RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Tomo LIV (2005), pp. 50-70

REMARKS ON CLASSIFICATIONS AND ADJUNCTIONS

GEORGE VOUTSADAKIS

Jon Barwise and Jerry Seligman introduced the category of classifications and infomorphisms to model information theoretical concepts and information flow processes. This is the dual category of the category of Chu spaces introduced by Michael Barr and Peter Chu and studied extensively by Vaughan Pratt, Gordon Plotkin and others at Stanford. It is also very closely related to the theory of institutions introduced in a different context by Joseph Goguen and Rod Burstall. Here some aspects of the theory are reviewed and some adjunctions between the category of classifications and other related categories are studied from a more abstract point of view.

1. Introduction.

Chu spaces were introduced in a purely categorical context by Peter Chu as the self-dual "completion" $\mathrm{Chu}(V, k)$ of a symmetric monoidal closed category V with pullbacks possessing a distinguished object k. An account of the construction is given in the appendix of [1]. In [2] Chu spaces were used to provide constructive models of linear logic [9]. [15] gives a very readable tutorial and reviews the most interesting and important recent developments in the theory of Chu spaces.

In a different context, Barwise and Seligman [4] offered a theoretical framework for the study of information flow between the components of a distributed information system. They called the basic structure upon which their

Keywords: Classifications, Chu Spaces, Infomorphisms, Information Flow, Functions, Relations, Partitions, Adjunctions.

1991 AMS Subject Classification: Primary: 94A14, 18B10 Secondary: 18A40, 03E75.

framework is developed a *classification.* Classifications are exactly the same objects as Chu spaces but mappings between classifications, so-called *infomorphisms,* point in the reverse direction that mappings between the corresponding Chu spaces. Thus, the category of classifications is the dual category of the category of Chu spaces. However, both categories are self-dual and, thus, classifications and Chu spaces form isomorphic categories. Barwise and Seligman pointed this out in [4] in the phrase "one could look at this book as an application of Chu spaces and Chu transformations to a theory of information."

In yet another context pertaining to the logics underlying the specification of programming languages, Goguen and Burstall [10, 11] introduced the notion of an *institution* to formalize the concept of a multisignature logical system. Institutions are much more complex structures than Chu spaces but it is worth pointing out that, if one restricts to trivial signature institutions, i.e., institutions whose signature category is the trivial one-object category, then one obtains a Chu space and institution morphisms between such institutions correspond to Chu morphisms. Thus the category of Chu spaces can be embedded in the category of institutions. In the few years passed since their introduction, institutions have been used in many contexts. The references [12, 13, 6, 7] are pointers in the recent computer science and model theoretical literature where institutions have played a central role.

In all aforementioned treatments categories and functors play a prominent role. As a consequence, investigating adjunctions between these and related categories that reveal the relationships between them is very important. Many adjunctions pertaining to the category of classifications were provided in [4]. In [8] some of these were revisited and some of the relationships between the category of classifications and related categories clarified. In this paper this work is briefly reviewed and some further relationships are established and investigated.

In Section 2, the definition of a classification and that of a state space are provided. Extensional classifications are those whose types are completely determined by their extensions, i.e., by the sets of tokes that they satisfy. These form a reflective subcategory of the category of classifications. A natural adjunction between classifications and state spaces, given in [4], is also reviewed here. Section 3 revisits the connection between this adjunction and the Boolean closure of a given classification. This is in some sense the minimal classification that extends the original one and possesses natural operations of negation, conjunction and disjunction on its types. Section 4 continues along the same lines but views classifications and, thus, Chu spaces as well, from a more "abstract" point of view. Classifications are really binary relations and state spaces are really set maps. A new category, the category of partitions, a subcategory of the category of functions, is introduced and several natural functors between this and the category of functions are introduced and studied. This brings the development close to a point in which a question posed in [8] may be investigated and answered. A functor between extensional classifications and partitions was introduced in [8] and the question of whether it has a left adjoint was asked. This functor in the present treatment is a composite of two left adjoints followed by a right adjoint. Hence the fact that it possesses a left adjoint is not immediate. However, it will be proved that it does have a left adjoint, which is given by the composite of the adjoints of its factors.

The "abstraction" that is supported in the last section of the paper is encouraged by the fact that Chu spaces, classifications and other related structures are much closer to the mathematical foundations than one is led to believe by their names and their disguises for different applications in computer science. It is therefore desirable that one take a step back and investigate some of their properties in the proper foundational setting and then return to the specific applications where these may be put to further use.

2. Background.

Some basic notions that are needed to understand the main results in this paper are presented in this section. For details on the categorical definitions the reader is referred to any of [14, 5, 3]. For a detailed development of classifications and their dual Chu spaces [4] and [15], respectively, may be used as the guiding sources and as pointers to other related references in the literature.

A **classification** $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$ is a triple consisting of a set $\Sigma_{\mathbf{A}} = \mathrm{typ}(\mathbf{A})$ of **types**, a set A = tok($\mathbf{A}$) of **tokens** and a relation $\models_{\mathbf{A}} \subseteq A \times \Sigma_{\mathbf{A}}$.

Given two classifications $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$, $\mathbf{B} = \langle \Sigma_{\mathbf{B}}, B, \models_{\mathbf{B}} \rangle$, a **classification morphism** (or **infomorphism**) $f : \mathbf{A} \to \mathbf{B}$ **from A to B** is a pair of set maps $\hat{f} : \mathrm{typ}(\mathbf{A}) \to \mathrm{typ}(\mathbf{B})$ and $\check{f} : \mathrm{tok}(\mathbf{B}) \to \mathrm{tok}(\mathbf{A})$, such that, for all $\alpha \in \mathrm{typ}(\mathbf{A})$ and all $b \in \mathrm{tok}(\mathbf{B})$,

$$\check{f}(b) \models_{\mathbf{A}} \alpha \quad \text{if and only if} \quad b \models_{\mathbf{B}} \hat{f}(\alpha).$$

$$\begin{tikzcd}
A \arrow[r, phantom, "\models_{\mathbf{A}}"] & \Sigma_{\mathbf{A}} \arrow[d, "\hat{f}"] \\
B \arrow[u, "\check{f}"] \arrow[r, phantom, "\models_{\mathbf{B}}"'] & \Sigma_{\mathbf{B}}
\end{tikzcd}$$

Classifications with infomorphisms between them form a category, called the **category of classifications**. This category will be denoted by **REL** since its objects are, simply, binary relations from the set of tokens to the set of types. (See Section 4, where this “abstract” point of view is adopted.)

Given a classification $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$ and $\alpha \in \Sigma_{\mathbf{A}}, a \in A$, define

$$\mathrm{tok}_{\mathbf{A}}(\alpha) = \{a \in A : a \models_{\mathbf{A}} \alpha\} \quad \text{and} \quad \mathrm{typ}_{\mathbf{A}}(a) = \{\alpha \in \Sigma_{\mathbf{A}} : a \models_{\mathbf{A}} \alpha\}.$$

Then, the equivalence relations $\equiv_{\mathbf{A}}$ and $\cong_{\mathbf{A}}$ are defined on $\Sigma_{\mathbf{A}}$ and A, respectively, by

$$\alpha \equiv_{\mathbf{A}} \beta \quad \text{if and only if} \quad \mathrm{tok}_{\mathbf{A}}(\alpha) = \mathrm{tok}_{\mathbf{A}}(\beta)$$

and

$$a \cong_{\mathbf{A}} b \quad \text{if and only if} \quad \mathrm{typ}_{\mathbf{A}}(a) = \mathrm{typ}_{\mathbf{A}}(b).$$

The classification $\mathbf{A}$ is called **extensional** if $\equiv_{\mathbf{A}} = \Delta_{\Sigma_{\mathbf{A}}}$, the identity relation on $\Sigma_{\mathbf{A}}$. By **EREL** is denoted the full subcategory of **REL** with objects all extensional classifications.

Given an arbitrary classification $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$ the structure

$$\mathrm{Ext}(\mathbf{A}) = \langle \Sigma_{\mathbf{A}}/{\equiv_{\mathbf{A}}}, A, \models_{\equiv_{\mathbf{A}}} \rangle,$$

where

$$a \models_{\equiv_{\mathbf{A}}} \alpha/{\equiv_{\mathbf{A}}} \quad \text{if and only if} \quad a \models_{\mathbf{A}} \alpha, \quad \text{for all } a \in A, \alpha \in \Sigma_{\mathbf{A}},$$

is an extensional classification. Moreover, given an infomorphism $f : \mathbf{A} \to \mathbf{B}$ between two classifications $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$, $\mathbf{B} = \langle \Sigma_{\mathbf{B}}, B, \models_{\mathbf{B}} \rangle$, the pair $\mathrm{Ext}(f) = \langle \mathrm{Ext}(\hat{f}), \mathrm{Ext}(\check{f}) \rangle$, where $\mathrm{Ext}(\hat{f}) : \Sigma_{\mathbf{A}}/{\equiv_{\mathbf{A}}} \to \Sigma_{\mathbf{B}}/{\equiv_{\mathbf{B}}}$ is defined by

$$\mathrm{Ext}(\hat{f})(\alpha/{\equiv_{\mathbf{A}}}) = \hat{f}(\alpha)/{\equiv_{\mathbf{B}}}, \quad \text{for all } \alpha \in \Sigma_{\mathbf{A}},$$

and $\mathrm{Ext}(\check{f}) : B \to A$ by

$$\mathrm{Ext}(\check{f})(b) = \check{f}(b), \quad \text{for all } b \in B,$$

is an infomorphism from $\mathrm{Ext}(\mathbf{A})$ to $\mathrm{Ext}(\mathbf{B})$.

These definitions of Ext on classifications and infomorphisms make Ext a functor from **REL** to **EREL**. Furthermore, $\mathrm{Ext} : \mathbf{REL} \to \mathbf{EREL}$ is a left adjoint to the inclusion functor Inc : **EREL**$\to$**REL**, i.e., the full subcategory **EREL** of extensional classifications is a reflective subcategory of the category **REL** of classifications.

$$\mathbf{REL} \underset{\text{Inc}}{\overset{\text{Ext}}{\rightleftarrows}} \mathbf{EREL}$$

A **state space** $\mathbf{S}$ is an object in the category Mor(**SET**) of arrows in the category **SET** of small sets and a **state space morphism** $f : \mathbf{S} \to \mathbf{T}$ from a state space $\mathbf{S}$ to a state space $\mathbf{T}$ is an arrow in the category Mor(**SET**) from the object $\mathbf{S}$ to the object $\mathbf{T}$, i.e., a pair of arrows $\langle \check{f}, \hat{f}\rangle$ in **SET**, with $\check{f} : \mathrm{dom}(\mathbf{S})\to\mathrm{dom}(\mathbf{T})$ and $\hat{f} : \mathrm{cod}(\mathbf{S})\to\mathrm{cod}(\mathbf{T})$, such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{dom}(\mathbf{S}) \arrow[r, "\mathbf{S}"] \arrow[d, "\check{f}"'] & \mathrm{cod}(\mathbf{S}) \arrow[d, "\hat{f}"] \\
\mathrm{dom}(\mathbf{T}) \arrow[r, "\mathbf{T}"'] & \mathrm{cod}(\mathbf{T})
\end{tikzcd}$$

Let $\mathbf{S}$ be a state space. Then the triple

$$\mathrm{Evt}(\mathbf{S}) = \langle \mathcal{P}(\mathrm{cod}(\mathbf{S})), \mathrm{dom}(\mathbf{S}), \models_{\mathrm{Evt}(\mathbf{S})} \rangle,$$

where $\models_{\mathrm{Evt}(\mathbf{S})} \subseteq \mathrm{dom}(\mathbf{S}) \times \mathcal{P}(\mathrm{cod}(\mathbf{S}))$ is defined by

$$s \models_{\mathrm{Evt}(\mathbf{S})} T \quad \text{if and only if} \quad \mathbf{S}(s) \in T, \quad \text{for all } s \in \mathrm{dom}(\mathbf{S}), T \subseteq \mathrm{cod}(\mathbf{S}),$$

is a classification. Moreover, if $f : \mathbf{S}\to\mathbf{T}$ is a state space morphism, then $\mathrm{Evt}(f) : \mathrm{Evt}(\mathbf{T})\to\mathrm{Evt}(\mathbf{S})$, defined as the pair of morphisms $\langle \hat{\mathrm{Evt}}(f), \check{\mathrm{Evt}}(f)\rangle$, with $\hat{\mathrm{Evt}}(f) : \mathcal{P}(\mathrm{cod}(\mathbf{T}))\to\mathcal{P}(\mathrm{cod}(\mathbf{S}))$, given by

$$\hat{\mathrm{Evt}}(f)(Y) = \hat{f}^{-1}(Y), \quad \text{for all } Y \subseteq \mathrm{cod}(\mathbf{T}),$$

and $\check{\mathrm{Evt}}(f) : \mathrm{dom}(\mathbf{S})\to\mathrm{dom}(\mathbf{T})$, given by

$$\check{\mathrm{Evt}}(f)(x) = \check{f}(x), \quad \text{for all } x \in \mathrm{dom}(\mathbf{S}),$$

is an infomorphism from the classification Evt($\mathbf{T}$) to the classification Evt($\mathbf{S}$).

Evt, defined as above on state spaces and state space morphisms, is a contravariant functor Evt : **FCT**$\to$**REL**$^{\mathrm{op}}$ between the category of state spaces **FCT** $=$ Mor(**SET**) and the category of classifications. Here **FCT** denotes the category of state spaces, since its objects are, simply, set mappings or functions. Again, Section 4 goes back to the roots stripping the objects of **FCT** from the fancier name state spaces[1].

[1] It is worth noting that, if a relation $\models_{\mathbf{A}} \subseteq A \times \Sigma_{\mathbf{A}}$ is replaced by its characteristic function

On the other hand, given a classification $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$, the function $\mathrm{Sp}(\mathbf{A}) : A \to \mathcal{P}(\Sigma_{\mathbf{A}})$, where

$$\mathrm{Sp}(\mathbf{A})(a) = \mathrm{typ}_{\mathbf{A}}(a), \quad \text{for all } a \in A,$$

is a state space. Moreover, if $f : \mathbf{A} \to \mathbf{B}$ is an infomorphism from the classification $\mathbf{A}$ to the classification $\mathbf{B}$, then $\mathrm{Sp}(f) : \mathrm{Sp}(\mathbf{B}) \to \mathrm{Sp}(\mathbf{A})$, defined to be the pair $\langle \check{\mathrm{Sp}(f)}, \hat{\mathrm{Sp}(f)} \rangle$, with $\check{\mathrm{Sp}(f)} : B \to A$, given by

$$\check{\mathrm{Sp}(f)}(b) = \check{f}(b), \quad \text{for all } b \in B,$$

and $\hat{\mathrm{Sp}(f)} : \mathcal{P}(\Sigma_{\mathbf{B}}) \to \mathcal{P}(\Sigma_{\mathbf{A}})$, given by

$$\hat{\mathrm{Sp}(f)}(\Gamma) = \hat{f}^{-1}(\Gamma), \quad \text{for all } \Gamma \subseteq \Sigma_{\mathbf{B}},$$

is a state space morphism.

Sp, defined as above on classifications and infomorphisms is a contravariant functor $\mathrm{Sp} : \mathbf{REL} \to \mathbf{FCT}^{\mathrm{op}}$ between the category of classifications and the category of state spaces[2].

THEOREM 1 ([4], Proposition 8.22) *The functor Evt* : $\mathbf{FCT} \to \mathbf{REL}^{\mathrm{op}}$ *is a contravariant right adjoint to the contravariant functor Sp* : $\mathbf{REL} \to \mathbf{FCT}^{\mathrm{op}}$.

$$\mathbf{REL} \underset{\mathrm{Evt}}{\overset{\mathrm{Sp}}{\rightleftarrows}} \mathbf{FCT}^{\mathrm{op}}$$

3. Boolean Classifications.

Let $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\mathbf{A}} \rangle$ be a classification. The **disjunctive power** of $\mathbf{A}$ is the classification $\vee\mathbf{A} = \langle \mathcal{P}(\Sigma_{\mathbf{A}}), A, \models_{\vee\mathbf{A}} \rangle$, where

$$a \models_{\vee\mathbf{A}} \Gamma \quad \text{if and only if} \quad a \models_{\mathbf{A}} \gamma, \text{ for some } \gamma \in \Gamma,$$

for all $a \in A, \Gamma \subseteq \Sigma_{\mathbf{A}}$. Moreover, given an infomorphism $f = \langle \hat{f}, \check{f} \rangle : \mathbf{A} \to \mathbf{B}$, the pair $\vee f = \langle \hat{\vee f}, \check{\vee f} \rangle$, where $\hat{\vee f} : \mathcal{P}(\Sigma_{\mathbf{A}}) \to \mathcal{P}(\Sigma_{\mathbf{B}})$ is given by

$$\hat{\vee f}(\Gamma) = \hat{f}(\Gamma), \quad \text{for all } \Gamma \subseteq \Sigma_{\mathbf{A}},$$

and $\check{\vee f} : B \to A$, given by

$$\check{\vee f}(b) = \check{f}(b), \quad \text{for all } b \in B,$$

$\chi_{\models_{\mathbf{A}}} : A \times \Sigma_{\mathbf{A}} \to \mathbf{2}$, then Evt sends a function $\mathbf{S} : S_1 \to S_2$ to the function $\chi_{\in} \circ (\mathbf{S} \times 1_{\mathbf{2}^{S_2}}) : S_1 \times \mathbf{2}^{S_2} \to \mathbf{2}$, where by $\mathbf{2}$ is denoted the 2-element set $\{0, 1\}$.

[2] Note that, similarly, Sp sends a relation $\chi_{\models_{\mathbf{A}}} : A \times \Sigma_{\mathbf{A}} \to \mathbf{2}$ to the curry $\mathrm{cur}(\chi_{\models_{\mathbf{A}}}) : A \to \mathbf{2}^{\Sigma_{\mathbf{A}}}$.

is also an infomorphism from the disjunctive power $\vee\mathbf{A}$ of $\mathbf{A}$ to the disjunctive power $\vee\mathbf{B}$ of $\mathbf{B}$.

$\vee$, whose action is defined above on classifications and infomorphisms, is an endofunctor on **REL**.

Similarly, the **conjunctive power** of $\mathbf{A}$ is the classification $\wedge\mathbf{A} = \langle \mathcal{P}(\Sigma_{\mathbf{A}}),\ A, \models_{\wedge\mathbf{A}} \rangle$, where

$$a \models_{\wedge\mathbf{A}} \Gamma \quad \text{if and only if} \quad a \models_{\mathbf{A}} \gamma,\ \text{for all } \gamma \in \Gamma,$$

for all $a \in A, \Gamma \subseteq \Sigma_{\mathbf{A}}$. Moreover, given an infomorphism $f = \langle \hat{f}, \check{f} \rangle : \mathbf{A} \to \mathbf{B}$, the pair $\wedge f = \langle \hat{\wedge f}, \check{\wedge f} \rangle$, where $\hat{\wedge f} : \mathcal{P}(\Sigma_{\mathbf{A}}) \to \mathcal{P}(\Sigma_{\mathbf{B}})$ is given by

$$\hat{\wedge f}(\Gamma) = \hat{f}(\Gamma), \quad \text{for all } \Gamma \subseteq \Sigma_{\mathbf{A}},$$

and $\check{\wedge f} : B \to A$, given by

$$\check{\wedge f}(b) = \check{f}(b), \quad \text{for all } b \in B,$$

is also an infomorphism from the conjunctive power $\wedge\mathbf{A}$ of $\mathbf{A}$ to the conjunctive power $\wedge\mathbf{B}$ of $\mathbf{B}$.

$\wedge$, whose action is defined above on classifications and infomorphisms, is, like $\vee$, an endofunctor on **REL**.

Finally, the **negation** of $\mathbf{A}$ is the classification $\neg\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \models_{\neg\mathbf{A}} \rangle$, with

$$a \models_{\neg\mathbf{A}} \alpha \quad \text{if and only if} \quad a \not\models_{\mathbf{A}} \alpha, \quad \text{for all } a \in A, \alpha \in \Sigma_{\mathbf{A}}.$$

In this case also, given an infomorphism $f = \langle \hat{f}, \check{f} \rangle : \mathbf{A} \to \mathbf{B}$, the pair $\neg f = \langle \hat{\neg f}, \check{\neg f} \rangle$, where $\hat{\neg f} : \Sigma_{\mathbf{A}} \to \Sigma_{\mathbf{B}}$ is given by

$$\hat{\neg f}(\alpha) = \hat{f}(\alpha), \quad \text{for all } \alpha \in \Sigma_{\mathbf{A}},$$

and $\check{\neg f} : B \to A$, given by

$$\check{\neg f}(b) = \check{f}(b), \quad \text{for all } b \in B,$$

is also an infomorphism from the negation $\neg\mathbf{A}$ of $\mathbf{A}$ to the negation $\neg\mathbf{B}$ of $\mathbf{B}$.

$\neg$, whose action is defined above on classifications and infomorphisms, is an involution endofunctor on **REL**.

A classification $\mathbf{A}$ is said to **admit disjunction** if the identity map $i_A : A \to A$ can be extended to an infomorphism $d : \vee\mathbf{A} \to \mathbf{A}$. In this case $\hat{d} : \mathcal{P}(\Sigma_{\mathbf{A}}) \to \Sigma_{\mathbf{A}}$ is called a **disjunction** on $\mathbf{A}$, sometimes denoted by $\vee$.

Similarly, $\mathbf{A}$ **admits conjunction** if the identity map $i_A : A \to A$ can be extended to an infomorphism $c : \wedge\mathbf{A} \to \mathbf{A}$ and, in this case, $\hat{c} : \mathcal{P}(\Sigma_{\mathbf{A}}) \to \Sigma_{\mathbf{A}}$ is called a **conjunction** on $\mathbf{A}$ and is denoted by $\wedge$.

Finally, $\mathbf{A}$ **admits negation** if $i_A : A \to A$ can be extended to an infomorphism $n : \neg\mathbf{A} \to \mathbf{A}$. In this case, $\hat{n} : \Sigma_{\mathbf{A}} \to \Sigma_{\mathbf{A}}$ is called a **negation** on $\mathbf{A}$ and is denoted by $\neg$.

A classification is said to be **Boolean** if it admits disjunction, conjunction and negation.

The following theorem ([4], Proposition 7.7) characterizes Boolean classifications

THEOREM 2. *A classification* $\mathbf{A}$ *is Boolean if and only if, for every set* X *of tokens closed under* $\cong_{\mathbf{A}}$, *there is a type* α, *such that* $X = typ_{\mathbf{A}}(\alpha)$.

There is a natural construction that produces a Boolean classification out of an arbitrary given classification $\mathbf{A}$. The **Boolean closure** of $\mathbf{A}$, denoted by $\mathrm{Boole}(\mathbf{A})$, is the classification

$$\mathrm{Boole}(\mathbf{A}) = \langle \mathcal{P}(\mathcal{P}(\Sigma_{\mathbf{A}})), A, \models_{\mathrm{Boole}(\mathbf{A})} \rangle,$$

where

$$a \models_{\mathrm{Boole}(\mathbf{A})} \mathcal{G} \quad \text{if and only if} \quad \mathrm{typ}_{\mathbf{A}}(a) \in \mathcal{G},$$

for all $a \in A$, $\mathcal{G} \subseteq \mathcal{P}(\Sigma_{\mathbf{A}})$.

THEOREM 3. ([4], Proposition 7.10) *Given a classification* $\mathbf{A}$, *the operations of union, intersection and complementation are a disjunction, conjunction and negation, respectively, on Boole*$(\mathbf{A})$.

It is not difficult to see that, for any classification $\mathbf{A}$, $\mathrm{Boole}(\mathbf{A}) = \mathrm{Evt}(\mathrm{Sp}(\mathbf{A}))$. Thus, in view of Theorem 1, the following holds

COROLLARY 4. ([4], Corollary 8.24) *Let* $\mathbf{A}$ *be a classification and* $\mathbf{S}$ *a state space. Every infomorphism* $f : \mathbf{A} \to Evt(\mathbf{S})$ *has a unique extension to an infomorphism* $f^* : Boole(\mathbf{A}) \to Evt(\mathbf{S})$.

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\eta_{\mathbf{A}}"] \arrow[dr, "f"'] & \mathrm{Boole}(\mathbf{A}) \arrow[d, "f^*"] \\ & \mathrm{Evt}(\mathbf{S}) \end{tikzcd}$$

More precisely, $f^* = \mathrm{Evt}(f^\dagger)$, where $f^\dagger : \mathbf{S} \to \mathrm{Sp}(\mathbf{A})$ is the unique state space morphism provided by the adjunction of Theorem 1 that makes the following triangle commute, where $\eta_{\mathbf{A}} : \mathbf{A} \to \mathrm{Evt}(\mathrm{Sp}(\mathbf{A}))$ is the unit of that adjunction.

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\eta_{\mathbf{A}}"] \arrow[dr, "f"'] & \mathrm{Evt}(\mathrm{Sp}(\mathbf{A})) \arrow[d, "\mathrm{Evt}(f^{\dagger})"] \\ & \mathrm{Evt}(\mathbf{S}) \end{tikzcd}$$

4. Abstract Classification Theory.

In this section the categories and the adjunctions that were described in the previous two sections are viewed from a more abstract perspective. This point of view has two advantages over the more "concrete" classification viewpoint. On the one hand it simplifies notation and terminology and brings the development closer to the foundations. And on the other it makes the context suitable for potential application to other areas and to possible generalizations.

A **relation from** a set A **to** a set Σ is, as usual, a subset $R \subseteq A \times \Sigma$. A **relation morphism** $f : R_1 \to R_2$ from a relation $R_1 \subseteq A_1 \times \Sigma_1$ to a relation $R_2 \subseteq A_2 \times \Sigma_2$ is a pair $f = \langle \check{f}, \hat{f} \rangle$ of set maps $\check{f} : A_2 \to A_1$ and $\hat{f} : \Sigma_1 \to \Sigma_2$, such that, for all $a_2 \in A_2, \sigma_1 \in \Sigma_1$,

$$\text{(1)} \qquad \check{f}(a_2) R_1 \sigma_1 \quad \text{if and only if} \quad a_2 R_2 \hat{f}(\sigma_1).$$

This condition may be expressed pictorially by the following diagram

$$\begin{tikzcd} A_1 \arrow[r, phantom, "R_1"] & \Sigma_1 \arrow[d, "\hat{f}"] \\ A_2 \arrow[u, "\check{f}"] \arrow[r, phantom, "R_2"] & \Sigma_2 \end{tikzcd}$$

Notice that this notion of a relation morphism is different from the one that is usually used in typed first-order logic and other logical contexts. A morphism there would be a covariant pair of arrows $\langle \check{g} : A_1 \to A_2, \hat{g} : \Sigma_1 \to \Sigma_2 \rangle$, such that, for every $a_1 \in A_1$ and $\sigma_1 \in \Sigma_1, a_1 R_1 \sigma_1$ implies $\check{g}(a_1) R_2 \hat{g}(\sigma_1)$. It is however very similar to the morphisms in institutions [10, 11] and the reader familiar with institutions will recognize in (1) an alias of the institutional satisfaction condition. A survey of the different kinds of institution morphisms that have been used in the literature and some of their relationships has been presented in [13].

Relations with relation morphisms between them form a category, the **category of relations**. This is our familiar category **REL** (whence the name), since classifications are simply relations, the domain A of a relation $R \subseteq A \times \Sigma$ corresponding to the set of tokens and the codomain Σ to the set of types. R itself corresponds to the classification relation and infomorphisms are the same as relation morphisms. *Abstract classification theory* is understood to mean the categorical study of the category **REL** of relations and other related categories.

By a **function** is meant a set morphism $\mathbf{S} : S_1 \rightarrow S_2$ from some set S_1 to some set S_2. A **function morphism** $f : \mathbf{S}_1 \rightarrow \mathbf{S}_2$ from a function $\mathbf{S}_1 : S_{11} \rightarrow S_{12}$ to a function $\mathbf{S}_2 : S_{21} \rightarrow S_{22}$ is a pair of morphism $f = \langle \check{f}, \hat{f} \rangle$, with $\check{f} : S_{11} \rightarrow S_{21}$ and $\hat{f} : S_{12} \rightarrow S_{22}$, such that the following diagram commutes

$$\begin{tikzcd} S_{11} \arrow[r, "\mathbf{S}_1"] \arrow[d, "\check{f}"'] & S_{12} \arrow[d, "\hat{f}"] \\ S_{21} \arrow[r, "\mathbf{S}_2"'] & S_{22} \end{tikzcd}$$

Functions with function morphisms between them form a category, the **category of functions**, which is well-known in category theory as the category of arrows in **SET**. This is our familiar category **FCT**, since, in fact, state spaces are, simply, functions in **SET**.

A **partition** is a collection of disjoint nonempty subsets of a set A whose union is A. Such a partition may also be viewed as a set A with an equivalence relation $\cong_{\mathbf{A}}$ on A, whose equivalence classes are the blocks of the partition. We take a hybrid point of view and denote a partition on A, that determines the equivalence relation $\cong_{\mathbf{A}}$ and is determined by it, by $A/{\cong_{\mathbf{A}}}$. A **partition morphism** $f : A/{\cong_{\mathbf{A}}} \rightarrow B/{\cong_{\mathbf{B}}}$ from a partition $A/{\cong_{\mathbf{A}}}$ to a partition $B/{\cong_{\mathbf{B}}}$ is a function $f : A \rightarrow B$, such that, for all $a_1, a_2 \in A$,

$$a_1 \cong_{\mathbf{A}} a_2 \quad \text{implies} \quad f(a_1) \cong_{\mathbf{B}} f(a_2),$$

i.e., $f(\cong_{\mathbf{A}}) \subseteq \; \cong_{\mathbf{B}}$. Partitions with partition morphisms between them form a category, a subcategory of the category **FCT** of functions, the **category of partitions**, denoted by **PAR**.

Recall that Theorem 1 has given an adjunction $\langle \mathrm{Sp}, \mathrm{Evt}, \eta, \epsilon \rangle : \mathbf{REL} \rightarrow \mathbf{FCT}^{\mathrm{op}}$.

$$\mathbf{REL} \underset{\mathrm{Evt}}{\overset{\mathrm{Sp}}{\rightleftarrows}} \mathbf{FCT}^{\mathrm{op}}$$

The functor $\mathrm{Sp} : \mathbf{REL} \to \mathbf{FCT}^{\mathrm{op}}$ maps a relation $R \subseteq A \times \Sigma$ to the function $\mathrm{Sp}(R) : A \to \mathcal{P}(\Sigma)$, with

$$\mathrm{Sp}(R)(a) = \{\sigma \in \Sigma : aR\sigma\}, \quad \text{for all } a \in A,$$

and maps a given relation morphism $f = \langle \check{f}, \hat{f} \rangle : R_1 \to R_2$, where $R_1 \subseteq A_1 \times \Sigma_1$ and $R_2 \subseteq A_2 \times \Sigma_2$, to the function morphism $\mathrm{Sp}(f) = \langle \check{f}, \hat{f}^{-1} \rangle : \mathrm{Sp}(R_2) \to \mathrm{Sp}(R_1)$.

The functor $\mathrm{Evt} : \mathbf{FCT} \to \mathbf{REL}^{\mathrm{op}}$ maps a function $\mathbf{S} : S_1 \to S_2$ to the relation $\mathrm{Evt}(\mathbf{S}) \subseteq S_1 \times \mathcal{P}(S_2)$, where

$$s_1 \mathrm{Evt}(\mathbf{S}) T \quad \text{if and only if} \quad \mathbf{S}(s_1) \in T,$$

for all $s_1 \in S_1, T \subseteq S_2$ and a given function morphism $f : \mathbf{S}_1 \to \mathbf{S}_2$ from a function $\mathbf{S}_1 : S_{11} \to S_{12}$ to a function $\mathbf{S}_2 : S_{21} \to S_{22}$ to the relation morphism $\mathrm{Evt}(f) = \langle \check{\mathrm{Evt}}(f), \hat{\mathrm{Evt}}(f) \rangle : \mathrm{Evt}(\mathbf{S}_2) \to \mathrm{Evt}(\mathbf{S}_1)$, with

$$\check{\mathrm{Evt}}(f) : S_{11} \to S_{21}; \quad \check{\mathrm{Evt}}(f) = \check{f}$$

and

$$\hat{\mathrm{Evt}}(f) : \mathcal{P}(S_{22}) \to \mathcal{P}(S_{12}); \quad \hat{\mathrm{Evt}}(f) = \hat{f}^{-1}.$$

The unit of this adjunction is the natural transformation $\eta : I_{\mathbf{REL}} \to \mathrm{Evt} \circ \mathrm{Sp}$ that is defined, for all $R \subseteq A \times \Sigma$, by $\eta_R : R \to \mathrm{Evt}(\mathrm{Sp}(R))$, with $\eta_R = \langle \check{\eta}_R, \hat{\eta}_R \rangle$, such that

$$\check{\eta}_R : A \to A; \quad \check{\eta}_R = i_A,$$

and

$$\hat{\eta}_R : \Sigma \to \mathcal{P}(\mathcal{P}(\Sigma)); \quad \hat{\eta}_R(\sigma) = \{X \subseteq \Sigma : \sigma \in X\}, \text{ for all } \sigma \in \Sigma.$$

This is a well-defined relation morphism since

$$\begin{tikzcd}
A & R & \Sigma \arrow[d, "\hat{\eta}_R"] \\
A \arrow[u, "\check{\eta}_R"] & \mathrm{Evt}(\mathrm{Sp}(R)) & \mathcal{P}(\mathcal{P}(\Sigma))
\end{tikzcd}$$

$$\begin{aligned} a\mathrm{Evt}(\mathrm{Sp}(R))\{X \subseteq \Sigma : \sigma \in X\} &\text{ iff } \mathrm{Sp}(R)(a) \in \{X \subseteq \Sigma : \sigma \in X\} \\ &\text{ iff } \{\sigma \in \Sigma : aR\sigma\} \in \{X \subseteq \Sigma : \sigma \in X\} \\ &\text{ iff } \sigma \in \{\sigma \in \Sigma : aR\sigma\} \\ &\text{ iff } aR\sigma. \end{aligned}$$

η is a natural transformation since, for all relation morphisms $f : R_1 \to R_2$, where $R_1 \subseteq A_1 \times \Sigma_1$ and $R_2 \subseteq A_2 \times \Sigma_2$, the following diagram commutes

$$\begin{tikzcd} R_1 \arrow[r, "f"] \arrow[d, "\eta_{R_1}"'] & R_2 \arrow[d, "\eta_{R_2}"] \\ \mathrm{Evt}(\mathrm{Sp}(R_1)) \arrow[r, "\mathrm{Evt}(\mathrm{Sp}(f))"'] & \mathrm{Evt}(\mathrm{Sp}(R_2)) \end{tikzcd}$$

Commutativity is shown as follows:

$$\begin{aligned} \check{f}(\check{\eta_{R_2}}(a_2)) &= \check{f}(a_2) \\ &= \mathrm{Evt}(\check{\mathrm{Sp}}(f))(a_2) \\ &= \check{\eta_{R_1}}(\mathrm{Evt}(\check{\mathrm{Sp}}(f))(a_2)) \end{aligned}$$

and

$$\begin{aligned} \hat{\eta_{R_2}}(\hat{f}(\sigma_1)) &= \{X \subseteq \Sigma_2 : \hat{f}(\sigma_1) \in X\} \\ &= \{X \subseteq \Sigma_2 : \sigma_1 \in \hat{f}^{-1}(X)\} \\ &= \{X \subseteq \Sigma_2 : \hat{f}^{-1}(X) \in \{X \subseteq \Sigma_1 : \sigma_1 \in X\}\} \\ &= \mathrm{Evt}(\hat{\mathrm{Sp}}(f))(\{X \subseteq \Sigma_1 : \sigma_1 \in X\}) \\ &= \mathrm{Evt}(\hat{\mathrm{Sp}}(f))(\hat{\eta_{R_1}}(\sigma_1)). \end{aligned}$$

The counit of the adjunction is the natural transformation $\epsilon : I_{\mathbf{FCT}} \to \mathrm{Sp} \circ \mathrm{Evt}$ that is defined, for all $\mathbf{S} : S_1 \to S_2$, by $\epsilon_{\mathbf{S}} : \mathbf{S} \to \mathrm{Sp}(\mathrm{Evt}(\mathbf{S}))$, with $\epsilon_{\mathbf{S}} = \langle \check{\epsilon_{\mathbf{S}}}, \hat{\epsilon_{\mathbf{S}}} \rangle$, such that

$$\check{\epsilon_{\mathbf{S}}} : S_1 \to S_1; \quad \check{\epsilon_{\mathbf{S}}} = i_{S_1},$$

and

$$\hat{\epsilon_{\mathbf{S}}} : S_2 \to \mathcal{P}(\mathcal{P}(S_2)); \quad \hat{\epsilon_{\mathbf{S}}}(s) = \{X \subseteq S_2 : s \in X\}, \quad \text{for all } s \in S_2.$$

This is a well-defined function morphism since, for all $s_1 \in S_1$,

$$\begin{tikzcd} S_1 \arrow[r, "\mathbf{S}"] \arrow[d, "\check{\epsilon_{\mathbf{S}}}"'] & S_2 \arrow[d, "\hat{\epsilon_{\mathbf{S}}}"] \\ S_1 \arrow[r, "\mathrm{Sp}(\mathrm{Evt}(\mathbf{S}))"'] & \mathcal{P}(\mathcal{P}(S_2)) \end{tikzcd}$$

$$\begin{aligned} \hat{\epsilon_{\mathbf{S}}}(\mathbf{S}(s_1)) &= \{X \subseteq S_2 : \mathbf{S}(s_1) \in X\} \\ &= \mathrm{Sp}(\mathrm{Evt}(\mathbf{S}))(s_1) \\ &= \mathrm{Sp}(\mathrm{Evt}(\mathbf{S}))(\check{\epsilon_{\mathbf{S}}}(s_1)). \end{aligned}$$

ϵ is a natural transformation since, for all function morphisms $f : \mathbf{S}_1 \to \mathbf{S}_2$, where $\mathbf{S}_1 : S_{11} \to S_{12}$ and $\mathbf{S}_2 : S_{21} \to S_{22}$, the following diagram commutes

$$\begin{tikzcd} \mathbf{S}_1 \arrow[r, "f"] \arrow[d, "\epsilon_{\mathbf{S}_1}"'] & \mathbf{S}_2 \arrow[d, "\epsilon_{\mathbf{S}_2}"] \\ \mathrm{Sp}(\mathrm{Evt}(\mathbf{S}_1)) \arrow[r, "\mathrm{Sp}(\mathrm{Evt}(f))"'] & \mathrm{Sp}(\mathrm{Evt}(\mathbf{S}_2)) \end{tikzcd}$$

Commutativity is shown as follows. For all $s_1 \in S_{11}$,

$$\begin{aligned} \check{\epsilon_{\mathbf{S}_2}}(\check{f}(s_1)) &= \check{f}(s_1) \\ &= \mathrm{Sp}(\check{\mathrm{Evt}}(f))(s_1) \\ &= \mathrm{Sp}(\check{\mathrm{Evt}}(f))(\check{\epsilon_{\mathbf{S}_1}}(s_1)) \end{aligned}$$

and, for all $s \in S_{12}$,

$$\begin{aligned} \hat{\epsilon_{\mathbf{S}_2}}(\hat{f}(s)) &= \{X \subseteq S_{22} : \hat{f}(s) \in X\} \\ &= \{X \subseteq S_{22} : s \in \hat{f}^{-1}(X)\} \\ &= \{X \subseteq S_{22} : s \in \mathrm{Evt}\hat{(}f)(X)\} \\ &= \mathrm{Evt}\hat{(}f)^{-1}(\{Y \subseteq S_{12} : s \in Y\}) \\ &= \mathrm{Sp}(\hat{\mathrm{Evt}}(f))(\{Y \subseteq S_{12} : s \in Y\}) \\ &= \mathrm{Sp}(\hat{\mathrm{Evt}}(f))(\hat{\epsilon_{\mathbf{S}_1}}(s)). \end{aligned}$$

Finally, it is not difficult to check that, for all $R \subseteq A \times \Sigma$ and $\mathbf{S} : S_1 \to S_2$, the following triangles commute

$$
\begin{tikzcd}
\mathrm{Sp}(\mathrm{Evt}(\mathrm{Sp}(R))) \arrow[r, "\mathrm{Sp}(\eta_R)"] & \mathrm{Sp}(R) \\
\mathrm{Sp}(R) \arrow[u, "\epsilon_{\mathrm{Sp}(R)}"] \arrow[ur, "i_{\mathrm{Sp}(R)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Evt}(\mathbf{S}) \arrow[r, "\eta_{\mathrm{Evt}(\mathbf{S})}"] \arrow[dr, "i_{\mathrm{Evt}(\mathbf{S})}"'] & \mathrm{Evt}(\mathrm{Sp}(\mathrm{Evt}(\mathbf{S}))) \arrow[d, "\mathrm{Evt}(\epsilon_{\mathbf{S}})"] \\
& \mathrm{Evt}(\mathbf{S})
\end{tikzcd}
$$

Now we set out to provide an adjunction $\langle \mathrm{Fct}, \mathrm{Prt}, \nu, \mu \rangle : \mathbf{PAR} \to \mathbf{FCT}$ from the category **PAR** of partitions to the category **FCT** of functions.

$$\mathbf{PAR} \underset{\mathrm{Prt}}{\overset{\mathrm{Fct}}{\rightleftarrows}} \mathbf{FCT}$$

Let $\mathbf{S} : S_1 \to S_2$ be a function. Define

$$\mathrm{Prt}(\mathbf{S}) = S_1/{\cong_{\mathbf{S}}} = \{\mathbf{S}^{-1}(s) : s \in \mathrm{Im}(\mathbf{S})\}.$$

Moreover, given a function morphism $f : \mathbf{S}_1 \to \mathbf{S}_2$, where $\mathbf{S}_1 : S_{11} \to S_{12}$ and $\mathbf{S}_2 : S_{21} \to S_{22}$, define the partition morphism $\mathrm{Prt}(f) : S_{11}/{\cong_{\mathbf{S}_1}} \to S_{21}/{\cong_{\mathbf{S}_2}}$ by

$$\mathrm{Prt}(f)(s_1) = \check{f}(s_1), \quad \text{for all} \quad s_1 \in S_{11}.$$

This is well-defined since, for all $s_1, s_1' \in S_{11}$,

$$
\begin{array}{rcl}
s_1 \cong_{\mathbf{S}_1} s_1' & \text{iff} & \mathbf{S}_1(s_1) = \mathbf{S}_1(s_1') \\
& \text{implies} & \hat{f}(\mathbf{S}_1(s_1)) = \hat{f}(\mathbf{S}_1(s_1')) \\
& \text{iff} & \mathbf{S}_2(\check{f}(s_1)) = \mathbf{S}_2(\check{f}(s_1')) \\
& \text{iff} & \check{f}(s_1) \cong_{\mathbf{S}_2} \check{f}(s_1').
\end{array}
$$

Prt defined as above on functions and function morphisms is a functor $\mathrm{Prt} : \mathbf{FCT} \to \mathbf{PAR}$ from the category **FCT** of functions to the category **PAR** of partitions.

Next, let $A/{\cong_{\mathbf{A}}}$ be a partition. Define $\mathrm{Fct}(A/{\cong_{\mathbf{A}}})$ to be the natural quotient map $q_{\mathbf{A}} : A \to A/{\cong_{\mathbf{A}}}$. Further, given a partition morphism $f : A/{\cong_{\mathbf{A}}} \to B/{\cong_{\mathbf{B}}}$, define a function morphism $\mathrm{Fct}(f) : q_{\mathbf{A}} \to q_{\mathbf{B}}$ by $\mathrm{Fct}(f) = \langle \check{\mathrm{Fct}}(f), \hat{\mathrm{Fct}}(f) \rangle$, where $\check{\mathrm{Fct}}(f) : A \to B$ is given by $\check{\mathrm{Fct}}(f) = f$ and $\hat{\mathrm{Fct}}(f) : A/{\cong_{\mathbf{A}}} \to B/{\cong_{\mathbf{B}}}$ is given by

$$\hat{\mathrm{Fct}}(f)(a/{\cong_{\mathbf{A}}}) = f(a)/{\cong_{\mathbf{B}}}, \quad \text{for all} \quad a \in A.$$

This is well defined since $a_1 \cong_{\mathbf{A}} a_2$ implies $f(a_1) \cong_{\mathbf{B}} f(a_2)$, for all $a_1, a_2 \in A$. It is a function morphism since, for all $a \in A$,

$$\begin{tikzcd} A \arrow[r, "q_{\mathbf{A}}"] \arrow[d, "\check{\mathrm{Fct}}(f)"'] & A/{\cong_{\mathbf{A}}} \arrow[d, "\hat{\mathrm{Fct}}(f)"] \\ B \arrow[r, "q_{\mathbf{B}}"'] & B/{\cong_{\mathbf{B}}} \end{tikzcd}$$

$$\begin{aligned} \hat{\mathrm{Fct}}(f)(q_{\mathbf{A}}(a)) &= \hat{\mathrm{Fct}}(f)(a/{\cong_{\mathbf{A}}}) \\ &= f(a)/{\cong_{\mathbf{B}}} \\ &= q_{\mathbf{B}}(f(a)) \\ &= q_{\mathbf{B}}(\check{\mathrm{Fct}}(f)(a)). \end{aligned}$$

The functor $\mathrm{Prt} : \mathbf{FCT} \to \mathbf{PAR}$ is a right adjoint to $\mathrm{Fct} : \mathbf{PAR} \to \mathbf{FCT}$. There is actually an underlying coreflection that can be uncovered by identifying a partition with its quotient map. The unit of the adjunction is the natural transformation $\nu : I_{\mathbf{PAR}} \to \mathrm{Prt} \circ \mathrm{Fct}$, such that, for all partitions $A/{\cong_{\mathbf{A}}}$, $\nu_{A/\cong_{\mathbf{A}}} : A/{\cong_{\mathbf{A}}} \to A/{\cong_{\mathbf{A}}}$ is the identity function

$$\nu_{A/\cong_{\mathbf{A}}} = i_A.$$

The counit of the adjunction is the natural transformation $\mu : \mathrm{Fct} \circ \mathrm{Prt} \to I_{\mathbf{FCT}}$, defined, for all functions $\mathbf{S} : S_1 \to S_2$ by $\mu_{\mathbf{S}} : q_{\cong_{\mathbf{S}}} \to \mathbf{S}$, where $\mu_{\mathbf{S}} = \langle \check{\mu_{\mathbf{S}}}, \hat{\mu_{\mathbf{S}}} \rangle$, with

$$\check{\mu_{\mathbf{S}}} : S_1 \to S_1; \quad \check{\mu_{\mathbf{S}}} = i_{S_1}$$

and

$$\hat{\mu_{\mathbf{S}}} : S_1/{\cong_{\mathbf{S}}} \to S_2; \quad \hat{\mu_{\mathbf{S}}}(s_1/{\cong_{\mathbf{S}}}) = \mathbf{S}(s_1), \quad \text{for all } s_1 \in S_1.$$

This is well defined since, for all $s_1 \in S_1$,

$$\begin{tikzcd} S_1 \arrow[r, "\mathrm{Fct}(\mathrm{Prt}(\mathbf{S}))"] \arrow[d, "\check{\mu_{\mathbf{S}}}"'] & S_1/{\cong_{\mathbf{S}}} \arrow[d, "\hat{\mu_{\mathbf{S}}}"] \\ S_1 \arrow[r, "\mathbf{S}"'] & S_2 \end{tikzcd}$$

$$\mathbf{S}(i_{S_1}(s_1)) = \mathbf{S}(s_1) = \hat{\mu_{\mathbf{S}}}(s_1/{\cong_{\mathbf{S}}}) = \hat{\mu_{\mathbf{S}}}(q_{\cong_{\mathbf{S}}}(s_1)).$$

It is a natural transformation since, for all function morphisms $f : \mathbf{S}_1 \to \mathbf{S}_2$, with $\mathbf{S}_1 : S_{11} \to S_{12}$ and $\mathbf{S}_2 : S_{21} \to S_{22}$ and $f = \langle \check{f}, \hat{f} \rangle$, the following diagram commutes

$$\begin{tikzcd}
q_{\cong_{\mathbf{S}_1}} \arrow[r, "\mu_{\mathbf{S}_1}"] \arrow[d, "\mathrm{Fct}(\mathrm{Prt}(f))"'] & \mathbf{S}_1 \arrow[d, "f"] \\
q_{\cong_{\mathbf{S}_2}} \arrow[r, "\mu_{\mathbf{S}_2}"'] & \mathbf{S}_2
\end{tikzcd}$$

$$\begin{aligned}
\check{f}(\check{\mu}_{\mathbf{S}_1}(s_1)) &= \check{f}(s_1) \\
&= \mathrm{Fct}(\check{\mathrm{Prt}}(f))(s_1) \\
&= \check{\mu}_{\mathbf{S}_2}(\mathrm{Fct}(\check{\mathrm{Prt}}(f))(s_1))
\end{aligned}$$

and

$$\begin{aligned}
\hat{f}(\hat{\mu}_{\mathbf{S}_1}(s_1/\cong_{\mathbf{S}_1}) &= \hat{f}(\mathbf{S}_1(s_1)) \\
&= \mathbf{S}_2(\check{f}(s_1)) \\
&= \hat{\mu}_{\mathbf{S}_2}(\check{f}(s_1)/\cong_{\mathbf{S}_2}) \\
&= \hat{\mu}_{\mathbf{S}_2}(\mathrm{Fct}(\hat{\mathrm{Prt}}(f))(s_1)).
\end{aligned}$$

Finally, it is not difficult to see that the adjunction triangles commute, since all morphisms below reduce to identities

$$\begin{tikzcd}
\mathrm{Prt}(\mathbf{S}) \arrow[r, "\nu_{\mathrm{Prt}(\mathbf{S})}"] \arrow[dr, "i_{\mathrm{Prt}(\mathbf{S})}"'] & \mathrm{Prt}(\mathrm{Fct}(\mathrm{Prt}(\mathbf{S}))) \arrow[d, "\mathrm{Prt}(\mu_{\mathbf{S}})"] \\
& \mathrm{Prt}(\mathbf{S})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fct}(A/\cong_{\mathbf{A}}) \arrow[r, "\mathrm{Fct}(\nu_{A/\cong_{\mathbf{A}}})"] \arrow[dr, "i_{\mathrm{Fct}(A/\cong_{\mathbf{A}})}"'] & \mathrm{Fct}(\mathrm{Prt}(\mathrm{Fct}(A/\cong_{\mathbf{A}}))) \arrow[d, "\mu_{\mathrm{Fct}(A/\cong_{\mathbf{A}})}"] \\
& \mathrm{Fct}(A/\cong_{\mathbf{A}})
\end{tikzcd}$$

The adjunction just described was to be expected since the functor Prt, that maps a given function to the induced partition of its domain "forgets" all the information that is carried along by the codomain of the function.

Collecting all adjunctions defined up to this point we obtain the following diagram of adjunctions. The arrows are pointing to the directions of the left adjoint functors which come first in the labelings

$$\mathbf{ERL}^{\mathrm{op}} \xrightarrow[\langle \mathrm{Inc}, \mathrm{Ext} \rangle]{} \mathbf{REL}^{\mathrm{op}} \xrightarrow[\langle \mathrm{Sp}, \mathrm{Evt} \rangle]{} \mathbf{FCT} \xleftarrow[\langle \mathrm{Fct}, \mathrm{Prt} \rangle]{} \mathbf{PAR}$$

Now, it is not difficult to check that the composite contravariant functor $\mathrm{Prt} \circ \mathrm{Sp} : \mathbf{REL} \to \mathbf{PAR}^{\mathrm{op}}$ from the category of relations to the opposite category of partitions corresponds, under appropriate identifications, to the functor $U : \mathbf{REL}\to\mathbf{FCT}$, defined in Theorem 35 of [8]. However, since this composition is a composition of a left adjoint with a right adjoint functor, this does not immediately yield any adjunction in the present setting. So it does not immediately verify the conjecture made in [8] (after the statement of Proposition 36) that the functor U must have a left adjoint. However, it is shown below that U does indeed have a left adjoint.

Specifically, we consider the functor $\mathrm{Qnt} : \mathbf{ERL}\to\mathbf{PAR}^{\mathrm{op}}$ which is defined as follows: Given an extensional relation $R \subseteq A \times \Sigma$, define the partition $\mathrm{Qnt}(R) = A/{\cong_R}$, where $\cong_R \subseteq A \times A$ is the equivalence relation on A defined by

$$a_1 \cong_R a_2 \quad \text{iff} \quad \{\sigma \in \Sigma : a_1 R\sigma\} = \{\sigma \in \Sigma : a_2 R\sigma\}, \quad \text{for all } a_1, a_2 \in A.$$

Moreover, given two extensional relations $R_1 \subseteq A_1 \times \Sigma_1$ and $R_2 \subseteq A_2 \times \Sigma_2$ and a relation morphism $f = \langle \check{f}, \hat{f}\rangle : R_1\to R_2$, define the partition morphism $\mathrm{Qnt}(f) : A_2/{\cong_{R_2}}\to A_1/{\cong_{R_1}}$ by $\mathrm{Qnt}(f) = \check{f}$. This is well defined, since, for all $a_1, a_2 \in A_2$, with $a_1 \cong_{R_2} a_2$, we have, for all $\sigma \in \Sigma_1$,

$$\begin{aligned} \check{f}(a_1)R_1\sigma \ &\text{iff}\ a_1 R_2 \hat{f}(\sigma) \\ &\text{iff}\ a_2 R_2 \hat{f}(\sigma) \\ &\text{iff}\ \check{f}(a_2)R_1\sigma, \end{aligned}$$

i.e., $\check{f}(a_1)R_1\check{f}(a_2)$. $\mathrm{Qnt} : \mathbf{ERL}\to\mathbf{PAR}^{\mathrm{op}}$, as defined above on extensional relations and relation morphisms is a functor[3] .

On the other hand, given a partition $A/{\cong_{\mathbf{A}}}$, define the extensional relation $\mathrm{Erl}(A/{\cong_{\mathbf{A}}}) \subseteq A \times \mathcal{P}(A/{\cong_{\mathbf{A}}})$, by stipulating that, for all $a \in A$, $X \subseteq A/{\cong_{\mathbf{A}}}$,

$$a \ \mathrm{Erl}(A/{\cong_{\mathbf{A}}})\ X \quad \text{iff} \quad a/{\cong_{\mathbf{A}}} \in X.$$

$\mathrm{Erl}(A/{\cong_{\mathbf{A}}})$ is extensional, since, for all $X, Y \subseteq A/{\cong_{\mathbf{A}}}$, if $X \neq Y$, then there exists $a/{\cong_{\mathbf{A}}} \in X - Y$ or $a/{\cong_{\mathbf{A}}} \in Y - X$. But then $\{a \in A : a/{\cong_{\mathbf{A}}} \in X\} \neq \{a \in A : a/{\cong_{\mathbf{A}}} \in Y\}$, i.e., $\{a \in A : a\mathrm{Erl}(A/{\cong_{\mathbf{A}}})X\} \neq \{a \in A : a\mathrm{Erl}(A/{\cong_{\mathbf{A}}})Y\}$. Finally, given a partition morphism $f : A/{\cong_{\mathbf{A}}}\to B/{\cong_{\mathbf{B}}}$, define the relation morphism $\mathrm{Erl}(f) : \mathrm{Erl}(B/{\cong_{\mathbf{B}}})\to\mathrm{Erl}(A/{\cong_{\mathbf{A}}})$, by $\mathrm{Erl}(f) = \langle \check{\mathrm{Erl}}(f), \hat{\mathrm{Erl}}(f)\rangle$, where $\check{\mathrm{Erl}}(f) : A\to B$ is given by $\check{\mathrm{Erl}}(f) = f$ and $\hat{\mathrm{Erl}}(f)$:

[3] With regards to Footnote 2, note that, if the function $\chi_R : A \times \Sigma\to\mathbf{2}$ is the characteristic function of an extensional relation R, then $\mathrm{cur}(\chi_R) : A\to\mathbf{2}^\Sigma$ may be neither an epi nor a mono. For instance, if $A = \{a, b\}$ and $\Sigma = \{0, 1\}$, with $\chi_R : (a, 0), (b, 0) \mapsto 0, (a, 1), (b, 1) \mapsto 1$, then R is extensional but $\mathrm{cur}(\chi_R)$ is neither an epi nor a mono.

$\mathcal{P}(B/\cong_{\mathbf{B}}) \to \mathcal{P}(A/\cong_{\mathbf{A}})$ by

$$\mathrm{Er}\hat{\mathrm{l}}(f)(Y) = \{a/\cong_{\mathbf{A}} : f(a)/\cong_{\mathbf{B}} \in Y\}, \quad \text{for all} \quad Y \subseteq B/\cong_{\mathbf{B}}.$$

This is well defined, since, on the one hand $f(\cong_{\mathbf{A}}) \subseteq \cong_{\mathbf{B}}$ and on the other, for all $a \in A$ and $Y \subseteq B/\cong_{\mathbf{B}}$,

$$\begin{aligned} f(a)\mathrm{Erl}(B/\cong_{\mathbf{B}})Y \;\; &\text{iff} \;\; f(a)/\cong_{\mathbf{B}} \in Y \\ &\text{iff} \;\; a/\cong_{\mathbf{A}} \in \mathrm{Er}\hat{\mathrm{l}}(f)(Y) \\ &\text{iff} \;\; a \;\mathrm{Erl}(A/\cong_{\mathbf{A}})\; \mathrm{Er}\hat{\mathrm{l}}(f)(Y). \end{aligned}$$

It is now shown that Erl : $\mathbf{PAR} \to \mathbf{REL}^{\mathrm{op}}$ is a left adjoint to Qnt : $\mathbf{REL} \to \mathbf{PAR}^{\mathrm{op}}$. To this end we exhibit an adjunction $\langle \mathrm{Erl}, \mathrm{Qnt}, \lambda, \chi \rangle$: $\mathbf{PAR} \to \mathbf{REL}^{\mathrm{op}}$. The unit λ of the adjunction is the natural transformation $\lambda : I_{\mathbf{PAR}} \to \mathrm{Qnt} \circ \mathrm{Erl}$, such that, for all partitions $A/\cong_{\mathbf{A}}$,

$$\lambda_{A/\cong_{\mathbf{A}}} : A/\cong_{\mathbf{A}} \to A/\cong_{\mathbf{A}}; \quad \lambda_{A/\cong_{\mathbf{A}}} = i_A.$$

The counit is the natural transformation $\chi : I_{\mathbf{ERL}} \to \mathrm{Erl} \circ \mathrm{Qnt}$, such that, for all extensional relations $R \subseteq A \times \Sigma$, $\chi_R = \langle \check{\chi}_R, \hat{\chi}_R \rangle$, where

$$\check{\chi}_R : A \to A; \quad \check{\chi}_R = i_A$$

and

$$\hat{\chi}_R : \Sigma \to \mathcal{P}(A/\cong_R); \quad \hat{\chi}_R(\sigma) = \{a/\cong_R : aR\sigma\}, \quad \text{for all } \sigma \in \Sigma.$$

This is well defined since, on the one hand $a_1 \cong_R a_2$ iff, for all $\sigma \in \Sigma$, $a_1 R\sigma$ iff $a_2 R\sigma$, and on the other, for all $a \in A, \sigma \in \Sigma$,

$$\begin{tikzcd} A \arrow[rr, phantom, "R"] & & \Sigma \arrow[d, "\hat{\chi}_R"] \\ A \arrow[u, "\check{\chi}_R"] \arrow[rr, phantom, "\mathrm{Erl}(\mathrm{Qnt}(R))"'] & & \mathcal{P}(A/\cong_R) \end{tikzcd}$$

$$\begin{aligned} \check{\chi}_R(a)R\sigma \;\; &\text{iff} \;\; aR\sigma \\ &\text{iff} \;\; a/\cong_R \in \hat{\chi}_R(\sigma) \\ &\text{iff} \;\; a \;\mathrm{Erl}(\mathrm{Qnt}(R))\; \hat{\chi}_R(\sigma). \end{aligned}$$

It is a natural transformation since, for all relation morphisms $f = \langle \check{f}, \hat{f} \rangle$: $R_1 \to R_2$ from $R_1 \subseteq A_1 \times \Sigma_1$, $R_2 \subseteq A_2 \times \Sigma_2$,

$$\begin{tikzcd} R_1 \arrow[r, "\chi_{R_1}"] \arrow[d, "f"'] & \mathrm{Erl}(\mathrm{Qnt}(R_1)) \arrow[d, "\mathrm{Erl}(\mathrm{Qnt}(f))"] \\ R_2 \arrow[r, "\chi_{R_2}"'] & \mathrm{Erl}(\mathrm{Qnt}(R_2)) \end{tikzcd}$$

$$\begin{aligned} \check{\chi_{R_1}}(\mathrm{Erl}(\check{\mathrm{Qnt}}(f))(a_2)) &= \mathrm{Erl}(\check{\mathrm{Qnt}}(f))(a_2) \\ &= \check{f}(a_2) \\ &= \check{f}(\check{\chi_{R_2}}(a_2)) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Erl}(\hat{\mathrm{Qnt}}(f))(\hat{\chi_{R_1}}(\sigma_1)) &= \mathrm{Erl}(\hat{\mathrm{Qnt}}(f))(\{a_1/\cong_{R_1} : a_1 R_1 \sigma_1\}) \\ &= \{a_2/\cong_{R_2} : \check{f}(a_2)/\cong_{R_1} \in \{a_1/\cong_{R_1} : a_1 R_1 \sigma_1\}\} \\ &= \{a_2/\cong_{R_2} : \check{f}(a_2) R_1 \sigma_1\} \\ &= \{a_2/\cong_{R_2} : a_2 R_2 \hat{f}(\sigma_1)\} \\ &= \hat{\chi_{R_2}}(\hat{f}(\sigma_1)). \end{aligned}$$

Finally, to see that the quadruple forms an adjunction, the following triangles must be shown to commute, for all partitions $A/\cong_{\mathbf{A}}$ and all extensional relations $R \subseteq A \times \Sigma$,

$$\begin{tikzcd} \mathrm{Qnt}(R) \arrow[r, "\lambda_{\mathrm{Qnt}(R)}"] \arrow[dr, "i_{\mathrm{Qnt}(R)}"'] & \mathrm{Qnt}(\mathrm{Erl}(\mathrm{Qnt}(R))) \arrow[d, "\mathrm{Qnt}(\chi_R)"] \\ & \mathrm{Qnt}(R) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Erl}(A/\cong_{\mathbf{A}}) \arrow[r, "\mathrm{Erl}(\lambda_{A/\cong_{\mathbf{A}}})"] \arrow[dr, "i_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}"'] & \mathrm{Erl}(\mathrm{Qnt}(\mathrm{Erl}(A/\cong_{\mathbf{A}}))) \arrow[d, "\chi_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}"] \\ & \mathrm{Erl}(A/\cong_{\mathbf{A}}) \end{tikzcd}$$

We have, for the first, for all $a \in A$,

$$\begin{aligned} \mathrm{Qnt}(\chi_R)(\lambda_{\mathrm{Qnt}(R)}(a)) &= \mathrm{Qnt}(\chi_R)(a) \\ &= a \\ &= i_{\mathrm{Qnt}(R)}(a) \end{aligned}$$

and, for the second, for all $a \in A$,

$$\begin{aligned}\check{\mathrm{Erl}}(\lambda_{A/\cong_{\mathbf{A}}})(\check{\chi}_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}(a)) &= \check{\mathrm{Erl}}(\lambda_{A/\cong_{\mathbf{A}}})(a)\\ &= \lambda_{A/\cong_{\mathbf{A}}}(a)\\ &= i_A(a)\\ &= \check{i}_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}\end{aligned}$$

and, for all $X \subseteq A/\cong_{\mathbf{A}}$,

$$\begin{aligned}\hat{\chi}_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}(\hat{\mathrm{Erl}}(\lambda_{A/\cong_{\mathbf{A}}})(X)) &= \hat{\chi}_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}(\{a/\cong_{\mathbf{A}}\colon \lambda_{A/\cong_{\mathbf{A}}}(a)/\cong_{\mathbf{A}} \in X\})\\ &= \{a/\cong_{\mathbf{A}} : a/\cong_{\mathbf{A}} \in \{a/\cong_{\mathbf{A}} : a/\cong_{\mathbf{A}} \in X\}\}\\ &= X\\ &= \hat{i}_{\mathrm{Erl}(A/\cong_{\mathbf{A}})}(X).\end{aligned}$$

Acknowledgements.

The author wishes to thank an anonymous referee whose insights and interesting remarks led to improvements in the presentation and to the few clarifying footnotes.

REFERENCES

[1] Barr, M., $*$-*Autonomous categories,* Lecture Notes in Mathematics, **752** (1979), Springer-Verlag.

[2] Barr, M., $*$-*Autonomous Categories and Linear Logic,* Mathematical Structures in Computer Science, **1** 2 (1991), 159-178.

[3] Barr, M., and Wells, C., *Category Theory for Computing Science,* Third Edition, Les Publications CRM, Montréal 1999.

[4] Barwise, J., and Seligman, J., *Information Flow The Logic of Distributed Systems,* Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 1997

[5] Borceux, F., *Handbook of Categorical Algebra,* Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, U.K., **50** (1994).

[6] Diaconescu, R., *Institution-Independent Ultraproducts*, Fundamenta Informaticae, **55** (2003), 321-348.

[7] Diaconescu, R., *An Institution-Independent Proof of Craig Interpolation Theorem*,Studia Logica, **77** (2004), 59-79.

[8] Gehrke, M., *Initial Report on Chu Spaces and Information Flows,* Technical Report, Physical Science Laboratory, New Mexico State University, April 2000

[9] Girard, J.-Y., *Linear Logic,* Theoretical Computer Science, **50** (1987), 1-102.

[10] Goguen, J.A., Burstall, R.M., *Introducing Institutions,* in Proceedings of the Logic of Programming Workshop, E. Clarke and D. Kozen, Eds., Lecture Notes in Computer Science, Springer-Verlag, New York, **164** (1984), 221-256.

[11] Goguen, J.A., Burstall, R.M., *Institutions: Abstract Model Theory for Specification and Programming,* Journal of the Association for Computing Machinery, **39** 1 (1992), 95-146.

[12] Goguen, J.A., and Diaconescu, R., *An Introduction to Category-Based Equational Logic*, AMAST, 1995, 91-126.

[13] Goguen, J.A., Roşu, G., *Institution Morphisms*, Formal Aspects of Computing, **13** 3-5 (2002), 274-307.

[14] Mac Lane, S., *Categories for the Working Mathematician,* Springer-Verlag, 1971.

[15] Pratt, V., *Chu Spaces,* Notes for the School on Category Theory and Applications, University of Coimbra, July 13-17, 1999.

Pervenuto il 23 gennaio 2002,
in forma modificata il 21 gennaio 2005.

School of Mathematics and Computer Science
Lake Superior State University