text{SEN}<sup>l</sup> \) is also in \( N, \)
is referred to as a category of natural transformations on \( \text{SEN}. \)

Consider an algebraic system \( F = (\text{Sign}, \text{SEN}, N) \), i.e., a triple consisting of

- a category \( \text{Sign} \), called the category of signatures;
- a functor \( \text{SEN} : \text{Sign} \to \text{Set} \), called the sentence functor;
- a category of natural transformations \( N \) on \( \text{SEN}. \)

A π-institution based on \( F \) is a pair \( I = (F, C) \), where \( C = \{ C_\Sigma \}_{\Sigma \in |\text{Sign}|} \)
is a closure system on \( \text{SEN} \), i.e., a \( |\text{Sign}| \)-indexed collection of closure operators \( C_\Sigma : \mathcal{P}(\text{SEN}(\Sigma)) \to \mathcal{P}(\text{SEN}(\Sigma)) \),
such that, for all \( \Sigma_1, \Sigma_2 \in |\text{Sign}| \), all \( f \in \text{Sign}(\Sigma_1, \Sigma_2) \) and all \( \Phi \in \text{SEN}(\Sigma_1) \),
\[
\text{SEN}(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\text{SEN}(f)(\Phi)).
\]

This condition is sometimes referred to as structurality. In this context, \( F \) is also referred to as the base algebraic system. Given a π-institution \( I, \) a theory family \( T = \{ T_\Sigma \}_{\Sigma \in |\text{Sign}|} \)
is a \( |\text{Sign}| \)-indexed collection of subsets \( T_\Sigma \subseteq \text{SEN}(\Sigma) \), closed under \( C_\Sigma \), i.e., such that \( C_\Sigma(T_\Sigma) = T_\Sigma \),
for all \( \Sigma \in |\text{Sign}| \). The collection of all theory families of \( I \) is denoted by \( \text{ThFam}(I) \).
Ordered by signature-wise inclusion, it forms a complete lattice \( \text{ThFam}(I) = (\text{ThFam}(I), \leq) \).

Note, also, that, given a base algebraic system \( F \), the collection of all closure systems based on \( F \) is closed under signature-wise intersections and, hence, forms a complete lattice under the signature-wise ordering \( \leq \):
\[
C^1 \leq C^2 \text{ iff for all } \Sigma \in |\text{Sign}| \text{ and all } \Phi \in \text{SEN}(\Sigma),
C^1_\Sigma(\Phi) \subseteq C^2_\Sigma(\Phi).
\]
3. Referential $\pi$-Institutions

We assume a base algebraic system $F = \langle \text{Sign}, \text{SEN}, N \rangle$. Consider also an $N$-algebraic system $A = \langle \text{Sign}', \text{SEN}', N' \rangle$, i.e., one such that there exists a surjective functor $': N \to N'$, preserving all projection natural transformations and, as a consequence, all arities of the natural transformations involved. We denote by $\sigma': \text{SEN}'^k \to \text{SEN}'$ the natural transformation in $N'$ that is the image of $\sigma: \text{SEN}^k \to \text{SEN}$ in $N$ under $'$.

More specifically, we want to focus on $N$-algebraic systems $A = \langle \text{Sign}', \text{SEN}', N' \rangle$, where $\text{SEN}'$ is a simple subfunctor (having the same domain) of the inverse powerset functor $\text{SEN}' : \text{Sign}' \to \text{Set}$ of a contravariant functor $\text{SEN} : \text{Sign} \to \text{Set}$. For $\Sigma \in |\text{Sign}'|$, the elements of $\text{SEN}'(\Sigma)$ in this context are referred to as $\Sigma$-reference or $\Sigma$-base points (see, e.g., [9]). An $N$-morphism $\langle F, \alpha \rangle: \text{SEN} \to \text{SEN}'$ will be viewed as a valuation of sentences of $\text{SEN}$ in the following way: For all $\Sigma \in |\text{Sign}|$ and all $\varphi \in \text{SEN}(\Sigma)$, $\varphi \in \text{SEN}(\Sigma)$ is true at $p \in \text{SEN}'(F(\Sigma))$ under $\langle F, \alpha \rangle$ iff $p \in \alpha_{\Sigma}(\varphi)$.

An $N$-algebraic system of this special form is called a referential $N$-algebraic system. By slightly abusing terminology, we use the same term to refer to an (interpreted) referential $N$-algebraic system, which is a pair $\mathcal{A} = \langle A, \langle F, \alpha \rangle \rangle$, with $\langle F, \alpha \rangle: F \to A$ an algebraic system morphism, also referred to as an $N$-morphism. We sometimes drop the subscript $s$ when referring to the subfunctor to make notation less cumbersome, provided that this is unlikely to cause any confusion.

Let $F = \langle \text{Sign}, \text{SEN}, N \rangle$ be a base algebraic system and $\mathcal{A} = \langle A, \langle F, \alpha \rangle \rangle$ an interpreted referential $N$-algebraic system. Then $\mathcal{A}$ determines a closure system $C^\mathcal{A}$ on $\text{SEN}$ (or on $F$) according to the following definition:

For all $\Sigma \in |\text{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$, $\varphi \in C^\mathcal{A}_\Sigma(\Phi)$ iff, for all $\Sigma' \in |\text{Sign}|$, $f \in \text{Sign}(\Sigma, \Sigma')$,

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\text{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\text{SEN}(f)(\varphi))$$

($\phi$ and $\varphi$, here, are intentionally different).

**Proposition 1** (Proposition 1 of [11]). Suppose $F = \langle \text{Sign}, \text{SEN}, N \rangle$ is a base algebraic system and $\mathcal{A} = \langle A, \langle F, \alpha \rangle \rangle$ an interpreted referential $N$-algebraic system. Then $C^\mathcal{A}$ is a closure system on $F$. 


Since $C^A$ is a closure system on $F$, the pair $I^A = \langle F, C^A \rangle$ is a $\pi$-institution. We call an institution having this form a referential $\pi$-institution. Such $\pi$-institutions correspond in the theory of categorical abstract algebraic logic (CAAL) to the referential propositional logics of Wójcicki [5].

Let $F = \langle \text{Sign}, \text{SEN}, N \rangle$ be a base algebraic system and $I = \langle F, C \rangle$ a $\pi$-institution based on $F$. We define the Frege equivalence system $\Lambda(I)$ of $I$ (see p. 37 of [7]), also known as the interderivability equivalence system, by setting, for all $\Sigma \in \text{Sign}$ and all $\varphi, \psi \in \text{SEN}(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \Lambda_{\Sigma}(I) \text{ if and only if } C_{\Sigma}(\varphi) = C_{\Sigma}(\psi).$$

The Tarski congruence system $\tilde{\Omega}(I)$ of $I$ ([3] for the universal algebraic notion and [10] for its categorical extension) is the largest $N$-congruence system on SEN that is compatible with every theory family $T \in \text{ThFam}(I)$.

Clearly, it is always the case that $\tilde{\Omega}(I) \leq \Lambda(I)$. We call the $\pi$-institution $I$ self-extensional if $\Lambda(I) \leq \tilde{\Omega}(I)$. In view of the preceding remark, $I$ is self-extensional if and only if $\Lambda(I) = \tilde{\Omega}(I)$.

A generalization to $\pi$-institutions of Wójcicki’s Theorem (see Theorem 2 of [5], but, also, Theorem 2.2 of [4] for a complete proof) provides a characterization of referential $\pi$-institutions

**Theorem 2** (Theorem 8 of [9]). A $\pi$-institution $I = \langle F, C \rangle$ is referential if and only if it is self-extensional.

### 4. Weakly Referential $\pi$-Institutions

We assume a base algebraic system $F = \langle \text{Sign}, \text{SEN}, N \rangle$. Recall that for any (interpreted) referential $N$-algebraic system $A = \langle A, \langle F, \alpha \rangle \rangle$, the pair $I^A = \langle F, C^A \rangle$ is a referential $\pi$-institution. We call a $\pi$-institution $I = \langle F, C \rangle$ a weakly referential $\pi$-institution if, for all $\Sigma \in \text{Sign}$,

$$C_{\Sigma}(\emptyset) = C_{\Sigma}^A(\emptyset),$$

for some referential $\pi$-institution $I^A$. Such $\pi$-institutions correspond in the theory of CAAL to the weakly referential propositional logics of Wójcicki [6].
Let \( \mathcal{F} = \langle \text{Sign}, \text{SEN}, N \rangle \) be a base algebraic system and \( \mathcal{I} = \langle \mathcal{F}, C \rangle \) a \( \pi \)-institution based on \( \mathcal{F} \). Let, also \( T \in \text{ThFam}(\mathcal{I}) \). The Leibniz congruence system \( \Omega(T) \) of \( T \) ([1] for the universal algebraic notion and p. 223 of [8] for its categorical extension) is the largest \( N \)-congruence system on \( \text{SEN} \) that is compatible with the theory family \( T \). We denote by \( \text{Thm} = \{ \text{Thm}_\Sigma \}_{\Sigma \in \text{Sign}} \) the theorem family of \( \mathcal{I} \), i.e., \( \text{Thm}_\Sigma = C_\Sigma(\emptyset) \), for all \( \Sigma \in \text{Sign} \).

We call the \( \pi \)-institution \( \mathcal{I} \) weakly self-extensional if, for all \( \Sigma \in \text{Sign} \) and all \( \varphi, \psi \in \text{SEN}(\Sigma) \),

\[
\varphi, \psi \in \text{Thm}_\Sigma \implies \langle \varphi, \psi \rangle \in \Omega(\text{Thm}_\Sigma).
\]

A generalization to \( \pi \)-institutions of Wójcicki’s Theorem (see the Theorem of [6]) provides a characterization of weakly referential \( \pi \)-institutions. This is the main result of the present work, formulated in Theorem 9. The value rests in both furnishing a more detailed proof based on the sketch provided in [6], and, also, in extending the scope of the result to encompass logics formalized as \( \pi \)-institutions. We start with the easy direction.

**Proposition 3.** If a \( \pi \)-institution \( \mathcal{I} = \langle \mathcal{F}, C \rangle \) is weakly referential, then it is weakly self-extensional.

**Proof.** Suppose that \( \mathcal{I} \) is weakly referential. Thus, there exists a referential \( N \)-algebraic system \( A \), such that \( C_\Sigma(\emptyset) = C_k^A(\emptyset) \), for all \( \Sigma \in \text{Sign} \). Let \( \Sigma \in \text{Sign} \) and \( \varphi, \psi \in \text{SEN}(\Sigma) \), such that \( \varphi, \psi \in C_\Sigma(\emptyset) = C_k^A(\emptyset) \). This implies that \( C_k^A(\varphi) = C_k^A(\psi) \), i.e., that \( \langle \varphi, \psi \rangle \in A_\Sigma(A^A) \).

Since \( A^A \) is referential, it is self-extensional by Theorem 2. Thus, we get \( \langle \varphi, \psi \rangle \in \Omega(\Sigma^A) \). Therefore, by the characterization theorem of the Tarski Operator in CAAL, Theorem 4 of [10], for all \( \sigma : \text{SEN}^k \to \text{SEN} \) in \( N \), all \( \Sigma' \in \text{Sen} \), all \( f \in \text{Sen}(\Sigma, \Sigma') \) and all \( \bar{\chi} \in \text{SEN}(\Sigma')^k \),

\[
C_k^A(\sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi})) = C_k^A(\sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi})).
\]

Thus, we obtain, for all \( \sigma : \text{SEN}^k \to \text{SEN} \) in \( N \), all \( \Sigma' \in \text{Sen} \), all \( f \in \text{Sen}(\Sigma, \Sigma') \) and all \( \bar{\chi} \in \text{SEN}(\Sigma')^k \),

\[
\sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi}) \in \text{Thm}_{\Sigma'} \iff \sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi}) \in \text{Thm}_{\Sigma'}.
\]

This shows that \( \langle \varphi, \psi \rangle \in \Omega(\text{Thm}) \).
Let $I = (F, C)$ be a weakly self-extensional $\pi$-institution, with theorem family $\text{Thm}$. Define the family $R = \{ R_\Sigma \}_{\Sigma \in |\text{Sign}|}$ by setting

$$R_\Sigma = \left\{ \sigma \in N : \frac{\sigma_\Sigma(\varphi, \chi)}{\sigma_\Sigma(\psi, \chi)} \varphi, \chi \in \text{SEN}(\Sigma)^k, \varphi, \psi \in \text{Thm}_\Sigma \right\},$$

where, following a common convention in CAAL, when we write $\sigma_\Sigma(\varphi, \chi)$, we mean that $\varphi, \psi$ may occupy any position in $\sigma$ and not just the first, as long as they occupy the same position in both the antecedent and the consequent of the rule.

Define on $F$ the operator family $C_{\text{Thm}, R} = \{ C_{\text{Thm}, R} \}_{\Sigma \in |\text{Sign}|}$, such that, for all $\Sigma \in |\text{Sign}|$, $C_{\text{Thm}, R} : P\text{SEN}(\Sigma) \to P\text{SEN}(\Sigma)$ is given, for all $\Phi \cup \{ \varphi \} \subseteq \text{SEN}(\Sigma)$, by

$$\varphi \in C_{\text{Thm}, R}(\Phi) \iff \varphi \text{ is } R_\Sigma\text{-provable from } \Phi \cup \text{Thm}_\Sigma.$$

Then, we can show that $C_{\text{Thm}, R}$ is a closure system on $F$:

**Lemma 4.** Let $I = (F, C)$ be a weakly self-extensional $\pi$-institution, with theorem family $\text{Thm}$. Then $C_{\text{Thm}, R}$ is a closure system on $F$.

**Proof.** By classical proof-theoretic arguments, one shows that $C_{\text{Thm}, R}$ is a closure operator on $\text{SEN}(\Sigma)$, for all $\Sigma \in |\text{Sign}|$. So it suffices to show that $C_{\text{Thm}, R}$ is structural. Suppose that $\Sigma \in |\text{Sign}|$ and $\Phi \cup \{ \varphi \} \subseteq \text{SEN}(\Sigma)$, such that $\varphi \in C_{\text{Thm}, R}(\Phi)$. This means that there exists an $R_\Sigma\text{-proof}$

$$\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$$

of $\varphi$ from $\Phi \cup \text{Thm}_\Sigma$. We must show that, for all $\Sigma' \in |\text{Sign}|$, $f \in \text{Sign}(\Sigma, \Sigma')$, $\text{SEN}(f)(\varphi) \in C_{\text{Thm}, R}(\text{SEN}(f)(\Phi))$. Consider the sequence of $\Sigma'\text{-sentences}$

$$\text{SEN}(f)(\varphi_0), \text{SEN}(f)(\varphi_1), \ldots, \text{SEN}(f)(\varphi_n) = \text{SEN}(f)(\varphi).$$

It suffices to show that this is a valid $R_{\Sigma'}\text{-proof}$ of $\text{SEN}(f)(\varphi)$ from hypotheses $\text{SEN}(f)(\Phi) \cup \text{Thm}_{\Sigma'}$. This is accomplished by induction on $0 \leq k \leq n$:

**Base:** If $k = 0$, then $\varphi_0$ must be a $\Sigma$-sentence in $\Phi \cup \text{Thm}_\Sigma$. But then, since the theorem family of any $\pi$-institution is a theory system, we get that $\text{SEN}(f)(\varphi_0)$ is in $\text{SEN}(f)(\Phi) \cup \text{Thm}_{\Sigma'}$. 
Hypothesis: Suppose, for all \( i < k \leq n \), \( SEN(f)(\varphi_i) \) is either in \( SEN(f)(\Phi) \cup \text{Thm}_\Sigma \) or follows from previous sentences in the sequence by a single application of an \( R\Sigma \)-rule.

Step: If \( \varphi_k \) is in \( \Phi \cup \text{Thm}_\Sigma \), then, as in the Base, it follows that \( SEN(f)(\varphi_k) \) is in \( SEN(f)(\Phi) \cup \text{Thm}_\Sigma \). Suppose, finally, that \( \varphi_k \) follows from \( \varphi_i, i < k \), by a single application of an \( R\Sigma \)-rule, i.e., there exists \( \sigma \) in \( N \) and \( \bar{\chi} \in SEN(\Sigma)^p \), such that \( \varphi_i = \sigma(\varphi, \bar{\chi}) \) and \( \varphi_k = \sigma(\psi, \bar{\chi}) \), for some \( \varphi, \psi \in \text{Thm}_\Sigma \). But, then, for the same \( \sigma \) in \( N \) and \( SEN(f)(\bar{\chi}) \in SEN(\Sigma')^p \), we have that \( SEN(f)(\varphi), SEN(f)(\psi) \in \text{Thm}_\Sigma \) and

\[
\begin{align*}
SEN(f)(\varphi_i) &= \sigma(SEN(f)(\varphi), SEN(f)(\bar{\chi})), \\
SEN(f)(\varphi_k) &= \sigma(SEN(f)(\psi), SEN(f)(\bar{\chi})).
\end{align*}
\]

Thus, \( SEN(f)(\varphi_k) \) follows from \( SEN(f)(\varphi_i) \) by an application of the \( R\Sigma \)-rule

\[
\sigma(SEN(f)(\varphi), SEN(f)(\bar{\chi})).
\]

This concludes the proof of structurality of \( C^{\text{Thm}, R} \). \( \square \)

Thus, \( I^{\text{Thm}, R} = \{ F, C^{\text{Thm}, R} \} \) is a \( \pi \)-institution. Let us denote by \( \text{Thm}^R = \{ \text{Thm}_\Sigma^R \}_{\Sigma \in \text{Sign}} \) the theorem system of \( I^{\text{Thm}, R} \). It turns out that the theorem system \( \text{Thm}^R \) coincides with the theorem system \( \text{Thm} \) of \( I \):

**Lemma 5.** Let \( I = \langle F, C \rangle \) be a weakly self-extensional \( \pi \)-institution, with theorem family \( \text{Thm} \). Then \( \text{Thm} = \text{Thm}^R \).

**Proof.** Clearly, by the definition of \( C^{\text{Thm}, R} \), \( \text{Thm} \leq \text{Thm}^R \).

For the converse, suppose that \( \Sigma \in |\text{Sign}| \) and \( \varphi \in \text{Thm}_\Sigma^R \). Thus, \( \varphi \in C^{\text{Thm}, R}_\Sigma(\emptyset) \). This means that there exists an \( R\Sigma \)-proof

\[
\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi
\]

of \( \phi \) from \( \text{Thm}_\Sigma \). We show by induction on \( k \leq n \) that \( \varphi_k \in \text{Thm}_\Sigma \).

Base: If \( k = 0 \), then \( \varphi_0 \) must be in \( \text{Thm}_\Sigma \) by hypothesis.

Hypothesis: Suppose that, for all \( i < k \leq n \), \( \varphi_i \in \text{Thm}_\Sigma \).

Step: If \( \varphi_k \in \text{Thm}_\Sigma \), then there is nothing to prove. Otherwise, \( \varphi_k \) follows from \( \varphi_i, i < k \), by an application of an \( R\Sigma \)-rule. Thus, for some \( \sigma \) in \( N \), some \( \bar{\chi} \in SEN(\Sigma)^p \) and some \( \varphi, \psi \in \text{Thm}_\Sigma \),

\[
\varphi_i = \sigma(\varphi, \bar{\chi}), \quad \varphi_k = \sigma(\psi, \bar{\chi}).
\]
By weak selfextensionality of \( \mathcal{I} \), we get \( \langle \varphi, \psi \rangle \in \Omega_\Sigma(\text{Thm}) \). Thus, since \( \Omega(\text{Thm}) \) is a congruence system, \( \langle \varphi_i, \varphi_k \rangle \in \Omega_\Sigma(\text{Thm}) \). Since, by the Induction Hypothesis, \( \varphi_i \in \text{Thm}_\Sigma \), by the compatibility of the Leibniz congruence system, we get \( \varphi_k \in \text{Thm}_\Sigma \).

This shows that \( \varphi \in \text{Thm}_\Sigma \). Therefore \( \text{Thm}^R \subseteq \text{Thm} \).

The next result shows that \( \mathcal{I}^{\text{Thm},R} \) is a self-extensional \( \pi \)-institution. Intuitively speaking, this feature is instilled to the \( \pi \)-institution by virtue of its definition.

**Lemma 6.** Let \( \mathcal{I} = (F, C) \) be a weakly self-extensional \( \pi \)-institution, with theorem family \( \text{Thm} \). Then \( \mathcal{I}^{\text{Thm},R} \) is a self-extensional \( \pi \)-institution.

**Proof.** Suppose \( \Sigma \in |\text{Sign}| \) and \( \varphi, \psi \in \text{SEN}(\Sigma) \) are such that

\[
C^{\text{Thm},R}_\Sigma(\varphi) = C^{\text{Thm},R}_\Sigma(\psi).
\]

Then \( \varphi \in C^{\text{Thm},R}_\Sigma(\psi) \). Let \( \sigma : \text{SEN}^k \rightarrow \text{SEN} \) in \( N, \Sigma' \in |\text{Sign}|, f \in \text{Sign}(\Sigma, \Sigma') \) and \( \bar{\chi} \in \text{SEN}(\Sigma')^k \) be fixed but arbitrary. Our goal is to show that \( \sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi}) \in C^{\text{Thm},R}_{\Sigma'}(\sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi})) \). By symmetry, it then follows

\[
C^{\text{Thm},R}_{\Sigma'}(\sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi})) = C^{\text{Thm},R}_{\Sigma'}(\sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi}))
\]

i.e., that \( \mathcal{I}^{\text{Thm},R} \) is self-extensional.

Suppose first that \( \varphi \in \text{Thm}_\Sigma \). Then, \( \psi \in \text{Thm}_\Sigma \) also. Hence \( \text{SEN}(f)(\varphi) \) and \( \text{SEN}(f)(\psi) \) are in \( \text{Thm}_\Sigma \). Therefore, \( \sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi}) \) follows by an application of a rule in \( R_{\Sigma'} \) from \( \sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi}) \). This proves that \( \sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi}) \in C^{\text{Thm},R}_{\Sigma'}(\sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi})) \).

Now we turn to the case where \( \varphi \notin \text{Thm}_\Sigma \). Since \( \varphi \in C^{\text{Thm},R}_\Sigma(\psi) \), there exists an \( R_{\Sigma'} \)-proof

\[
\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi
\]

of \( \varphi \) from premises \( \{ \psi \} \cup \text{Thm}_\Sigma \). Consider the sequence

\[
\varphi'_0, \varphi'_1, \ldots, \varphi'_n,
\]

defined by induction on \( k \leq n \) as follows:

- If \( \varphi_k = \psi \), then \( \varphi'_k = \sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi}) \).
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- If \( \varphi_k \in \text{Thm}_\Sigma \), then \( \varphi'_k = \text{SEN}(f)(\varphi_k) \).

- If \( \varphi_k \) follows from \( \varphi_i \), \( i < k \), by an application of the \( R_\Sigma \)-rule \( \tau_{\Sigma}(\zeta, \eta) \), we set:
  
  - \( \varphi'_k = \text{SEN}(f)(\varphi_k) \), if \( \varphi'_i = \text{SEN}(f)(\varphi_i) \);
  
  - \( \varphi'_k = \sigma_\Sigma'(\text{SEN}(f)(\varphi_k), \bar{\chi}) \), if \( \varphi'_i = \sigma_\Sigma'(\text{SEN}(f)(\varphi_i), \bar{\chi}) \).

Our goal is to show that this is a valid \( R_\Sigma' \)-proof of \( \sigma_\Sigma'(\text{SEN}(f)(\varphi), \bar{\chi}) \) from premises \( \{ \sigma_\Sigma'(\text{SEN}(f)(\psi), \bar{\chi}) \} \cup \text{Thm}_\Sigma' \). We do this by employing induction on \( k \leq n \) to show that the sequence

\[ \varphi'_0, \varphi'_1, \ldots, \varphi'_k \]

is an \( R_\Sigma' \)-proof of \( \varphi'_k \) from premises \( \{ \sigma_\Sigma'(\text{SEN}(f)(\psi), \bar{\chi}) \} \cup \text{Thm}_\Sigma' \).

**Base:** If \( k = 0 \), we have two cases:

- If \( \varphi_0 = \psi \), then \( \varphi'_0 = \sigma_\Sigma'(\text{SEN}(f)(\psi), \bar{\chi}) \) follows by hypothesis.

- If \( \varphi_0 \in \text{Thm}_\Sigma \), then \( \varphi'_0 = \text{SEN}(f)(\varphi_0) \in \text{Thm}_\Sigma' \) also follows by hypothesis.

**Hypothesis:** Assume that, for all \( i < k \leq n \),

\[ \varphi'_0, \varphi'_1, \ldots, \varphi'_i \]

is a valid \( R_\Sigma' \)-proof of \( \varphi'_i \) from premises \( \{ \sigma_\Sigma'(\text{SEN}(f)(\psi), \bar{\chi}) \} \cup \text{Thm}_\Sigma' \).

**Step:** If \( \varphi_k = \psi \) or \( \varphi_k \in \text{Thm}_\Sigma \), then we replicate the reasoning in the Base.

Suppose that \( \varphi_k \) follows from \( \varphi_i \), \( i < k \), by an application of the \( R_\Sigma \)-rule \( \tau_{\Sigma}(\zeta, \eta) \), where \( \zeta, \eta \in \text{Thm}_\Sigma \).

- If \( \varphi'_i = \text{SEN}(f)(\varphi_i) \), then \( \varphi'_k = \text{SEN}(f)(\varphi_k) \). Since \( \zeta, \xi \in \text{Thm}_\Sigma \), \( \text{SEN}(f)(\zeta) \), \( \text{SEN}(f)(\xi) \in \text{Thm}_\Sigma' \). Thus, this step in the proof is justified by the fact that

\[
\frac{\varphi'_i}{\varphi'_k} = \frac{\text{SEN}(f)(\varphi_i)}{\text{SEN}(f)(\varphi_k)} = \frac{\tau_{\Sigma'}(\text{SEN}(f)(\zeta), \text{SEN}(f)(\xi))}{\tau_{\Sigma'}(\text{SEN}(f)(\xi), \text{SEN}(f)(\eta))}
\]

is a valid \( R_{\Sigma'} \)-rule.
If \( \varphi'_i = \sigma_{\Sigma'}(\text{SEN}(f)(\varphi_i), \bar{\chi}) \), then \( \varphi'_k = \sigma_{\Sigma'}(\text{SEN}(f)(\varphi_k), \bar{\chi}) \). Once more, since \( \zeta, \xi \in \text{Thm}_{\Sigma} \), we get \( \text{SEN}(f)(\zeta), \text{SEN}(f)(\xi) \in \text{Thm}_{\Sigma'} \). Thus, this step in the proof is justified by the fact that

\[
\frac{\sigma_{\Sigma'}(\text{SEN}(f)(\varphi_i), \bar{\chi})}{\sigma_{\Sigma'}(\text{SEN}(f)(\varphi_k), \bar{\chi})} = \frac{\sigma_{\Sigma'}(\tau_{\Sigma'}(\text{SEN}(f)(\zeta), \text{SEN}(f)^\mu(\eta)), \bar{\chi})}{\sigma_{\Sigma'}(\tau_{\Sigma'}(\text{SEN}(f)(\xi), \text{SEN}(f)^\mu(\eta)), \bar{\chi})}
\]

is a valid \( R_{\Sigma'} \)-rule.

By symmetry, interchanging the roles of \( \varphi, \psi \) in the preceding reasoning, we get that, for all \( \sigma : \text{SEN}^k \to \text{SEN} \) in \( N \), all \( \Sigma' \in |\text{Sign}| \), all \( f \in \text{Sign}(\Sigma, \Sigma') \) and all \( \bar{\chi} \in \text{SEN}(\Sigma')^k \),

\[
C^\text{Thm,} \sigma_{\Sigma'}(\text{SEN}(f)(\varphi), \bar{\chi}) = C^\text{Thm,} \sigma_{\Sigma'}(\text{SEN}(f)(\psi), \bar{\chi}).
\]

By the CAAL characterization theorem of the Tarski congruence system of a \( \pi \)-institution (Theorem 4 of [10]), we get that \( (\varphi, \psi) \in \Omega_{\Sigma}(I^\text{Thm,} R) \). This proves that \( I^\text{Thm,} R \) is a selfextensional \( \pi \)-institution.

**Corollary 7.** Let \( I = (F, C) \) be a weakly self-extensional \( \pi \)-institution, with theorem family \( \text{Thm} \). Then \( I^\text{Thm,} R \) is a referential \( \pi \)-institution.

**Proof.** By Lemma 6 and Theorem 2 (Theorem 8 of [9]).

**Proposition 8.** If a \( \pi \)-institution \( I = (F, C) \) is weakly self-extensional, then it is weakly referential.

**Proof.** Let \( I \) be weakly self-extensional. Denote by \( \text{Thm} \) its theorem family. Construct the \( \pi \)-institution \( I^\text{Thm,} \sigma_{\Sigma'} \) and denote by \( \text{Thm}^R \) its theorem family. By Corollary 7, \( I^\text{Thm,} R \) is referential and, by Lemma 5, \( \text{Thm} = \text{Thm}^R \). Therefore, \( I \) is weakly referential.

**Theorem 9.** A \( \pi \)-institution \( I = (F, C) \) is weakly referential if and only if it is weakly self-extensional.

**Proof.** The left-to-right implication is Proposition 3. The right-to-left implication is Proposition 8.
References


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