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Malinowski modalization, modalization through fibring and the Leibniz hierarchy*

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Abstract

We show how various modal systems considered by Malinowski as extensions of classical propositional calculus may be obtained as fibrings of classical propositional calculus and corresponding implicative modal logics, using the fibring framework for combining logics of Fernández and Coniglio. Taking advantage of this construction and known results of Malinowski, we draw some useful conclusions concerning some limitations of the fibring process. Finally, Malinowski's constructions are extended to obtain some modal extensions of arbitrary equivalential logics in the context of abstract algebraic logic. These are studied with respect to their algebraic character.

Keywords: Abstract algebraic logic, sentential logics, Leibniz hierarchy, modal logic, protoalgebraic logics, equivalential logics, modalization of logic, fibring, combination of logics.

1 Introduction

The research presented in this paper is related to three different developments in mathematical logic. First, it uses the framework of *fibring of logical systems*, as presented by Fernández and Coniglio in [11], as a means of combining logical systems. It is shown that the *modal systems* of Malinowski [18], which were originally defined as structural deductive systems with a given set of theorems, including all classical tautologies, and closed under modus ponens, can be obtained as fibrings of classical propositional logic with appropriately defined modal implicative logics. Theorems 4 and 5 of Section 4 are results asserting these relationships for the modal systems $\vec{\mathrm{E}}$ and $\vec{\mathrm{K}}$ of Malinowski (these are denoted by $\mathcal{S}_{\mathrm{E}}$ and $\mathcal{S}_{\mathrm{K}}$, respectively, in this article). The goal of these results, besides providing additional examples of the usefulness and broad applicability of the fibring process, is to study this process with respect to the Leibniz (algebraic) hierarchy of logics, as developed over the past several years by many researchers in the field of *abstract algebraic logic*, see, e.g. [3, 8, 12, 14]. More precisely, in [11], it has been shown that fibring is possible inside the categories of protoalgebraic [2] and equivalential [6, 7] deductive systems. Moreover, in [16] the authors consider combinations of algebraizable logics.

In Section 5, we extend the constructions of Malinowski to apply not only to extensions of classical propositional calculus, but, also, of any arbitrary equivalential logic $\mathcal{S}$ in the sense of abstract algebraic logic, that has the deduction detachment theorem with respect to an implication system forming part of its equivalence system. For instance, classical propositional calculus falls under this framework, since it has the deduction–detachment theorem with respect to $\{x \to y\}$ as well

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as being equivalential with respect to the equivalence system $\{x \to y, y \to x\}$. Such systems have been studied and characterized both in terms of their intrinsic properties and in terms of their algebraic character and their equivalent algebraic semantics in [9]. In the main result of Section 5 and one of the main results of the article, Theorem 11, one of the theorems of Malinowski (Theorem II.4 of [18]) asserting that $\vec{\mathrm{K}}$ is equivalential is extended to show that a similarly defined logic over an arbitrary equivalential deductive system with the deduction–detachment theorem (and not just over classical propositional calculus) is equivalential.

Finally, in the results that are presented in Section 6, we show that, even though the results of [11] seem to present a satisfactory state of affairs when fibring of protoalgebraic logics is considered, they have some drawbacks when it comes to fibring equivalential logics.

A more detailed review of the results of Fernández and Coniglio [11] that will be used in this article appears in Section 3. Malinowski's constructions [18] of some of his equivalential modal systems are described in some detail in Section 2. Finally, for all unexplained categorical notions and accompanying notation, the reader is encouraged to consult any of the standard references in general category theory [1, 4, 17].

2 The systems of Malinowski

A **sentential language** $\mathcal{L} = \langle \Lambda, \rho \rangle$ consists of a set Λ of at most countably many finitary connectives together with an arity function $\rho : \Lambda \to \omega$, assigning to every connective $\lambda \in \Lambda$ its arity $\rho(\lambda)$.

Let V be a countably infinite fixed set of **propositional variables**. The set $\mathrm{Fm}_{\mathcal{L}}(V)$ of **formulas of type** $\mathcal{L}$ (or $\mathcal{L}$-**formulas**) **over the set of variables** V is defined as the smallest set, such that

1. $V \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,
2. if $\lambda \in \Lambda$, with $\rho(\lambda) = n$, $\phi_1, \ldots, \phi_n \in \mathrm{Fm}_{\mathcal{L}}(V)$, then $\lambda(\phi_1, \ldots, \phi_n) \in \mathrm{Fm}_{\mathcal{L}}(V)$.

The structure of an algebra can be introduced on $\mathrm{Fm}_{\mathcal{L}}(V)$ by associating with each n-ary $\lambda \in \Lambda$ an n-ary operation $\lambda^{\mathbf{Fm}_{\mathcal{L}}(V)}$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ defined by $\lambda^{\mathbf{Fm}_{\mathcal{L}}(V)}(\phi_1, \ldots, \phi_n) = \lambda(\phi_1, \ldots, \phi_n)$. The resulting algebra $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V), \mathcal{L}^{\mathbf{Fm}_{\mathcal{L}}(V)} \rangle$ is in fact the absolutely free $\mathcal{L}$-algebra over the set V. An endomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ is called a **substitution**.

A **sentential logic** $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$ consists of a sentential language $\mathcal{L}$ and a structural consequence operation $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$ on the set of $\mathcal{L}$-formulas, i.e. such that, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

1. $\Gamma \subseteq C_{\mathcal{S}}(\Gamma)$;
2. $C_{\mathcal{S}}(\Gamma) \subseteq C_{\mathcal{S}}(\Delta)$, if $\Gamma \subseteq \Delta$;
3. $C_{\mathcal{S}}(C_{\mathcal{S}}(\Gamma)) = C_{\mathcal{S}}(\Gamma)$;
4. $h(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(h(\Gamma))$, for every substitution h.

Moreover, we say that $\mathcal{S}$ is **finitary** if

$$C_{\mathcal{S}}(\Gamma) = \bigcup_{\Delta \subseteq_\omega \Gamma} C_{\mathcal{S}}(\Delta),$$

where $\subseteq_\omega$ denotes the finite subset relation.

A formula ϕ is called a **theorem** of $\mathcal{S}$ if $\phi \in C_{\mathcal{S}}(\emptyset)$. The set of all theorems is denoted by $\mathrm{Thm}(\mathcal{S})$. A set T of formulas is called a **theory** of $\mathcal{S}$ if it is closed under the consequence operation, that is, if $C_{\mathcal{S}}(T) \subseteq T$. The set of all theories of $\mathcal{S}$ is denoted by $\mathrm{Th}(\mathcal{S})$. The set $\mathrm{Th}(\mathcal{S})$ forms a complete lattice $\mathbf{Th}(\mathcal{S}) = \langle \mathrm{Th}(\mathcal{S}), \cap, \vee^{\mathcal{S}} \rangle$ (which is algebraic whenever $\mathcal{S}$ is finitary), where the meet operation is the intersection and the join operation is defined in the following way: for any $T, T' \in \mathrm{Th}(\mathcal{S})$, $T \vee^{\mathcal{S}} T' = \bigcap\{R \in \mathrm{Th}(\mathcal{S}) : T \cup T' \subseteq R\}$. The largest theory is the set $\mathrm{Fm}_{\mathcal{L}}(V)$ and the smallest theory is the set $\mathrm{Thm}(\mathcal{S})$. It is not difficult to see that $T \vee^{\mathcal{S}} T' = C_{\mathcal{S}}(T \cup T')$. A theory T of $\mathcal{S}$ is **finitely axiomatized** if $T = C_{\mathcal{S}}(\Gamma)$ for some finite $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$.

An **inference rule** is a pair $\langle \Gamma, \phi \rangle$ $\big($also written as $\dfrac{\Gamma}{\phi}\big)$ where Γ is a finite set of formulas (the **premises** of the rule) and ϕ is a single formula (the **conclusion** of the rule). An **axiom** is an inference rule with $\Gamma = \emptyset$, i.e. a pair $\langle \emptyset, \phi \rangle$, usually just denoted by ϕ. The rules of this type are called **Hilbert-style** rules of inference. We say that a rule $\langle \Gamma, \phi \rangle$ holds in $\mathcal{S}$ if $\phi \in C_{\mathcal{S}}(\Gamma)$.

Let Ax be a set of axioms and IR a set of inference rules. We say that a formula ϕ is **directly derivable from** a set Γ of formulas **by** or **via** the inference rule $\langle \Delta, \psi \rangle$ if there is a substitution h such that $h(\psi) = \phi$ and $h(\Delta) \subseteq \Gamma$.

We say that ψ is **derivable** from Γ by the set Ax and the set IR, in symbols $\Gamma \vdash_{\mathrm{Ax,IR}} \psi$, if there is a **proof of** ψ **from** Γ **based on** Ax and IR, i.e. a finite sequence of formulas, $\psi_0, \ldots, \psi_{n-1}$ such that $\psi_{n-1} = \psi$, and for each $i < n$ one of the following conditions hold:

1. $\psi_i \in \Gamma$,
2. ψ_i is a substitution instance of a formula in Ax, or
3. ψ_i is directly derivable from $\{\psi_j : j < i\}$ by one of the inference rules in IR.

The pair $\langle \mathcal{L}, \vdash_{\mathrm{Ax,IR}} \rangle$ is called a **deductive system** (or simply a **logic**) with the set of axioms Ax and the set of inference rules IR.

A deductive system gives rise to a finitary sentential logic by defining the consequence operation by $\phi \in C_{\mathrm{Ax,IR}}(\Gamma)$ iff $\Gamma \vdash_{\mathrm{Ax,IR}} \phi$, for all $\Gamma \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$. This identification will be used in some examples.

In general, a pair $\langle \mathrm{Ax}, \mathrm{IR} \rangle$ of axioms and inference rules such that $C_{\mathcal{S}} = C_{\mathrm{Ax,IR}}$ is called a **presentation** or an **axiomatization** of $\mathcal{S}$. If both the set of axioms and the set of inference rules are finite then $\langle \mathrm{Ax}, \mathrm{IR} \rangle$ is called a **finite presentation**. Of course, a deductive system may have many axiomatizations. However, for any finitary sentential logic $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$, there is always an obvious axiomatization, namely

$$\begin{array}{rcl} \mathrm{Ax} & = & \{\phi : \phi \in C_{\mathcal{S}}(\emptyset)\} \\ \mathrm{IR} & = & \{\langle \Gamma, \phi \rangle : \phi \in C_{\mathcal{S}}(\Gamma) \text{ and } \Gamma \text{ finite}\}. \end{array}$$

Let $\wedge$ be a binary operation symbol, either primitive or derived, i.e. defined by an $\mathcal{L}$-term. We say that $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$ has the **Property of Conjunction** (PC) **with respect to** $\wedge$, if for every $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$C_{\mathcal{S}}(\phi \wedge \psi) = C_{\mathcal{S}}(\phi, \psi),$$

i.e. if $\wedge$ behaves like the ordinary classical conjunction. The following are well known characterizations of a logic having the conjunction property with respect to $\wedge$ (see, e.g. [12]).

1. The logic $\mathcal{S}$ has the (PC) iff, for every $\mathcal{S}$-theory T and all formulas $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\phi \wedge \psi \in T$ iff $\phi \in T$ and $\psi \in T$.
2. The logic $\mathcal{S}$ has the (PC) iff the following rules hold in $\mathcal{S}$:

$$\frac{\phi \wedge \psi}{\phi}, \quad \frac{\phi \wedge \psi}{\psi} \quad \text{and} \quad \frac{\phi, \psi}{\phi \wedge \psi}.$$

A logic $\mathcal{S}$ is called **protoalgebraic** if there exists a set $E(x,y)$ of formulas in $\mathrm{Fm}_{\mathcal{L}}(V)$ in two variables x,y, called an **implication system** or a **protoalgebraizator**, such that

1. $E(x,x)\subseteq C_{\mathcal{S}}(\emptyset)$; (Reflexivity)
2. $y\in C_{\mathcal{S}}(x,E(x,y))$. (Modus Ponens or Detachment)

The logic $\mathcal{S}$ is called (**finitely**) **equivalential** if there exists a (finite) set $E(x,y)$, as above, called an **equivalence system**, such that $E(x,y)$ is an implication system and, in addition,

3. $E(y,x)\subseteq C_{\mathcal{S}}(E(x,y))$; (Symmetry)
4. $E(x,z)\subseteq C_{\mathcal{S}}(E(x,y),E(y,z))$; (Transitivity)
5. For every $\lambda\in\Lambda$, with $\rho(\lambda)=n$, $E(\lambda(x_0,\dots,x_{n-1}),\lambda(y_0,\dots,y_{n-1}))\subseteq C_{\mathcal{S}}(E(x_0,y_0),\dots,E(x_{n-1},y_{n-1}))$. (Replacement)

Given two sentential logics $\mathcal{S}=\langle\mathcal{L},C_{\mathcal{S}}\rangle$ and $\mathcal{S}'=\langle\mathcal{L},C_{\mathcal{S}'}\rangle$ over the same language type $\mathcal{L}$, $\mathcal{S}'$ is called a **strengthening** of $\mathcal{S}$, in symbols $\mathcal{S}\leq\mathcal{S}'$, if, for all $\Phi\subseteq\mathrm{Fm}_{\mathcal{L}}(V)$, $C_{\mathcal{S}}(\Phi)\subseteq C_{\mathcal{S}'}(\Phi)$. Such a strengthening $\mathcal{S}'$ of $\mathcal{S}$ is said to be **axiomatic** if there exists a set $\Gamma\subseteq\mathrm{Fm}_{\mathcal{L}}(V)$, such that, for all $\Phi\subseteq\mathrm{Fm}_{\mathcal{L}}(V)$,

$$C_{\mathcal{S}'}(\Phi)=C_{\mathcal{S}}(\Gamma\cup\Phi).$$

Malinowski [18] states the following results concerning equivalential logics and their strengthenings:

Proposition 1 (Corollary I.12 of [6])
If $\mathcal{S}$ is an equivalential logic, then so is every strengthening $\mathcal{S}'$ of $\mathcal{S}$. Moreover, every equivalence system for $\mathcal{S}$ is also an equivalence system for $\mathcal{S}'$.

Proposition 2 ([19])
Let $\mathcal{S}=\langle\mathcal{L},C_{\mathcal{S}}\rangle$ be a logic and let $E_1(x,y),E_2(x,y)\subseteq\mathrm{Fm}_{\mathcal{L}}(V)$.

(i) If $C_{\mathcal{S}}(E_1(x,y))=C_{\mathcal{S}}(E_2(x,y))$, then $E_1(x,y)$ is an equivalence system for $\mathcal{S}$ iff so is $E_2(x,y)$.
(ii) If $E_1(x,y),E_2(x,y)$ are equivalence systems for $\mathcal{S}$, then $C_{\mathcal{S}}(E_1(x,y))=C_{\mathcal{S}}(E_2(x,y))$.

For the proof of the following proposition see Corollary I.7 of [18].

Proposition 3
Let $\mathcal{S}$ be a finitary equivalential logic and $E(x,y)$ an equivalence system for $\mathcal{S}$. If $\mathcal{S}'$ is a finitary finitely equivalential strengthening of $\mathcal{S}$, then there exists a finite $E'(x,y)\subseteq E(x,y)$, that is an equivalence system for $\mathcal{S}'$.

Next, we recall the definitions of the classical and normal modal systems of Malinowski, since they constitute the starting points of our own investigations relating modalized logics with the fibring process of Fernández and Coniglio [11] and, also, with some of the levels of the Leibniz hierarchy in abstract algebraic logic [8, 14] (more precisely, with protoalgebraic [2] and equivalential logics [6, 7]).

Let $\mathcal{L}$ be the language $\{\vee,\wedge,\neg\}$ (with $\vee$ and $\wedge$ binary and $\neg$ unary) and $\mathcal{L}_\Box$ the language $\{\Box,\vee,\wedge,\neg\}$ (with $\Box$ unary). Consider a set Φ of formulas over $\mathcal{L}$ and a set IR of rules of inference over $\mathcal{L}_\Box$ (which might include axioms as a special case of rules). We denote by $\mathrm{Sb}(\Phi,\mathrm{IR})$ the least substitution invariant set of formulas in the language $\mathcal{L}_\Box$, that includes Φ and is closed under the inference rules in IR. Now set $\phi\to\psi:=\neg\phi\vee\psi$ and $\phi\leftrightarrow\psi:=(\phi\to\psi)\wedge(\psi\to\phi)$, for all

$\phi, \psi \in \mathrm{Fm}_{\mathcal{L}_\Box}(V)$, and define the following:

$$
\begin{array}{llll}
\mathrm{CL} & = & \text{the least substitution invariant set in } \mathcal{L}_\Box \text{ that contains} & \\
 & & \text{all classical tautologies} & \\
(\mathrm{MP}) & & \dfrac{\phi, \phi \to \psi}{\psi} & (\text{Modus Ponens}) \\
(\mathrm{RE}) & & \dfrac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi} & (\text{Extensionality}) \\
(\mathrm{NR}) & & \dfrac{\phi}{\Box\phi} & (\text{Necessitation}) \\
(\mathrm{K}) & & \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) & (\text{Gödel's Axiom})
\end{array}
$$

Finally, let

$$
\begin{array}{lll}
\mathrm{E} & = & \mathrm{Sb}(\mathrm{CL}, (\mathrm{MP}), (\mathrm{RE})) \\
\mathrm{K} & = & \mathrm{Sb}(\mathrm{CL}, (\mathrm{K}), (\mathrm{MP}), (\mathrm{NR}))
\end{array}
$$

By a **modal system** L is understood any substitution invariant set of $\mathcal{L}_\Box$-formulas containing all classical tautologies and closed under (MP). A modal system L is called **classical** if $\mathrm{E} \subseteq \mathrm{L}$ and L is closed under (MP) and (RE). A modal system L is called **normal** if $\mathrm{K} \subseteq \mathrm{L}$ and L is closed under (MP) and (NR). In the literature, an equivalent definition of normal modal logic has also been used, in which Gödel's axiom (K) is replaced by the so called Kripke's axiom (K′) $\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$ (see [13, 15]).

If L is a modal system, $\mathcal{S}_\mathrm{L} = \langle \mathcal{L}_\Box, C_\mathrm{L} \rangle$ denotes the finitary sentential logic with the modal system L as its set of axioms and modus ponens (MP) as its only inference rule. This logic $\mathcal{S}_\mathrm{L}$ is denoted by $\vec{\mathrm{L}}$ in [18]. Another finitary sentential logic, stronger than $\vec{\mathrm{L}}$, which is very often considered in this setting is $\mathrm{L}^{\to,\Box}$, having the same axioms as $\vec{\mathrm{L}}$ but with two inference rules (MP) and (NR) (cf. [8]). A useful property of both of these sentential logics, when L is normal, is the following meta-rule:

$$\phi \in C(\Gamma) \quad \text{implies} \quad \Box\phi \in C(\Box\Gamma).$$

Malinowski proves in [18] that the logic $\mathcal{S}_\mathrm{E}$ (denoted by $\vec{\mathrm{E}}$) is not equivalential. He achieves this by showing that the logic

$$\mathrm{RE} = \mathrm{Sb}(\mathrm{E}, (\mathrm{MP}), (\mathrm{NR}), (\mathrm{RE})),$$

which is such that $\mathcal{S}_\mathrm{RE}$ is a strengthening of $\mathcal{S}_\mathrm{E}$, is not equivalential and, then, taking into account Proposition 1. This, in turn, he proves by showing that its class of reduced matrices is not closed under submatrices, a property known to hold for equivalential logics [6] (see, also, [8]).

Finally, in another interesting result, Malinowski shows that the logic $\mathcal{S}_\mathrm{K}$ ($\vec{\mathrm{K}}$ in [18]) is equivalential, with the infinite set $\{\Box^n(x \leftrightarrow y) : n \in \omega\}$ as its equivalence system ($\Box^n\alpha$ is an abbreviation for the formula $\Box\Box\cdots\Box\alpha$, in which the modal operator $\Box$ appears n times). Moreover, it is not finitely equivalential, i.e. it does not possess any finite equivalence system. This latter result is obtained, using Proposition 3, by first showing that the logic $\mathcal{S}_\mathrm{T}$, where

$$\mathrm{T} = \mathrm{Sb}(\mathrm{K}, (\mathrm{T}), (\mathrm{K}), (\mathrm{MP}), (\mathrm{NR})),$$

with

$$\text{(T)} \qquad \Box\phi \rightarrow \phi,$$

is not finitely equivalential, since, clearly, $\mathcal{S}_{\mathrm{T}}$ is a strengthening of $\mathcal{S}_{\mathrm{K}}$.

3 Fibring à la Fernández and Coniglio

Fernández and Coniglio define in [11] fibring of logics by revisiting previous ideas on categorical fibring (see, e.g. [20] and [5]). In this section we review this setting since, in the next section, it will be shown that the modal logics of Malinowski discussed in Section 2 can be obtained as special cases of fibred logics. Moreover, in Section 5, it will be shown that some more abstractly defined modal systems, which we call *generalized modal systems* and which are originally axiomatically defined, may also be obtained as combinations of underlying deductive systems and appropriate sentential modal systems via the fibring method.

Before proceeding further, it would be appropriate to emphasize here the fact that, even though the framework of Fernández and Coniglio, employed in [11], is adequate for our goals in the present work, it is but a very special instance of a much more general and powerful framework for the combination of logical systems via fibring perceived as an abstract categorial construction as outlined carefully in [20].

The reader should also be informed at this point that, in order to avoid an unpleasant shift of notation as compared with that of Section 2, we slightly modify the original notation of [11] in order to present the material in [18] and [11], in which slightly different notations were originally used, under a unified common notation.

The underlying category **Sig** of **signatures** (or **sentential languages**) over which fibring is supposed to take place has objects signatures, i.e. ω-indexed families of connectives Λ_k, $k\in\omega$, where the set Λ_k contains the k-ary connectives of $\mathcal{L}=\{\Lambda_k\}_{k\in\omega}$. Morphisms in **Sig** are mappings from one signature to another that preserve the arities of the connectives. A **logic** $\mathcal{S}=\langle\mathcal{L},\vdash\rangle$ over a signature $\mathcal{L}$ is a finitary and structural consequence relation $\vdash\subseteq\mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))\times\mathrm{Fm}_{\mathcal{L}}(V)$ on the set of formulas $\mathrm{Fm}_{\mathcal{L}}(V)$ that are formed by using variables from a fixed denumerable set V and the connectives from the signature $\mathcal{L}$. The category **Cons** of **logics** has as objects all logics and as morphisms $f:\langle\mathcal{L},\vdash\rangle\rightarrow\langle\mathcal{L}',\vdash'\rangle$ **Sig**-morphisms $f:\mathcal{L}\rightarrow\mathcal{L}'$, such that the induced function $\hat{f}:\mathrm{Fm}_{\mathcal{L}}(V)\rightarrow\mathrm{Fm}_{\mathcal{L}'}(V)$ is a **translation**, i.e. it satisfies

$$\Gamma\vdash\phi \quad \text{implies} \quad \hat{f}(\Gamma)\vdash'\hat{f}(\phi),\ \text{for all } \Gamma\cup\{\phi\}\subseteq\mathrm{Fm}_{\mathcal{L}}(V).$$

Since the category **Cons** is small cocomplete (see [11]), one may take the coproduct $\mathcal{S}\circledast\mathcal{S}'$ of two logics $\mathcal{S}$ and $\mathcal{S}'$ in **Cons**. The resulting logic is termed the **unconstrained fibring** of the constituent logics $\mathcal{S}$ and $\mathcal{S}'$. This construction results in a new logical system in which, intuitively speaking, no connectives of $\mathcal{S}$ and $\mathcal{S}'$ are intended to be identified. For our own purposes, however, and the studies in the subsequent section, more interesting is the *constrained fibring* of two logics in the category **Cons**. In this type of fibring, the intention is to have some of the connectives of the constituent logics identified in the fibred logic resulting from the process. We describe this type of fibring in some detail and also refer the reader to the original exposition in Section 3 of [11], as well as to the dissertation [10].

Let **C** and **D** be two categories and $F:\mathbf{C}\rightarrow\mathbf{D}$ a functor. An F-**costructured morphism with codomain** $d\in|\mathbf{D}|$, is a pair $\langle c,f\rangle$, where $c\in|\mathbf{C}|$ and $f:F(c)\rightarrow d$ is in **D**. A **cocartesian lifting of**

an F-costructured morphism $\langle c,f\rangle$ is a morphism $f^*:c\to c'$ in $\mathbf{C}$, such that $F(f^*)=f$ and such that the following universal property is satisfied:

For every $g:c\to c''$ in $\mathbf{C}$, and every $h:d\to F(c'')$ in $\mathbf{D}$, such that $h\circ f=F(g)$,

$$\begin{tikzcd} c \arrow[r, "f^*"] \arrow[dr, "g"'] & c' \arrow[d, "h^*"] \\ & c'' \end{tikzcd} \qquad \begin{tikzcd} F(c) \arrow[r, "f"] \arrow[dr, "F(g)"'] & d \arrow[d, "h"] \\ & F(c'') \end{tikzcd}$$

there exists a unique $h^*:c'\to c''$ in $\mathbf{C}$, such that $F(h^*)=h$ and $h^*\circ f^*=g$.

The functor F is called a **cofibration** if every F-costructured morphism admits a cocartesian lifting.

In [11], it is shown that the natural forgetful functor $N:\mathbf{Cons}\to\mathbf{Sig}$ (that forgets the consequence relation and keeps only the signature of the logic) is a cofibration and the **constrained fibring** of two logics $\mathcal{S}$ and $\mathcal{S}'$ constrained by the sharing D, where D is a diagram formed by two given monomorphisms $j:\bar{\mathcal{L}}\to N(\mathcal{S})$ and $j':\bar{\mathcal{L}}\to N(\mathcal{S}')$ in $\mathbf{Sig}$,

$$N(\mathcal{S}) \xleftarrow{\ j\ } \bar{\mathcal{L}} \xrightarrow{\ j'\ } N(\mathcal{S}')$$

is the codomain $\mathcal{S}\circledast_D\mathcal{S}'$ of the cocartesian lifting of the coequalizer $q:N(\mathcal{S}\circledast\mathcal{S}')\to\hat{\mathcal{L}}$ in $\mathbf{Sig}$ of the diagram

$$\begin{tikzcd} \bar{\mathcal{L}} \arrow[r, shift left, "N(i)\circ j"] \arrow[r, shift right, "N(i')\circ j'"'] & N(\mathcal{S}\circledast\mathcal{S}') \end{tikzcd}$$

where $i:\mathcal{S}\to\mathcal{S}\circledast\mathcal{S}'$ and $i':\mathcal{S}'\to\mathcal{S}\circledast\mathcal{S}'$ are the canonical injections of the coproduct in $\mathbf{Cons}$ of $\mathcal{S}$ and $\mathcal{S}'$.

A more intuitive description of the main idea of this process, also borrowed from [11], is as follows: At the start, two logics $\mathcal{S}=\langle\mathcal{L},\vdash\rangle$ and $\mathcal{S}'=\langle\mathcal{L}',\vdash'\rangle$ are given, together with a signature $\bar{\mathcal{L}}$ and two monomorphisms $j:\bar{\mathcal{L}}\to\mathcal{L}$ and $j':\bar{\mathcal{L}}\to\mathcal{L}'$, which are used to enforce the identification of some of the connectives in the two signatures $\mathcal{L}$ and $\mathcal{L}'$. Then, the coproduct $\mathcal{S}\circledast\mathcal{S}'$, with canonical injections $i:\mathcal{S}\to\mathcal{S}\circledast\mathcal{S}'$ and $i':\mathcal{S}'\to\mathcal{S}\circledast\mathcal{S}'$, and the coequalizer $q:\mathcal{L}\circledast\mathcal{L}'\to\hat{\mathcal{L}}$ is considered in $\mathbf{Sig}$. The object $\mathcal{L}\circledast\mathcal{L}'=N(\mathcal{S}\circledast\mathcal{S}')$ is in the image of the cofibration N, whence, there exists a cocartesian lifting $q^*:\mathcal{S}\circledast\mathcal{S}'\to\hat{\mathcal{S}}$ of the N-costructured morphism $\langle\mathcal{S}\circledast\mathcal{S}',q\rangle$. This logic $\hat{\mathcal{S}}$ is defined to be the fibring of the logics $\mathcal{S}$ and $\mathcal{S}'$ as constrained by the diagram

$$\mathcal{L} \xleftarrow{\ j\ } \bar{\mathcal{L}} \xrightarrow{\ j'\ } \mathcal{L}'$$

4 The systems of Malinowski as fibred logics

In this section, the framework described in Section 3 is used to show how the modal logics studied by Malinowski in [18] and discussed in Section 2 may be obtained as fibrations starting from classical propositional calculus and appropriately selected modal implicative logics.

Keeping with the notation used in Section 3, we consider the three signatures $\mathcal{L}_{\mathrm{CPC}}=\{\wedge,\vee,\to,\neg\}$, $\mathcal{L}_{\mathrm{E}'}=\{\to,\Box\}$ and $\mathcal{L}_{\to}=\{\to\}$ (where, of course, $\wedge,\vee,\to$ are binary and $\neg$ and $\Box$ are unary).

Over the signatures $\mathcal{L}_{\mathrm{CPC}}$ and $\mathcal{L}_{\mathrm{E}'}$ we consider, respectively, classical propositional calculus $\mathcal{S}_{\mathrm{CPC}} = \langle \mathcal{L}_{\mathrm{CPC}}, \vdash_{\mathrm{CPC}} \rangle$ and $\mathcal{S}_{\mathrm{E}'} = \langle \mathcal{L}_{\mathrm{E}'}, \vdash_{\mathrm{E}'} \rangle$, where $\mathrm{E}' = \mathrm{Sb}(\mathrm{CL}', (\mathrm{MP}), (\mathrm{RE})')$, where CL' is the least substitution invariant set over the signature $\mathcal{L}_{\mathrm{E}'}$ containing all classical $\{\rightarrow\}$-tautologies and closed under

$$(\mathrm{RE})' \qquad \frac{\phi \rightarrow \psi,\ \psi \rightarrow \phi}{\Box\phi \rightarrow \Box\psi}.$$

Finally, let $j : \mathcal{L}_{\rightarrow} \rightarrow \mathcal{L}_{\mathrm{CPC}}$ and $j' : \mathcal{L}_{\rightarrow} \rightarrow \mathcal{L}_{\mathrm{E}'}$ be the signature morphisms that inject $\rightarrow$ as a binary connective into the signatures $\mathcal{L}_{\mathrm{CPC}}$ and $\mathcal{L}_{\mathrm{E}'}$, that both contain $\rightarrow$ as a binary connective.

In the next proposition, it is shown that the logical system $\mathcal{S}_{\mathrm{E}}$ of Malinowski (i.e. $\vec{\mathrm{E}}$ in the notation of [18]) is the fibring of the two logics $\mathcal{S}_{\mathrm{CPC}}$ and $\mathcal{S}_{\mathrm{E}'}$ constrained by the identification of $\rightarrow$ in the two signatures $\mathcal{L}_{\mathrm{CPC}}$ and $\mathcal{L}_{\mathrm{E}'}$.

Proposition 4
The constrained fibring $\mathcal{S}_{\mathrm{E}}^* = \langle \mathcal{L}_{\mathrm{E}}^*, \vdash_{\mathrm{E}}^* \rangle$ of the two logics $\mathcal{S}_{\mathrm{CPC}} = \langle \mathcal{L}_{\mathrm{CPC}}, \vdash_{\mathrm{CPC}} \rangle$ and $\mathcal{S}_{\mathrm{E}'} = \langle \mathcal{L}_{\mathrm{E}'}, \vdash_{\mathrm{E}'} \rangle$ constrained by the sharing

$$\mathcal{L}_{\mathrm{CPC}} \xleftarrow{\ j\ } \mathcal{L}_{\rightarrow} \xrightarrow{\ j'\ } \mathcal{L}_{\mathrm{E}'}$$

coincides with the modal logic $\mathcal{S}_{\mathrm{E}} = \langle \mathcal{L}_{\mathrm{E}}, \vdash_{\mathrm{E}} \rangle$ of Malinowski.

Proof. The following is a coequalizer diagram in **Sig**,

$$\begin{tikzcd}
 & \{\wedge, \vee, \rightarrow, \neg\} \arrow[dr, "i"] & & \\
\{\rightarrow\} \arrow[ur, "j"] \arrow[dr, "j'"'] & & \{\wedge, \vee, \rightarrow, \neg, \rightarrow', \Box\} \arrow[r, "q"] & \{\wedge, \vee, \rightarrow, \neg, \Box\} \\
 & \{\rightarrow, \Box\} \arrow[ur, "i'"'] & &
\end{tikzcd}$$

where q maps all symbols to themselves and $\rightarrow' \overset{q}{\mapsto} \rightarrow$, which shows that $\mathcal{L}_{\mathrm{E}}^* = \mathcal{L}_{\mathrm{E}}$.

Let $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}_{\mathrm{E}}}(V)$. We must show that $\Phi \vdash_{\mathrm{E}} \phi$ iff $\Phi \vdash_{\mathrm{E}}^* \phi$.

1. Assume, first, that $\Phi \vdash_{\mathrm{E}} \phi$. Because of the definition of $\vdash_{\mathrm{E}}$, to see that $\Phi \vdash_{\mathrm{E}}^* \phi$, it suffices to show that $\mathrm{E} \subseteq \mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$, the set of theorems of $\mathcal{S}_{\mathrm{E}}^*$, and that $\vdash_{\mathrm{E}}^*$ is closed under (MP).
   Since $\vdash_{\mathrm{E}}^*$ is substitution invariant in the fibred signature, for the first demonstration, it suffices to show that $\mathrm{CL}, (\mathrm{MP})$ and (RE) are all rules of $\mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$.
   - That $\mathrm{CL} \subseteq \mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$ follows from $\mathrm{CL} \subseteq \mathrm{Thm}(\mathcal{S}_{\mathrm{CPC}})$, the fact that all morphisms involved are translations and that all logics involved are structural in their respective signatures.
   - To see that $\mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$ is closed under (MP), suppose that $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}_{\mathrm{E}}}(V)$, with $\phi \rightarrow \psi, \phi \in \mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$. Then, since $\mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$ is closed under all substitution instances of rules of $\mathcal{S}_{\mathrm{CPC}}$, in particular the corresponding modus ponens in CPC, we get that $\psi \in \mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$.
   - Finally, to show that $\mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$ is also closed under (RE), suppose that $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}_{\mathrm{E}}}(V)$, with $\phi \leftrightarrow \psi \in \mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$. Then $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$ are in $\mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$, whence, since $\mathrm{Thm}(\mathcal{S}_{\mathrm{E}}^*)$ is closed under all substitution instances of rules in E', in particular of the rule $(\mathrm{RE})'$, we get that

$\Box\phi \to \Box\psi$ and $\Box\psi \to \Box\phi$ are in $\mathrm{Thm}(\mathcal{S}^*_{\mathrm{E}})$. But $\mathrm{Thm}(\mathcal{S}^*_{\mathrm{E}})$ is also closed under all substitution instances of rules in $\mathcal{S}_{\mathrm{CPC}}$, whence we obtain that $\Box\phi \leftrightarrow \Box\psi \in \mathrm{Thm}(\mathcal{S}^*_{\mathrm{E}})$, which shows that $\mathrm{Thm}(\mathcal{S}^*_{\mathrm{E}})$ is closed under (RE).

Finally, $\vdash^*_{\mathrm{E}}$ is closed under (MP), since this closure property is inherited both by $\mathcal{S}_{\mathrm{CPC}}$ and by $\mathcal{S}_{\mathrm{E}'}$, given the structurality of all logics involved.

2. Assume, conversely, that $\Phi \vdash^*_{\mathrm{E}} \phi$. Because of the construction of $\vdash^*_{\mathrm{E}}$, to see that $\Phi \vdash_{\mathrm{E}} \phi$ we reason categorically as follows: First, by applying the coproduct property in **Cons**, we get the following morphism $[f,g]: \mathcal{S}_{\mathrm{CPC}} \circledast \mathcal{S}_{\mathrm{E}'} \to \mathcal{S}_{\mathrm{E}}$, where $f: \mathcal{S}_{\mathrm{CPC}} \to \mathcal{S}_{\mathrm{E}}$ and $g: \mathcal{S}_{\mathrm{E}'} \to \mathcal{S}_{\mathrm{E}}$ are the translations injecting $\mathcal{S}_{\mathrm{CPC}}$ and $\mathcal{S}_{\mathrm{E}'}$, respectively, to the stronger logic $\mathcal{S}_{\mathrm{E}}$ of Malinowski:

$$\begin{tikzcd} \mathcal{S}_{\mathrm{CPC}} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{S}_{\mathrm{CPC}} \circledast \mathcal{S}_{\mathrm{E}'} \arrow[d, "{[f,g]}"] & \mathcal{S}_{\mathrm{E}'} \arrow[l, "i'"'] \arrow[dl, "g"] \\ & \mathcal{S}_{\mathrm{E}} & \end{tikzcd}$$

Then, by applying the universal mapping property of the cocartesian lifting $q^*: \mathcal{S}_{\mathrm{CPC}} \circledast \mathcal{S}_{\mathrm{E}'} \to \mathcal{S}^*_{\mathrm{E}}$ of the coequalizer $q: \mathcal{L}_{\mathrm{CPC}} \circledast \mathcal{L}_{\mathrm{E}'} \to \mathcal{L}_{\mathrm{E}}$ to the pair of morphisms $[f,g]: \mathcal{S}_{\mathrm{CPC}} \circledast \mathcal{S}_{\mathrm{E}'} \to \mathcal{S}_{\mathrm{E}}$ and $i_{\mathcal{L}_{\mathrm{E}}}: \mathcal{L}_{\mathrm{E}} \to N(\mathcal{S}_{\mathrm{E}})$ in **Sig**, we get

$$\begin{tikzcd} \mathcal{S}_{\mathrm{CPC}} \circledast \mathcal{S}_{\mathrm{E}'} \arrow[r, "q^*"] \arrow[dr, "{[f,g]}"'] & \mathcal{S}^*_{\mathrm{E}} \arrow[d, "i^*_{\mathcal{L}_{\mathrm{E}}}"] \\ & \mathcal{S}_{\mathrm{E}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}_{\mathrm{CPC}} \circledast \mathcal{L}_{\mathrm{E}} \arrow[r, "q"] \arrow[dr, "{N([f,g])}"'] & \mathcal{L}_{\mathrm{E}} \arrow[d, "i_{\mathcal{L}_{\mathrm{E}}}"] \\ & \mathcal{L}_{\mathrm{E}} \end{tikzcd}$$

the morphism $i^*_{\mathcal{L}_{\mathrm{E}}}: \mathcal{S}^*_{\mathrm{E}} \to \mathcal{S}_{\mathrm{E}}$ in **Cons**, such that $N(i^*_{\mathcal{L}_{\mathrm{E}}}) = i_{\mathcal{L}_{\mathrm{E}}}$ and making the triangle on the left commute. The fact that this morphism is a translation in **Cons** yields the desired conclusion.

∎

It should be remarked here, for the benefit of the interested reader, that there is an alternative proof of Proposition 4 that does not rely directly on the categorical definition of a cofibration. It is based, instead, on a characterization, given in Fernández' thesis (Teorema 2.4.27 in Capítulo 2, page 54 of [10]), of the specific form that the codomain $\mathcal{S}' = \langle \mathcal{L}', \vdash' \rangle$ of the cocartesian lifting of an N-costructured morphism $(\mathcal{S}, f)$ of a logic $\mathcal{S} = \langle \mathcal{L}, \vdash \rangle$ assumes in the specific context of the categories **Sig**, **Cons** and the forgetful functor $N: \mathbf{Cons} \to \mathbf{Sig}$. According to this characterization, the codomain $\mathcal{S}'$ of the cocartesian lifting is characterized by the following properties:

- The signature $\mathcal{L}'$ is $f(\mathcal{L})$;
- The consequence relation $\vdash'$ is the smallest element in the lattice of consequence relations on $\mathrm{Fm}_{\mathcal{L}'}(V)$, such that $\hat{f}$ is a translation.

Therefore, the alternative proof, alluded to above, consists of showing that $\vdash_{\mathrm{E}}$ is the smallest consequence relation on the signature $\{\wedge, \vee, \to, \neg, \Box\}$ that makes $\hat{q}$ a translation.

We consider, next, the signatures $\mathcal{L}_{\mathrm{CPC}}$, $\mathcal{L}_{\mathrm{K}'} = \{\to, \Box\}$ and $\mathcal{L}_{\to} = \{\to\}$. Over the signatures $\mathcal{L}_{\mathrm{CPC}}$ and $\mathcal{L}_{\mathrm{K}'}$, we consider, respectively, classical propositional calculus $\mathcal{S}_{\mathrm{CPC}}$, as before, and $\mathcal{S}_{\mathrm{K}'} = \langle C_{\mathrm{K}'}, \vdash_{\mathrm{K}'}\rangle$, where $\mathrm{K}' = \mathrm{Sb}(\mathrm{CL}', (\mathrm{K})', (\mathrm{MP}), (\mathrm{NR}))$, where CL' is the least substitution invariant set on $\mathcal{L}_{\mathrm{K}'}$ containing all classical $\{\to\}$-tautologies, and

$$(\mathrm{K})' \qquad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi).$$

Finally, let $j : \mathcal{L}_{\to} \to \mathcal{L}_{\mathrm{CPC}}$ and $j' : \mathcal{L}_{\to} \to \mathcal{L}_{\mathrm{K}'}$ be the signature morphisms that inject $\to$ as a binary connective into the signatures $\mathcal{L}_{\mathrm{CPC}}$ and $\mathcal{L}_{\mathrm{K}'}$, that both contain $\to$ as a binary connective.

Proposition 5
The constrained fibring $\mathcal{S}^*_{\mathrm{K}} = \langle \mathcal{L}^*_{\mathrm{K}}, \vdash^*_{\mathrm{K}}\rangle$ of the two logics $\mathcal{S}_{\mathrm{CPC}} = \langle \mathcal{L}_{\mathrm{CPC}}, \vdash_{\mathrm{CPC}}\rangle$ and $\mathcal{S}_{\mathrm{K}'} = \langle \mathcal{L}_{\mathrm{K}'}, \vdash_{\mathrm{K}'}\rangle$ constrained by the sharing

$$\mathcal{L}_{\mathrm{CPC}} \xleftarrow{\;j\;} \mathcal{L}_{\to} \xrightarrow{\;j'\;} \mathcal{L}_{\mathrm{K}'}$$

coincides with the modal logic $\mathcal{S}_{\mathrm{K}} = \langle \mathcal{L}_{\mathrm{K}}, \vdash_{\mathrm{K}}\rangle$ of Malinowski.

Proof. This proof is very similar to the proof of Proposition 4. So the details will be omitted. ∎

5 Generalized modal systems

Let $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}}\rangle$ be a sentential logic with (PC) with respect to $\wedge$, i.e. such that $C_{\mathcal{S}}(\phi \wedge \psi) = C_{\mathcal{S}}(\phi, \psi)$, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Assume that $\mathcal{S}$ is protoalgebraic with a finite implication system $I(x,y)$ and, also, finitely equivalential, with finite equivalence system $E(x,y) = I(x,y) \cup I(y,x)$. Let $\mathcal{L}_{\Box} = \mathcal{L} \cup \{\Box\}$, where $\Box$ is a unary connective.

Given a set Φ of formulas in $\mathrm{Fm}_{\mathcal{L}}(V)$ and a family IR of rules of inference over $\mathcal{L}_{\Box}$ (which may include axioms as a special case), let $\mathrm{Sb}(\Phi, \mathrm{IR})$ denote the least substitution invariant set of formulas in the language $\mathcal{L}_{\Box}$, that includes Φ and is closed under the inference rules in IR. Define further

BS	$=$	the least substitution invariant set in $\mathcal{L}_{\Box}$ that contains all $\mathcal{S}$-theorems	
(I-MP)		$\dfrac{\phi, I(\phi,\psi)}{\psi}$	(I-Modus Ponens)
(E-RE)		$\dfrac{E(\phi,\psi)}{E(\Box\phi,\Box\psi)}$	(E-Extensionality)
(NR)		$\dfrac{\phi}{\Box\phi}$	(Necessitation)
(I-K)		$I(\Box I(\phi,\psi), I(\Box\phi,\Box\psi))$	

Note that (E-RE) stands for the finite set of rules $\dfrac{E(\phi,\psi)}{\epsilon(\Box\phi,\Box\psi)}$, for all $\epsilon(x,y) \in E(x,y)$, and, similarly, (I-K) represents a finite set of schemes, namely, $\epsilon(\Box \bigwedge I(\phi,\psi), \bigwedge I(\Box\phi,\Box\psi))$, for all $\epsilon(x,y) \in I(x,y)$.

Finally, let

$$\begin{aligned} \mathrm{E}^{\mathcal{S}} &= \mathrm{Sb}(\mathrm{BS}, (\mathrm{I\text{-}MP}), (\mathrm{E\text{-}RE})) \\ \mathrm{K}^{\mathcal{S}} &= \mathrm{Sb}(\mathrm{BS}, (\mathrm{I\text{-}K}), (\mathrm{I\text{-}MP}), (\mathrm{NR})) \end{aligned}$$

We note that for the study involving $\mathrm{K}^{\mathcal{S}}$ that is to follow, under the hypothesis that $\mathcal{S}$ has the Deduction–Detachment Theorem with respect to I (see Definition 9 for the precise meaning), axiom (I-K) above could be equivalently replaced by a generalized version of Kripke's axiom:

$$\text{(I-K}'\text{)} \qquad E(\Box(\phi\wedge\psi), \Box\phi\wedge\Box\psi),$$

where, as before, (I-K′) represents the finite set of schemes $\epsilon(\Box(\phi\wedge\psi),\ \Box\phi\wedge\Box\psi)$, for all $\epsilon(x,y)\in E(x,y)$.

A **generalized modal system based on** $\mathcal{S}$ is a substitution invariant set of $\mathcal{L}_\Box$-formulas containing all formulas in BS and closed under (I-MP). In case $\mathcal{S}$ is clear from context, explicit reference to $\mathcal{S}$ might be omitted. A generalized modal system L is called **basic** if $\mathrm{E}^{\mathcal{S}}\subseteq \mathrm{L}$ and L is closed under (I-MP) and (E-RE). A generalized modal system L is called **normal** if $\mathrm{K}^{\mathcal{S}}\subseteq \mathrm{L}$ and L is closed under (I-MP) and (NR). If L is a generalized modal system, $\mathcal{S}_{\mathrm{L}}=\langle\mathcal{L}_\Box, C_{\mathrm{L}}\rangle$ denotes the sentential logic having L as its set of axioms and (I-MP) as its unique rule of inference.

It is clear that every modal system in the sense of Malinowski [18] is a generalized modal system and that, moreover, if the base system $\mathcal{S}$ is taken to be the classical propositional calculus $\mathcal{S}_{\mathrm{CPC}}$, then basic and normal generalized modal systems correspond to classical and normal modal systems in the sense of [18], respectively. Moreover, since the intuitionistic propositional calculus does satisfy all the hypotheses imposed on our base system $\mathcal{S}$, modal intuitionistic systems, analogous to the classical systems of Malinowski, also fall under the purview of the framework studied here. Finally, relying once more on Malinowski's results, it is immediate that the logic $\mathcal{S}_{\mathrm{E}^{\mathcal{S}}}$ is not equivalential in general. In fact, by Corollary II.3 of [18], we get:

Corollary 6
The logic $\mathcal{S}_{\mathrm{E}^{\mathcal{S}}}$ is not equivalential in general.

We would also like to mention an alternative generalization of the framework of Malinowksi that is actually a further abstraction of the system $\mathcal{S}_{\mathrm{L}}$ presented above. Assuming that $\mathcal{S}$ is finitely axiomatizable and starting from a generalized modal system L (based on $\mathcal{S}$), we define $\mathcal{S}^s_{\mathrm{L}}=\langle\mathcal{L}_\Box, C^s_{\mathrm{L}}\rangle$ to be the sentential logic having L as its set of axioms and as its rules of inference all rules of inference of $\mathcal{S}$ together with (I-MP). First, note that, since classical propositional calculus may be formalized with modus ponens as its only rule of inference (the same actually holds for all logics considered in Proposition 10 and Theorem 11, by results of [9]; see discussion before Proposition 10), both $\mathcal{S}_{\mathrm{L}}$ and $\mathcal{S}^s_{\mathrm{L}}$ specialize in these particular cases to the logics considered by Malinowski. Therefore, it is not necessary for those results related to Malinowski's modalization of classical propositional calculus to consider the additional abstraction encompassed by $\mathcal{S}^s_{\mathrm{L}}$. On the other hand, it seems to be the case that, even though $\mathcal{S}_{\mathrm{L}}$ itself cannot be obtained as a fibred logic along the lines of Proposition 4, an analogue of Proposition 4 does indeed hold for $\mathcal{S}^s_{\mathrm{L}}$. In fact in Proposition 7, we sketch how our generalized modal logics $\mathcal{S}^s_{\mathrm{L}}$ may be obtained as fibrations, starting from the underlying logical system $\mathcal{S}=\langle\mathcal{L}, C_{\mathcal{S}}\rangle$, having the (PC) with respect to $\wedge$, and from some appropriately selected modal generalized implicative logics. In addition to satisfying the (PC) with respect to $\wedge$, we assume that $\mathcal{S}$ is protoalgebraic with a finite implication system $I(x,y)$ and finitely equivalential, with the finite equivalence system $E(x,y)=I(x,y)\cup I(y,x)$. We assume that $\mathcal{L}_I$ is the least subsignature of $\mathcal{L}$, that includes all the primitive connectives adequate to express the formulas in the implication system I. Under this notation, the term 'modal generalized implicative logics' in this context will refer to a logic over the signature $\mathcal{L}_I\cup\{\Box\}$.

Turning to a more detailed account, consider the three signatures $\mathcal{L}$, $\mathcal{L}_I$ and $\mathcal{L}_{\mathrm{E}'}=\mathcal{L}_I\cup\{\Box\}$. Over the signatures $\mathcal{L}$ and $\mathcal{L}_{\mathrm{E}'}$ we consider, respectively, the sentential logic $\mathcal{S}=\langle\mathcal{L},\vdash\rangle$ and $\mathcal{S}_{\mathrm{E}'}=\langle\mathcal{L}_{\mathrm{E}'},\vdash_{\mathrm{E}'}\rangle$, where $\mathrm{E}' = \mathrm{Sb}(\mathrm{Thm}_I(\mathcal{S}), (\mathrm{MP}), (\mathrm{RE})')$, where $\mathrm{Thm}_I(\mathcal{S})$ is the least substitution invariant set over the signature $\mathcal{L}_{\mathrm{E}'}$ containing all $\mathcal{S}$-theorems over $\mathcal{L}_I$, and closed under

$$(\mathrm{RE})' \qquad \frac{I(\phi,\psi),\ I(\psi,\phi)}{\epsilon(\Box\phi,\Box\psi)}, \quad \text{for all } \epsilon \in I.$$

Finally, let $j : \mathcal{L}_I \to \mathcal{L}$ and $j' : \mathcal{L}_I \to \mathcal{L}_{\mathrm{E}'}$ be the signature morphisms that inject all the connectives of $\mathcal{L}_I$ into the signatures $\mathcal{L}$ and $\mathcal{L}_{\mathrm{E}'}$, respectively, that both contain those connectives.

In the next proposition, it is shown that the generalized modal logic $\mathcal{S}^s_{\mathrm{E}\mathcal{S}}$ defined above is the fibring of the two logics $\mathcal{S}$ and $\mathcal{S}_{\mathrm{E}'}$ constrained by the identification of all the primitive connectives of $\mathcal{L}$ in $\mathcal{L}_I$ in the two signatures $\mathcal{L}$ and $\mathcal{L}_{\mathrm{E}'}$. Proposition 7 is an extension of Proposition 4 for modal logics based on an arbitrary sentential logic $\mathcal{S}$ (satisfying the appropriate hypotheses) rather than just on $\mathcal{S}_{\mathrm{CPC}}$. We also mention, without formulating it in detail, that an analogue of Proposition 5 along the lines of Proposition 7 also holds.

Proposition 7
The constrained fibring $\mathcal{S}^*_{\mathrm{E}\mathcal{S}} = \langle \mathcal{L}^*_\Box, \vdash^*_{\mathrm{E}\mathcal{S}} \rangle$ of the two logics $\mathcal{S} = \langle \mathcal{L}, \vdash \rangle$ and $\mathcal{S}_{\mathrm{E}'} = \langle \mathcal{L}_{\mathrm{E}'}, \vdash_{\mathrm{E}'} \rangle$ constrained by the sharing

$$\mathcal{L} \xleftarrow{\ j\ } \mathcal{L}_I \xrightarrow{\ j'\ } \mathcal{L}_{\mathrm{E}'}$$

coincides with the generalized modal system $\mathcal{S}^s_{\mathrm{E}\mathcal{S}} = \langle \mathcal{L}_\Box, \vdash^s_{\mathrm{E}\mathcal{S}} \rangle$.

Proof. The following is a coequalizer diagram in **Sig**,

$$\begin{tikzcd}
& \mathcal{L} \arrow[dr, "i"] & & \\
\mathcal{L}_I \arrow[ur, "j"] \arrow[dr, "j'"'] & & \mathcal{L} \uplus \mathcal{L}_I \uplus \{\Box\} \arrow[r, "q"] & \mathcal{L}_\Box \\
& \mathcal{L}_{\mathrm{E}} \arrow[ur, "i'"'] & &
\end{tikzcd}$$

where q maps all symbols to themselves and injects every distinct copy of a connective in $\mathcal{L}_I$ into its original progenitor in $\mathcal{L}$. It is, therefore, easy to see that $\mathcal{L}^*_\Box = \mathcal{L}_\Box$. As mentioned in the remark following the proof of Proposition 4, it now suffices to show that the logic $\mathcal{S}^s_{\mathrm{E}\mathcal{S}}$ is the smallest $\mathcal{L}_\Box$-logic that makes $\hat{q}$ a translation, i.e. that satisfies

$$\Gamma \vdash_{\mathcal{S} \circledast \mathcal{S}_{\mathrm{E}'}} \phi \quad \text{implies} \quad \hat{q}(\Gamma) \vdash_{\mathcal{S}^s_{\mathrm{E}\mathcal{S}}} \hat{q}(\phi),$$

for all $\Gamma \cup \phi \in \mathrm{Fm}_{\mathcal{L} \uplus \mathcal{L}_I \uplus \{\Box\}}(V)$. We omit the details, but invite the reader to notice that, for $\hat{q}$ to be a translation the postulated closure of the logic $\mathcal{S}^s_{\mathrm{L}}$, not just under (I-MP), but, rather, under all rules of inference of $\mathcal{S}$ is necessary. That is, it would not be sufficient to consider $\mathcal{S}_{\mathrm{L}}$ for this role. ∎

In the next result, we show that a normal generalized modal system satisfies a form of a general necessitation meta-rule that is also satisfied by any classical normal modal logic.

Proposition 8

Let $\mathcal{S}=\langle\mathcal{L},C_{\mathcal{S}}\rangle$ be a sentential logic with (PC) with respect to $\wedge$. Suppose that $\mathcal{S}$ has a finite implication system $I(x,y)$ and an equivalence system $E(x,y)=I(x,y)\cup I(y,x)$ and that $\mathcal{S}_{\mathrm{L}}=\langle\mathcal{L}_{\Box},C_{\mathrm{L}}\rangle$ is a normal generalized modal system over $\mathcal{S}$. Then, for all $\Gamma\cup\{\phi\}\subseteq\mathrm{Fm}_{\mathcal{L}}(V)$,

$$\phi\in C_{\mathrm{L}}(\Gamma) \quad \text{implies} \quad \Box\phi\in C_{\mathrm{L}}(\Box\Gamma).$$

Proof. By the definition of $\mathcal{S}_{\mathrm{L}}$, if $\phi\in C_{\mathrm{L}}(\Gamma)$, there exists a proof $\phi_0,\ldots,\phi_n=\phi$ of ϕ from hypotheses Γ, such that, for all $i\leq n$, ϕ_i is in Γ, ϕ_i is in L or ϕ_i follows from $\{\phi_0,\ldots,\phi_{i-1}\}$ by an application of (I-MP). We show by induction, that $\Box\phi_0,\ldots,\Box\phi_n=\Box\phi$ is a valid proof of $\Box\phi$ from hypotheses $\Box\Gamma$.

For the base case, if $\phi_0\in\Gamma$, then $\Box\phi_0\in\Box\Gamma$ and, if $\phi_0\in\mathrm{L}$, $\Box\phi_0\in\mathrm{L}$, since L is closed under (NR).

Assume, as the induction hypothesis, that $\Box\phi_0,\ldots,\Box\phi_{i-1}$ is a valid proof from hypotheses $\Box\Gamma$. If $\phi_i\in\Gamma$, then $\Box\phi_i\in\Box\Gamma$. If $\phi_i\in\mathrm{L}$, then, since L is closed under (NR), we get that $\Box\phi_i\in\mathrm{L}$. Finally, if ϕ_i follows from $\{\phi_0,\ldots,\phi_{i-1}\}$ by (I-MP), we have that $I(\phi,\phi_i)\cup\{\phi\}\subseteq\{\phi_0,\ldots,\phi_{i-1}\}$, for some $\phi\in\mathrm{Fm}_{\mathcal{L}_{\Box}}(V)$. Thus, taking into account that $I(\Box I(\phi,\phi_i),I(\Box\phi,\,\Box\phi_i))\subseteq\mathrm{K}^{\mathcal{S}}\subseteq\mathrm{L}$, we get that

$$I(\Box I(\phi,\phi_i),I(\Box\phi,\Box\phi_i)), \quad \Box I(\phi,\phi_i), \quad \Box\phi$$

have already been proven, whence, applying (I-MP) twice, $\Box\phi_i$ is also provable from hypotheses $\Box\Gamma$. ∎

In our final result, we show that, under some mild assumptions that are satisfied, e.g. by $\mathcal{S}_{\mathrm{CPC}}$, the techniques of [18] can be employed to show that, for every basic finitely equivalential system $\mathcal{S}$, with a finite equivalence $E(x,y)$, over a language $\mathcal{L}$ with conjunction $\wedge$, the logic $\mathcal{S}_{\mathrm{K}^{\mathcal{S}}}$ is equivalential with equivalence system $\{\Box^n E(x,y): n\in\omega\}$. Here $\Box^n E(x,y)$ stands for the formula $\Box^n\bigwedge E(x,y):=\Box^n\bigwedge_{\epsilon\in E(x,y)}\epsilon(x,y)$. Of course, $\mathcal{S}_{\mathrm{K}^{\mathcal{S}}}$ is not finitely equivalential in general, since the special case considered in [18] does not have this property.

Definition 9

A logic $\mathcal{S}=\langle\mathcal{L},\vdash_{\mathcal{S}}\rangle$ is said to **have the deduction–detachment theorem** (DDT) with respect to a set $I(x,y)$ of $\mathcal{L}$-formulas in two variables x,y if, for all $\Phi\cup\{\phi,\psi\}\subseteq\mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Phi,\phi\vdash_{\mathcal{S}}\psi \quad \text{iff} \quad \Phi\vdash_{\mathcal{S}} I(\phi,\psi). \tag{1}$$

The left-to-right implication of this equivalence forms the **deduction theorem** and the converse is the **detachment property** or **modus ponens** with respect to $I(x,y)$.

In establishing Proposition 10 and Theorem 11, we will be working with a sentential logic $\mathcal{S}$ that will be assumed to have (PC) with respect to a binary $\wedge$, to possess the DDT with respect to a finite implication system $I(x,y)$ and to be finitely equivalential with equivalence system $E(x,y)=I(x,y)\cup I(y,x)$. By Lemma 62 of [9], such logics are Fregean and it follows, further, by Theorem 61 of [9], that they are also regularly algebraizable. Since the availability of the connective $\wedge$ with respect to which $\mathcal{S}$ has the (PC) and the finiteness of I imply that $\mathcal{S}$ has the DDT with respect to a single formula, Theorem 63 of [9] completely determines the logic $\mathcal{S}$ as an axiomatic extension of a given logical system having three axioms and modus ponens as its only rule of inference. Finally, Theorem 66 of [9] asserts that $\mathcal{S}$ is strongly regularly algebraizable, i.e. belongs to the highest level in the abstract algebraic (Leibniz) hierarchy of sentential logics and Theorem 68 of [9] characterizes its equivalent algebraic semantics, i.e. the class of algebras serving, in the sense of abstract algebraic logic, as its equivalent class of algebras, which, by strong algebraizability, is always a variety and not just a quasi-variety (the default case). For more details on these important connections with the field of *abstract algebraic logic* we refer the reader to [9], since a further exposition would lead to a significant detour.

Proposition 10
Let $\mathcal{S}=\langle\mathcal{L},\vdash_{\mathcal{S}}\rangle$ be a logic over a language $\mathcal{L}$ with a binary conjunction $\wedge$. Assume that $\mathcal{S}$ has the deduction-detachment theorem with respect to a finite implication system $I(x,y)$ and that it is finitely equivalential with equivalence system $E(x,y)=I(x,y)\cup I(y,x)$. Then for all $\Gamma\cup\{\phi\}\subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, such that $|\Gamma|<\omega$, we have

$$\Gamma\vdash_{\mathcal{S}}\phi \quad \text{implies} \quad \Gamma\vdash_{\mathcal{S}_{K\mathcal{S}}}\phi.$$

Proof. Indeed, we have

$$\begin{array}{rll} \Gamma\vdash_{\mathcal{S}}\phi & \text{iff} & \vdash_{\mathcal{S}} I(\Gamma,\phi) \quad (\mathcal{S} \text{ has the (DDT)}) \\ & \text{implies} & \vdash_{\mathcal{S}_{K\mathcal{S}}} I(\Gamma,\phi) \quad (\mathrm{Thm}(\mathcal{S})\subseteq\mathrm{Thm}(\mathcal{S}_{K\mathcal{S}})) \\ & \text{implies} & \Gamma\vdash_{\mathcal{S}_{K\mathcal{S}}}\phi \quad \text{(by (I-MP))} \end{array}$$

Here, we are implicitly assuming an order in Γ, say $\Gamma:=\{\psi_0,\dots,\psi_{n-1}\}$, and $I(\Gamma,\phi)$ stands for $I(\psi_0,\dots,I(\psi_{n-1},\phi)\dots)$.

∎

Theorem 11
Let $\mathcal{S}=\langle\mathcal{L},\vdash_{\mathcal{S}}\rangle$ be a logic over a language $\mathcal{L}$ with a binary conjunction $\wedge$. Assume that $\mathcal{S}$ has the deduction-detachment theorem with respect to a finite implication system $I(x,y)$ and that it is finitely equivalential with equivalence system $E(x,y)=I(x,y)\cup I(y,x)$. Then, the logic $\mathcal{S}_{\mathrm{K}\mathcal{S}}$ is equivalential with infinite equivalence system $\{\Box^n E(x,y):n\in\omega\}$.

Proof. Besides the Equivalence (1), since, by hypothesis, $E(x,y)$ is an equivalence system for $\mathcal{S}$, the following five properties also hold for $\mathcal{S}$:

1. $\vdash_{\mathcal{S}} E(x,x)$;
2. $E(x,y)\vdash_{\mathcal{S}} E(y,x)$;
3. $E(x,y),E(y,z)\vdash_{\mathcal{S}} E(x,z)$;
4. $E(x_0,y_0),\dots,E(x_{n-1},y_{n-1})\vdash_{\mathcal{S}} E(\lambda(\vec{x}),\lambda(\vec{y}))$, for every n-ary $\lambda\in\Lambda$;
5. $x,E(x,y)\vdash_{\mathcal{S}} y$.

Consequently, by Proposition 10, we have

1. $\vdash_{\mathcal{S}_{\mathrm{K}\mathcal{S}}} E(x,x)$;
2. $E(x,y)\vdash_{\mathcal{S}_{\mathrm{K}\mathcal{S}}} E(y,x)$;
3. $E(x,y),E(y,z)\vdash_{\mathcal{S}_{\mathrm{K}\mathcal{S}}} E(x,z)$;
4. $E(x_0,y_0),\dots,E(x_{n-1},y_{n-1})\vdash_{\mathcal{S}_{\mathrm{K}\mathcal{S}}} E(\lambda(\vec{x}),\lambda(\vec{y}))$, for every n-ary $\lambda\in\Lambda$;

By applying Proposition 8 we have that these four conditions also hold for the sentential logic $\mathcal{S}_{\mathrm{K}\mathcal{S}}$ with the system $\{\Box^n E(x,y):n\in\omega\}$ in place of $E(x,y)$. Note that, by clause 4 above, the replacement condition holds for $\lambda\in\Lambda$. To show that it also holds for $\Box$, it suffices to prove that, for all $n\in\omega$,

$$\{\Box^k E(x,y):k\in\omega\}\vdash_{\mathcal{S}_{\mathrm{K}\mathcal{S}}}\Box^n E(\Box x,\Box y).$$

First, by applying (I-K′) and (I-MP), we get $\Box E(x,y) \vdash_{\mathcal{S}_{\mathrm{K}}\mathcal{S}} E(\Box x, \Box y)$. Then, by applying repeatedly Proposition 8, we get, for all $n \in \omega$,

$$\{\Box^k E(x,y) : k \in \omega\} \vdash_{\mathcal{S}_{\mathrm{K}}\mathcal{S}} \Box^n E(\Box x, \Box y).$$

Finally, since (I-MP) is a rule of $\mathcal{S}_{\mathrm{K}}\mathcal{S}$, we have

$$x, \{\Box^n E(x,y) : n \in \omega\} \vdash_{\mathcal{S}_{\mathrm{K}}\mathcal{S}} y,$$

whence modus ponens also holds in $\mathcal{S}_{\mathrm{K}}\mathcal{S}$ with respect to the system $\{\Box^n E(x, y) : n \in \omega\}$ and this concludes the proof that $\{\Box^n E(x,y) : n \in \omega\}$ is an equivalence system. ∎

6 On protoalgebraicity and equivalentiality

In sections 4 and 5 of [11], the constructions of unconstrained and constrained fibring are examined with regards to the algebraic character of the fibred logic resulting from fibring two protoalgebraic or two equivalential logics, respectively, in the categories **Prot** and **Equiv**. (The interested reader may also want to consult [16] for closely related work.) The category **Prot** of protoalgebraic logics is defined as the subcategory of **Cons** with objects all protoalgebraic logics and morphisms all morphisms in **Cons** that send a protoalgebraizator of the source logic onto a protoalgebraizator of the target logic. It turns out that for all protoalgebraic logics $\mathcal{S}$ and $\mathcal{S}'$, every translation $f : \mathcal{S} \to \mathcal{S}'$ in **Cons** must send a protoalgebraizator of $\mathcal{S}$ onto a protoalgebraizator of $\mathcal{S}'$ and, hence, it must be a morphism in **Prot**. Thus, **Prot** is the full subcategory of **Cons** with objects all protoalgebraic logics (Proposition 4.3 of [11]). Theorem 4.4 of [11] asserts that the category **Prot** has both unconstrained and constrained fibring and, moreover, its proof shows that this fibred logic is exactly the same logic that results by the straightforward fibring process in **Cons**.

Based on this result and the fact that both $\mathcal{S}_{\mathrm{CPC}}$ and $\mathcal{S}_{\mathrm{E}'}$ are protoalgebraic with protoalgebraizator $\{x \to y\}$, we obtain, based on our Propositions 4 and 5, an alternative proof of the relatively obvious result that both $\mathcal{S}_{\mathrm{E}}$ and $\mathcal{S}_{\mathrm{K}}$ are protoalgebraic logics with protoalgebraizator $\{x \to y\}$:

Corollary 12
The modal logics $\mathcal{S}_{\mathrm{E}}$ and $\mathcal{S}_{\mathrm{K}}$ of Malinowski are protoalgebraic with $\{x \to y\}$ as an implication system.

According to Theorem 5.14 of [11], the category **Equiv** of equivalential logics, which is a subcategory (but not a full one) of **Cons** also has both unconstrained and constrained fibring. But Proposition 5.8 of [11] and its proof show that, in general, the coproduct of two logics in **Cons** is different than their coproduct in **Equiv**. Thus, it is reasonable to expect that when two logics are fibred, their fibring (unconstrained or constrained) will be different depending on which category this fibring is taking place in. In fact, one of the vulnerable points of the theory developed by Fernández and Coniglio in [11] is that the fact that the (unconstrained and constrained) fibrings of pairs of equivalential logics in their context is also equivalential does not follow by any appropriately selected properties of the constituent logics or of the fibring process in **Cons**, but is rather forced in **Equiv** from the fact that in this category all logics are equivalential and all morphisms are equivalence-preserving morphisms. In this context, once fibring is shown to exist, the fibres cannot but be equivalential ‘by ambience’.

Another vulnerability of the setting of [11], also relating to the case of fibring of equivalential logics, is revealed by the following result

Corollary 13
Even though the logic $\mathcal{S}_{\mathrm{K}}$ is equivalential and it is the constrained fibring in **Cons** of the equivalential logics $\mathcal{S}_{\mathrm{CPC}}$ and $\mathcal{S}_{\mathrm{K}'}$, it is not the fibring of $\mathcal{S}_{\mathrm{CPC}}$ and $\mathcal{S}_{\mathrm{K}'}$ in **Equiv**.

In fact, Malinowski shows that $\mathcal{S}_{\mathrm{K}}$ is equivalential, with an equivalence system $\{\Box^n(x\leftrightarrow y)\colon n\in\omega\}$, consisting of an infinite number of equivalence formulas, and, in addition, it is not finitely equivalential, i.e. it does not have any finite equivalence system. Thus, there cannot be any morphism in **Equiv** mapping the equivalence system of $\mathcal{S}_{\mathrm{CPC}}$ onto the infinite equivalence system of $\mathcal{S}_{\mathrm{E}}$.

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