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Categorical abstract algebraic logic: The criterion for deductive equivalence

George Voutsadakis*

School of Mathematics and Computer Science, Lake Superior State University,
650 W. Easterday Avenue, Sault Sainte Marie, MI 49783, U. S. A.

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Equivalent deductive systems were introduced in [4] with the goal of treating 1-deductive systems and algebraic 2-deductive systems in a uniform way. Results of [3], appropriately translated and strengthened, show that two deductive systems over the same language type are equivalent if and only if their lattices of theories are isomorphic via an isomorphism that commutes with substitutions. Deductive equivalence of π-institutions [14, 15] generalizes the notion of equivalence of deductive systems. In [15, Theorem 10.26] this criterion for the equivalence of deductive systems was generalized to a criterion for the deductive equivalence of term π-institutions, forming a subclass of all π-institutions that contains those π-institutions directly corresponding to deductive systems. This criterion is generalized here to cover the case of arbitrary π-institutions.

1 Introduction

In [3], following work of [5] and [2], the notion of algebraizability for deductive systems was introduced and studied. Given a class K of $\mathcal{L}$-algebras, an *algebraic* 2*-deductive system* $\mathcal{S}_K$ over $\mathcal{L}$ is one whose equational consequence relation is the model-theoretic consequence relation of the class K. An $\mathcal{L}$-deductive system $\mathcal{S}$ is said to be *algebraizable* if there exists a class K of $\mathcal{L}$-algebras such that $\mathcal{S}$ and $\mathcal{S}_K$ are interpretable in one another and the interpretations are inverse of each other, in the sense that applying the composition of the two interpretations to an $\mathcal{L}$-formula or an $\mathcal{L}$-equation, as appropriate, results in a set of formulas or equations, respectively, that are interderivable in $\mathcal{S}$ or $\mathcal{S}_K$, respectively, with the original formula or equation. Subsequently, the work of [3] was generalized in [10, 11, 12] to cover infinitary structural sentential logics and in a different direction in [7].

To unify this setting, without explicitly making the distinction between 1- and 2-deductive systems, the notion of *equivalence* of deductive systems was introduced in [4]. In this unifying setting algebraizability of a k-deductive system $\mathcal{S}$ may be simply restated as the equivalence of $\mathcal{S}$ with an algebraic deductive system $\mathcal{S}_K$ for some class K of $\mathcal{L}$-algebras. One of the main results of [3] is that a deductive system $\mathcal{S}$ is equivalent to an algebraic 2-deductive system $\mathcal{S}_K$ if and only if the lattice of theories $\mathsf{Th}(\mathcal{S})$ is isomorphic to the lattice of theories $\mathsf{Th}(\mathcal{S}_K)$ via an isomorphism that commutes with substitutions. This result may be generalized to the level of equivalence of arbitrary deductive systems. In fact, it is shown in [4] that two $\mathcal{L}$-deductive systems are equivalent if and only if their lattices of theories are isomorphic via an isomorphism that commutes with substitutions.

This framework of algebraizability for k-deductive systems does not work in an entirely satisfactory way when dealing with multiple signature logics with quantifiers, such as equational and first-order logic (for more details see the Introduction in [16]). To provide a platform for handling these logics with the same success, the theory of algebraizability for deductive systems was generalized in [14] to cover the algebraizability of institutions and π-institutions [6].

* e-mail: gvoutsad@lssu.edu

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Roughly speaking, an *algebraic institution* is one whose signature category is a full subcategory K of the Kleisli category $\mathsf{C}_{\mathbf{T}}$ of an algebraic theory $\mathbf{T}$ in monoid form in some category C, its model categories are very closely related to a subcategory Q of the Eilenberg-Moore category of algebras of $\mathbf{T}$ and its model-theoretic satisfaction relations are equational in nature. An *algebraic π-institution* is a π-institution associated with some algebraic institution in the sense of [6]. A π-institution is said to be *algebraizable* if it is deductively equivalent to an algebraic π-institution. Deductive equivalence of π-institutions was defined in [15] as a generalization of the notion of equivalence of deductive systems. Two π-institutions are called *deductively equivalent* if they are interpretable in one another and the two interpretations are inverse of each other in a sense very similar to the one described for deductive systems but slightly more involved to handle the increased complexity of the π-institution framework as opposed to the deductive system framework. A result, analogous to the previously stated characterization of equivalence in terms of the isomorphism of the theory lattices, was also proved for the case of *term π-institutions* in [15]. It states that two term π-institutions are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence between their categories of theories that commutes with substitutions. Term π-institutions constitute a class of π-institutions that contains all π-institutions naturally associated to k-deductive systems, called *deductive institutions* in [16]. However, this class is not large enough to contain other very important institutions such as the ones capturing equational and first-order logic that were presented in some detail in [16]. So the characterization result of [15] cannot be applied to those institutions.

In this paper, this result, characterizing deductive equivalence of term π-institutions in terms of the existence of an adjoint equivalence between their categories of theories, is generalized to cover the case of arbitrary π-institutions.

2 The framework and the main theorem

Some of the basic notions that are necessary to understand the proofs in the paper are now reviewed. The reader is referred to [1] and [13] for all unexplained categorical notation.

Fiadeiro and Sernadas [6] modified the notion of an institution, introduced by Goguen and Burstall [8, 9], to free the structure from the model theoretic satisfaction relations and bring it closer in spirit to the deductive system framework. The model theoretic deductions were replaced by logical closure operators. The emerging structures were termed π-institutions.

Definition 1 ([6]) A *π-institution* $\mathcal{I} = \langle \mathsf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}|}\rangle$ consists of

(i) a category Sign whose objects are called *signatures*,

(ii) a functor $\mathrm{SEN} : \mathsf{Sign} \longrightarrow \mathsf{Set}$ from the category Sign of signatures into the category Set of sets, called the *sentence functor* and giving, for each signature Σ a set whose elements are called *sentences over that signature* Σ or *Σ-sentences* and

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \longrightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathsf{Sign}|$, called *$\Sigma$-closure*, such that

(a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathsf{Sign}|$ and $A \subseteq \mathrm{SEN}(\Sigma)$,

(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$ for all $\Sigma \in |\mathsf{Sign}|$ and $A \subseteq \mathrm{SEN}(\Sigma)$,

(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$ for all $\Sigma \in |\mathsf{Sign}|$ and $A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$ for all $\Sigma_1, \Sigma_2 \in |\mathsf{Sign}|$, $f \in \mathsf{Sign}(\Sigma_1, \Sigma_2)$, and $A \subseteq \mathrm{SEN}(\Sigma_1)$.

From now on, given a π-institution $\mathcal{I} = \langle \mathsf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}|}\rangle$ a signature Σ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we will use the simplified notation Φ^{c} to denote $C_\Sigma(\Phi)$. Usually the signature Σ is clear from context and therefore this simplified notation does not cause any confusion. Also, when the "$^{\mathrm{c}}$" symbol is used, its scope will be the largest possible well-formed expression to its left. For instance, in $\mathrm{SEN}(f)(\Phi)^{\mathrm{c}}$ the scope of "$^{\mathrm{c}}$" is $\mathrm{SEN}(f)(\Phi)$ and not just (Φ), and in $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^{\mathrm{c}}))^{\mathrm{c}}$ the scope of the second "$^{\mathrm{c}}$" is $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^{\mathrm{c}}))$ and not just $\mathrm{SEN}(f)^{-1}(\Phi^{\mathrm{c}})$.

Let $\mathcal{I} = \langle \mathsf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}|}\rangle$ be a π-institution. Following [6] we define its *category of theories* $\mathsf{Th}(\mathcal{I})$ as follows: The objects of $\mathsf{Th}(\mathcal{I})$ are pairs $\langle \Sigma, T\rangle$, where $\Sigma \in |\mathsf{Sign}|$ and $T \subseteq \mathrm{SEN}(\Sigma)$ with $T^{\mathrm{c}} = T$. The morphisms $f : \langle \Sigma_1, T_1\rangle \longrightarrow \langle \Sigma_2, T_2\rangle$ are Sign-morphisms $f : \Sigma_1 \longrightarrow \Sigma_2$ such that $\mathrm{SEN}(f)(T_1) \subseteq T_2$. Let $\pi_2 : |\mathsf{Th}(\mathcal{I})| \longrightarrow \mathsf{Set}$ denote the projection onto the second coordinate. Now, we define a functor $\mathrm{SIG} : \mathsf{Th}(\mathcal{I}) \longrightarrow \mathsf{Sign}$ by $\mathrm{SIG}(\langle \Sigma, T\rangle) = \Sigma$ for every $\langle \Sigma, T\rangle \in |\mathsf{Th}(\mathcal{I})|$ and $\mathrm{SIG}(f) = f$ for every $f : \langle \Sigma_1, T_1\rangle \longrightarrow \langle \Sigma_2 T_2\rangle \in \mathrm{Mor}(\mathsf{Th}(\mathcal{I}))$.

The following relations between the categories of theories of two π-institutions will be useful in what follows.

Definition 2 Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}_2|}\rangle$ be two π-institutions. A functor $F : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ will be called *signature-respecting* if there exists a functor $F^\dagger : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$ such that the following rectangle commutes:

(1)
$$\begin{tikzcd}
\mathsf{Th}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathsf{Th}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\
\mathsf{Sign}_1 \arrow[r, "F^\dagger"'] & \mathsf{Sign}_2
\end{tikzcd}$$

If this is the case it is easy to verify that $F^\dagger$ is necessarily unique. F is said to be (*strongly*) *monotonic* if, for all $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathsf{Th}(\mathcal{I}_1)|$, $T_1 \subseteq T_1'$ (if and) only if $\pi_2(F(\langle \Sigma_1, T_1\rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1'\rangle))$. F is called *join-continuous* if, for all $\Sigma_1 \in |\mathsf{Sign}_1|$ and $\Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$, $(\bigcup_{\varphi \in \Phi} \pi_2(F(\langle \Sigma_1, \{\varphi\}^{\mathrm{c}}\rangle)))^{\mathrm{c}} = \pi_2(F(\langle \Sigma_1, \Phi^{\mathrm{c}}\rangle))$. A signature-respecting functor $F : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ will be said to *commute with substitutions* if, for every $f : \Sigma_1 \longrightarrow \Sigma_1' \in \mathrm{Mor}(\mathsf{Sign}_1)$, $\mathrm{SEN}_2(F^\dagger(f))(\pi_2(F(\langle \Sigma_1, T_1\rangle)))^{\mathrm{c}} = \pi_2(F(\langle \Sigma_1', \mathrm{SEN}_1(f)(T_1)^{\mathrm{c}}\rangle))$ for every $\langle \Sigma_1, T_1\rangle \in |\mathsf{Th}(\mathcal{I}_1)|$, where $F^\dagger : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$ is the (necessarily unique) functor of diagram (1).

The properties above may be extended to the case where the two categories of theories $\mathsf{Th}(\mathcal{I}_1)$ and $\mathsf{Th}(\mathcal{I}_2)$ are related via an adjunction. The following definition then applies:

Definition 3 An adjunction $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ will be called *signature-respecting* if both F and G are signature-respecting. It is said to be (*strongly*) *monotonic* if both F and G are (strongly) monotonic. It is said to be *join-continuous* if both F and G are join-continuous. Finally, a signature-respecting adjunction will be said to *commute with substitutions* if both F and G commute with substitutions.

Next, relations between π-institutions are reviewed with the goal of comparing relations between π-institutions with corresponding relations between their categories of theories.

Definition 4 Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}_2|}\rangle$ be two π-institutions. A *translation of* $\mathcal{I}_1$ *in* $\mathcal{I}_2$ is a pair $\langle F, \alpha\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$ consisting of a functor $F : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$ and a natural transformation $\alpha : \mathrm{SEN}_1 \longrightarrow \mathcal{P}\mathrm{SEN}_2 F$. A translation $\langle F, \alpha\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$ is an *interpretation of* $\mathcal{I}_1$ *in* $\mathcal{I}_2$ if, for all $\Sigma_1 \in |\mathsf{Sign}_1|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

(2) $\qquad \varphi \in \Phi^{\mathrm{c}}$ if and only if $\alpha_{\Sigma_1}(\varphi) \subseteq \alpha_{\Sigma_1}(\Phi)^{\mathrm{c}}$.

Using these notions the relation of deductive equivalence on π-institutions can be defined.

Definition 5 Let $\mathcal{I}_1$, $\mathcal{I}_2$ be two π-institutions, as above. $\mathcal{I}_1$ and $\mathcal{I}_2$ will be said to be *deductively equivalent* if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \longrightarrow \mathcal{I}_1$ such that

1. $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$ is an adjoint equivalence and
2. for all $\Sigma_1 \in |\mathsf{Sign}_1|$ and $\varphi \in \mathrm{SEN}_1(\Sigma_1)$,

(3) $\qquad \mathrm{SEN}_1(\eta_{\Sigma_1})(\varphi)^{\mathrm{c}} = \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\varphi))^{\mathrm{c}}$

and, for all $\Sigma_2 \in |\mathsf{Sign}_2|$ and $\psi \in \mathrm{SEN}_2(\Sigma_2)$,

(4) $\qquad \mathrm{SEN}_2(\varepsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))^{\mathrm{c}} = \{\psi\}^{\mathrm{c}}$.

To formulate the main result that will be the focus of our present study the definition of a term π-institution has to be introduced.

Definition 6 Let $\mathcal{I} = \langle \mathsf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}|}\rangle$ be a π-institution, $A \in |\mathsf{Sign}|$ and $p \in \mathrm{SEN}(A)$. $\langle A, p\rangle$ is called a *source signature-variable pair* if there exists a function

$$f : \{\langle \Sigma, \varphi\rangle : \Sigma \in |\mathsf{Sign}|,\ \text{and}\ \varphi \in \mathrm{SEN}(\Sigma)\} \longrightarrow |(A \,|\, \mathsf{Sign})|$$

such that, for all $\Sigma \in |\mathsf{Sign}|$ and for all $\varphi \in \mathrm{SEN}(\Sigma)$ we have $f_{\langle \Sigma, \varphi\rangle} : A \longrightarrow \Sigma$ and $\mathrm{SEN}(f_{\langle \Sigma, \varphi\rangle})(p) = \varphi$, and for all $\Sigma' \in |\mathsf{Sign}|$ and for all $g : \Sigma \longrightarrow \Sigma'$ we have $g f_{\langle \Sigma, \varphi\rangle} = f_{\langle \Sigma', \mathrm{SEN}(g)(\varphi)\rangle}$. A π-institution is called *term* if it has a source signature-variable pair. A Sign-object such as A will be called a *source signature* and a sentence such as p will be called a *source variable* or, simply, a *variable.*

The following diagrams illustrate the definition:

$$\begin{tikzcd}
A \arrow[rr, "f_{\langle\Sigma,\varphi\rangle}"] \arrow[dr, "f_{\langle\Sigma',\mathrm{SEN}(g)(\varphi)\rangle}"'] & & \Sigma \arrow[dl, "g"] \\
& \Sigma' &
\end{tikzcd}
\qquad
\begin{tikzcd}
p \arrow[rr, "\mathrm{SEN}(f_{\langle\Sigma,\varphi\rangle})"] \arrow[dr, "\mathrm{SEN}(f_{\langle\Sigma',\mathrm{SEN}(g)(\varphi)\rangle})"'] & & \varphi \arrow[dl, "\mathrm{SEN}(g)"] \\
& \mathrm{SEN}(g)(\varphi) &
\end{tikzcd}$$

The following constitutes one of the main theorems of [14, 15].

Theorem 7 *Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent, then there exists a signature-respecting adjoint equivalence $\langle F^\#, G^\#, \eta^\#, \varepsilon^\#\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ that commutes with substitutions. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are term and there exists a signature-respecting adjoint equivalence $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$, then $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent.*

In the next section, Theorem 7 will be strengthened by getting rid of the assumption that the two π-institutions be term in the last statement. Along the way, Theorems 8.17 and 9.21 of [15] will also be strengthened. The following will be finally proved, thus providing a complete analog of the characterization theorem for deductive systems of Blok and Pigozzi in the categorical abstract algebraic logic framework.

Theorem 8 *Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence between $\mathsf{Th}(\mathcal{I}_1)$ and $\mathsf{Th}(\mathcal{I}_2)$ that commutes with substitutions.*

3 Proof of theorem 8

Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions and let $F : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ be a signature-respecting functor. Let $F^\dagger : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$ be the functor of diagram (1). Define $\alpha^F : \mathrm{SEN}_1 \longrightarrow \mathcal{P}\mathrm{SEN}_2 F^\dagger$ for $\Sigma_1 \in |\mathsf{Sign}_1|$ by $\alpha^F_{\Sigma_1} : \mathrm{SEN}_1(\Sigma_1) \longrightarrow \mathcal{P}\mathrm{SEN}_2(F^\dagger(\Sigma_1))$ with

$$\alpha^F_{\Sigma_1}(\varphi) = \pi_2(F(\langle \Sigma_1, \{\varphi\}^{\mathrm{c}}\rangle)), \quad \text{for } \varphi \in \mathrm{SEN}_1(\Sigma_1). \tag{5}$$

This is well-defined since $\alpha^F_{\Sigma_1}(\varphi) \in \mathcal{P}\mathrm{SEN}_2(F^\dagger(\Sigma_1))$.

The following lemma modifies [15, Theorem 8.16(ii)]. More precisely, it modifies its assumption by eliminating the requirement that $\mathcal{I}_1$ be term at the expense of adding the condition that F commutes with substitutions.

Lemma 9 *Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions and $F : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ a signature-respecting functor that commutes with substitutions. Then $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$ is a translation.*

P r o o f. It suffices to show that $\alpha^F : \mathrm{SEN}_1 \longrightarrow \mathcal{P}\mathrm{SEN}_2 F^\dagger$ is a natural transformation. To this end, let $f : \Sigma_1 \longrightarrow \Sigma_1' \in \mathrm{Mor}(\mathsf{Sign}_1)$ and $\varphi \in \mathrm{SEN}_1(\Sigma)$. Then

$$\begin{tikzcd}
\mathrm{SEN}_1(\Sigma_1) \arrow[r, "\alpha^F_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}_1(f)"'] & \mathcal{P}\mathrm{SEN}_2(F^\dagger(\Sigma_1)) \arrow[d, "\mathcal{P}\mathrm{SEN}_2(F^\dagger(f))"] \\
\mathrm{SEN}_1(\Sigma_1') \arrow[r, "\alpha^F_{\Sigma_1'}"'] & \mathcal{P}\mathrm{SEN}_2(F^\dagger(\Sigma_1'))
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{P}\mathrm{SEN}_2(F^\dagger(f))(\alpha^F_{\Sigma_1}(\varphi)) &= \mathcal{P}\mathrm{SEN}_2(F^\dagger(f))(\pi_2(F(\langle\Sigma_1,\{\varphi\}^{\mathrm{c}}\rangle))) && \text{(by Definition (5)}\\
&= \pi_2(F(\langle\Sigma_1',\mathrm{SEN}_1(f)(\{\varphi\}^{\mathrm{c}})^{\mathrm{c}}\rangle)) && \text{(by commutativity with substitutions)}\\
&= \pi_2(F(\langle\Sigma_1',\mathrm{SEN}_1(f)(\varphi)^{\mathrm{c}}\rangle)) && \text{(by [15, Corollary 2.4])}\\
&= \alpha^F_{\Sigma_1'}(\mathrm{SEN}_1(f)(\varphi)) && \text{(by Definition (5)).}
\end{aligned}$$

□

The next lemma strengthens [15, Theorem 8.17(ii)].

Lemma 10 *Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions and let $F : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ be a strongly monotonic, join-continuous, signature-respecting functor that commutes with substitutions. Then $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$ is an interpretation.*

P r o o f . By Lemma 9, it suffices to show that, for all $\Sigma_1 \in |\mathsf{Sign}_1|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\varphi \in \Phi^{\mathrm{c}} \text{ if and only if } \alpha^F_{\Sigma_1}(\varphi) \subseteq \alpha^F_{\Sigma_1}(\Phi)^{\mathrm{c}}.$$

It is first shown that for all $\Sigma_1 \in |\mathsf{Sign}_1|$ and $\Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\alpha^F_{\Sigma_1}(\Phi)^{\mathrm{c}} = \pi_2(F(\langle\Sigma_1,\Phi^{\mathrm{c}}\rangle)). \tag{6}$$

Indeed,

$$\begin{aligned}
\alpha^F_{\Sigma_1}(\Phi)^{\mathrm{c}} &= (\textstyle\bigcup_{\varphi\in\Phi} \alpha^F_{\Sigma_1}(\varphi))^{\mathrm{c}}\\
&= (\textstyle\bigcup_{\varphi\in\Phi} \pi_2(F(\langle\Sigma_1,\{\varphi\}^{\mathrm{c}}\rangle)))^{\mathrm{c}} && \text{(by Definition (5))}\\
&= \pi_2(F(\langle\Sigma_1,\Phi^{\mathrm{c}}\rangle)) && \text{(by join-continuity).}
\end{aligned}$$

We have

$$\begin{aligned}
\alpha^F_{\Sigma_1}(\varphi) \subseteq \alpha^F_{\Sigma_1}(\Phi)^{\mathrm{c}} &\text{ iff } \alpha^F_{\Sigma_1}(\varphi)^{\mathrm{c}} \subseteq \alpha^F_{\Sigma_1}(\Phi)^{\mathrm{c}}\\
&\text{ iff } \pi_2(F(\langle\Sigma_1,\{\varphi\}^{\mathrm{c}}\rangle)) \subseteq \pi_2(F(\langle\Sigma_1,\Phi^{\mathrm{c}}\rangle)) && \text{by (6)}\\
&\text{ iff } \{\varphi\}^{\mathrm{c}} \subseteq \Phi^{\mathrm{c}} && \text{by strong monotonicity,}\\
&\text{ iff } \varphi \in \Phi^{\mathrm{c}}.
\end{aligned}$$

□

Next, recall from [15, Lemma 9.18] that, given two π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ and a signature-respecting adjunction $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$, then $\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,T_1'\rangle})$ for all $\langle\Sigma_1,T_1\rangle$, $\langle\Sigma_1,T_1'\rangle \in |\mathsf{Th}(\mathcal{I}_1)|$, i. e., the unit theory morphisms at the signature level are independent of the second component of a theory and only depend on the signature of the theory. The same holds for the counit morphisms. This allows the definition of $\eta^\dagger : I_{\mathsf{Sign}_1} \longrightarrow G^\dagger F^\dagger$ and $\varepsilon^\dagger : F^\dagger G^\dagger \longrightarrow I_{\mathsf{Sign}_2}$ by letting $\eta^\dagger_{\Sigma_1} : \Sigma_1 \longrightarrow G^\dagger(F^\dagger(\Sigma_1)$ denote the common value of $\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle})$ for all Σ_1-theories $\langle\Sigma_1,T_1\rangle$, and similarly for $\varepsilon^\dagger$. [15, Lemma 9.19] then shows that $\langle F^\dagger, G^\dagger, \eta^\dagger, \varepsilon^\dagger\rangle : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$ is an adjunction that is, in fact, an adjoint equivalence if $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ is a signature-respecting adjoint equivalence.

Following now mutatis mutandis the proof of [15, Theorem 9.21(ii)], but using Lemma 10 instead of [15, Theorem 8.17], [15, Theorem 9.21(ii)] may be strengthened to

Lemma 11 *Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions and let $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ be a strongly monotonic, join-continuous, signature-respecting adjoint equivalence that commutes with substitutions. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent via the interpretations $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$, $\langle G^\dagger, \beta^G\rangle : \mathcal{I}_2 \longrightarrow \mathcal{I}_1$, and the adjoint equivalence $\langle F^\dagger, G^\dagger, \eta^\dagger, \varepsilon^\dagger\rangle : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$.*

But, by [15, Lemma 10.25 and Lemma 10.24], it is known that a signature-respecting adjoint equivalence $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ that commutes with substitutions is necessarily strongly monotonic and join-continuous, respectively, whence the following theorem, a strengthening of [15, Theorem 10.26], is obtained:

Theorem 12 *Let $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathsf{Sign}_2|}\rangle$ be two π-institutions and $\langle F, G, \eta, \varepsilon\rangle : \mathsf{Th}(\mathcal{I}_1) \longrightarrow \mathsf{Th}(\mathcal{I}_2)$ a signature-respecting adjoint equivalence that commutes with substitutions. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent via the interpretations $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$, $\langle G^\dagger, \beta^G\rangle : \mathcal{I}_2 \longrightarrow \mathcal{I}_1$, and the adjoint equivalence $\langle F^\dagger, G^\dagger, \eta^\dagger, \varepsilon^\dagger\rangle : \mathsf{Sign}_1 \longrightarrow \mathsf{Sign}_2$.*

Combining this with [15, Theorem 9.21(i)], the strengthened criterion for the deductive equivalence of two π-institutions in terms of the existence of an adjoint equivalence between their categories of theories is obtained:

Theorem 13 *Let* $\mathcal{I}_1 = \langle \mathsf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}_1|} \rangle$ *and* $\mathcal{I}_2 = \langle \mathsf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathsf{Sign}_2|} \rangle$ *be two* π*-institutions. Then* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence between* $\mathsf{Th}(\mathcal{I}_1)$ *and* $\mathsf{Th}(\mathcal{I}_2)$ *that commutes with substitutions.*

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