CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: GENTZEN (π)-INSTITUTIONS

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Abstract. Josep Maria Font and Ramon Jansana, inspired by previous work of Czelakowski, Blok and Pigozzi and of other members of the Barcelona algebraic logic group, studied the interaction between the algebraization of deductive systems in the sense of Tarski and the algebraization of Gentzen systems, connected with the deductive systems in various ways. Only recently, did the author extend the notion of a Gentzen system to the π-institution level and this extension provides the framework for the extension of some of the results of Font and Jansana to the categorical abstract algebraic logic level.

1 Introduction In [12] Font and Jansana used abstract logics, introduced by Suszko and his collaborators in [5] and [7], to provide an alternative algebraic semantics for sentential logics to the traditionally used logical matrix semantics (see, e.g., [3, 4]). An abstract logic $\mathbb{L} = \langle \mathbf{A}, C\rangle$ consists of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ together with a closure operator C on the universe A of the algebra, as opposed to the single set $F \subseteq A$, the set of designated elements or filter, of a logical matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$. As a consequence, in the theory of [12], the well-known Leibniz operator of Blok and Pigozzi [3], as applied to filters over an algebra, is replaced by the Tarski operator, which is applied to closure systems over a given algebra. Logics are classified, depending on different properties possessed by the Leibniz operator on their models, into several steps of an algebraic hierarchy. The major classes consist of the protoalgebraic [2], the equivalential [8] and the algebraizable logics [3]. An overview of the hierarchy may be found in Czelakowski's book [9] and in the survey article [13]. The theories of the Tarski operator and that of the Leibniz operator give identical results when applied to protoalgebraic sentential logics, which are at the lowest end of the algebraic hierarchy. These are generally understood to be the most primitive algebraically, but yet rich enough to be amenable to algebraic logic techniques. However, the theory of abstract logics and the Tarski operator give new insights in the case of non-protoalgebraic logics and reveal some of the ties that exist between logics in different steps of the hierarchy that are obscure when the matrix models and the Leibniz operator alone are used. Chapter 5 of [12] contains many applications of this theory to different specific sentential logics.

One of the main concepts in [12] is that of a full model of a sentential logic $\mathcal{S}$. A full model of $\mathcal{S}$ is an abstract logic $\mathbb{L} = \langle \mathbf{A}, \mathcal{C}\rangle$ whose Tarski quotient $\mathbb{L}^* = \langle \mathbf{A}^*, \mathcal{C}^*\rangle$ has as its closure set system the entire collection $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}^*$ of $\mathcal{S}$-filters on the algebra $\mathbf{A}^*$. The collection of all full models of $\mathcal{S}$ on $\mathbf{A}$ is denoted by $\mathrm{FMod}_{\mathcal{S}}\mathbf{A}$. Of a particular interest to us, in the present context, is the use by Font and Jansana of Gentzen systems to investigate properties of sentential logics (see Chapter 4 of [12]). Two main considerations lead them

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to the introduction of Gentzen systems as auxiliary tools for the study of sentential logics. The first is the general observation that the definition of full models involves second-order reasoning. This property hints to the use of Gentzen systems, whose deduction is a higher order deduction. The second, related to the first, is the specific question of whether the full models of a specific logic may be characterized exactly as the models of a given Gentzen system that is somehow related to the sentential logic.

To put the investigations on a sound basis, Font and Jansana introduce the notion of an abstract logic being a model of a Gentzen system. Roughly speaking, $\mathbb{L} = \langle \mathbf{A}, C \rangle$ is a model of the Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mid\!\sim_{\mathcal{G}} \rangle$ if the entailment $\mid\!\sim_{\mathcal{G}}$ can be interpreted into the entailment of C-entailments. The two notions that serve for relating sentential logics with Gentzen systems are that of a Gentzen system $\mathcal{G}$ being adequate for a sentential logic $\mathcal{S}$ and that of a Gentzen system $\mathcal{G}$ being fully adequate[1] for a sentential logic $\mathcal{S}$. Again roughly speaking, $\mathcal{G}$ is adequate for $\mathcal{S}$ when $\vdash_{\mathcal{S}}$ is exactly the logic defined by $\mid\!\sim_{\mathcal{G}}$ in the sense that its entailments are exactly the theorem sequents of $\mathcal{G}$. On the other hand, $\mathcal{G}$ is fully adequate for $\mathcal{S}$ when the full models of $\mathcal{S}$ correspond to the models of $\mathcal{G}$. Of course, if $\mathcal{G}$ is fully adequate for $\mathcal{S}$, then $\mathcal{G}$ is also adequate for $\mathcal{S}$.

Font and Jansana define the notion of an $\mathcal{S}$-algebra and of a $\mathcal{G}$-algebra. An algebra $\mathbf{A}$ is an $\mathcal{S}$-algebra if the abstract logic consisting of all the $\mathcal{S}$-filters on $\mathbf{A}$ is reduced. This is tantamount to saying that $\mathbf{A}$ is the algebraic reduct of a reduced full model of $\mathcal{S}$. The class of all $\mathcal{S}$-algebras is denoted by $\mathbf{Alg}\mathcal{S}$. The collection of all $\mathbf{Alg}\mathcal{S}$-congruences on an algebra $\mathbf{A}$, i.e., congruences on $\mathbf{A}$ whose quotient algebras lie in $\mathbf{Alg}\mathcal{S}$, is denoted, as usual, by $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\mathbf{A}$. In the Isomorphism Theorem 2.30 of [12], it is shown that, given an algebra $\mathbf{A}$, the Tarski operator is an order-isomorphism between $\langle \mathrm{FMod}_{\mathcal{S}}\mathbf{A}, \leq \rangle$ and $\langle \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\mathbf{A}, \subseteq \rangle$. On the other hand an algebra $\mathbf{A}$ is a $\mathcal{G}$-algebra if it is the algebraic reduct of a reduced model of $\mathcal{G}$. $\mathbf{Alg}\mathcal{G}$ denotes the class of all $\mathcal{G}$-algebras.

In terms of the classes of algebras $\mathbf{Alg}\mathcal{S}$ and $\mathbf{Alg}\mathcal{G}$ associated with $\mathcal{S}$ and with $\mathcal{G}$, respectively, it is shown that, $\mathcal{G}$ being adequate for $\mathcal{S}$ implies that $\mathbf{Alg}\mathcal{G} \subseteq \mathbf{Alg}\mathcal{S}$, whereas in the case of full adequacy the two classes are identical.

The present work is continuing work presented in a series of papers by the author adapting aspects of the theory of Font and Jansana to make it suitable for handling institutional logics. The origin of the concept of institution lies in the work of Goguen and Burstall [14, 15] in the domain of specifications of programming languages. The idea of using it in the framework of a categorical theory of algebraization originated with Diskin [10]. It was more rigorously pursued by the author in the dissertation [19], written under the supervision of Don Pigozzi (see also [20, 21, 22]). The categorical theory uses the notion of a π-institution, introduced by Fiadeiro and Sernadas in [11]. π-institutions possess a general entailment system rather than a semantical entailment, as is the case with institutions. Gentzen π-institutions, corresponding to Gentzen systems, were introduced in [23] and were used to give a characterization of those π-institutions having a specific form of the Deduction-Detachment Theorem. The first paper that deals with a generalization of the Tarski operator to the institutional level is [24] (see also [26]). Introduction of logical congruence systems and of the Tarski congruence system, in particular, which is the largest logical congruence of a π-institution, allows the development of a model theory for π-institutions that parallels the model theory of sentential logics and is inspired by methods and results of universal algebra. The notions of a basic full model and of a full model from sentential logics were adapted to the π-institution framework in [25]. There, they were called min and full models, respectively[2]. An order preserving correspondence between

[1] "Fully adequate" was originally called "strongly adequate" in [12] but was later renamed by Font, Jansana and Pigozzi [13] to the more suggestive term.

[2] The difference in terminology is not deep or philosophical. The author thinks that it is more suggestive

full models and logical congruence systems in the style of the correspondence between full models and $\mathcal{S}$-congruences of a sentential logic $\mathcal{S}$ ([12], Theorem 2.30) is provided in [27].

In the present work, Gentzen π-institutions are used in the institutional framework to provide an adaptation of the part of the theory of [12] that explores the connections of sentential logics with Gentzen systems. More specifically, the notion of a π-institution $\mathcal{I}$ serving as a model of a Gentzen π-institution $\mathcal{G}$ is introduced. In analogy with sentential logics, this roughly means that the closure system of $\mathcal{G}$ may be interpreted into that of $\mathcal{I}$. Similarly, the notions of a Gentzen π-institution being adequate and fully adequate for a given π-institution are introduced and results paralleling those governing the relationship of a sentential logic and of a Gentzen system adequate or fully adequate for it are proved.

Throughout the paper, by **Set** will be denoted the category of all small sets, by $\mathcal{P}$: **Set** $\rightarrow$ **Set** the power set functor and by $\_^2$: **Set** $\rightarrow$ **Set** the cartesian square functor. For all other categorical concepts and unexplained categorical notation the reader is referred to any of [1],[6] or [17].

Note also that the present line of work is continued further in [28].

2 Gentzen π-Institutions Recall from [11] that a **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ is a triple consisting of

(i) a category **Sign**, whose objects are called **signatures** and whose morphisms are called **assignments**,

(ii) a functor SEN : **Sign** $\rightarrow$ **Set** from the category of signatures to the category of small sets, giving, for each $\Sigma \in |\mathbf{Sign}|$, the set of **$\Sigma$-sentences** $\mathrm{SEN}(\Sigma)$ and mapping an assignment $f : \Sigma_1 \rightarrow \Sigma_2$ to a **substitution** $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma_1) \rightarrow \mathrm{SEN}(\Sigma_2)$,

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called **$\Sigma$-closure,** such that

(a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

A family $C = \{C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))\}_{\Sigma \in |\mathbf{Sign}|}$ will be referred to as a **closure system** on SEN : **Sign** $\rightarrow$ **Set** if it satisfies (iii)(a)-(d) above. Note that a closure system in the present sense consists of a family of closure operators rather than a family of closure systems in the traditional sense, that, in addition to the three closure conditions (iii)(a)-(c), also satisfy the structurality condition (iii)(d).

It is well-known that, given an institution $\mathcal{I}$, a π-institution $\pi(\mathcal{I})$ results by taking the semantic closure relations of $\mathcal{I}$ as the closure relations of $\pi(\mathcal{I})$ [11]. Therefore the abundance of examples of institutions in the literature (see, for instance, [14, 15, 16, 18, 20, 21, 22]) immediately yields, via this construction, many important examples of logics formulated as π-institutions. We will not present any more examples here.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ be two π-institutions. A **translation** $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ from $\mathcal{I}$ to $\mathcal{I}'$ consists of a functor $F : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \rightarrow \mathcal{P}\mathrm{SEN}'F$ [19, 20]. It is said to be **singleton**, denoted $\langle F, \alpha \rangle : \mathcal{I} \rightarrow^s \mathcal{I}'$,

for mnemonical purposes to use "min models" than "basic full models".

if $|\alpha_\Sigma(\phi)| = 1$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$. A translation $\langle F, \alpha\rangle : \mathcal{I} \to \mathcal{I}'$ is called a **semi-interpretation**, denoted $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!- \mathcal{I}'$, if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{implies} \quad \alpha_\Sigma(\phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Finally, it is called an **interpretation**, denoted by $\langle F, \alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$, if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

A π-institution $\mathcal{I}$ is said to be **semi-interpretable** in a π-institution $\mathcal{I}'$ if there exists a semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!- \mathcal{I}'$. Similarly, $\mathcal{I}$ is said to be **interpretable** in $\mathcal{I}'$ if there exists an interpretation $\langle F, \alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$.

In the following definition the notation $\mathcal{P}\mathrm{SEN}^2$ will be used to denote the functor $(\mathcal{P}\mathrm{SEN})^2$ (and *not* the functor $\mathcal{P}(\mathrm{SEN}^2)$).

Definition 1 *A* **Gentzen π-institution** $\mathcal{I}$ *is a π-institution*

$$\mathcal{I} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle,$$

where SEN : **Sign** $\to$ **Set** *is a functor and C is a closure system on* $\mathcal{P}\mathrm{SEN}^2$*, such that*

<u>*Axiom*</u> $\langle \Phi, \Phi\rangle \in C_\Sigma(\emptyset)$, *for all* $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$,

<u>*Weakening*</u> $\langle \Gamma \cup \Psi, \Phi\rangle \in C_\Sigma(\langle \Gamma, \Phi\rangle)$, *for all* $\Sigma \in |\mathbf{Sign}|, \Gamma, \Psi, \Phi \subseteq \mathrm{SEN}(\Sigma)$,

<u>*Cut*</u> $\langle \Gamma, \Psi\rangle \in C_\Sigma(\langle \Gamma, \Phi\rangle, \langle \Gamma \cup \Phi, \Psi\rangle)$, *for all* $\Sigma \in |\mathbf{Sign}|, \Gamma, \Psi, \Phi \subseteq \mathrm{SEN}(\Sigma)$.

<u>*Entailment*</u> $\langle \Phi, \{\psi\}\rangle \in C_\Sigma(\mathbf{\Phi})$, *for all* $\psi \in \Psi$, *implies that* $\langle \Phi, \Psi\rangle \in C_\Sigma(\mathbf{\Phi})$, *for all* $\Sigma \in |\mathbf{Sign}|$, $\Phi, \Psi \subseteq \mathrm{SEN}(\Sigma), \mathbf{\Phi} \subseteq \mathcal{P}\mathrm{SEN}(\Sigma)^2$ *and* $\phi \in \mathrm{SEN}(\Sigma)$.

In the present setting, the following notational conventions will be used: By a capital Greek letter, like Γ, Φ, etc, will be denoted subsets of $\mathrm{SEN}(\Sigma)$. By vectored capital Greek letters, like $\vec{\Gamma}, \vec{\Phi}$, etc., will be denoted pairs of subsets of $\mathrm{SEN}(\Sigma)$, with $\vec{\Gamma} = \langle \Gamma_1, \Gamma_2\rangle, \vec{\Phi} = \langle \Phi_1, \Phi_2\rangle$, etc. By boldfaced capital Greek letters, such as $\mathbf{\Gamma}, \mathbf{\Phi}$, etc. will be denoted collections of pairs of subsets of $\mathrm{SEN}(\Sigma)$. Finally, we often adopt the notation $\Gamma_1 \vdash_\Sigma \Gamma_2$ for $\langle \Gamma_1, \Gamma_2\rangle$ when $\langle \Gamma_1, \Gamma_2\rangle \in \mathcal{P}\mathrm{SEN}(\Sigma)^2$ and $\mathbf{\Phi} \mid\!\sim_\Sigma \vec{\Phi}$ for $\vec{\Phi} \in C_\Sigma(\mathbf{\Phi})$. These are all well-known notational conventions from the theory of ordinary Gentzen systems (see, for instance, Chapter 4 of [12]). With these conventions in place, the axioms of Axiom, Weakening and Cut above take, respectively, the forms

- $\mid\!\sim_\Sigma \Phi \vdash_\Sigma \Phi$,
- $\Gamma \vdash_\Sigma \Phi \mid\!\sim_\Sigma \Gamma, \Psi \vdash_\Sigma \Phi$,
- $\Gamma \vdash_\Sigma \Phi;\ \Gamma, \Phi \vdash_\Sigma \Psi \mid\!\sim_\Sigma \Gamma \vdash_\Sigma \Psi$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathcal{P}\mathrm{SEN}'^2, C'\rangle$ be two Gentzen π-institutions. A **Gentzen translation** $\langle F, \alpha\rangle : \mathcal{I} \to^G \mathcal{I}'$ from $\mathcal{I}$ to $\mathcal{I}'$ consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{SEN}'$. It is said to be **singleton**, denoted $\langle F, \alpha\rangle : \mathcal{I} \to^{Gs} \mathcal{I}'$, if $|\alpha_\Sigma(\phi)| = 1$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$. A Gentzen translation $\langle F, \alpha\rangle : \mathcal{I} \to^G \mathcal{I}'$ is called a **Gentzen semi-interpretation**, denoted $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^G \mathcal{I}'$, if, for all $\Sigma \in |\mathbf{Sign}|$, $\mathbf{\Phi} \cup \{\vec{\Phi}\} \subseteq \mathcal{P}(\mathcal{P}\mathrm{SEN}(\Sigma)^2)$,

$$\vec{\Phi} \in C_\Sigma(\mathbf{\Phi}) \quad \text{implies} \quad \alpha^2_\Sigma(\vec{\Phi}) \subseteq C'_{F(\Sigma)}(\alpha^2_\Sigma(\mathbf{\Phi})).$$

Finally, it is called a **Gentzen interpretation**, denoted by $\langle F, \alpha\rangle : \mathcal{I} \vdash^G \mathcal{I}'$, if, for all $\Sigma \in |\mathbf{Sign}|, \mathbf{\Phi} \cup \{\vec{\Phi}\} \subseteq \mathcal{P}(\mathcal{P}\mathrm{SEN}(\Sigma)^2)$,

$$\vec{\Phi} \in C_\Sigma(\mathbf{\Phi}) \quad \text{iff} \quad \alpha^2_\Sigma(\vec{\Phi}) \subseteq C'_{F(\Sigma)}(\alpha^2_\Sigma(\mathbf{\Phi})).$$

Gentzen semi-interpretable and **Gentzen interpretable** are defined very similarly to the terms semi-interpretable and interpretable, respectively, except that they refer to Gentzen interpretations and Gentzen semi-interpretations.

Note that the notion of a Gentzen translation differs from the notion of translation between two arbitrary π-institutions as introduced previously in [19, 20]. Also note that in the definitions above, $\alpha_\Sigma(\Phi) := \{\alpha_\Sigma(\phi) : \phi \in \Phi\}$, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$, extends the notation to sets of sentences and $\alpha^2_\Sigma(\vec{\Phi}) = \langle \alpha_\Sigma(\Phi_1), \alpha_\Sigma(\Phi_2)\rangle$, for all $\vec{\Phi} = \langle \Phi_1, \Phi_2\rangle$, further extends the notation to pairs of sets of sentences. Finally, $\alpha^2_\Sigma(\mathbf{\Phi}) := \{\alpha^2_\Sigma(\vec{\Phi}) : \vec{\Phi} \in \mathbf{\Phi}\}$. This abuse of notation should not cause any confusion since the intended meaning will, hopefully, be clear from context.

The following small table summarizes the various translation concepts and the associated symbols:

	General	Gentzen
translation	$\rightarrow$	$\rightarrow^G$
semi-interpretation	$\succ$	$\succ^G$
interpretation	$\vdash$	$\vdash^G$

Given a Gentzen π-institution $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, C\rangle$, there is a natural way to construct a closure system on SEN that reflects some of the properties of the closure system C of $\mathcal{G}$.

Proposition 2 *Let $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a Gentzen π-institution. Define $\mathcal{G}^s = \langle \mathbf{Sign}, \mathrm{SEN}, \{C^s_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$, such that, for all $\Sigma \in |\mathbf{Sign}|, \Gamma \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,*

$$\phi \in C^s_\Sigma(\Gamma) \quad \textit{iff} \quad \langle \Gamma, \{\phi\}\rangle \in C_\Sigma(\emptyset) \quad \textit{i.e., iff} \quad \vdash\!\!\sim_\Sigma \Gamma \vdash_\Sigma \phi.$$

Then $\mathcal{G}^s$ is also a π-institution.

Proof:

Let $\Sigma \in |\mathbf{Sign}|$. Axiom and Weakening for Gentzen π-institutions combined yield $\phi \in C^s_\Sigma(\Gamma)$, for all $\phi \in \Gamma$.

Now suppose that $\phi \in C^s_\Sigma(C^s_\Sigma(\Gamma))$. Then $C^s_\Sigma(\Gamma) \vdash_\Sigma \phi \in C_\Sigma(\emptyset)$. But, by Entailment, $\Gamma \vdash_\Sigma C^s_\Sigma(\Gamma) \in C_\Sigma(\emptyset)$, which, together with $C^s_\Sigma(\Gamma) \vdash_\Sigma \phi \in C_\Sigma(\emptyset)$, imply, by Cut, that $\Gamma \vdash_\Sigma \phi \in C_\Sigma(\emptyset)$, whence $\phi \in C^s_\Sigma(\Gamma)$.

If $\Gamma \subseteq \Delta$ and $\phi \in C^s_\Sigma(\Gamma)$, then $\Gamma \vdash_\Sigma \phi \in C_\Sigma(\emptyset)$, whence, since $\Delta \vdash_\Sigma \Gamma \in C_\Sigma(\emptyset)$, by Axiom and Weakening, we get $\Delta \vdash_\Sigma \phi \in C_\Sigma(\emptyset)$, by Cut. Hence $\phi \in C^s_\Sigma(\Delta)$.

Finally, suppose $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and let $\phi \in C^s_{\Sigma_1}(\Gamma)$. Then $\Gamma \vdash_{\Sigma_1} \phi \in C_{\Sigma_1}(\emptyset)$, whence, by the structurality of C, we get that $\mathrm{SEN}(f)(\Gamma) \vdash_{\Sigma_2} \mathrm{SEN}(f)(\phi) \in C_{\Sigma_2}(\emptyset)$, and, therefore, $\mathrm{SEN}(f)(\phi) \in C^s_{\Sigma_2}(\mathrm{SEN}(f)(\Gamma))$, i.e., C^s is also a closure system. ■

If $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ is a Gentzen π-institution, as above, the institution $\mathcal{G}^s = \langle \mathbf{Sign}, \mathrm{SEN}, \{C^s_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ will be called the **weak sentential counterpart** or the **theorem counterpart** of $\mathcal{G}$. The construction of $\mathcal{G}^s$ parallels the construction of the sentential logic $\langle \mathcal{L}, \vdash_{\mathfrak{G}}\rangle$, defined by a given Gentzen system $\mathfrak{G}$, in the theory of sentential logics (see Definition 4.2 of [12]). The name "sentential counterpart" was chosen because of this analogy. The same analogy motivates the definition of adequacy of a Gentzen π-institution $\mathcal{G}$ for a π-institution $\mathcal{I}$. Roughly speaking, it means that $\mathcal{G}$ defines $\mathcal{I}$ via its sentential counterpart.

Definition 3 *Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$, a Gentzen π-institution $\mathcal{G} = \langle \mathbf{Sign}', \mathcal{P}\mathrm{SEN}'^2, \{C'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}\rangle$ is said to be* **adequate for** $\mathcal{I}$ *if $\mathcal{I}$ is interpretable in $\mathcal{G}^s$, written $\mathcal{I} \vdash \mathcal{G}^s$.*

In the opposite way, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, one may construct a Gentzen closure on $\mathcal{P}\mathrm{SEN}^2$ reflecting some of the properties of C.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. Construct $\mathcal{I}^g = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C^g_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, C^g_Σ be the smallest closure operator on $\mathcal{P}\mathrm{SEN}(\Sigma)^2$ satisfying

- $\Gamma \vdash_\Sigma \Phi \in C^g_\Sigma(\emptyset)$, for all $\Gamma \cup \Phi \subseteq \mathrm{SEN}(\Sigma)$, such that $\Phi \subseteq C_\Sigma(\Gamma)$,
- $\Gamma \cup \Psi \vdash_\Sigma \Phi \in C^g_\Sigma(\Gamma \vdash_\Sigma \Phi)$, for all $\Gamma, \Psi, \Phi \subseteq \mathrm{SEN}(\Sigma)$,
- $\Gamma \vdash_\Sigma \Psi \in C^g_\Sigma(\Gamma \vdash_\Sigma \Phi, \Gamma \cup \Phi \vdash_\Sigma \Psi)$, for all $\Gamma, \Psi, \Phi \subseteq \mathrm{SEN}(\Sigma)$.

Proposition 4 *$\mathcal{I}^g$ is a Gentzen π-institution, for every π-institution $\mathcal{I}$.*

Proof:

Since, by definition, C^g_Σ is a closure operator, for every $\Sigma \in |\mathbf{Sign}|$, it suffices to show that C^g is a closure system on $\mathcal{P}\mathrm{SEN}^2$. But this follows from the definition of C^g_Σ and the fact that all closure conditions involved in that definition are structural, i.e., invariant under the application of $\mathcal{P}\mathrm{SEN}(f)^2$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$. ■

Given a π-institution $\mathcal{I}$, the Gentzen π-institution $\mathcal{I}^g$ will be called the **weak Gentzenization** of $\mathcal{I}$. In the next proposition, it is shown that it is adequate for $\mathcal{I}$.

Proposition 5 *Given a π-institution $\mathcal{I}$, the weak Gentzenization $\mathcal{I}^g$ of $\mathcal{I}$ is a Gentzen π-institution adequate for $\mathcal{I}$.*

Proof:

It must be shown that $\mathcal{I} \vdash \mathcal{I}^{gs}$. We show that $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{I} \to \mathcal{I}^{gs}$, where $\mathrm{I}_{\mathbf{Sign}}$ is the identity functor on $\mathbf{Sign}$ and $\iota : \mathrm{SEN} \to \mathrm{SEN}$ is the identity natural transformation, is an interpretation of $\mathcal{I}$ into $\mathcal{I}^{gs}$. It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \phi \in C^{gs}_\Sigma(\Phi).$$

Suppose $\phi \in C_\Sigma(\Phi)$. Then $\Phi \vdash_\Sigma \phi \in C^g_\Sigma(\emptyset)$, whence $\phi \in C^{gs}_\Sigma(\Phi)$. Suppose, conversely, that $\phi \in C^{gs}_\Sigma(\Phi)$. Then $\Phi \vdash_\Sigma \phi \in C^g_\Sigma(\emptyset)$. Now, note that $\{\Phi \vdash_\Sigma \Psi : \Psi \subseteq C_\Sigma(\Phi)\}$ is the smallest Σ-theory of $\mathcal{I}^g$, whence $C^g_\Sigma(\emptyset) = \{\Phi \vdash_\Sigma \Psi : \Psi \subseteq C_\Sigma(\Phi)\}$, and, therefore $\phi \in C_\Sigma(\Phi)$. ■

Out of a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, another construction of a Gentzen counterpart may also be carried out that reflects properties of the closure system C more closely than does the weak Gentzenization of $\mathcal{I}$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. Define the π-institution $\mathcal{I}^G = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C^G_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ by letting

$$C^G_\Sigma : \mathcal{P}(\mathcal{P}\mathrm{SEN}(\Sigma)^2) \to \mathcal{P}(\mathcal{P}\mathrm{SEN}(\Sigma)^2)$$

be defined, for all $\Sigma \in |\mathbf{Sign}|$, by $\vec{\Phi} \in C^G_\Sigma(\mathbf{\Gamma})$ iff, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\begin{array}{l} \mathrm{SEN}(f)(\Gamma_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Gamma_1)), \quad \text{for all} \quad \vec{\Gamma} = \langle \Gamma_1, \Gamma_2\rangle \in \mathbf{\Gamma}, \\ \text{implies} \quad \mathrm{SEN}(f)(\Phi_2) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi_1)). \end{array} \tag{1}$$

In Proposition 2.1 of [23], it is shown that $\mathcal{I}^G$ is a π-institution and it is called the (**strong**) **Gentzenization** of $\mathcal{I}$. Condition (1) will sometimes be abbreviated in the form

$$\mathrm{SEN}(f)(\mathbf{\Gamma}) \subseteq C_{\Sigma'} \quad \text{implies} \quad \mathrm{SEN}(f)(\vec{\Phi}) \in C_{\Sigma'}.$$

3 Models of Gentzen π-Institutions

Definition 6 *Suppose that* $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ *is a Gentzen* π*-institution. A* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}', \mathrm{SEN}', \{C'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}\rangle$ *is said to be a* **model** *of* $\mathcal{G}$ *if* $\mathcal{G}$ *is Gentzen semi-interpretable into the Gentzenization* $\mathcal{I}^G$ *of* $\mathcal{I}$*, in symbols* $\mathcal{G}\rangle\!\!-^G\mathcal{I}^G$.

Let $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a Gentzen π-institution. A collection $T = \{\langle\Sigma, T_\Sigma\rangle : \Sigma\in|\mathbf{Sign}|\}$ of theories of $\mathcal{G}$, such that

$$\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}, \quad \text{for all } \Sigma_1,\Sigma_2\in|\mathbf{Sign}|, f\in\mathbf{Sign}(\Sigma_1,\Sigma_2),$$

is called a **theory system of** $\mathcal{G}$.

Note that the collection $\mathrm{Th} = \{\langle\Sigma,\mathrm{Th}_\Sigma\rangle : \Sigma\in|\mathbf{Sign}|\}$, where $\mathrm{Th}_\Sigma = C_\Sigma(\emptyset)$ is the set of all Σ-theorems, is a theory system of $\mathcal{G}$, which will be called the **theorem system** of $\mathcal{G}$.

Given a theory system T of $\mathcal{G}$ as above, define a triple $\mathcal{G}^T = \langle\mathbf{Sign}, \mathrm{SEN}, \{C^T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ by setting, for all $\Sigma\in|\mathbf{Sign}|, \Gamma\subseteq\mathrm{SEN}(\Sigma)$,

$$C^T_\Sigma(\Gamma) = \{\phi\in\mathrm{SEN}(\Sigma) : \Gamma\vdash_\Sigma\phi\in T_\Sigma\}.$$

It is not difficult to check that $\mathcal{G}^T$, thus defined, is a π-institution.

Given a model $\mathcal{I} = \langle \mathbf{Sign}', \mathrm{SEN}', \{C'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}\rangle$ of a Gentzen π-institution $\mathcal{G}$ via the Gentzen semi-interpretation $\langle F,\alpha\rangle : \mathcal{G}\rangle\!\!-^G\mathcal{I}^G$, define

$$T^{\mathcal{I}}_\Sigma = \{\Gamma\vdash_\Sigma\Phi : \alpha_\Sigma(\Gamma\vdash_\Sigma\Phi)\subseteq C'^G_{F(\Sigma)}(\emptyset)\}, \quad \text{for all} \quad \Sigma\in|\mathbf{Sign}|,$$

and, let

$$T^{\mathcal{I}} = \{T^{\mathcal{I}}_\Sigma : \Sigma\in|\mathbf{Sign}|\}.$$

In the next result a way of obtaining a model of a Gentzen π-institution, given one of its theory systems, is exhibited. Conversely, it is also shown how a given model of a Gentzen π-institution gives rise to one of its theory systems.

Proposition 7 *Let* $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ *be a Gentzen* π*-institution.*

1. *If* $T = \{\langle\Sigma, T_\Sigma\rangle : \Sigma\in|\mathbf{Sign}|\}$ *is a theory system of* $\mathcal{G}$*,* $\mathcal{G}^T$ *is a model of* $\mathcal{G}$*.*

2. *Conversely, if* $\mathcal{I} = \langle \mathbf{Sign}', \mathrm{SEN}', \{C'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}\rangle$ *is a model of* $\mathcal{G}$*, then* $T^{\mathcal{I}}$ *is a theory system of* $\mathcal{G}$*.*

Proof:

1. It will be shown that the identity $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{G}\rangle\!\!-^G\mathcal{G}^{TG}$ is a Gentzen semi-interpretation of $\mathcal{G}$ into $\mathcal{G}^{TG}$. It suffices, to this end, to show that, for all $\Sigma\in|\mathbf{Sign}|, \mathbf{\Gamma}\cup\{\vec{\Phi}\}\subseteq\mathcal{P}\mathrm{SEN}(\Sigma)^2$,
$$\Phi_1\vdash_\Sigma\Phi_2\in C_\Sigma(\mathbf{\Gamma}) \quad \text{implies} \quad \Phi_1\vdash_\Sigma\Phi_2\in C^{TG}_\Sigma(\mathbf{\Gamma}).$$
We have $\vec{\Phi}\in C_\Sigma(\mathbf{\Gamma})$ implies, by structurality of C, that for all $\Sigma'\in|\mathbf{Sign}|, f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f)(\vec{\Phi})\in C_{\Sigma'}(\mathrm{SEN}(f)(\mathbf{\Gamma}))$, whence, if $\mathrm{SEN}(f)(\mathbf{\Gamma})\subseteq T_{\Sigma'}$, then we get that $\mathrm{SEN}(f)(\vec{\Phi})\in T_{\Sigma'}$. Thus, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f)(\mathbf{\Gamma})\subseteq C^T_{\Sigma'}$ implies $\mathrm{SEN}(f)(\vec{\Phi})\in C^T_{\Sigma'}$, i.e., $\Phi_1\vdash_\Sigma\Phi_2\in C^{TG}_\Sigma(\mathbf{\Gamma})$.

2. Now suppose that $\mathcal{I} = \langle \mathbf{Sign}', \mathrm{SEN}', \{C'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}\rangle$ is a model of $\mathcal{G}$ via the Gentzen semi-interpretation $\langle F, \alpha\rangle : \mathcal{G} \rangle\!\!-^G \mathcal{I}^G$. Clearly, $T^{\mathcal{I}}_\Sigma \subseteq \mathcal{P}\mathrm{SEN}(\Sigma)^2$. Suppose that $\vec{\Phi} \in C_\Sigma(T^{\mathcal{I}}_\Sigma)$. Then $\alpha^2_\Sigma(\vec{\Phi}) \subseteq C'^G_{F(\Sigma)}(\alpha^2_\Sigma(T^{\mathcal{I}}_\Sigma))$. Thus $\alpha^2_\Sigma(\vec{\Phi}) \subseteq C'^G_{F(\Sigma)}(\emptyset)$, and, hence, $\vec{\Phi} \in T^{\mathcal{I}}_\Sigma$, which proves that $T^{\mathcal{I}}_\Sigma$ is a Σ-theory, for every $\Sigma \in |\mathbf{Sign}|$.

   Next suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$. Then

$$\mathrm{SEN}(f)(T^{\mathcal{I}}_{\Sigma_1}) =$$

$$\begin{array}{cl} = & \mathrm{SEN}(f)(\{\vec{\Phi} : \alpha^2_{\Sigma_1}(\vec{\Phi}) \subseteq C'^G_{F(\Sigma_1)}(\emptyset)\}) \\ = & \{\mathrm{SEN}(f)(\vec{\Phi}) : \alpha^2_{\Sigma_1}(\vec{\Phi}) \subseteq C'^G_{F(\Sigma_1)}(\emptyset)\} \\ \subseteq & \{\mathrm{SEN}(f)(\vec{\Phi}) : \mathrm{SEN}'(F(f))(\alpha^2_{\Sigma_1}(\vec{\Phi})) \subseteq C'^G_{F(\Sigma_2)}(\emptyset)\} \end{array}$$

$$\begin{tikzcd} \mathcal{P}\mathrm{SEN}(\Sigma_1)^2 \arrow[r, "\alpha_{\Sigma_1}"] \arrow[d, "\mathcal{P}\mathrm{SEN}(f)^2"'] & \mathcal{P}(\mathcal{P}\mathrm{SEN}'(F(\Sigma_1))^2) \arrow[d, "\mathcal{P}\mathrm{SEN}'(F(f))^2"] \\ \mathcal{P}\mathrm{SEN}(\Sigma_2)^2 \arrow[r, "\alpha_{\Sigma_2}"'] & \mathcal{P}(\mathcal{P}\mathrm{SEN}'(F(\Sigma_2))^2) \end{tikzcd}$$

$$\begin{array}{cl} = & \{\mathrm{SEN}(f)(\vec{\Phi}) : \alpha^2_{\Sigma_2}(\mathrm{SEN}(f)(\vec{\Phi})) \subseteq C'^G_{F(\Sigma_2)}(\emptyset)\} \\ \subseteq & \{\vec{\Psi} : \alpha^2_{\Sigma_2}(\vec{\Psi}) \subseteq C'^G_{F(\Sigma_2)}(\emptyset)\} \\ = & T^{\mathcal{I}}_{\Sigma_2}. \end{array}$$

   Therefore $T^{\mathcal{I}} = \{T^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is indeed a theory system of $\mathcal{G}$.

■

It is now shown that a surjective singleton interpretation between two π-institutions induces a Gentzen interpretation between their Gentzenizations.

Lemma 8 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be two π-institutions and $\langle F, \alpha\rangle : \mathcal{I} \vdash^s \mathcal{I}'$ be a surjective singleton interpretation. Then $\langle F, \alpha\rangle : \mathcal{I}^G \rightarrow^G \mathcal{I}'^G$ is a Gentzen interpretation $\langle F, \alpha\rangle : \mathcal{I}^G \vdash^G \mathcal{I}'^G$.*

Proof:
Suppose that $\Sigma \in |\mathbf{Sign}|, \mathbf{\Phi} \cup \{\vec{\Phi}\} \subseteq \mathcal{P}\mathrm{SEN}(\Sigma)^2$. Then $\vec{\Phi} \in C^G_\Sigma(\mathbf{\Phi})$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\mathbf{\Phi}) \subseteq C_{\Sigma'} \quad \text{implies} \quad \mathrm{SEN}(f)(\vec{\Phi}) \in C_{\Sigma'}$$

iff, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\mathbf{\Phi})) \subseteq C'_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\Phi})) \in C'_{F(\Sigma')}$$

iff, by commutativity of

$$\begin{tikzcd} \mathrm{SEN}(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'(F(\Sigma)) \arrow[d, "\mathrm{SEN}'(F(f))"] \\ \mathrm{SEN}(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}'(F(\Sigma')) \end{tikzcd}$$

for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}'(F(f))(\alpha_\Sigma(\mathbf{\Phi})) \subseteq C'_{F(\Sigma')} \quad \text{implies} \quad \mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\Phi})) \in C'_{F(\Sigma')}$$

iff, by surjectivity, for all $\Sigma' \in |\mathbf{Sign}'|, f' \in \mathbf{Sign}'(F(\Sigma), \Sigma')$,

$$\mathrm{SEN}'(f')(\alpha_\Sigma(\mathbf{\Phi})) \subseteq C'_{\Sigma'} \quad \text{implies} \quad \mathrm{SEN}'(f')(\alpha_\Sigma(\vec{\Phi})) \in C'_{\Sigma'}$$

iff $\alpha_\Sigma(\vec{\Phi}) \in C'^{G}_{F(\Sigma)}(\alpha_\Sigma(\mathbf{\Phi}))$. ■

Finally, it is shown how the passage from theory systems to models and from models to theory systems of Proposition 7 relates models and theories of a given Gentzen π-institution very closely. This result forms a partial analog of Proposition 4.4 of [12] for π-institutions.

Proposition 9 *Let $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a Gentzen π-institution.*

1. *If $\mathcal{I} = \langle \mathbf{Sign}', \mathrm{SEN}', \{C'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}\rangle$ is a model of $\mathcal{G}$, then $T^{\mathcal{I}}$ is a theory system of $\mathcal{G}$ and $\mathcal{G}^{T^{\mathcal{I}}} \vdash \mathcal{I}$.*

   If, conversely, $T^{\mathcal{I}}$ is a theory system of $\mathcal{G}$ and $\mathcal{G}^{T^{\mathcal{I}}} \vdash \mathcal{I}$ via a surjective singleton interpretation, then $\mathcal{I}$ is a model of $\mathcal{G}$.

2. *T is a theory system of $\mathcal{G}$ if and only if $\mathcal{G}^T$ is a model of $\mathcal{G}$ and $T = T^{\mathcal{G}^T}$.*

Proof:

1. Suppose $\mathcal{I} = \langle \mathbf{Sign}', \mathrm{SEN}', \{C'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}\rangle$ is a model of $\mathcal{G}$, i.e., there exists a Gentzen semi-interpretation $\alpha : \mathcal{G} \rangle\!\!-^{G} \mathcal{I}^G$. Then, by Proposition 7, Part 1, $T^{\mathcal{I}}$ is a theory system of $\mathcal{G}$. Thus, it suffices to show that $\mathcal{G}^{T^{\mathcal{I}}} \vdash \mathcal{I}$. In fact, it is shown that $\alpha : \mathcal{G}^{T^{\mathcal{I}}} \vdash \mathcal{I}$.

   To this end, let $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then we have

   $$\begin{array}{lll} \phi \in C_\Sigma^{T^{\mathcal{I}}}(\Phi) & \text{iff} & \Phi \vdash_\Sigma \phi \in T_\Sigma^{\mathcal{I}} \\ & \text{iff} & \alpha_\Sigma(\Phi \vdash_\Sigma \phi) \subseteq C'^{G}_{F(\Sigma)}(\emptyset) \\ & \text{iff} & \alpha_\Sigma(\phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \end{array}$$

   Suppose, conversely, that $T^{\mathcal{I}}$ is a theory system of $\mathcal{G}$ and that $\mathcal{G}^{T^{\mathcal{I}}} \vdash \mathcal{I}$ via a surjective singleton interpretation. Then, by Proposition 7, Part 1, $\mathcal{G}^{T^{\mathcal{I}}}$ is a model of $\mathcal{G}$, whence $\mathcal{G} \rangle\!\!-^{G} \mathcal{G}^{T^{\mathcal{I}}G} \vdash^G \mathcal{I}^G$, where the Gentzen interpretation involved is provided by Lemma 8. Therefore $\mathcal{I}$ is a model of $\mathcal{G}$.

2. Suppose that T is a theory system of $\mathcal{G}$. Then, by Proposition 7, Part 1, we have that $\mathcal{G}^T$ is a model of $\mathcal{G}$ via the identity Gentzen semi-interpretation $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{G} \rangle\!\!-^{G} \mathcal{G}^{TG}$, whence, it suffices to show that $T = T^{\mathcal{G}^T}$. We have

   $$\begin{array}{lll} \Phi_1 \vdash_\Sigma \Phi_2 \in T_\Sigma^{\mathcal{G}^T} & \text{iff} & \Phi_1 \vdash_\Sigma \Phi_2 \in C_\Sigma^{TG}(\emptyset) \\ & \text{iff} & \Phi_2 \subseteq C_\Sigma^T(\Phi_1) \\ & \text{iff} & \Phi_1 \vdash_\Sigma \Phi_2 \in T_\Sigma. \end{array}$$

   Suppose, conversely, that $\mathcal{G}^T$ is a model of $\mathcal{G}$ and $T = T^{\mathcal{G}^T}$. Then , by Proposition 7, Part 2, $T^{\mathcal{G}^T}$ is a theory system of $\mathcal{G}$, whence T is also a theory system of $\mathcal{G}$.

■

The following proposition shows that the weak sentential counterpart $\mathcal{G}^s$ of a given Gentzen π-institution $\mathcal{G}$ is identical with $\mathcal{G}^{\rm Th}$ and characterizes the closure system of $\mathcal{G}$ in terms of the closure systems of its models.

Proposition 10 *Let $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}{\rm SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a Gentzen π-institution.*

1. *The weak sentential counterpart $\mathcal{G}^s$ of $\mathcal{G}$ is a model of $\mathcal{G}$ and it coincides with $\mathcal{G}^{\rm Th}$.*
2. *$\vec{\Phi} \in C_\Sigma(\mathbf{\Phi})$ if and only if, for all $\mathcal{I} = \langle \mathbf{Sign}', {\rm SEN}', \{C'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}\rangle$, with $\langle F, \alpha\rangle : \mathcal{G}\rangle\!\!-^G \mathcal{I}^G$, and all $\Sigma' \in |\mathbf{Sign}'|, f \in \mathbf{Sign}'(F(\Sigma), \Sigma')$,*

$$ {\rm SEN}(f)(\alpha_\Sigma(\mathbf{\Phi})) \subseteq C'^G_{\Sigma'} \quad \textit{implies} \quad {\rm SEN}(f)(\alpha_\Sigma(\vec{\Phi})) \subseteq C'^G_{\Sigma'}. $$

Proof:

1. That $\mathcal{G}^s = \mathcal{G}^{\rm Th}$ is obvious, since, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq {\rm SEN}(\Sigma)$,

$$ \begin{array}{lll} \phi \in C^s_\Sigma(\Phi) & \text{iff} & \Phi \vdash_\Sigma \phi \in C_\Sigma(\emptyset) \\ & \text{iff} & \Phi \vdash_\Sigma \phi \in {\rm Th}_\Sigma \\ & \text{iff} & \phi \in C^{\rm Th}_\Sigma(\Phi). \end{array} $$

   $\mathcal{G}^{\rm Th}$ is a model of $\mathcal{G}$, by Proposition 9, Part 2.

2. The only if is obvious by the definition of a model. The if part follows by Part 1 if one uses the model $\mathcal{G}^s$ and the identity interpretation $\langle {\rm I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{G} \vdash^G \mathcal{G}^{sG}$.

■

Finally, it is shown that every model of a Gentzen π-institution $\mathcal{G}$, adequate for a π-institution $\mathcal{I}$, is also a model of $\mathcal{I}$. To this end, two preliminary lemmas are needed. The first asserts that a Gentzen semi-interpretation from a Gentzen π-institution $\mathcal{G}$ to a Gentzen π-institution $\mathcal{G}'$ induces a semi-interpretation from $\mathcal{G}^s$ into $\mathcal{G}'^s$.

Lemma 11 *Suppose that $\mathcal{G}, \mathcal{G}'$ are Gentzen π-institutions and that $\langle F, \alpha\rangle : \mathcal{G}\rangle\!\!-^G \mathcal{G}'$ a Gentzen semi-interpretation. Then $\langle F, \alpha\rangle : \mathcal{G}^s\rangle\!\!- \mathcal{G}'^s$ is a (ordinary) semi-interpretation.*

Proof:

Let $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}{\rm SEN}^2, C\rangle$ and $\mathcal{G}' = \langle \mathbf{Sign}', \mathcal{P}{\rm SEN}'^2, C'\rangle$ be two Gentzen π-institutions and $\langle F, \alpha\rangle : \mathcal{G}\rangle\!\!-^G \mathcal{G}'$ be a Gentzen semi-interpretation. Consider $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq {\rm SEN}(\Sigma)$. We have

$$ \begin{array}{lcl} \phi \in C^s_\Sigma(\Phi) & \text{iff} & \Phi \vdash_\Sigma \phi \in C_\Sigma(\emptyset) \\ & \text{implies} & \alpha_\Sigma(\Phi \vdash_\Sigma \phi) \in C'_{F(\Sigma)}(\emptyset) \\ & \text{iff} & \alpha_\Sigma(\Phi) \vdash'_{F(\Sigma)} \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\emptyset) \\ & \text{iff} & \alpha_\Sigma(\phi) \subseteq C'^s_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \end{array} $$

Therefore $\langle F, \alpha\rangle : \mathcal{G}^s\rangle\!\!- \mathcal{G}'^s$ is indeed a semi-interpretation. ■

The second auxiliary lemma asserts that, for every π-institution $\mathcal{I}$, we have that $\mathcal{I}^{Gs} = \mathcal{I}$.

Lemma 12 *$\mathcal{I}^{Gs} = \mathcal{I}$, for every π-institution $\mathcal{I}$.*

Proof:
Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then

$$\begin{array}{rll} \phi \in C^{Gs}_\Sigma(\Phi) & \text{iff} & \Phi \vdash_\Sigma \phi \in C^G_\Sigma(\emptyset) \\ & \text{iff} & \mathrm{SEN}(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \end{array}$$

for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, iff $\phi \in C_\Sigma(\Phi)$. Therefore $\mathcal{I}^{Gs} = \mathcal{I}$. ■

Finally, the main result, showing that every model of a Gentzen π-institution $\mathcal{G}$, adequate for a π-institution $\mathcal{I}$, is also a model of $\mathcal{I}$, is presented.

Proposition 13 *Let $\mathcal{G}$ be a Gentzen π-institution adequate for the π-institution $\mathcal{I}$. Then, every model $\mathcal{M}$ of $\mathcal{G}$ is a model of $\mathcal{I}$, i.e., $\mathcal{G}\rangle\!\!-^G\mathcal{M}^G$ implies $\mathcal{I}\rangle\!\!-\mathcal{M}$.*

Proof:
Since $\mathcal{G}$ is adequate for $\mathcal{I}$, we have, by definition, $\mathcal{I} \vdash \mathcal{G}^s$. Now suppose that $\mathcal{M}$ is a model of $\mathcal{G}$, i.e., $\mathcal{G}\rangle\!\!-^G\mathcal{M}^G$. Then, by Lemma 11, $\mathcal{G}^s\rangle\!\!-\mathcal{M}^{Gs}$, and, therefore, $\mathcal{I}\rangle\!\!-\mathcal{M}^{Gs}$. But, by Lemma 12, $\mathcal{M}^{Gs} = \mathcal{M}$, whence, $\mathcal{I}\rangle\!\!-\mathcal{M}$. ■

4 Models, Algebraic Systems and Full Adequacy It is now shown that, for a π-institution, the property of being a model of a Gentzen π-institution is invariant (in a restricted sense) under surjective singleton interpretations. As a corollary, it follows that the property of being a model is invariant under appropriate bilogical morphisms.

Proposition 14 *Let $\mathcal{G}$ be a Gentzen π-institution, $\mathcal{I}', \mathcal{I}''$ two π-institutions, $\langle M, \mu\rangle : \mathcal{G} \to^G \mathcal{I}'^G, \langle K, \kappa\rangle : \mathcal{G} \to^G \mathcal{I}''^G$ two Gentzen translations and $\langle F, \alpha\rangle : \mathcal{I}' \to \mathcal{I}''$ a surjective singleton interpretation, that make the following triangle commute:*

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "\langle M,\mu\rangle"'] \arrow[dr, "\langle K,\kappa\rangle"] & \\ \mathcal{I}'^G \arrow[rr, "\langle F,\alpha\rangle"'] & & \mathcal{I}''^G \end{tikzcd}$$

Then $\mathcal{I}'$ is a model of $\mathcal{G}$ via $\langle M, \mu\rangle$ if and only if $\mathcal{I}''$ is a model of $\mathcal{G}$ via $\langle K, \kappa\rangle$.

Proof:
First, by Lemma 8, the interpretation $\langle F, \alpha\rangle : \mathcal{I}' \vdash \mathcal{I}''$ induces a Gentzen interpretation $\langle F, \alpha\rangle : \mathcal{I}'^G \vdash^G \mathcal{I}''^G$.

Now it is obvious that if $\langle M, \mu\rangle : \mathcal{G}\rangle\!\!-^G\mathcal{I}'^G$ and $\langle F, \alpha\rangle : \mathcal{I}'^G \vdash^G \mathcal{I}''^G$, then $\langle K, \kappa\rangle = \langle F, \alpha\rangle\langle M, \mu\rangle : \mathcal{G}\rangle\!\!-^G\mathcal{I}''^G$.

If, conversely, $\langle K, \kappa\rangle : \mathcal{G}\rangle\!\!-^G\mathcal{I}''^G$ and $\langle F, \alpha\rangle : \mathcal{I}'^G \vdash^G \mathcal{I}''^G$, then we have, for all $\Sigma \in |\mathbf{Sign}|, \mathbf{\Phi} \cup \{\vec{\Phi}\} \subseteq \mathcal{P}\mathrm{SEN}(\Sigma)^2$,

$$\begin{array}{rll} \vec{\Phi} \in C_\Sigma(\mathbf{\Phi}) & \text{implies} & \alpha_{M(\Sigma)}(\mu_\Sigma(\vec{\Phi})) \subseteq C''^G_{F(M(\Sigma))}(\alpha_{M(\Sigma)}(\mu_\Sigma(\mathbf{\Phi}))) \\ & \text{iff} & \mu_\Sigma(\vec{\Phi}) \subseteq C'^G_{M(\Sigma)}(\mu_\Sigma(\mathbf{\Phi})), \end{array}$$

whence $\langle M, \mu\rangle : \mathcal{G}\rangle\!\!-^G\mathcal{I}'^G$. ■

Recall from [24] the definition of a category of natural transformations: Given a category $\mathbf{Sign}$ and a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ the clone of all natural transformations on SEN is the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}$.

A subcategory of this category containing all objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, is referred to as a category of natural transformations on SEN.

Moreover, given two π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ and categories of natural transformations N, N', respectively, on SEN, SEN$'$, a singleton translation (semi-interpretation or interpretation) $\langle F, \alpha\rangle$ from $\mathcal{I}$ to $\mathcal{I}'$ is said to be (N, N')-homomorphic if, for every natural transformation $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, there exists a natural transformation $\sigma : \mathrm{SEN}'^k \to \mathrm{SEN}'$ in N', such that, for every $\Sigma \in |\mathbf{Sign}|$ and every $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\begin{array}{ccc} \mathrm{SEN}(\Sigma)^k & \xrightarrow{\alpha_\Sigma^k} & \mathrm{SEN}'(F(\Sigma))^k \\ {\scriptstyle\tau_\Sigma}\downarrow & & \downarrow{\scriptstyle\sigma_{F(\Sigma)}} \\ \mathrm{SEN}(\Sigma) & \xrightarrow[\alpha_\Sigma]{} & \mathrm{SEN}'(F(\Sigma)) \end{array}$$

$$\alpha_\Sigma(\tau_\Sigma(\vec{\phi})) = \sigma_{F(\Sigma)}(\alpha_\Sigma^k(\vec{\phi})). \tag{2}$$

It is said to be (N, N')-epimorphic if it is (N, N')-homomorphic and, in addition, for every $\sigma : \mathrm{SEN}'^k \to \mathrm{SEN}'$ in N', there exists $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, such that Equation (2) holds, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^k$. Corresponding notions are transferred in a straightforward way to Gentzen translations (semi-interpretations or interpretations).

Finally, recall from [25], Lemma 4.2, that a π-institution $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a model of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ if $\mathcal{I}$ is semi-interpretable in $\mathcal{I}'$. If N, N' are categories of natural transformations on SEN, SEN$'$, respectively and $\mathcal{I}'$ is a model of $\mathcal{I}$ via an (N, N')-epimorphic semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \rightarrowtail \mathcal{I}'$, then $\mathcal{I}'$ is said to be an (N, N')-model of $\mathcal{I}$.

Combining the statement of Proposition 14 with the relevant definitions, we easily obtain

Corollary 15 *Let $\mathcal{G}$ be a Gentzen π-institution, $\mathcal{I}', \mathcal{I}''$ two π-institutions, and N, N', N'' categories of natural transformations on* SEN, SEN$'$, SEN$''$, *respectively. Suppose, also, that $\langle M, \mu\rangle : \mathcal{G} \to^G \mathcal{I}'^G$ is an (N, N')-epimorphic Gentzen translation, $\langle K, \kappa\rangle : \mathcal{G} \to^G \mathcal{I}''^G$ an (N, N'')-epimorphic Gentzen translation and $\langle F, \alpha\rangle : \mathcal{I}' \to \mathcal{I}''$ an (N', N'')-bilogical morphism, such that the following triangle commutes:*

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "\langle M,\mu\rangle"'] \arrow[dr, "\langle K,\kappa\rangle"] & \\ \mathcal{I}'^G \arrow[rr, "\langle F,\alpha\rangle"'] & & \mathcal{I}''^G \end{tikzcd}$$

Then $\mathcal{I}'$ is an (N, N')-model of $\mathcal{G}$ via $\langle M, \mu\rangle$ if and only if $\mathcal{I}''$ is an (N, N'')-model of $\mathcal{G}$ via $\langle K, \kappa\rangle$.

Recall, now, from [27] the definition of an $(\mathcal{I}, N)$-algebraic system. Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a category N of natural transformations on SEN, a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ was said to be an $(\mathcal{I}, N)$-algebraic system if there exists a category N' of natural transformations on SEN$'$ and a singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min model $\mathcal{I}'$ of $\mathcal{I}$ on SEN$'$ is N'-reduced. In analogy with that definition, the notion of a $(\mathcal{G}, N)$-algebraic system, for a Gentzen π-institution $\mathcal{G}$, is now introduced.

Definition 16 *Let $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a Gentzen π-institution,* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *a functor and N, N' categories of natural transformations on* SEN, SEN$'$, *respectively.* SEN' *will be said to be a* $(\mathcal{G}, N)$-**algebraic system** *if there exists a closure system* C' *on* SEN' *such that* $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ *is an* N'*-reduced* (N, N')*-model of* $\mathcal{G}$.

The collection of all $(\mathcal{G}, N)$*-algebraic systems of* $\mathcal{G}$ *is denoted by* $\mathrm{Alg}^N(\mathcal{G})$.

Recall from Definition 3 that a Gentzen π-institution $\mathcal{G}$ was said to be adequate for a π-institution $\mathcal{I}$ if $\mathcal{I}$ is interpretable in $\mathcal{G}^s$. If the interpretation is an (N, N')-epimorphic interpretation, for some categories N, N' of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, then $\mathcal{G}$ will be said to be (N, N')-adequate for $\mathcal{I}$.

It will now be shown that, if $\mathcal{G}'$ is a Gentzen π-institution (N, N')-adequate for a π-institution $\mathcal{I}$, then every (N', N'')-model of $\mathcal{G}'$ is an (N, N'')-model of $\mathcal{I}$ and, as a consequence, the class of all $(\mathcal{G}', N')$-algebraic systems forms a subclass of all $(\mathcal{I}, N)$-algebraic systems. To this end a technical lemma is needed, showing that a Gentzen semi-interpretation between two Gentzen π-institutions lifts to a logical morphism between their weak sentential counterparts.

Lemma 17 *Suppose that* $\mathcal{G} = \langle \mathbf{Sign}, \mathcal{P}\mathrm{SEN}^2, C \rangle, \mathcal{G}' = \langle \mathbf{Sign}', \mathcal{P}\mathrm{SEN}'^2, C' \rangle$ *are two Gentzen* π*-institutions and suppose that* $\langle F, \alpha \rangle : \mathcal{G} \rangle\!\!-^G \mathcal{G}'$ *is a Gentzen* (N, N')*-semi-interpretation. Then* $\langle F, \alpha \rangle : \mathcal{G}^s \rangle\!\!-^{se} \mathcal{G}'^s$ *is an* (N, N')*-logical morphism.*

Proof:
Very similar to the proof of Lemma 11. ■

Proposition 18 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *be a* π*-institution,* $\mathcal{G}' = \langle \mathbf{Sign}', \mathcal{P}\mathrm{SEN}'^2, C' \rangle$ *a Gentzen* π*-institution and* N, N' *categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$, *respectively. If* $\mathcal{G}'$ *is* (N, N')*-adequate for* $\mathcal{I}$*, then*

1. *every* (N', N'')*-model of* $\mathcal{G}'$ *is an* (N, N'')*-model of* $\mathcal{I}$ *and*
2. $\mathrm{Alg}^{N'}(\mathcal{G}') \subseteq \mathrm{Alg}^N(\mathcal{I})$.

Proof:

1. Suppose that $\mathcal{I}''$ is an (N', N'')-model of $\mathcal{G}'$. Then, by definition, we have that $\mathcal{G}' \rangle\!\!-^G \mathcal{I}''^G$, whence, by Lemma 17, $\mathcal{G}'^s \rangle\!\!-^{se} \mathcal{I}''^{Gs}$, i.e., $\mathcal{G}'^s \rangle\!\!-^{se} \mathcal{I}''$. Since $\mathcal{G}'$ is adequate for $\mathcal{I}$, we obtain, by definition, that $\mathcal{I} \vdash^{se} \mathcal{G}'^s$. Combining the previous two relations, we get $\mathcal{I} \rangle\!\!-^{se} \mathcal{I}''$, whence $\mathcal{I}''$ is an (N, N'')-model of $\mathcal{I}$.
2. Suppose that $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set} \in \mathrm{Alg}^{N'}(\mathcal{G})$ via $\langle F, \alpha \rangle : \mathcal{G} \rangle\!\!-^G \langle \mathbf{Sign}'', \mathrm{SEN}'', C'' \rangle^G$, where $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C'' \rangle$ is N''-reduced. Then, by Part 1, $\mathcal{I} \rangle\!\!-^{se} \mathcal{I}''$ and $\mathcal{I}''$ is N''-reduced. Therefore, by Proposition 6 of [27], $\mathrm{SEN}'' \in \mathrm{Alg}^N(\mathcal{I})$.

■

Next, the definition of full adequacy of a Gentzen π-institution $\mathcal{G}'$ for a π-institution $\mathcal{I}$ is formulated. In the proposition following the definition, it will be shown that this is indeed a stronger notion than that of adequacy (justifying the name full adequacy).

Definition 19 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *be a* π*-institution,* $\mathcal{G}' = \langle \mathbf{Sign}', \mathcal{P}\mathrm{SEN}'^2, C' \rangle$ *a Gentzen* π*-institution and* N, N' *categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$*, respectively.* $\mathcal{G}'$ *is said to be* (N, N')-**fully adequate** *for* $\mathcal{I}$ *if, there exists a singleton* (N, N')*-epimorphic translation* $\langle F, \alpha \rangle : \mathcal{I} \to^{se} \mathcal{G}'^s$*, such that, for every* π*-institution* $\mathcal{I}''$*,*

- *if $\mathcal{I}''$ is an (N', N'')-model of $\mathcal{G}'$ via $\langle M, \mu\rangle : \mathcal{G}' \rangle\!\!-^G \mathcal{I}''^G$, then $\mathcal{I}''$ is an (N, N'')-full model of $\mathcal{I}$ via $\langle M, \mu\rangle\langle F, \alpha\rangle$,*

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "\langle F,\alpha\rangle"] \arrow[dr, "\langle M,\mu\rangle\langle F,\alpha\rangle"'] & & \mathcal{G}'^s \arrow[dl, "\langle M,\mu\rangle"] \\ & \mathcal{I}'' & \end{tikzcd}$$

- *if $\mathcal{I}''$ is an (N, N'')-full model of $\mathcal{I}$ via $\langle K, \kappa\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}''$, then $\mathcal{I}''$ is a model of $\mathcal{G}'$ via some $\langle M, \mu\rangle : \mathcal{G}' \rangle\!\!-^G \mathcal{I}''^G$, such that $\langle K, \kappa\rangle = \langle M, \mu\rangle\langle F, \alpha\rangle$.*

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "\langle F,\alpha\rangle"] \arrow[dr, "\langle K,\kappa\rangle"'] & & \mathcal{G}'^s \arrow[dl, "\langle M,\mu\rangle"] \\ & \mathcal{I}'' & \end{tikzcd}$$

Proposition 20 shows that, if a Gentzen π-institution is fully adequate for a π-institution $\mathcal{I}$, then it is adequate for $\mathcal{I}$.

Proposition 20 *If $\mathcal{G}'$ is a Gentzen π-institution (N, N')-fully adequate for a π-institution $\mathcal{I}$, then $\mathcal{G}'$ is (N, N')-adequate for $\mathcal{I}$.*

Proof:

Suppose that $\mathcal{G}'$ is fully adequate for $\mathcal{I}$ via the singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathcal{G}'^s$. It suffices to show that $\langle F, \alpha\rangle$ is an interpretation. Recall that, by Proposition 10, $\mathcal{G}'^s$ is a model of $\mathcal{G}'$ via $\langle \mathrm{I}_{\mathbf{Sign}'}, \iota\rangle : \mathcal{G}' \rangle\!\!-^G \mathcal{G}'^{sG}$ whence, by the definition of full adequacy, $\mathcal{G}'^s$ is an (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{G}'^s$. Finally, to show that this semi-interpretation is an interpretation, consider $\mathcal{I}$ as an (N, N)-full model of $\mathcal{I}$ via $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{I} \vdash^{se} \mathcal{I}$. Then, by full adequacy, there exists $\langle M, \mu\rangle : \mathcal{G}' \rangle\!\!-^G \mathcal{I}^G$, such that $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle = \langle M, \mu\rangle\langle F, \alpha\rangle$. Hence,

$$\begin{array}{ll} \multicolumn{2}{l}{\alpha_\Sigma(\phi) \in C'^s_{F(\Sigma)}(\alpha_\Sigma(\Phi))} \\ \text{iff} & \alpha_\Sigma(\Phi) \vdash'_{F(\Sigma)} \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\emptyset) \\ \text{implies} & \mu_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \vdash^G_{M(F(\Sigma))} \mu_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in C^G_{M(F(\Sigma))}(\emptyset) \\ \text{iff} & \Phi \vdash^G_\Sigma \phi \in C^G_\Sigma(\emptyset) \\ \text{iff} & \phi \in C_\Sigma(\Phi) \end{array}$$

and $\langle F, \alpha\rangle : \mathcal{I} \vdash \mathcal{G}^s$ is an interpretation. ■

Finally, it is shown that (N, N')-full adequacy is equivalent to a strong correspondence between $(\mathcal{I}, N)$-algebraic systems and $(\mathcal{G}', N')$-algebraic systems.

Proposition 21 *Let $\mathcal{G}' = \langle \mathbf{Sign}', \mathcal{P}\mathrm{SEN}'^2, C'\rangle$ be a Gentzen π-institution, $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ a π-institution and N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively. $\mathcal{G}'$ is fully adequate for $\mathcal{I}$ via the singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathcal{G}'^s$ iff, for all functors SEN'' and categories of natural transformations N'' on SEN'',*

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "\langle F,\alpha\rangle"] \arrow[dr, "\langle M,\mu\rangle\langle F,\alpha\rangle"'] & & \mathcal{G}'^s \arrow[dl, "\langle M,\mu\rangle"] \\ & \mathrm{SEN}'' & \end{tikzcd}$$

$\mathrm{SEN}'' \in \mathrm{Alg}^{N'}(\mathcal{G}')$ *via the* (N', N'')*-epimorphic translation* $\langle M, \mu\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ *if and only if* $\mathrm{SEN}'' \in \mathrm{Alg}^{N}(\mathcal{I})$ *via the* (N, N'')*-epimorphic translation* $\langle M, \mu\rangle\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}''$.

Proof:

First, suppose that $\mathcal{G}'$ is fully adequate for $\mathcal{I}$ via the singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathcal{G}'^{s}$. Then $\mathrm{SEN}'' \in \mathrm{Alg}^{N'}(\mathcal{G}')$ via the (N', N'')-epimorphic translation $\langle M, \mu\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ if and only if there exists a closure system C'' on SEN'', such that $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ is an N''-reduced (N, N'')-model of $\mathcal{G}'$ via $\langle M, \mu\rangle : \mathcal{G}' \succ^{G} \mathcal{I}''^{G}$ if and only if, by the definition of full adequacy, $\mathcal{I}''$ is an N''-reduced (N', N'')-full model of $\mathcal{I}$ via $\langle M, \mu\rangle\langle F, \alpha\rangle : \mathcal{I} \succ \mathcal{I}''$ if and only if $\mathrm{SEN}'' \in \mathrm{Alg}^{N}(\mathcal{I})$ via $\langle M, \mu\rangle\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}''$.

Suppose, conversely, that $\mathrm{SEN}'' \in \mathrm{Alg}^{N'}(\mathcal{G}')$ via the (N', N'')-epimorphic translation $\langle M, \mu\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ if and only if $\mathrm{SEN}'' \in \mathrm{Alg}^{N}(\mathcal{I})$ via the (N, N'')-epimorphic translation $\langle M, \mu\rangle\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}''$.

If $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ is an (N', N'')-model of $\mathcal{G}'$ via $\langle M, \mu\rangle : \mathcal{G}' \succ^{G} \mathcal{I}''^{G}$, then $\mathcal{I}''^{N''}$ is an $\overline{N''}$-reduced $(N', \overline{N''})$-model of $\mathcal{G}'$ via $\langle M, \pi_M^{N''}\mu\rangle$, whence $\mathcal{I}''^{N''}$ is an $\overline{N''}$-reduced $(N, \overline{N''})$-full model of $\mathcal{I}$ via $\langle F, \alpha\rangle\langle M, \pi_M^{N''}\mu\rangle$ and, therefore, by Corollary 5.10 of [25], $\mathcal{I}''$ is an (N, N'')-full model of $\mathcal{I}$ via $\langle M, \mu\rangle\langle F, \alpha\rangle$.

If $\mathcal{I}''$ is an (N, N'')-full model of $\mathcal{I}$ via $\langle K, \kappa\rangle = \langle M, \mu\rangle\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}''$, then $\mathcal{I}''^{N''}$ is an $\overline{N''}$-reduced $(N, \overline{N''})$-full model of $\mathcal{I}$ via $\langle M, \pi_M^{N''}\mu\rangle\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}''^{N''}$, whence $\mathcal{I}''^{N''}$ is an $\overline{N''}$-reduced $(N', \overline{N''})$-model of $\mathcal{G}'$ via $\langle M, \pi_M^{N''}\mu\rangle : \mathcal{G}' \succ^{G} \mathcal{I}''^{N''G}$ and, therefore, by Corollary 15, $\mathcal{I}''$ is an (N', N'')-model of $\mathcal{G}'$ via some $\langle M, \mu\rangle : \mathcal{G}' \succ^{G} \mathcal{I}''^{G}$. ■

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