A CATEGORICAL CONSTRUCTION OF A VARIETY OF CLONE ALGEBRAS*

GEORGE VOUTSADAKIS†

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ABSTRACT.

A category **Cln** representing free algebras of clones of operations of finite but arbitrary arities is constructed together with an adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{Set} \to \mathbf{Cln}$. This gives rise to an algebraic theory $\mathbf{T}$ over $\mathbf{Set}$. A single-sorted variety $\mathcal{V}$ of clone algebras is, then, equationally defined inspired by the multi-sorted construction of Taylor [20]. It is shown that the Eilenberg-Moore category of $\mathbf{T}$-algebras is isomorphic to the category $\vec{\mathcal{V}}$ corresponding to the variety $\mathcal{V}$.

1 Introduction In algebraic logic one studies the classes of algebras that form the so-called *algebraic semantics* of deductive systems ([2, 3]). Along these lines several attempts have been made to define algebras that would be appropriate for algebraizing equational logic. Some of these attempts were focusing on ordinary, single-sorted, algebras, whereas others were using many-sorted algebras. The general theory of this latter type of algebras has been developed independently in [14, 15],[9] and [1]. Some of these attempts are P. Hall's notion of clone (see [6]), which gives a partial single-sorted algebra, B.H. Neumann and E.C. Wiegold's representation of varieties in terms of semigroups [18], W.D. Neumann's substitution algebras [17], having infinitary substitution operations, W. Lawvere's algebraic theories [10, 11] (see also [12, 19]), W. Taylor's heterogeneous variety of substitution algebras [20] and, finally, N. Feldman's polynomial substitution algebras [8] (see also [5]). In a similar direction Czelakowski and Pigozzi [7] view equational logic as a 2-deductive system in the sense of [3] and propose its algebraization via another 2-deductive system, based on [8], which they call *hyperequational logic.*

The common feature underlying all these algebraizations is the a priori choice of the basic operations of the class of algebras that is chosen as the algebraizing class. For instance, in [18] the identity, repetition, deletion and transposition operators are taken as constants and composition of operators as an associative binary operation, in [17] projections and infinitary substitutions are the basic operations, in [20] n-ary projections and m-ary to n-ary substitutions, for all $m, n \geq 1$, are chosen as basic, whereas in [8] projections and one-place substitutions are basic.

In [21, 23], a general framework for the algebraization of multi-signature logical systems, based on the notion of equivalence of institutions [22, 24] was introduced. This framework suggests another approach to the algebraization of equational logic (see [25]) closer in spirit to [11], based on the categorical algebraic notion of an algebraic theory ([4, 13, 16]). Namely,

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†School of Mathematics and Computer Science, Lake Superior State University, 650 W. Easterday Ave., Sault Sainte Marie, MI 49783, USA, gvoutsad@lssu.edu. Partially supported by National Science Foundation grant CCR - 9593168

the algebraizing class of algebras is not presented in the traditional way by choosing a priori basic operations and relating them via equational axioms. Rather, an adjunction is constructed based on the original logical system that is chosen to represent equational logic over multiple signatures. This adjunction gives rise in the standard way to an algebraic theory in monoid form. The Eilenberg-Moore algebras of this algebraic theory constitute the algebraizing class of algebras. Thus, basic operations are not given. Instead all clone operations are assigned equal weight. On the other hand, in an attempt to connect this approach to the traditional one, in Section 5, a variety $\mathcal{V}$ of algebras is constructed based on a similar construction in [20]. The algebras of $\mathcal{V}$ correspond to clones of algebras with operations of arbitrary finite arities. It is then shown, in Section 6, that the category of the Eilenberg-Moore algebras of the aforementioned algebraic theory is isomorphic to the category $\vec{\mathcal{V}}$ of this variety $\mathcal{V}$.

2 Basic Constructions A countably infinite **set of variables** V is fixed in advance and well-ordered and by **Set** is denoted the category of all small sets. Given a set X, we define the set of X-terms with variables in the set V.

Definition 1 *Let $X \in |\mathbf{Set}|$. We define the* **set of X-terms** $\mathrm{Tm}_X(V) \in |\mathbf{Set}|$, *to be the smallest set with*

(i) $V \subseteq \mathrm{Tm}_X(V)$ *and*

(ii) If $x \in X, n \in \omega$ *and* $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)$, *with* $t_{n-1} \neq v_{n-1}$, *then*

$$x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V).$$

The definitions of simultaneous substitution of terms for variables in a term and that of the extension of a given set map $f : X \to \mathrm{Tm}_Y(V)$ to a map $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ are given next.

Definition 2 *Let $X \in |\mathbf{Set}|$, as before. Define a function*

$$R_X : \mathrm{Tm}_X(V) \times \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)^k \to \mathrm{Tm}_X(V)$$

by $R_X : \mathrm{Tm}_X(V) \times \mathrm{Tm}_X(V)^0 \to \mathrm{Tm}_X(V); (t, \langle\rangle) \mapsto t$, *and, otherwise, by recursion on the structure of X-terms as follows:*

(i)

$$R_X(v_i, \langle s_0, \ldots, s_{m-1}\rangle) = \begin{cases} s_i, & i < m \\ v_i, & i \geq m \end{cases}$$

for every $m \in \omega, s_0, \ldots, s_{m-1} \in \mathrm{Tm}_X(V)$,

(ii) $R_X(x(t_0, \ldots, t_{n-1}), \vec{s}) = x(R_X(t_0, \vec{s}), \ldots, R_X(t_{n-1}, \vec{s}), s_n, \ldots, s_{m-1})$, *for every* $x \in X, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$, *and every* $m \in \omega, \vec{s} \in \mathrm{Tm}_X(V)^m$.

It is understood that the last, say k-th, term inside the parenthesis on the right, i.e., $R_X(t_{k-1}, \vec{s}), 0 \leq k < n$, *if* $m \leq n$, *and either* $R_X(t_{k-1}, \vec{s})$ *or* $s_{k-1}, 0 \leq k < m$, *if* $n < m$, *must be the last term that is not equal to the variable* v_{k-1}.

Definition 3 *Let* $X, Y \in |\mathbf{Set}|$ *and* $f : X \to \mathrm{Tm}_Y(V)$. *Define* $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ *by recursion on the structure of X-terms as follows:*

(i) $f^*(v) = v$, *for every* $v \in V$,

(ii) $f^*(x(t_0, \ldots, t_{n-1})) = R_Y(f(x), \langle f^*(t_0), \ldots, f^*(t_{n-1})\rangle)$, *for every* $x \in X, n \in \omega$, $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$.

In the sequel, we write $f : X \rightharpoondown Y$ to denote a **Set**-map $f : X \to \mathrm{Tm}_Y(V)$, as above. The choice follows an analogous notation used in [16] for Kleisli morphisms. It is used to anticipate the fact that these morphisms will turn out to be morphisms in the Kleisli category of the algebraic theory **T** in **Set** that will be constructed in the fourth section. Given two such maps $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$, their **composition** $g \circ f : X \rightharpoondown Z$ is defined to be

$$g \circ f = g^* f.$$

This definition is also reminiscent of the composition in a Kleisli category of an algebraic theory. We denote by **Cln** the category having as collection of objects $|\mathbf{Set}|$ and as its collections of morphisms

$$\mathbf{Cln}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{Set}(X, \mathrm{Tm}_Y(V))\},$$

for every $X, Y \in |\mathbf{Set}|$. Composition in **Cln** is the composition $\circ$ as defined above and the identity arrows $j_X : X \rightharpoondown X$ are the set maps $j_X : X \to \mathrm{Tm}_X(V)$, with

$$j_X(x) = x(), \quad \text{for every} \quad x \in X.$$

Given two **Cln**-maps $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$, a **Set**-map from $\mathrm{Tm}_X(V)$ into $\mathrm{Tm}_Z(V)$ may be obtained either by taking the extension $(g \circ f)^*$ of $g \circ f$ to X-terms or by composing the extensions f^* and g^*. It is now shown that the outcomes are the same both ways. Two lemmas are needed first.

Lemma 4 *Let* $f : X \rightharpoondown Y, k, m \in \omega, t \in \mathrm{Tm}_X(V), \vec{u} \in \mathrm{Tm}_X(V)^k$ *and* $\vec{s} \in \mathrm{Tm}_X(V)^m$. *Then*

$$R_X(R_X(t, \vec{u}), \vec{s}) = R_X(t, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle).$$

Proof:
By recursion on the structure of t.
If $t = v_i \in V$,

$$R_X(R_X(v_i, \vec{u}), \vec{s}) = \left\{ \begin{array}{ll} R_X(u_i, \vec{s}), & i < k \\ R_X(v_i, \vec{s}), & i \geq k \end{array} \right\} = \left\{ \begin{array}{ll} R_X(u_i, \vec{s}), & i < k \\ s_i, & k \leq i < m \\ v_i, & m \leq i \end{array} \right\} =$$

$$= R_X(v_i, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle).$$

Next, if $x \in X, n \in \omega$ and $t_0, \dots, t_{n-1} \in \mathrm{Tm}_X(V), t_{n-1} \neq v_{n-1}$,

$$R_X(R_X(x(t_0, \dots, t_{n-1}), \vec{u}), \vec{s}) =$$

$$\begin{array}{rl} = & R_X(x(R_X(t_0, \vec{u}), \dots, R_X(t_{n-1}, \vec{u}), u_n, \dots, u_{k-1}), \vec{s}) \\ & \text{(by definition of } R_X) \\ = & x(R_X(R_X(t_0, \vec{u}), \vec{s}), \dots, R_X(R_X(t_{n-1}, \vec{u}), \vec{s}), \\ & R_X(u_n, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}) \text{ (by definition of } R_X) \\ = & x(R_X(t_0, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle), \dots, \\ & R_X(t_{n-1}, \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle), R_X(u_n, \vec{s}), \dots, \\ & R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}) \text{ (by the induction hypothesis)} \\ = & R_X(x(t_0, \dots, t_{n-1}), \langle R_X(u_0, \vec{s}), \dots, R_X(u_{k-1}, \vec{s}), s_k, \dots, s_{m-1}\rangle), \\ & \text{(by definition of } R_X). \end{array}$$

■

The proofs of Lemmas 5 and 6 below are also by induction on the structure of the term t and will be omitted. Lemma 4 is used in the proof of the inductive step in Lemma 5 and Lemma 5 in the inductive step of Lemma 6.

Lemma 5 *Let* $f : X \rightharpoondown Y, m \in \omega, t \in \mathrm{Tm}_X(V), \vec{s} \in \mathrm{Tm}_X(V)^m$. *Then*

$$f^*(R_X(t, \vec{s})) = R_Y(f^*(t), f^*(\vec{s})).$$

Lemma 6 *Let* $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ *be two* **Cln**-*maps. Then*

$$(g \circ f)^* = g^* f^*.$$

3 The Adjunction We are now ready to proceed with the construction of the promised adjunction

$$\langle F, U, \eta, \epsilon \rangle : \mathbf{Set} \rightarrow \mathbf{Cln}.$$

First, define a functor $F : \mathbf{Set} \rightarrow \mathbf{Cln}$ by

$$F(X) = X, \quad \text{for every} \quad X \in |\mathbf{Set}|,$$

and, if $f : X \rightarrow Y \in \mathrm{Mor}(\mathbf{Set})$,

$$F(f) = j_Y f : X \rightharpoondown Y.$$

If $f : X \rightarrow Y, g : Y \rightarrow Z \in \mathrm{Mor}(\mathbf{Set})$, then

$$F(gf) = j_Z(gf) = (j_Z g)^*(j_Y f) = F(g)^* F(f) = F(g) \circ F(f),$$

i.e., F is a functor.

Now define a functor $U : \mathbf{Cln} \rightarrow \mathbf{Set}$ by

$$U(X) = \mathrm{Tm}_X(V), \quad \text{for every} \quad X \in |\mathbf{Cln}|,$$

and, if $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Cln})$,

$$U(f) = f^* : \mathrm{Tm}_X(V) \rightarrow \mathrm{Tm}_Y(V).$$

Then, if $f : X \rightharpoondown Y, g : Y \rightharpoondown Z \in \mathrm{Mor}(\mathbf{Cln})$, we have

$$\begin{array}{rcl} U(g \circ f) & = & (g \circ f)^* \\ & = & g^* f^* \quad \text{(by Lemma 6)} \\ & = & U(g)U(f), \end{array}$$

i.e., U is also a functor.

Finally, define natural transformations $\eta : I_{\mathbf{Set}} \rightarrow UF$ by $\eta_X : X \rightarrow \mathrm{Tm}_X(V)$ with $\eta_X = j_X$, for every $X \in |\mathbf{Set}|$, and $\epsilon : FU \rightarrow I_{\mathbf{Cln}}$ by $\epsilon_X : \mathrm{Tm}_X(V) \rightharpoondown X$ with $\epsilon_X = i_{\mathrm{Tm}_X(V)}$, for every $X \in |\mathbf{Cln}|$. It is now shown that η and ϵ are indeed natural transformations.

To this end, let $f : X \rightarrow Y \in \mathrm{Mor}(\mathbf{Set})$. Then, for every $x \in X$,

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & U(F(X)) \arrow[d, "U(F(f))"] \\ Y \arrow[r, "\eta_Y"'] & U(F(Y)) \end{tikzcd}$$

$$U(F(f))(\eta_X(x)) = U(F(f))(x()) = (j_Y f)^*(x()) =$$

$$= R_Y((j_Y f)(x), \langle\rangle) = j_Y f(x) = \eta_Y(f(x)).$$

Next, let $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Cln})$. Then, for every $t \in \mathrm{Tm}_X(V)$,

$$\begin{tikzcd} F(U(X)) \arrow[r, "\epsilon_X"] \arrow[d, "F(U(f))"'] & X \arrow[d, "f"] \\ F(U(Y)) \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

$$(f \circ \epsilon_X)(t) = f^*(\epsilon_X(t)) = f^*(t) = \epsilon_Y^*(j_{\mathrm{Tm}_Y(V)}(f^*(t))) = (\epsilon_Y \circ F(U(f)))(t).$$

Finally, if $t \in \mathrm{Tm}_X(V)$, then

$$i^*_{\mathrm{Tm}_X(V)}(\eta_{\mathrm{Tm}_X(V)}(t)) = i^*_{\mathrm{Tm}_X(V)}(t()) = t,$$

and, if $y \in Y$,

$$i^*_{\mathrm{Tm}_X(V)}(\eta_{\mathrm{Tm}_Y(V)}(\eta_Y(y))) = i^*_{\mathrm{Tm}_X(V)}(y()()) = y() = \eta_Y(y),$$

i.e., the following triangles commute

$$\begin{tikzcd} \mathrm{Tm}_X(V) \arrow[r, "\eta_{\mathrm{Tm}_X(V)}"] \arrow[dr, "i_{\mathrm{Tm}_X(V)}"'] & \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[d, "i^*_{\mathrm{Tm}_X(V)}"] \\ & \mathrm{Tm}_X(V) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta_{\mathrm{Tm}_Y(V)}\eta_Y"] \arrow[dr, "\eta_Y"'] & \mathrm{Tm}_Y(V) \arrow[d, "i_{\mathrm{Tm}_Y(V)}"] \\ & Y \end{tikzcd}$$

which proves

Theorem 7 $\langle F, U, \eta, \epsilon\rangle : \mathbf{Set} \to \mathbf{Cln}$ *is an adjunction.*

4 The Theory of the Adjunction It is well-known ([13, 16, 4]) that an adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{Set} \to \mathbf{Cln}$ gives rise to an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in monoid form over $\mathbf{Set}$, with $T = UF$ and $\mu = U\epsilon_F$. Moreover there exists a unique functor $K : \mathbf{Set_T} \to \mathbf{Cln}$ from the Kleisli category of the theory to $\mathbf{Cln}$, called the *Kleisli comparison functor of the adjunction,* that makes the F- and U-paths of the following diagrams commute.

$$\begin{tikzcd} \mathbf{Set_T} \arrow[rr, "K"] \arrow[dr, "F_\mathbf{T}"'] & & \mathbf{Cln} \arrow[dl, "F"] \\ & \mathbf{Set} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Set_T} \arrow[rr, "K"] \arrow[dr, "U_\mathbf{T}"'] & & \mathbf{Cln} \arrow[dl, "U"] \\ & \mathbf{Set} & \end{tikzcd}$$

It is easy to verify that, in this case $\mathbf{Set_T} = \mathbf{Cln}$ and $K = I_{\mathbf{Cln}}$. Therefore $\mathbf{Cln}$ is the category of all free algebras of the algebraic theory $\mathbf{T}$ in $\mathbf{Set}$.

Also recall that a $\mathbf{T}$-algebra $\langle X, \xi\rangle$ consists of a set X together with a map $\xi : T(X) \to X$, i.e., $\xi : \mathrm{Tm}_X(V) \to X$, such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[dr, "i_X"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[r, "(j_X\xi)^*"] \arrow[d, "i^*_{\mathrm{Tm}_X(V)}"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ \mathrm{Tm}_X(V) \arrow[r, "\xi"'] & X \end{tikzcd}$$

5 Clone Algebras In this section a variety of algebras $\mathcal{V}$ is equationally defined whose members are called **clone algebras.** The construction is inspired by W. Taylor's construction of a multi-sorted analog in [20]. In the next section, it will be shown that the category $\vec{\mathcal{V}}$ of this variety is isomorphic to the Eilenberg-Moore category $\mathbf{Set}^{\mathbf{T}}$ of the algebraic theory $\mathbf{T}$ in $\mathbf{Set}$, that was constructed in the previous section.

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be the language type defined as follows.

$$\Lambda = \{\mathrm{v}_i, \mathrm{C}_i : i \in \omega\}, \quad \text{with} \quad \rho(\mathrm{v}_i) = 0, \rho(\mathrm{C}_i) = i + 1.$$

Definition 8 *A* **clone algebra A** *is an $\mathcal{L}$-algebra that satisfies the following identities, for every $n, m \in \omega$,*

- $\mathrm{C}_0(x) = x$
- $\mathrm{C}_n(x, y_0 \ldots, y_{n-2}, \mathrm{v}_{n-1}) = \mathrm{C}_{n-1}(x, y_0 \ldots, y_{n-2})$
- $$\mathrm{C}_n(\mathrm{v}_m, x_0, \ldots, x_{n-1}) = \begin{cases} x_m, & \textit{if } m < n \\ \mathrm{v}_m, & \textit{otherwise} \end{cases}$$
- $\mathrm{C}_n(z, \mathrm{C}_n(y_0, \vec{x}), \ldots, \mathrm{C}_n(y_{m-1}, \vec{x}), x_m, \ldots, x_{n-1}) = \mathrm{C}_n(C_m(z, \vec{y}), \vec{x})$

Let $\mathcal{V}$ be the variety of all clone algebras and denote by $\vec{\mathcal{V}}$ the category associated with $\mathcal{V}$.

It will now be shown that a functor P can be constructed from the category $\vec{\mathcal{V}}$, associated with the variety $\mathcal{V}$, to the category $\mathbf{Set}^{\mathbf{T}}$ of all $\mathbf{T}$-algebras in $\mathbf{Set}$. The object part of P is constructed first.

Let $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ be a clone algebra. Define $\mathbf{A}^* = \langle A, \xi_{\mathbf{A}^*} \rangle$ as follows: $\xi_{\mathbf{A}^*} : \mathrm{Tm}_A(V) \to A$ is defined by recursion on the structure of A-terms, by

- $\xi_{\mathbf{A}^*}(v_i) = \mathrm{v}_i^{\mathbf{A}}$, for every $i \in \omega$,
- If $a \in A, n \in \omega, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_A(V), t_{n-1} \neq v_{n-1}$,
$$\xi_{\mathbf{A}^*}(a(t_0, \ldots, t_{n-1})) = \mathrm{C}_n^{\mathbf{A}}(a, \xi_{\mathbf{A}^*}(t_0), \ldots, \xi_{\mathbf{A}^*}(t_{n-1})).$$

Lemma 9 *Let $\mathbf{A} \in \mathcal{V}, \mathbf{A}^* = \langle A, \xi_{\mathbf{A}^*} \rangle$. Then, for every $t \in \mathrm{Tm}_A(V), m \in \omega, \vec{s} \in \mathrm{Tm}_A(V)^m$,*

$$\xi_{\mathbf{A}^*}(R_A(t, \vec{s})) = \mathrm{C}_m^{\mathbf{A}}(\xi_{\mathbf{A}^*}(t), \xi_{\mathbf{A}^*}(\vec{s})).$$

Proof:
By induction on the structure of t.
If $t = v_i \in V$, then

$$\begin{aligned} \xi_{\mathbf{A}^*}(R_A(v_i, \vec{s})) &= \left\{ \begin{array}{ll} \xi_{\mathbf{A}^*}(s_i), & \text{if } i < m \\ \xi_{\mathbf{A}^*}(v_i), & \text{if } i \geq m \end{array} \right\} = \left\{ \begin{array}{ll} \xi_{\mathbf{A}^*}(s_i), & \text{if } i < m \\ \mathrm{v}_i^{\mathbf{A}}, & \text{if } i \geq m \end{array} \right\} \\ &= \mathrm{C}_m^{\mathbf{A}}(\mathrm{v}_i^{\mathbf{A}}, \xi_{\mathbf{A}^*}(\vec{s})) \quad \text{(by the third axiom)} \\ &= \mathrm{C}_m^{\mathbf{A}}(\xi_{\mathbf{A}^*}(v_i), \xi_{\mathbf{A}^*}(\vec{s})). \end{aligned}$$

If $a \in A, n \in \omega, \vec{t} \in \mathrm{Tm}_A(V)^n, t_{n-1} \neq v_{n-1}$, then

$$\xi_{\mathbf{A}^*}(R_A(a(\vec{t}), \vec{s})) =$$

$$\begin{aligned}
&= \xi_{\mathbf{A}^*}(a(R_A(t_0,\vec{s}),\ldots,R_A(t_{n-1},\vec{s}),s_n,\ldots,s_{m-1})) \quad \text{(by definition of } R_A)\\
&= \mathrm{C}^{\mathbf{A}}_m(a,\xi_{\mathbf{A}^*}(R_A(t_0,\vec{s})),\ldots,\xi_{\mathbf{A}^*}(R_A(t_{n-1},\vec{s})),\xi_{\mathbf{A}^*}(s_n),\ldots,\xi_{\mathbf{A}^*}(s_{m-1}))\\
&\quad \text{(by definition of } \xi_{\mathbf{A}^*})\\
&= \mathrm{C}^{\mathbf{A}}_m(a,\mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^*}(t_0),\xi_{\mathbf{A}^*}(\vec{s})),\ldots,\mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^*}(t_{n-1}),\xi_{\mathbf{A}^*}(\vec{s})),\xi_{\mathbf{A}^*}(s_n),\ldots,\xi_{\mathbf{A}^*}(s_{m-1}))\\
&\quad \text{(by the induction hypothesis)}\\
&= \mathrm{C}^{\mathbf{A}}_m(\mathrm{C}^{\mathbf{A}}_n(a,\xi_{\mathbf{A}^*}(\vec{t})),\xi_{\mathbf{A}^*}(\vec{s})) \quad \text{(by the fourth axiom)}\\
&= \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^*}(a(\vec{t})),\xi_{\mathbf{A}^*}(\vec{s})). \quad \text{(by definition of } \xi_{\mathbf{A}^*})
\end{aligned}$$

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Lemma 10 *Let* $\mathbf{A} \in \mathcal{V}$. *Then* $\mathbf{A}^* = \langle A, \xi_{\mathbf{A}^*}\rangle \in |\mathbf{Set}^{\mathbf{T}}|$.

Proof:
We need to show that the following diagrams commute

$$\begin{tikzcd}
A \arrow[r, "j_A"] \arrow[dr, "i_A"'] & \mathrm{Tm}_A(V) \arrow[d, "\xi_{\mathbf{A}^*}"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Tm}_{\mathrm{Tm}_A(V)}(V) \arrow[r, "(j_A\xi_{\mathbf{A}^*})^*"] \arrow[d, "i^*_{\mathrm{Tm}_A(V)}"'] & \mathrm{Tm}_A(V) \arrow[d, "\xi_{\mathbf{A}^*}"] \\
\mathrm{Tm}_A(V) \arrow[r, "\xi_{\mathbf{A}^*}"'] & A
\end{tikzcd}$$

For the triangle, we have, for every $a \in A$,

$$\begin{aligned}
\xi_{\mathbf{A}^*}(j_A(a)) &= \xi_{\mathbf{A}^*}(a()) \quad \text{(by definition of } j_A)\\
&= \mathrm{C}^{\mathbf{A}}_0(a) \quad \text{(by definition of } \xi_{\mathbf{A}^*})\\
&= a \quad \text{(by the first axiom)}\\
&= i_A(a).
\end{aligned}$$

For the rectangle, we proceed by induction on the structure of a $\mathrm{Tm}_A(V)$-term t. If $t = v_i \in V$, then

$$\xi_{\mathbf{A}^*}((j_A\xi_{\mathbf{A}^*})^*(v_i)) = \xi_{\mathbf{A}^*}(v_i) = \xi_{\mathbf{A}^*}(i^*_{\mathrm{Tm}_A(V)}(v_i)).$$

If $s \in \mathrm{Tm}_A(V), n \in \omega, \vec{t} \in \mathrm{Tm}_{\mathrm{Tm}_A(V)}(V)^n, t_{n-1} \neq v_{n-1}$, then

$$\begin{aligned}
\xi_{\mathbf{A}^*}((j_A\xi_{\mathbf{A}^*})^*(s(\vec{t}))) &= \xi_{\mathbf{A}^*}(R_A((j_A\xi_{\mathbf{A}^*})(s),(j_A\xi_{\mathbf{A}^*})^*(\vec{t}))) \quad \text{(by definition of } (j_A\xi_{\mathbf{A}^*})^*)\\
&= \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^*}(j_A(\xi_{\mathbf{A}^*}(s))),\xi_{\mathbf{A}^*}((j_A\xi_{\mathbf{A}^*})^*(\vec{t}))) \quad \text{(by Lemma 9)}\\
&= \mathrm{C}^{\mathbf{A}}_m(\xi_{\mathbf{A}^*}(s),\xi_{\mathbf{A}^*}(i^*_{\mathrm{Tm}_A(V)}(\vec{t})))\\
&\quad \text{(by commutativity of triangle and the induction hypothesis)}\\
&= \xi_{\mathbf{A}^*}(R_A(s,i^*_{\mathrm{Tm}_A(V)}(\vec{t}))) \quad \text{(by Lemma 9)}\\
&= \xi_{\mathbf{A}^*}(i^*_{\mathrm{Tm}_A(V)}(s(\vec{t}))). \quad \text{(by definition of } i^*_{\mathrm{Tm}_A(V)})
\end{aligned}$$

■

Next suppose that $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle, \mathbf{B} = \langle B, \mathcal{L}^{\mathbf{B}}\rangle \in \mathcal{V}$ and $h : \mathbf{A} \to \mathbf{B} \in \vec{\mathcal{V}}(\mathbf{A},\mathbf{B})$. We show that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Tm}_A(V) \arrow[r, "(j_Bh)^*"] \arrow[d, "\xi_{\mathbf{A}^*}"'] & \mathrm{Tm}_B(V) \arrow[d, "\xi_{B^*}"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

i.e., that $h \in \mathbf{Set}^{\mathbf{T}}(\mathbf{A}^*, \mathbf{B}^*)$.

We work by induction on the structure of an A-term t.

If $t = v_i \in V$, then

$$\begin{array}{rcl} \xi_{\mathbf{B}^*}((j_B h)^*(v_i)) & = & \xi_{\mathbf{B}^*}(v_i) \quad \text{(by definition of } (j_B h)^*) \\ & = & \mathrm{v}_i^{\mathbf{B}} \quad \text{(by definition of } \xi_{\mathbf{B}^*}) \\ & = & h(\mathrm{v}_i^{\mathbf{A}}) \quad \text{(since } h \in \vec{\mathcal{V}}(\mathbf{A}, \mathbf{B})) \\ & = & h(\xi_{\mathbf{A}^*}(v_i)). \quad \text{(by definition of } \xi_{\mathbf{A}^*}) \end{array}$$

If $a \in A, n \in \omega, \vec{t} \in \mathrm{Tm}_A(V)^n, t_{n-1} \neq v_{n-1}$,

$$\begin{array}{rcl} \xi_{\mathbf{B}^*}((j_B h)^*(a(\vec{t}))) & = & \xi_{\mathbf{B}^*}(R_B((j_B h)(a), (j_B h)^*(\vec{t}))) \quad \text{(by definition of } (j_B h)^*) \\ & = & \mathrm{C}_n^{\mathbf{B}}(\xi_{\mathbf{B}^*}(j_B(h(a))), \xi_{\mathbf{B}^*}((j_B h)^*(\vec{t}))) \quad \text{(by Lemma 9)} \\ & = & \mathrm{C}_n^{\mathbf{B}}(h(a), h(\xi_{\mathbf{A}^*}(\vec{t}))) \quad \text{(by comm. of triangle and the ind. hyp.)} \\ & = & h(\mathrm{C}_n^{\mathbf{A}}(a, \xi_{\mathbf{A}^*}(\vec{t}))) \quad \text{(since } h \in \vec{\mathcal{V}}(\mathbf{A}, \mathbf{B})) \\ & = & h(\xi_{\mathbf{A}^*}(a(\vec{t}))). \quad \text{(by definition of } \xi_{\mathbf{A}^*}) \end{array}$$

Thus, it is possible to define the functor $P : \vec{\mathcal{V}} \to \mathbf{Set}^{\mathbf{T}}$ by

$$P(\mathbf{A}) = \mathbf{A}^*, \quad \text{for every} \quad \mathbf{A} \in \mathcal{V},$$

and, given $h \in \vec{\mathcal{V}}(\mathbf{A}, \mathbf{B}), P(h) \in \mathbf{Set}^{\mathbf{T}}(\mathbf{A}^*, \mathbf{B}^*)$, by

$$P(h) = h.$$

6 The Equivalence In this section, a functor $Q : \mathbf{Set}^{\mathbf{T}} \to \vec{\mathcal{V}}$ in the opposite direction is defined and it is shown that P and Q are inverses of each other. Therefore the two categories $\mathbf{Set}^{\mathbf{T}}$ and $\vec{\mathcal{V}}$ are isomorphic categories.

Let $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle$ be a $\mathbf{T}$-algebra. Define an $\mathcal{L}$-algebra $\mathbf{A}^{\#} = \langle A, \mathcal{L}^{\mathbf{A}^{\#}} \rangle$ as follows:

- $\mathrm{v}_i^{\mathbf{A}^{\#}} = \xi_{\mathbf{A}}(v_i)$, for every $i \in \omega$,
- $\mathrm{C}_n^{\mathbf{A}^{\#}}(a, a_0, \ldots, a_{n-1}) = \xi_{\mathbf{A}}(R_A(j_A(a), \langle j_A(a_0), \ldots, j_A(a_{n-1}) \rangle))$, for every $n \in \omega, a, a_0, \ldots, a_{n-1} \in A$.

Lemma 11 *Let* $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle \in |\mathbf{Set}^{\mathbf{T}}|$. *Then* $j_A \xi_{\mathbf{A}} = (j_A \xi_{\mathbf{A}})^* j_{\mathrm{Tm}_A(V)}$.

Proof:

Let $t \in \mathrm{Tm}_A(V)$. Then

$$\begin{array}{rcl} (j_A \xi_{\mathbf{A}})^*(j_{\mathrm{Tm}_A(V)}(t)) & = & (j_A \xi_{\mathbf{A}})^*(t()) \quad \text{(by definition of } j_{\mathrm{Tm}_A(V)}) \\ & = & j_A \xi_{\mathbf{A}}(t). \quad \text{(by definition of } (j_A \xi_{\mathbf{A}})^*) \end{array}$$

■

Lemma 12 *Let* $\mathbf{A} \in |\mathbf{Set}^{\mathbf{T}}|$. *Then* $\mathbf{A}^{\#} \in \mathcal{V}$.

Proof:

We need to verify that the identities of Definition 8 hold. For the first one,

$$\mathrm{C}_0^{\mathbf{A}^{\#}}(a) = \xi_{\mathbf{A}}(R_A(j_A(a), \langle \rangle)) = \xi_{\mathbf{A}} j_A(a) = a.$$

For the second, we have

$$\mathrm{C}_n^{\mathbf{A}^\#}(a, b_0, \ldots, b_{n-2}, \mathrm{v}_{n-1}^{\mathbf{A}^\#}) =$$

$$\begin{array}{rl}
= & \mathrm{C}_n^{\mathbf{A}^\#}(a, \vec{b}, \xi_{\mathbf{A}}(v_{n-1})) \quad \text{(by definition of } \mathrm{v}_{n-1}^{\mathbf{A}^\#}) \\
= & \xi_{\mathbf{A}}(R_A(j_A(a), \langle j_A(\vec{b}), j_A(\xi_{\mathbf{A}}(v_{n-1}))\rangle)) \quad \text{(by defin. of } \mathrm{C}_n^{\mathbf{A}^\#}) \\
= & \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(j_A(a))), \langle j_A(\xi_{\mathbf{A}}(j_A(\vec{b}))), j_A(\xi_{\mathbf{A}}(v_{n-1}))\rangle)) \\
& \text{(by } \xi_{\mathbf{A}} j_A = i_A) \\
= & \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(a())), \langle j_A(\xi_{\mathbf{A}}(\vec{b}())), j_A(\xi_{\mathbf{A}}(v_{n-1}))\rangle)) \\
& \text{(by definition of } j_A) \\
= & \xi_{\mathbf{A}}(R_A((j_A\xi_{\mathbf{A}})(a()), \langle (j_A\xi_{\mathbf{A}})^*(\vec{b}()()), (j_A\xi_{\mathbf{A}})^*(v_{n-1}()())\rangle)) \\
& \text{(by Lemma 11)} \\
= & \xi_{\mathbf{A}}((j_A\xi_{\mathbf{A}})^*(a()(\vec{b}()(), v_{n-1}()))) \quad \text{(by defin. of } (j_A\xi_{\mathbf{A}})^*) \\
= & \xi_{\mathbf{A}}(i^*_{\mathrm{Tm}_A(V)}(a()(\vec{b}()(), v_{n-1}()))) \\
& \text{(since } \xi_{\mathbf{A}}(j_A\xi_{\mathbf{A}})^* = \xi_{\mathbf{A}} i^*_{\mathrm{Tm}_A(V)}) \\
= & \xi_{\mathbf{A}}(R_A(a(), \langle \vec{b}(), v_{n-1}\rangle)) \quad \text{(by definition of } i^*_{\mathrm{Tm}_A(V)}) \\
= & \xi_{\mathbf{A}}(R_A(a(), \vec{b}())), \quad \text{(by definition of } R_A) \\
= & \ldots \quad \text{(reverse all the steps in the deduction above)} \\
= & \mathrm{C}_{n-1}^{\mathbf{A}^\#}(a, \vec{b}).
\end{array}$$

The third and the fourth identities can be proved similarly. Lemma 11 is used in the proof of both. In the proof of the fourth, Lemma 4 is also used. ■

Next, let $\mathbf{A} = \langle A, \xi_{\mathbf{A}}\rangle, \mathbf{B} = \langle B, \xi_{\mathbf{B}}\rangle \in |\mathbf{Set^T}|$ and $h \in \mathbf{Set^T}(\mathbf{A}, \mathbf{B})$, i.e., the following diagram commutes

$$\begin{tikzcd}
\mathrm{Tm}_A(V) \arrow[r, "(j_B h)^*"] \arrow[d, "\xi_{\mathbf{A}}"'] & \mathrm{Tm}_B(V) \arrow[d, "\xi_{\mathbf{B}}"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

We show that $h \in \vec{\mathcal{V}}(\mathbf{A}^\#, \mathbf{B}^\#)$. To this end, we need to verify the following two equations

- $h(\mathrm{v}_i^{\mathbf{A}^\#}) = \mathrm{v}_i^{\mathbf{B}^\#}$, for every $i \in \omega$, and
- $h(\mathrm{C}_n^{\mathbf{A}^\#}(a, a_0, \ldots, a_{n-1})) = \mathrm{C}_n^{\mathbf{B}^\#}(h(a), h(a_0), \ldots, h(a_{n-1}))$, for every $n \in \omega, a, a_0, \ldots, a_{n-1} \in A$.

We have

$$\begin{array}{rcl}
h(\mathrm{v}_i^{\mathbf{A}^\#}) & = & h(\xi_{\mathbf{A}}(v_i)) \quad \text{(by definition of } \mathrm{v}_i^{\mathbf{A}^\#}) \\
& = & \xi_{\mathbf{B}}((j_B h)^*(v_i)) \quad \text{(by commutativity of rectangle)} \\
& = & \xi_{\mathbf{B}}(v_i) \quad \text{(by definition of } (j_B h)^*) \\
& = & \mathrm{v}_i^{\mathbf{B}^\#}, \quad \text{(by definition of } \mathrm{v}_i^{\mathbf{B}^\#})
\end{array}$$

and

$$\begin{array}{rcl}
h(\mathrm{C}_n^{\mathbf{A}^\#}(a, \vec{a})) & = & h(\xi_{\mathbf{A}}(R_A(j_A(a), j_A(\vec{a})))) \quad \text{(by definition of } \mathrm{C}_n^{\mathbf{A}^\#}) \\
& = & \xi_{\mathbf{B}}((j_B h)^*(R_A(j_A(a), j_A(\vec{a})))) \quad \text{(by commut. of rectangle)} \\
& = & \xi_{\mathbf{B}}(R_B((j_B h)^*(j_A(a)), (j_B h)^*(j_A(\vec{a})))) \quad \text{(by Lemma 5)} \\
& = & \xi_{\mathbf{B}}(R_B((j_B h)(a), (j_B h)(\vec{a}))) \\
& = & \mathrm{C}_n^{\mathbf{B}^\#}(h(a), h(\vec{a})). \quad \text{(by definition of } \mathrm{C}_n^{\mathbf{B}^\#})
\end{array}$$

Therefore, we can define a functor $Q : \mathbf{Set}^{\mathbf{T}} \to \vec{\mathcal{V}}$, by

$$Q(\mathbf{A}) = \mathbf{A}^{\#}, \quad \text{for every} \quad \mathbf{A} \in |\mathbf{Set}^{\mathbf{T}}|,$$

and, given $h \in \mathbf{Set}^{\mathbf{T}}(\mathbf{A}, \mathbf{B}), Q(h) \in \vec{\mathcal{V}}(\mathbf{A}^{\#}, \mathbf{B}^{\#})$, by

$$Q(h) = h.$$

We finally proceed to show that $QP = I_{\vec{\mathcal{V}}}$ and $PQ = I_{\mathbf{Set}^{\mathbf{T}}}$. To this end, let $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle \in \mathcal{V}$. We have

$$\mathrm{v}_i^{\mathbf{A}^{*\#}} = \xi_{\mathbf{A}^*}(v_i) = \mathrm{v}_i^{\mathbf{A}}$$

and, for every $n \in \omega, a, a_0, \ldots, a_{n-1} \in A$,

$$\begin{array}{rcl}
\mathrm{C}_n^{\mathbf{A}^{*\#}}(a, \vec{a}) & = & \xi_{\mathbf{A}^*}(R_A(j_A(a), j_A(\vec{a}))) \quad (\text{by definition of } \mathrm{C}_n^{\mathbf{A}^{*\#}}) \\
& = & \xi_{\mathbf{A}^*}(a(j_A(\vec{a}))) \quad (\text{by definition of } R_A) \\
& = & \mathrm{C}_n^{\mathbf{A}}(a, \xi_{\mathbf{A}^*}(j_A(\vec{a}))) \quad (\text{by definition of } \xi_{\mathbf{A}^*}) \\
& = & \mathrm{C}_n^{\mathbf{A}}(a, \vec{a}). \quad (\text{by } \xi_{\mathbf{A}^*} j_A = i_A)
\end{array}$$

Finally, let $\mathbf{A} = \langle A, \xi_{\mathbf{A}} \rangle \in |\mathbf{Set}^{\mathbf{T}}|$. We have

$$\xi_{\mathbf{A}^{\#*}}(v_i) = \mathrm{v}_i^{\mathbf{A}^{\#}} = \xi_{\mathbf{A}}(v_i)$$

and, for every $a \in A, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_A(V), t_{n-1} \neq v_{n-1}$,

$$\begin{array}{rcl}
\xi_{\mathbf{A}^{\#*}}(a(\vec{t})) & = & \mathrm{C}_n^{\mathbf{A}^{\#}}(a, \xi_{\mathbf{A}^{\#*}}(\vec{t})) \quad (\text{by definition of } \xi_{\mathbf{A}^{\#*}}) \\
& = & \xi_{\mathbf{A}}(R_A(j_A(a), j_A(\xi_{\mathbf{A}}(\vec{t})))) \\
& & (\text{by definition of } \mathrm{C}_n^{\mathbf{A}^{\#}} \text{ and the ind. hyp.}) \\
& = & \xi_{\mathbf{A}}(R_A(j_A(\xi_{\mathbf{A}}(j_A(a))), (j_A \xi_{\mathbf{A}})^*(\vec{t}()))) \\
& & (\text{by } \xi_{\mathbf{A}} j_A = i_A \text{ and Lemma 11}) \\
& = & \xi_{\mathbf{A}}((j_A \xi_{\mathbf{A}})^*(a()(\vec{t}()))) \quad (\text{by definition of } R_A) \\
& = & \xi_{\mathbf{A}}(i^*_{\mathrm{Tm}_A(V)}(a()(\vec{t}()))) \quad (\text{since } \xi_{\mathbf{A}}(j_A \xi_{\mathbf{A}})^* = \xi_{\mathbf{A}} i^*_{\mathrm{Tm}_A(V)}) \\
& = & \xi_{\mathbf{A}}(R_A(a(), \vec{t})) \quad (\text{by definition of } i^*_{\mathrm{Tm}_A(V)}) \\
& = & \xi_{\mathbf{A}}(a(\vec{t})). \quad (\text{by definition of } R_A)
\end{array}$$

Thus, the following theorem holds

Theorem 13 $\vec{\mathcal{V}} \cong \mathbf{Set}^{\mathbf{T}}$.

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School of Mathematics and Computer Science, Lake Superior State University, 650 W.Easterday Avenue, Sault Saine Marie, MI 49783, U.S.A.
E-mail: gvoutsad@lssu.edu