Advances and Applications in Mathematical Sciences
Volume …, Issue …, 2014, Pages …

CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: CRYPTOFIBRING OF LOGICAL SYSTEMS

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Abstract

This work is the result of cross-fertilization between the area of logic dealing with combining logical systems, that has its main roots in computing science considerations, and the area of categorical abstract algebraic logic, where focus shifts from propositional to more complex systems, potentially involving multiple signatures and quantifiers. More specifically, on the former side, it is inspired by work of Caleiro and Ramos on cryptofibring, a method for combining semantically logical systems that provides a solution to the collapsing problem present in the more traditional method of fibring. On the side of algebraic logic, it is inspired by work previously carried out by the author in formalizing logical systems by representing their algebraic signatures as categories of natural transformations and replacing ordinary matrix semantics by the so-called matrix system semantics. This line of work goes back to the work of Lawvere on algebraic theories. In the present work the method of cryptofibring is extended to cover these more complex logical systems and several of the results of Caleiro and Ramos related to soundness, completeness and conservativeness are considered in this more general context.

1. Introduction

The need for versatility and adaptability of logical systems used in various diverse applications of computer science and artificial intelligence has led to a reinvigoration of the area of logic dealing with combinations of logical systems. Recently, various authors have revisited the methodology of fibring, first introduced by Gabbay [17, 18], in order to either provide a more widely

2010 Mathematics Subject Classification: Primary: 03B22, 03G27; Secondary: 03B35, 03G25, 03G30.

Keywords: combining logics, fibring, collapsing problem, cryptofibring, soundness, completeness, conservativeness, representability, institutions, categories of natural transformations, closure systems, abstract algebraic logic.

Received December 20, 2013

applicable and versatile mechanism of combining logics based on the original idea [25, 9, 6] or to study various logical properties of specific systems [7, 28] or classes of systems [5, 29] that result from applications of fibring. In the most important and interesting case where some of the connectives of the component systems are identified, there has been a realization that semantic fibring sometimes fails to yield a conservative extension of the constituent logical systems and this leads to the collapsing problem (see, e.g., [24, 8]).

The paradigmatic example of the collapsing problem has been the combination of intuitionistic and classical propositional logics. If the two implications, intuitionistic and classical, are identified, then in the "syntactic" fibring the two implications collapse into a single classical implication. However, as shown in [8], even if the two implications are kept separate and fibring is handled semantically, it may still happen that all models of the resulting logic interpret both resulting implication connectives as classical, i.e., some form of collapsing still occurs, thus, defeating, at least in part, the intended purpose of constructing the fibred logical system.

Several approaches have been devised to cope with this phenomenon. They strive to create an alternative logical system that avoids the collapsing problem. Among those techniques, that either depart substantially from fibring or modify it in a less radical way, one should mention the attempt of [13] at combining intuitionistic and classical logics by imposing syntactic restrictions on the instantiation of the axioms employed, the method of modulated fibring [24], replacing fibring by meet-combination of logics [26, 27], which, however, leads to a logic in which the shared connectives preserve only those properties that are common in both constituent logics, and cryptofibring [8], which is the work at the focus of our current investigations.

Since the treatment of cryptofibring by Caleiro and Ramos [8] constitutes the primary inspiration and the main starting point of the present work, we provide, next, a summary of its main ideas and results, together with some intuition of how they came about in the attempts to devise solutions for the collapsing problem.

In [8] Caleiro and Ramos deal with propositional logics that are defined either by a collection of rules of inference (including axioms) Hilbert-style or by classes of logical matrices over a fixed propositional signature. Their basic syntactic apparatus consists of two propositional languages C' and C'' that share a subset $\hat{C} = C' \cap C''$ of connectives and their union $\hat{C} = C' \cap C''$. In other words, it is assumed that the naming has been appropriately arranged a priori to take the intended sharing into account. We will follow an analogous understanding, since it simplifies the presentation and, modulo renaming, does not harm the generality. On the syntactic side the fibring consists essentially of taking the union $\mathcal{R}' \cup \mathcal{R}''$ of the rules of inference $\mathcal{R}'$ and $\mathcal{R}''$ determining the constituent propositional logics over C' and C'' respectively, Hilbert-style. The semantic fibring of a class $\mathcal{M}'$ of C'-logical matrices and of a class $\mathcal{M}''$ of C''-logical matrices gives rise to a class $\mathcal{M}' * \mathcal{M}''$ of C-logical matrices that have as algebraic components C-algebras that extend $C'-$ and C''-algebras over a common universe and that share the same sets of designated elements.

Using [4, 6] as their starting points, Caleiro and Ramos remind the reader in [8] that, if one starts with sound logical systems, then the syntactic fibring is sound with respect to the fibred semantics. More importantly, under fullness, a condition that ensures availability of enough models, completeness is also preserved. The drawback here, which is one of the main points of Caleiro and Ramos' analysis, is that, depending on context, collapsing may occur. This, moreover, happens in a natural formalization of the fibring of intuitionistic and classical propositional calculus, due to the requirement that algebraic universes and designated sets of elements of the fibred logical matrices must be shared.

The collapsing phenomenon motivates modifying the framework of fibring to obtain crypto-fibring, a more general way of combining models that circumvents the restrictions that are responsible for causing collapsing. On the syntactic side crypto-fibring does not entail any changes. Thus, the system obtained has still the combined signature $C = C' \cup C''$ with the intended sharing of $\hat{C} = C' \cap C''$ (again taking place automatically by appropriate arrangement). But on the semantic side, the crypto-fibred class $\mathcal{M}' \otimes \mathcal{M}''$ contains many more models than $\mathcal{M}' * \mathcal{M}''$ in fact, so many more that sometimes it is necessary to consider only its subclass of sound models $\mathcal{M}'_{\circledS} \mathcal{M}'' \subseteq \mathcal{M}' \otimes \mathcal{M}''$ to preserve soundness of the resulting logic with respect to this class. The clever natural device used in passing from $\mathcal{M}' * \mathcal{M}''$ to $\mathcal{M}' \otimes \mathcal{M}''$ is that, instead of just allowing combinations of models with the same universes and designated sets of elements, Caleiro and Ramos allow also $C'-$ and C''- model morphisms from $C'-$ and C''- models, respectively, to the $C'-$ and C''- reducts of C-models that preserve the designated sets of elements in a precise technical sense.

Without adding excessive conceptual complexity, this abstraction reaps a wealth of advantages. First, it preserves soundness. Second, subject to a representability condition on pairs of models from the constituent logical systems, it ensures conservativeness of the crypto-fibred logic with respect to both components and, therefore, it also preserves completeness. Representability is a technical condition that, roughly speaking, postulates the availability of enough models, in analogy with the fullness of the fibring context. Finally, the collapsing problem is avoided in some important examples of combinations, including the paradigmatic combination of intuitionistic and classical implicational propositional calculi, using crypto-fibring.

Caleiro and Ramos conclude their work in [8] by presenting an elegant necessary and sufficient condition for the representability of pairs of models based on an interesting construction of chains of congruences whose unions must respect and be compatible with designated sets. This latter condition is inspired by the fundamental role that compatibility plays in the theory of abstract algebraic logic, particularly emphasized both by the founders Wim Blok and Don Pigozzi in their seminal "Memoirs monograph" [2] and by other pioneers, such as Josep Maria Font and Ramon Jansana [15] and Janusz Czelakowski [10, 11] (see, also [16, 12]).

This connection of the field of combinations of logics with abstract algebraic logic has provided the motivation and the impetus for the work of the author in abstracting the methodology and the techniques employed in combining logical systems to a level general enough to encompass logical systems treated in *categorical abstract algebraic logic* [30, 31]. Those systems are formalized as institutions [19, 20] and/or π-institutions [14], since the institutional framework can accommodate successfully logics with multiple signatures and quantifiers rather than being restricted to only propositional logics. This line of research was already initiated in [35], where the work of Sernadas, Sernadas and Rasga [27] on meet-combinations of logics was lifted to encompass more general classes of logical systems. The targeted abstraction has become possible through the techniques pertaining to congruence systems, matrix systems and generalized compatibility, corresponding to ordinary congruences, matrices and compatibility in the propositional context (see, e.g., the very recent work [34]).

The paper is structured as follows: In Section 2 we recall the basic definitions pertaining to the underlying structures over which we build the theory. The most fundamental is that of a logical system or logic, which consists of a signature category, a sentence functor equipped with a category of natural transformations, a collection of inference rules that defines a proof system, as well as a class of matrix system models that defines a closure system on the sentence functor semantically. The sentence functor and natural transformation setting follow similar concepts employed in categorical abstract algebraic logic, that have their origins in work of Lawvere on algebraic theories [21] (see, also, [23]). The use of inference rules in this setting goes back to [33], whereas the matrix system models form an adaptation of the ordinary logical matrices from abstract algebraic logic that better fit the categorical framework. In Section 3 we begin our work towards the study of cryptofibring by first introducing a type of fibring of logical systems and showing that the fibring of two sound logical systems is sound and that the fibring of two full logical systems is full, which implies that it is also complete. These results form analogs in this more abstract setting of corresponding results established for propositional logics by Caleiro and Ramos in Section 3 of [8]. In Section 4 several key concepts at the heart of crypto-fibring are defined and studied. The concept of crypto-morphism between algebraic systems, as employed in categorical abstract algebraic logic, is introduced, taking after the corresponding concept of Caleiro and Ramos. Using crypto-morphisms, we define the concept of a crypto-extension of two matrix system models and, in turn, the crypto-fibring of two logical systems. As is ordinarily the case, the original notion of Caleiro and Ramos is easily seen to be a special case of this more general categorical notion. Since the special case of [8] already exhibits the potential of spoiling soundness in crypto-fibring, we also introduce the sound crypto-fibring of logical systems at the present level, expecting it to have similar taming influence in avoiding overextending the collection of crypto-fibred models in a way that adversely affects soundness. In Section 5, the notion of representability of pairs of models is adapted to the categorical context and it is shown that, subject to the condition that every pair of models from the constituent logical systems be represented in the crypro-fibred system, the latter's semantic entailment relation is a conservative extension of the semantic entailments of both original systems. The most interesting and challenging results, inspired by corresponding results of [8] for propositional logics, are encountered in Section 6. Namely, two important necessary conditions for the representability of a pair of models in the crypto-fibred systems are formulated. Roughly speaking, the first asserts that the morphisms of the represented models must respect the corresponding families of the designated sets of elements: a formula is assigned a designated value in one system iff it is assigned a designated value in the other. Necessity arises from the fact that crypto-morphisms are stipulated to preserve designated sets of elements. Again roughly speaking, the second necessary condition exhibits congruence relations on the represented models that "mesh" nicely and that are compatible with the corresponding designated systems. "Meshing nicely" here expresses the fact that, in some sense, the pairs in each congruence include the pairs in the other under intended identifications. The remaining three sections of the paper exploit this necessary condition to unveil how it can be more intrinsically formulated in terms of the represented systems, i.e., without recourse to a specific cryptoextension, and to use the ensuing intrinsic formulation to obtain a sufficient condition for representability. Sections 7 and 8 are rather technical and prepare the prerequisite groundwork, while the main work, culminating in Proposition 17, are presented in Section 9.

2. The Underlying Framework

For standard categorical notation that will mostly remain unexplained, we refer the reader to any of the standard references [1, 3, 22].

In the sequel we consider an arbitrary but fixed category **Sign**, called the **category of signatures**, and an arbitrary but fixed **Set**-valued functor $\mathrm{SEN}: \mathbf{Sign} \to \mathbf{Set}$, called the **sentence functor**. Also into the picture in a critical way will be one or more categories of natural transformations on SEN, usually denoted by the letter N, with possible intonation signs, sub-or super-scripts, hats, etc. Such a category N, which, roughly speaking, is intended to represent the clone of all algebraic operations on SEN of interest in a specific context, is defined as follows (see, e.g., [34]). The **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$, $i < k$, $k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

An N-**rule of inference** or simply an N-**rule** is a pair of the form $\langle \{\sigma^0, \ldots, \sigma^{n-1}\}, \tau \rangle$, sometimes written $\dfrac{\sigma^0, \ldots, \sigma^{n-1}}{\tau}$, where $\sigma^0, \ldots, \sigma^{n-1}, \tau$ are natural transformations in N. The elements σ^i, $i < n$, are called the **premises** and τ the **conclusion** of the rule.

An N-**Hilbert calculus** $\mathcal{R}$ is a set of N-rules. Using the N-rules in $\mathcal{R}$, one may define **derivations** of a natural transformation σ in N from a set Δ of natural transformations in N. Such a derivation is denoted by $\Delta \vdash^{\mathcal{R}} \sigma$. If the calculus $\mathcal{R}$ is fixed and clear in a particular context, we might simply write $\Delta \vdash \sigma$.

Given two functors $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with categories of natural transformations N, N' on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, a pair $\langle F, \alpha \rangle$, where $F : \mathbf{Sign} \to \mathbf{Sign}'$ is a functor and $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$ is a natural transformation, is called a **translation** from SEN to SEN′ Moreover, it is said to be $(N,\ N')$- **epimorphic** if, there exists a correspondence $\sigma \mapsto \sigma'$ between the natural transformations in N and N' that preserves projections (and, thus, also arities), such that, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$, all $\Sigma \in |\,\mathbf{Sign}\,|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha_\Sigma^k(\vec{\phi})).$$

An $(N,\ N')$-epimorphic translation from SEN to SEN′ will be denoted by $\langle F,\ \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$, with the relevant categories N, N' of natural transformations on SEN, SEN′, respectively, understood from context.

Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a category N of natural transformations on SEN, an N-**algebraic system** $\mathcal{A} = \langle \mathrm{SEN}',\ \langle F,\ \alpha \rangle \rangle$ consists of

- a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ with a category N' of natural transformations on SEN′;
- an $(N,\ N')$- epimorphic translation $\langle F,\ \alpha \rangle : \mathbf{SEN} \to \mathbf{SEN}'$.

An N-**matrix system** or, simply, N-**matrix** $\mathfrak{A} = \langle \mathcal{A},\ T \rangle$ is a pair consisting of

- an N-algebraic system $\mathcal{A} = \langle \mathrm{SEN}',\ \langle F,\ \alpha \rangle \rangle$;
- an axiom family $T \in \mathrm{AxFam}(\mathrm{SEN}')$ on SEN′ i.e., a collection $T = \{T_\Sigma\}_{\Sigma \in |\,\mathbf{Sign}'\,|}$ of subsets $T_\Sigma \subseteq \mathrm{SEN}'(\Sigma)$, $\Sigma|\,\mathbf{Sign}'\,|$.

We perceive of the elements of $\mathrm{SEN}'(F(\Sigma))$ as **truth values** for evaluating the natural transformations in N and those of $T_{F(\Sigma)}$ as being the

designatedones. An N-**matrix** (**system**) **semantics** $\mathcal{M}$ is a class of N-matrix systems. Given a natural transformation $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, we set

$$\sigma_{\Sigma'}(f(\vec{\phi})) := \sigma_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi})),$$

where $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$. The matrix $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ **satisfies** σ **at** $\vec{\phi} \in SEN(\Sigma)^k$ **under** $f \in \mathbf{Sign}(\Sigma, \Sigma')$, written $\mathfrak{A} \vDash_{\Sigma} \sigma[\vec{\phi}, f]$, if $\alpha_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi}))) \in T_{F(\Sigma')}$. A N-rule $\dfrac{\sigma^0, \ldots, \sigma^{n-1}}{\tau}$ is a **rule of** an N-matrix semantics $\mathcal{M}$, written

$$\sigma^0, \ldots, \sigma^{n-1} \vDash^{\mathcal{M}} \tau,$$

if $\mathfrak{A} \vDash_{\Sigma} \sigma^i[\vec{\phi}, f]$, for all $i < n$, implies $\mathfrak{A} \vDash_{\Sigma} \tau[\vec{\phi}, f]$, for every N-matrix $\mathfrak{A} \in \mathcal{M}$, all $\Sigma \in |\mathbf{Sign}|$, all Σ-assignments $\vec{\phi}$ in $\mathfrak{A}$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$. If the semantics is clear from context, we simply write $\sigma^0, \ldots, \sigma^{n-1} \vDash \tau$.

In the remainder of this paper, by a **logical system**, or simply a **logic**, we understand a pentuple $\mathcal{L} = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M} \rangle$, where

- **Sign** is a category;
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a sentence functor (with domain **Sign**);
- N is a category of natural transformations on SEN;
- $\mathcal{R}$ is an N-Hilbert calculus and
- $\mathcal{M}$ is a N-matrix system semantics.

3. Fibring of Logical Systems

Throughout our discussion, we are dealing with an arbitrary but fixed category **Sign** of signatures and an arbitrary but fixed sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$. Given two arbitrary categories N' and N'' of natural transformations on SEN, we denote by $N' \cap N''$ the category of natural transformations on SEN consisting, for all $k, l \in \omega$, of all those natural transformations $\mathrm{SEN}^k \to \mathrm{SEN}^l$ that are in both N' and N''. It is not difficult to show that this definition makes sense, because the included morphisms do satisfy the axioms of a category of natural transformations.

We consider two logical systems $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ and $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$. We let $\hat{N} = N' \cap N''$ and denote by $N = \langle N', N'' \rangle$ the least category of natural transformations on SEN including N' and N'', which we call the **category of natural transformations on** SEN generated by N' and N'' :

$$\begin{tikzcd} & N & \\ N' \arrow[ur] & & N'' \arrow[ul] \\ & \widehat{N} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{1}$$

This definition also makes sense because there exists a largest category of natural transformations on SEN and the class of all categories of natural transformations on SEN is closed under intersection.

The intended goal of N is to model in this abstract framework of clones the concept of disjoint union of the signatures N' and N'' subject to sharing of the common sub-signature $\hat{N}$, as is done in constrained fibring and cryptofibring in the sentential context (see, e.g., [25, 8]).

The following lemma provides an alternative (constructive) characterization of the category $\langle N', N'' \rangle$.

Lemma 1. *Suppose that* **Sign** *is a category,* $SEN : \mathbf{Sign} \to \mathbf{Set}$ *is a functor and* N', N'' *are categories of natural transformations on* SEN. *Then, the category* $N = \langle N', N'' \rangle$ *of natural transformations on* SEN *generated by* N' *and* N'' *has, for all* $k, l \in \omega$, *as its morphisms* $\mathrm{SEN}^k \to \mathrm{SEN}^l$, *all* l*-tuples of natural transformations* $\mathrm{SEN}^k \to \mathrm{SEN}$, *that are built recursively according to the following rules*:

1. *All natural transformations* $\mathrm{SEN}^k \to \mathrm{SEN}$, *in* N' *or* N'' *are in* N;

2. *For all* $\tau^0, \ldots, \tau^{m-1} : \mathrm{SEN}^k \to \mathrm{SEN}$ *in* N, $\langle \tau^0, \ldots, \tau^{m-1} \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^m$, *is also in* N;

3. *For all* $\sigma' : \mathrm{SEN}^m \to \mathrm{SEN}$ *in* N' *and all* $\tau^0, \ldots, \tau^{m-1} : \mathrm{SEN}^k \to \mathrm{SEN}$ *in* N, $\sigma'(\tau^0, \ldots, \tau^{m-1}) : \mathrm{SEN}^k \to \mathrm{SEN}$ *is in* N;

4. *For all* $\sigma'' : \mathrm{SEN}^m \to \mathrm{SEN}$ *in* N'' *and all* $\tau^0, \ldots, \tau^{m-1} : \mathrm{SEN}^k \to \mathrm{SEN}$ *in* N, $\sigma''(\tau^0, \ldots, \tau^{n-1}) : \mathrm{SEN}^k \to \mathrm{SEN}$ *is in* N.

Proof. The category constructed recursively as above is a category of natural transformations on SEN and it contains N' and N''. Thus, it includes $\langle N', N'' \rangle$. On the other hand, since $\langle N', N'' \rangle$ includes all natural transformations in N' and all natural transformations in N'' and is closed under compositions and formation of tuples, it must certainly include all natural transformations built recursively according to the Rules 1-4 enumerated above.

The **fibring** of $\mathcal{L}'$ and $\mathcal{L}''$ **constrained by the sharing** of $\hat{N}$ (as in Diagram (1)) or, more simply, the **constrained fibring** of $\mathcal{L}'$ and $\mathcal{L}''$, is the logical system

$$\mathcal{L}' * \mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}^* \rangle$$

defined as follows:

The **signature category** and the **sentence functor** are the common signature category and sentence functor of the two systems $\mathcal{L}'$ and $\mathcal{L}''$. The **category of natural transformations** is $N = \langle N', N'' \rangle$.

The **set of** N-**rules** $\mathcal{R}$ is the union $\mathcal{R}' \cup \mathcal{R}''$ of the sets of N'-rules $\mathcal{R}'$ and N''-rules $\mathcal{R}''$, which may be viewed as N-rules because of the inclusion of N' and N'', respectively, in N.

Suppose, now, that $\mathfrak{A}' = \langle \mathcal{A}', T \rangle \in \mathcal{M}'$ and $\mathfrak{A}'' = \langle \mathcal{A}'', T \rangle \in \mathcal{M}''$ are matrix system models of $\mathcal{L}'$ and $\mathcal{L}''$, respectively, such that

- $\mathcal{A}' = \langle SEN^{\mathcal{A}}, \langle F, \alpha \rangle \rangle$, with a category $N'^{\mathcal{A}}$ of natural transformations on $\mathrm{SEN}^{\mathcal{A}}$, is an N'-algebraic system, and $\langle F, \alpha \rangle$ an $(N', N'^{\mathcal{A}})$-algebraic morphism;

- $\mathcal{A}'' = \langle SEN^{\mathcal{A}}, \langle F, \alpha\rangle\rangle$, with a category $N''^{\mathcal{A}}$ of natural transformations on $\mathrm{SEN}^{\mathcal{A}}$, is an N''-algebraic system, and $\langle F, \alpha\rangle$ an $(N'', N''^{\mathcal{A}})$-algebraic morphism;

- there exists a category $N^{\mathcal{A}}$ of natural transformations on $\mathrm{SEN}^{\mathcal{A}}$, extending both $N'^{\mathcal{A}}$ and $N''^{\mathcal{A}}$, such that $\mathcal{A} = \langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha\rangle\rangle$, with $N^{\mathcal{A}}$ on $\mathrm{SEN}^{\mathcal{A}}$, is an N-algebraic system, and $\langle F, \alpha\rangle$ an $(N, N^{\mathcal{A}})$-algebraic morphism.

In this case we call $\mathfrak{A}' \in \mathcal{M}'$ and $\mathfrak{A}'' \in \mathcal{M}''$ **compatible models**. Given two such models, we let $\mathfrak{A}' * \mathfrak{A}'' = \langle \mathcal{A}, T\rangle$. We then define

$$\mathcal{M}^* = \{\mathfrak{A}' * \mathfrak{A}'' : \mathfrak{A} \in \mathcal{M}' \text{ and } \mathfrak{A}'' \in \mathcal{M}'' \text{ compatible}\}.$$

A straightforward lemma relates satisfiability in $\mathfrak{A}'$ (and $\mathfrak{A}''$) with satisfiability in $\mathfrak{A}' * \mathfrak{A}''$, in case $\mathfrak{A}'$ and $\mathfrak{A}''$ are compatible matrix systems in $\mathcal{M}'$ and $\mathcal{M}''$ respectively.

Lemma 2. *Suppose* $\sigma = \sigma'(\tau^0, \ldots, \tau^{m-1})$ *is a natural transformation in* N, *such that* σ' *is in* N'. *Then, for all compatible matrix systems* $\mathfrak{A}' \in \mathcal{M}'$ *and* $\mathfrak{A}'' \in \mathcal{M}''$, *all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, *and all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\mathfrak{A}' * \mathfrak{A}'' \vDash_{\Sigma} \sigma[\vec{\phi}, f] \textit{ iff } \mathfrak{A}' \vDash_{\Sigma} \sigma'[\tau_{\Sigma}(\vec{\phi}), f],$$

where $\tau_{\Sigma}(\vec{\phi}) = \langle \tau_{\Sigma}^0(\vec{\phi}), \ldots, \tau_{\Sigma}^{m-1}(\vec{\phi})\rangle$.

Proof.

$$\begin{aligned}
\mathfrak{A}' * \mathfrak{A}'' \vDash_{\Sigma} \sigma[\vec{\phi}, f] &\text{ iff } \alpha_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi}))) \in T_{F(\Sigma')} \\
&\text{ iff } \alpha_{\Sigma'}(\sigma'_{\Sigma'}(\tau_{\Sigma'}^0(f(\vec{\phi})), \ldots, \tau_{\Sigma'}^{m-1}(f(\vec{\phi})))) \in T_{F(\Sigma')} \\
&\text{ iff } \alpha_{\Sigma'}(\sigma'_{\Sigma'}(f(\tau_{\Sigma}^0(\vec{\phi})), \ldots, f(\tau_{\Sigma}^{m-1}(\vec{\phi})))) \in T_{F(\Sigma')} \\
&\text{ iff } \mathfrak{A}' \vDash_{\Sigma} \sigma'[\tau_{\Sigma}(\vec{\phi}), f].
\end{aligned}$$

Lemma 3. *Suppose* $\sigma = \sigma''(\tau^0, \ldots, \tau^{m-1})$ *is a natural transformation in* N, *such that* σ'' *is in* N''. *Then, for all compatible matrix systems* $\mathfrak{A}' \in \mathcal{M}'$ *and* $\mathfrak{A}'' \in \mathcal{M}''$, *all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, *and all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\mathfrak{A}' * \mathfrak{A}'' \vDash_{\Sigma} \sigma[\vec{\phi}, f] \text{ iff } \mathfrak{A}'' \vDash_{\Sigma} \sigma''[\tau_{\Sigma}(\vec{\phi}), f].$$

Proof. Following similar steps as in the proof of Lemma 2.

It is not difficult to show that the constrained fibring $\mathcal{L}' * \mathcal{L}''$ of two sound logical systems is also a sound logical system. This is an analog of Proposition 3.3 of [8].

Proposition 4. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems over the same sentence functor* SEN *and* $N = \langle N', N'' \rangle$. *If* $\mathcal{L}', \mathcal{L}''$ *are sound, then the system* $\mathcal{L}' * \mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}^* \rangle$ *is also sound.*

Proof. It suffices to show that every instance of every axiom and of every rule of inference in $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$ is sound with respect to all matrix systems in $\mathcal{M}^*$. We show this only for rules of inference in $\mathcal{R}'$, since using an analogous argument yields a proof for axioms and a completely symmetric argument establishes the corresponding statement for axioms and rules in $\mathcal{R}''$.

Suppose that $\dfrac{\sigma^0, \ldots, \sigma^{n-1}}{\sigma}$ is an N'-rule in $\mathcal{R}'$. Consider an instance of this rule $\dfrac{\sigma^0(\tau), \ldots, \sigma^{n-1}(\tau)}{\sigma(\tau)}$ in $\mathcal{R}$, i.e., $\tau = \langle \tau^0, \ldots, \tau^{k-1} \rangle$ is a k-tuple of natural transformations $\mathrm{SEN}^l \to \mathrm{SEN}$ in N. Assume that $\mathfrak{A}' * \mathfrak{A}'' \in \mathcal{M}^*$, $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^l$, such that $\mathfrak{A}' * \mathfrak{A}'' \vDash_{\Sigma} \sigma^i(\tau)[\vec{\phi}, f]$, for all $i < n$. Then, by Lemma 2, $\mathfrak{A}' \vDash_{\Sigma} \sigma^i[\tau_{\Sigma}(\vec{\phi}), f]$, for all $i < n$, whence, since $\dfrac{\sigma^0(\tau), \ldots, \sigma^{n-1}(\tau)}{\sigma(\tau)}$ is in $\mathcal{R}$, we get that $\mathfrak{A}' \vDash_{\Sigma} \sigma[\tau_{\Sigma}(\vec{\phi}), f]$. A new application of Lemma 2 yields that $\mathfrak{A}' * \mathfrak{A}'' \vDash_{\Sigma} \sigma[\vec{\phi}, f]$. Therefore, the given N-instance $\dfrac{\sigma^0(\tau), \ldots, \sigma^{n-1}(\tau)}{\sigma(\tau)}$ of the N'-rule $\dfrac{\sigma^0, \ldots, \sigma^{n-1}}{\sigma}$ in $\mathcal{R}'$ is indeed sound for every model $\mathfrak{A}' * \mathfrak{A}'' \in \mathcal{M}^*$, showing that $\mathcal{L}' * \mathcal{L}''$ is a sound logical system.

In the sequel we focus on a special class of logical systems, called Lindenbaum systems. Roughly speaking, they are characterized by the weak completeness property that, every N-rule that is valid in all Lindenbaum matrix system models of $\mathcal{R}$ is $\mathcal{R}$-provable. We then define fullness for logical systems as a means to obtain that a Lindenbaum logical system is complete with respect to the class of its matrix system models. Finally, we show that fullness is preserved by fibring, thus, obtaining an indirect, but quite elegant way of ensuring that the fibring of two sound and full Lindenbaum logical systems is a sound and full and, hence, also a complete logical system.

Consider a logical system $\mathcal{L} = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}\rangle$. The collection $\mathcal{R}$ of N-inference rules specifies a closure system $C^{\mathcal{R}}$ on SEN (see, e.g., [33, 34] for details). We consider the theory families $\mathrm{ThFam}(\mathcal{I}^{\mathcal{R}})$ of the π-institution $\mathcal{I}^{\mathcal{R}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathcal{R}}\rangle$ and form the collection of Lindenbaum models

$$\mathcal{M}^{\mathcal{R}} = \{\mathfrak{A}^T : T \in \mathrm{ThFam}(\mathcal{I}^{\mathcal{R}})\},$$

where $\mathfrak{A}^T = \langle\langle SEN, \langle I_{\mathbf{Sign}}, \iota\rangle\rangle, T\rangle$. (Here $I_{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}$ is the identity functor and $\iota : \mathrm{SEN} \to \mathrm{SEN}$ is the identity natural transformation). We call the logical system $\mathcal{L}$ a **Lindenbaum system** and say that it satisfies the **Lindenbaum property**, if, for all $\Delta \cup \{\sigma\}$ in N,

$$\Delta \vDash^{\mathcal{M}^{\mathcal{R}}} \sigma \text{ implies } \Delta \vdash^{\mathcal{R}} \sigma. \tag{2}$$

A logical system $\mathcal{L} = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}\rangle$ is called **full** if it contains all N-matrix systems $\mathfrak{A} = \langle\langle \mathrm{SEN}^{\mathcal{A}}, \langle \mathrm{F}, \alpha\rangle\rangle, T\rangle$ satisfying all N-rules in $\mathcal{R}$. It is not very difficult to show that, due to the fact that fullness forces a logical system to contain all Lindenbaum models, any full Lindenbaum logical system is also complete.

Proposition 5. *A full Lindenbaum logical system* $\mathcal{L} = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}\rangle$ *is complete.*

Proof. Consider again the theory families $\mathrm{ThFam}(\mathcal{I}^{\mathcal{R}})$ of the π-institution $\mathcal{I}^{\mathcal{R}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathcal{R}}\rangle$, where $C^{\mathcal{R}}$ is the closure system on SEN specified by $\mathcal{R}$, and form the collection of Lindenbaum models

$$\mathcal{M}^{\mathcal{R}} = \{\mathfrak{A}^T : T \in \mathrm{ThFam}(\mathcal{I}^{\mathcal{R}})\},$$

where $\mathfrak{A}^T = \langle\langle \mathrm{SEN}, \langle I_{\mathbf{Sign}}, \iota\rangle\rangle, T\rangle$. Since, for all $T \in \mathrm{ThFam}(\mathcal{I}^{\mathcal{R}})$, T is closed under $\mathcal{R}$, we have, by fullness, that $\mathfrak{A}^T \in \mathcal{M}$. Therefore, $\mathcal{M}^{\mathcal{R}} \subseteq \mathcal{M}$.

If $\Delta \cup \{\sigma\}$ is a set of natural transformations in N, such that $\Delta \nvdash^{\mathcal{R}} \sigma$, then, by Condition (2), there exists $T \in \mathrm{ThFam}(\mathcal{I}^{\mathcal{R}})$, $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, such that $\mathfrak{A}^T \vDash_{\Sigma} \Delta[\vec{\phi}, f]$ and $\mathfrak{A}^T \nvDash_{\Sigma} \sigma[\vec{\phi}, f]$. Thus, $\mathfrak{A}^T \nvDash^{\mathcal{M}^{\mathcal{R}}} \sigma$ and, a fortiori, $\mathfrak{A}^T \nvDash^{\mathcal{M}} \sigma$. This shows that $\mathcal{L}$ is a complete logical system.

In addition, fullness has the attractive property of being preserved under fibring, i.e., as the following lemma asserts, the fibring of two full logical systems is also a full logical system.

Lemma 6. *Suppose* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}'\rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}''\rangle$ *are two logical systems over the same sentence functor* SEN *and* $N = \langle N', N''\rangle$. *If* $\mathcal{L}', \mathcal{L}''$ *are full, then* $\mathcal{L}' * \mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}^*\rangle$ *is also full.*

Proof. Suppose that $\mathfrak{A} = \langle\langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha\rangle\rangle, T\rangle$ is an N-matrix system that is sound for $\mathcal{R}' \cup \mathcal{R}''$. Then, clearly, $\mathfrak{A}$ is sound for both $\mathcal{R}'$ and $\mathcal{R}''$. Therefore, by the fullness of $\mathcal{L}'$ and of $\mathcal{L}''$, we get that the $N'^{\mathcal{A}}$-reduct $\mathfrak{A}' = \langle\langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha\rangle\rangle, T\rangle$ of $\mathfrak{A}$ is in $\mathcal{M}'$ and that the $N''^{\mathcal{A}}$-reduct $\mathfrak{A}'' = \langle\langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha\rangle\rangle, T\rangle$ of $\mathfrak{A}$ is in $\mathcal{M}''$. This, however, shows that $\mathfrak{A} = \mathfrak{A}' * \mathfrak{A}'' \in \mathcal{M}^*$ and, hence, $\mathcal{L}' * \mathcal{L}''$ is also a full logical system.

Lemma 6, put together with Proposition 5, yields the following result to the effect that the fullness of the components ensures the completeness of the fibred system, subject to the requirement that it be a Lindenbaum system.

Proposition 7. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems over the same sentence functor* SEN *and* $N = \langle N', N'' \rangle$. *If* $\mathcal{L}', \mathcal{L}''$ *are full, then, if the logic* $\mathcal{L}' * \mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N$ $\mathcal{R}, \mathcal{M}^* \rangle$ *satisfies the Lindenbaum property, it is complete.*

Proof. By Lemma 6, $\mathcal{L}' * \mathcal{L}''$ is full and, therefore, since it is Lindenbaum, by Proposition 5, it is complete.

4. Cryptofibring of Logical Systems

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a functor. Consider a category N'' of natural transformations on SEN and a subcategory N' of N'', which is itself a category of natural transformations on SEN. Given an N'-matrix system $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ with $\mathcal{A}' = \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle$, and an N''-matrix system $\mathfrak{A}'' = \langle \mathcal{A}'', T'' \rangle$, with $\mathcal{A}'' = \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle$, a **cryptomorphism** $\langle F, \alpha \rangle : \mathfrak{A}' \to \mathfrak{A}''$ is an N'-algebraic morphism $\langle F, \alpha \rangle : \mathcal{A}' \to \mathcal{A}''$, satisfying the additional property that $T' = \alpha^{-1}(T'')$. Note that $\langle F, \alpha \rangle : \mathcal{A}' \to \mathcal{A}''$ being an N'-algebraic morphism involves by definition the commutativity of the following diagram:

$$\begin{tikzcd}
 & \mathrm{SEN} \arrow[dl, "{\langle F', \alpha' \rangle}"'] \arrow[dr, "{\langle F'', \alpha'' \rangle}"] & \\
\mathrm{SEN}' \arrow[rr, "{\langle F, \alpha \rangle}"'] & & \mathrm{SEN}''
\end{tikzcd}$$

The morphism $\langle F, \alpha \rangle : \mathcal{A}' \to \mathcal{A}''$ being an N'-algebraic morphism makes sense, since N' is assumed to be a subcategory of natural transformations of the category N'' of natural transformations and $\mathcal{A}''$ is assumed to be an N''-algebraic system. Moreover, recall that the condition $T' = \alpha^{-1}(T'')$ means that, for all $\Sigma' \in |\mathbf{Sign}'|$, $T'_{\Sigma'} = \alpha^{-1}_{\Sigma'}(T''_{F(\Sigma')})$.

Once more, we consider a fixed but arbitrary category of signatures **Sign** and an arbitrary but fixed functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$. Let $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ and $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ be two logical systems and let $N = \langle N', N'' \rangle$ be, as before, the category of natural transformations on SEN generated by N' and N'' subject to sharing $\hat{N} = N' \cap N''$. We define the **crypto-fibring** of $\mathcal{L}'$ and $\mathcal{L}''$ **constrained by the sharing** of $\hat{N}$ or, more simply, the **constrained crypto-fibring** of $\mathcal{L}'$ and $\mathcal{L}''$ as the logical system

$$\mathcal{L}' \otimes \mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}^{\otimes} \rangle$$

defined as follows:

The **signature category** and the **sentence functor** are the common signature category and sentence functor of the two systems $\mathcal{L}'$ and $\mathcal{L}''$. The **category of natural transformations** is $N = \langle N', N'' \rangle$.

The **set of N-rules** $\mathcal{R}$ is the union $\mathcal{R}' \cup \mathcal{R}''$ of the sets of N'-rules $\mathcal{R}'$ and N''-rules $\mathcal{R}''$. Again, recall that, as was the case in constrained fibring, these rules may be viewed as N-rules because of the inclusion of N' and N'' in N.

Let $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle \in \mathcal{M}'$ and $\mathfrak{A}'' = \langle \mathcal{A}'', T'' \rangle \in \mathcal{M}''$ be models of $\mathcal{L}'$ and $\mathcal{L}''$, respectively, such that

- $\mathcal{A}' = \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle$, with $N'^{\mathcal{A}}$ a category of natural transformations on SEN', is an N'-algebraic system, and $\langle F', \alpha' \rangle$ an $(N', N'^{\mathcal{A}})$-algebraic morphism and

- $\mathcal{A}'' = \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle$, with $N''^{\mathcal{A}}$ a category of natural transformations on SEN'', is an N''-algebraic system, and $\langle F'', \alpha'' \rangle$ an $(N'', N''^{\mathcal{A}})$-algebraic morphism.

$$\begin{tikzcd} \mathrm{SEN}' & & \mathrm{SEN}'' \\ & \mathrm{SEN} \arrow[ul, "{\langle F',\alpha' \rangle}"] \arrow[ur, "{\langle F'',\alpha'' \rangle}"'] & \end{tikzcd}$$

Then, a **crypto-extension of** $\mathfrak{A}', \mathfrak{A}''$ is a triple $\langle \mathfrak{A}, \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$, where

- $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ is an N-matrix system, such that $\mathcal{A} = \langle SEN^{\mathcal{A}}, \langle F, \alpha \rangle \rangle$, with $N^{\mathcal{A}}$ a category of natural transformations on $\mathrm{SEN}^{\mathcal{A}}$ is an N-algebraic system, and $\langle F, \alpha \rangle$ an $(N, N^{\mathcal{A}})$-algebraic morphism;

- $\langle G', \beta' \rangle : \mathfrak{A}' \to \mathfrak{A}$ and $\langle G'', \beta'' \rangle : \mathfrak{A}'' \to \mathfrak{A}$ are crypto-morphisms which, by definition, have to make the following diagram commute:

$$\begin{tikzcd} & \mathrm{SEN}^{\mathcal{A}} & \\ \mathrm{SEN}' \arrow[ur, "\langle G', \beta' \rangle"] & & \mathrm{SEN}'' \arrow[ul, "\langle G'', \beta'' \rangle"'] \\ & \mathrm{SEN} \arrow[ul, "\langle F', \alpha' \rangle"] \arrow[ur, "\langle F'', \alpha'' \rangle"'] \arrow[uu, "\langle F, \alpha \rangle"] & \end{tikzcd} \tag{3}$$

We set $\mathfrak{A}' \otimes \mathfrak{A}'' = \{\mathfrak{A} : \langle \mathfrak{A}, \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$ is a crypto-extension of $\mathfrak{A}', \mathfrak{A}''\}$, the class of all matrix system components of crypto-extensions of $\mathfrak{A}'$ and $\mathfrak{A}''$, and, moreover, we set

$$\mathcal{M}^{\otimes} = \bigcup \{\mathfrak{A}' \otimes \mathfrak{A}'' : \mathfrak{A}' \in \mathcal{M}', \mathfrak{A}'' \in \mathcal{M}''\}.$$

As the observant reader might have expected, it is the case, as was also in the context of ordinary crypto-fibring of sentential logics of [8], that the class of all crypto-models includes the class of all fibred models. The following proposition forms an analog of Proposition 4.3 of [8].

Proposition 8. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems and* $N = \langle N', N'' \rangle$. *Then, the class of models* $\mathcal{M}^*$ *is a subclass of the class of models* $\mathcal{M}^{\otimes}$.

Proof. If $\mathfrak{A}' \in \mathcal{M}'$ and $\mathfrak{A}'' \in \mathcal{M}''$ are compatible, and $\mathfrak{A}' * \mathfrak{A}'' \in \mathcal{M}^*$, with $\mathfrak{A}' * \mathfrak{A}'' = \langle \langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha \rangle \rangle, T \rangle$, then the following diagram

$$\begin{tikzcd}
& \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}^{\mathcal{A}} \arrow[ur, "\langle I,\iota\rangle"] & & \mathrm{SEN}^{\mathcal{A}} \arrow[ul, "\langle I,\iota\rangle"'] \\
& \mathrm{SEN} \arrow[ul, "\langle F,\alpha\rangle"] \arrow[ur, "\langle F,\alpha\rangle"'] \arrow[uu, "\langle F,\alpha\rangle"] &
\end{tikzcd}$$

is a special case of Diagram (3), showing that $\mathfrak{A}' * \mathfrak{A}'' = \mathcal{M}^{\otimes}$.

As is pointed out in the discussion following Proposition 4.3 in [8], in general, the class $\mathcal{M}^{\otimes}$ is in fact much larger than the class $\mathcal{M}^{*}$ and, sometimes, so large that soundness of $\mathcal{L}' * \mathcal{L}''$ is not inherited by the soundness of the constituent logical systems. Thus, to preserve soundness, one has to restrict the class of models $\mathcal{M}^{\otimes}$ to consist of only those models with respect to which the N-rules in $\mathcal{R}$ are sound.

Given two logical systems $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ and $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ and $N = \langle N', N'' \rangle$, we define the **sound crypto-fibring** of $\mathcal{L}'$ and $\mathcal{L}''$, **constrained by the sharing** of $\hat{N} = N' \cap N''$ or, more simply, the **constrained sound crypto-fibring** of $\mathcal{L}'$ and $\mathcal{L}''$, as the logical system

$$\mathcal{L}'_{\circledS}\ \mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N, \mathcal{R}, \mathcal{M}^{\circledS} \rangle,$$

where $\mathcal{M}^{\circledS}$ is the subclass of $\mathcal{M}^{\otimes}$ consisting of all those models that satisfy all N-rules in $\mathcal{R}$.

With this latter definition, the sound crypto-fibring of two logical systems becomes by default a sound logical system. Moreover, as is shown next, in an analog of Proposition 4.5 of [8], if the original logical systems are full, then the sound crypto-fibring is rich enough to also be full and, as a consequence, completeness is also preserved under these circumstances, under the proviso that the crypto-fibred system is Lindenbaum (see Proposition 5).

Proposition 9. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ and $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two sound logical systems and* $N = \langle N', N'' \rangle$. *If both* $\mathcal{L}'$ *and* $\mathcal{L}''$, *are full, then the class* $\mathcal{M}^{\circledS}$ *is a subclass of* $\mathcal{M}^{*}$.

Proof. If $\mathfrak{A} \in \mathcal{M}^{\circledS}$, then, by definition, $\mathfrak{A}$ satisfies all N-rules of $\mathcal{R}$. Therefore, since, by Lemma 6, $\mathcal{M}^*$ is full, we get that there exist $\mathfrak{A}' \in \mathcal{M}'$ and $\mathfrak{A}'' \in \mathcal{M}''$ such that $\mathfrak{A} = \mathfrak{A}' * \mathfrak{A}'' \in \mathcal{M}^*$.

5. Representability and Conservativeness

In the case of crypto-fibring, the concept of representability of a pair of models plays a key role for ensuring conservativeness, similar to the role that fullness plays in ensuring completeness. Because of its importance, most of the remainder of the paper will be dedicated in its detailed study and in taking advantage of it, whenever possible, to endow crypto-fibred systems with the ensuing desirable properties. For sentential logics, where our inspiration originates from, we refer the reader to Section 5 of [8].

Let $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ and $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ be two logical systems and $N = \langle N', N'' \rangle$. Assume that $\mathfrak{A}' \in \mathcal{M}'$ and $\mathfrak{A}'' \in \mathcal{M}''$. The pair $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is said to be **represented in** $\mathcal{M}^{\otimes}$ if there exists an N-matrix system $\mathfrak{A}' \in \mathcal{M}^{\otimes}$, such that $\mathfrak{A} \in \mathfrak{A}' \otimes \mathfrak{A}''$, i.e., if there exists a cryptoextension $\langle \mathfrak{A}', \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$ of $\mathfrak{A}'$ and $\mathfrak{A}''$.

Proposition 10. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems and* $N = \langle N', N'' \rangle$. *If, for all* $\mathfrak{A}' \in \mathcal{M}'$, *there exists* $\mathfrak{A}'' \in \mathcal{M}''$, *such that* $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$, *and vice-versa, then* $\vDash^{\otimes}$ *is a conservative extension of both* $\vDash'$ *and* $\vDash''$.

Proof. Suppose that $\Delta \cup \{\sigma\}$ is a collection of natural transformations in N', such that $\Delta \vDash' \sigma$, and that $\langle \mathfrak{A}, \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$, with $\mathfrak{A} = \langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha \rangle \rangle, T \rangle$, is a crypto-extension of $\mathfrak{A}' \in \mathcal{M}'$ and $\mathfrak{A}'' \in \mathcal{M}''$,

$$\begin{tikzcd} & \mathrm{SEN}^{\mathcal{A}} & \\ \mathrm{SEN}' \arrow[ur, "{\langle G', \beta' \rangle}"] & & \mathrm{SEN}'' \arrow[ul, "{\langle G'', \beta'' \rangle}"'] \\ & \mathrm{SEN} \arrow[ul, "{\langle F', \alpha' \rangle}"] \arrow[ur, "{\langle F'', \alpha'' \rangle}"'] \arrow[uu, "{\langle F, \alpha \rangle}"] & \end{tikzcd}$$

such that, for some $\Sigma, \Sigma' \in |\mathbf{Sign}| f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\mathfrak{A} \models_{\Sigma} \Delta[\vec{\phi}, f].$$

Then $\alpha_{\Sigma'}(\Delta_{\Sigma'}(f(\vec{\phi}))) \subseteq T_{F(\Sigma')}$. This implies that

$$\beta'_{F'(\Sigma')}(\alpha'_{\Sigma'}(\Delta_{\Sigma'}(f(\vec{\phi})))) \subseteq T_{G'(F'(\Sigma'))},$$

i.e., that $\alpha'_{\Sigma'}(\Delta_{\Sigma'}(f(\vec{\phi}))) \subseteq \beta'^{-1}_{F'(\Sigma')}(T_{G'(F'(\Sigma'))}) = T'_{F'(\Sigma')}$. Thus, since $\Delta \models' \sigma$, we get that $\alpha'_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi}))) \in T'_{F'(\Sigma')} = \beta'^{-1}_{F'(\Sigma')}(T_{G'(F'(\Sigma'))})$, i.e.,

$$\beta'_{F'(\Sigma')}(\alpha'_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi})))) \in T_{G'(F'(\Sigma'))}.$$

Equivalently, $\alpha_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi}))) \in T_{F(\Sigma')}$. This shows that $\Delta \models^{\otimes} \sigma$.

Conversely, suppose $\Delta \cup \{\sigma\}$ is a collection of natural transformations in N', such that $\Delta \not\models' \sigma$. Then, there exists $\mathfrak{A}' \in \mathcal{M}'$, $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, such that $\mathfrak{A}' \models'_{\Sigma} \Delta[\vec{\phi}, f]$, and $\mathfrak{A}' \not\models'_{\Sigma} \sigma[\vec{\phi}, f]$, i.e., such that $\alpha'_{\Sigma'}(\Delta_{\Sigma'}(f(\vec{\phi}))) \subseteq T'_{F'(\Sigma')}$ and $\alpha'_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi}))) \notin T'_{F'(\Sigma')}$. By hypothesis, there exists, then, $\mathfrak{A}'' \in \mathcal{M}''$, such that $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is represented in $\mathcal{M}^{\otimes}$, say, via the diagram

$$\begin{tikzcd}
 & \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "{\langle G', \beta' \rangle}"] & & \mathrm{SEN}'' \arrow[ul, "{\langle G'', \beta'' \rangle}"'] \\
 & \mathrm{SEN} \arrow[ul, "{\langle F', \alpha' \rangle}"] \arrow[ur, "{\langle F'', \alpha'' \rangle}"'] \arrow[uu, "{\langle F, \alpha \rangle}"] &
\end{tikzcd}$$

where $\langle \mathfrak{A}, \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$, with $\mathfrak{A} = \langle \langle SEN^{\mathcal{A}}, \langle F, \alpha \rangle \rangle, T \rangle$, is a crypto-extension of $\mathfrak{A}' \in \mathcal{M}'$ and $\mathfrak{A}'' \in \mathcal{M}''$. Since $\beta'^{-1}(T) = T'$, we get that

$$\alpha_{\Sigma'}(\Delta_{\Sigma'}(f(\vec{\phi}))) = \beta'_{F'(\Sigma')}(\alpha'_{\Sigma'}(\Delta_{\Sigma'}(f(\vec{\phi})))) \subseteq T_{F(\Sigma')}$$

but

$$\alpha_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi}))) = \beta'_{F'(\Sigma')}(\alpha'_{\Sigma'}(\sigma_{\Sigma'}(f(\vec{\phi})))) \notin T_{F(\Sigma')}.$$

This shows that $\mathfrak{A} \vDash^{\otimes}_{\Sigma} \Delta[\vec{\phi}, f]$ and $\mathfrak{A} \nvDash^{\otimes}_{\Sigma} \sigma[\vec{\phi}, f]$, i.e., that $\Delta \nvDash^{\otimes} \sigma$.

A symmetric argument shows that $\vDash^{\otimes}$ is a conservative extension of $\vDash''$.

6. Necessary Conditions for Representability

In this section, we abstract the two necessary conditions for representability of pairs of models of sentential logics provided in Lemmas 5.3 and 5.4 of [8] to obtain similar, but more general, conditions for the representability of pairs of matrix systems in the abstract framework of crypto-fibring of logical systems, as investigated in the present work. The first result, Proposition 11, corresponds to Lemma 5.3 of [8]. It expresses the fact that, due to the availability of crypto-extensions, the model morphisms of the matrix systems of the constituent logical systems must respect, in some sense, the families of designated sets of elements.

Proposition 11. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems*, $N = \langle N', N'' \rangle$ *and* $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$. If $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$, *then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha'_{\Sigma}(\phi) \in T'_{F'(\Sigma)} \text{ iff } \alpha''_{\Sigma}(\phi) \in T''_{F''(\Sigma)}.$$

Proof. Suppose that $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is represented in $\mathcal{M}^{\otimes}$ by the crypto-extension $\langle \mathfrak{A}, \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$, with $\mathfrak{A} = \langle \langle \mathrm{SEN}^{\mathcal{A}}, \langle F, \alpha \rangle \rangle, T \rangle$:

$$\begin{tikzcd}
& \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "{\langle G',\beta'\rangle}"] & & \mathrm{SEN}'' \arrow[ul, "{\langle G'',\beta''\rangle}"'] \\
& \mathrm{SEN} \arrow[ul, "{\langle F',\alpha'\rangle}"] \arrow[ur, "{\langle F'',\alpha''\rangle}"'] \arrow[uu, "{\langle F,\alpha\rangle}"] &
\end{tikzcd}$$

Then, we have

$$\beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\phi)) = \alpha_{\Sigma}(\phi) = \beta''_{F''(\Sigma)}(\alpha''_{\Sigma}(\phi)).$$

Therefore, we obtain

$$\begin{aligned}\alpha'_{\Sigma}(\phi) \in T'_{F'(\Sigma)} &\text{ iff } \beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\phi)) \in T_{F(\Sigma)}\\ &\text{ iff } \beta''_{F''(\Sigma)}(\alpha''_{\Sigma}(\phi)) \in T_{F(\Sigma)}\\ &\text{ iff } \alpha''_{\Sigma}(\phi) \in T''_{F''(\Sigma)},\end{aligned}$$

which is the required equivalence in the conclusion.

Proposition 12 formalizes an analog of Lemma 5.4 of [8]. Roughly speaking, it asserts that the existence of a crypto-extension for a pair of matrix system models implies the existence of congruence systems, one on each model of the pair, that are in some sense intertwined and, moreover, each is compatible with the corresponding family of designated sets of elements. This latter condition is inspired by the cornerstone compatibility conditions of abstract algebraic logic [2, 15, 16] (see also, e.g., [32] for categorical analogs, more intimately connected to the present context).

Proposition 12. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}'\rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}''\rangle$ *be two logical systems,* $N = \langle N', N''\rangle$ *and* $\mathfrak{A}' = \langle\langle \mathrm{SEN}', \langle F', \alpha'\rangle\rangle, T'\rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle\langle \mathrm{SEN}'', \langle F'', \alpha''\rangle\rangle, T''\rangle \in \mathcal{M}''$. If $\langle \mathfrak{A}', \mathfrak{A}''\rangle$ *is represented in* $\mathcal{M}^{\otimes}$, *then, there exist an* N'*-congruence system* $\sim$ *on* SEN' *and an* N''*-congruence system* $\approx$ *on* SEN'', *such that*

- $\alpha'(\alpha''^{-1}(\approx)) \leq \sim$ and $\alpha''(\alpha'^{-1}(\sim)) \leq \approx$;
- $\sim$ *is compatible with* T' *and* $\approx$ *is compatible with* T''.

Proof. Suppose, once more, that $(\mathfrak{A}', \mathfrak{A}'')$ is represented in $\mathcal{M}^{\otimes}$, via the following diagram:

$$\begin{tikzcd} & \mathrm{SEN}^{\mathcal{A}} & \\ \mathrm{SEN}' \arrow[ur, "\langle G', \beta'\rangle"] & & \mathrm{SEN}'' \arrow[ul, "\langle G'', \beta''\rangle"'] \\ & \mathrm{SEN} \arrow[ul, "\langle F', \alpha'\rangle"] \arrow[ur, "\langle F'', \alpha''\rangle"'] \arrow[uu, "\langle F, \alpha\rangle"] & \end{tikzcd}$$

We consider the N'-congruence system $\sim = \mathrm{Ker}(\langle G', \beta' \rangle)$ on SEN' and the N''-congruence system $\approx = \mathrm{Ker}(\langle G'', \beta'' \rangle)$ on SEN''.

To see that the first listed conditions are satisfied, suppose that $\alpha''^{-1}_{\Sigma}(\approx_{F''(\Sigma)})$, i.e., that $\alpha''_{\Sigma}(\phi) \approx_{F''(\Sigma)} (\alpha''_{\Sigma}(\psi))$. Thus,

$$\beta''_{F''(\Sigma)}(\alpha''_{\Sigma}(\phi)) = \beta''_{F''(\Sigma)}(\alpha''_{\Sigma}(\psi)).$$

By the commutativity of the previous diagram, we get that $\beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\phi)) = \beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\phi))$, yielding $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\psi)$. Hence, $\alpha'_{\Sigma}(\alpha''^{-1}_{\Sigma}(\approx_{F''(\Sigma)})) \subseteq \sim_{F'(\Sigma)}$. Since $\Sigma \in |\mathbf{Sign}|$ was arbitrary, we get that $\alpha'(\alpha''^{-1}(\approx)) \leq \sim$.

The proof that $\alpha''(\alpha'^{-1}(\sim)) \leq \approx$ is symmetric.

To demonstrate the second of the listed conditions, let $\Sigma' \in |\mathrm{Sign}'|$ and $\phi', \psi' \in \mathrm{SEN}'(\Sigma')$, such that $\phi' \sim_{\Sigma'} \psi'$ and $\phi' \in T'_{\Sigma'} = \beta'^{-1}_{\Sigma'}(T_{G'(\Sigma')})$. Thus, $\beta'_{\Sigma}(\phi') = \beta'_{\Sigma'}(\psi')$ and $\beta'_{\Sigma'}(\phi') \in T_{G'(\Sigma')}$. Hence $\beta'_{\Sigma'}(\psi') \in T_{G'(\Sigma')}$ and, therefore, $\psi' \in \beta'^{-1}_{\Sigma'}(T_{G'(\Sigma')} {=}{\in}\, T'_{\Sigma'}$, showing that $\sim$ is compatible with T'. That $\approx$ is compatible with T'' may be shown similarly.

7. Technical Lemmas on Congruence Systems

Motivated by Proposition 12, we start in this section an investigation on how one can characterize intrinsically the congruences $\sim$ and $\approx$ without prior knowledge of the morphisms $\langle G', \beta' \rangle$ and $\langle G'', \beta'' \rangle$ of a crypto-extension $\langle \mathfrak{A}, \langle G', \beta' \rangle, \langle G'', \beta'' \rangle \rangle$ of the pair $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$. To do this, we need to be able to construct congruence systems $\sim$ on SEN' and $\approx$ on SEN'' that will be shown to satisfy the requisite conditions in case such a crypto-extension exists, but without explicitly referring to it. Moreover, in the last section of the paper, taking further advantage of these constructions, it will be shown that, in an important special case, if the constructed congruences satisfy those conditions, then such a crypto-extension may be constructed a posteriori, i.e., those conditions turn out to also be sufficient for a crypto-extension to exist.

To begin with, we show that, given a congruence system on one of the matrix systems $\mathfrak{A}'$ or $\mathfrak{A}''$, one may pull it back to obtain a corresponding congruence system on SEN and that, under this inverse image construction, inclusion of the congruence systems in kernels of appropriate morphisms is preserved. This is not yet an intrinsic condition, but it is a property that will be proven useful later in constructing intrinsically an intertwined pair of congruences on the matrix system models.

Lemma 13. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems,* $N = \langle N', N'' \rangle$ *and* $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$, *such that* $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$ *as follows*:

$$
\begin{tikzcd}
& \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "\langle G', \beta' \rangle"] & & \mathrm{SEN}'' \arrow[ul, "\langle G'', \beta'' \rangle"'] \\
& \mathrm{SEN} \arrow[ul, "\langle F', \alpha' \rangle"] \arrow[ur, "\langle F'', \alpha'' \rangle"'] \arrow[uu, "\langle F, \alpha \rangle"] &
\end{tikzcd}
$$

(i) *If* $\sim$ *is an* N'*-congruence system on* SEN', *then* $\alpha'^{-1}(\sim)$ *is an* N'*-congruence system on* SEN. *Moreover, if* $\sim\, \leq \mathrm{Ker}\,(\langle G', \beta' \rangle)$, *then* $\alpha'^{-1}(\sim) \leq \mathrm{Ker}\,(\langle F, \alpha \rangle)$;

(ii) *If* $\approx$ *is an* N''*-congruence system on* SEN'', *then* $\alpha''^{-1}(\approx)$ *is an* N''*-congruence system on* SEN. *Moreover, if* $\approx\, \leq \mathrm{Ker}\,(\langle G'', \beta'' \rangle)$, *then* $\alpha''^{-1}(\approx) \leq \mathrm{Ker}\,(\langle F, \alpha \rangle)$.

Proof. It suffices to prove Part (i). The second part follows, then, by symmetry.

First, it is not difficult to show that, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})$ is an equivalence relation on $\mathrm{SEN}(\Sigma)$. In fact, for all $\phi, \chi, \psi \in \mathrm{SEN}(\Sigma)$,

- by reflexivity of $\sim_{F'(\Sigma)}$, $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\phi)$, whence $\phi \; \alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)}) \; \phi$;

- if $\phi\,\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\chi$, then $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\chi)$, whence, by the symmetry of $\sim_{F'(\Sigma)}$, $\alpha'_{\Sigma}(\mathcal{X}) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\phi)$, and, therefore, $\chi\,\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\phi$;

- if $\phi\,\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\chi$ and $\chi\,\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\psi$, then $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)}, \alpha'_{\Sigma}(\chi)$ and $\alpha'_{\Sigma}(\chi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\psi)$, whence, by the transitivity of $\sim_{F'(\Sigma)}$, we obtain $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\psi)$, and, therefore, $\phi\,\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\psi$;

Moreover, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})$ is an N'-congruence relation on $\mathrm{SEN}(\Sigma)$. Indeed, if σ is a natural transformation in N', and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^k$, such that $\vec{\phi}\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})^k\,\vec{\psi}$, then $\alpha'^k_{\Sigma}(\vec{\phi}) \sim^k_{F'(\Sigma)}, \alpha'^k_{\Sigma}(\vec{\psi})$, whence, since $\sim_{F'(\Sigma)}$, is an N'-congruence relation, we get that $\sigma_{F'(\Sigma)}(\alpha'^k_{\Sigma}(\vec{\phi})) \sim_{F'(\Sigma)} \sigma_{F'(\Sigma)}(\alpha'^k_{\Sigma}(\vec{\psi}))$. This is equivalent to $\alpha'_{\Sigma}(\sigma_{\Sigma}(\vec{\phi})) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\sigma_{\Sigma}(\vec{\psi}))$, i.e., $\sigma_{\Sigma}(\vec{\phi})\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\,\sigma_{\Sigma}(\vec{\psi})$.

Next, the collection $\alpha'^{-1}(\sim) = \{\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\}_{\Sigma\in|\mathbf{Sign}|}$ is an N'-congruence system, since, if $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\langle\phi, \psi\rangle \in \alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})$, then $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\psi)$, whence, by the system property of $\sim$,

$$\mathrm{SEN}(F(f))(\alpha'_{\Sigma}(\phi)) \sim_{F'(\Sigma')} \mathrm{SEN}(F(f))(\alpha'_{\Sigma}(\psi)).$$

Thus, by the natural transformation property of α, $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \sim_{F'(\Sigma')} \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, giving, finally, $\mathrm{SEN}(f)^2(\langle\phi, \psi\rangle) \in \alpha'^{-1}_{\Sigma'}(\sim_{F'(\Sigma')})$.

It only remains to show $\sim\ \leq \mathrm{Ker}(\langle G', \beta'\rangle)$ implies $\alpha'^{-1}(\sim) \leq \mathrm{Ker}(\langle F, \alpha\rangle)$. If $\phi\alpha'^{-1}_{\Sigma}(\sim_{F'(\Sigma)})\psi$, then $\alpha'_{\Sigma}(\phi) \sim_{F'(\Sigma)} \alpha'_{\Sigma}(\psi)$, whence, by hypothesis,

$$\beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\phi)) = \beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\psi)),$$

i.e., $\alpha_{\Sigma}(\phi) = \alpha_{\Sigma}(\psi)$, showing that $\langle\phi, \psi\rangle \in \mathrm{Ker}_{\Sigma}(\langle F, \alpha\rangle)$.

As mentioned at the outset, Part (ii) follows by symmetry.

We concentrate, next, in showing that, given either an N''- or an N'-congruence system on SEN, we can push it forward obtaining a relation family on SEN' or SEN'' respectively, that generates an N'- or N''-congruence system on the matrix system $\mathfrak{A}'$ or the matrix system $\mathfrak{A}''$, respectively, and that, under this forward congruence generation, inclusion of the congruence systems in kernels of appropriate morphisms is preserved. We obtain, as before, a property that will prove handy in constructing intrinsically an intertwined pair of congruences on the matrix system models.

Lemma 14. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems*, $N = \langle N', N'' \rangle$ *and* $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$, *such that* $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$ *via the following diagram*:

$$\begin{tikzcd}
 & \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "{\langle G', \beta' \rangle}"] & & \mathrm{SEN}'' \arrow[ul, "{\langle G'', \beta'' \rangle}"'] \\
 & \mathrm{SEN} \arrow[ul, "{\langle F', \alpha' \rangle}"] \arrow[uu, "{\langle F, \alpha \rangle}"] \arrow[ur, "{\langle F'', \alpha'' \rangle}"'] &
\end{tikzcd}$$

(i) *If* $\overset{\frown}{\frown}$ *is an* N''-*congruence system on* SEN, *such that* $\overset{\frown}{\frown} \leq \mathrm{Ker}(\langle F, \alpha \rangle)$, *then the* N'-*congruence system* $\sim$ *on* SEN' *generated by the image* $\alpha''(\overset{\frown}{\frown}) = \{\alpha''_{\Sigma}(\overset{\frown}{\frown}_{\Sigma})\}_{\Sigma \in |\mathbf{Sign}|}$ *satisfies* $\sim \leq \mathrm{Ker}(\langle G', \beta' \rangle)$;

(ii) *If* $\frown$ *is an* N'-*congruence system on* SEN, *such that* $\frown \leq \mathrm{Ker}(\langle F, \alpha \rangle)$, *then the* N''-*congruence system* $\approx$ *on* SEN'' *generated by* $\alpha''(\frown) = \{\alpha''_{\Sigma}(\frown_{\Sigma})\}_{\Sigma \in |\mathbf{Sign}|}$ *satisfies* $\approx \leq \mathrm{Ker}(\langle G'', \beta'' \rangle)$.

Proof. It suffices to prove Part (i). The second part follows, then, by symmetry. Since $\langle G', \beta' \rangle : \mathrm{SEN}' \to \mathrm{SEN}^{\mathcal{A}}$ is an N'-morphism, it follows that $\mathrm{Ker}(\langle G', \beta' \rangle)$ is an N'-congruence system on SEN'. Therefore, to conclude the proof of Part (i), it suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_{\Sigma}(\overset{\frown}{\frown}_{\Sigma}) \subseteq \mathrm{Ker}_{F'(\Sigma)}(\langle G', \beta' \rangle)$. Suppose, to this end, that $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \overset{\frown}{\frown}_{\Sigma} \psi$. Then, by hypothesis, $\alpha_{\Sigma}(\phi) = \alpha_{\Sigma}(\psi)$, whence $\beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\phi)) = \beta'_{F'(\Sigma)}(\alpha'_{\Sigma}(\psi))$. Thus, $\langle \alpha'_{\Sigma}(\phi), \alpha'_{\Sigma}(\psi) \rangle \in \mathrm{Ker}_{F'(\Sigma)}(\langle G', \beta' \rangle)$, showing that $\alpha'_{\Sigma}(\overset{\frown}{\frown}_{\Sigma}) \subseteq \mathrm{Ker}_{F'(\Sigma)}(\langle G', \beta' \rangle)$.

8. Building Up Congruence Systems

In this section, we use the technical lemmas of Section 7 to construct congruence systems on the matrix systems $\mathfrak{A}'$ and $\mathfrak{A}''$ of a pair in $\mathcal{M}' \times \mathcal{M}''$, without the prior assumption that the pair $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is represented in $\mathcal{M}^{\otimes}$. This is what exactly was meant by the use of the word "intrinsic". If the pair is in fact represented in $\mathcal{M}^{\otimes}$, then we show that the intrinsically constructed congruence systems satisfy the necessary conditions outlined in Section 6. In Section 9, we will show that, in addition, these two conditions do imply representability in a special case that is, however, general enough to capture the important case of categorical Lindenbaum models. In this way, we close, at least partially, the full circle of necessity and sufficiency.

Let us fix, again, two logical systems $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ and $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ set $N = \langle N', N'' \rangle$ and $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$:

$$\begin{tikzcd}
\mathrm{SEN}' & & \mathrm{SEN}'' \\
& \mathrm{SEN} \arrow[ul, "{\langle F', \alpha' \rangle}"] \arrow[ur, "{\langle F'', \alpha'' \rangle}"'] &
\end{tikzcd}$$

Inspired by analogous constructions presented in Sections 6 and 7 of Caleiro and Ramos [8] for the propositional framework, we define two sequences

$$\overset{\frown}{\frown}{}^{0} \leq \frown^{1} \leq \frown^{2} \leq \ldots \text{ and } \overset{\frown}{\frown}{}^{0} \leq \overset{\frown}{\frown}{}^{1} \leq \overset{\frown}{\frown}{}^{2} \leq \ldots$$

of N'- and N''-congruence systems, respectively, on SEN, a sequence

$$\overset{\frown}{\frown}{}^{0} \leq \frown^{1} \leq \frown^{2} \leq \ldots$$

of N'-congruence systems on SEN' and a sequence

$$\approx^{0} \leq \approx^{1} \leq \approx^{2} \leq \ldots$$

of N''-congruence systems on SEN'' by joint induction on superscripts as follows:

- First, we set $\sim^0 = \Delta^{\mathrm{SEN}'}$ and $\approx^0 = \Delta^{\mathrm{SEN}''}$, the identity congruence systems on SEN' and SEN'', respectively.

- Next, assuming that $\frown^n$ and $\stackrel{\frown}{\frown}{}^0$ have been defined on SEN' and SEN'', respectively, define $\sim^n = \alpha'^{-1}(\sim^n)$ and $\approx^n = \alpha''^{-1}(\approx^n)$;

- Finally, assuming that $\sim^n$ and $\approx^n$ have been defined on SEN, we define the N'-congruence system $\sim^{n+1}$ on SEN' and the N''-congruence system $\approx^{n+1}$ on SEN'' as the N'- and N''-congruence systems, respectively, generated by $\alpha'(\stackrel{\frown}{\frown}{}^n)$ and $\alpha''(\frown^n)$, respectively.

Lemma 15 asserts that these constructions, applied inductively starting from identity congruence systems in increasing superscript order, produce increasing chains of congruence systems on the corresponding sentence functors.

Lemma 15. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems,* $N = \langle N', N'' \rangle$ *and* $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$, *such that* $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$.

$$\begin{tikzcd}
\mathrm{SEN}' & & \mathrm{SEN}'' \\
& \mathrm{SEN} \arrow[ul, "{\langle F', \alpha' \rangle}"] \arrow[ur, "{\langle F'', \alpha'' \rangle}"'] &
\end{tikzcd}$$

The sequences $\frown\frown^0 \leq \frown^1 \leq \frown^2 \leq \ldots$, $\stackrel{\frown}{\frown}{}^0 \leq \stackrel{\frown}{\frown}{}^1 \leq \stackrel{\frown}{\frown}{}^2 \leq \ldots$, $\sim^0 \leq \sim^1 \leq \sim^2 \leq \ldots$ *and* $\approx^0 \leq \approx^1 \leq \approx^2 \leq \ldots$ *are increasing sequences of* N'-, N''-, N'- *and* N''- *congruence systems, the first two on* SEN *and the last two on* SEN' *and* SEN'', *respectively.*

Proof. Suppose that $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is represented in $\mathcal{M}^{\otimes}$ as follows:

$$\begin{tikzcd}
 & \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "{\langle G', \beta' \rangle}"] & & \mathrm{SEN}'' \arrow[ul, "{\langle G'', \beta'' \rangle}"'] \\
 & \mathrm{SEN} \arrow[ul, "{\langle F', \alpha' \rangle}"] \arrow[ur, "{\langle F'', \alpha'' \rangle}"'] \arrow[uu, "{\langle F, \alpha \rangle}"] &
\end{tikzcd}$$

We use a joint induction on the superscripts of the sequences involved. First, note that $\frown^0$ and $\overset{\frown}{\frown}{}^0$ being identity equivalence systems, are valid N'- and N''-congruence systems on SEN' and SEN'', respectively, and, in addition, they satisfy $\sim^0 \leq \mathrm{Ker}(\langle G', \beta' \rangle)$ and $\approx^0 \leq \mathrm{Ker}(\langle G'', \beta'' \rangle)$. Therefore, by Lemma 13, both $\frown^0$ and $\overset{\frown}{\frown}{}^0$, are N'- and N''-congruence systems, respectively, on SEN and they satisfy $\frown^0, \overset{\frown}{\frown}{}^0 \leq \mathrm{Ker}(\langle F, \alpha \rangle)$. This concludes the basis of the induction.

Suppose, next, that $\frown^n, \overset{\frown}{\frown}{}^n$ are already defined N'- and N''-congruence systems, respectively, on SEN, such that $\frown^n, \overset{\frown}{\frown}{}^n \leq \mathrm{Ker}(\langle F, \alpha \rangle)$. Then, by Lemma 14, $\sim^{n+1}$ is an N'-congruence system on SEN' and $\approx^{n+1}$, is an N''-congruence system on SEN'' such that $\sim^{n+1} \leq \mathrm{Ker}(\langle G', \beta' \rangle)$ and $\approx^{n+1} \leq \mathrm{Ker}(\langle G'', \beta'' \rangle)$.

Now assume that $\sim^n, \approx^n$ are already defined N'- and N''-congruence systems on SEN' and SEN'', respectively, satisfying $\sim^n \leq \mathrm{Ker}(\langle G', \beta' \rangle)$ and $\approx^n \leq \mathrm{Ker}(\langle G'', \beta'' \rangle)$. Then, by Lemma 13, $\frown^n$ is an N'- and $\overset{\frown}{\frown}{}^n$ an N''-congruence system on SEN and they satisfy $\frown^n, \overset{\frown}{\frown}{}^n \leq \mathrm{Ker}(\langle F, \alpha \rangle)$. This concludes the inductive step.

As far as the inclusions go, note, first, that $\sim^0 \leq \sim^1$ and that $\approx^0 \leq \approx^1$. But we also have that $\sim^n \leq \sim^{n+1}$ implies $\frown^n, \overset{\frown}{\frown}{}^{n+1}$, $\approx^n \leq \approx^{n+1}$ implies $\overset{\frown}{\frown}{}^n$ $\overset{\frown}{\frown}{}^{n+1}$, as well as $\frown^n \leq \frown^{n+1}$ implies $\approx^{n+1} \leq \approx^{n+2}$, and $\asymp^n \; \asymp^{n+1}$ implies $\sim^{n+1} \leq \sim^{n+2}$. Using these implications, together with an easy induction on superscripts, we may show that the postulated chains of inclusions of congruence systems hold as claimed.

At last, Lemma 15 justifies the following definitions: Let $\sim$ be the N'-congruence system on SEN' defined by

$$\sim \; = \left(\bigcup_{n \in \omega} \sim^n \right)$$

and, similarly, $\approx$ be the N''-congruence system on SEN'' defined by

$$\approx \; = \left(\bigcup_{n \in \omega} \approx^n \right).$$

The next lemma, an analog of Lemma 5.4 of [8], asserts that the congruence systems $\sim$ and $\approx$ satisfy both necessary conditions outlined for the congruences $\sim$ and $\approx$ of Proposition 12, under the proviso that the pair $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is represented in $M^{\otimes}$. The main difference and the main gain in this section, as compared to the context of Section 6, is that the present section's $\sim$ and $\approx$ have been constructed intrinsically, whereas those of Section 6 were constructed extrinsically on SEN' and SEN'' respectively, as kernels of the algebraic morphisms $\langle G', \beta' \rangle$ and $\langle G'', \beta'' \rangle$, that were assumed known a priori.

Proposition 16. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems,* $N = \langle N', N'' \rangle$ *and* $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle F', \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}'$, $\mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle F'', \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$. *It is always the case that*

$$\alpha'(\alpha''^{-1}(\approx)) \leq \sim \textit{ and } \alpha''(\alpha'^{-1}(\sim)) \leq \approx .$$

Moreover, if $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$, *then we also have that* $\sim$ *is compatible with* T' *and* $\approx$ *is compatible with* T''.

Proof. For the first statement, we have

$$\begin{aligned}
\alpha'(\alpha''^{-1}(\approx)) &= \alpha'\left(\alpha''^{-1}\left(\bigcup_{n=0}^{\infty}\approx^n\right)\right)\\
&= \alpha'\left(\bigcup_{n=0}^{\infty}\alpha''^{-1}(\approx^n)\right)\\
&= \alpha'\left(\bigcup_{n=0}^{\infty}\overset{\frown}{\frown}{}^n\right)\\
&\leq \bigcup_{n=1}^{\infty}\sim^n\\
&\leq \sim.
\end{aligned}$$

Similarly, one obtains also that $\alpha''(\alpha'^{-1}(\sim)) \leq \approx$.

Next, assume that the representation diagram is given by

$$\begin{tikzcd}
 & \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "\langle G', \beta' \rangle"] & & \mathrm{SEN}'' \arrow[ul, "\langle G'', \beta'' \rangle"'] \\
 & \mathrm{SEN} \arrow[ul, "\langle F', \alpha' \rangle"] \arrow[ur, "\langle F'', \alpha'' \rangle"'] \arrow[uu, "\langle F, \alpha \rangle"] &
\end{tikzcd}$$

Since, by definition, $\sim\, = (\bigcup_{n\in\omega} \sim^n)$ and $\approx\, = (\bigcup_{n\in\omega} \approx^n)$ and since compatibility is preserved under unions of chains, it suffices to show that, for all $n \in \omega$, the N'-congruence system $\sim^n$ on SEN and the N''-congruence system $\approx^n$ on SEN'' are compatible with T' and T'', respectively.

We do this for $\sim^n$, since the case of $\approx^n$ follows along similar lines using symmetry. Recall that, according to Lemma 14, $\sim^n \leq \mathrm{Ker}(\langle G', \beta' \rangle)$. Let $\Sigma' \in |\mathbf{Sign}'|$ and $\phi', \psi' \in \mathrm{SEN}'(\Sigma')$, such that $\phi' \sim^n_{\Sigma'} \psi'$ and $\phi' \in T'_{\Sigma}$.

Therefore, $\beta'_{\Sigma'}(\phi') = \beta'_{\Sigma'}(\psi')$ and $\beta'_{\Sigma'}(\phi') \in T_{G'(\Sigma')}$. But, then, we also have $\beta'_{\Sigma'}(\psi') \in T_{G'(\Sigma')}$ and, hence, $\psi' \in T'_{\Sigma'}$ showing that $\sim^n$ is in fact compatible with T'.

9. A Result Concerning Sufficiency

In this section, we tackle the sufficiency of the conditions established in Propositions 11 and 12 for the representability of a pair of matrix system models of two logical systems $\mathcal{L}'$ and $\mathcal{L}''$ in their crypto-fibring in the special case in which the functor components of the algebraic morphisms are identities (or more generally, isomorphisms). Even though this condition is restrictive, it does allow handling a variety of important special cases. Among these are the framework of sentential logics, handled in [8]. To accommodate sentential logics one has to consider only trivial one element signature categories and, therefore, they fit in the present categorical framework as trivial special cases. Another important special case that our framework is general enough to handle is that of the categorical Lindenbaum models, where the underlying algebraic systems of the matrix systems are the "formula" algebraic systems (or sometimes their canonical quotients). In this case, all algebraic morphisms are either identities or natural projections with identity functor components and, hence, equally amenable to the techniques of this section.

Proposition 17. *Let* $\mathcal{L}' = \langle \mathbf{Sign}, \mathrm{SEN}, N', \mathcal{R}', \mathcal{M}' \rangle$ *and* $\mathcal{L}'' = \langle \mathbf{Sign}, \mathrm{SEN}, N'', \mathcal{R}'', \mathcal{M}'' \rangle$ *be two logical systems and* $N = \langle N', N'' \rangle$. *Consider two sentence functors* $\mathrm{SEN}', \mathrm{SEN}'' : \mathbf{Sign} \to \mathbf{Set}$ *and* $\mathfrak{A}' = \langle \langle \mathrm{SEN}', \langle I_{\mathbf{Sign}}, \alpha' \rangle \rangle, T' \rangle \in \mathcal{M}', \mathfrak{A}'' = \langle \langle \mathrm{SEN}'', \langle I_{\mathbf{Sign}}, \alpha'' \rangle \rangle, T'' \rangle \in \mathcal{M}''$, *such that, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha'_{\Sigma'}(\phi) \in T'_{F'(\dot{\Sigma}')} \textit{ iff } \alpha''_{\Sigma'}(\phi) \in T''_{F''(\dot{\Sigma}')}. \tag{4}$$

If the N'*-congruence system* $\sim$ *is compatible with* T' *and the* N''*-congruence system* $\approx$ *is compatible with* T'', *then* $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ *is represented in* $\mathcal{M}^{\otimes}$.

Proof. We first define a new sentence functor $\mathrm{SEN}^{\mathcal{A}} : \mathbf{Sign} \to \mathbf{Set}$ as follows: Let $\mathrm{SEN}^{+} : \mathbf{Sign} \to \mathbf{Set}$ be defined as the free N-algebraic system on the disjoint union of SEN' and SEN'' i.e., for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}^{+}(\Sigma) = \mathrm{Tm}^{N}(\mathrm{SEN}'(\Sigma) \uplus \mathrm{SEN}''(\Sigma))$$

the collection of all N-terms build on the disjoint union of $\mathrm{SEN}'(\Sigma)$ and $\mathrm{SEN}''(\Sigma)$, and, given $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\mathrm{SEN}^{+}(f)(\phi) = \begin{cases} \mathrm{SEN}'(f)(\phi), \text{ if } \phi \in \mathrm{SEN}'(\Sigma_1) \\ \mathrm{SEN}''(f)(\phi), \text{ if } \phi \in \mathrm{SEN}''(\Sigma_1) \end{cases}$$

and, if $\sigma(t_0, \ldots, t_{k-1}) \in \mathrm{Tm}^{N}(\mathrm{SEN}'(\Sigma) \uplus \mathrm{SEN}''(\Sigma))$,

$$\mathrm{SEN}^{+}(f)(\sigma(t_0, \ldots, t_{k-1})) = \sigma(\mathrm{SEN}^{+}(f)(t_0), \ldots, \mathrm{SEN}^{+}(f)(t_{k-1})).$$

Next, define on SEN^{+} the N-congruence system $\equiv = \{\equiv_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ generated by (i.e., smallest N-congruence system including) the following relation systems:

(i) Equality of N'-terms in $\mathrm{Tm}^{N'}(\mathrm{SEN}')$ as evaluated in $\mathfrak{A}'$ and equality of N''-terms in $\mathrm{Tm}^{N''}(\mathrm{SEN}'')$ as evaluated in $\mathfrak{A}''$;

(ii) The N'-congruence system $\sim$ on SEN' and the N''-congruence system $\approx$ on SEN'';

(iii) The relation system $\bowtie = \{\bowtie_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$, defined, for all $\Sigma \in |\mathbf{Sign}|$ by

$$\bowtie = \{\langle \alpha'_{\Sigma}(\phi), \alpha''_{\Sigma}(\phi) \rangle : \phi \in \mathrm{SEN}(\Sigma)\}.$$

Set $\mathrm{SEN}^{\mathcal{A}} : \mathbf{Sign} \to \mathbf{Set}$ to be the quotient functor $\mathrm{SEN}^{\mathcal{A}} := \mathrm{SEN}^{+}/\equiv$ and define

- the N'-morphism $\langle I_{\mathbf{Sign}}, \beta' \rangle : \mathrm{SEN}' \to \mathrm{SEN}^{\mathcal{A}}$ by setting $\beta'_{\Sigma}(\phi) = \phi/\equiv_{\Sigma}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$, and

- the N''-morphism $\langle I_{\mathbf{Sign}}, \beta''\rangle : \mathrm{SEN}'' \to \mathrm{SEN}^{\mathcal{A}}$ by setting $\beta''_{\Sigma}(\phi) = \phi/{\equiv_{\Sigma}}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}''(\Sigma)$.

Finally, we set $T = \{T_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ be the family defined by $T_{\Sigma} = (T'_{\Sigma} \uplus T''_{\Sigma})/{\equiv_{\Sigma}}$, for all $\Sigma \in |\mathbf{Sign}|$. Noting that, because $\bowtie \leq \equiv$ we have, for all $\Sigma \in |\mathbf{Sign}|$, $\beta'_{\Sigma} \circ \alpha'_{\Sigma} = \beta''_{\Sigma} \circ \alpha''_{\Sigma}$ and denoting this composition by α_{Σ}, we define

$$\mathfrak{A} = \langle\langle \mathrm{SEN}^{\mathcal{A}}, \langle I_{\mathbf{Sign}\,\alpha}\rangle\rangle, T\rangle.$$

We prove that $\langle \mathfrak{A}', \mathfrak{A}''\rangle$ is represented in $\mathcal{M}^{\otimes}$ by the crypto-extension $\langle \mathfrak{A}, \langle I_{\mathbf{Sign}}, \beta'\rangle, \langle I_{\mathbf{Sign}}, \beta''\rangle\rangle$:

$$\begin{tikzcd}
 & \mathrm{SEN}^{\mathcal{A}} & \\
\mathrm{SEN}' \arrow[ur, "{\langle I_{\mathbf{Sign}}, \beta'\rangle}"] & & \mathrm{SEN}'' \arrow[ul, "{\langle I_{\mathbf{Sign}}, \beta''\rangle}"'] \\
 & \mathrm{SEN} \arrow[ul, "{\langle I_{\mathbf{Sign}}, \alpha'\rangle}"] \arrow[ur, "{\langle I_{\mathbf{Sign}}, \alpha''\rangle}"'] \arrow[uu, "{\langle I_{\mathbf{Sign}}, \alpha\rangle}"] &
\end{tikzcd} \tag{5}$$

First, note that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}'(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}'(\Sigma) \arrow[r, "\mathrm{SEN}'(f)"] \arrow[d, "\beta'_{\Sigma}"'] & \mathrm{SEN}'(\Sigma') \arrow[d, "\beta'_{\Sigma'}"] \\
\mathrm{SEN}^{\mathcal{A}}(\Sigma) \arrow[r, "\mathrm{SEN}^{\mathcal{A}}(f)"'] & \mathrm{SEN}^{\mathcal{A}}(\Sigma')
\end{tikzcd}$$

$$\begin{aligned}
\beta'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) &= \mathrm{SEN}'(f)(\phi)/{\equiv_{\Sigma'}} \quad \text{(by definition)} \\
&= \mathrm{SEN}^{+}(f)(\phi)/{\equiv_{\Sigma'}} \quad \text{(by definition of }^{+}\text{)} \\
&= \mathrm{SEN}^{+}(f)/{\equiv})(\phi/{\equiv_{\Sigma}}) \quad (\equiv \text{ a congruence system}) \\
&= \mathrm{SEN}^{\mathcal{A}}(f)(\beta'_{\Sigma'}(\phi)). \quad \text{(by definition)}
\end{aligned}$$

If σ' is a k-ary natural transformation in N, $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \ldots, \phi_{k-1} \in \mathrm{SEN}'(\Sigma)$, with $\psi = \sigma'_{\Sigma}(\phi_0, \ldots, \phi_{k-1}) \in \mathrm{SEN}'(\Sigma)$,

$$\begin{aligned}\beta'_\Sigma(\sigma'_\Sigma(\phi_0, \ldots, \phi_{k-1})) &= \beta'_\Sigma(\psi) \text{ (since } \psi = \sigma'_\Sigma(\phi_0, \ldots, \phi_{k-1}))\\ &= \psi/{\equiv_\Sigma} \text{ (by definition)}\\ &= \sigma'_\Sigma(\phi_0, \ldots, \phi_{k-1})/{\equiv_\Sigma} \text{ (by Condition (i))}\\ &= \sigma'^{\equiv}_\Sigma(\phi_0/{\equiv_\Sigma}, \ldots, \phi_{k-1}/{\equiv_\Sigma}) \text{ (quotient)}\\ &= \sigma'^{\equiv}_\Sigma(\beta'_\Sigma(\phi_0), \ldots, \beta'_\Sigma(\phi_{k-1})). \text{ (by definition)}\end{aligned}$$

Thus, $\langle I_{\mathbf{Sign}}, \beta' \rangle : \mathrm{SEN}' \to \mathrm{SEN}^{\mathcal{A}}$ is an N'-algebraic morphism. One shows similarly that $\langle I_{\mathbf{Sign}}, \beta'' \rangle : \mathrm{SEN}'' \to \mathrm{SEN}^{\mathcal{A}}$ is an N''-algebraic morphism.

To conclude the proof that $\langle \mathfrak{A}', \langle I_{\mathbf{Sign}}, \beta' \rangle, \langle I_{\mathbf{Sign}}, \beta'' \rangle \rangle$ is a crypto-extension of $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ it suffices now to show that the collections of designated sets of elements are preserved by β'^{-1} and by β''^{-1}. We provide a detailed proof only for β'^{-1} since the corresponding statement for β''^{-1} follows then by a symmetric argument. The goal is to show that, for all $\Sigma \in |\mathbf{Sign}|$ we have

$$\beta'^{-1}_\Sigma(T_\Sigma) = T'_\Sigma.$$

Suppose first that $\phi \in T'_\Sigma$. Then $\beta'_\Sigma(\phi) = \phi/{\equiv_\Sigma} \in T'_\Sigma/{\equiv_\Sigma} \subseteq T_\Sigma$, by the definition of T_Σ. Thus, $T'_\Sigma \subseteq \beta'^{-1}_\Sigma(T_\Sigma)$. The reverse inclusion is slightly more challenging, but it follows, generally, along lines similar to the proof of the corresponding sentential result, presented in Proposition 5.5 of [8]. Suppose that $\beta'_\Sigma(\phi) = \phi/{\equiv_\Sigma} \in T_\Sigma$. This implies that either there exists $\psi \in T'_\Sigma$, such that $\phi \equiv_\Sigma \psi$, or there exists $\chi \in T''_\Sigma$, such that $\phi \equiv_\Sigma \chi$. In the first case we must have, by Condition (ii), $\phi \sim_\Sigma \psi$, whence, by the postulated compatibility of $\sim$ with T', we get that $\phi \in T'_\Sigma$. In the second case we must have, by Condition (iii), that there exists $\theta \in \mathrm{SEN}(\Sigma)$, such that $\phi = \alpha'_\Sigma(\theta) \equiv_\Sigma \alpha''_\Sigma(\theta) = \chi$. Thus, by the fact that $\alpha''_\Sigma(\theta) = \chi \in T''_\Sigma$ and the postulated Equivalence (4), we get that $\phi = \alpha'_\Sigma(\theta) \in T'_\Sigma$.

Since, by symmetry, we get also that $\beta''^{-1}_{\Sigma}(T_{\Sigma}) = T''_{\Sigma}$, for all $\Sigma \in |\mathbf{Sign}|$, this concludes the proof that $\langle \mathfrak{A}', \mathfrak{A}'' \rangle$ is represented in $\mathcal{M}^{\otimes}$.

Acknowledgements

I would like to acknowledge the critical influence that the work of Caleiro and Ramos [8], along with some previous work of the Portuguese group working in the area of combinations of logics in universal logic, has had on the work presented here. I would also like to acknowledge, once more, the critical and deep personal and scientific debt that I owe to Don Pigozzi of Iowa State (currently of Oakland), Charles Wells of Case Western Reserve (currently of St. Paul), and Josep Maria Font and Ramon Jansana of Barcelona. I thank them all, together with Giora Slutzki of Iowa State, for offering encouragement and support throughout many years.

References

[1] M. Barr and C. Wells, Category Theory for Computing Science, Third Edition, Les Publications CRM, Montréal 1999.

[2] W. J. Blok and D. Pigozzi, Algebraizable Logics, Memoirs of the American Mathematical Society 77(396) (1989).

[3] F. Borceux, Handbook of Categorical Algebra, Encyclopedia of Mathematics and its Applications, Vol. 50, Cambridge University Press, Cambridge U. K., 1994.

[4] C. Caleiro, Combining Logics, Ph.D. Thesis, IST, TU Lisbon, Portugal, 2000.

[5] C. Caleiro, W. A. Carnielli, M. E. Coniglio, A. Sernadas and C. Sernadas, Fibring Non-Truth-Functional Logics: Completeness Preservation, Journal of Logic, Language and Information 12(2) (2003), 183-211.

[6] C. Caleiro, W. A. Carnielli, J. Rasga and C. Sernadas, Fibring of Logics as a Universal Construction, In D. Gabbay and F. Guenthner, Eds., Handbook of Philosophical Logic, 2nd Edition, Volume 13, Springer, Dordrecht (2005), 123-187.

[7] C. Caleiro and J. Ramos, Combining Classical and Intuitionistic Logics, Technical Report, Department of Mathematics, IST, TU Lisboa 2006.

[8] C. Caleiro and J. Ramos, From Fibring to Cryptofibring, A Solution to the Collapsing Problem, Logica Universalis 1 (2007), 71-92.

[9] M. E. Coniglio, A. Sernadas and C. Sernadas, Fibring Logics with Topos Semantics, Journal of Logic and Computation 13(4) (2003), 595-624.

[10] J. Czelakowski, Equivalential Logics I, Studia Logica 40 (1981), 227-236.

[11] J. Czelakowski, Equivalential Logics II, Studia Logica, 40 (1981), 355-372.

[12] J. Czelakowski, Protoalgebraic Logics, Trends in Logic-Studia Logica Library 10, Kluwer, Dordrecht 2001.

[13] L. Farinas del Cerro and A. Herzig, Combining Classical and Intuitionistic Logic, In F. Baader and K. Schulz, Eds., Frontiers of Combining Systems, Vol. 3, Kluwer Academic Publishers, Dordrecht, (1996), 93-102.

[14] J. Fiadeiro and A. Sernadas, Structuring Theories on Consequence, in Recent Trends in Data Type Specification, Donald Sannella and Andrzej Tarlecki, Eds., Lecture Notes in Computer Science, Vol. 332, Springer-Verlag, New York (1988), 44-72.

[15] J. M. Font and R. Jansana, A General Algebraic Semantics for Sentential Logics, Lecture Notes in Logic, Springer-Verlag, Berlin, Heidelberg 332(7) (1996).

[16] J. M. Font, R. Jansana and D. Pigozzi, A Survey of Abstract Algebraic Logic, Studia Logica, 74(1/2) (2003), 13-97.

[17] D. Gabbay, Fibred Semantics and the Weaving of Logics: Part I, Journal of Symbolic Logic, 61(4) (1996), 1057-1120.

[18] D. Gabbay, Fibring Logics, Oxford University Press, New York 1999.

[19] J. A. Goguen and R. M. Burstall, Introducing Institutions, in Proceedings of the Logic of Programming Workshop, E. Clarke and D. Kozen, Eds., Lecture Notes in Computer Science, Vol. 164, Springer-Verlag, New York (1984), 221-256.

[20] J. A. Goguen and R. M. Burstall, Institutions: Abstract Model Theory for Specification and Programming, Journal of the Association for Computing Machinery 39(1) (1992), 95-146.

[21] F. W. Lawvere, Algebraic Theories, Algebraic Categories, and Algebraic Functors, Theory of Models, North-Holland, Amsterdam (1965), 413-418.

[22] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, 1971.

[23] E. G. Manes, Algebraic Theories, Springer-Verlag, 1976.

[24] A. Sernadas, J. Rasga and W. A. Carnielli, Modulated Fibring and the Collapsing Problem, Journal of Symbolic Logic 67(4) (2002), 1541-1569.

[25] A. Sernadas, C. Sernadas and A. Caleiro, Fibring of Logics as a Categorial Construction, Journal of Logic and Computation 9(2) (1999), 149-179.

[26] A. Sernadas, C. Sernadas and J. Rasga, On Combined Connectives, Logica Universalis 5(2) (2011), 205-224.

[27] A. Sernadas, C. Sernadas and J. Rasga, On Meet-Combination of Logics, Journal of Logic and Computation, 22(6) (2012), 1453-1470.

[28] A. Sernadas, C. Sernadas and A. Zanardo, Fibring Modal First-Order Logics: Completeness Preservation, Logic Journal of the IGPL 10(4) (2002), 413-451.

[29] A. Zanardo, A. Sernadas and C. Sernadas, Fibring: Completeness Preservation, Journal of Symbolic Logic 66(1) (2001), 414-439.

[30] G. Voutsadakis, Categorical Abstract Algebraic Logic: Algebraizable Institutions, Applied Categorical Structures 10(6) (2002), 531-568.

[31] G. Voutsadakis, Categorical Abstract Algebraic Logic: Equivalent Institutions, Studia Logica 74 (2003), 275-311.

[32] G. Voutsadakis, Categorical Abstract Algebraic Logic: Models of pi-Institutions, Notre Dame Journal of Formal Logic 46(4) (2005), 439-460.

[33] G. Voutsadakis, Categorical Abstract Algebraic Logic: Bloom's Theorem for Rule-Based pi-Institutions, Logic Journal of the IGPL, 16(3) (2008), 233-248.

[34] G. Voutsadakis, Categorical Abstract Algebraic Logic: Algebraic Semantics for pi-Institutions, To appear in Mathematical Logic Quarterly.

[35] G. Voutsadakis, Categorical Abstract Algebraic Logic: Meet-Combinations of Logical Systems, To appear in the Journal of Mathematics.