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Categorical Abstract Algebraic Logic: Equivalent Institutions

Abstract. A category theoretic generalization of the theory of algebraizable deductive systems of Blok and Pigozzi is developed. The theory of institutions of Goguen and Burstall is used to provide the underlying framework which replaces and generalizes the universal algebraic framework based on the notion of a deductive system. The notion of a term π-institution is introduced first. Then the notions of quasi-equivalence, strong quasi-equivalence and deductive equivalence are defined for π-institutions. Necessary and sufficient conditions are given for the quasi-equivalence and the deductive equivalence of two term π-institutions, based on the relationship between their categories of theories. The results carry over without any complications to institutions, via their associated π-institutions. The π-institution associated with a deductive system and the institution of equational logic are examined in some detail and serve to illustrate the general theory.

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1. Introduction

Recent work in algebraic logic has focused on abstracting the classical Tarski-Lindenbaum algebraization process by which a class of algebras is associated with certain deductive systems. In the prototypical example of classical propositional calculus the presence of a biconditional induces a congruence on the algebra of formulas whose quotient gives a Boolean algebra. In the absence of an explicit biconditional one has to discover alternative ways of determining whether a reasonable algebraic counterpart exists for a given deductive system $\mathcal{S}$ and of finding a class of algebras serving this purpose if such a class exists. To address this problem, Blok and Pigozzi [4] developed the theory of algebraizable deductive systems. The focus was now on the equational deductive system associated with a class of algebras rather than being on the algebras themselves. The possible ways of interpreting the deductive apparatus of a deductive system into the equational deductive apparatus of a class of algebras and vice-versa play a key role in this theory. A class of algebras K is said to be an *equivalent algebraic semantics*

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for a deductive system $\mathcal{S}$ if there exists an interpretation of the entailment relation of $\mathcal{S}$ in the equational entailment of K and vice-versa and the two interpretations are inverses of one another in a natural sense. A deductive system is then called *algebraizable* if it has an equivalent algebraic semantics. In [4] a characterization of algebraizability is obtained in terms of the existence of an isomorphism between the theory lattice of $\mathcal{S}$ and the equational theory lattice of K, and a second characterization is given in terms of the abstract properties of the Leibniz operator which maps $\mathcal{S}$-theories to congruences of the formula algebra of $\mathcal{S}$. In [13, 14, 15] the theory was generalized to encompass infinitary deductive systems. In [10] the notion of Leibniz operator was modified and the notion of Tarski operator was obtained which is applied to generalized matrices (called abstract logics in [10]) rather than ordinary logical matrices of the deductive system. In [6] a further abstraction of the algebraization process is obtained. Algebraizability is now viewed as a specific example of the notion of equivalence of deductive systems. It is the equivalence of the algebraizable deductive system $\mathcal{S}$ with a very special deductive system, namely the 2-dimensional deductive system that is associated with its equivalent algebraic semantics.

In a slightly different direction Andréka, Németi and Sain introduced a process of algebraization of logics with semantics in [1, 2, 3]. The two approaches are different but have much in common. See [9] for a detailed comparison.

The theory developed in [4, 6] together with its generalizations and refinements [13] and [10] serve well the algebraization of propositional-like logics over a fixed similarity type. However they are rather inefficient in handling more complex deductive systems where similarity types (or signatures) vary, like equational logic (the theory of abstract clones) and first-order logic. These systems may be handled by the methods of abstract algebraic logic only after the logical system is transformed into a propositional logic in a rather artificial way; see, e.g., Appendix C in [4]. On the other hand the notion of institution introduced in [12] has proven very successful in formalizing deductive systems with varying signatures. Roughly speaking, an *institution* consists of an arbitrary category **Sign** of *signatures* together with two functors SEN and MOD that give, respectively, for each signature object Σ, a set of Σ-*sentences* and a category of Σ-*models*. For each signature object Σ, sentences and models are related via a Σ-*satisfaction relation.* The main axiom formalizes the slogan that "truth is invariant under change of notation" (see [11]). Motivated by Goguen and Burstall's work, Fiadeiro and Sernadas [8] introduced the notion of π-institution. The main modification is that, instead of having a semantical satisfaction relation as the basis for the deduction, the focus is shifted towards a syntactic consequence relation in the spirit of Tarski [19]. Thus, a π-*institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ is a triple, where **Sign** and SEN are the same as before but C_Σ is a closure operator on the set of Σ-sentences. The notion of a π-institution may be viewed as the natural generalization of the notion of a deductive system on which a categorical theory of algebraizability, generalizing the theory of [4] may be based. It would thus be desirable to extend the notions of interpretation and equivalence to π-institutions. The notion of an institution morphism that is used in [11, 12] to connect two institutions may be appropriately modified to serve the purposes of categorical abstract algebraic logic.

A *translation* will now be defined to be a pair $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$, where $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is a functor and α is a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F$. I.e., single sentences of the source institution get mapped not to single sentences of the target institution but, rather, to sets of sentences in accordance with the notion of translation for deductive systems in [6]. A translation is an *interpretation* if, for all $\Sigma \in |\mathbf{Sign}_1|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{if and only if} \quad \alpha_\Sigma(\phi) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Following [8] and [11, 12], the *category of theories* of a π-institution and that of an institution are considered, i.e., the category with objects theories (closed sets of sentences) with respect to either the sentence closures, in the π-institution framework, or the induced consequence relations, in the institution framework. This category plays the role of the theory lattice of a deductive system in this broader context.

Inspired by [4, 6, 7], the notions of *quasi-equivalence* and *deductive equivalence* for two π-institutions are then defined. Generally speaking, two π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ are *quasi-equivalent* if the sentence closure operators of the first can be interpreted in the corresponding closure operators of the second in a natural way and vice versa. This notion of quasi-equivalence generalizes the notion of equivalence for deductive systems introduced in [6]. Attention is subsequently restricted to a special, but yet wide, class of π-institutions, the, so-called, *term π-institutions.* Using the theory categories of π-institutions, necessary and sufficient conditions for the quasi-equivalence and the deductive equivalence of two term π-institutions are given. Namely, it is proved in Theorem 9.4 that two term π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ are quasi-equivalent if and only if their categories of theories are adjoint categories via an adjunction satisfying some additional, relatively simple and quite natural, conditions. A similar characterization for deductive equivalence is also provided. More precisely, it is shown in Theorem 10.5 that two term π-institutions are deductively equivalent if and only if their categories of theories are naturally equivalent via an equivalence satisfying some of the same conditions. These results carry over without any complications to the institution framework.

Finally, as an illustration of the theory, a π-institution $\mathcal{I}_{\mathcal{S}}$ that naturally represents a k-deductive system $\mathcal{S}$ in the sense of [6] and an institution $\mathcal{EQ}$ that corresponds to a version of equational logic with varying signatures are considered. If the k-deductive system is algebraizable [4], the associated π-institution $\mathcal{I}_{\mathcal{S}}$ is easily seen to have an algebraic counterpart. Similarly, $\mathcal{EQ}$ is shown to be deductively equivalent to an algebraic counterpart $\mathcal{EA}$.

2. Institutions and π-Institutions

Definition 2.1 (Goguen and Burstall). An **institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ consists of

(i) A category **Sign** whose objects are called **signatures.**

(ii) A functor SEN : $\mathbf{Sign} \to \mathbf{Set}$, from the category **Sign** of signatures into the category **Set** of sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or Σ**-sentences.**

(iii) A functor MOD : $\mathbf{Sign} \to \mathbf{CAT}^{\mathrm{op}}$ from the category of signatures into the opposite of the category of categories, called the **model functor** and giving, for each signature Σ, a category whose objects are called Σ**-models** and whose morphisms are called Σ**-morphisms**.

(iv) A relation $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times \mathrm{SEN}(\Sigma)$, for each $\Sigma \in |\mathbf{Sign}|$, called Σ**-satisfaction,** such that for every morphism $f : \Sigma_1 \to \Sigma_2$ in **Sign** the **satisfaction condition**

$$m_2 \models_{\Sigma_2} \mathrm{SEN}(f)(\phi_1) \quad \text{if and only if} \quad \mathrm{MOD}(f)(m_2) \models_{\Sigma_1} \phi_1$$

holds, for every $m_2 \in |\mathrm{MOD}(\Sigma_2)|$ and every $\phi_1 \in \mathrm{SEN}(\Sigma_1)$.

The defining categories and functors of an institution together with their interconnections are illustrated by the following diagram:

$$\begin{tikzcd}
 & \mathbf{Set} \\
\mathbf{Sign} \arrow[ur, "\mathrm{SEN}"] \arrow[dr, "\mathrm{MOD}"'] & \models \\
 & \mathbf{CAT}^{\mathrm{op}}
\end{tikzcd}$$

Furthermore, the satisfaction condition can be given pictorially as follows:

If $f : \Sigma_1 \to \Sigma_2$ is a morphism in **Sign**, then

$$\begin{tikzcd}
\mathrm{MOD}(\Sigma_1) \arrow[r, phantom, "\models_{\Sigma_1}"] & \mathrm{SEN}(\Sigma_1) \arrow[d, "\mathrm{SEN}(f)"] \\
\mathrm{MOD}(\Sigma_2) \arrow[u, "\mathrm{MOD}(f)"] \arrow[r, phantom, "\models_{\Sigma_2}"'] & \mathrm{SEN}(\Sigma_2)
\end{tikzcd}$$

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ and $\Sigma \in |\mathbf{Sign}|$, we define, for all $\Phi \subseteq \mathrm{SEN}(\Sigma)$ and $M \subseteq |\mathrm{MOD}(\Sigma)|$,

$$\Phi^* = \{m \in |\mathrm{MOD}(\Sigma)| : m \models_\Sigma \phi \quad \text{for every} \quad \phi \in \Phi\}$$

and

$$M^* = \{\phi \in \mathrm{SEN}(\Sigma) : m \models_\Sigma \phi \quad \text{for every} \quad m \in M\}.$$

Moreover we set $\Phi^c = \Phi^{**}$ and $M^c = M^{**}$.

From now on, when the "c" symbol is used, its scope will be the largest possible well-formed expression to its left. For instance, in $\mathrm{SEN}(f)(\Phi)^c$ the scope of "c" is $\mathrm{SEN}(f)(\Phi)$ and not just Φ and in $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))^c$ the scope of the second "c" is $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))$ and not just $\mathrm{SEN}(f)^{-1}(\Phi^c)$.

Goguen and Burstall [12], prove the following very useful lemma that is used below to obtain the π-institution associated with a given institution $\mathcal{I}$.

LEMMA 2.2. [Closure Lemma] *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ be an institution, $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign})$ and $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$. Then*

$$\mathrm{SEN}(f)(\Phi^c) \subseteq \mathrm{SEN}(f)(\Phi)^c.$$

DEFINITION 2.3 (Fiadeiro and Sernadas). A **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ consists of

(i) A category **Sign** whose objects are called **signatures.**

(ii) A functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, from the category **Sign** of signatures into the category **Set** of sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or **Σ-sentences.**

(iii) A mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called **$\Sigma$-closure,** such that

(a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), A \subseteq \mathrm{SEN}(\Sigma_1)$.

Note that the Σ-closure operator of a π-institution is not required to be finitary. Also in Definition 2.3 condition (iii)(d) generalizes the structurality condition of deductive systems and Corollary 2.4 and Lemma 2.5 below are generalizations of well-known properties of the closure operators of deductive systems to the present institutional context.

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$, define

$$\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle,$$

by setting

$$C_\Sigma(\Phi) = \Phi^c, \text{ for all } \Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma).$$

It is easy to verify, using Lemma 2.2, that $\pi(\mathcal{I})$ is a π-institution. We will refer to $\pi(\mathcal{I})$ as to the **π-institution associated with** the institution $\mathcal{I}$.

From now on, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$, a signature Σ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we will use the simplified notation Φ^c to denote $C_\Sigma(\Phi)$. Usually the signature Σ is clear from context and therefore this simplified notation does not cause any confusion.

Corollary 2.4. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution. Then, for all $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign}), \Phi \subseteq \mathrm{SEN}(\Sigma_1)$,*

$$\mathrm{SEN}(f)(\Phi^c)^c = \mathrm{SEN}(f)(\Phi)^c.$$

Proof. Clearly $\mathrm{SEN}(f)(\Phi)^c \subseteq \mathrm{SEN}(f)(\Phi^c)^c$. For the reverse inclusion

$$\mathrm{SEN}(f)(\Phi^c)^c \subseteq (\mathrm{SEN}(f)(\Phi)^c)^c = \mathrm{SEN}(f)(\Phi)^c,$$

the inclusion being valid by (iii)(d) of Definition 2.3. ∎

Another lemma will also be of utmost importance for our subsequent considerations.

Lemma 2.5. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution, $f : \Sigma_1 \to \Sigma_2$ a morphism in* **Sign** *and $\Phi \subseteq \mathrm{SEN}(\Sigma_2)$. Then*

$$\mathrm{SEN}(f)^{-1}(\Phi^c)^c = \mathrm{SEN}(f)^{-1}(\Phi^c).$$

Proof. Clearly, $\mathrm{SEN}(f)^{-1}(\Phi^c) \subseteq \mathrm{SEN}(f)^{-1}(\Phi^c)^c$. For the reverse inclusion, let

$$\phi \in \mathrm{SEN}(f)^{-1}(\Phi^c)^c.$$

Then $\mathrm{SEN}(f)(\phi) \in \mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c)^c)$, whence by (d) of Definition 2.3 $\mathrm{SEN}(f)(\phi) \in \mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))^c$, and therefore $\mathrm{SEN}(f)(\phi) \in (\Phi^c)^c$, i.e., $\mathrm{SEN}(f)(\phi) \in \Phi^c$. Hence $\phi \in \mathrm{SEN}(f)^{-1}(\Phi^c)$, as required. ∎

Corollary 2.6. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution, $f : \Sigma_1 \to \Sigma_2$ an isomorphism in* **Sign** *and $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$. Then*

$$\mathrm{SEN}(f)(\Phi^c)^c = \mathrm{SEN}(f)(\Phi^c).$$

The notions of institution and π-institution are too general to allow the formulation of any useful general results. Usually one has to focus on a particular class of institutions by imposing appropriate conditions on the signature category, the model categories or the sentence closures of the institution (see, e.g., [18]). The development of a general theory of algebraizability of institutions may be achieved by restricting to institutions that satisfy a categorical analog of a property of deductive systems that plays a crucial role in the classical theory of algebraizability. This is the property of having "real" (recursively defined) terms and "real" (also recursively defined) substitutions of terms for variables in other terms. The generalized institutional property is, thus, called the *term property.* Its precise definition follows and π-institutions that satisfy this property are called *term π-institutions.* Some examples follow in the next section.

Definition 2.7. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution, $A \in |\mathbf{Sign}|$ and $p \in \mathrm{SEN}(A)$. $\langle A, p\rangle$ is called a **source signature - variable pair** if there exists a function $f : \{\langle \Sigma, \phi\rangle : \Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)\} \to |(A \downarrow \mathbf{Sign})|$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and for all $\phi \in \mathrm{SEN}(\Sigma)$, $f_{\langle \Sigma,\phi\rangle} : A \to \Sigma$ and $\mathrm{SEN}(f_{\langle \Sigma,\phi\rangle})(p) = \phi$ and

$$(\forall \Sigma' \in |\mathbf{Sign}|\ \forall g : \Sigma \to \Sigma'(g f_{\langle \Sigma,\phi\rangle} = f_{\langle \Sigma',\mathrm{SEN}(g)(\phi)\rangle})).$$

A π-institution is called **term** if it has a source signature-variable pair. An institution $\mathcal{I}$ is called **term** if its associated π-institution $\pi(\mathcal{I})$ is a term π-institution. A **Sign**-object such as A will be called a **source signature** and a sentence such as p will be called a **source variable** or, simply, a **variable.**

The following diagrams illustrate the definition:

$$\begin{tikzcd}
A \arrow[rr, "f_{\langle \Sigma \phi \rangle}"] \arrow[dr, swap, "f_{\langle \Sigma', \mathrm{SEN}(g)(\phi) \rangle}"] & & \Sigma \arrow[dl, "g"] \\
& \Sigma' &
\end{tikzcd}
\qquad
\begin{tikzcd}
p \arrow[rr, "\mathrm{SEN}(f_{\langle \Sigma \phi \rangle})"] \arrow[dr, swap, "\mathrm{SEN}(f_{\langle \Sigma', \mathrm{SEN}(g)(\phi) \rangle})"] & & \phi \arrow[dl, "\mathrm{SEN}(g)"] \\
& \mathrm{SEN}(g)(\phi) &
\end{tikzcd}$$

3. Examples

An example of a π-institution and one of an institution will now be sketched. More details and proofs of relevant statements will be presented elsewhere. The π-institution is a very simple one. Its signature category is a one-object category. It naturally represents a deductive system in the sense of [4, 6]. The institution, on the other hand, naturally represents a version of equational logic with varying types. The types are the objects of its signature category and its morphisms are interpretations between the types. Thus, in this case, the structure of the signature category is much more complex. These two examples provide an illustration of the fact that deductive systems may be viewed as special cases of π-institutions and that the π-institution and institution formalisms can handle much more complex logical systems. For some additional examples see [11, 12] and [20].

k-Deductive Systems

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a propositional language and V a countable set of variables. $\mathrm{Fm}_{\mathcal{L}}(V)$ denotes the set of formulas constructed by recursion using variables in V and connectives in $\mathcal{L}$ in the usual way. An assignment of formulas to variables is a mapping $f : V \to \mathrm{Fm}_{\mathcal{L}}(V)$. It will be denoted by $f : V \rightharpoondown V$. Such an assignment can be extended uniquely to a substitution, i.e., an endomorphism of the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$, denoted by $f^* : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a k-deductive system over $\mathcal{L}$ in the sense of [6]. We construct the π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{Sign}_{\mathcal{S}}, \mathrm{SEN}_{\mathcal{S}}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_{\mathcal{S}}|} \rangle$ as follows:

(i) $\mathbf{Sign}_{\mathcal{S}}$ is the one-object category with object V and morphisms all assignments $f : V \rightharpoondown V$. The identity morphism is the inclusion $i_V : V \to \mathrm{Fm}_{\mathcal{L}}(V)$. Composition $g \circ f$ of two assignments f and g is defined by $g \circ f = g^* f$.

(ii) $\mathrm{SEN}_{\mathcal{S}} : \mathbf{Sign}_{\mathcal{S}} \to \mathbf{Set}$ maps V to $\mathrm{Fm}_{\mathcal{L}}(V)^k$ and $f : V \rightharpoondown V$ to $(f^*)^k : \mathrm{Fm}_{\mathcal{L}}(V)^k \to \mathrm{Fm}_{\mathcal{L}}(V)^k$. It is easy to see that $\mathrm{SEN}_{\mathcal{S}}$ is a functor.

(iii) Finally, $C_V : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)^k) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)^k)$ is the standard closure operator $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)^k) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)^k)$ associated with the k-deductive system $\mathcal{S}$, i.e.,

$$C_V(\Phi) = \{\phi \in \mathrm{Fm}_{\mathcal{L}}(V)^k : \Phi \vdash_{\mathcal{S}} \phi\}, \quad \text{for all} \quad \Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V)^k.$$

C_V, defined in this way, satisfies conditions (iii)(a)-(d) of Definition 2.3. In fact, as already noted, (iii)(d) of Definition 2.3 in this case is the structurality property of a deductive system that plays a central role in the classical theory of algebraizability. Abstracting structurality in the context of multi-signature logical systems is one of the basic motivations for the introduction of categorical abstract algebraic logic. $\mathcal{I}_{\mathcal{S}}$ is thus a π-institution. It will be called the **π-institution associated with** the k-deductive system $\mathcal{S}$. Note that $\mathcal{I}_{\mathcal{S}}$ is a term π-institution for any k-deductive system $\mathcal{S}$. Indeed, the pair $\langle V, p\rangle$, where p is a k-variable, is a source signature-variable pair for $\mathcal{I}_{\mathcal{S}}$. Note also that for $k = 1$ we obtain the π-institutions associated with deductive systems in the sense of [4] and for $k = 2$ we obtain all π-institutions associated with 2-deductive systems including those associated with the semantically defined equational 2-deductive systems $\mathcal{S}_K$ whose consequence relations $C_K : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)^2) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)^2)$ are the equational consequence relations determined by some class K of $\mathcal{L}$-algebras.

In the next section an institution representing equational logic with varying similarity types is developed. Comparison with $\mathcal{I}_{\mathcal{S}}$ makes it obvious that one has to deal with a significantly more complex signature structure due to the different type of structurality present in morphisms relating different similarity types. An institution representing equational logic with varying signatures was also developed in [11]. The one given here is different in many ways from the one in [11]. Goguen and Burstall's system is multi-sorted whereas the present system is single-sorted. On the other hand the present system handles substitutions of derived operations of one type for basic operations of another whereas Goguen and Burstall restrict to substitutions of basic operations for other basic operations. The added generality at this point is essential for appropriately handling structurality in the context of abstract algebraic logic. It is the main feature that allows the generation of an algebraic theory via its Kleisli adjunction serving to algebraize the equational institution.

Equational Logic

An ω- **indexed set** or, simply, ω-**set** A is a family of sets $A = \{A_k : k \in \omega\}$. An ω-**indexed set morphism** or, simply, ω-**set morphism** $f : A \to B$, from an ω-set A to an ω-set B, is a collection of set maps $f = \{f_k : A_k \to B_k : k \in \omega\}$. Given two ω-set morphisms $f : A \to B, g : B \to C$, define their **composite** $gf : A \to C$ by $gf = \{g_k f_k : A_k \to C_k : k \in \omega\}$. With this composition, the collection of ω-sets with ω-set morphisms between them forms a category, called the **category of** ω**-sets** and denoted by $\Omega\mathbf{Set}$.

An ω-set $V = \{V_k : k < \omega\}$, with $V_k = \{v_{ki} : i < k\}$, called ω**-set of variables,** is fixed in advance. Given an ω-set X, the ω**-set of** X**-terms** $\mathrm{Tm}_X(V) = \{\mathrm{Tm}_X(V)_k : k \in \omega\}$ is defined by letting $\mathrm{Tm}_X(V)_k$ be the smallest set with

- $v_{ki} \in \mathrm{Tm}_X(V)_k, i < k$,
- $x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V)_k$, for all $n \in \omega, x \in X_n, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$.

Given $X, Y \in |\Omega\mathbf{Set}|, f : X \to \mathrm{Tm}_Y(V) \in \mathrm{Mor}(\Omega\mathbf{Set})$, let $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ be the $\Omega\mathbf{Set}$-morphism such that f^*_k leaves $v_{ki}, i < k$, fixed, for all $k \in \omega$, and$f^*_k(t)$ is the Y-term obtained from t by recursively replacing each subterm $x(t_0, \ldots, t_{n-1})$ of t by $f_n(x)(f^*_k(t_0), \ldots, f^*_k(t_{n-1}))$. (See also [21, 22] for a formal definition and proofs.) We write $f : X \rightharpoondown Y$ to denote an $\Omega\mathbf{Set}$-map $f : X \to \mathrm{Tm}_Y(V)$. Given two such morphisms $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$ their **composition** $g \circ f : X \rightharpoondown Z$ is defined to be the $\Omega\mathbf{Set}$-map $g \circ f = g^* f$. With this composition, the collection of ω-sets with the harpoon morphisms between them forms a category, denoted by **EQSIG**. Identities in **EQSIG** are the morphisms $j^{\mathrm{EQ}}_X : X \rightharpoondown X$, with $j^{\mathrm{EQ}}_{X_k}(x) = x(v_{k0}, \ldots, v_{k,k-1})$, for all $k \in \omega, x \in X_k$. The category **EQSIG** will be the signature category of the institution for equational logic.

Next, define the sentence functor $\mathrm{EQSEN} : \mathbf{EQSIG} \to \mathbf{Set}$ by

$$\mathrm{EQSEN}(X) = \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)^2_k, \quad \text{for every} \quad X \in |\mathbf{EQSIG}|,$$

and, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{EQSIG})$, $\mathrm{EQSEN}(f) : \mathrm{EQSEN}(X) \to \mathrm{EQSEN}(Y)$, is given by

$$\mathrm{EQSEN}(f)(\langle s, t\rangle) = \langle f^*_k(s), f^*_k(t)\rangle, \quad \text{if } s, t \in \mathrm{Tm}_X(V)_k,$$

for all $\langle s, t\rangle \in \mathrm{EQSEN}(X)$. EQSEN is well-defined, because, if $s \in \mathrm{Tm}_X(V)_k \cap \mathrm{Tm}_X(V)_l, k \neq l$, then it can be shown that $s \in \mathrm{Tm}_X(V)_0$ and $f^*_k(s) = f^*_0(s)$, for all $k \in \omega$. We call an $\langle s, t\rangle \in \mathrm{EQSEN}(X)$ an X**-equation** and denote it by $s \approx t$.

The model functor EQMOD : $\mathbf{EQSIG} \to \mathbf{CAT}^{\mathrm{op}}$ of the equational institution is described next. Given a set A, by $\mathrm{Cl}(A)$ is denoted the ω-set whose k-th level $\mathrm{Cl}_k(A)$ consists of all functions $f : A^k \to A$. Given an ω-set X, an X-**algebra** $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$ is a pair consisting of a set A together with an $\Omega\mathbf{Set}$-morphism $X^{\mathbf{A}} : X \to \mathrm{Cl}(A)$. If $x \in X_k$, following common usage, we write $x^{\mathbf{A}}$ for $X_k^{\mathbf{A}}(x) \in \mathrm{Cl}_k(A)$. Given two X-algebras $\mathbf{A}$ and $\mathbf{B}$, an X-**algebra homomorphism** $h : \mathbf{A} \to \mathbf{B}$ is a set map $h : A \to B$, such that, for all $n \in \omega, x \in X_n, \vec{a} \in A^n$,

$$h(x^{\mathbf{A}}(\vec{a})) = x^{\mathbf{B}}(h(\vec{a})).$$

X-algebras with X-algebra homomorphisms between them form a category, denoted by $\mathrm{EQMOD}(X)$. Given an X-algebra $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$, define an $\Omega\mathbf{Set}$-morphism ${}^{\mathbf{A}} : \mathrm{Tm}_X(V) \to \mathrm{Cl}(A)$ by letting $v_{ki}^{\mathbf{A}} : A^k \to A$ be the i-th projection function in k variables and $x(t_0, \ldots, t_{n-1})^{\mathbf{A}} = x^{\mathbf{A}}(t_0^{\mathbf{A}}, \ldots, t_{n-1}^{\mathbf{A}})$, for all $n \in \omega, x \in X_n, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$. Then, it is not difficult to define EQMOD at the morphism level. To this end, let $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{EQSIG})$. Then $\mathrm{EQMOD}(f)(\langle A, Y^{\mathbf{A}} \rangle) = \langle A, X^{\mathrm{EQMOD}(f)(\mathbf{A})} \rangle$, for all $\langle A, Y^{\mathbf{A}} \rangle \in |\mathrm{EQMOD}(Y)|$, where $x^{\mathrm{EQMOD}(f)(\mathbf{A})} = f_k(x)^{\mathbf{A}}$, for all $k \in \omega, x \in X_k$, and, if $h : \langle A, Y^{\mathbf{A}} \rangle \to \langle B, Y^{\mathbf{B}} \rangle \in \mathrm{Mor}(\mathrm{EQMOD}(Y))$, then $\mathrm{EQMOD}(f)(h) = h : \langle A, X^{\mathrm{EQMOD}(f)(\mathbf{A})} \rangle \to \langle B, X^{\mathrm{EQMOD}(f)(\mathbf{B})} \rangle$. h may be shown to be an X-algebra homomorphism and, hence, $\mathrm{EQMOD}(f)$ is well-defined at the morphism level.

Finally, for the satisfaction relation, we have, for every $X \in |\mathbf{EQSIG}|$,

$$\mathbf{A} \models_X s \approx t \quad \text{iff} \quad s^{\mathbf{A}} = t^{\mathbf{A}},$$

for all $\mathbf{A} \in |\mathrm{EQMOD}(X)|, s \approx t \in \mathrm{EQSEN}(X)$. It is not very hard to verify that the satisfaction condition holds and that $\mathcal{EQ} = \langle \mathbf{EQSIG}, \mathrm{EQSEN}, \mathrm{EQMOD}, \models \rangle$ is an institution. It is worth pointing out that $\mathcal{EQ}$ does not possess a source signature-variable pair and, hence, is not a term institution. The main reason is that, roughly speaking, sentence morphisms work level-wise. So it is not possible starting from a single sentence (the source variable of Definition 2.7) at a specific level to reach sentences of arbitrary levels.

One may develop a π-institution corresponding to first-order logic in a very similar way. Details of that construction are given in [23].

4. The Category of Theories and the Theory Functor

In [4] algebraizability of a deductive system was characterized via the existence of an isomorphism connecting the theory lattice of the deductive system with the congruence lattice of its algebraic counterpart that commutes with substitutions. In [6] this setting was generalized to include the characterization of the equivalence of an arbitrary k-deductive system with an arbitrary l-deductive system. Lattices of theories are playing a key role in this characterization as well. Since then, they have proven to be very effective model theoretic tools in the whole of abstract algebraic logic. The notion corresponding to the theory lattice in the present categorical context is that of the category of theories of a π-institution [8]. Thus, it is only natural that the category of theories of a π-institution will play a key role in categorical abstract algebraic logic in general. We summarize below the main facts concerning the category of theories of a π-institution. For a more detailed treatment see [8].

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution. Define its **category of theories** $\mathbf{TH}(\mathcal{I})$, as follows:

The objects of $\mathbf{TH}(\mathcal{I})$ are pairs $\langle \Sigma, T\rangle$, where $\Sigma \in |\mathbf{Sign}|$ and $T \subseteq \mathrm{SEN}(\Sigma)$ with $T^c = T$. The morphisms $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle$ are **Sign**-morphisms $f : \Sigma_1 \to \Sigma_2$, such that $\mathrm{SEN}(f)(T_1) \subseteq T_2$.

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, define

$$\mathbf{TH}(\mathcal{I}) = \mathbf{TH}(\pi(\mathcal{I})),$$

i.e., its **category of theories** is the category of theories of its associated π-institution. It is straightforward to verify that this notion coincides with the notion defined directly in [12]. Note also that the category of theories $\mathbf{TH}(\mathcal{I}_\mathcal{S})$ of the π-institution $\mathcal{I}_\mathcal{S}$ associated with a deductive system $\mathcal{S}$ has as its objects all pairs $\langle V, T\rangle$, where T is an $\mathcal{S}$-theory in the sense of [4], and as morphisms between $\langle V, T_1\rangle$ and $\langle V, T_2\rangle$ all assignments $f : V \rightharpoondown V$, such that $f^*(T_1) \subseteq T_2$. In particular, the wide subcategory of $\mathbf{TH}(\mathcal{I}_\mathcal{S})$ with morphisms all theory morphisms induced by the identity assignment is isomorphic to the category associated with the theory lattice of $\mathcal{S}$ in the standard way.

Now, coming back to the π-institution framework, define a functor $\mathrm{SIG} : \mathbf{TH}(\mathcal{I}) \to \mathbf{Sign}$ by

$$\mathrm{SIG}(\langle \Sigma, T\rangle) = \Sigma, \quad \text{for every} \quad \langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|,$$

and by letting $\mathrm{SIG}(f) : \Sigma_1 \to \Sigma_2$ denote the underlying **Sign**-morphism of f, for every $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$. Then the following holds.

Lemma 4.1. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution and $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$ an isomorphism. Then*

$$\mathrm{SEN}(\mathrm{SIG}(f))(T_1) = T_2.$$

PROOF. Since $f : \langle \Sigma_1, T_1 \rangle \to \langle \Sigma_2, T_2 \rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, $\mathrm{SEN}(\mathrm{SIG}(f))(T_1) \subseteq T_2$. Since $f^{-1} : \langle \Sigma_2, T_2 \rangle \to \langle \Sigma_1, T_1 \rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, we also have

$$\mathrm{SEN}(\mathrm{SIG}(f^{-1}))(T_2) \subseteq T_1.$$

Thus, $T_2 \subseteq \mathrm{SEN}(\mathrm{SIG}(f))(T_1)$, whence $\mathrm{SEN}(\mathrm{SIG}(f))(T_1) = T_2$, as was to be shown. ∎

Next, define a functor $\mathrm{THY} : \mathbf{Sign} \to \mathbf{TH}(\mathcal{I})$ by

$$\mathrm{THY}(\Sigma) = \langle \Sigma, \emptyset^c \rangle, \quad \text{for every} \quad \Sigma \in |\mathbf{Sign}|,$$

and $\mathrm{THY}(f) : \langle \Sigma_1, \emptyset^c \rangle \to \langle \Sigma_2, \emptyset^c \rangle$, with

$$\mathrm{SIG}(\mathrm{THY}(f)) = f, \quad \text{for every} \quad f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign}),$$

which is well-defined since $\mathrm{SEN}(f)(\emptyset^c)^c \subseteq \mathrm{SEN}(f)(\emptyset)^c = \emptyset^c$, by (d) of Definition 2.3.

Finally, denoting by $I_{\mathbf{Sign}}, I_{\mathbf{TH}(\mathcal{I})}$ the identity functors of $\mathbf{Sign}, \mathbf{TH}(\mathcal{I})$, respectively, define natural transformations $\eta : I_{\mathbf{Sign}} \to \mathrm{SIG} \circ \mathrm{THY}$ by

$$\eta_\Sigma : \Sigma \to \mathrm{SIG}(\mathrm{THY}(\Sigma)) \in \mathrm{Mor}(\mathbf{Sign}),$$

with

$$\eta_\Sigma = i_\Sigma, \quad \text{for every} \quad \Sigma \in |\mathbf{Sign}|,$$

and $\epsilon : \mathrm{THY} \circ \mathrm{SIG} \to I_{\mathbf{TH}(\mathcal{I})}$ by $\epsilon_{\langle \Sigma, T \rangle} : \langle \Sigma, \emptyset^c \rangle \to \langle \Sigma, T \rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, with

$$\mathrm{SIG}(\epsilon_{\langle \Sigma, T \rangle}) = i_\Sigma, \quad \text{for every} \quad \langle \Sigma, T \rangle \in |\mathbf{TH}(\mathcal{I})|.$$

Then, the following theorem ([8], Proposition 3.32) holds.

THEOREM 4.2. $\langle \mathrm{THY}, \mathrm{SIG}, \eta, \epsilon \rangle : \mathbf{Sign} \to \mathbf{TH}(\mathcal{I})$ *is an adjunction.*

Recall that if $\mathcal{S}$ is a deductive system, T is an $\mathcal{S}$-theory and $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ a substitution, $\sigma^{-1}(T)$ is always an $\mathcal{S}$-theory whereas $\sigma(T)$ is not in general an $\mathcal{S}$-theory. Thus, in dealing with the effect of substitutions on theories one has to define (see [4]) an induced operator $\sigma_{\mathcal{S}} : \mathrm{Th}_{\mathcal{S}} \to \mathrm{Th}_{\mathcal{S}}$, on the set of $\mathcal{S}$-theories $\mathrm{Th}_{\mathcal{S}}$, such that

$$\sigma_{\mathcal{S}}(T) = C_{\mathcal{S}}(\sigma(T)), \quad \text{for every} \quad T \in \mathrm{Th}_{\mathcal{S}}.$$

In the present categorical context the place of $\sigma_{\mathcal{S}}$ is taken by the morphism part of a functor $\mathrm{THS} : \mathbf{Sign} \to \mathbf{Set}$. It is defined formally as follows:

$$\mathrm{THS}(\Sigma) = \{\langle \Sigma, T \rangle : T \subseteq \mathrm{SEN}(\Sigma), T^c = T\}, \quad \text{for every } \Sigma \in |\mathbf{Sign}|,$$

i.e., $\mathrm{THS}(\Sigma)$ is the set of all Σ-theories. Given $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign})$, $\mathrm{THS}(f) : \mathrm{THS}(\Sigma_1) \to \mathrm{THS}(\Sigma_2)$ is defined by

$$\mathrm{THS}(f)(\langle \Sigma_1, T_1 \rangle) = \langle \Sigma_2, \mathrm{SEN}(f)(T_1)^c \rangle, \quad \text{for every } \langle \Sigma_1, T_1 \rangle \in \mathrm{THS}(\Sigma_1).$$

Recall that by our adopted scoping convention for "c" in Section 2, in "$\mathrm{SEN}(f)(T_1)^c$" "c" applies to $\mathrm{SEN}(f)(T_1)$ and not only to T_1. $\mathrm{THS} : \mathbf{Sign} \to \mathbf{Set}$ is indeed a functor, since, if $f : \Sigma_1 \to \Sigma_2, g : \Sigma_2 \to \Sigma_3 \in \mathrm{Mor}(\mathbf{Sign})$ and $\langle \Sigma_1, T_1 \rangle \in \mathrm{THS}(\Sigma_1)$,

$$\begin{aligned} \mathrm{THS}(gf)(\langle \Sigma_1, T_1 \rangle) &= \langle \Sigma_3, \mathrm{SEN}(gf)(T_1)^c \rangle \\ &= \langle \Sigma_3, \mathrm{SEN}(g)(\mathrm{SEN}(f)(T_1))^c \rangle \\ &= \langle \Sigma_3, \mathrm{SEN}(g)(\mathrm{SEN}(f)(T_1)^c)^c \rangle \\ &= \mathrm{THS}(g)(\langle \Sigma_2, \mathrm{SEN}(f)(T_1)^c \rangle) \\ &= \mathrm{THS}(g)(\mathrm{THS}(f)(\langle \Sigma_1, T_1 \rangle)). \end{aligned}$$

We call $\mathrm{THS} : \mathbf{Sign} \to \mathbf{Set}$ the **theory functor.**

5. Relating Categories of Theories

A key role in the theory of algebraizable deductive systems [4] is played by special properties of the Leibniz operator mapping theories to congruences on the formula algebra. For instance, monotonicity of the Leibniz operator is equivalent to protoalgebraicity and injectivity together with join-continuity is equivalent to algebraizability. In the categorical context, where the correspondence between theory lattices assumes the form of a functor between categories of theories of institutions, it is natural to look for special properties that this functor may possess and expect that they will be crucial in characterizing equivalence of institutions. Properties of this kind are studied in this and subsequent sections.

Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|} \rangle$, $\mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|} \rangle$ be two π-institutions. We now introduce properties of functors relating the categories of theories $\mathbf{TH}(\mathcal{I}_1)$ and $\mathbf{TH}(\mathcal{I}_2)$, that will be used in the sequel to give the main characterization theorems of the relations of quasi-equivalence and deductive equivalence between the π-institutions themselves.

Denote by $\pi_2 : |\mathbf{TH}(\mathcal{I}_1)| \to |\mathbf{Set}|$ the second projection, defined by

$$\pi_2(\langle \Sigma_1, T_1 \rangle) = T_1, \quad \text{for every} \quad \langle \Sigma_1, T_1 \rangle \in |\mathbf{TH}(\mathcal{I}_1)|,$$

and, similarly, $\pi_2 : |\mathbf{TH}(\mathcal{I}_2)| \to |\mathbf{Set}|$, given by

$$\pi_2(\langle \Sigma_2, T_2 \rangle) = T_2, \quad \text{for every} \quad \langle \Sigma_2, T_2 \rangle \in |\mathbf{TH}(\mathcal{I}_2)|.$$

DEFINITION 5.1. A functor $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be called

(i) **signature-respecting** if there exists a functor $F^\dagger : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, such that the following rectangle commutes

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{Sign}_1 \arrow[r, "F^\dagger"'] & \mathbf{Sign}_2 \end{tikzcd}$$

If this is the case, it is easy to verify that $F^\dagger$ is necessarily unique.

(ii) **(strongly) monotonic** if, for all $\langle \Sigma_1, T_1 \rangle, \langle \Sigma_1, T_1' \rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$T_1 \subseteq T_1' \quad \text{(if and) only if} \quad \pi_2(F(\langle \Sigma_1, T_1 \rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1' \rangle)),$$

(iii) **join-respecting** if, for all $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$(\bigcup_{\phi \in \Phi} \pi_2(F(\langle \Sigma_1, \{\phi\}^c \rangle)))^c = \pi_2(F(\langle \Sigma_1, \Phi^c \rangle)).$$

Finally, a signature-respecting functor $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be said to **commute with substitutions** if, for every $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{Sign}_1)$,

$$\mathrm{THS}_2(F^\dagger(f))(F(\langle \Sigma_1, T_1 \rangle)) = F(\mathrm{THS}_1(f)(\langle \Sigma_1, T_1 \rangle)), \tag{i}$$

for every $\langle \Sigma_1, T_1 \rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, where $F^\dagger : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is the (necessarily unique) functor of part (i).

Let $\mathcal{S}$ be an algebraizable deductive system with equivalent algebraic semantics K. In [4] it is shown that the mapping $\Omega : \mathrm{Th}_{\mathcal{S}} \to \mathrm{Th}_K$ sending an $\mathcal{S}$-theory T to the largest congruence on the formula algebra of $\mathcal{S}$ that is compatible with T, which is called the Leibniz operator, is actually an isomorphism from the lattice of $\mathcal{S}$-theories onto the lattice of K-congruences. This isomorphism induces a functor $\Omega : \mathbf{TH}(\mathcal{I}_{\mathcal{S}}) \to \mathbf{TH}(\mathcal{I}_{\mathcal{S}_K})$ by defining $\Omega(\langle V, T \rangle) = \langle V, \Omega(T) \rangle$, for every $\langle V, T \rangle \in |\mathbf{TH}(\mathcal{I}_{\mathcal{S}})|$, and

$$\Omega(f) = (f^*)^2 : \langle V, \Omega(T_1) \rangle \to \langle V, \Omega(T_2) \rangle,$$

for all $f : \langle V, T_1 \rangle \to \langle V, T_2 \rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_{\mathcal{S}}))$. In other words, a theory morphism $\sigma_{\mathcal{S}}$ induced by a substitution σ is mapped to the congruence morphism σ_K induced by the same substitution. In this case $\Omega^\dagger$ is the identity signature functor and commutativity with substitutions assumes the familiar form in [4]

$$\sigma_K(\Omega(T_1)) = \Omega(\sigma_{\mathcal{S}}(T_1)).$$

On the other hand, the join-respecting property has nothing to do with the join-continuity of [4]. In [4], $\Omega : \mathrm{Th}_{\mathcal{S}} \to \mathrm{Th}_K$ is said to be join-continuous if, for every family $T_i, i \in I$, of $\mathcal{S}$-theories, $\Omega(\bigvee^{\mathcal{S}}_{i\in I} T_i) = \bigvee^{K}_{i\in I} \Omega(T_i)$. In the case of the π-institution associated with a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ it assumes the form

$$\bigvee\nolimits^{K}_{\phi\in\Phi} \Omega(C_{\mathcal{S}}(\phi)) = \Omega(C_{\mathcal{S}}(\Phi)), \quad \text{for every} \quad \Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V).$$

Using the definition of the theory functors $\mathrm{THS}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ and $\mathrm{THS}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$, Equation (i) may be rewritten in the form

$$\mathrm{SEN}_2(F^\dagger(f))(\pi_2(F(\langle \Sigma_1, T_1\rangle)))^c = \pi_2(F(\langle \Sigma_1', \mathrm{SEN}_1(f)(T_1)^c\rangle)).$$

The properties above may be extended to the case where the two categories of theories $\mathbf{TH}(\mathcal{I}_1)$ and $\mathbf{TH}(\mathcal{I}_2)$ are related via an adjunction. The following definition then applies

Definition 5.2. An adjunction $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be called

(i) **signature-respecting** if both F and G are signature-respecting,

(ii) **(strongly) monotonic** if both F and G are (strongly) monotonic,

(iii) **join-respecting** if both F and G are join-respecting.

Finally, a signature-respecting adjunction will be said to **commute with substitutions** if both F and G commute with substitutions.

6. Relating Institutions

In [6], the notions of interpretation and of translation between two deductive systems were introduced. The notion of equivalence of deductive systems was then given. This allowed for an elegant and symmetric reformulation of the notion of algebraizability. In this section the notion of a translation and that of an interpretation are extended to fit the π-institution framework. Based on these notions, the relations of quasi-equivalence, strong quasi-equivalence and deductive equivalence, increasing in strength, can be defined between two π-institutions. These relations provide the necessary means for comparing their deductive apparatuses. The weakest notion is introduced first and the rest are then developed in increasing order of strength. Characterizations of these relations will be provided in the next sections of the paper, in terms of the strength of the ties that they impose between the categories of theories of the two π-institutions they relate.

DEFINITION 6.1. Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|} \rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|} \rangle$ be two π-institutions.

- A **translation** of $\mathcal{I}_1$ **in** $\mathcal{I}_2$ is a pair $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$ consisting of
  - (i) a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ and
  - (ii) a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2\ F$.
- A translation $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an **interpretation of** $\mathcal{I}_1$ **in** $\mathcal{I}_2$ if, for all $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\phi \in \Phi^c \quad \text{if and only if} \quad \alpha_{\Sigma_1}(\phi) \subseteq \alpha_{\Sigma_1}(\Phi)^c. \tag{ii}$$

Note that if $\mathcal{S}$ is an algebraizable deductive system and K is its equivalent algebraic semantics, a translation $\langle F, \alpha \rangle : \mathcal{I}_\mathcal{S} \to \mathcal{I}_{\mathcal{S}_K}$ is always taken to be the identity on signatures in [4]. In that case $\alpha_V : \mathrm{Fm}_\mathcal{L}(V) \to \mathcal{P}(\mathrm{Fm}_\mathcal{L}(V)^2)$ is the well-known system of defining equations $\delta \approx \epsilon$, which is always finite. In particular (ii) assumes the form

$$\Phi \vdash_\mathcal{S} \phi \quad \text{iff} \quad \{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Phi\} \models_K \delta(\phi) \approx \epsilon(\phi), \tag{iii}$$

which is the condition defining K as an algebraic semantics for $\mathcal{S}$ and is the first condition for algebraizability. The existence of an interpretation in the other direction gives the second condition for algebraizability in the context of deductive systems. The remaining two (third and fourth) conditions dictate that the two interpretations must be inverses of one another. (iii) also appears in a more general form as the definition of an interpretation from a general k-deductive system to a general l-deductive system in [6].

Using the notion of interpretation for π-institutions the following relations on π-institutions can be defined.

DEFINITION 6.2. Let $\mathcal{I}_1, \mathcal{I}_2$ be two π-institutions, as above.

- $\mathcal{I}_1$ will be said to be **interpretable in** $\mathcal{I}_2$ if there exists an interpretation $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$.

- $\mathcal{I}_1$ will be said to be **left quasi-equivalent to** $\mathcal{I}_2$ and $\mathcal{I}_2$ is **right quasi-equivalent to** $\mathcal{I}_1$ if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that

  1. $\langle F, G, \eta, \epsilon\rangle$: $\mathbf{Sign}_1 \to \mathbf{Sign}_2$ is an adjunction, for some natural transformations η, ϵ,
  2. for all $\Sigma_1 \in |\mathbf{Sign}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1)$,
  $$\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)^c \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c \tag{iv}$$
  and, for all $\Sigma_2 \in |\mathbf{Sign}_2|, \psi \in \mathrm{SEN}_2(\Sigma_2)$,
  $$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))^c \subseteq \{\psi\}^c. \tag{v}$$

  In this case $\langle F, \alpha\rangle$ is a **left quasi-inverse** of $\langle G, \beta\rangle$ and $\langle G, \beta\rangle$ a **right quasi-inverse** of $\langle F, \alpha\rangle$.

- $\mathcal{I}_1$ will be said to be **strongly left quasi-equivalent to** $\mathcal{I}_2$ and $\mathcal{I}_2$ **strongly right quasi-equivalent to** $\mathcal{I}_1$ if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that 1 and 2 above hold, but in 2 the inclusions are replaced by equalities. In this case $\langle F, \alpha\rangle$ is a **strong left quasi-inverse** of $\langle G, \beta\rangle$ and $\langle G, \beta\rangle$ a **strong right quasi-inverse** of $\langle F, \alpha\rangle$.

- $\mathcal{I}_1$ and $\mathcal{I}_2$ are **deductively equivalent** if there exist an interpretation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and an interpretation $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that $\langle F, \alpha\rangle$ and $\langle G, \beta\rangle$ are **inverses** of one another meaning that $\langle F, \alpha\rangle$ is a strong left quasi-inverse of $\langle G, \beta\rangle$ and in 1 above the adjunction is replaced by an adjoint equivalence.

Note that if $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent and $\langle F, \alpha\rangle$, $\langle G, \beta\rangle$ are inverses of each other, then each is both left and right strongly quasi-equivalent to the other and the unit and counit of the quasi-invertibility relations are natural isomorphisms.

Coming back to the case of algebraizable deductive systems, it is interesting to note that (iv) and (v) replace the invertibility relations of [4] and the generalized invertibility relations of [6]. But in the present setting, because of the additional complexity induced by varying signatures, the additional condition that the signature adjunction $\langle F, G, \eta, \epsilon\rangle$ be an adjoint equivalence is necessary to obtain full deductive equivalence. Note also that, if $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and

$\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, then, for all $\Sigma_2 \in |\mathbf{Sign}_2|$ and $\psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$\{\psi\}^c = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi))^c), \tag{vi}$$

and, for all $\Sigma_1 \in |\mathbf{Sign}_1|$ and $\phi \in \mathrm{SEN}_1(\Sigma_1)$,

$$\{\phi\}^c = \mathrm{SEN}_1(\eta_{\Sigma_1})^{-1}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c). \tag{vii}$$

In this case (vi) and (vii) are equivalent to the equality versions of (v) and (iv), respectively, in view of Corollaries 2.4 and 2.6 and the fact that η_{Σ_1} and ϵ_{Σ_2} are isomorphisms.

We define the corresponding notions for institutions using their associated π-institutions.

DEFINITION 6.3. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two institutions.

- $\mathcal{I}_1$ is **interpretable in** $\mathcal{I}_2$ if $\pi(\mathcal{I}_1)$ is interpretable in $\pi(\mathcal{I}_2)$.
- $\mathcal{I}_1$ is **(strongly) left quasi-equivalent to** $\mathcal{I}_2$ if $\pi(\mathcal{I}_1)$ is (strongly) left quasi-equivalent to $\pi(\mathcal{I}_2)$ and, similarly, for **(strong) right quasi-equivalence.**
- $\mathcal{I}_1$ and $\mathcal{I}_2$ are **deductively equivalent** if $\pi(\mathcal{I}_1)$ and $\pi(\mathcal{I}_2)$ are deductively equivalent.

A technical lemma that will be used very often in what follows is given first.

LEMMA 6.4. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two π-institutions and* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *an interpretation. Then*

$$\alpha_{\Sigma_1}(\Phi^c)^c = \alpha_{\Sigma_1}(\Phi)^c, \quad \textit{for all} \quad \Sigma_1 \in |\mathbf{Sign}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1). \tag{viii}$$

PROOF. Clearly, $\alpha_{\Sigma_1}(\Phi)^c \subseteq \alpha_{\Sigma_1}(\Phi^c)^c$. Since α is an interpretation, $\alpha_{\Sigma_1}(\Phi^c) \subseteq \alpha_{\Sigma_1}(\Phi)^c$, whence $\alpha_{\Sigma_1}(\Phi^c)^c \subseteq (\alpha_{\Sigma_1}(\Phi)^c)^c$, i.e., $\alpha_{\Sigma_1}(\Phi^c)^c \subseteq \alpha_{\Sigma_1}(\Phi)^c$, as required. ∎

Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle$, $\mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ be two π-institutions such that there exist translations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and an adjunction $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. It is routine to check that in case any of the relations (iv)-(vii) holds, then the same relation is valid with the single sentence ϕ or ψ replaced by a set of sentences. This fact will be used repeatedly in what follows without being explicitly stated.

Moreover, it can be shown (see [20]) that the existence of the two translations and of the adjoint equivalence together with conditions (ii) and (vi) are sufficient for the deductive equivalence of $\mathcal{I}_1$ and $\mathcal{I}_2$. This parallels Corollary 2.9 of [4], where an analogous result is proved for algebraizable deductive systems.

7. Equational Algebra

In this section, an algebraic counterpart $\mathcal{EA}$ of the institution $\mathcal{EQ}$, given in Section 3 is constructed. This is an institution based on an algebraic theory in monoid form in an appropriately chosen category. The definition of deductive equivalence, presented in the previous section, will provide the necessary framework in which $\mathcal{EQ}$ will be shown to be related to its algebraic counterpart $\mathcal{EA}$.

The construction of the institution $\mathcal{EA} = \langle \mathbf{EASIG}, \mathrm{EASEN}, \mathrm{EAMOD}, \models\rangle$ is outlined first. This institution is based on an algebraic theory $\mathbf{T}_{\mathrm{EQ}} = \langle T_{\mathrm{EQ}}, \eta_{\mathrm{EQ}}, \mu_{\mathrm{EQ}}\rangle$ in monoid form in the category $\Omega\mathbf{Set}$ of ω-sets. It may be shown that the equational institution $\mathcal{EQ}$ and $\mathcal{EA}$ are deductively equivalent institutions. Details of the construction and the equivalence will be provided elsewhere.

First, an adjunction $\langle F^{\mathrm{EQ}}, U^{\mathrm{EQ}}, \eta^{\mathrm{EQ}}, \epsilon^{\mathrm{EQ}}\rangle : \Omega\mathbf{Set} \to \mathbf{EQSIG}$ is constructed as follows: The functor $F^{\mathrm{EQ}} : \Omega\mathbf{Set} \to \mathbf{EQSIG}$ is the identity on objects and sends an $\Omega\mathbf{Set}$-morphism $f : X \to Y$ to the morphism $j_Y^{\mathrm{EQ}} f : X \rightharpoondown Y$ in $\mathbf{EQSIG}$ obtained by composing f with the embedding of Y in $\mathrm{Tm}_Y(V)$, taking the k-ary basic operation $y \in Y_k$ to the Y-term $y(v_{k,0}, \ldots, v_{k,k-1})$. The functor $U^{\mathrm{EQ}} : \mathbf{EQSIG} \to \Omega\mathbf{Set}$ sends an object X of $\mathbf{EQSIG}$ to the ω-set $\mathrm{Tm}_X(V)$ of X-terms with variables in V and an $\mathbf{EQSIG}$-morphism $f : X \rightharpoondown Y$ to the ω-set map $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ obtained by extending f to X-terms (see also Section 3). The natural transformation $\eta^{\mathrm{EQ}} : I_{\Omega\mathbf{Set}} \to U^{\mathrm{EQ}}F^{\mathrm{EQ}}$ from the identity functor on $\Omega\mathbf{Set}$ to the composite of F^{EQ} and U^{EQ} assigns to each ω-set X the standard embedding j_X^{EQ} of X into $\mathrm{Tm}_X(V)$ sending $x \in X_k$ to $x(v_{k,0}, \ldots, v_{k,k-1})$. Finally, the natural transformation $\epsilon^{\mathrm{EQ}} : F^{\mathrm{EQ}}U^{\mathrm{EQ}} \to I_{\mathbf{EQSIG}}$ from the composite of U^{EQ} and F^{EQ} to the identity functor on $\mathbf{EQSIG}$ assigns to each object X of $\mathbf{EQSIG}$ the identity map $i_{\mathrm{Tm}_X(V)}$ viewed as an $\mathbf{EQSIG}$-morphism from $\mathrm{Tm}_X(V)$ to X.

This adjunction gives rise in a standard way to an algebraic theory $\mathbf{T}_{\mathrm{EQ}} = \langle T_{\mathrm{EQ}}, \eta_{\mathrm{EQ}}, \mu_{\mathrm{EQ}}\rangle$ in $\Omega\mathbf{Set}$ by setting $T_{\mathrm{EQ}} = U^{\mathrm{EQ}}F^{\mathrm{EQ}}, \eta_{\mathrm{EQ}} = \eta^{\mathrm{EQ}}$

and $\mu_{\mathrm{EQ}} = U^{\mathrm{EQ}}\epsilon^{\mathrm{EQ}}_{F^{\mathrm{EQ}}}$ (see, e.g., [16], p.134). It turns out that the Kleisli category $\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}$ coincides with **EQSIG**. A $\mathbf{T}_{\mathrm{EQ}}$-algebra in $\Omega\mathbf{Set}$ is now a pair $\mathbf{X} = \langle X, \xi\rangle$ consisting of an ω-set X together with an $\Omega\mathbf{Set}$-morphism $\xi : \mathrm{Tm}_X(V) \to X$, such that the following diagrams commute

$$\begin{tikzcd}
X \arrow[r, "\eta_{\mathrm{EQ}_X}"] \arrow[dr, "i_X"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[r, "T_{\mathrm{EQ}}(\xi)"] \arrow[d, "\mu_{\mathrm{EQ}_X}"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\
\mathrm{Tm}_X(V) \arrow[r, "\xi"'] & X
\end{tikzcd}$$

The ω-set X is called the **carrier** or **universe** of the algebra $\mathbf{X}$ and the $\Omega\mathbf{Set}$-morphism ξ is called the **structure map** of $\mathbf{X}$. Given two $\mathbf{T}_{\mathrm{EQ}}$-algebras $\mathbf{X} = \langle X, \xi\rangle$ and $\mathbf{Y} = \langle Y, \zeta\rangle$, a $\mathbf{T}_{\mathrm{EQ}}$-algebra homomorphism is an $\Omega\mathbf{Set}$-morphism $h : X \to Y$ that preserves the algebra structure, i.e., such that the following rectangle commutes.

$$\begin{tikzcd}
\mathrm{Tm}_X(V) \arrow[r, "T_{\mathrm{EQ}}(h)"] \arrow[d, "\xi"'] & \mathrm{Tm}_Y(V) \arrow[d, "\zeta"] \\
X \arrow[r, "h"'] & Y
\end{tikzcd}$$

The collection of all $\mathbf{T}_{\mathrm{EQ}}$-algebras together with $\mathbf{T}_{\mathrm{EQ}}$-algebra homomorphisms between them forms a category $\Omega\mathbf{Set}^{\mathbf{T}_{\mathrm{EQ}}}$, known as the Eilenberg-Moore category of $\mathbf{T}_{\mathrm{EQ}}$-algebras in $\Omega\mathbf{Set}$. $\Omega\mathbf{Set}^{\mathbf{T}_{\mathrm{EQ}}}$ is usually thought of as a class of algebras of the same similarity type that is equationally defined, since Eilenberg-Moore categories of algebras for algebraic theories in **Set** roughly correspond to universal algebraic varieties of algebras [17]. Thus, the passage from $\mathcal{EQ}$ to $\mathcal{EA}$, as defined below, is not a trivial process. Models of $\mathcal{EQ}$ are first-order models over varying algebraic signatures whereas models of $\mathcal{EA}$ come from algebras of the same similarity type that, in addition, belong to some equationally defined class of algebras corresponding to the chosen algebraic theory $\mathbf{T}_{\mathrm{EQ}}$. The significance of this passage in the general theory of categorical abstract algebraic logic and its key role in the algebraization of a multi-sorted logical system was emphasized in [20] and will be further studied in [21], where a further step towards the formulation of a general theory of algebraizability along the lines of [4, 6] will be taken. Namely, based on the notion of an algebraic theory, the general notion of *algebraic institution* will be defined, which roughly corresponds to a 2-deductive system with an equational consequence relation. Algebraic institutions will then be used in conjunction with the graded notions of equivalence, that were defined in Section 6, to give a definition of corresponding graded notions of *algebraizability* for institutions.

Recall from Section 3.2 that given a set A, by $\mathrm{Cl}(A)$ is denoted the ω-set whose k-th level consists of all k-ary operations on A. Given a set A, one may construct a $\mathbf{T}_{\mathrm{EQ}}$-algebra $\mathbf{A}^* = \langle \mathrm{Cl}(A), \xi_A\rangle$, with universe $\mathrm{Cl}(A)$, whose structure map reflects the way clone operations compose in $\mathrm{Cl}(A)$. Letting $\mathcal{Q}_{\mathrm{EQ}}$ be the full subcategory of $\Omega\mathbf{Set}^{\mathbf{T}_{\mathrm{EQ}}}$ with objects $\{\mathbf{A}^* = \langle \mathrm{Cl}(A), \xi_A\rangle : A \in |\mathbf{Set}|\}$, construct the institution $\mathcal{EA} = \langle \mathbf{EASIG}, \mathrm{EASEN}, \mathrm{EAMOD}, \models\rangle$ as follows:

(i) $\mathbf{EASIG} = \mathbf{EQSIG}$

(ii) $\mathrm{EASEN} = \mathrm{EQSEN}$

(iii) For every $X \in |\mathbf{EASIG}|, \mathrm{EAMOD}(X)$ is the category with objects pairs $\langle \mathbf{A}^*, f\rangle, \mathbf{A}^* \in |\mathcal{Q}_{\mathrm{EQ}}|, f : X \rightharpoondown \mathrm{Cl}(A) \in \mathrm{Mor}(\mathbf{EASIG})$, and morphisms $h : \langle \mathbf{A}^*, f\rangle \to \langle \mathbf{B}^*, g\rangle$, $\mathbf{T}_{\mathrm{EQ}}$-algebra homomorphisms $h : \mathbf{A}^* \to \mathbf{B}^*$, such that $g = h \circ f$. Moreover, given $k : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{EASIG})$, $\mathrm{EAMOD}(k) : \mathrm{EAMOD}(Y) \to \mathrm{EAMOD}(X)$ is the functor that maps an object $\langle \mathbf{A}^*, f\rangle \in |\mathrm{EAMOD}(Y)|$ to $\langle \mathbf{A}^*, f \circ k\rangle \in |\mathrm{EAMOD}(X)|$ and a morphism $h : \langle \mathbf{A}^*, f\rangle \to \langle \mathbf{B}^*, g\rangle \in \mathrm{Mor}(\mathrm{EAMOD}(Y))$ to the morphism $\mathrm{EAMOD}(k)(h) : \langle \mathbf{A}^*, f \circ k\rangle \to \langle \mathbf{B}^*, g \circ k\rangle$, with $\mathrm{EAMOD}(f)(h) = h$.

(iv) Finally, satisfaction in $\mathcal{EA}$ is defined, for every $X \in |\mathbf{EASIG}|$, by

$$\langle \mathbf{A}^*, f\rangle \models_X s \approx t \quad \text{iff} \quad \xi_A(f^*(s)) = \xi_A(f^*(t)),$$

for all $\langle \mathbf{A}^*, f\rangle \in |\mathrm{EAMOD}(X)|, s \approx t \in \mathrm{EASEN}(X)$.

It is not hard to see that $\mathcal{EA}$ is an institution and that $\mathcal{EA}$ and $\mathcal{EQ}$ are deductively equivalent, via the interpretations $\langle I_{\mathbf{EQSIG}}, \alpha\rangle : \mathcal{EQ} \to \mathcal{EA}$, and $\langle I_{\mathbf{EASIG}}, \beta\rangle : \mathcal{EA} \to \mathcal{EQ}$, where

$$\alpha_X(s \approx t) = \{s \approx t\}, \qquad \text{for all } X \in |\mathbf{EQSIG}|, s \approx t \in \mathrm{EQSEN}(X) \quad \text{and}$$

$$\beta_X(s \approx t) = \{s \approx t\}, \qquad \text{for all } X \in |\mathbf{EASIG}|, s \approx t \in \mathrm{EASEN}(X).$$

The key observation for this equivalence is that to an X-algebra $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle$ one may associate the $\mathbf{T}_{\mathrm{EQ}}$-algebra $\langle \mathrm{Cl}(A), \xi_A\rangle \in |\mathcal{Q}_{\mathrm{EQ}}|$ and a morphism $f : X \rightharpoondown \mathrm{Cl}(A)$, such that $f_k(x) = \eta^{\mathrm{EQ}}_{\mathrm{Cl}(A)_k}(x^{\mathbf{A}})$, for all $k \in \omega, x \in X_k$, and to each model $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, f\rangle$ one may associate an X-algebra $\mathbf{A} = \langle A, \xi_A f\rangle$, such that the two members in each of these pairs satisfy the same equations in their respective institutions.

As an example, let $X = \{X_k : k \in \omega\}$, with $X_0 = X_1 = \emptyset, X_2 = \{\cdot\}$ and $X_n = \emptyset$, for all $n \geq 3$, i.e., X is the signature of groupoids. To a groupoid $\langle A, \cdot^{\mathbf{A}}\rangle \in |\mathrm{EQMOD}(X)|$ one associates the $\mathbf{T}_{\mathrm{EQ}}$-algebra $\langle \mathrm{Cl}(A), \xi_A\rangle$ with the morphism $\eta_{\mathrm{Cl}(A)} X^{\mathbf{A}} : X \rightharpoondown \mathrm{Cl}(A)$, with $\eta_{\mathrm{Cl}(A)}(X^{\mathbf{A}}(\cdot)) = v_{20} \cdot^{\mathbf{A}} v_{21}$, and to $\langle\langle \mathrm{Cl}(A), \xi_A\rangle, g\rangle \in |\mathrm{EAMOD}(X)|$ one associates the groupoid $\langle A, \xi_A(g(\cdot))\rangle$. Then the assertion expresses the fact that, for any groupoid term t,

$$t^{\mathbf{A}} = \xi_A((\eta^{\mathrm{EQ}}_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*(t)) \quad \text{and} \quad t^{\mathbf{A}} = \xi_A(g^*(t)).$$

$$\mathrm{Tm}_X(V) \xrightarrow{(\eta^{\mathrm{EQ}}_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*} \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \xrightarrow{\xi_A} \mathrm{Cl}(A)$$

$$\mathrm{Tm}_X(V) \xrightarrow{g^*} \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \xrightarrow{\xi_A} \mathrm{Cl}(A)$$

8. Interpretability

In this section a characterization of interpretability of a term π-institution $\mathcal{I}_1$ in a π-institution $\mathcal{I}_2$ is provided. Namely, it will be shown that a term π-institution $\mathcal{I}_1$ is interpretable in a π-institution $\mathcal{I}_2$ if and only if there exists a strongly monotonic, join-respecting, signature-respecting functor from $\mathbf{TH}(\mathcal{I}_1)$ into $\mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions. A characterization of the existence of a translation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$, from a term π-institution $\mathcal{I}_1$ to a π-institution $\mathcal{I}_2$ is given first. This will just require and will be guaranteed by the existence of a signature-respecting functor from $\mathbf{TH}(\mathcal{I}_1)$ into $\mathbf{TH}(\mathcal{I}_2)$.

Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ be two π-institutions and $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ a translation. Define $F^{\#} : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ as follows.

$$F^{\#}(\langle \Sigma_1, T_1\rangle) = \langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle, \quad \text{for every} \quad \langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|,$$

and, given $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1', T_1'\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_1))$,

$$F^{\#}(f) : \langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle \to \langle F(\Sigma_1'), \alpha_{\Sigma_1'}(T_1')^c\rangle$$

is determined by

$$\mathrm{SIG}(F^{\#}(f)) = F(\mathrm{SIG}(f)),$$

where SIG : $\mathbf{TH}(\mathcal{I}_1) \to \mathbf{Sign}_1$ and SIG : $\mathbf{TH}(\mathcal{I}_2) \to \mathbf{Sign}_2$ denote the forgetful functors from theories to signatures, defined formally in Section 4.

SIG being faithful, $F^{\#}(f)$ is well-defined, and it is a theory morphism since

$$\begin{array}{rcl}\mathrm{SEN}_2(\mathrm{SIG}(F^{\#}(f)))(\alpha_{\Sigma_1}(T_1)^c)^c & = & \mathrm{SEN}_2(F(\mathrm{SIG}(f)))(\alpha_{\Sigma_1}(T_1))^c \\ & & \text{(by Corollary 2.4)} \\ & = & \alpha_{\Sigma_1'}(\mathrm{SEN}_1(\mathrm{SIG}(f))(T_1))^c \\ & & \text{(since } \alpha \text{ is a natural transf.)} \\ & \subseteq & \alpha_{\Sigma_1'}(T_1')^c \\ & & \text{(since } f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1', T_1'\rangle).\end{array}$$

$F^{\#} : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is a functor, since, if $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1', T_1'\rangle, g : \langle \Sigma_1', T_1'\rangle \to \langle \Sigma_1'', T_1''\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_1))$, we have

$$\begin{array}{rcl}\mathrm{SIG}(F^{\#}(gf)) & = & F(\mathrm{SIG}(gf)) \\ & = & F(\mathrm{SIG}(g))F(\mathrm{SIG}(f)) \\ & = & \mathrm{SIG}(F^{\#}(g))\mathrm{SIG}(F^{\#}(f)) \\ & = & \mathrm{SIG}(F^{\#}(g)F^{\#}(f))\end{array}$$

and therefore $F^{\#}(gf) = F^{\#}(g)F^{\#}(f)$. Finally $F^{\#}$ is signature-respecting. In fact, the following diagram commutes:

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F^{\#}"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{Sign}_1 \arrow[r, "F"'] & \mathbf{Sign}_2 \end{tikzcd}$$

For every $\langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$\begin{array}{rcl}\mathrm{SIG}(F^{\#}(\langle \Sigma_1, T_1\rangle)) & = & \mathrm{SIG}(\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle) \\ & = & F(\Sigma_1) \\ & = & F(\mathrm{SIG}(\langle \Sigma_1, T_1\rangle)),\end{array}$$

and, for every $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1', T_1'\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_1))$, we have, by definition of $F^{\#}$,

$$\mathrm{SIG}(F^{\#}(f)) = F(\mathrm{SIG}(f)).$$

Suppose, next, that $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle$ is a term π-institution, $\mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ is a π-institution and $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ a signature-respecting functor. Then, by Definition

5.1(i), there exists a (necessarily unique) functor $F^\dagger : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, such that the following rectangle commutes

$$\begin{tikzcd}
\mathbf{TH}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\
\mathbf{Sign}_1 \arrow[r, "F^\dagger"'] & \mathbf{Sign}_2
\end{tikzcd}$$

Since $\mathcal{I}_1$ is term, there exists a source signature-variable pair $\langle A, p\rangle$ for $\mathcal{I}_1$. Set

$$\Theta = \pi_2(F(\langle A, \{p\}^c\rangle)).$$

Define $\alpha^F : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F^\dagger$ by $\alpha^F_{\Sigma_1} : \mathrm{SEN}_1(\Sigma_1) \to \mathcal{P}(\mathrm{SEN}_2(F^\dagger(\Sigma_1)))$, with

$$\alpha^F_{\Sigma_1}(\phi) = \mathcal{P}\mathrm{SEN}_2(F^\dagger(f_{\langle\Sigma_1,\phi\rangle}))(\Theta), \quad \text{for all} \quad \Sigma_1 \in |\mathbf{Sign}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1).$$

To show that the pair $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is a translation, it suffices to show that $\alpha^F : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F^\dagger$ is a natural transformation, i.e., that the following diagram commutes, for every $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{Sign}_1)$.

$$\begin{tikzcd}
\mathrm{SEN}_1(\Sigma_1) \arrow[r, "\alpha^F_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}_1(f)"'] & \mathcal{P}\mathrm{SEN}_2(F^\dagger(\Sigma_1)) \arrow[d, "\mathcal{P}\mathrm{SEN}_2(F^\dagger(f))"] \\
\mathrm{SEN}_1(\Sigma_1') \arrow[r, "\alpha^F_{\Sigma_1'}"'] & \mathcal{P}\mathrm{SEN}_2(F^\dagger(\Sigma_1'))
\end{tikzcd}$$

For every $\phi \in \mathrm{SEN}_1(\Sigma_1)$, we have

$$\begin{array}{rcl}
\mathcal{P}\mathrm{SEN}_2(F^\dagger(f))(\alpha^F_{\Sigma_1}(\phi)) & = & \mathcal{P}\mathrm{SEN}_2(F^\dagger(f))(\mathcal{P}\mathrm{SEN}_2(F^\dagger(f_{\langle\Sigma_1,\phi\rangle}))(\Theta)) \\
& & \text{(by definition of } \alpha^F_{\Sigma_1}) \\
& = & \mathcal{P}\mathrm{SEN}_2(F^\dagger(ff_{\langle\Sigma_1,\phi\rangle}))(\Theta) \\
& & \text{(since } \mathcal{P}\mathrm{SEN}_2 F^\dagger \text{ is a functor)} \\
& = & \mathcal{P}\mathrm{SEN}_2(F^\dagger(f_{\langle\Sigma_1',\mathrm{SEN}_1(f)(\phi)\rangle}))(\Theta) \\
& & \text{(by the term property)} \\
& = & \alpha^F_{\Sigma_1'}(\mathrm{SEN}_1(f)(\phi)) \text{ (by definition of } \alpha^F_{\Sigma_1'}),
\end{array}$$

as required. Thus, $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is indeed a translation. This establishes the following

THEOREM 8.1. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions.*

(i) *If* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *is a translation, then* $F^\# : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a signature-respecting functor.*

(ii) *If* $\mathcal{I}_1$ *is term and* $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a signature-respecting functor, then* $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *is a translation.*

With the help of Theorem 8.1, it is not difficult to obtain a similar characterization for the existence of an interpretation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ from a term π-institution $\mathcal{I}_1$ to a π-institution $\mathcal{I}_2$. Namely, we have

THEOREM 8.2. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions.*

(i) *If* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *is an interpretation, then* $F^\# : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a strongly monotonic, join-respecting, signature-respecting functor that commutes with substitutions.*

(ii) *If* $\mathcal{I}_1$ *is term and* $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a strongly monotonic, join-respecting, signature-respecting functor that commutes with substitutions, then* $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *is an interpretation.*

PROOF. (i) By part (i) of Theorem 8.1, it suffices to show that $F^\# : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is strongly monotonic, join-respecting and commutes with substitutions. To this end, let $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$. Then

$$
\begin{array}{lll}
T_1 \subseteq T_1' & \text{iff} & \alpha_{\Sigma_1}(T_1)^c \subseteq \alpha_{\Sigma_1}(T_1')^c \text{ (since } \alpha \text{ is an interpretation)}\\
 & \text{iff} & \pi_2(F^\#(\langle \Sigma_1, T_1\rangle)) \subseteq \pi_2(F^\#(\langle \Sigma_1, T_1'\rangle))\\
 & & \text{(by the definition of } F^\#, \pi_2).
\end{array}
$$

To show that $F^\#$ is join-respecting, let $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$. Then

$$
\begin{array}{lll}
(\bigcup_{\phi\in\Phi} \pi_2(F^\#(\langle \Sigma_1, \{\phi\}^c\rangle)))^c & = & (\bigcup_{\phi\in\Phi} \alpha_{\Sigma_1}(\{\phi\}^c)^c)^c\\
 & & \text{(by the definitions of } F^\#, \pi_2)\\
 & = & (\bigcup_{\phi\in\Phi} \alpha_{\Sigma_1}(\phi)^c)^c \text{ (by Lemma 6.4)}\\
 & = & (\bigcup_{\phi\in\Phi} \alpha_{\Sigma_1}(\phi))^c\\
 & = & \alpha_{\Sigma_1}(\Phi)^c\\
 & = & \alpha_{\Sigma_1}(\Phi^c)^c \text{ (by Lemma 6.4)}\\
 & = & \pi_2(F^\#(\langle \Sigma_1, \Phi^c\rangle))\\
 & & \text{(by the definitions of } F^\#, \pi_2).
\end{array}
$$

Finally, for commutativity with substitutions, we have, for all $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{Sign}_1)$,

$$\begin{array}{rcl}
\mathrm{SEN}_2(F(f))(\pi_2(F^{\#}(\langle \Sigma_1, T_1\rangle)))^c & = & \mathrm{SEN}_2(F(f))(\alpha_{\Sigma_1}(T_1)^c)^c \\
 & & \text{(by the definitions of } F^{\#}, \pi_2) \\
 & = & \mathrm{SEN}_2(F(f))(\alpha_{\Sigma_1}(T_1))^c \text{ (by Cor. 2.4)} \\
 & = & \alpha_{\Sigma_1'}(\mathrm{SEN}_1(f)(T_1))^c \\
 & & \text{(since } \alpha \text{ is a natural transformation)} \\
 & = & \alpha_{\Sigma_1'}(\mathrm{SEN}_1(f)(T_1)^c)^c \text{ (by Lemma 6.4)} \\
 & = & \pi_2(F^{\#}(\langle \Sigma_1', \mathrm{SEN}_1(f)(T_1)^c\rangle)) \\
 & & \text{(by definitions of } F^{\#}, \pi_2).
\end{array}$$

(ii) By part (ii) of Theorem 8.1, it suffices to show that, for every $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$, we have

$$\phi \in \Phi^c \quad \text{iff} \quad \alpha^F_{\Sigma_1}(\phi) \subseteq \alpha^F_{\Sigma_1}(\Phi)^c.$$

To this end, we first prove that

$$\pi_2(F(\langle \Sigma_1, \Phi^c\rangle)) = \alpha^F_{\Sigma_1}(\Phi)^c, \quad \text{for all} \quad \Sigma_1 \in |\mathbf{Sign}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1). \quad \text{(ix)}$$

In fact, we have

$$\begin{array}{rcl}
\alpha^F_{\Sigma_1}(\Phi)^c & = & (\bigcup_{\phi\in\Phi} \alpha^F_{\Sigma_1}(\phi))^c \\
 & = & (\bigcup_{\phi\in\Phi} \mathcal{P}\mathrm{SEN}_2(F^{\dagger}(f_{\langle\Sigma_1,\phi\rangle}))(\Theta))^c \text{ (by the definition of } \alpha^F_{\Sigma_1}) \\
 & = & (\bigcup_{\phi\in\Phi} \mathcal{P}\mathrm{SEN}_2(F^{\dagger}(f_{\langle\Sigma_1,\phi\rangle}))(\pi_2(F(\langle A, \{p\}^c\rangle))))^c \\
 & & \text{(by definition of } \Theta) \\
 & = & (\bigcup_{\phi\in\Phi} \mathcal{P}\mathrm{SEN}_2(F^{\dagger}(f_{\langle\Sigma_1,\phi\rangle}))(\pi_2(F(\langle A, \{p\}^c\rangle)))^c)^c \\
 & = & (\bigcup_{\phi\in\Phi} \pi_2(F(\langle \Sigma_1, \mathrm{SEN}_1(f_{\langle\Sigma_1,\phi\rangle})(p)^c\rangle)))^c \\
 & & \text{(by commutativity with substitutions)} \\
 & = & (\bigcup_{\phi\in\Phi} \pi_2(F(\langle \Sigma_1, \{\phi\}^c\rangle)))^c \text{ (by the term property)} \\
 & = & \pi_2(F(\langle \Sigma_1, \Phi^c\rangle)) \text{ (by join-continuity).}
\end{array}$$

Finally, let $\Sigma_1 \in |\mathbf{Sign}_1|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$. Then

$$\begin{array}{rcl}
\alpha^F_{\Sigma_1}(\phi) \subseteq \alpha^F_{\Sigma_1}(\Phi)^c & \text{iff} & \alpha^F_{\Sigma_1}(\phi)^c \subseteq \alpha^F_{\Sigma_1}(\Phi)^c \\
 & \text{iff} & \pi_2(F(\langle \Sigma_1, \{\phi\}^c\rangle)) \subseteq \pi_2(F(\langle \Sigma_1, \Phi^c\rangle)) \\
 & & \text{by Equation (ix),} \\
 & \text{iff} & \{\phi\}^c \subseteq \Phi^c \ \text{ by strong monotonicity,} \\
 & \text{iff} & \phi \in \Phi^c,
\end{array}$$

as required. ■

9. Quasi-Equivalence

In this section the relation of quasi-equivalence between two term π-institutions $\mathcal{I}_1$ and $\mathcal{I}_2$ is characterized. As a corollary, a characterization of strong quasi-equivalence is obtained. This also yields a characterization of deductive equivalence by looking at the special case where the adjunction between the signature categories happens to be an adjoint equivalence. However, in the main result of the next section, Theorem 10.5, it will be shown that in this special case, the additional requirement that the unit and counit of the adjunction be natural isomorphisms can simplify the conditions imposed significantly.

It is worth pointing out that the main characterization theorem of this section, Theorem 9.4, and its proof are slightly more complicated than the corresponding result for deductive systems because in the present context one has to deal with the increased complexity of the signature categories of the π-institutions involved. The reader may wish at this point to compare the two examples presented in Section 3 as an illustration of this fact.

Lemma 9.1. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions and* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *a signature-respecting adjunction. Then, for all* $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, $\langle \Sigma_2, T_2\rangle, \langle \Sigma_2, T_2'\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$,

$$\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,T_1'\rangle}) \quad \textit{and} \quad \mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2\rangle}) = \mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2'\rangle}).$$

Proof. We show that, for all $\Sigma_1 \in |\mathbf{Sign}_1|, \langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,\emptyset^c\rangle}).$$

To this end, consider the theory morphism $i : \langle \Sigma_1, \emptyset^c\rangle \to \langle \Sigma_1, T_1\rangle$, such that $\mathrm{SIG}(i) = i_{\Sigma_1}$. This morphism agrees on signatures with the morphism $i_{\langle\Sigma_1,\emptyset^c\rangle} : \langle \Sigma_1, \emptyset^c\rangle \to \langle \Sigma_1, \emptyset^c\rangle$, that is also the identity on signatures, by definition. Thus, by signature-respectivity,

$$\begin{array}{rcl}\mathrm{SIG}(F(i)) & = & \mathrm{SIG}(F(i_{\langle\Sigma_1,\emptyset^c\rangle}))\\ & = & \mathrm{SIG}(i_{F(\langle\Sigma_1,\emptyset^c\rangle)}).\end{array}$$

Similarly, by signature-respectivity, the above equation yields

$$\begin{array}{rcl}\mathrm{SIG}(G(F(i))) & = & \mathrm{SIG}(G(i_{F(\langle\Sigma_1,\emptyset^c\rangle)}))\\ & = & \mathrm{SIG}(i_{G(F(\langle\Sigma_1,\emptyset^c\rangle))})\\ & = & i_{\mathrm{SIG}(G(F(\langle\Sigma_1,\emptyset^c\rangle)))},\end{array}$$

and, therefore, the following diagram commutes, by the naturality of η :

$$\begin{tikzcd}
\mathrm{SIG}(\langle\Sigma_1,\emptyset^c\rangle) \arrow[r, "\mathrm{SIG}(\eta_{\langle\Sigma_1,\emptyset^c\rangle})"] \arrow[d, "i_{\Sigma_1}"'] & \mathrm{SIG}(G(F(\langle\Sigma_1,\emptyset^c\rangle))) \arrow[d, "i_{\mathrm{SIG}(G(F(\langle\Sigma_1,\emptyset^c\rangle)))}"] \\
\mathrm{SIG}(\langle\Sigma_1,T_1\rangle) \arrow[r, "\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle})"'] & \mathrm{SIG}(G(F(\langle\Sigma_1,T_1\rangle)))
\end{tikzcd}$$

This shows that $\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}) = \mathrm{SIG}(\eta_{\langle\Sigma_1,\emptyset^c\rangle})$, as required. The corresponding relation for the counit ϵ can be proved similarly. ■

Recall from Section 5 that, given a signature-respecting functor $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$, by $F^\dagger : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is denoted the induced signature functor such that $\mathrm{SIG} \circ F = F^\dagger \circ \mathrm{SIG}$. We denote by $\eta^\dagger_{\Sigma_1} : \Sigma_1 \to G^\dagger(F^\dagger(\Sigma_1))$ the common value $\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle})$, for all Σ_1-theories $\langle\Sigma_1,T_1\rangle$, and by $\epsilon^\dagger_{\Sigma_2} : F^\dagger(G^\dagger(\Sigma_2)) \to \Sigma_2$ the common value $\mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2\rangle})$, for all Σ_2-theories $\langle\Sigma_2,T_2\rangle$.

Using Lemma 9.1, it is not hard to see that the following holds

LEMMA 9.2. *Let* $\mathcal{I}_1 = \langle\mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle\mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions.*

(i) *If* $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a signature-respecting adjunction, then* $\langle F^\dagger, G^\dagger, \eta^\dagger, \epsilon^\dagger\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ *is an adjunction.*

(ii) *Moreover, if* $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a signature-respecting adjoint equivalence then* $\langle F^\dagger, G^\dagger, \eta^\dagger, \epsilon^\dagger\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ *is an adjoint equivalence.*

The following definition will be used in the characterization of strong quasi-equivalence.

DEFINITION 9.3. Let $\mathcal{I}_1 = \langle\mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle\mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ be two π-institutions. An adjunction $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be said to be **strong** if the following hold

(i) $\mathrm{SEN}_1(\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}))(T_1)^c = \pi_2(G(F(\langle\Sigma_1,T_1\rangle)))$, for every $\langle\Sigma_1,T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, and

(ii) $\mathrm{SEN}_2(\mathrm{SIG}(\epsilon_{\langle\Sigma_2,T_2\rangle}))(\pi_2(F(G(\langle\Sigma_2,T_2\rangle))))^c = T_2$, for every $\langle\Sigma_2,T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$.

THEOREM 9.4. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions.*

(i) *If* $\mathcal{I}_1$ *is left quasi-equivalent (respectively, strongly left quasi-equivalent, deductively equivalent) to* $\mathcal{I}_2$ *via the interpretations* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjunction (respectively, adjunction, adjoint equivalence)* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, *then* $\langle F^\#, G^\#, \eta^\#, \epsilon^\#\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a strongly monotonic, join-respecting, signature-respecting adjunction (respectively, strong adjunction, adjoint equivalence) that commutes with substitutions, where*

$$\mathrm{SIG}(\eta^\#_{\langle\Sigma_1,T_1\rangle}) = \eta_{\Sigma_1} \quad \textit{and} \quad \mathrm{SIG}(\epsilon^\#_{\langle\Sigma_2,T_2\rangle}) = \epsilon_{\Sigma_2},$$

for all $\langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|, \langle \Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$.

(ii) *If* $\mathcal{I}_1, \mathcal{I}_2$ *are term and* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is a strongly monotonic, join-respecting, signature-respecting adjunction (respectively, strong adjunction, adjoint equivalence) that commutes with substitutions, then* $\mathcal{I}_1$ *is left quasi-equivalent (respectively, strongly left quasi-equivalent, deductively equivalent) to* $\mathcal{I}_2$ *via the interpretations* $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G^\dagger, \alpha^G\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjunction (respectively, adjoint equivalence)* $\langle F^\dagger, G^\dagger, \eta^\dagger, \epsilon^\dagger\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$.

PROOF. We restrict our attention to the case of quasi-equivalence. The remaining cases may be handled similarly.

(i) Let $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ be the two interpretations and $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ the adjunction witnessing the quasi-equivalence relation between $\mathcal{I}_1$ and $\mathcal{I}_2$. By part (i) of Theorem 8.2 there exist strongly monotonic, join-respecting, signature-respecting functors $F^\# : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2), G^\# : \mathbf{TH}(\mathcal{I}_2) \to \mathbf{TH}(\mathcal{I}_1)$, that commute with substitutions. Define $\eta^\# : I_{\mathbf{TH}(\mathcal{I}_1)} \to G^\# F^\#$ and $\epsilon^\# : F^\# G^\# \to I_{\mathbf{TH}(\mathcal{I}_2)}$ as in the statement of the theorem. Since, by the definition of left quasi-equivalence,

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(T_1)^c \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c$$

and

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(T_2)))^c \subseteq T_2,$$

both $\eta^\#_{\langle\Sigma_1,T_1\rangle}$ and $\epsilon^\#_{\langle\Sigma_2,T_2\rangle}$ are well-defined theory morphisms and it is clear that $\langle F^\#, G^\#, \eta^\#, \epsilon^\#\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is an adjunction. Since both $F^\#$ and $G^\#$ are strongly monotonic, join-respecting, signature-respecting and commute with substitutions, $\langle F^{\#}, G^{\#}, \eta^{\#}, \epsilon^{\#}\rangle$ is also strongly monotonic, join-respecting, signature-respecting and commutes with substitutions.

(ii) By Theorem 8.2, $\langle F^\dagger, \alpha^F\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G^\dagger, \alpha^G\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ are interpretations. Furthermore, since $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is signature-respecting, $\eta^\dagger : I_{\mathbf{Sign}_1} \to G^\dagger F^\dagger$ and $\epsilon^\dagger : F^\dagger G^\dagger \to I_{\mathbf{Sign}_2}$ are such that $\langle F^\dagger, G^\dagger, \eta^\dagger, \epsilon^\dagger\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is an adjunction. Thus, it only remains to show that

$$\mathrm{SEN}_1(\eta^\dagger_{\Sigma_1})(\phi)^c \subseteq \alpha^G_{F^\dagger(\Sigma_1)}(\alpha^F_{\Sigma_1}(\phi))^c,$$

for all $\Sigma_1 \in |\mathbf{Sign}_1|$ and all $\phi \in \mathrm{SEN}_1(\Sigma_1)$ and

$$\mathrm{SEN}_2(\epsilon^\dagger_{\Sigma_2})(\alpha^F_{G^\dagger(\Sigma_2)}(\alpha^G_{\Sigma_2}(\psi)))^c \subseteq \{\psi\}^c,$$

for all $\Sigma_2 \in |\mathbf{Sign}_2|$ and $\psi \in \mathrm{SEN}_2(\Sigma_2)$. We show the first.

$$\begin{aligned}
\alpha^G_{F^\dagger(\Sigma_1)}(\alpha^F_{\Sigma_1}(\phi))^c &= \alpha^G_{F^\dagger(\Sigma_1)}(\alpha^F_{\Sigma_1}(\phi)^c)^c \text{ (by Lemma 6.4)}\\
&= \alpha^G_{F^\dagger(\Sigma_1)}(\alpha^F_{\Sigma_1}(\{\phi\}^c)^c)^c \text{ (by Lemma 6.4)}\\
&= \alpha^G_{F^\dagger(\Sigma_1)}(\pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c \text{ (by Equation (ix))}\\
&= \pi_2(G(\langle F^\dagger(\Sigma_1), \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle))\rangle)) \text{ (by Eqt. (ix))}\\
&= \pi_2(G(F(\langle\Sigma_1, \{\phi\}^c\rangle)))\\
&\quad \text{(since } F(\langle\Sigma_1, \{\phi\}^c\rangle) = \langle F^\dagger(\Sigma_1), \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle))\rangle)\\
&\supseteq \mathrm{SEN}_1(\eta^\dagger_{\Sigma_1})(\{\phi\}^c)^c \text{ (by Lemma 9.1)}\\
&= \mathrm{SEN}_1(\eta^\dagger_{\Sigma_1})(\phi)^c.
\end{aligned}$$

The second may be shown analogously. ∎

10. Deductive Equivalence

The notion of deductive equivalence was defined for π-institutions in Section 6 and a characterization was obtained for the deductive equivalence of two term π-institutions in terms of their categories of theories in Theorem 9.4 of the preceding section as a special case of a similar characterization for the more general notion of quasi-equivalence. In this section, we exploit the special additional features present in the case of a deductive equivalence, more precisely, the fact that units and counits of the adjunctions involved are natural isomorphisms, to obtain a refinement of Theorem 9.4 for the case of deductive equivalence.

LEMMA 10.1. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions. A signature-respecting adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is monotonic.*

PROOF. Suppose $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ is signature-respecting and let $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, with $T_1 \subseteq T_1'$. Then, the identity on Σ_1 induces a theory morphism $i : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1, T_1'\rangle$. This morphism agrees on signatures with the identity $i_{\langle \Sigma_1, T_1\rangle} : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_1, T_1\rangle$, whence, by signature-respectivity,

$$\begin{aligned}\mathrm{SIG}(F(i)) &= \mathrm{SIG}(F(i_{\langle \Sigma_1, T_1\rangle}))\\ &= \mathrm{SIG}(i_{F(\langle \Sigma_1, T_1\rangle)})\\ &= i_{\mathrm{SIG}(F(\langle \Sigma_1, T_1\rangle))}.\end{aligned}$$

Thus, $F(i) : F(\langle \Sigma_1, T_1\rangle) \to F(\langle \Sigma_1, T_1'\rangle)$ is the identity on signatures, showing that

$$\pi_2(F(\langle \Sigma_1, T_1\rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1'\rangle)).$$

By symmetry, for all $\langle \Sigma_2, T_2\rangle, \langle \Sigma_2, T_2'\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$, with $T_2 \subseteq T_2'$,

$$\pi_2(G(\langle \Sigma_2, T_2\rangle)) \subseteq \pi_2(G(\langle \Sigma_2, T_2'\rangle)),$$

as required. ∎

LEMMA 10.2. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions. A signature-respecting adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is injective on* Σ_1*-theories, i.e., for all* $\Sigma_1 \in |\mathbf{Sign}_1|, \langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$\langle \Sigma_1, T_1\rangle \neq \langle \Sigma_1, T_1'\rangle \quad \textit{implies} \quad F(\langle \Sigma_1, T_1\rangle) \neq F(\langle \Sigma_1, T_1'\rangle)$$

and the same holds for Σ_2*-theories, for every* $\Sigma_2 \in |\mathbf{Sign}_2|$.

PROOF. Let $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$. If $F(\langle \Sigma_1, T_1\rangle) = F(\langle \Sigma_1, T_1'\rangle)$, then, by signature-respectivity and Lemma 9.1,

$$\begin{aligned}&\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle \Sigma_1, T_1\rangle}))(\pi_2(G(F(\langle \Sigma_1, T_1\rangle)))) =\\ &\qquad\qquad \mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle \Sigma_1, T_1'\rangle}))(\pi_2(G(F(\langle \Sigma_1, T_1'\rangle)))),\end{aligned}$$

whence, by Lemma 4.1, $T_1 = T_1'$. An analogous argument can be used for G. ∎

LEMMA 10.3. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions. A signature-respecting adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is join-respecting.*

PROOF. Let $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1)$. Since, by Lemma 10.1, $\langle F, G, \eta, \epsilon\rangle$ is monotonic,

$$\pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)) \subseteq \pi_2(F(\langle\Sigma_1, \Phi^c\rangle)), \quad \text{for every} \quad \phi \in \Phi,$$

whence

$$(\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c \subseteq \pi_2(F(\langle\Sigma_1, \Phi^c\rangle)).$$

Suppose that the inclusion is proper, i.e., that

$$(\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c \subset \pi_2(F(\langle\Sigma_1, \Phi^c\rangle)).$$

Then, by Lemmas 10.1 and 10.2, if $\Sigma_2 = \mathrm{SIG}(F(\langle\Sigma_1, \Phi^c\rangle))$, we have

$$\begin{array}{rl} \pi_2(G(\langle\Sigma_2, (\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c\rangle)) \subset & \\ \subset & \pi_2(G(\langle\Sigma_2, \pi_2(F(\langle\Sigma_1, \Phi^c\rangle))\rangle)) \\ = & \pi_2(G(F(\langle\Sigma_1, \Phi^c\rangle))), \end{array}$$

whence, since $\eta_{\langle\Sigma_1,\Phi^c\rangle}$ is an isomorphism,

$$\begin{array}{l} \mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(\langle\Sigma_2, (\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c\rangle))) \subset \\ \qquad \mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(F(\langle\Sigma_1, \Phi^c\rangle)))) \end{array}$$

i.e., by Lemma 4.1,

$$\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(\langle\Sigma_2, (\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c\rangle))) \subset \Phi^c. \qquad \text{(x)}$$

Now, note that

$$\pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)) \subseteq (\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c,$$

for every $\phi \in \Phi$, whence, by Lemma 10.1,

$$\pi_2(G(F(\langle\Sigma_1, \{\phi\}^c\rangle))) \subseteq \pi_2(G(\langle\Sigma_2, (\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c\rangle)),$$

and, hence,

$$(\bigcup_{\phi\in\Phi} \pi_2(G(F(\langle\Sigma_1, \{\phi\}^c\rangle))))^c \subseteq \pi_2(G(\langle\Sigma_2, (\bigcup_{\phi\in\Phi} \pi_2(F(\langle\Sigma_1, \{\phi\}^c\rangle)))^c\rangle)).$$

Thus, by Lemma 9.1,

$$\mathrm{SEN}_1(\eta^{\dagger^{-1}}_{\Sigma_1})((\bigcup_{\phi\in\Phi}\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle))))^c)\subseteq$$
$$\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\Phi^c\rangle}))(\pi_2(G(\langle\Sigma_2,(\bigcup_{\phi\in\Phi}\pi_2(F(\langle\Sigma_1,\{\phi\}^c\rangle)))^c\rangle))),$$

where recall that

$$\eta^{\dagger}_{\Sigma_1}=\mathrm{SIG}(\eta_{\langle\Sigma_1,T_1\rangle}),\quad \text{for every}\quad \Sigma_1\text{-theory }\langle\Sigma_1,T_1\rangle\in|\mathbf{TH}(\mathcal{I}_1)|.$$

Therefore, by (x), and Corollaries 2.6 and 2.4, we have

$$\mathrm{SEN}_1(\eta^{\dagger^{-1}}_{\Sigma_1})(\bigcup_{\phi\in\Phi}\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle))))^c\subset\Phi^c,$$

i.e., by Lemma 9.1,

$$(\bigcup_{\phi\in\Phi}\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,\{\phi\}^c\rangle}))(\pi_2(G(F(\langle\Sigma_1,\{\phi\}^c\rangle)))))^c\subset\Phi^c,$$

whence, by Lemma 4.1,

$$(\bigcup_{\phi\in\Phi}\{\phi\}^c)^c\subset\Phi^c,\ \text{i.e.,}\ \Phi^c\subset\Phi^c,$$

a contradiction. ∎

LEMMA 10.4. *Let* $\mathcal{I}_1=\langle\mathbf{Sign}_1,\mathrm{SEN}_1,\{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle,\mathcal{I}_2=\langle\mathbf{Sign}_2,\mathrm{SEN}_2,\{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions and* $\langle F,G,\eta,\epsilon\rangle:\mathbf{TH}(\mathcal{I}_1)\to\mathbf{TH}(\mathcal{I}_2)$ *a signature-respecting adjoint equivalence. Then, for all* $\langle\Sigma_1,T_1\rangle,\langle\Sigma_1,T_1'\rangle\in|\mathbf{TH}(\mathcal{I}_1)|$,

$$T_1\subseteq T_1'\quad \textit{iff}\quad \pi_2(F(\langle\Sigma_1,T_1\rangle))\subseteq\pi_2(F(\langle\Sigma_1,T_1'\rangle)),$$

and, similarly, for G.

PROOF. The “only if” holds by Lemma 10.1.

For the “if” direction, assume that $\pi_2(F(\langle\Sigma_1,T_1\rangle))\subseteq\pi_2(F(\langle\Sigma_1,T_1'\rangle))$. Then we must have, by Lemma 10.1,

$$\pi_2(G(F(\langle\Sigma_1,T_1\rangle)))\subseteq\pi_2(G(F(\langle\Sigma_1,T_1'\rangle))),$$

and, therefore, by Lemma 9.1,

$$\mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,T_1\rangle}))(\pi_2(G(F(\langle\Sigma_1,T_1\rangle)))) \subseteq \mathrm{SEN}_1(\mathrm{SIG}(\eta^{-1}_{\langle\Sigma_1,T_1'\rangle}))(\pi_2(G(F(\langle\Sigma_1,T_1'\rangle)))),$$

i.e., by Lemma 4.1, $T_1 \subseteq T_1'$, as required. ■

THEOREM 10.5. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two term* π*-institutions.* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence* $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

PROOF. A stronger "only if" was proved in part (i) of Theorem 9.4.

For the "if" part, it suffices, by part (ii) of Theorem 9.4, to show that the signature-respecting adjoint equivalence $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ that commutes with substitutions is also strongly monotonic and join-continuous. But this was shown in Lemmas 10.4 and 10.3, respectively. ■

Theorem 10.5 is a direct generalization of Theorems 3.7 of [4] and V3.5 of [6] modulo the signature-respectivity condition which is needed here to tame the much more complex structure of the category of signatures of an arbitrary π-institution as compared to the one associated with a deductive system $\mathcal{S}$ in the sense of [4, 6][1].

Since the notions of deductive equivalence and the category of theories for institutions were defined in terms of the corresponding notions on the associated π-institutions, Theorem 10.5 can be reformulated to fit in the institution framework as follows:

COROLLARY 10.6. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \mathrm{MOD}_1, \models^1\rangle$, $\mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \mathrm{MOD}_2, \models^2\rangle$ *be two term institutions.* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are deductively equivalent if and only if there exists a signature-respecting adjoint equivalence* $\langle F,G,\eta,\epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

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[1]The author believes that he was, very recently, able to further generalize Theorem 10.5 to arbitrary (not necessarily term) π-institutions [24].

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